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N3LO corrections to zero-jettiness soft function

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We report on the progress of calculating NNNLO corrections for the zero-jettiness soft function in QCD. We present results for one-loop corrections to double-emission contributions and review techniques for calculating the remaining triple-real emission contributions.

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1. Introduction

To make high-precision differential predictions for modern colliders experiments, we need to push theoretical predictions to the NNNLO level. This is significant because at this level, we can account for the most intricate details of the particle interactions, leading to more accurate predictions.

Due to infrared and collinear divergencies, cross sections are divergent at high perturbative orders, and we need to develop a scheme to cancel singularities between contributions with different numbers of additional soft or collinear emissions and contributions with virtual corrections.

Schemes working with colorful final-state particles are especially important. One of the most straightforward solutions is generalizing the slicing scheme with a slicing variable N-jettiness [1–4] for N colorful external emissions for the collision of two partons.

To describe the cross-section in the singular region, we can use factorization and up-to-power corrections in the slicing variable τ to express it in the following form

$$\lim_{\tau\to 0} \mathrm{d}\sigma(O) = B_\tau \otimes B_\tau \otimes S_\tau \otimes J_\tau \otimes \cdots \otimes J_\tau \otimes H_\tau \otimes \mathrm{d}\sigma_{\mathrm{LO}} + O(\tau). \tag{1}$$

The zero-jettiness case considered in the proceedings corresponds to the color singlet final state production in hadronic collisions or two hard jet event production in the electron-positron annihilation or Higgs decay. At the NNNLO level, beam(B_τ) and jet(J_τ) functions are known [5–11], and the soft function(S_τ) is the only missing part.

2. Computation

To calculate the NNNLO contribution to the soft function, we must include all diagrams from Figure 1. These diagrams appear after taking the soft limit of the amplitude squared for the process of the electron-positron annihilation into jets with two hard jets in the final state. In all these diagram sets, we need to insert an appropriate measurement function defining our slicing variable and do phase-space integration.

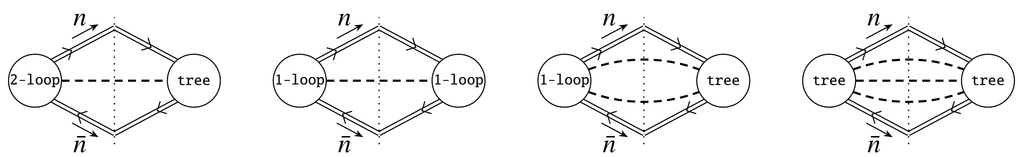

Figure 1: Contributions required for the NNNLO soft function calculation.

We can simplify the calculation by considering amplitudes in the soft limit from the beginning using eikonal Feynman rules for hard parton lines. Measurement delta function for the zero-jettiness slicing variable τ in case of m-soft emissions with momenta k_i

$$\tau = \sum_{i=1}^{m} \min_{q\in\{n,\bar{n}\}} \left[\frac{2q\cdot k_i}{n\cdot\bar{n}}\right] = \sum_{i=1}^{m} \min\{k_i\cdot n, k_i\cdot\bar{n}\}, \tag{2}$$

can be represented in a form better suited for phase-space integration if we replace a minimum function with a set of configurations with an explicit minimum value

$$\delta\left(\tau-\sum_{i=1}^{m}\min\{\alpha_i,\beta_i\}\right)=\delta(\tau-\alpha_1-\alpha_2-\ldots)\theta(\beta_1-\alpha_1)\theta(\beta_2-\alpha_2)\ldots$$
$$+\,\delta(\tau-\alpha_1-\beta_2-\ldots)\theta(\beta_1-\alpha_1)\theta(\alpha_2-\beta_2)\ldots \tag{3}$$

For two-loop diagrams with single emission integration over phase-space is trivial, and the only non-trivial part of the NNLO result is contained in the soft current, which is known [12].

To simplify the problem of the phase space integration for contributions with more emissions, we first apply a modified IBP reduction technique suitable for integrals with theta functions [13] to obtain a smaller set of so-called master integrals and then perform the integrations. Our approach for calculating master integrals can be summarized in two main rules:

- The direct integration technique can be used to calculate a large set of non-trivial integrals after subtracting all possible divergences from the integrands to make integrations convergent.
- We aim to use well-developed techniques in multi-loop calculations to simplify the problem, e.g., the IBP reduction and/or the method of differential equations to reduce the number of integrals requiring direct integration or their complexity.

3. One-loop corrections to double-real emission contribution

At the NNNLO order, one-loop corrections with double emission include diagrams with two gluons or a quark anti-quark pair in the final state. In our recent paper [14], we have calculated both contributions and confirmed previous calculations with final state gluons [15]. The result with a quark pair is new.

We use a technique developed in the paper [13] to reduce integrals with theta functions to the minimal set of master integrals. The main feature of the IBP reduction applied to integrals with theta functions is the appearance of integrals with some of the theta functions replaced by delta functions, making the reduction inhomogeneous. From the definition of the measurement function (3), we start from integrals with two theta functions. After the IBP reduction, we have master integrals with two, one, and zero theta functions.

Many master integrals can be computed by directly integrating over phase space after inserting the appropriate one-loop integral expression. However, we do not have a concise representation of the pentagon integral shown in Figure 2, so we have chosen to use a method of differential equations that applies to all needed integrals.

When writing down differential equations, we consider a set of auxiliary integrals that depend on a set of parameters. Our original integrals can then be obtained from the auxiliary integrals by integrating over these parameters

$$I=\int \mathrm{d}z_1\ldots\mathrm{d}z_n\,J(z_1,\ldots,z_n). \tag{4}$$

For integrals with theta functions we insert integral representations for each theta function $\theta(b-a)=\int b\delta(zb-a)\mathrm{d}z$ and consider each integration variable as a new auxiliary integral parameter. For

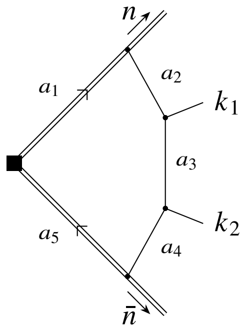

Figure 2: Complicated integrals with five-point and four-point function with $a_3 = 0$ as one-loop integral.

integrals without theta functions we construct auxiliary integrals by insertion of the delta function $\int \mathrm{d}\,(k_1 \cdot k_2)\,\delta\left(k_1 \cdot k_2 - \frac{x}{2}\right)$. Since all auxiliary integrals are free from theta functions, we can apply available tools for IBP reduction [16] to construct differential equations in parameters.

It is essential to transform the obtained differential equations into a ε-form [17]. This form allows us to construct solutions in terms of generalized polylogarithms immediately, and it is beneficial for constructing subtraction terms for integration. A minimal set of required boundary conditions can be calculated by considering the expansion of auxiliary integrals around singular points, where their calculation drastically simplifies.

4. Triple-real emission contribution

Triple-emission contribution consists of two hemisphere contributions. Same-hemisphere contribution with soft gluon emission was calculated before [13, 18]. Our current work focuses on calculating the contribution from the configuration with soft gluon emissions in different hemispheres. We note that the diagrams involving the emission of one gluon plus quark pair, which are also a part of the final answer, can be calculated similarly to the triple gluon emission part.

Calculating emissions in different hemispheres is more challenging than calculating emissions in the same hemisphere. In the same hemisphere case, during IBP reduction, all integrals with three theta functions were reduced to integrals with at least one theta function replaced with a delta function. However, in the present calculation, this is not the case.

As before, the most challenging step is calculating integrals with complicated angle dependence between emitted soft partons contained in integrals with the propagator dependent on all three soft momenta. To overcome this difficulty, we consider a class of auxiliary integrals with massive version of the propagator

$$\frac{1}{(k_1 + k_2 + k_3)^2} \to \frac{1}{(k_1 + k_2 + k_3)^2 + m^2}, \tag{5}$$

and construct a differential equation in m^2. We can calculate all other integrals by direct integration with appropriate subtractions to make integrations finite.

Boundary conditions for the differential equation solution can be calculated at the point $m^2 \to \infty$. Three different regions contribute: one trivial region, where the massive propagator is effectively removed completely, and two non-trivial regions, with contributions proportional to $(m^2)^{-\varepsilon}$ and $(m^2)^{-2\varepsilon}$. For non-trivial regions, the propagator (5) simplifies after expansion, and it becomes possible to perform remaining integrations directly, similar to the case of integrals without such propagator.

With calculated boundary conditions, the system of differential equations can be solved numerically as a sequence of high-precision series expansions between regular points[13, 19, 20] inside the radius of convergence of the corresponding series until we reach the last regular point in the radius of convergence of the series around point $m^2 = 0$. The high-precision numerical result of the integral with $m^2 = 0$ we are interested in is extracted from the specific branch of the generalized expansion constructed around singular point $m^2 = 0$.

Another problem in calculating triple emission contribution, which was already encountered during the same hemisphere configuration calculation [18], is the presence of unregulated divergencies in dimensional regularization. An additional regulator ν was introduced to make integrals well-defined, but due to the additional variable, the IBP reduction of such integrals is more complicated. To overcome this difficulty, we have considered three different strategies to integrals reduction:

1. Complete ν-dependent reduction of the IBP equations system
2. Reduction of the filtered IBP system with all $\frac{1}{\nu}$ divergent integrals removed
3. Reduction of the IBP system for integrals coefficients of the ν-expansion

Each of the three suggested strategies has its benefits and downsides. Full reduction is very time-consuming but provides exact, unexpanded results. Reducing the filtered system containing only well-defined integrals is the fastest option, but it potentially leaves some integrals unreduced. Another option for reduction is to insert ansatz for integrals as the series in ν into the IBP equation system and consider solution of the system for expansion coefficients. Since all the integrals considered have a natural lower bound on the maximal depth of the $1/\nu$ poles, we can extend the IBP system for expansion coefficients with a set of boundary equations that put all deeper expansion coefficients to zero.

The solution of such a system of expansion coefficients is faster than complete system reduction. However, there is a price to pay: Different integral expansion orders act as integrals from different topologies, making the system inhomogeneous. In the problem we are solving, this is not a great difficulty since, from the beginning, we are dealing with a highly inhomogeneous system of IBP equations. Also, during reduction, a more extensive set of master integrals can be produced, containing pole parts of some intermediate steps integrals requiring additional inspection. Most of such pole parts are zero, except the minimal set of integrals we need to calculate with additional regulator.

5. Summary and Outlook

In the proceeding, we summarize the techniques used and present the results of the complete set of one-loop corrections to the NNNLO zero-jettiness soft function. For the triple-emission contribution to the NNNLO zero-jettiness soft function, we provide details of the techniques used to make the calculation of the final result possible. We have finished the calculation of all required master integrals and proceeded with extensive numerical checks of the obtained results, which are also very challenging due to the complicated divergencies structure of considered integrals.

Acknowledgments

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