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Periodic waveguides revisited: Radiation conditions, limiting absorption principles, and the space of bounded solutions

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We study the Helmholtz equation with periodic coefficients in a closed waveguide. A functional analytic approach is used to formulate and solve the radiation problem in a self-contained exposition. In this context, we simplify the non-degeneracy assumption on the frequency. Limiting absorption principles (LAPs) are studied, and the radiation condition corresponding to the chosen LAP is derived; we include an example to show different LAPs lead, in general, to different solutions of the radiation problem. Finally, we characterize the set of all bounded solutions to the homogeneous problem.

KEYWORDS
Helmholtz equation, limiting absorption principle, radiation condition, waveguide

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1 | INTRODUCTION

This paper is devoted to the study of equations modelling waves in a periodic waveguide. We consider

$$-\Delta u - k^2 n u = f \quad \text{in} \quad \Omega := \mathbb{R} \times S. \tag{1.1}$$

The domain is an unbounded cylinder with a cross-section S; we assume that $S \subset \mathbb{R}^{d-1}$ is a bounded Lipschitz domain, and $d \geq 2$ is the dimension of the waveguide. The wave-number $k \in \mathbb{C}$ is prescribed and satisfies $\text{Im}\ k \geq 0$; the coefficient function $n : \Omega \to \mathbb{R}$ is assumed to be 2π-periodic in x_1; the right-hand side $f \in L^2(\Omega)$ is assumed to have compact support or, more general, decay properties; see (1.3). We treat the homogeneous Dirichlet boundary condition $u = 0$ on $\mathbb{R} \times \partial S$. We are interested in solutions u to (1.1) that satisfy, additionally, a radiation condition.

In this article, we show existence and uniqueness results for (1.1); we investigate different limiting absorption principles (LAPs), and we characterize function spaces that are related to (1.1). Regarding the LAPs, we show that a vanishing absorption can yield, indeed, a (radiating) solution to the original problem; we additionally show that different damping mechanisms in the LAP can lead to different radiation conditions and, hence, select different solutions to (1.1).

In this work, we treat only the case of a strictly periodic coefficient $n = n(x)$. Nonetheless, we mention that our work has implications for the case that the medium is only periodic outside a compact set; we comment on this connection at the end of this introduction.

We always use a weak solution concept. Solutions to (1.1) are functions $u \in H^1_{\text{loc}}(\bar{\Omega}) := \{u : \Omega \to \mathbb{C} \,|\, u|_{(-R,R)\times S} \in H^1((-R,R)\times S) \text{ for every } R > 0\}$, and (1.1) is interpreted in the weak sense: We demand that

$$\int_\Omega \left\{-k^2 n u \bar{\varphi} + \nabla u \cdot \nabla \bar{\varphi}\right\} = \int_\Omega f \bar{\varphi} \tag{1.2}$$

holds for all $\varphi \in H^1_0(\Omega)$, and $u = 0$ on $\mathbb{R} \times \partial S$ in the sense of traces. We assume that the right-hand side is in the space

$$L^2_*(\Omega) := \left\{f \,|\, x \mapsto (1 + |x_1|^2) f(x) \in L^2(\Omega)\right\} \tag{1.3}$$

with the corresponding norm.

In the first step, we construct solutions $u \in H^1(\Omega)$. For wave numbers $k \in \mathbb{C}$ with $\text{Im } k > 0$, the existence of such solutions follows for all f from the Lax–Milgram theorem. We show that, for real values of k, the existence of $H^1(\Omega)$ solutions can be obtained with the Floquet–Bloch transform for right-hand sides f that satisfy an orthogonality condition (we will write g instead of f for sources with this property). This construction of solutions $u \in H^1(\Omega)$ can be used to show existence and uniqueness of a radiating solution $u \in H^1_{\text{loc}}(\bar{\Omega})$ for a general right-hand side f.

In Section 4, we turn to LAPs. In a first result, we consider a real number $k > 0$ and use the wave-number $k + i\eta$ in the equation. We find that as $\eta > 0$ tends to zero, solutions u^η tend to solutions of the limit problem with $\eta = 0$. It is interesting to compare this result with other mechanisms of a small absorption: We show that different absorption terms lead, in general, to different limit solutions. We can characterize the radiation condition for different absorption mechanisms.

The starting point for all these results is the Floquet–Bloch transform. It allows to transform the original Equation (1.1) to a family of problems on the bounded domain $W := (0, 2\pi) \times S$. The family of problems is parametrized by a parameter $\alpha \in I := [-1/2, 1/2]$. Equation (1.1) has then to be solved on W for all $\alpha \in I$, demanding the α-quasi-periodicity of solutions on the lateral boundaries $\{0\}\times S$ and $\{2\pi\}\times S$. To obtain an equivalent formulation of the problem, it is important to impose, additionally, certain boundedness properties of solutions with respect to the parameter α.

When a fixed parameter $k \in \mathbb{R}$ is considered, we obtain a one-parameter family of problems (α is the only parameter). For a wave number of the form $k + i\eta$, we will deal with a two-parameter family of problems, where $\eta > 0$ is a second parameter.

1.1 Known results and literature

The Helmholtz equation is an old and intensively treated research subject. Classical contributions concern homogeneous media and treat the appropriate radiation conditions in different (unbounded) geometries, the development of appropriate numerical schemes, and the field of inverse scattering. Here, we refrain from citing any of the corresponding results.

The two simplest cases for heterogeneous media are (a) periodic media and (b) compact perturbations of periodic media. The methods for the two cases are closely related. In particular, in both cases, one can exploit the tool of the Floquet–Bloch transform [1, 2]. Within this setting, the simplest geometry is that of a closed waveguide. An important contribution is [3], where the appropriate radiation condition was specified and an existence and uniqueness proof was presented. A related work is [4], where a LAP for the periodic waveguide was shown. In [5], the focus is on equivalent descriptions with Dirichlet-to-Neumann maps, which are useful also in numerical approaches. Such an approach was also used to study, for example, waveguides with different periodic geometries in the two directions. We refer to [6] for a typical result and further references.

All of the above articles are based on complex integrals to invert operators or operator families. Another route to existence and uniqueness results was developed with [7, 8] based on an idea taken from [9]. Essentially, after a Floquet–Bloch transform of the equations, one has to deal with a family of operators that are, except for a discrete set of exceptional points, invertible. With an application of the implicit function theorem, one can construct bounded families of solutions. These provide solutions in periodic waveguides without advanced operator theory. In the paper at hand, we will use this method. We note that the method can be used also for an analysis of Maxwell's equation; see [10].

While all of the above approaches are based, in one way of the other, on the Floquet–Bloch transform, [11] is not using it; an existence result is shown, and in a more general geometry, a Fredholm alternative, the proofs use only energy methods. The only assumption that is made in [11] is that of a non-degeneracy of ω (which is essentially k in the article at hand). With Section 6, we show that our quite classical Assumption 3.5 implies the non-degeneracy that was assumed in [11]. We note that similar ideas allow to introduce a different radiation condition; see [12], and for a numerical scheme, see [13].

Let us close this overview by mentioning some results beyond closed waveguides. Perturbed periodic geometries in two dimensions are considered in [14–19]. For open waveguides, by which we mean here a domain that is unbounded in more than one direction, one has to introduce radiation conditions also in the additional direction; we refer to [20, 21] for formulations of such conditions.

1.2 | Outline of this article and further implications

The present work provides a simplified, direct approach to the Helmholtz equation in periodic media. This comes with a simplification of the non-degeneracy assumption on frequencies; see Assumption 3.5.

We collect and prove basic facts about the Floquet–Bloch transform in Section 2. Using these tools and a functional analysis theorem, we show a result on existence and uniqueness for the radiation problem in Section 3. Section 4 is devoted to limiting absorption principles and the radiation conditions that are obtained from the LAPs. The simple examples of Section 5 demonstrate how different damping approaches lead to different limit solutions. Section 6 is devoted to the analysis of two important spaces of homogeneous solutions.

Let us make a comment on the result of Section 6. Let Y be the span of all quasiperiodic homogeneous solutions and let B be the space of all bounded homogeneous solutions; for precise definitions, see (6.1) and (6.2). We show $Y = B$ in Theorem 6.2. This means that every bounded homogeneous solution is a linear combination of quasiperiodic solutions.

The result $Y = B$ has an important implication: Under our simple non-degeneracy assumption in 3.5, also, the more complex non-degeneracy assumption of [11] is satisfied. This means, in particular, that under Assumption 3.5, the existence result of [11] is valid; it yields an existence result in the case that two different periodic media are given in the left and in the right half-waveguide.

2 | FLOQUET–BLOCH TRANSFORM OF THE EQUATION

This section is devoted to the application of the Floquet–Bloch transform to (1.1). We emphasize that, here, we only study coefficients that are x_1-periodic in all of Ω. Only homogeneous Dirichlet conditions are treated here, but we note that, for example, homogeneous Neumann conditions can be treated with only notational changes in the proofs. Also, operators of the form $u \mapsto -\nabla \cdot (a\nabla u) - k^2 u$ with strictly positive and 2π-periodic $a \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ can be treated with only notational changes.

2.1 | The Floquet–Bloch transform

We perform the Floquet–Bloch transformation only in the x_1-variable. We recall that the interval for the parameter α is $I = [-1/2, 1/2]$ and that the periodicity cell is $W = (0, 2\pi) \times S$. The transformation is a bounded linear map

$$\mathcal{F}_{\mathrm{FB}} : L^2(\Omega) \to L^2(W \times I), \qquad u \mapsto \hat{u}. \tag{2.1}$$

For smooth functions u with compact support, writing $x = (x_1, \tilde{x})$ for the argument, the transformation is defined by

$$\hat{u}((x_1, \tilde{x}), \alpha) := \sum_{\ell \in \mathbb{Z}} u((x_1 + 2\pi\ell, \tilde{x}))e^{-i\ell 2\pi\alpha}, \tag{2.2}$$

for every $x_1 \in (0, 2\pi)$, $\tilde{x} \in S$, $\alpha \in \mathbb{R}$. The map $\mathcal{F}_{\mathrm{FB}}$ of (2.1) is defined as the continuous extension of this map. Proofs regarding properties of the map $\mathcal{F}_{\mathrm{FB}}$ are given in Appendix A.

We say that a function $u \in H^1(W)$ is α-quasiperiodic when $u(2\pi, \cdot) = e^{2\pi i\alpha}u(0, \cdot)$ holds in the sense of traces. We define the space $H^1_\alpha(W)$ as the subspace of $H^1(W)$ that consists α-quasiperiodic functions. From the definition of $\mathcal{F}_{\mathrm{FB}}$ in (2.2), it is clear that, for almost every α, the function $\hat{u}(\cdot, \alpha)$ is α-quasiperiodic.

A direct consequence of definition (2.2) is that the transformation respects derivatives in the sense that $\mathcal{F}_{\mathrm{FB}}(\partial_k u) = \partial_k(\mathcal{F}_{\mathrm{FB}} u)$ for $u \in H^k(\Omega)$ and $k \le n$. This fact implies that we can interpret $\mathcal{F}_{\mathrm{FB}}$ also as a map from $H^1(\Omega)$ onto

$$L^2\left(I, H^1_\alpha(W)\right) := \left\{ u \in L^2\left(I, H^1(W)\right) \,\middle|\, \begin{array}{l} u(\cdot,\alpha) \text{ is } \alpha \text{ -quasiperiodic} \\ \text{for almost all } \alpha \end{array} \right\}.$$

Two remarks should be made at this point. One regards a notational difficulty: The target space $H^1_\alpha(W)$ depends on the parameter α; hence, $L^2\left((-1/2,1/2), H^1_\alpha(W)\right)$ is not a Bochner-space. Nevertheless, it is a closed subspace of $L^2\left((-1/2,1/2), H^1(W)\right)$ and carries the topology of that ambient space. Our second remark is that $H^1_\alpha(W)$ does not include a boundary condition on $\mathbb{R} \times \partial S$ but a boundary condition can and will be included later on.

With the above space, the transformation map $\mathcal{F}_{\mathrm{FB}}$ has a bounded inverse

$$\mathcal{F}_{\mathrm{FB}}^{-1} : L^2\left(I, H^1_\alpha(W)\right) \to H^1(\Omega). \tag{2.3}$$

The construction of $\mathcal{F}_{\mathrm{FB}}^{-1}$ with its easy formula (A4) is provided in Appendix A for convenience of the reader. The method is quite standard; for generalized approaches, we refer to [1, 2].

2.2 | A family of operators

We exploit the Floquet–Bloch transform to analyze equation (1.1). In this subsection, the wave-number can also be complex; we treat an arbitrary $k \in \mathbb{C}$. The right-hand side is denoted by $g \in L^2_*(\Omega)$ and not by f; the reason is that, in this first step, we construct $H^1(\Omega)$-solutions u for right-hand sides g with a particular structure. Later on, we treat general right-hand sides $f \in L^2_*(\Omega)$. We consider, as in (1.1),

$$-\Delta u - k^2 n u = g \quad \text{in } \Omega = \mathbb{R} \times S, \tag{2.4}$$

with the weak form as in (1.2),

$$\int_\Omega \left\{ -k^2 n u \bar{\varphi} + \nabla u \cdot \nabla \bar{\varphi} \right\} = \int_\Omega g \bar{\varphi} \tag{2.5}$$

for all $\varphi \in H^1_0(\Omega)$. We always impose Dirichlet boundary conditions without further mentioning: $u(\cdot) = 0$ on $\mathbb{R} \times \partial S$ for (2.4) and later on, $\hat{u}(\cdot,\alpha) = 0$ on $(0, 2\pi) \times \partial S$ for almost every $\alpha \in I$.

With the interval $I = [-1/2, 1/2]$, the Floquet–Bloch transform can be applied to $g \in L^2(\Omega)$; it provides $\hat{g} := \mathcal{F}_{\mathrm{FB}}(g) \in L^2(I, L^2(W))$. A solution u is transformed to $\hat{u} := \mathcal{F}_{\mathrm{FB}}(u)$. At least formally, the transformed equation reads

$$-\Delta \hat{u}(\cdot,\alpha) - k^2 n \hat{u}(\cdot,\alpha) = \hat{g}(\cdot,\alpha) \tag{2.6}$$

for almost every $\alpha \in I$. At this point, we exploit the periodicity of the coefficient n, which implies that $\mathcal{F}_{\mathrm{FB}}(nu) = n\mathcal{F}_{\mathrm{FB}}(u)$. We additionally demand $\hat{u}(\cdot,\alpha) \in H^1_\alpha(W)$ for almost every $\alpha \in I$ (and vanishing boundary conditions). A weak solution $\hat{u}$ is characterized by the equality

$$\int_W \left\{ -k^2 n(x)\, \hat{u}(x,\alpha)\overline{\phi(x)} + \nabla \hat{u}(x,\alpha) \cdot \nabla \overline{\phi(x)} \right\} dx = \int_W \hat{g}(x,\alpha)\overline{\phi(x)}\, dx \tag{2.7}$$

for every $\phi \in H^1_\alpha(W)$ that vanishes on $(0, 2\pi) \times \partial S$ and for almost every $\alpha \in I$.

Indeed, the original problem (2.4) is equivalent to the Floquet–Bloch transformed system (2.6) in the following sense.

Lemma 2.1 (Equivalent equation with Floquet–Bloch transform).

(1) Let $u \in H^1_0(\Omega)$ be a weak solution of (2.4). Then the Floquet–Bloch transform $\hat{u} := \mathcal{F}_{FB}(u)$ is an element of $L^2(I, H^1_\alpha(W))$; in particular, $\hat{u}(\cdot,\alpha) \in H^1_\alpha(W)$ for almost every $\alpha \in I$. The functions $\hat{u}(\cdot,\alpha)$ are weak solutions of (2.6).

(2) If $\hat{u} \in L^2(I, H^1_\alpha(W))$ and $\hat{u}(\cdot,\alpha)$ is a weak solution of (2.6) with homogeneous Dirichlet conditions for almost all $\alpha \in I$, then the inverse Floquet–Bloch transform $u := \mathcal{F}_{FB}^{-1}(\hat{u}) = \int_I \hat{u}(\cdot,\alpha)\, d\alpha$ is in $H^1_0(\Omega)$, and it is a weak solution of (2.4).

Proof.

(1) Let $u \in H_0^1(\Omega)$ be a weak solution of (2.4). Our aim is to derive (2.7) for $\hat{u} := \mathcal{F}_{\mathrm{FB}}(u)$. To this end, let $\phi \in H_\alpha^1(W)$ be a test-function; we write ϕ in the form $\phi(x) = \psi(x)e^{i\alpha x_1}$ with a function $\psi \in H^1(W)$ that is periodic with respect to x_1. Additionally, we choose a number $m \in \mathbb{Z}$.
We can now construct a test-function for u: We define $\varphi \in H_0^1(\Omega)$ as the inverse Floquet–Bloch transform of the function $\hat{\varphi}(\alpha, x) := \phi(x)e^{i\alpha 2\pi m} = \psi(x)e^{i\alpha(2\pi m + x_1)}$. By the unitarity of the Floquet–Bloch transform, see (A2), in the integral equation (2.5), the Ω-integrals transform into $I \times W$-integrals, and we obtain

$$\int_I \int_W \left\{ -k^2 n(x)\, \hat{u}(x,\alpha)\, \overline{\hat{\varphi}(x,\alpha)} + \nabla \hat{u}(x,\alpha) \cdot \nabla \overline{\hat{\varphi}(x,\alpha)} \right\} dx\, d\alpha = \int_I \int_W \hat{g}(x,\alpha)\, \overline{\hat{\varphi}(x,\alpha)}\, dx\, d\alpha\,. \tag{2.8}$$

Substituting $\hat{\varphi}(x,\alpha) = \psi(x)e^{i\alpha(2\pi m + x_1)}$ yields

$$\int_I \left[\int_W \left[-k^2 n\, \hat{u}(x,\alpha) \overline{\psi(x)e^{i\alpha x_1}} + \nabla \hat{u}(x,\alpha) \cdot \nabla \overline{(\psi(x)e^{i\alpha x_1})} \right] dx \right] e^{-i2\pi m\alpha}\, d\alpha = \int_I \left[\int_W \hat{g}(x,\alpha) \overline{\psi(x)e^{i\alpha x_1}}\, dx \right] e^{-i2\pi m\alpha}\, d\alpha\,.$$

Since m was arbitrary, all Fourier coefficients of the two terms in squared brackets coincide. This implies that the squared brackets coincide for almost every $\alpha \in I$. Because of $\phi(x) = \psi(x)e^{i\alpha x_1}$, this is (2.7).

(2) Let $\hat{u} \in L^2(I, H_\alpha^1(W))$ be a solution of (2.7) for almost every $\alpha \in I$. We consider an arbitrary test-function $\varphi \in H_0^1(\Omega)$. Using $\phi = \hat{\varphi}(\cdot, \alpha)$ in (2.7) and integrating with respect to α yields (2.8). Again, by the unitarity of the Floquet–Bloch transform, this relation is equivalent to (2.5) for u. The unitarity also provides $u \in H^1(\Omega)$; see (2.3). □

For the further development of the theory, it is useful to have a target space that is independent of parameters. We introduce

$$X := H_{\mathrm{per}}^1(W) := \left\{ u \in H^1(W) | u = 0 \ \text{ on } \ \mathbb{R} \times \partial S \ \text{ and } \ u|_{x_1=0} = u|_{x_1=2\pi} \right\}. \tag{2.9}$$

We denote the canonical inner product in $X = H_{\mathrm{per}}^1(W)$ by $\langle \cdot, \cdot \rangle_X$. Note that we included the Dirichlet boundary condition into the space $H_{\mathrm{per}}^1(W)$. We exploit the following equivalence for $U \in H^1(W)$:

$$[x \mapsto U(x)] \ \alpha\text{-periodic in } \ x_1 \iff [x \mapsto U(x)e^{-i\alpha x_1}] \ \text{periodic in } \ x_1\,. \tag{2.10}$$

It allows to map $H_\alpha^1(W)$-functions to $H_{\mathrm{per}}^1(W)$-functions and vice versa. Replacing $\hat{u}(x,\alpha)$ by $v(x,\alpha)e^{i\alpha x_1}$ and $\phi(x)$ by $\varphi(x)e^{i\alpha x_1}$, we can rewrite the problem described in (2.7) as a family of problems in the space $X = H_{\mathrm{per}}^1(W)$: We seek for $v \in L^2(I, X)$ such that

$$\int_W \left[-k^2 n(x) v(x,\alpha) \overline{\varphi(x)} + \nabla \left(v(x,\alpha)e^{i\alpha x_1} \right) \cdot \nabla \overline{\left(\varphi(x)e^{i\alpha x_1} \right)} \right] dx = \int_W \hat{g}(x,\alpha) \overline{\varphi(x)e^{i\alpha x_1}}\, dx \ \text{for every } \ \varphi \in X, \tag{2.11}$$

for almost every $\alpha \in I$.

For fixed $\alpha \in I$, we can consider the right-hand side of (2.11) as a function of φ, defining a functional on X. We can also, for fixed v, consider the left-hand side of (2.11) as a functional on X. By the Riesz representation theorem, there exist $y_\alpha \in X$ and $L_\alpha v \in X$ with

$$\langle L_\alpha v, \varphi \rangle_X = \int_W \left[-k^2 n(x) v(x) \overline{\varphi(x)} + \nabla (v(x)e^{i\alpha x_1}) \cdot \nabla \left(\overline{\varphi(x)e^{i\alpha x_1}} \right) \right] dx, \tag{2.12}$$

$$\langle y_\alpha, \varphi \rangle_X = \int_W \hat{g}(x,\alpha) \overline{\varphi(x)e^{i\alpha x_1}}\, dx \tag{2.13}$$

for every $\varphi \in X$. With these representations, using Lemma 2.1 (b), the original problem (2.4) is solved when we find, for almost every $\alpha \in I$, a solution $v(\cdot, \alpha) \in X = H_{\mathrm{per}}^1(W)$ of

$$L_\alpha v(\cdot, \alpha) = y_\alpha\,, \tag{2.14}$$

and if this family of solution satisfies $v \in L^2(I, X)$.

It is not obvious how to solve (2.14). Indeed, structural assumptions on g will be necessary in order to solve the equation. The reason for this restriction is that we are looking for solutions u of the original problem in the space $H^1(\Omega)$, that is, for solutions with decay properties.

On the other hand, some structural properties of L_α follow immediately from the definition. For fixed α, the operator L_α is a linear bounded operator from $X = H^1_{\text{per}}(W)$ into itself. The form of L_α shows that L_α is self-adjoint and that we can write $L_\alpha = \text{id} + K_\alpha$, where K_α is a compact linear operator. Accordingly, every operator L_α is a Fredholm operator with index 0. Additionally, the definition of L_α extends, for $\varepsilon > 0$, to the increased interval $I_\varepsilon := (-1/2 - \varepsilon, 1/2 + \varepsilon)$. The operators depend continuously differentiable and even analytically on α.

On the structure of critical values, the definition of L_α and the fact that n and k are real implies the following: $v \in \ker L_\alpha$ if and only if $\bar{v} \in \ker L_{-\alpha}$. For the particular value, $\alpha = 1/2$ holds: $\ker L_{1/2} \subset H^1_{\text{per}}(W)$ and $\ker L_{-1/2} \subset H^1_{\text{per}}(W)$ represent the same set (in the sense that $v \in H^1_{\text{per}}(W)$ represents $w(x) = v(x)e^{i\alpha x_1}$ in $H^1_\alpha(W)$), namely, the space of homogeneous solutions w with $w|_{x_1=2\pi} = -w|_{x_1=0}$. We therefore do not want to consider the two values $\alpha = 1/2$ and $\alpha = -1/2$ separately.

A remark on notation. When k is replaced by $k + i\eta$ with $k > 0$ and $\eta \geq 0$, then we write L^η_α to indicate the dependence on the second parameter η.

3 | EXISTENCE AND UNIQUENESS

In this section, we consider the case of a real wave number $k > 0$ and Equation (1.1) for arbitrary $f \in L^2_*(\Omega)$; see (1.3) for the function space. Our approach will be the following. In the first step, we search for solutions $u \in H^1(\Omega)$; we can find such solutions only when the right-hand side $f = g$ has certain orthogonality properties. Roughly speaking, g must be orthogonal to the space of quasiperiodic solutions of the homogeneous equation. For such g, we will show the existence of a solution u by a functional analytic singular perturbation theorem which we learned from [9]. In the second step, we allow general $f \in L^2_*(\Omega)$, but we search for a solution in a larger class of functions u satisfying a radiation condition.

3.1 | Functional analysis for one-parameter families

Definition 3.1 (C^1-families of operators and regular C^1-families)**.** Let X be a Banach space and let $I \subset \mathbb{R}$ be the unit interval $I := [-1/2, 1/2]$. We say that $(L_\alpha)_\alpha$ is a *C^1-family of operators* when there exists $\varepsilon > 0$ and a C^1-map $I_\varepsilon := (-1/2 - \varepsilon, 1/2 + \varepsilon) \ni \alpha \mapsto L_\alpha \in \mathcal{L}(X, X)$ such that, for every $\alpha \in I$, the operator L_α is a Fredholm operator with index 0.

We say that $(L_\alpha)_\alpha$ is a *regular C^1-family of operators* when additionally the following two conditions are satisfied for every $\alpha \in I$ for which L_α is not invertible: (i) The operator L_α has Riesz number 1, i.e., $\mathcal{N} := \ker(L_\alpha) = \ker(L^2_\alpha)$. (ii) With the range $\mathcal{R} := L_\alpha(X) \subset X$ and the projection P onto $\mathcal{N}$ corresponding to $X = \mathcal{N} \oplus \mathcal{R}$, the operator

$$M := \partial_\alpha P L_\alpha|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N} \tag{3.1}$$

is invertible.

Remark 3.1.

1. We demand that every operator L_α is a Fredholm operator with index 0. This implies that, for every α and $L = L_\alpha$, the subspace $\mathcal{N} := \ker(L)$ has finite dimension and the subspace $\mathcal{R} := L(X)$ is closed and has finite co-dimension; the latter agrees with the dimension of $\mathcal{N}$ since the index is 0. Together with the requirement $\ker(L) = \ker(L^2)$, we conclude that the space possesses the decomposition $X = \mathcal{N} \oplus \mathcal{R}$ and corresponding continuous projections $P : X \to X$ onto $\mathcal{N}$ and $Q = (\text{id} - \mathcal{P})$ onto $\mathcal{R}$. We recall the easy argument why the intersection is trivial: $u \in \mathcal{N} \cap \mathcal{R}$ implies $u = Lx$ and $Lu = 0$; hence, $L^2x = 0$, and thus, $Lx = 0$, we find $x \in \mathcal{N}$ and $u = Lx = 0$.
2. When X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$ and L is self-adjoint, it has Riesz number 1. Indeed, $L^2x = 0$ implies $\langle Lx, Lx \rangle_X = \langle L^2x, x \rangle_X = 0$, and thus, $x \in \mathcal{N}$.

Theorem 3.2 (Functional analysis I)**.** *Let $(L_\alpha)_\alpha$ be a regular C^1-family of operators. There holds the following:*

(1) The set $\mathcal{A}$ of critical numbers

$$\mathcal{A} := \{\alpha \in I \,|\, \ker(L_\alpha) \neq \{0\}\} \tag{3.2}$$

is finite.

(2) *Let $I_\varepsilon \ni \alpha \mapsto y_\alpha$ be a C^1-family of right-hand sides such that $y_\alpha \in L_\alpha(X)$ holds for every $\alpha \in \mathcal{A}$. Then the family of solutions*

$$I_\varepsilon \backslash \mathcal{A} \ni \alpha \mapsto u_\alpha := (L_\alpha)^{-1}(y_\alpha)$$

can be continued to a C^0-family on I_ε. With C independent of the family $(y_\alpha)_\alpha$, there holds

$$\sup_{\alpha \in I} \|u_\alpha\|_X \le C \sup_{\alpha \in I} \left[\|y_\alpha\|_X + \|y'_\alpha\|_X \right] . \tag{3.3}$$

According to assertion (1) of the theorem, the set $\mathcal{A}$ is a finite collection of values $\alpha \in I$. As indicated above, in our application, we do not want to enumerate the values $-1/2$ and $1/2$ independently of each other. We therefore enumerate a (possibly) reduced set as follows:

$$\mathcal{A}_* := \mathcal{A} \cap (-1/2, 1/2] := \{\alpha_j \,|\, 0 < j \le J\} . \tag{3.4}$$

The indices j are natural numbers, J is the number of critical values, and $J = 0$ is allowed. In the case $J = 0$, the set $\mathcal{A}_*$ is empty.

Proof.

Step 1: An equivalent form of the system.

We start the proof by investigating a point $\alpha_0 \in I$ with $\ker(L_{\alpha_0}) \neq \{0\}$. It is no loss of generality to assume $\alpha_0 = 0$. The critical (non-invertible) operator is $L := L_0$. We use $X = \mathcal{N} \times \mathcal{R}$ with projections P and Q. We emphasize that these subspaces and projections are chosen for L and independent of α in the following. For α close to α_0, we write the operator L_α as

$$L_\alpha = \begin{bmatrix} PL_\alpha|_\mathcal{N} & PL_\alpha|_\mathcal{R} \\ QL_\alpha|_\mathcal{N} & QL_\alpha|_\mathcal{R} \end{bmatrix} : \mathcal{N} \times \mathcal{R} \to \mathcal{N} \times \mathcal{R} . \tag{3.5}$$

For $\alpha \neq \alpha_0 = 0$, the equation $L_\alpha u_\alpha = y_\alpha$ for $u_\alpha = (u_\alpha^N, u_\alpha^R) \in \mathcal{N} \times \mathcal{R}$ is equivalent to the following set of equations:

$$\tilde{L}_\alpha u_\alpha := \begin{bmatrix} \frac{1}{\alpha} PL_\alpha\Big|_\mathcal{N} & \frac{1}{\alpha} PL_\alpha\Big|_\mathcal{R} \\ QL_\alpha|_\mathcal{N} & QL_\alpha|_\mathcal{R} \end{bmatrix} \begin{pmatrix} u_\alpha^N \\ u_\alpha^R \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha} P y_\alpha \\ Q y_\alpha . \end{pmatrix} \tag{3.6}$$

Relation (3.6) defines linear operators $\tilde{L}_\alpha : X \to X$ for $\alpha \neq 0$.

We want to extend this family of operators to the point $\alpha = 0$. With $L' := (\partial_\alpha L_\alpha)|_{\alpha=0}$ and $M = PL'|_\mathcal{N}$ of (3.1) we set for arbitrary $u = (u^N, u^R) \in \mathcal{N} \times \mathcal{R} = X$,

$$\tilde{L}_0 u := \begin{bmatrix} M & PL'|_\mathcal{R} \\ 0 & QL|_\mathcal{R} \end{bmatrix} \begin{pmatrix} u^N \\ u^R \end{pmatrix} . \tag{3.7}$$

We claim that the new operator family $(-\varepsilon, \varepsilon) \ni \alpha \mapsto \tilde{L}_\alpha \in \mathcal{L}(X, X)$ is continuous. This is clear by definition in all points $\alpha \in (-\varepsilon, \varepsilon) \backslash \{0\}$. Regarding $\alpha = 0$, we note that the operators of (3.6) can be written as difference quotients: Because of $L|_\mathcal{N} \equiv 0$, there holds $\frac{1}{\alpha} PL_\alpha\Big|_\mathcal{N} = \frac{1}{\alpha} P(L_\alpha - L)\Big|_\mathcal{N}$. Since we extended with $PL'|_\mathcal{N} = M$ for $\alpha = 0$, the resulting family is continuous in α. The same argument can be performed for the second entry of the matrix: Because of $PL|_\mathcal{R} \equiv 0$, we can write $\frac{1}{\alpha} PL_\alpha\Big|_\mathcal{R} = \frac{1}{\alpha} P(L_\alpha - L)\Big|_\mathcal{R}$. The limit operator is given by the derivative that is used in (3.7). Finally, regarding the third entry, we note that $QL|_\mathcal{N} = 0$ by the definition of $\mathcal{N}$. We obtain that the family $\tilde{L}_\alpha$ is continuous in α.

We next observe that the operator $\tilde{L}_0$ is invertible: The operator $M : \mathcal{N} \to \mathcal{N}$ is invertible by the definition of a regular family. The operator $QL|_\mathcal{R} : \mathcal{R} \to \mathcal{R}$ is invertible by definition of $\mathcal{R}$ and Q. As a triagonal matrix, $\tilde{L}_0$ is invertible.

Continuity of the family $\tilde{L}_\alpha$ together with invertibility of $\tilde{L}_0$ yields the invertibility of $\tilde{L}_\alpha \in \mathcal{L}(X, X)$ for $\alpha \in (-\varepsilon, \varepsilon)$, upon possibly choosing a smaller $\varepsilon > 0$.

Step 2: Assertion (1).

Since $I = [-1/2, 1/2] \subset \mathbb{R}$ is compact, it is sufficient to show the following claim: For every $\alpha \in I$, there exists $\varepsilon > 0$ such that $\mathcal{A} \cap (\alpha - \varepsilon, \alpha + \varepsilon)$ contains at most one point.

For $\alpha \notin \mathcal{A}$, the claim holds, since small perturbations of invertible operators are invertible.

We consider now $\alpha_0 \in \mathcal{A}$ and investigate α in a neighborhood of α_0. To simplify notation and without loss of generality, we assume $\alpha_0 = 0$. In Step 1, we obtained that the equation $L_\alpha u_\alpha = y_\alpha$ has the equivalent form (3.6) and that $\tilde{L}_\alpha$ is invertible for every $\alpha \in (-\varepsilon, \varepsilon)$. This yields that L_α is invertible for every $\alpha \in (-\varepsilon, \varepsilon)\backslash\{0\}$.

Step 3: Assertion (2).

We have to consider again the situation of Step 2, with u_α solving $L_\alpha u_\alpha = y_\alpha$ (or, equivalently, (3.6)) for $\alpha \in (-\varepsilon, \varepsilon)\backslash\{0\}$. Regarding the right-hand side y_α, we have imposed the property $y_{\alpha_j} \in L_{\alpha_j}(X)$ for all j. In the local situation and with our assumption that the critical point is $\alpha_0 = 0$, we have $y_0 \in L(X) = \mathcal{R}$. This implies $Py_0 = 0$, and we can write the first entry of the right-hand side of (3.6) as $\frac{1}{\alpha}Py_\alpha = \frac{1}{\alpha}P(y_\alpha - y_0)$, which can be extended continuously with $Py'|_0$ for $\alpha = 0$.

The fact that the family $\tilde{L}_\alpha$ is a continuous family of invertible operators on $\alpha \in (-\varepsilon, \varepsilon)$ together with the fact that the right-hand sides of (3.6) can be extended continuously to $(-\varepsilon, \varepsilon)$ shows that the family u_α can be extended continuously. The proof also provides (3.3). □

Remark 3.3 (Functional analysis with two parameters). Definition 3.1 can be adapted to define C^1-families of operators L_α^η depending on two parameters, $\alpha \in [-1/2, 1/2]$ and $\eta \geq 0$. Regarding the definition of a regular C^1-family, requirement (ii) of Definition 3.1 has to be replaced by the requirement that for every α for which L_α^0 is not invertible and any direction vector $0 \neq \xi \in \mathbb{R}^2$ with $\xi_2 \geq 0$, the operator

$$\partial_\xi P L_\alpha^0|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N} \tag{3.8}$$

is invertible; here $\partial_\xi PL_\alpha^0 = \xi_1 \partial_\alpha PL_\alpha^0 + \xi_2 \partial_\eta PL_\alpha^0$ denotes the directional derivative.

With these adaptations, the assertion of Theorem 3.2 holds in a slightly weaker form: For some $\varepsilon > 0$, the family of solutions

$$(I_\varepsilon \times [0, \varepsilon)) \backslash (\mathcal{A} \times \{0\}) \ni (\alpha, \eta) \mapsto u_\alpha^\eta := (L_\alpha^\eta)^{-1}(y_\alpha^\eta)$$

is bounded.

To show this result, one considers, for fixed direction ξ, parameters along a semiray: $(\alpha, \eta) = \tau\xi$ with $\tau > 0$. The arguments of Theorem 3.2 can be repeated upon replacing the parameter α with the new parameter τ.

3.2 | Regularity of the C^1-family of operators L_α

We now consider the one-parameter family L_α of (2.12). This family is a C^1-family because of the smooth dependence of L_α on α. Using the equivalence (2.10), the kernel $\mathcal{N}_\alpha := \ker(L_\alpha) \subset X$ is given by $\mathcal{N}_\alpha = \{e^{-i\alpha x_1} u \,|\, u \in Y^\alpha\}$ with

$$Y^\alpha := \{u \in H^1_\alpha(W) \,|\, (\Delta + k^2 n)u = 0 \text{ in } W \text{ and } u = 0 \text{ on } \mathbb{R} \times \partial S\}. \tag{3.9}$$

Since each L_α is a Fredholm operator, the kernel $\mathcal{N}_\alpha$ is finite dimensional, and hence, also, Y^α is finite dimensional. We are interested in the set of critical points, defined in (3.2), $\mathcal{A} = \{\alpha \in I \,|\, \ker(L_\alpha) \neq \{0\}\}$ with $I = [-1/2, 1/2]$. Without further assumptions, the set $\mathcal{A}$ can be finite or infinite. Theorem 3.2 yields that $\mathcal{A}$ is finite when we can show that L_α is a regular C^1-family. This is what we will obtain under a certain assumption.

We define a sesqui-linear form E by setting, for $u, v \in H^1(W)$,

$$E(u, v) := i \int_W u\,\partial_1 \bar{v} - \bar{v}\,\partial_1 u. \tag{3.10}$$

We emphasize that, typically, the arguments of E are α-periodic functions but not necessarily elements of $X = H^1_{\text{per}}(W)$. We observe that E is hermitian; thus, $E(u, u) \in \mathbb{R}$ for all u.

The form E is related to energy fluxes through sections of the form $\Gamma_t := \{t\} \times S \subset \Omega$ for $t \in \mathbb{R}$. Indeed, when u and v are two solutions, $(\Delta + k^2 n)u = 0 = (\Delta + k^2 n)v$, then an application of Green's theorem in $W_{s,t} := (s, t) \times S$ for arbitrary $s < t$ yields

$$\int_{\Gamma_t} \{u\partial_1\bar{v} - \bar{v}\partial_1 u\} - \int_{\Gamma_s} \{u\partial_1\bar{v} - \bar{v}\partial_1 u\} = \int_{\partial W_{s,t}} \{u\partial_\nu\bar{v} - \bar{v}\partial_\nu u\} = \int_{W_{s,t}} \left\{u(\Delta + k^2 n)\bar{v} - \bar{v}(\Delta + k^2 n)u\right\} = 0,$$

where we denoted with ∂_ν the normal derivatives into the exterior of $W_{s,t}$. The calculation shows that the flux quantity

$$F_{u,v,t} := i\int_{\Gamma_t} \{u\,\partial_1\bar{v} - \bar{v}\,\partial_1 u\} \tag{3.11}$$

is independent of $t \in \mathbb{R}$. In particular, there holds $E(u,v) = \int_0^{2\pi} F_{u,v,t}\,dt = 2\pi F_{u,v,s}$ for any $s \in \mathbb{R}$.

We obtain easily that, for different values of $\alpha \in (-1/2, 1/2]$, the spaces Y^α are orthogonal with respect to the above sesquilinear form.

Lemma 3.4 (Orthogonality for different quasimoments). *Let $\alpha, \beta \in (-1/2, 1/2]$ with $\alpha \neq \beta$ be two quasimoments and let $u \in Y^\alpha$ and $v \in Y^\beta$ be two solutions of the homogeneous equation. Then $E(u,v) = 0$.*

Proof. For quasiperiodic u and v as in the lemma, the expression of (3.11) satisfies, by its definition, $F_{u,v,t+2\pi} = e^{2\pi i\alpha}e^{-2\pi i\beta}F_{u,v,t}$. On the other hand, as noted above, $F_{u,v,t}$ is independent of t. Because of $|\alpha - \beta| < 1$, we conclude $F_{u,v,t} = 0$, and thus, $E(u,v) = 0$. □

We can show that L_α is a regular C^1-family under the following assumption.

Assumption 3.5 (Non-degeneracy assumption). For every $\alpha \in \mathcal{A}$, the sesquilinear form E is non-degenerate on Y^α in the following sense: For every $0 \neq \phi \in Y^\alpha$, the map $E(\phi,\cdot) : Y^\alpha \to \mathbb{C}$ is a nontrivial form.

Lemma 3.6 (Regularity of the Floquet–Bloch transformed equation). *Let L_α be the C^1-family of operators constructed in (2.12) and let Assumption 3.5 hold. Then L_α is a regular C^1-family of operators in the sense of Definition 3.1.*

Proof. We fix $\alpha \in \mathcal{A}$ and consider the operator $L := L_\alpha$ with kernel $\mathcal{N} := \ker(L)$ and derivative $L' := \partial_\alpha L_\alpha$. We have to verify that $M := PL'|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is invertible, where P is the projection onto $\mathcal{N}$. In the subsequent calculation, the definition of L_α in (2.12) yields the first equality; we use here that $e^{i\alpha x_1}e^{-i\alpha x_1} = 1$ is independent of α. In the second equality, we use that when the derivatives are applied to $u(x)e^{i\alpha x_1}$ and to $\varphi(x)e^{i\alpha x_1}$, but not on x_1, the terms from the first and second terms cancel.

$$\begin{aligned}
\langle L'u, \varphi\rangle_X &= i\int_W \nabla\left(u(x)x_1\, e^{i\alpha x_1}\right)\cdot\nabla\left(\overline{\varphi(x)\, e^{i\alpha x_1}}\right) - \nabla\left(u(x)e^{i\alpha x_1}\right)\cdot\nabla\left(\overline{\varphi(x)x_1\, e^{i\alpha x_1}}\right)dx\\
&= i\int_W u(x)e^{i\alpha x_1}\,\partial_1\left(\overline{\varphi(x)e^{i\alpha x_1}}\right) - \partial_1\left(u(x)e^{i\alpha x_1}\right)\overline{\varphi(x)\,e^{i\alpha x_1}}\,dx\\
&= E\left(u\,e^{i\alpha x_1}, \varphi\, e^{i\alpha x_1}\right).
\end{aligned}$$

From this calculation, we can conclude that $PL'|_{\mathcal{N}}$ is invertible. Indeed, let $u \in \mathcal{N}$ satisfy $PL'u = 0$. Since $L = L_\alpha$ is self-adjoint, $\mathcal{N}$ and $\mathcal{R}$ are orthogonal. In this situation, $PL'u = 0$ implies that $\varphi \mapsto \langle L'u, \varphi\rangle_X$ is the trivial form on $\mathcal{N}$. The above calculation, together with the fact that E is non-degenerate, implies that this is possible only for $u\,e^{i\alpha x_1} = 0$ and thus for $u = 0$. We obtain that the kernel of $PL'|_{\mathcal{N}}$ is trivial and hence that M of (3.1) is invertible. □

Corollary 3.7 (The spaces Y_j and basis functions). *We consider a Helmholtz equation for which Assumption 3.5 holds. In this situation, the family L_α constructed in (2.12) is a regular C^1-family of operators. There is a finite (possibly empty) set of values $\mathcal{A}_* = \{\alpha_j | 0 < j \leq J\} \subset (-1/2, 1/2]$ such that*

$$Y_j := \{u \in H^1_{\alpha_j}(W) \,|\, (\Delta + k^2 n)u = 0 \;\; in \;\; W \;\; and \;\; u = 0 \;\; on \;\; \mathbb{R}\times\partial S\} \tag{3.12}$$

is nontrivial. Every space Y_j has a finite dimension $m_j \in \mathbb{N}$, and the spaces Y_j are orthogonal with respect to E. We introduce the direct sum

$$Y := \bigoplus_{j=1}^{J} Y_j \subset H^1(W). \tag{3.13}$$

We choose, for every space Y_j, an inner product $\langle\cdot,\cdot\rangle_{Y_j}$, and solve the self-adjoint eigenvalue problem

$$E(\phi,\psi) = \lambda\langle\phi,\psi\rangle_{Y_j} \; for\; all\;\; \psi \in Y_j \tag{3.14}$$

for $\lambda \in \mathbb{R}$ and $\phi \in Y_j$. This provides an orthogonal basis of Y_j consisting of eigenfunctions $\phi_{\ell,j}$, $\ell = 1, \dots, m_j$. The value $\lambda = 0$ is not an eigenvalue.

Proof. Lemma 3.6 provides that L_α is a regular family. The functional analysis Theorem 3.2 provides that the set of critical α-values is finite. Because of Lemma 3.4, the spaces Y_j are also orthogonal to each other (with respect to E). Assumption 3.5 guarantees that $\lambda = 0$ is not an eigenvalue. The other assertions repeat the definitions and follow from the Fredholm assumption on the family of operators. The solutions ϕ of (3.14) are orthogonal to each other in Y_j by construction. □

3.3 | $H^1(\Omega)$-solutions

We turn to our first existence result for the Helmholtz equation. We characterize the right-hand sides g such that equation (2.4) has a solution in $H^1(\Omega)$.

Theorem 3.8 (Existence of $H^1(\Omega)$ solutions with Floquet–Bloch theory)**.** *We consider the Helmholtz equation (2.4) with fixed S (geometry), fixed k and n (coefficients), and fixed $g \in L^2_*(\Omega) \subset L^2(\Omega)$. We demand that Assumption 3.5 is satisfied.*

Existence: Let the Floquet–Bloch transform $\hat{g}(\cdot,\alpha)$ have the cell-wise orthogonality property

$$\langle \hat{g}(\cdot,\alpha_j), \phi\rangle_{L^2(W)} = 0 \text{ for all } j \in \{1, \dots, J\},\ \phi \in Y_j. \tag{3.15}$$

Then (2.4) has a solution $u \in H^1(\Omega)$ with $\|u\|_{H^1(\Omega)} \le C\|g\|_{L^2_*(\Omega)}$ for some constant $C = C(S,k,n)$.

Uniqueness: When $u \in H^1(\Omega)$ is a solution of (2.4), then the orthogonality (3.15) holds. Furthermore, the solution u is uniquely defined.

Proof. Existence. Using the Floquet–Bloch transform, we have shown that Equation (2.4) is equivalent to the family of equations $L_\alpha v(\cdot,\alpha) = y_\alpha$ of (2.14), $\alpha \in I = [-1/2, 1/2]$. In particular, it is sufficient to find a family $v(\cdot,\alpha)$ of solutions to (2.14) and to verify that $v \in L^2\left(I, H^1_{\mathrm{per}}(W)\right)$. By definition of the critical values $\mathcal{A}$ in (3.2), a unique solution $v(\cdot,\alpha)$ exists for every $\alpha \in I\backslash\mathcal{A}$. We claim that this family of solutions extends continuously to all of I.

We consider one of the critical values, $\alpha = \alpha_j \in \mathcal{A}$, and a small interval $\tilde{I} = [\alpha_j - \varepsilon, \alpha_j + \varepsilon]$ that contain no other critical value. We want to use the functional analysis result of Theorem 3.2. We use the space X of (2.9), the family of operators L_α of (2.12), and the family of right-hand sides y_α of (2.13).

We have to check the assumptions of Theorem 3.2. The operators L_α depend smoothly on α, and they are invertible for all $\alpha \in \tilde{I}\backslash\mathcal{A}$. We turn to the condition $y_{\alpha_j} \in \mathcal{R} = L_{\alpha_j}(X)$. For an arbitrary element $\varphi \in \mathcal{N} := \ker(L_{\alpha_j}) \subset X$, we note that there holds $\phi(x) := \varphi(x)e^{i\alpha_j x_1} \in Y_j$, and by definition of y_α,

$$\langle y_{\alpha_j}, \varphi\rangle_X = \int_W \hat{g}(x,\alpha_j)e^{-i\alpha_j x_1}\,\overline{\varphi(x)}\,dx = \int_W \hat{g}(\cdot,\alpha_j)\bar{\phi} = 0 \tag{3.16}$$

by the orthogonality assumption (3.15). This shows that y_{α_j} is orthogonal to $\mathcal{N}$. Since L_{α_j} is self-adjoint, the subspaces $\mathcal{N}$ and $\mathcal{R}$ are orthogonal. Since L_{α_j} is also Fredholm with index 0, the space X is the orthogonal direct sum $\mathcal{N} \oplus \mathcal{R}$. Since we have shown that y_{α_j} is orthogonal to $\mathcal{N}$, we have found $y_{\alpha_j} \in \mathcal{R}$.

Lemma 3.6 provides that L_α is a regular family of operators in the sense of Definition 3.1; hence, Theorem 3.2 can be applied. We find that $I \ni \alpha \mapsto v(\cdot,\alpha)$ is continuous; hence, in particular, $v \in L^2(I, H^1_{\mathrm{per}}(W))$. This provides a $H^1_0(\Omega)$-solution of (2.4).

We note that the orthogonality (3.15) is not explicitly demanded in $\alpha = -1/2$ when this is a point in $\mathcal{A}$. On the other hand, $\hat{g}(\cdot,-1/2+\beta) = \hat{g}(\cdot,1/2+\beta)$ for all $\beta \in \mathbb{R}$ and $Y^{-1/2} = Y^{1/2}$, so that the orthogonality holds also in $\alpha = -1/2$.

We turn to the estimate for the solution. The right-hand side is an element $g \in L^2_*(\Omega)$. With the functions $g_\ell : W \to \mathbb{C}$, $g_\ell(x_1,\tilde{x}) := g(x_1 + 2\pi\ell, \tilde{x})$, we can estimate the corresponding norm as $\|g\|^2_{L^2_*(\Omega)} = \sum_{\ell\in\mathbb{Z}}\int_W |g_\ell(x)|^2[1 + (x_1 + 2\pi\ell)^2]^2 dx \ge c\sum_{\ell\in\mathbb{Z}}(1+\ell^2)^2\|g_\ell\|^2_{L^2(W)}$. This allows to calculate, for arbitrary $m < M$, the norm of a finite sum, which is related to the derivative $\partial_\alpha\hat{g}(\cdot,\alpha)$ of the Floquet–Bloch transform of g with respect to α;

compare (2.2):

$$\left\| \sum_{m\le|\ell|\le M} \ell\, g(x_1+2\pi\ell,\tilde{x})\, e^{-i\ell 2\pi\alpha} \right\|_{L^2(W)} \le \sum_{m\le|\ell|\le M} |\ell|\, \|g_\ell\|_{L^2(W)} \le \sum_{m\le|\ell|\le M} \frac{1}{\sqrt{1+\ell^2}}(1+\ell^2)\,\|g_\ell\|_{L^2(W)}$$
$$\le \sqrt{\sum_{m\le|\ell|\le M} \frac{1}{1+\ell^2}} \sqrt{\sum_{m\le|\ell|\le M} (1+\ell^2)^2\, \|g_\ell\|^2_{L^2(W)}} \le C_{m,M}\|g\|_{L^2_*(\Omega)},$$

where $C_{m,M}$ is independent of g and tends to zero as $m \to \infty$. The Cauchy argument shows that $\partial_\alpha \hat{g}(\cdot,\alpha)$ is well-defined in $L^2(W)$ and is bounded by $C\|g\|_{L^2_*(\Omega)}$ for some $C > 0$. We conclude that, for some $C > 0$, there holds $\|\hat{g}(\cdot,\alpha)\|_{L^2(W)} + \|\partial_\alpha \hat{g}(\cdot,\alpha)\|_{L^2(W)} \le C\|g\|_{L^2_*(\Omega)}$ for all α. Theorem 3.2 provides estimate (3.3) for solutions, which is a bound for $\hat{u} \in C^0(I, H^1(W))$; hence, in particular, for $\hat{u} \in L^2(I, H^1(W))$. This yields the bound for $u \in H^1(\Omega)$, namely, $\|u\|_{H^1(\Omega)} \le C\|g\|_{L^2_*(\Omega)}$.

Uniqueness. In order to show unique solvability of (2.4), it is sufficient to show the unique solvability of (2.6) for almost every α. For every $\alpha \notin \mathcal{A}$, Equation (2.6) can be solved uniquely by definition of the critical α-values. This already shows the uniqueness of the solution.

Let $u \in H^1(\Omega)$ be a solution of (2.4). We have to show that the orthogonality (3.15) holds. We use equation (2.7), which is a consequence of (2.4):

$$\langle \hat{g}(\cdot,\alpha_j), \phi\rangle_{L^2(W)} = \int_W \left\{ -k^2 n(x)\,\hat{u}(x,\alpha)\,\overline{\phi(x)} + \nabla\hat{u}(x,\alpha)\cdot\nabla\overline{\phi(x)} \right\} dx = -\langle \hat{u}(\cdot,\alpha_j), (\Delta + k^2 n)\phi\rangle_{L^2(W)} = 0,$$

where we exploited that, for every α, integration by parts holds without boundary terms for two functions in the space $H^1_\alpha(W)$. This concludes the proof of the theorem. □

Lemma 3.9 (Orthogonality criterion)**.** *The orthogonality condition (3.15) is formulated in terms of the Floquet–Bloch transform of g. With the original function g and the space Y of (3.13), an equivalent condition is*

$$\int_\Omega g(x)\,\overline{\phi(x)}\,dx = 0 \text{ for all } \phi \in Y. \tag{3.17}$$

Proof. We fix $j \in \{1, \dots, J\}$, set $\alpha := \alpha_j$, and choose a function ϕ in Y_j. We identify ϕ with its α-quasiperiodic extension, which satisfies $\phi(x+2\pi\ell e_1) = \phi(x)e^{i\ell 2\pi\alpha}$ with the unit vector e_1 in x_1-direction. We calculate for $\hat{g} = \mathcal{F}_{\mathrm{FB}}(g)$

$$\langle \hat{g}(\cdot,\alpha), \phi\rangle_{L^2(W)} = \left\langle \sum_{\ell\in\mathbb{Z}} g(\cdot + 2\pi\ell e_1)\, e^{-i\ell 2\pi\alpha}, \phi \right\rangle_{L^2(W)} = \left\langle \sum_{\ell\in\mathbb{Z}} g(\cdot + 2\pi\ell e_1), \phi(\cdot + 2\pi\ell e_1) \right\rangle_{L^2(W)} = \int_\Omega g(x)\,\overline{\phi(x)}\,dx.$$

We note that the series in the definition of the Floquet–Bloch transform is well-defined because of $g \in L^2_*(\Omega)$. □

3.4 | The radiation problem

In the previous subsection, we have obtained a solution u to (2.4) where g satisfies the orthogonality condition (3.17). This is not the kind of solution that is typically observed. In the physical problem, we have to consider the equation with a general right-hand side f and obtain solutions that are, approximately, far away from the origin, linear combinations of outgoing waves. Such solutions are not in the space $H^1(\Omega)$. We recall that we impose $f \in L^2_*(\Omega)$.

In order to define the radiation condition, we use two cut-off functions.

Definition 3.10 (Cut-off functions $\rho_\pm$)**.** We say that $\rho_+, \rho_- \in C^2(\mathbb{R}, \mathbb{R})$ are admissible cut-off functions when they satisfy $\rho_\pm(x_1) \in [0,1]$ for every $x_1 \in \mathbb{R}$ and when the limiting behavior is given by $\rho_\pm(x_1) \to \frac{1}{2} \pm \frac{1}{2}$ for $x_1 \to \infty$ and $\rho_\pm(x_1) \to \frac{1}{2} \mp \frac{1}{2}$ for $x_1 \to -\infty$. We additionally demand, for some $C > 0$, the decay properties $1 - \rho_+(x_1) \le C/|x_1|$ and $\rho_-(x_1) \le C/|x_1|$ for $x_1 > 1$, and $\rho_+(x_1) \le C/|x_1|$ and $1 - \rho_-(x_1) \le C/|x_1|$ for $x_1 < -1$.

3.4.1 | Remark on cut-off functions

Formally, the radiation condition formulated below depends on the choice of $\rho_\pm$. But we will show later on that the solution u of the radiation problem does not depend on the choice of $\rho_\pm$.

The requirement $\rho_\pm \in C^2(\mathbb{R}, \mathbb{R})$ can be replaced by $\rho_\pm \in C^{0,1}(\mathbb{R}, \mathbb{R})$, that is, Lipschitz continuity of the cut-off functions (we keep the property of the rate of decay). One can argue as follows: The existence result below is performed for cut-off functions of class C^2. Remark 2 after Theorem 3.12 can provide that the constructed solutions are also solutions for arbitrary cut-off functions of class $C^{0,1}$. Formally, our proof does not cover this case since we demand $g \in L^2(\Omega)$ in the uniqueness statement below and therefore need $w \in H^2(\Omega)$. In order to resolve this obstacle, one has to use weak solution concepts in all equations to conclude that $\rho_\pm \in C^{0,1}(\mathbb{R}, \mathbb{R})$ is sufficient.

From now on, we use the spaces Y_j and the basis functions $\phi_{\ell,j}$ as chosen in Corollary 3.7. We slightly change notation at this point: We now collect all basis functions $\phi_{\ell,j}$ as a new family with only one index and write $(\phi_\ell)_\ell$, where now $1 \le \ell \le L := \sum_{j=1}^{J} m_j$. We recall that we have orthogonality with respect to the hermitian sesqui-linear form E, that is, $E(\phi_\ell, \phi_{\ell'}) = \delta_{\ell,\ell'} E(\phi_\ell, \phi_\ell)$ for all ℓ, ℓ'.

Definition 3.11 (Propagating part and radiation condition)**.** We fix admissible cut-off functions $\rho_\pm$ as in Definition 3.10. For every $\ell \le L$, the mode ϕ_ℓ is called right-going when $E(\phi_\ell, \phi_\ell) > 0$; it is called left-going when $E(\phi_\ell, \phi_\ell) < 0$. Note that when E is non-degenerate, these are the only possible cases. For every ℓ such that ϕ_ℓ is right-going, we set $\rho_\ell := \rho_+$; for every ℓ for which ϕ_ℓ is left-going, we set $\rho_\ell := \rho_-$.

(i) Propagating part. For complex coefficients $(a_\ell)_{1\le\ell\le L}$, we say that

$$w = \sum_{\ell=1}^{L} a_\ell \, \rho_\ell \, \phi_\ell \tag{3.18}$$

is the propagating wave function corresponding to $a \in \mathbb{C}^L$.

(ii) Radiation condition. We say that a solution $u \in H^1_{\text{loc}}(\Omega)$ of (1.1) satisfies the radiation condition, when there exists $a \in \mathbb{C}^L$ such that, with the corresponding propagating wave function w, there holds

$$v := u - w \in H^1(\Omega). \tag{3.19}$$

Definition 3.11 allows to show an existence result with our previously developed methods: We solve the radiation problem (1.1) by constructing $v = u - w \in H^1_0(\Omega)$ with Theorem 3.8. We can write the equation for v as

$$-\Delta v - k^2 n v = g := f + (\Delta w + k^2 n w). \tag{3.20}$$

We note that the expression $\Delta w + k^2 n w$ has bounded support. This implies $g \in L^2_*(\Omega)$. The function g depends on the vector of coefficients $a \in \mathbb{C}^L$. We will construct $a \in \mathbb{C}^L$ such that (3.20) has a solution $v \in H^1_0(\Omega)$.

We note that, by definition of the radiation condition in Definition 3.11, there is an equivalence of the solution concepts. Existence: When we find $a \in \mathbb{C}^L$ such that (3.20) has a solution $v \in H^1_0(\Omega)$, then $u = w + v \in H^1_{\text{loc}}(\bar{\Omega})$ is a solution of (1.1) with radiation condition. Uniqueness: When $u \in H^1_{\text{loc}}(\bar{\Omega})$ is a nontrivial solution of (1.1) with radiation condition, then there exists $a \in \mathbb{C}^L$ and a solution $v \in H^1_0(\Omega)$ of (3.20) such that a or v are nontrivial.

Theorem 3.12 (Existence of radiating solutions)**.** *Let S, k, n, and f be as above and let $\rho_\pm$ be fixed. We demand that Assumption 3.5 is satisfied. Then (1.1) has a unique solution $u \in H^1_{loc}(\bar{\Omega})$ satisfying the radiation condition. With w, v, and a from the radiation condition, there holds*

$$\|v\|_{H^1(\Omega)} + \|w\|_{H^1(W)} \le C \|f\|_{L^2_*(\Omega)} \tag{3.21}$$

with $C = C(S, k, n, \rho_\pm)$. The coefficients a_ℓ for $\ell \in \{1, \dots, L\}$ are given by

$$a_\ell = \frac{2\pi i}{|E(\phi_\ell, \phi_\ell)|} \langle f, \phi_\ell \rangle_{L^2(\Omega)}. \tag{3.22}$$

Proof. Existence. We want to determine $a \in \mathbb{C}^L$ in the definition of w such that g of (3.20) satisfies the orthogonality condition (3.17). Using a basis function $\phi_{\ell'} \in Y_j$ for some j and extending this basis function to an α_j-quasiperiodic function on Ω, we can calculate, using (3.17) in the first equality,

$$-\langle f, \phi_{\ell'} \rangle_{L^2(\Omega)} = \langle (\Delta + k^2 n) w, \phi_{\ell'} \rangle_{L^2(\Omega)} = \sum_{\ell=1}^{L} a_\ell \, \langle (\Delta + k^2 n)(\rho_\ell \, \phi_\ell), \phi_{\ell'} \rangle_{L^2(\Omega)} \,. \tag{3.23}$$

We evaluate

$$\begin{aligned}(\Delta + k^2 n)(\rho_\ell \, \phi_\ell) &= \rho_\ell (\Delta + k^2 n)\phi_\ell + \nabla \rho_\ell \cdot \nabla \phi_\ell + \nabla \cdot (\phi_\ell \, \nabla \rho_\ell) \\ &= \rho_\ell' \, \partial_1 \phi_\ell + \partial_1 (\phi_\ell \, \rho_\ell') \,.\end{aligned}$$

The scalar product can therefore be evaluated with an integration by parts,

$$\Big\langle (\Delta + k^2 n)(\rho_\ell \, \phi_\ell), \phi_{\ell'} \Big\rangle_{L^2(\Omega)} = \Big\langle \rho_\ell' \, \partial_1 \phi_\ell + \partial_1 (\phi_\ell \, \rho_\ell'), \phi_{\ell'} \Big\rangle_{L^2(\Omega)} = \int_\Omega \bar{\phi}_{\ell'} \, \rho_\ell' \, \partial_1 \phi_\ell - \partial_1 \bar{\phi}_{\ell'} \, \rho_\ell' \, \phi_\ell \, = i \int_{\mathbb{R}} \rho_\ell'(t) F_{\phi_\ell, \phi_{\ell'}, t} \, dt$$

with the flux quantity $F_{\phi_\ell, \phi_{\ell'}, t}$ of (3.11). The flux is independent of t and coincides with $\frac{1}{2\pi} E(\phi_\ell, \phi_{\ell'}) = \frac{1}{2\pi} E(\phi_\ell, \phi_\ell) \delta_{\ell, \ell'}$. We evaluate the right-hand side for a right-going wave ϕ_ℓ, that is, for ρ_ℓ with $\rho_\ell(-\infty) = 0$ and $\rho_\ell(+\infty) = 1$:

$$i \int_{\mathbb{R}} \rho_\ell'(t) F_{\phi_\ell, \phi_{\ell'}, t} \, dt \, = \, \frac{i}{2\pi} E(\phi_\ell, \phi_\ell) \delta_{\ell, \ell'} \,.$$

For a left-going wave ϕ_ℓ, $\rho_\ell(+\infty) - \rho_\ell(-\infty) = -1$ introduces a negative prefactor. We find that the orthogonality condition (3.23) is

$$-\langle f, \phi_{\ell'} \rangle_{L^2(\Omega)} \, = \, a_{\ell'} \, \frac{i}{2\pi} \, |E(\phi_{\ell'}, \phi_{\ell'})| \,.$$

This condition is identical to (3.22).

The above calculation also shows that, choosing $(a_\ell)_\ell$ according to (3.22), the orthogonality condition (3.15) is satisfied for g. We can therefore solve for v with Theorem 3.8. With C depending on $\rho_\pm$, we have the estimate $\|v\|_{H^1(\Omega)} \le C\|g\|_{L^2_*(\Omega)} \le C(\|f\|_{L^2_*(\Omega)} + \|(\Delta + k^2 n)w\|_{L^2(\Omega)}) \le C(\|f\|_{L^2_*(\Omega)} + |a|_{\mathbb{C}^L}) \le C\|f\|_{L^2_*(\Omega)}$. This implies (3.21).

Uniqueness. Let u be a solution of the radiation problem with $f = 0$. Our goal is to show that u vanishes. Theorem 3.8 implies that the right-hand side $g = -(\Delta + k^2 n)w$ of the equation for v satisfies the orthogonality condition (3.17). The existence part of the proof implies that the coefficients $a \in \mathbb{C}^L$ for which the orthogonality condition is satisfied are uniquely determined; hence, by $f = 0$, we conclude $a = 0$. Together with the uniqueness statement of Theorem 3.8, we find $a = 0$ and $v = 0$. This shows that u vanishes. □

Remark 3.4.

1. We note that the decomposition of the propagating modes ϕ_ℓ into left-going and right-going modes is not needed from the mathematical point of view. Indeed, the proof works also for the case that we decompose $\{1, \dots, L\}$ into $\{1, \dots, L\} = \mathcal{L}^+ \cup \mathcal{L}^-$ for disjoint sets $\mathcal{L}^\pm$ and set $\rho_\ell = \rho_+$ for $\ell \in \mathcal{L}^+$ and $\rho_\ell = \rho_-$ for $\ell \in \mathcal{L}^-$.
   A particular choice would be to use $\mathcal{L}^+ := \{1, \dots, L\}$ and $\mathcal{L}^- := \emptyset$. With this choice, we impose that no propagating modes (neither left-going nor right-going) can be used on the right, but all propagating modes (not only outgoing / left-going) can be used on the left.
2. Above, we have constructed, for given $\rho_\pm$, solutions $u = v + w$. In order to investigate well-posedness of the radiation condition, let us consider the consequences of choosing another set of admissible cut-off functions; we denote them as $\tilde{\rho}_\pm$.
   We denote the corresponding solutions as $u = v + w$ and $\tilde{u} = \tilde{v} + \tilde{w}$. We write

$$u - \tilde{u} \, = \, v - \tilde{v} + \sum_\ell \tilde{a}_\ell \, (\rho_\ell - \tilde{\rho}_\ell) \phi_\ell + \sum_\ell (a_\ell - \tilde{a}_\ell) \, \rho_\ell \, \phi_\ell \,.$$

   We observe that $v - \tilde{v} + \sum_\ell \tilde{a}_\ell \, (\rho_\ell - \tilde{\rho}_\ell) \phi_\ell$ is in $H^1(\Omega)$. We emphasize that, at this point, we exploited the decay rate of the cut-off functions that was demanded in Definition 3.10. Therefore, $u - \tilde{u}$ satisfies not only the homogeneous Helmholtz equation but also the radiation condition with coefficients $(a_\ell - \tilde{a}_\ell)_\ell$. The uniqueness result of Theorem 3.12 implies that $u = \tilde{u}$ and $a_\ell = \tilde{a}_\ell$ for all ℓ. In this sense, the choice of the cut-off functions has no influence on the solution.

3. The radiation condition depends on the choice of the inner product chosen in Y^α. Regarding this point, it is very illustrative to study a simple example.

Example 3.13 (The standard example)**.** In the two-dimensional case, $d = 2$, we use the cross-section $S = (0, \pi)$, and the coefficient $n \equiv 1$, considered as a 2π-periodic function with respect to x_1. Since we are interested in eigenspaces with dimension larger than 1, we choose a specific wave number k in the following.

For $\alpha \in I = [-1/2, 1/2]$ chosen below, we consider

$$\phi_1(x) := e^{i\alpha x_1} \sin(2x_2) \text{ and } \phi_2(x) := e^{i(\alpha-2)x_1} \sin x_2 .$$

The two functions satisfy $\Delta\phi_1 + (\alpha^2 + 4)\phi_1 = 0$ and $\Delta\phi_2 + ((\alpha - 2)^2 + 1)\phi_2 = 0$. It is possible to choose α such that the two factors coincide, $\alpha^2 + 4 = (\alpha - 2)^2 + 1$, namely, $\alpha = 1/4$. Accordingly, we define the wave number to be $k = \sqrt{\alpha^2 + 4} = \sqrt{65}/4$. With these choices, we have found two linearly independent, α-quasiperiodic solutions of $\Delta\phi + k^2\phi = 0$. Indeed, for $\alpha = 1/4$, there holds $Y^\alpha = \mathrm{span}(\phi_1, \phi_2)$.

The fluxes of ϕ_1 and ϕ_2 are

$$E(\phi_1, \phi_1) = i \int_W \phi_1 \partial_1 \bar{\phi}_1 - \bar{\phi}_1 \partial_1 \phi_1 = i(-i\alpha)2 \int_W \sin^2(2x_2)\, dx = 2\alpha\pi^2 > 0,$$
$$E(\phi_2, \phi_2) = i(-i(\alpha - 2))2 \int_W \sin^2(x_2)\, dx = 2(\alpha - 2)\pi^2 < 0.$$

We have therefore found a right-going wave ϕ_1 and a left-going wave ϕ_2.

Let us check the orthogonality and normalization. We have $E(\phi_1, \phi_2) = 0$ with $\|\phi_j\|_{L^2(W)} = \pi$ and $\langle \phi_1, \phi_2 \rangle_{L^2(W)} = 0$. Therefore, $\phi_1/\sqrt{\pi}$ and $\phi_2/\sqrt{\pi}$ are the normalized eigenfunctions of the two-dimensional eigenvalue problem (3.14) with $\lambda_1 = 2\alpha\pi^2$ and $\lambda_2 = 2(\alpha - 2)\pi^2$ when $\langle \cdot, \cdot \rangle_{L^2(W)}$ is chosen as the inner product in Y^α. However, if one chooses a different inner product in Y^α (for which ϕ_1 and ϕ_2 are not orthogonal), then one gets a different basis $\tilde{\phi}_1, \tilde{\phi}_2$. This changes the radiation condition.

We will continue the above analysis in Example 5.1 where we show that, indeed, different absorption mechanisms can lead to different inner products, hence to different basis functions, and hence to different radiation conditions.

4 | LAPS

4.1 | The operator family in the case with absorption

In the classical LAP, one replaces the real wave-number $k > 0$ by the complex number $k_\eta := k + i\eta$ with $\eta > 0$ and studies the equation

$$-\Delta u^\eta - (k + i\eta)^2 n u^\eta = f \quad \text{in } \Omega. \tag{4.1}$$

The boundary condition $u^\eta = 0$ on $\partial\Omega$ remains unchanged. It is well-known that this equation is uniquely solvable in $H^1(\Omega)$ for every $\eta > 0$. This can be shown with an application of the Lax–Milgram theorem; the positivity of η implies that the bilinear form corresponding to (4.1) is coercive.

The rewriting of the equation with the Floquet–Bloch transform can be performed with only minimal notational changes: Because of $f \in L^2_*(\Omega)$ and $u^\eta \in H^1(\Omega)$, the Floquet–Bloch transformed functions $\hat{u}^\eta = \mathcal{F}_{\mathrm{FB}}(u^\eta) \in L^2((-1/2, 1/2), H^1_\alpha(W))$ and $\hat{f} = \mathcal{F}_{\mathrm{FB}}(f) \in L^2((-1/2, 1/2), L^2(W))$ are well-defined and satisfy, for $\eta > 0$,

$$-\Delta \hat{u}^\eta(\cdot, \alpha) - (k + i\eta)^2 n \hat{u}^\eta(\cdot, \alpha) = \hat{f}(\cdot, \alpha) \quad \text{in } W, \tag{4.2}$$

with boundary condition $\hat{u}^\eta(\cdot, \alpha) = 0$ on $(0, 2\pi) \times \partial S$.

We use again the space $X = H^1_{\text{per}}(W)$ of (2.9) and the equivalence (2.10); the operator $L^\eta_\alpha \in \mathcal{L}(X, X)$ and the element $y_\alpha \in X$ are defined by

$$\langle L^\eta_\alpha u, \varphi \rangle_{H^1(W)} := -(k + i\eta)^2 \int_W n\, u\, \bar{\varphi} + \int_W \nabla \left(u(x) e^{i\alpha x_1} \right) \cdot \nabla \left(\overline{\varphi(x) e^{i\alpha x_1}} \right) dx \tag{4.3}$$

$$\langle y_\alpha, \varphi \rangle_{H^1(W)} := \int_W \hat{f}(x, \alpha)\, \overline{\varphi(x) e^{i\alpha x_1}}\, dx \tag{4.4}$$

for $u, \varphi \in X$. Then (4.2) is equivalent to $L^\eta_\alpha u^\eta_\alpha = y_\alpha$ for $u^\eta_\alpha(x) = \hat{u}^\eta(x, \alpha) e^{-i\alpha x_1}$. We note that the operators L^η_α are invertible from $X = H^1_{\text{per}}(W)$ onto itself for all $(\alpha, \eta) \in (I \times [0, \varepsilon]) \backslash \{(\alpha_j, 0) \,|\, j = 1, \dots, J\}$.

Since the operators L^η_α depend on two parameters, we need the partial derivatives with respect to both parameters. The α-derivative is calculated as in the case $\eta = 0$:

$$\begin{aligned}
\partial_\alpha \langle L^\eta_\alpha u, \varphi \rangle_{H^1(W)} &= i \int_W \nabla \left(u(x) x_1 e^{i\alpha x_1} \right) \cdot \nabla \left(\overline{\varphi(x) e^{i\alpha x_1}} \right) - \nabla \left(u(x) e^{i\alpha x_1} \right) \cdot \nabla \left(\overline{\varphi(x) x_1 e^{i\alpha x_1}} \right) dx \\
&= i \int_W u(x) e^{i\alpha x_1}\, \partial_1 \left(\overline{\varphi(x) e^{i\alpha x_1}} \right) - \partial_1 \left(u(x) e^{i\alpha x_1} \right) \overline{\varphi(x) e^{i\alpha x_1}} \\
&= E \left(u\, e^{i\alpha x_1}, \varphi\, e^{i\alpha x_1} \right) .
\end{aligned}$$

Taking the derivative of (4.3) with respect to η provides

$$\partial_\eta \langle L^\eta_\alpha u, \varphi \rangle_{H^1(W)} = -2i(k + i\eta) \int_W n\, u\, \bar{\varphi}\, dx .$$

We introduce two operators, essentially given by the two derivatives of L^η_α. For a given $\alpha_j \in \mathcal{A}$, we consider the kernel $\mathcal{N} = \ker \left(L^0_{\alpha_j} \right) = \{ \phi e^{-i\alpha_j x_1} \,|\, \phi \in Y_j \}$, the operator $M_\eta := iP \partial_\eta L^0_{\alpha_j}|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$, and the operator $M_\alpha := P \partial_\alpha L^0_{\alpha_j}|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$.

We note that, by the above formulas, M_η is self-adjoint and positive definite (it can be identified with a multiplication with $2kn$), and $M_\alpha := P \partial_\alpha L^0_{\alpha_j}|_{\mathcal{N}}$ is self-adjoint and one-to-one provided E is non-degenerate on Y_j.

4.2 | Functional analysis for two-parameter families

Our aim is now to extend the one-parameter theory of the last section to a theory for two-parameter families.

Definition 4.1 (Two-parameter family of operators)**.** We consider a Banach space X and the unit interval $I = [-1/2, 1/2] \subset \mathbb{R}$. We say that (L^η_α) is a *two-parameter family of Fredholm operators* when there exists $\varepsilon > 0$ and a C^2-map

$$(-1/2 - \varepsilon, 1/2 + \varepsilon) \times [0, \varepsilon) \ni (\alpha, \eta) \mapsto L^\eta_\alpha \in \mathcal{L}(X, X), \tag{4.5}$$

such that every operator L^η_α is a Fredholm operator with index 0, and for every $\alpha \in I$ for which L^0_α is not invertible, the operator $L := L^0_\alpha$ has Riesz number 1, $\ker(L) = \ker(L^2)$.

Remark 4.1.

1. We actually need less than the C^2 property of the operator family. The proof works when in $\mathcal{L}(X, X)$ the following approximation property holds:

$$\left\| L^\eta_\alpha - \left[L^{\eta_0}_{\alpha_0} + (\alpha - \alpha_0) \partial_\alpha L^{\eta_0}_{\alpha_0} + (\eta - \eta_0) \partial_\eta L^{\eta_0}_{\alpha_0} \right] \right\| \le c \left[(\alpha - \alpha_0)^2 + (\eta - \eta_0)^2 \right] .$$

   Here, the norm is the operator norm in $\mathcal{L}(X, X)$.
2. An illustrative example is $X = \mathbb{C}$ and $L^\eta_\alpha = \alpha - i\eta$ (this will actually be, for $\alpha_j = 0$, the essential action of L^η_α on the kernel of L^0_0). For the family of right-hand sides $y^\eta_\alpha = 1$, we find the solutions

$$u^\eta_\alpha = (L^\eta_\alpha)^{-1}(y^\eta_\alpha) = \frac{1}{\alpha - i\eta} . \tag{4.6}$$

We observe that u_α^η has a singularity in $(\alpha, \eta) = (0, 0)$. This singular behavior was somehow to be expected, since y_α^η is not vanishing in $(\alpha, \eta) = (0, 0)$. Let us therefore look at a right-hand side that vanishes in the singular point; we investigate $y_\alpha^\eta = \alpha$ with $y_0^0 = 0$. The solution for this right-hand side is

$$u_\alpha^\eta = (L_\alpha^\eta)^{-1}(y_\alpha^\eta) = \frac{\alpha}{\alpha - i\eta}\,. \tag{4.7}$$

We observe that the solution is bounded. On the other hand, the solution family is not continuous at $(0, 0)$. Indeed, along the two coordinate axes, we find the following: $u_\alpha^0 = 1$ for all α and $u_0^\eta = 0$ for all η.

The following theorem considers the local situation with only one critical value α. Once more, without loss of generality, we choose the critical point to be $\alpha = 0$.

Theorem 4.2 (Functional analysis II). *Let X be a Hilbert space and L_α^η be a two-parameter family of Fredholm operators in the sense of Definition 4.1. Let $I \ni \alpha \mapsto y_\alpha \in X$ be a family of right-hand sides that depend on Lipschitz continuously on $\alpha \in I$. Let the following properties be satisfied:*

(a) $L_\alpha^\eta : X \to X$ is invertible for all $(\alpha, \eta) \in ((-\varepsilon, \varepsilon) \times [0, \varepsilon))\backslash(0, 0)$.
(b) With $\mathcal{N} := \ker(L_0^0)$ and $\mathcal{R} := L_0^0(X)$ and $P \in \mathcal{L}(\mathcal{N}, \mathcal{N})$ the projection onto $\mathcal{N}$ corresponding to $X = \mathcal{N} + \mathcal{R}$, the operator $M_\eta := iP\partial_\eta L_0^0|_\mathcal{N} \in \mathcal{L}(\mathcal{N}, \mathcal{N})$ is self-adjoint and positive definite and $M_\alpha := P\partial_\alpha L_0^0|_\mathcal{N} \in \mathcal{L}(\mathcal{N}, \mathcal{N})$ is self-adjoint and invertible.

Let $u_\alpha^\eta \in X$ be the unique solution of $L_\alpha^\eta u_\alpha^\eta = y_\alpha$ for all $(\alpha, \eta) \in ((-\varepsilon, \varepsilon) \times [0, \varepsilon))\backslash(0, 0)$. Then there exists $\varepsilon_1 \in (0, \varepsilon)$ such that u_α^η has the form

$$u_\alpha^\eta = v_\alpha^\eta + \sum_{\ell=1}^m \frac{\langle Py_0, \phi_\ell\rangle_X}{\lambda_\ell \alpha - i\eta}\,\phi_\ell \text{ for } (\alpha, \eta) \in ((-\varepsilon_1, \varepsilon_1) \times [0, \varepsilon_1))\backslash(0, 0). \tag{4.8}$$

In this representation, $\|v_\alpha^\eta\|_X$ is uniformly bounded with respect to (α, η). The family $\{\phi_\ell \mid \ell = 1, \dots, m\}$, $m = \dim \mathcal{N}$, is an orthonormal eigensystem with eigenvalues $\{\lambda_\ell \mid \ell = 1, \dots, m\}$ of the following generalized eigenvalue problem in the finite dimensional space $\mathcal{N}$:

$$M_\alpha \phi_\ell = \lambda_\ell M_\eta \phi_\ell \text{ in } \mathcal{N} \text{ with normalization } \langle M_\eta \phi_\ell, \phi_{\ell'}\rangle_X = \delta_{\ell,\ell'} \tag{4.9}$$

for $\ell, \ell' = 1, \dots, m$.

Remark 4.2. The difference to Theorem 3.2 is—except of the appearance of the second parameter η—that we do not assume $y_0 \in \mathcal{R}$. This gives the singular behavior of the solution u_α^η when (α, η) tends to $(0, 0)$.

Proof. We obtain the singular part of the solution as the highest order approximation. Considering only the kernel $\mathcal{N}$ and the Taylor expansion $PL_\alpha^\eta|_\mathcal{N} \sim \alpha M_\alpha - i\eta M_\eta$, we solve

$$(\alpha M_\alpha - i\eta M_\eta) w(\alpha, \eta) = Py_0 \tag{4.10}$$

in $\mathcal{N}$. The right-hand side can be expanded with the orthonormal basis; we write $Py_0 = \sum_{\ell=1}^m \langle Py_0, \phi_\ell\rangle_X \phi_\ell$. The unique solution $w(\alpha, \eta)$ is given by

$$w(\alpha, \eta) = \sum_{\ell=1}^m \frac{\langle Py_0, \phi_\ell\rangle_X}{\lambda_\ell \alpha - i\eta}\,\phi_\ell\,, \tag{4.11}$$

as can be checked by inserting into (4.10).

Similar to the proof of Theorem 3.2, we write u_α^η in the form $u_\alpha^\eta = w(\alpha, \eta) + u^N(\alpha, \eta) + u^R(\alpha, \eta)$, where $u^N(\alpha, \eta) \in \mathcal{N}$ and $u^R(\alpha, \eta) \in \mathcal{R}$ for every η and α. The equation $L_\alpha^\eta u_\alpha^\eta = y_\alpha$ is then equivalent to

$$\begin{bmatrix} PL_\alpha^\eta|_\mathcal{N} & PL_\alpha^\eta|_\mathcal{R} \\ QL_\alpha^\eta|_\mathcal{N} & QL_\alpha^\eta|_\mathcal{R} \end{bmatrix} \begin{pmatrix} w(\alpha, \eta) + u^N(\alpha, \eta) \\ u^R(\alpha, \eta) \end{pmatrix} = \begin{pmatrix} Py_\alpha \\ Qy_\alpha \end{pmatrix} \text{ in } \mathcal{N} \times \mathcal{R}\,. \tag{4.12}$$

The second line can be written as

$$QL_\alpha^\eta u^R(\alpha,\eta) = -QL_\alpha^\eta w(\alpha,\eta) - QL_\alpha^\eta u^N(\alpha,\eta) + Qy_\alpha\,. \tag{4.13}$$

The operator $QL_0^0|_\mathcal{R}$ is an isomorphism from $\mathcal{R}$ onto itself. This implies that, for sufficiently small η and $|\alpha|$, the inverse operators $[QL_\alpha^\eta|_\mathcal{R}]^{-1}$ exist and are bounded from $\mathcal{R}$ onto itself. Furthermore, they depend twice continuously differentiable on η and α for sufficiently small η and $|\alpha|$. We claim that the first term on the right-hand side of (4.13) is bounded. Indeed, using $w(\alpha,\eta) \in \mathcal{N}$, we have

$$QL_\alpha^\eta w(\alpha,\eta) = Q[L_\alpha^\eta - L_0^0]w(\alpha,\eta) = O(|(\alpha,\eta)|)\,\|w(\alpha,\eta)\| = O(1)$$

by the differentiability of L_α^η and the fact that $\|w(\alpha,\eta)\| = O(|(\alpha,\eta)|^{-1})$. This implies that (4.13) can be solved with $u^R(\alpha,\eta)$ of the form

$$u^R(\alpha,\eta) = -[QL_\alpha^\eta|_\mathcal{R}]^{-1}QL_\alpha^\eta u^N(\alpha,\eta) + u_1^R(\alpha,\eta), \tag{4.14}$$

with a bounded family $u_1^R(\alpha,\eta) \in \mathcal{R}$, which depends only on y_0 and y_α.

Substituting u^R into the first equation of (4.12) yields

$$\begin{aligned}\left(PL_\alpha^\eta - PL_\alpha^\eta[QL_\alpha^\eta|_\mathcal{R}]^{-1}QL_\alpha^\eta\right)u^N(\alpha,\eta) &= Py_\alpha - PL_\alpha^\eta w(\alpha,\eta) - PL_\alpha^\eta u_1^R(\alpha,\eta)\\ &= Py_\alpha - Py_0 - O(|(\alpha,\eta)|^2)\,\|w(\alpha,\eta)\| - O(|(\alpha,\eta)|) = O(|(\alpha,\eta)|).\end{aligned} \tag{4.15}$$

In the second equality, we used $PL_\alpha^\eta - (PL_0^0 + \alpha M_\alpha - i\eta M_\eta) = O(|(\alpha,\eta)|^2)$ and the construction of $w(\alpha,\eta)$. Furthermore, for the last term, we exploited $PL_0^0 = 0$ from the definition of P and the differentiability of the family L_α^η.

Equation (4.15) has the form $\tilde{L}_\alpha^\eta u^N(\alpha,\eta) = \tilde{y}_\alpha^\eta$ with an operator $\tilde{L}_\alpha^\eta$ from $\mathcal{N}$ into itself, with $\tilde{L}_0^0 = 0$ and $\tilde{y}_0^0 = 0$. We claim that the partial derivative $\partial_\xi \tilde{L}_0^0$ is invertible for every $0 \neq \xi \in \mathbb{R}^2$ with $\xi_2 \geq 0$. Indeed, differentiating the second part of $\tilde{L}_\alpha^\eta$ with the chain rule gives three terms. Differentiating the first or second factor leaves the third factor QL_0^0 unchanged, and this third factor is the trivial map on $\mathcal{N}$. Differentiating the third factor leaves the first two factors $PL_0^0\left[QL_0^0|_\mathcal{R}\right]^{-1}$ unchanged, but this operator vanishes because of $PL_0^0 = 0$. Therefore, there remains only the derivative of the first term: $\partial_\xi \tilde{L}_0^0 = \partial_\xi PL_0^0 = \xi_1 M_\alpha - i\xi_2 M_\eta$, which is invertible (as seen already in (4.11)).

A theorem like Theorem 3.2 with two parameters (see Remark 3.3) implies that the solution family $u^N(\alpha,\eta)$ is bounded. We note that it cannot be expected that the solution family is continuous; see the example in (4.7). □

4.3 | Application of the functional analysis result

We want to apply Theorem 4.2 to Equation (4.2), which we write again in the form $L_\alpha^\eta u_\alpha^\eta = y_\alpha$ for $u_\alpha^\eta(x) = \hat{u}^\eta(x,\alpha)e^{-i\alpha x_1}$. We consider a fixed parameter $\alpha_j \in I$ for some $j \in \{1,\ldots,J\}$ and recall that $\ker(L_{\alpha_j}^0) = \{\phi e^{-i\alpha_j x_1}\,|\,\phi \in Y_j\}$ where Y_j has been defined in (3.12). Shifting the critical value $\alpha = 0$ in Theorem 4.2 to $\alpha = \alpha_j$ yields the following decomposition.

Proposition 4.3 (Representation of solutions in Floquet–Bloch space). *Let Assumption 3.5 hold, let $j \in \{1,\ldots,J\}$ be fixed and let $f \in L_*^2(\Omega)$ be given. Then there exists $\varepsilon_1 \in (0,\varepsilon)$ such that for $\eta \in (0,\varepsilon_1)$ and $|\alpha - \alpha_j| < \varepsilon_1$, the unique solution $\hat{u}^\eta(\cdot,\alpha) \in H_\alpha^1(W)$ of (4.2) has a decomposition in the form*

$$\hat{u}^\eta(x,\alpha) = v_j^\eta(x,\alpha) + \sum_{\ell=1}^{m_j} \frac{\langle \hat{f}(\cdot,\alpha_j), \phi_{\ell,j}\rangle_{L^2(W)}}{\lambda_{\ell,j}(\alpha-\alpha_j) - i\eta}\,\phi_{\ell,j}(x)\,e^{i(\alpha-\alpha_j)x_1}, \tag{4.16}$$

for almost every $x \in W$. Here, $\|v_j^\eta(\cdot,\alpha)\|_{H^1(W)}$ is uniformly bounded with respect to (α,η), and $\left\{\phi_{\ell,j}\,|\,\ell = 1,\ldots,m_j\right\}$, $m_j = \dim Y_j$, is an orthonormal eigensystem with eigenvalues $\left\{\lambda_{\ell,j}\,|\,\ell = 1,\ldots,m_j\right\}$ of the following generalized eigenvalue problem in the finite dimensional space Y_j:

$$E(\phi_{\ell,j},\psi) = \lambda_{\ell,j}\,2k\int_W n\,\phi_{\ell,j}\,\bar{\psi} \text{ for all } \psi \in Y_j \tag{4.17}$$

with normalization $2k\int_W n\,\phi_{\ell,j}\,\overline{\phi_{\ell',j}} = \delta_{\ell,\ell'}$.

Proof. In the end of subsection 4.1, we have obtained characterizations for M_α and M_η; they show that the abstract eigenvalue problem (4.9) reduces to the problem to determine λ_ℓ and $\phi_\ell \in \ker(L^0_{\alpha_j})$ with $E\left(\phi_\ell e^{i\alpha x_1}, \varphi e^{i\alpha x_1}\right) = \lambda_\ell 2k \int_W n \phi_\ell \bar{\varphi}$ for all $\varphi \in \ker(L^0_{\alpha_j})$ which coincides with (4.17) when replacing $\phi_\ell e^{i\alpha x_1}$ and $\varphi e^{i\alpha x_1}$ by $\phi_{\ell,j} \in Y_j$ and $\psi \in Y_j$, respectively.

Formula (4.8) of Theorem 4.2 (for singularity at α_j instead of 0) yields the representation

$$\hat{u}^\eta(x,\alpha)e^{-i\alpha x_1} = u^\eta_\alpha(x) = v^\eta_\alpha(x) + \sum_{\ell=1}^{m_j} \frac{\langle y_{\alpha_j}, \phi_{\ell,j} e^{-i\alpha_j x_1}\rangle_{H^1(W)}}{\lambda_{\ell,j}(\alpha-\alpha_j) - i\eta} \phi_{\ell,j}(x) e^{-i\alpha_j x_1}$$

for $x \in W$. The identity $\langle y_{\alpha_j}, \phi_{\ell,j} e^{-i\alpha_j x_1}\rangle_{H^1(W)} = \langle \hat{f}(\cdot,\alpha_j), \phi_{\ell,j}\rangle_{L^2(W)}$ follows from the definition of y_α for $\varphi(x) = \phi_{\ell,j}(x) e^{-i\alpha_j x_1}$. □

The inverse Floquet–Bloch transform

With (4.16), we have found an expression for the Floquet–Bloch transform $\hat{u}^\eta$ of the solution u^η. Using the inverse transform yields an expression for u^η.

For the subsequent theorem, let $\rho_\pm$ be two admissible cut-off functions as described in Definition 3.10

Theorem 4.4 (LAP). *We consider solutions $u^\eta \in H^1_0(\Omega)$ of (4.1) for the right-hand side $f \in L^2_*(\Omega)$. Let Assumption 3.5 be satisfied. We use the eigenvalues and eigenfunctions $\lambda_{\ell,j}$ and $\phi_{\ell,j}$ of Proposition 4.3. Then, as η tends to zero, $u^\eta \in H^1_0(\Omega)$ converge to a solution $u \in H^1_{loc}\left(\overline{\Omega}\right)$ of (4.1) with $\eta = 0$. Denoting cut-off functions as $\rho_{\ell,j} := \rho_{sign(\lambda_{\ell,j})}$, the limit u can be written as*

$$u(x) = v(x) + \sum_{j=1}^{J}\sum_{\ell=1}^{m_j} a_{\ell,j}\, \rho_{\ell,j}(x_1)\, \phi_{\ell,j}(x) \text{ with } a_{\ell,j} = 2\pi i \frac{\langle f, \phi_{\ell,j}\rangle_{L^2(\Omega)}}{|\lambda_{\ell,j}|} \tag{4.18}$$

and $v \in H^1(\Omega)$. The convergence $u^\eta \to u$ is a local convergence: For every $R > 0$ and $\Omega_R := \{x \in \Omega \,|\, |x_1| < R\}$, the restricted functions converge strongly in $H^1(\Omega_R)$.

Remark 4.3. We will derive the result for a specific pair of cut-off functions, namely, for some suitably chosen $\varepsilon > 0$,

$$\rho_\pm(x) := \frac{1}{2} \pm \frac{1}{\pi}\int_0^{\varepsilon x_1} \frac{\sin t}{t}\, dt. \tag{4.19}$$

We note that the integral term behaves like $\int_0^{x_1} \frac{\sin t}{t}\, dt = \pm\frac{\pi}{2} + \mathcal{O}(1/|x_1|)$ as $\pm x_1 \to \infty$. This implies that the two functions $\rho_\pm$ have the required properties of cut-off functions of Definition 3.10.

By Remark 2 after Theorem 3.12, the solution u is independent of the choice of the cut-off functions. This implies the following: When we verify that the limit solution u satisfies (4.18) with the cut-off functions of (4.19), then u satisfies (4.18) for every choice of admissible cut-off functions.

Proof. The solution u^η is the inverse Floquet–Bloch transform of $\hat{u}^\eta$; hence, it is given by an integral over the interval $I = [-1/2, 1/2]$; see (A4).

We decompose the interval I in the form $I = \bigcup_{j=1}^{J}(\alpha_j - \varepsilon, \alpha_j + \varepsilon) \cup U$ where $U := I \setminus \bigcup_{j=1}^{J}(\alpha_j - \varepsilon, \alpha_j + \varepsilon)$ and where $\varepsilon > 0$ is chosen such that the intervals $(\alpha_j - \varepsilon, \alpha_j + \varepsilon)$ do not intersect each other and allow the representation (4.16). We have for $x \in \Omega$

$$\begin{aligned} u^\eta(x) &= \int_{-1/2}^{1/2} \hat{u}^\eta(x,\alpha)\, d\alpha = \int_U \hat{u}^\eta(x,\alpha)\, d\alpha + \sum_{j=1}^{J}\int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} \hat{u}^\eta(x,\alpha)\, d\alpha \\ &= \int_U \hat{u}^\eta(x,\alpha)\, d\alpha + \sum_{j=1}^{J}\int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} v^\eta_j(x,\alpha)\, d\alpha + \sum_{j=1}^{J}\sum_{\ell=1}^{m_j} \langle \hat{f}(\cdot,\alpha_j), \phi_{\ell,j}\rangle_{L^2(W)} \int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} \frac{e^{i(\alpha-\alpha_j)x_1}}{\lambda_{\ell,j}(\alpha-\alpha_j) - i\eta}\, d\alpha\, \phi_{\ell,j}(x). \end{aligned}$$

We now consider $\eta \to 0$ in the different terms.

On U we have convergence in the space $C^0\left(U, H^1(W)\right)$ of $\hat{u}^\eta$ to some function $\hat{w} \in C^0\left(U, H^1(W)\right)$. Therefore, $\int_U \hat{u}^\eta(x,\alpha)\, d\alpha$ converges to $w(x) := \int_U w(x,\alpha)\, d\alpha$ in $H^1(\Omega)$ by the boundedness of the inverse Floquet-Bloch transform. In particular, $w \in H^1(\Omega)$.

For fixed $j \in \{1, \dots, J\}$, we next treat the integral $\int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} v_j^\eta(x,\alpha)\,d\alpha$. The integrand v_j^η tends to v_j^0 in $L^2\left((\alpha_j-\varepsilon,\alpha_j+\varepsilon), H^1(W)\right)$ by Lebesgue's theorem of dominated convergence because $v_j^\eta(\cdot,\alpha)$ tends to $v_j^0(\cdot,\alpha)$ in $H^1(W)$ for every $\alpha \neq \alpha_j$ and is uniformly bounded with respect to α and η. Again, the boundedness of the inverse Floquet-Bloch transform yields convergence of $\int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} v_j^\eta(x,\alpha)\,d\alpha$ to $\int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} v_j^0(x,\alpha)\,d\alpha$ in $H^1(\Omega)$.

Finally, we consider the integral in the last term for fixed j and ℓ. With a parameter transformation, we write the integral as

$$\int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} \frac{e^{i(\alpha-\alpha_j)x_1}}{\lambda_{\ell,j}(\alpha-\alpha_j)-i\eta}\,d\alpha = \int_{-\varepsilon}^{\varepsilon} \frac{e^{i\alpha x_1}}{\lambda_{\ell,j}\alpha - i\eta}\,d\alpha\,. \tag{4.20}$$

In the appendix, see (B1); we show that, for $\rho_\pm$ from (4.19), this integral converges to $\frac{2\pi i}{|\lambda_{\ell,j}|}\rho_{\mathrm{sign}(\lambda_{\ell,j})}(x_1)$, uniformly with respect to $|x_1| \le R$ for every $R > 0$. Altogether, we have shown the local convergence of u^η to

$$u(x) = v(x) + 2\pi i \sum_{j=1}^{J}\sum_{\ell=1}^{m_j} \frac{\langle \hat f(\cdot,\alpha_j), \phi_{\ell,j}\rangle_{L^2(W)}}{|\lambda_{\ell,j}|}\rho_{\ell,j}(x_1)\,\phi_{\ell,j}(x)$$

for some $v \in H^1(\Omega)$. It remains to note that $\langle \hat f(\cdot,\alpha_j), \phi_{\ell,j}\rangle_{L^2(W)} = \langle f, \phi_{\ell,j}\rangle_{L^2(\Omega)}$, which was stated and shown in the proof of Lemma 3.9, exploiting the quasi-periodicity of $\phi_{\ell,j}$. □

5 | ALTERNATIVE DAMPING APPROACHES

With Equation (4.1), we have analyzed the LAP for a specific absorption term: k was replaced by $k + i\eta$. Other damping mechanisms are also physically relevant, for example, nonhomogeneous damping in the k-part or damping in the elliptic-part. We investigate here the LAP for these alternative damping mechanisms.

Nonhomogeneous damping in the k-part

We choose a nonnegative real valued function $p \in L^\infty(\Omega)$ that is 2π-periodic with respect to x_1 and with a positive lower bound, $p \ge p_0 > 0$ on Ω. We consider

$$-\Delta u^\eta - k^2(n + i\eta p)u^\eta = f \ \text{ in } \Omega \tag{5.1}$$

with the usual boundary condition $u^\eta = 0$ on $\partial\Omega$. This is a modification of the homogeneous damping of (4.1). Once more, an application of the Lax–Milgram theorem yields that the equation is uniquely solvable in $H^1(\Omega)$ for every $\eta > 0$. The variational form of the Floquet–Bloch transformed equation is equivalent to $L_\alpha^\eta u_\alpha^\eta = y_\alpha$ for $u_\alpha^\eta(x) = \hat u^\eta(x,\alpha)e^{-i\alpha x_1}$, where y_α is given by (4.4) and L_α^η by (4.3), with $k + i\eta$ replaced by k and with the refractive index n replaced by $n + i\eta p$.

The operator M_η is given by a partial derivative of L_α^η with respect to η. We calculate it to be

$$\langle M_\eta u, \varphi\rangle := i\partial_\eta \langle L_\alpha^\eta u, \varphi\rangle_{H^1(W)} = k^2 \int_W p\,u\,\bar\varphi\,.$$

Therefore, the eigenvalue problem (4.17) has to be replaced by

$$E(\phi_{\ell,j}, \psi) = \lambda_{\ell,j} k^2 \int_W p\,\phi_{\ell,j}\,\bar\psi \ \text{ for all } \ \psi \in Y_j\,. \tag{5.2}$$

Nonhomogeneous damping in the elliptic part

As a second form of damping, we consider, for $p \in L^\infty(\Omega)$ as above,

$$-\nabla\cdot((1 - i\eta p)\nabla u^\eta) - k^2 n u^\eta = f \ \text{ in } \Omega, \tag{5.3}$$

with the usual boundary condition $u^\eta = 0$ on $\partial\Omega$. The variational form is to find $u^\eta \in H_0^1(\Omega)$ with

$$\int_\Omega (1 - i\eta p)\nabla u^\eta \cdot \nabla\overline{\varphi} - k^2 n u^\eta \bar\varphi = \int_W f\,\overline{\varphi} \quad \text{for all } \ \varphi \in H_0^1(\Omega)\,.$$

The theorem by Lax–Milgram yields existence and uniqueness. The periodic form $u_\alpha^\eta(x) = \hat{u}^\eta(x,\alpha)e^{-i\alpha x_1}$ of the Floquet–Bloch transform satisfies $L_\alpha^\eta u_\alpha^\eta = y_\alpha$, where y_α is again given by (4.4) and L_α^η by

$$\langle L_\alpha^\eta u, \varphi\rangle_{H^1(W)} = -k^2 \int_W n\, u\, \bar{\varphi} + \int_W (1 - i\eta p) \nabla \left(u(x)e^{i\alpha x_1}\right) \cdot \nabla \left(\overline{\varphi(x)e^{i\alpha x_1}}\right) dx$$

for $u, \varphi \in H^1_{\mathrm{per}}(W)$. The operator M_η is now

$$\langle M_\eta u, \varphi\rangle := i\partial_\eta \langle L_\alpha^\eta u, \varphi\rangle_{H^1(W)} = \int_W p \nabla \left(u(x)e^{i\alpha x_1}\right) \cdot \nabla \left(\overline{\varphi(x)e^{i\alpha x_1}}\right) dx. \tag{5.4}$$

Therefore, the eigenvalue problem (4.17) has to be replaced by

$$E(\phi_{\ell,j}, \psi) = \lambda_{\ell,j} \int_W p \nabla \phi_{\ell,j} \cdot \nabla \overline{\psi} \quad \text{for all } \psi \in Y_j. \tag{5.5}$$

Example 5.1 (The standard example continued)**.** We continue Example 3.13, where we have found two linearly independent eigenfunctions ϕ_1 and ϕ_2 spanning Y^α for $\alpha = 1/4$. The wave ϕ_1 is right-going, and the wave ϕ_2 is left-going.

We now investigate different eigenvalue problems that are generated by different LAPs. The abstract eigenvalue problem is stated in (4.9); it uses the positive definite operator $M_\eta := iP\partial_\eta L_0^0|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ and the self-adjoint operator $M_\alpha := P\partial_\alpha L_0^0|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$.

For the standard absorption mechanism of (4.1), M_η and M_α are given, loosely speaking, by a multiplication operator (factor $2kn$) and by the form E, respectively. The eigenvalue problem was calculated to be (4.17). For our concrete example, ϕ_1 and ϕ_2 are indeed eigenfunctions for this problem. The eigenvalues are

$$\lambda_j = \frac{E(\phi_j, \phi_j)}{2k\|\phi_j\|^2_{L^2(W)}}, \text{ hence } \lambda_1 = \frac{\alpha}{k} > 0 \text{ and } \lambda_2 = \frac{\alpha - 2}{k} < 0. \tag{5.6}$$

For a solution $u = v + w$ of the radiation problem, the propagating function w has the form $w = a_1\rho_+\phi_1 + a_2\rho_-\phi_2$. In particular, when $\rho'_\pm$ has support in $(-L, L)$, the function w coincides with a multiple of ϕ_1 for $x_1 \geq L$ and with a multiple of ϕ_2 for $x_1 \leq -L$.

Let us now choose a different absorption principle. Referring to (5.2), we consider $\langle u, v\rangle_p = k^2 \int_W p\, u\, \bar{v}\, dx$ with some positive function $p \in L^\infty(W)$. The eigenvalue problem (3.14) takes the form $E(\tilde{\phi}, \phi_j) = \tilde{\lambda}\langle\tilde{\phi}, \phi_j\rangle_p$ for $j = 1, 2$. Making the ansatz $\tilde{\phi} = a_1\phi_1 + a_2\phi_2$ leads to the generalized eigenvalue problem

$$\begin{bmatrix} E(\phi_1, \phi_1) & 0 \\ 0 & E(\phi_2, \phi_2) \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \tilde{\lambda} \begin{bmatrix} \langle\phi_1, \phi_1\rangle_p & \langle\phi_1, \phi_2\rangle_p \\ \langle\phi_2, \phi_1\rangle_p & \langle\phi_2, \phi_2\rangle_p \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Two normalized orthogonal solutions to this problem are given by two complex vectors (a_1, a_2) and (b_1, b_2). Accordingly, we find new eigenfunctions $\tilde{\phi}_1 = a_1\phi_1 + a_2\phi_2$ and $\tilde{\phi}_2 = b_1\phi_1 + b_2\phi_2$. This means that the wave that is outgoing to the right is, for example, $\tilde{\phi}_1 = a_1\phi_1 + a_2\phi_2$. This function is, for a generic coefficient p, neither a multiple of ϕ_1 nor a multiple of ϕ_2. The limiting absorption process then provides a radiating solution of the limit problem that uses on the right the function $\rho_+(a_1\phi_1 + a_2\phi_2)$. It is hence different from the previously obtained limit solution.

We obtain that the radiation condition indeed depends on the choice of the inner product or, in other words, on the damping mechanism.

6 | TWO SPACES OF HOMOGENEOUS SOLUTIONS

Let us recall the spaces that were used in the above constructions: The space Y_j of (3.12) consists of α_j-quasiperiodic homogeneous solutions,

$$Y_j = Y^{\alpha_j} = \left\{ u \in H^1_{\alpha_j}(\Omega) \,|\, (\Delta + k^2 n)u = 0 \text{ in } \Omega, u = 0 \text{ on } \mathbb{R} \times \partial S \right\}.$$

We recall that $\{\alpha_j \,|\, j = 1, \ldots, J\} \subset [-1/2, 1/2]$ are the quasi-moments that correspond to nontrivial spaces Y^α. In the above formula, we identified $H^1_{\alpha_j}(W)$ with $H^1_{\alpha_j}(\Omega)$; the canonical identification is given by the α_j-quasiperiodic extension of a function in $H^1_{\alpha_j}(W)$ (and, vice versa, the restriction to a function on W). We furthermore introduced in (3.13) the space

$$Y = \bigoplus_{j=1}^{J} Y_j \subset H^1(W), \text{ identified with } Y \subset H^1_{\text{loc}}\left(\bar{\Omega}\right). \tag{6.1}$$

It has a basis $\{\phi_\ell \,|\, \ell = 1, \ldots, L\}$ with orthogonality $E(\phi_\ell, \phi_{\ell'}) = 0$ for $\ell \neq \ell'$.

Let us consider another space, the space B of bounded solutions. That space was extensively used in [11] (where it was named X). In order to impose a boundedness property, we introduce the norm $\|U\|_{sL} := \sup_{\ell \in \mathbb{Z}} \|U|_{W_\ell}\|_{L^2(W_\ell)}$ for functions $U \in L^2_{\text{loc}}(\Omega)$, where $W_\ell := (2\pi\ell, 2\pi\ell + 2\pi) \times S$. The space of bounded homogeneous solutions is defined as

$$B := \left\{ U \in H^1_{\text{loc}}\left(\bar{\Omega}\right) \,|\, (\Delta + k^2 n)U = 0 \text{ in } \Omega,\, U = 0 \text{ on } \mathbb{R} \times \partial S,\, \|U\|_{sL} < \infty \right\}. \tag{6.2}$$

It is clear that every quasiperiodic homogeneous solutions is a bounded homogeneous solution; hence, $Y \subset B$. Our aim is to show that the spaces Y and B actually coincide.

Before we formulate the corresponding result, we note that an equivalent norm is obtained when we measure the H^1-norm in every cell.

Lemma 6.1 (Equivalent norms). *There exists a constant $C > 0$ such that*

$$\sup_{\ell \in \mathbb{Z}} \|U|_{W_\ell}\|_{H^1(W_\ell)} \leq C\|U\|_{sL} = C \sup_{\ell \in \mathbb{Z}} \|U|_{W_\ell}\|_{L^2(W_\ell)} \text{ for all } U \in B. \tag{6.3}$$

Proof. The lemma follows from Caccioppoli's inequality for solutions of elliptic problems. □

We can now give the characterization of B.

Theorem 6.2 (Every bounded homogeneous solution is a linear combination of quasiperiodic homogeneous solutions). *When Assumption 3.5 holds, then the spaces Y of (6.1) and B of (6.2) coincide,*

$$Y = B. \tag{6.4}$$

The proof is given in the next subsection. We provide the proof in a more abstract setting such that it covers, e.g., compact perturbations of periodic media. If the reader wants to see the proof of Theorem 6.2 immediately: It is possible to jump to the proof of Theorem 6.5 and to read it as a proof of Theorem 6.2.

6.1 | A generalized setting

We write A for the underlying self-adjoint differential operator of second order, defined on some domain $\Omega \subset \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$. In the main part of this text, we treat $A = -\Delta - k^2 n$. By contrast, the next result holds also for compact perturbations of this operator, for example $A = -\Delta - k^2(n + q)$ where q has bounded support, or $A = -\nabla \cdot ((I + Q)\nabla) - k^2$ where I is the identity and Q has bounded support. We always assume that the operator is everywhere uniformly elliptic. The domain Ω is assumed to be cylindrical outside a compact set: For some bounded set $S \subset \mathbb{R}^{d-1}$ and some $M > 0$, there holds $\Omega \cap \{x \,|\, |x_1| > M\} = (\mathbb{R} \times S) \cap \{x \,|\, |x_1| > M\}$. We always assume that the coefficients are 2π-periodic in x_1 in the cylindrical parts, more precisely: We assume that there exists a self-adjoint operator $\hat{A}$ of second order in $\mathbb{R} \times S$ with

2π-periodic coefficients (in x_1) which coincides with A in $\Omega \cap \{x \,|\, |x_1| > M\}$. The space in which we look for solutions is $H^1_{\text{loc}}(\bar{\Omega})$.

We consider the space B corresponding to the elliptic operator A, here defined with the norm $\|U\|_{sH} := \sup_{\ell \in \mathbb{Z}} \|U\|_{H^1(W_\ell)}$:

$$B := \left\{ u \in H^1_{\text{loc}}(\bar{\Omega}) \,|\, Au = 0 \text{ in } \Omega,\, u = 0 \text{ on } \partial\Omega,\, \|u\|_{sH} < \infty \right\}.$$

We emphasize that, due to the equivalence of norms of Lemma 6.1, in the setting of the last subsection, the definition of B was not changed with respect to (6.2).

In the following, we assume that cut-off functions $\rho_\pm \in C^2(\mathbb{R})$ with $\rho_\pm(x_1) = 1$ for $\pm x_1 \geq 1$ and $\rho_\pm(x_1) = 0$ for $\pm x_1 \leq -1$ are chosen. Let $\{\phi_\ell \,|\, \ell = 1, \ldots, L\}$ be quasiperiodic homogeneous solutions to the unperturbed operator $\hat{A}$ in $\mathbb{R} \times S$ with homogeneous Dirichlet conditions on $\mathbb{R} \times \partial S$. For two disjoint sets $\mathcal{L}^+$ and $\mathcal{L}^-$ with $\mathcal{L}^+ \cup \mathcal{L}^- = \{1, \ldots, L\}$, we set $\rho_\ell = \rho_+$ for $\ell \in \mathcal{L}^+$ and $\rho_\ell = \rho_-$ for $\ell \in \mathcal{L}^-$.

Assumption 6.3 (An abstract existence and uniqueness result)**.** We assume the following on the operator A. For every right-hand side $f \in L^2_*(\Omega)$, there exist uniquely determined functions $v \in H^1_0(\Omega)$ and $w = \sum_{\ell=1}^L \rho_\ell a_\ell \phi_\ell$ such that $u = v + w \in H^1_{\text{loc}}(\bar{\Omega})$ satisfies $Au = f$. The map $L^2_*(\Omega) \ni f \mapsto (a_\ell)_{\ell=1}^L \in \mathbb{C}^L$ is linear and continuous.

We note that Assumption 6.3 is verified in the standard setting of this contribution: For $A = -\Delta - k^2 n$ on the domain $\Omega = \mathbb{R} \times S$ with $S \subset \mathbb{R}^{d-1}$ a bounded Lipschitz domain, Assumption 3.5 implies Assumption 6.3. This is shown in Theorem 3.12.

For cylindrical domains and periodic coefficients, the space Y is defined in (3.13). When we treat compact perturbations of this setting (as described above), we have to define the space Y in a different way. We construct as follows: Let $\theta \in C^2(\mathbb{R})$ be any function with $\theta(x_1) = 1$ for $|x_1| \geq M + 1$ and $\theta(x_1) = 0$ for $|x_1| \leq M$. For fixed $\ell \in \{1, \ldots, L\}$, we define the incident field $u^{\text{inc}}(x) := \theta(x_1)\phi_\ell(x)$ and seek for a solution ϕ^t_ℓ (total field) of $A\phi^t_\ell = 0$ in the form $\phi^t_\ell = u^{\text{inc}} + \phi^s_\ell$; here ϕ^s_ℓ is the scattered field, which has to satisfy the radiation condition. Assumption 6.3 allows to solve for $u = \phi^s_\ell$, since $A\phi^s_\ell = f := -A(\theta\phi_\ell)$ has compact support. Performing the construction of ϕ^t_ℓ for every ℓ, we can define

$$Y := \text{span}\left\{ \phi^t_\ell \,|\, \ell = 1, \ldots, L \right\}. \tag{6.5}$$

The following lemma provides that the dimension of Y is L.

Lemma 6.4 (Dimension of Y in compactly perturbed setting)**.** *The total fields $(\phi^t_\ell)_{1 \leq \ell \leq L}$ are linearly independent; there holds* $\dim Y = L$.

Proof. Let $\sum_\ell c_\ell \phi^t_\ell \equiv 0$ be a linear combination of the trivial function. We can consider the incident field $u^{\text{inc}} := \sum_\ell c_\ell \theta \phi_\ell$ and solve for the corresponding total field u^t: By linearity of the equation, we find $u^t := \sum_\ell c_\ell \phi^t_\ell \equiv 0$ with the scattered field $u^s := \sum_\ell c_\ell \phi^s_\ell$ satisfying $0 = u^t = u^{\text{inc}} + u^s$.

On this basis, the principle argument is simple: Up to a $H^1_0(\Omega)$-function v_ℓ, each function ϕ^s_ℓ is a linear combination of the outgoing fields, $\phi^s_\ell = v_\ell + \sum_{\ell'} a_{\ell,\ell'} \rho_{\ell'} \phi_{\ell'}$; hence, also, u^s is essentially a linear combination of the outgoing fields. On the other hand, $u^{\text{inc}} = \sum_\ell c_\ell \theta \phi_\ell$ contains each field with a factor c_ℓ. Let us study $\ell \in \mathcal{L}^-$ and a large (positive) position x_1: In the left-hand side of $-u^{\text{inc}} = u^s$, the prefactor of ϕ_ℓ is c_ℓ; in the right-hand side, it is vanishing. This shows that $c_\ell = 0$. Similarly, one argues for $\ell \in \mathcal{L}^+$ by considering positions $x_1 < 0$.

We formalize this argument as follows: With $v := \sum_\ell c_\ell v_\ell$, we calculate

$$\begin{aligned}
-\sum_\ell c_\ell \theta \phi_\ell = -u^{\text{inc}} = u^s = \sum_\ell c_\ell \phi^s_\ell &= v + \sum_\ell \sum_{\ell'} c_\ell a_{\ell,\ell'} \rho_{\ell'} \phi_{\ell'} \\
&= v + \sum_{\ell'} \left[\sum_\ell c_\ell a_{\ell,\ell'} \right] \rho_{\ell'} \phi_{\ell'} = v + \sum_\ell d_\ell \rho_\ell \phi_\ell,
\end{aligned}$$

where $d_\ell := \sum_{\ell'} a_{\ell',\ell} c_{\ell'}$. For $z = (z_1, \tilde{z}) \in \Omega$ and sufficiently large $m \in \mathbb{N}$, we have $z_1 + 2\pi m > M + 1$. Therefore, using the quasi-periodicity of ϕ_ℓ and the evaluation point $z = (z_1 + 2\pi m, \tilde{z})$, we have

$$-\sum_\ell c_\ell e^{2\pi i m \alpha_\ell} \phi_\ell(z) = v(z_1 + 2\pi m, \tilde{z}) + \sum_{\ell \in \mathcal{L}^+} d_\ell e^{2\pi i m \alpha_\ell} \phi_\ell(z).$$

For a subsequence $m \to \infty$, the factors $e^{2\pi i m \alpha_\ell}$ converge to some $e^{i\gamma_\ell}$, and $v(z_1 + 2\pi m, \tilde{z})$ converges to zero. Therefore,

$$-\sum_{\ell \in \mathcal{L}^+} c_\ell \, e^{i\gamma_\ell} \, \phi_\ell - \sum_{\ell \in \mathcal{L}^-} c_\ell \, e^{i\gamma_\ell} \, \phi_\ell = \sum_{\ell \in \mathcal{L}^+} d_\ell \, e^{i\gamma_\ell} \, \phi_\ell \, .$$

Since the ϕ_ℓ are linearly independent, we obtain $\sum_{\ell \in \mathcal{L}^-} c_\ell \, e^{i\gamma_\ell} \, \phi_\ell = 0$, and hence, $c_\ell = 0$ for $\ell \in \mathcal{L}^-$. Analogously, for $m \to -\infty$, we conclude that $c_\ell = 0$ for $\ell \in \mathcal{L}^+$. □

The subsequent theorem provides, in particular, Theorem 6.2.

Theorem 6.5 ($Y = B$ in the abstract setting)**.** *When the existence and uniqueness property of Assumption 6.3 hold, then $Y = B$.*

Proof. The inclusion $Y \subset B$ is clear. We know that Y has dimension $\dim Y = L$. In order to show $B \subset Y$, it suffices to show $\dim B \leq L$.

In this proof we use, for arbitrary $R > M$, the piecewise affine cut-off function $\vartheta_R : \mathbb{R} \to [0,1]$ with $\vartheta_R(s) = 1$ for every $s \in [-R, R]$, $\vartheta_R(s) = 0$ for $|s| \geq R+1$, affine on $[-R-1, -R]$ and on $[R, R+1]$. We interpret ϑ_R also as a function on Ω by setting $\vartheta_R(x) := \vartheta_R(x_1)$.

Step 1: A representation for the coefficients a_ℓ.
Since every coefficient map $L^2_*(\Omega) \ni f \mapsto a_\ell \in \mathbb{C}$ is linear and continuous, we can represent this map by an element $\xi_\ell \in L^2_*(\Omega)$. We find a family $(\xi_\ell)_{1 \leq \ell \leq L}$ such that, for every $f \in L^2_*(\Omega)$,

$$a_\ell = \langle f \, , \xi_\ell \rangle_{L^2_*(\Omega)} = \langle f(x) \, , \xi_\ell(x)(1 + |x_1|^2)^2 \rangle_{L^2(\Omega)} \, . \tag{6.6}$$

Step 2: A scalar product with $U \in B$.
We consider an arbitrary element $U \in B$. We want to calculate, for arbitrary $f \in L^2_*(\Omega)$, the inner product $\langle f, U \rangle_{L^2(\Omega)}$. With this aim, we use the solution $u = v + w \in H^1_{\mathrm{loc}}(\bar{\Omega})$ of $Au = f$ in Ω; see Assumption 6.3 (or in the concrete setting of Theorems 6.2 and 3.12). We write, for $R \to \infty$,

$$\langle f, U \rangle_{L^2(\Omega)} \leftarrow \langle f, U\vartheta_R \rangle_{L^2(\Omega)} = \langle Au, U\vartheta_R \rangle_{L^2(\Omega)} = \langle Av, U\vartheta_R \rangle_{L^2(\Omega)} + \langle Aw, U\vartheta_R \rangle_{L^2(\Omega)} \, ,$$

and evaluate the terms separately. By the self-adjointness of A,

$$\langle Av, U\vartheta_R \rangle_{L^2(\Omega)} = \langle v, A(U\vartheta_R) \rangle_{L^2(\Omega)} \to 0 \tag{6.7}$$

as $R \to \infty$. The convergence follows from $AU = 0$, the boundedness of ∇U in the cells W_ℓ, and the decay property of v. The function $w = \sum_{\ell=1}^{L} a_\ell \, \rho_\ell \, \phi_\ell$ satisfies, for $U \in B$ and R sufficiently large:

$$\langle Aw, U\vartheta_R \rangle_{L^2(\Omega)} = \sum_{\ell=1}^{L} a_\ell \, c_\ell \ \text{ with } \ c_\ell = \langle A(\rho_\ell \phi_\ell), U \rangle_{L^2(\Omega)} \, . \tag{6.8}$$

We therefore obtain

$$\langle f, U \rangle_{L^2(\Omega)} = \sum_{\ell=1}^{L} c_\ell \, a_\ell \, . \tag{6.9}$$

Step 3: Conclusion.
It remains to insert the representation (6.6) of a_ℓ into (6.9). We find

$$\langle f, U \rangle_{L^2(\Omega)} = \sum_{\ell=1}^{L} c_\ell \, \langle f \, , \xi_\ell(x)(1 + |x_1|^2)^2 \rangle_{L^2(\Omega)} \, . \tag{6.10}$$

Since f was arbitrary, we find

$$U(x) = \sum_{\ell=1}^{L} c_\ell \, \xi_\ell(x) \, (1 + |x_1|^2)^2 \tag{6.11}$$

for all $x \in \Omega$. We have therefore represented an arbitrary element $U \in B$ with the L functions $\xi_\ell(x)(1+|x_1|^2)^2$. This implies $\dim B \le L$ and hence the theorem.

□

6.2 | Finite dimension of B in other settings

We return here to the geometry of the main part of this paper, $\Omega = \mathbb{R} \times S$ with S bounded. We note that the space B can be defined for any (positive) refractive index $n \in L^\infty(\Omega)$ without the assumption of periodicity. We ask: Does B have a finite dimension? We do not know the answer in the general case.

One particular case can be treated with the above methods. When $n \in L^\infty(\Omega)$ coincides with a periodic function n^+ for $x_1 \ge M$ and with another periodic function n^- for $x_1 \le -M$ (for some $M > 0$), then B can be characterization much as in the previous subsection: B is spanned by the solutions of scattering problems with incident fields $\phi_\ell^\pm$ (the right-going modes for index n^-) and $\phi_\ell^\mp$ (the left-going modes for index n^+). In particular, in this case, B is finite dimensional.

Another case that allows to show finite dimensionality of B is the following: Let $n \in L^\infty(\Omega)$ be of the form $n(x_1, \tilde{x}) = n_1(x_1) + n_2(\tilde{x})$ for $x_1 \in \mathbb{R}$ and $\tilde{x} \in S$. In this case, we can use separation of variables techniques. Let $\lambda_j \in \mathbb{R}$ and $\phi_j \in H^2(S)$ be the eigenvalues and eigenfunctions, respectively, of the self-adjoint operator $-\tilde{\Delta} - k^2 n_2$, that is,

$$-\tilde{\Delta}\phi_j(\tilde{x}) - k^2 n_2(\tilde{x})\phi_j(\tilde{x}) = \lambda_j \phi_j(\tilde{x}) \ \text{ in } \ S, \ \ \phi_j(\tilde{x}) = 0 \ \text{ on } \ \partial S.$$

Let $U \in B$ be an arbitrary element. For every $x_1 \in \mathbb{R}$, the function $U(x_1, \cdot)$ can be expanded as

$$U(x_1, \tilde{x}) = \sum_{j=1}^{\infty} u_j(x_1)\phi_j(\tilde{x})$$

with some coefficients $u_j(x_1)$. Inserting this expansion in the differential equation $\Delta U + k^2 n_1 U + k^2 n_2 U = 0$ yields

$$u_j''(x_1) + \left(k^2 n_1(x_1) - \lambda_j\right) u_j(x_1) = 0 \text{ for } \ x_1 \in \mathbb{R}.$$

We know that $\lambda_j \to \infty$ as $j \to \infty$. Therefore, there exists $j_0 \in \mathbb{N}$ such that $k^2 n_1(x_1) - \lambda_j \le -1$ for all $j \ge j_0$. Since the equation $u''(x_1) - a(x_1)u(x_1) = 0$ does not allow any bounded solutions if $a > 0$, we conclude that only a finite sum appears in the expansion of U; there holds $U \in \operatorname{span}\left\{u_j(x_1)\phi_j(\tilde{x}) \,|\, j = 1, \dots, j_0\right\}$. Since the ansatz functions are independent of U, we conclude that B has finite dimension.

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A. Kirsch: Conceptualization; methodology; validation; investigation; formal analysis; writing—original draft. **B. Schweizer:** Conceptualization; methodology; formal analysis; validation; investigation; writing—original draft.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflict of interest.

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APPENDIX A: FORMULAS FOR THE FLOQUET-BLOCH TRANSFORM

We treat here only the one-dimensional Floquet–Bloch transform and write $x \in \mathbb{R}$ for the variable. With $W = (0, 2\pi)$ and $I = [-1/2, 1/2]$, the transformation $\mathcal{F}_{\mathrm{FB}} : L^2(\mathbb{R}) \to L^2(W \times I)$, $u \mapsto \hat{u}$, was defined in (2.2) as the continuous extension of

$$\hat{u}(x, \alpha) := \sum_{\ell \in \mathbb{Z}} u(x + 2\pi\ell)\, e^{-i\ell 2\pi\alpha}, \tag{A1}$$

for $x \in W$ and $\alpha \in I$. An elementary calculation shows that $\mathcal{F}_{\text{FB}}$ is an unitary transformation to its image:

$$\begin{aligned}\int_I \langle \hat{u}(\cdot,\alpha), \hat{v}(\cdot,\alpha)\rangle_{L^2(W)}\, d\alpha &= \int_I\int_W \sum_{\ell,k\in\mathbb{Z}} u(x+2\pi\ell)\overline{v(x+2\pi k)}\, e^{-i(\ell-k)2\pi\alpha}\, dx\, d\alpha \\ &= \int_W \sum_{\ell\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \delta_{k\ell}\, u(x+2\pi\ell)\overline{v(x+2\pi k)}\, dx \\ &= \int_W \sum_{\ell\in\mathbb{Z}} u(x+2\pi\ell)\overline{v(x+2\pi\ell)}\, dx = \int_{\mathbb{R}} u\bar{v} = \langle u, v\rangle_{L^2(\mathbb{R})}.\end{aligned} \tag{A2}$$

This also shows that $\mathcal{F}_{\text{FB}}$ is well-defined on $L^2(\mathbb{R})$.

Vice-versa, for $\hat{u} \in L^2(W \times I)$, we define, for $x \in W$ and $k \in \mathbb{Z}$,

$$u(x+2\pi k) := \int_I \hat{u}(x,\beta)\, e^{ik2\pi\beta}\, d\beta. \tag{A3}$$

We claim that this operation defines an inverse $\mathcal{F}_{\text{FB}}^{-1} : \hat{u} \mapsto u$. We start by showing $\mathcal{F}_{\text{FB}}^{-1}\circ\mathcal{F}_{\text{FB}} = \text{id}$. Let $u \in L^2(\mathbb{R})$ be arbitrary and let $\hat{u}$ be defined by (A1). Then, for every $k \in \mathbb{Z}$,

$$\begin{aligned}\int_I \hat{u}(x,\beta)\, e^{ik2\pi\beta}\, d\beta &= \int_I \sum_{\ell\in\mathbb{Z}} u(x+2\pi\ell)\, e^{-i\ell2\pi\beta}\, e^{ik2\pi\beta}\, d\beta \\ &= \sum_{\ell\in\mathbb{Z}} \delta_{k\ell}\, u(x+2\pi\ell) = u(x+2\pi k);\end{aligned}$$

hence the transformation of (A3) indeed recovers the original function.

It remains to show that $\mathcal{F}_{\text{FB}}^{-1}$ of (A3) also defines a right inverse, $\mathcal{F}_{\text{FB}}\circ\mathcal{F}_{\text{FB}}^{-1} = \text{id}$. To this end, we consider an arbitrary function $\hat{u} \in L^2(W \times I)$. We fix a point $x \in W$ and denote the ℓ-th Fourier coefficient of $\hat{u}(x,\cdot)$ by $c_\ell \in \mathbb{C}$ such that, for almost every x, there holds $\hat{u}(x,\alpha) = \sum_{\ell\in\mathbb{Z}} c_\ell e^{-i\ell2\pi\alpha}$. We consider such a point $x \in W$ and evaluate $\mathcal{F}_{\text{FB}}(u)$ for u given by (A3),

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}} u(x+2\pi\ell)\, e^{-i\ell2\pi\alpha} &= \sum_{\ell\in\mathbb{Z}} \int_I \hat{u}(x,\beta)\, e^{i\ell2\pi\beta}\, d\beta\, e^{-i\ell2\pi\alpha} \\ &= \sum_{\ell\in\mathbb{Z}} c_\ell e^{-i\ell2\pi\alpha} = \hat{u}(x,\alpha).\end{aligned}$$

This shows, in particular, that $\mathcal{F}_{\text{FB}} : L^2(\mathbb{R}) \to L^2(W \times I)$ is surjective. We conclude that $\mathcal{F}_{\text{FB}}$ is an isometry and that the inverse is given by (A3).

We close this section with a simplified formula for $\mathcal{F}_{\text{FB}}^{-1}$. When $\hat{u}(\cdot,\beta)$ is interpreted as a β-quasiperiodic function on $\mathbb{R}$, there holds $\hat{u}(x+2\pi k,\beta) = \hat{u}(x,\beta)e^{ik2\pi\beta}$ for every $k \in \mathbb{Z}$. With this extension of $\hat{u}(\cdot,\beta)$, formula (A3) for the inverse yields, for arbitrary $y = x + 2\pi k \in \mathbb{R}$,

$$u(y) := \int_I \hat{u}(y,\beta)\, d\beta. \tag{A4}$$

APPENDIX B: EVALUATION OF A COMPLEX INTEGRAL

This appendix deals with an integral that appears in an inverse Floquet–Bloch transformation; see (4.20). For the following calculations, $\varepsilon > 0$ is an arbitrary number. We calculate

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon} \frac{e^{i\alpha x_1}}{\lambda\alpha - i\eta}\, d\alpha &= \int_{-\varepsilon}^{\varepsilon} \frac{[\cos(\alpha x_1) + i\sin(\alpha x_1)][\lambda\alpha + i\eta]}{\lambda^2\alpha^2 + \eta^2}\, d\alpha \\ &= 2i\eta \int_0^{\varepsilon} \frac{\cos(\alpha x_1)}{\lambda^2\alpha^2 + \eta^2}\, d\alpha + 2i\lambda \int_0^{\varepsilon} \frac{\alpha\sin(\alpha x_1)}{\lambda^2\alpha^2 + \eta^2}\, d\alpha,\end{aligned}$$

where we used that the integral over odd integrands vanishes. Let us start with an analysis of the first term, using the substitution $\alpha = t\eta/|\lambda|$,

$$2i\eta \int_0^{\varepsilon} \frac{\cos(\alpha x_1)}{\lambda^2\alpha^2 + \eta^2}\, d\alpha = \frac{2i\eta^2}{|\lambda|} \int_0^{\varepsilon|\lambda|/\eta} \frac{\cos(t\eta x_1/|\lambda|)}{t^2\eta^2 + \eta^2}\, dt = \frac{2i}{|\lambda|} \int_0^{\varepsilon|\lambda|/\eta} \frac{\cos(t\eta x_1/|\lambda|)}{1 + t^2}\, dt.$$

In the limit $\eta \to 0$, we therefore find, for this term,

$$2i\eta \int_0^{\varepsilon} \frac{\cos(\alpha x_1)}{\lambda^2\alpha^2 + \eta^2}\, d\alpha \to \frac{2i}{|\lambda|} \int_0^{\infty} \frac{1}{1 + t^2}\, dt = \frac{\pi i}{|\lambda|}.$$

The convergence is uniform in x_1 on compact subsets of $\mathbb{R}$. The second integral satisfies, as $\eta \to 0$,

$$2i\lambda \int_0^{\varepsilon} \frac{\alpha \sin(\alpha x_1)}{\lambda^2\alpha^2 + \eta^2}\, d\alpha \to \frac{2i}{\lambda} \int_0^{\varepsilon} \frac{\sin(\alpha x_1)}{\alpha}\, d\alpha = \frac{2i}{\lambda} \int_0^{\varepsilon x_1} \frac{\sin t}{t}\, dt.$$

We obtain, as $\eta \to 0$,

$$\int_{-\varepsilon}^{\varepsilon} \frac{e^{i\alpha x_1}}{\lambda\alpha - i\eta}\, d\alpha \to \frac{2\pi i}{|\lambda|} \left[\frac{1}{2} + \operatorname{sign}(\lambda) \frac{1}{\pi} \int_0^{\varepsilon x_1} \frac{\sin t}{t}\, dt \right]. \tag{B1}$$

The convergence is uniform with respect to $|x_1| \le R$ for every $R > 0$.