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On the Gromov–Hausdorff limits of compact surfaces with boundary

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Abstract
In this work we investigate Gromov–Hausdorff limits of compact surfaces carrying length metrics. More precisely, we consider the case where all surfaces have the same Euler characteristic. We give a complete description of the limit spaces and study their topological properties. Our investigation builds on the results of a previous work which treats the case of closed surfaces.

Keywords Gromov–Hausdorff limit · Surface · Length space · Generalized cactoid · Peano space · Continuum

Mathematics Subject Classification 51F99 · 53C20 · 54F15

1 Introduction

Let X be a simply connected compact absolute neighborhood retract (ANR) carrying a length metric and M be a closed connected smooth manifold of dimension larger than two. From a result by Ferry and Okun it follows that X can be obtained as the Gromov–Hausdorff limit of length spaces that are homeomorphic to M (cf. [5, p. 1866]). In dimension two this is not the case. For example a sequence of length spaces that are homeomorphic to the 2-sphere cannot converge to a space that is homeomorphic to the 3-disc (cf. [2, p. 269]).

We recall that a surface is denoted as *closed* if it is compact and its boundary is empty. The aforementioned observation naturally leads to the following question: What do the Gromov–Hausdorff limits of length spaces that are homeomorphic to a fixed closed surface look like? An answer was given by the author in [4, pp. 13, 15].

In the present paper we completely describe the Gromov–Hausdorff limits of length spaces that are homeomorphic to compact surfaces of fixed Euler characteristic. In particular, we allow the possibility that the surfaces have non-empty boundary and thus generalize the main result of the previous work [4]. Our investigation builds on the results of [4] and extends the central concept of a generalized cactoid. As an additional technical difficulty, the limit of the boundaries may display a rather wild behavior. Even the statement of the main result is more complicated since new topological quantities appear.

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It will turn out that the limit spaces satisfy the following topological properties:

Theorem 1.1 *Let X be a space that can be obtained as the Gromov–Hausdorff limit of length spaces that are homeomorphic to a fixed compact surface. Then the following statements apply:*

(1) *X is at most 2-dimensional.*
(2) *X is locally simply connected.*
(3) *There are finitely many compact surfaces $S_1, \ldots, S_n$ and $k \in \mathbb{N}_0$ such that $\pi_1(X)$ is isomorphic to the free product $\pi_1(S_1) * \ldots * \pi_1(S_n) * \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{k\text{-times}}$.*

In the 1920s, Whyburn founded the *cyclic element theory*, the main subject of which is a certain decomposition of Peano spaces (cf. [11, p. 337]). With regard to the local description of the limit spaces, we introduce a key concept of this theory: Let X be a Peano space and $A \subset X$. If every pair of points in A can be connected by a simple closed curve in A, then we denote A as *cyclicly connected*. Moreover we say that A is *maximal cyclic* provided it is nondegenerate (i.e., contains more than one point), cyclicly connected and no proper subset of a cyclicly connected subset in X.

We have the following local description of the limit spaces:

Theorem 1.2 *Let X be a space that can be obtained as the Gromov–Hausdorff limit of length spaces that are homeomorphic to a fixed compact surface. Then every point of X admits an open neighborhood that is homeomorphic to an open subset of some Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc.*

In the aforementioned theorem, as in the rest of this paper, the term “2-disc” refers to the closed 2-disc.

Now we introduce further definitions for the global description:

A Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere is called a *cactoid*. According to Whyburn, the limit of length spaces that are homeomorphic to the 2-sphere is always a cactoid (cf. [19, p. 419]). This result motivates to consider generalizations of cactoids: Let X be a Peano space whose maximal cyclic subsets are compact surfaces and $C \subset X$ be a subcontinuum. Then C is denoted as *admissible* in X provided $T \cap C$ is a point or a boundary component of T for every maximal cyclic subset $T \subset X$.

Definition 1.3 Let X be a Peano space. Then X is called a *generalized cactoid* if the following statements apply:

(1) All maximal cyclic subsets are compact surfaces and only finitely many of them are not homeomorphic to the 2-sphere or the 2-disc.
(2) There are finitely many disjoint admissible subcontinua $C_1, \ldots, C_n \subset X$ such that the boundary components of the maximal cyclic subsets of X are covered by the subcontinua.

There exists a natural choice $C_1, \ldots, C_n$ of the admissible subcontinua as above which is uniquely defined by the following property: The number n is minimal and the union $\cup_{i=1}^n C_i$ is maximal among all choices with n admissible subcontinua (see Lemma 2.16). We define the *boundary* of X as $\cup_{i=1}^n C_i$ and denote it by ∂X. Further we say that C_i is a *boundary component* of X.

Especially we will see that the boundary components are Peano spaces whose maximal cyclic subsets are homeomorphic to the 1-sphere (see Lemma 2.13). According to the original definition of a generalized cactoid, all maximal cyclic subsets are supposed to be closed

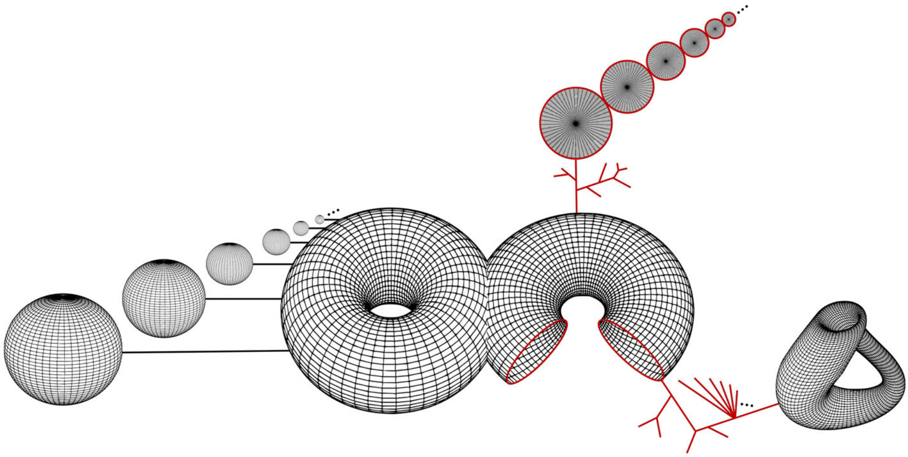

Fig. 1 A generalized cactoid with three boundary components. The boundary of the generalized cactoid is shown in red. Moreover the space has infinitely many maximal cyclic subsets and all but one of them are orientable. The non-orientable maximal cyclic subset is homeomorphic to the Klein bottle which is represented using a parametrization from [16, p. 141]

surfaces (cf. [15, p. 854]). The extension presented here, including the definition of the boundary, is completely new.

Compact surfaces are simple examples of generalized cactoids. For compact surfaces, the usual definition of the boundary coincides with the definition for generalized cactoids. A more advanced example of a generalized cactoid is shown in Fig. 1.

The limit spaces we study are closely related to generalized cactoids. Our description makes use of the following concepts: A space that is isometric to a metric quotient of X whose underlying equivalence relation identifies exactly two points is called a *metric 2-point identification* of X. If we consider a space that can be obtained by a successive application of $k > 0$ metric 2-point identifications to some generalized cactoid X and $p_1, \ldots, p_k$ denotes a choice of the corresponding projection maps, then p_i is called a *boundary identification* provided it identifies two points of $(p_{i-1} \circ \ldots \circ p_0)(\partial X)$ where $p_0 := id_X$.

Finally we want to assign a topological quantity to each generalized cactoid:

The *connectivity number* of a compact surface S is defined as $2 - \chi(S)$. Roughly speaking, this quantity can be calculated by adding the number of boundary components, the number of "cross-caps" and twice the number of "holes". If we subtract the number of boundary components, we get the definition of the *reduced connectivity number* of S.

Example 1.4 Let S be a surface that can be obtained by removing two disjoint topological open discs from the Klein bottle. Then S has two boundary components, two "cross-caps" and no "holes". Hence the connectivity number of S is equal to four and its reduced connectivity number is equal to two.

We define the *connectivity number* of a generalized cactoid as the sum of the reduced connectivity numbers of its maximal cyclic subsets and the number of its boundary components. For compact surfaces, the aforementioned definition of the connectivity number coincides with the definition for generalized cactoids. The generalized cactoid in Fig. 1 has a connectivity number of seven.

For the sake of simplicity, we call a surface carrying a length metric a *length surface*.

The main result of this work completely describes the Gromov–Hausdorff closure of the class of compact length surfaces whose connectivity number is fixed:

Main Theorem *Let $c \in \mathbb{N}_0$ and X be a compact length space. Then the following statements are equivalent:*

(1) *X can be obtained as the Gromov–Hausdorff limit of compact length surfaces whose connectivity number is equal to c.*
(2) *There are $k, k_0 \in \mathbb{N}_0$ and a geodesic generalized cactoid Y such that the following statements apply:*
    (a) *X can be obtained by a successive application of k metric 2-point identifications to Y such that k_0 of them are boundary identifications.*
    (b) *We have $c_0 - k_0 + 2k \leq c$, where c_0 denotes the connectivity number of Y.*

In general the choice of the generalized cactoid Y in the second statement is not unique and the quantity $c_0 - k_0 + 2k$ highly depends on this choice. Provided the converging surfaces are closed, the main result of [4] implies that the boundary of Y is always empty and hence $k_0 = 0$. The possible appearance of a non-empty boundary in Y marks the key difference between the main result of the present paper and that of [4].

For a better understanding of the Main Theorem, we want to discuss a simple example:

Example 1.5 Let $D \subset \mathbb{R}^2$ be the 2-disc and X_n be a metric quotient obtained by identifying $n \in \mathbb{N}$ distinct points of the boundary with the center of D. Then D is the only choice for the generalized cactoid Y. In particular, we have

$$c_0 - k_0 + 2k = 1 - (n - 1) + 2n = n + 2.$$

As a consequence of the Main Theorem, X_n can be obtained as the Gromov–Hausdorff limit of compact length surfaces whose connectivity number is equal to $n + 2$. Moreover it follows that $n + 2$ is the smallest value for which the statement is true.

A more advanced example is illustrated in Fig. 2.

Also some modifications of the Main Theorem are possible: Restricting the first statement of the Main Theorem to smooth Riemannian or polyhedral 2-manifolds, does not effect the validity of the equivalence. This is a direct consequence of the fact that every compact length surface S can be obtained as the limit of smooth Riemannian 2-manifolds that are homeomorphic to S and also of polyhedral surfaces that are homeomorphic to S (cf. [12, p. 1674], [14, p. 77]). Furthermore we will investigate how the Main Theorem changes if we restrict the first statement to orientable or non-orientable surfaces (see Theorem 3.12 and Theorem 4.10).

Beyond the main result of [4], we are only aware of the following predecessors: In the 1930s, Whyburn described the limits of length spaces that are homeomorphic to the 2-disc (see Theorem 2.3). Further Gromov states the first statement of Theorem 1.1 for orientable surfaces without proof and attributes it to Ivanov (cf. [8, p. 103]). From a result by Cassorla follows that every compact length space can be obtained as the limit of closed length surfaces (cf. [3, p. 505]).

The latter result especially implies that the bound on the connectivity number is essential to our investigation.

This paper is organized as follows: In the preliminary notes we provide results on Gromov–Hausdorff convergence and the limits of closed length surfaces. Further we deal with the topology of Peano spaces, generalized cactoids and compact surfaces. In particular, we show

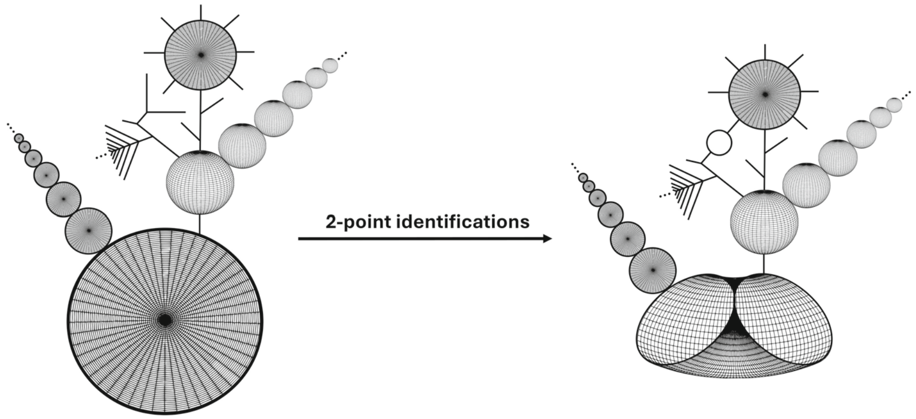

Fig. 2 An illustration of the Main Theorem. On the left hand side we see a geodesic generalized cactoid with two boundary components whose maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc. We note that it has a connectivity number of two. A successive application of three metric 2-point identifications yields the space shown on the right hand side. In particular, we can do this using only one boundary identification. As a consequence of the Main Theorem, the space on the right hand side can be obtained as the Gromov–Hausdorff limit of compact length surfaces whose connectivity number is equal to seven

that the boundary of a generalized cactoid is well-defined and describe the topology of the boundary.

The aim of the third section is to show that the first statement of the Main Theorem implies the second. To do this, we first look at sequences with additional topological control. We also prove the first two statements of Theorem 1.1 and give a proof of Theorem 1.2.

In Sect. 4 we treat the remaining direction of the Main Theorem. At the end of the section we show the third statement of Theorem 1.1.

We note that the final results of Sect. 3 and 4 particularly describe how the Main Theorem changes if we restrict the first statement to orientable or non-orientable surfaces.

2 Preliminaries

2.1 Gromov–Hausdorff convergence

This subsection provides results on Gromov–Hausdorff convergence. Basic definitions and results regarding the Gromov–Hausdorff distance can be found in [2, pp. 251–270]. A corresponding notion of convergence for maps is discussed in [13, pp. 401–402].

With regard to Gromov–Hausdorff convergence, we note the following: For every Gromov–Hausdorff convergent sequence of compact metric spaces there are isometric embeddings of the spaces and their limit into some compact metric space. Moreover the embedded sequence Hausdorff converges to the embedding of the limit. (cf. [7, pp. 64–65])

For the sake of simplicity, we will apply this statement and identify corresponding sets without mentioning the underlying space.

Next we want to characterize Gromov–Hausdorff convergence. For this we introduce the concept of almost isometries: Let $f: X \to Y$ be a map between metric spaces. Then its *distortion* is defined by:

$$dis(f) := \sup_{x_1, x_2 \in X} \{|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)|\}.$$

Further we call f an ε-*isometry* provided $dis(f) \leq \varepsilon$ and $f(X)$ is an ε-net in Y.

We have the following convergence criterion:

Proposition 2.1 (cf. [2, p. 260]) *Let X be a compact metric space and* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of compact metric spaces. Then the following statements are equivalent:*

(1) *The sequence converges to* X.
(2) *For every* $n \in \mathbb{N}$ *there is an* ε_n*-isometry* $f_n : X_n \to X$ *and* $\varepsilon_n \to 0$.

Moreover the equivalence remains true if we interchange X_n *and* X *in the second statement.*

The property of being a length space is stable under Gromov–Hausdorff convergence:

Proposition 2.2 (cf. [2, p. 265]) *Let X be a space that can be obtained as the Gromov–Hausdorff limit of compact length spaces. Then X is a compact length space.*

As already mentioned, Whyburn already described the Gromov–Hausdorff limits of 2-discs:

Theorem 2.3 (cf. [19, p. 422]) *Let X be a space that can be obtained as the Gromov–Hausdorff limit of length spaces* $(X_n)_{n\in\mathbb{N}}$ *that are homeomorphic to the 2-disc. Moreover we assume that* $(\partial X_n)_{n\in\mathbb{N}}$ *is convergent. Then X is a compact length space satisfying the following properties:*

(1) *The maximal cyclic subsets of X are homeomorphic to the 2-sphere or the 2-disc.*
(2) *X is a generalized cactoid with at most one boundary component.*
(3) *The sequence* $(\partial X_n)_{n\in\mathbb{N}}$ *converges to a subset of* ∂X.

2.2 Limits of closed surfaces

As already mentioned, our investigation builds on the results of the previous work [4] which deals with the limits of closed length surfaces. The aim of this subsection is to summarize some of its key results.

The first proposition states two topological properties of the limit spaces and is a special case of Theorem 1.1. Throughout this work the term "dimension" refers to the covering dimension.

Proposition 2.4 (cf. [4, p. 1]) *Let X be a space that can be obtained as the Gromov–Hausdorff limit of length spaces that are homeomorphic to a fixed closed surface. Then the following statements apply:*

(1) *X is at most 2-dimensional.*
(2) *X is locally simply connected.*

Our next result contains a special case of one direction of the Main Theorem:

Theorem 2.5 (cf. [4, p. 13]) *Let X be a space that can be obtained as the Gromov–Hausdorff limit of closed length surfaces* $(X_n)_{n\in\mathbb{N}}$ *whose connectivity number is equal to c. Then X can be obtained by a successive application of k metric 2-point identifications to some geodesic generalized cactoid Y. Moreover the following statements apply:*

(1) *The connectivity number of* Y *is less or equal to* $c - 2k$ *and its boundary is empty.*
(2) *If* X_n *is orientable for infinitely many* $n \in \mathbb{N}$*, then the maximal cyclic subsets of* Y *are orientable.*
(3) *If* X_n *is non-orientable for infinitely many* $n \in \mathbb{N}$ *and the maximal cyclic subsets of* Y *are orientable, then the connectivity number of* Y *is less than* c*.*

Now we consider sequences with additional topological control: Throughout this work we call a simple closed curve a *Jordan curve*. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed length surfaces. Then the sequence is denoted as *regular* provided $\inf\{diam(J_n) : n \in \mathbb{N}\}$ is positive for every sequence $(J_n)_{n\in\mathbb{N}}$ such that J_n is a non-contractible Jordan curve in X_n.

If we restrict the last theorem to regular sequences, then we derive the following result:

Proposition 2.6 (cf. [4, p. 12]) *Let* X *be a space that can be obtained as the Gromov–Hausdorff limit of closed length surfaces* $(X_n)_{n\in\mathbb{N}}$ *whose connectivity number is equal to* $c > 0$*. If the sequence is regular, then* X *is a compact length space satisfying the following property: All but one maximal cyclic subsets are homeomorphic to the 2-sphere and one maximal cyclic subset is homeomorphic to* X_n *for all but finitely many* $n \in \mathbb{N}$*.*

We will treat the case of compact surfaces with non-empty boundary in Proposition 3.9.

2.3 Topology of Peano spaces

In this subsection we discuss results on the topology of Peano spaces. As main source on this topic we used [20].

We call a compact connected metric space X a *continuum*. Further we denote a subset of X that is a continuum as a *subcontinuum* of X. If X is also locally connected, then we say that X is a Peano space.

Compact length spaces are examples of Peano spaces. Moreover the following converse statement applies:

Proposition 2.7 (cf. [1, p. 1109]) *Every Peano space is homeomorphic to a compact length space.*

As already mentioned in the introduction, the main subject of the cyclic element theory is a certain decomposition of Peano spaces. First we want to make this statement more precise: Let X be a Peano space and $x \in X$. Then we call x a *separating point* of X if $X \setminus \{x\}$ is disconnected. Provided the boundaries of arbitrarily small open neighborhoods of x are singletons, we denote x as an *endpoint* of X. For example the set of separating points in $[0, 1] \subset \mathbb{R}$ is given by $(0, 1)$ and the set of its endpoints equals $\{0, 1\}$.

The aforementioned decomposition is provided by the following result:

Proposition 2.8 (cf. [20, pp. 64, 79]) *Let* X *be a Peano space. Then every point of* X *is either a separating point, an endpoint or a point of a unique maximal cyclic subset.*

The maximal cyclic subsets and all singletons consisting of separating points or endpoints are called the *cyclic elements* of X.

Next we state further results of the theory:

Lemma 2.9 (cf. [20, pp. 65, 69, 71, 79]) *Let* X *be a Peano space and* $T \subset X$ *be a maximal cyclic subset. Then the following statements apply:*

(1) *There are only countably many maximal cyclic subsets in X.*
(2) *There are only countably many connected components in $X \setminus T$.*
(3) *If $(C_n)_{n\in\mathbb{N}}$ is a sequence of pairwise distinct connected components of $X \setminus T$, then $diam(C_n) \to 0$.*
(4) *If $(T_n)_{n\in\mathbb{N}}$ is a sequence of pairwise distinct maximal cyclic subsets of X, then $diam(T_n) \to 0$.*
(5) *If C is a connected component of $X \setminus T$, then there is some $x \in T$ such that $\partial C = \{x\}$.*
(6) *If $T_1, T_2 \subset X$ are distinct maximal cyclic subsets, then $|T_1 \cap T_2| \leq 1$.*

Let X be a continuum. Then we say that $A \subset X$ separates $x, y \in X$ if the points lie in distinct connected components of $X \setminus A$. Moreover we call two distinct points of X *conjugate* to each other provided no point of X separates them. For example in a wedge sum of two circles only distinct points of the same circle are conjugate to each other.

Lemma 2.10 (cf. [20, pp. 65, 67, 79]) *Let X be a Peano space. Then the following statements apply:*

(1) *Every pair of conjugate points in X can be connected by a Jordan curve in X.*
(2) *Every non-degenerate cyclicly connected subset of X lies in some maximal cyclic subset of X.*
(3) *If $T \subset X$ is a maximal cyclic subset and $A \subset T$ separates two points in T, then A also separates these points in X.*

Furthermore we have the following characterization of cyclic connectedness:

Proposition 2.11 (cf. [20, p. 79]) *Let X be a Peano space. Then X is cyclicly connected if and only if there is no separating point in X.*

Provided all maximal cyclic subsets of a Peano space are homeomorpic to the 1-sphere, we denote the space as a *1-cactoid*.

The following criterion for 1-cactoids by Whyburn will be a helpful tool:

Proposition 2.12 (cf. [19, p. 417]) *Let X be a continuum. Then the following statements are equivalent:*

(1) *X is a 1-cactoid.*
(2) *For every pair of conjugate points $x, y \in X$ the subset $X \setminus \{x, y\}$ is disconnected.*

2.4 Generalized cactoids

In this subsection we deal with generalized cactoids.

First we describe the topology of admissible subcontinua:

Lemma 2.13 *Let X be a Peano space whose maximal cyclic subsets are compact surfaces and $C \subset X$ be an admissible subcontinuum. Then C is a 1-cactoid.*

Proof Let $x, y \in C$ be conjugate. Then the points are also conjugate in X. Due to Lemma 2.10 the points lie in some maximal cyclic subset $T \subset X$. Since C is admissible, there is some boundary component $b \subset T$ such that $T \cap C = b$. Further we find an arc $\gamma \subset T$ satisfying the following property: The intersection of γ with b is given by $\{x, y\}$ and γ separates two points of b in T.

By Lemma 2.10 the arc also separates these points in X. It follows that $C \setminus \{x, y\}$ is disconnected. From Proposition 2.12 we derive that C is a 1-cactoid. □

We have the following technical lemma:

Lemma 2.14 *Let X be a Peano space whose maximal cyclic subsets are compact surfaces. Then the following statements apply:*

(1) *If $C_1, C_2 \subset X$ are admissible subcontinua intersecting exactly once, then $C := C_1 \cup C_2$ is an admissible subcontinuum.*
(2) *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of subcontinua in X which intersect every maximal cyclic subset at most once. If the sequence Hausdorff converges to some $C \subset X$, then C is a subcontinuum which intersects every maximal cyclic subset at most once.*

Proof (1) It directly follows that C is a continuum. Let $T \subset X$ be a maximal cyclic subset and $x, y \in T \cap C$ be distinct points. Then we may assume that $x \in C_1 \setminus C_2$.
For the sake of contradiction, we further assume that $y \in C_2 \setminus C_1$. By Lemma 2.13 the subsets C_1 and C_2 are arcwise connected. Hence there is an arc $\gamma \subset C$ connecting x and y which is the union of non-degenerate arcs $\gamma_1 \subset C_1$ and $\gamma_2 \subset C_2$. From Lemma 2.9 we get $\gamma \subset T$. Because C_1 and C_2 are admissible, there are boundary component $b_1, b_2 \subset T$ with $T \cap C_1 = b_1$ and $T \cap C_2 = b_2$. Since γ is connected, we get $b_1 = b_2$. This contradicts the fact that $|C_1 \cap C_2| = 1$.
We conclude that $T \cap C = T \cap C_1$. Due to the fact that C_1 is admissible, it follows that $T \cap C$ is a boundary component of T. Therefore we derive that C is admissible.

(2) We directly get that C is a continuum. For the sake of contradiction, we assume the existence of a maximal cyclic subset $T \subset X$ and distinct points $x_1, x_2 \in T \cap C$. Then there is a sequence $(x_{i,n})_{n\in\mathbb{N}}$ converging to x_i such that $x_{i,n} \in C_n$ and $x_{1,n} \neq x_{2,n}$. By Proposition 2.7 we may assume that X is geodesic and we find a geodesic $\gamma_{i,n} \subset X$ connecting $x_{i,n}$ and x_i. Since $x_1 \neq x_2$, we may assume that the geodesics do not intersect. From Lemma 2.13 we derive that C_n is arcwise connected. Hence there is a non-degenerate arc $\alpha_n \subset C_n$ connecting $x_{1,n}$ and $x_{2,n}$. Due to the fact that $\gamma_{1,n}$ and $\gamma_{2,n}$ do not intersect, we may assume that α_n intersects $\gamma_{1,n} \cup \gamma_{2,n}$ only twice. Then $\gamma_n := \gamma_{1,n} \cup \alpha_n \cup \gamma_{2,n}$ is an arc and Lemma 2.9 yields $\gamma_n \subset T$. This contradicts the fact that C_n intersects every maximal cyclic subset at most once. □

Next we introduce some definitions: Let X be a Peano space whose maximal cyclic subsets are compact surfaces. Further let $C_1, \ldots, C_n \subset X$ be disjoint admissible subcontinua such that the boundary components of the maximal cyclic subsets are covered by the subcontinua. If every C_i contains a boundary component of some maximal cyclic subset, then we denote $\cup_{i=1}^{n} C_i$ as a *pre-boundary* of X. Provided the number n is minimal among all pre-boundaries of X, we say that the pre-boundary is *minimal*.

We note that the second property of Definition 1.3 can be restated as the existence of a pre-boundary. The following example illustrates this property:

Example 2.15 We consider a subset $X \subset \mathbb{R}^3$ which is the union of the 2-sphere and disjoint subsets $(D_n)_{n\in\mathbb{N}}$ that are homeomorphic to the 2-disc. Further we assume that $diam(D_n) \to 0$ and that D_n intersects the 2-sphere exactly once for every $n \in \mathbb{N}$. Then X is a Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc. Hence the first property of Definition 1.3 is satisfied. But X is not a generalized cactoid since there is no pre-boundary in X.

For our second example we replace the 2-sphere in the construction above with a further subset $D \subset \mathbb{R}^3$ that is homeomorphic to the 2-disc. We denote this new subset by Y. Then Y is a Peano space whose maximal cyclic subsets are homeomorphic to the 2-disc. Hence the first property of Definition 1.3 is again satisfied. Moreover Y is a generalized cactoid if and only if ∂D_n intersects ∂D for all but finitely many $n \in \mathbb{N}$.

Finally we show that the boundary of a generalized cactoid is well-defined:

Lemma 2.16 *Let X be a generalized cactoid. Then the union of all minimal pre-boundaries is a minimal pre-boundary.*

Proof Let $P \subset X$ be a minimal pre-boundary. We denote the set of all subcontinua in X which intersect every maximal cyclic subset at most once by $\mathcal{T}$. Further we define P_0 as the union of P with all subcontinua in $\mathcal{T}$ which intersect P exactly once.

We show that P_0 is a minimal pre-boundary: Since P covers the boundary components of the maximal cyclic subsets, the same applies to P_0. Moreover P_0 has at most as many connected components as P.

We prove that every connected component $C \subset P_0$ is compact: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in P_0 and $x \in X$ with $x_n \to x$. By construction there is a subcontinuum $A_n \in \mathcal{T}$ which contains x_n and intersects P exactly once. After passing to a subsequence, we may assume that the subcontinua $(A_n)_{n\in\mathbb{N}}$ Hausdorff converge to some subcontinuum $A \subset X$.

Since P is compact, the intersection of A and P is non-empty. From Lemma 2.14 it follows that A intersects every maximal cyclic subset at most once. In particular, A is admissible and hence arcwise connected by Lemma 2.13. Therefore we find an arc $\gamma \subset A$ connecting x and P which intersects P exactly once. Especially we have $\gamma \in \mathcal{T}$ and we deduce $\gamma \subset P_0$. This yields $x \in P_0$.

We conclude that P_0 is closed. Because X is compact, we get that P_0 is compact. Hence C is also compact.

Due to the fact that the subcontinua in $\mathcal{T}$ and the connected components of P are admissible, Lemma 2.14 yields that C is admissible. We conclude that P_0 is a minimal pre-boundary.

Next we show that every pre-boundary $A \subset X$ lies in P_0: For the sake of contradiction, we assume that A is not a subset of P_0. Then we find some $p \in A \setminus P_0$. We denote the connected component of A containing p by C. The subset C contains a boundary component of some maximal cyclic subset. Moreover P covers the boundary components of the maximal cyclic subsets. Therefore C intersects P. Since C is admissible, it is arcwise connected. Hence there is an arc $\gamma \subset C$ starting in p whose endpoint e is the only intersection point of γ with P.

We note that $\gamma \setminus \{e\}$ does not intersect boundary components of maximal cyclic subsets. Due to the fact that C is admissible, we have $\gamma \in \mathcal{T}$. Finally we deduce $\gamma \subset P_0$ and therefore $p \in P_0$. A contradiction. □

We remark that the boundary components of a generalized cactoid are 1-cactoids by Lemma 2.13.

2.5 Curves in compact surfaces

This subsection is devoted to the classification of curves in compact surfaces.

Let γ be an arc in a compact surface S. Then γ is called *simple* if its endpoints lie on boundary components and the interior of γ does not. Moreover we denote γ as *separating* provided $S \setminus \gamma$ is disconnected.

The next two results yield a classification of simple arcs in compact surfaces:

Proposition 2.17 (cf. [10, pp. 54–55]) *Let S be a compact surface of connectivity number c. Further let $\gamma \subset S$ be a separating simple arc which does not form a contractible Jordan curve together with a subarc of some boundary component.*

Then there are $c_1, c_2 \in \mathbb{N}_{\geq 2}$ with $c_1 + c_2 = c + 1$ and a compact surface S_i of connectivity number c_i such that the topological quotient S/γ is a wedge sum of S_1 and S_2. Moreover the wedge point lies in $\partial S_1 \cap \partial S_2$.

At least one of the surfaces is non-orientable if and only if S is non-orientable.

Proposition 2.18 (cf. [10, pp. 54–55]) *Let S be a compact surface of connectivity number c and $\gamma \subset S$ be a non-separating simple arc.*

Then there is a compact surface S_1 of connectivity number $c - 1$ such that S/γ is a topological 2-point identification of S_1. Moreover the glued points lie in ∂S_1.

If S is orientable, then S_1 is orientable.

We say that a Jordan curve $J \subset S$ is *simple* provided we have $|J \cap \partial S| \leq 1$. There is also a classification of non-contractible simple Jordan curves in compact surfaces:

Proposition 2.19 (cf. [10, pp. 54–55]) *Let S be a compact surface of connectivity number c and $J \subset S$ be a non-contractible simple Jordan curve. Then the topological quotient $X{:=}S/J$ can be described in one of the following ways:*

(1) *There are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$ and a compact surface S_i of connectivity number c_i such that X is a wedge sum of S_1 and S_2. Moreover at least one of the surfaces is non-orientable if and only if S is non-orientable.*
(2) *There is a compact surface of connectivity number $c - 2$ such that X is a topological 2-point identification of it. Moreover the surface is orientable if S is orientable.*
(3) *X is a compact surface of connectivity number $c - 1$ and S is non-orientable.*

2.6 Fundamental group formulas

We provide two fundamental group formulas.

First we consider locally simply connected Peano spaces. In the previous work [4] the author showed the following result:

Proposition 2.20 (cf. [4, p. 10]) *Let X be a locally simply connected Peano space and $(T_n)_{n\in\mathbb{N}}$ be an enumeration of its maximal cyclic subsets. Then $\pi_1(X)$ is isomorphic to $\pi_1(T_1) * \ldots * \pi_1(T_n)$ for all but finitely many $n \in \mathbb{N}$.*

Further we consider topological 2-point identifications. The next proposition is a consequence of the HNN-Seifert-van Kampen Theorem in [6, p. 1435]:

Proposition 2.21 *Let X be a locally simply connected and path-connected topological space. Further let Y be a topological 2-point identification of X. Then $\pi_1(Y)$ is isomorphic to $\pi_1(X) * \mathbb{Z}$.*

Notation

$\mathcal{M}$	The class of compact metric spaces.
$\mathcal{S}(c)$	The class of compact length surfaces whose connectivity is equal to c.
$\mathcal{S}(c, b)$	The class of compact length surfaces with b boundary components whose reduced connectivity number is equal to c.
$\mathcal{G}(c)$	The class of geodesic generalized cactoids whose connectivity number is equal c.
$\mathcal{G}(c, b)$	The class of geodesic generalized cactoids with b boundary components such that the reduced connectivity numbers of their maximal cyclic subsets sum up to c.

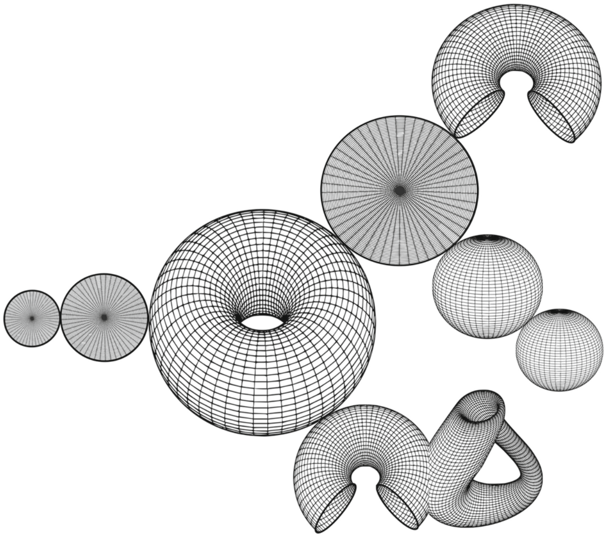

Fig. 3 A successive metric wedge sum of eight compact length surfaces. Since every wedge point is only shared by two of the surfaces, the space lies in $\mathcal{W}_0$

$\mathcal{W}$ The class of successive metric wedge sums of non-degenerate cyclicly connected compact length spaces and finite metric trees.

$\mathcal{W}_0$ The class of successive metric wedge sums of compact length surfaces such that every wedge point is only shared by two of their surfaces.

With regard to the last two notations, we note the following: In every construction step of a successive metric wedge sum we allow a change of the wedge point. An example of a space in $\mathcal{W}_0$ is illustrated in Fig. 3.

3 The limit spaces

In this chapter we prove that the first statement of the Main Theorem implies the second. We also show the first two statements of Theorem 1.1 and give a proof of Theorem 1.2.

3.1 Topological properties

First we prove that the limit spaces are at most 2-dimensional and locally simply connected. For this we introduce some notations:

Notation 3.1 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c, b)$, where $b > 0$, and $X \in \mathcal{M}$ with $X_n \to X$.

We denote the metric gluing of $X_n \sqcup X_n$ along ∂X_n by $2X_n$. Then the sequence $(2X_n)_{n\in\mathbb{N}}$ is convergent and we denote its limit by $2X$. Especially there are subsets $X^\pm \subset 2X$ and maps $\tau^\pm : 2X \to X^\pm$ such that the following statements apply:

(1) $X^+ \cup X^- = 2X$.
(2) $X^\pm$ is isometric to X.
(3) The restriction of $\tau^\pm$ to $X^\pm$ is the identity map and the restriction to $X^\mp$ is an isometry.

(4) The restriction of $\tau^{\pm} \circ \tau^{\mp}$ to $X^{\pm}$ is the identity map.

We fix some isometries as in the second statement. For every $A \subset X$ we denote its corresponding subset of $X^{\pm}$ by $A^{\pm}$. After passing to a subsequence, we may and will assume that $(\partial X_n)_{n\in\mathbb{N}}$ is convergent and we denote its limit by $\partial^{\infty} X$. Finally also the following property is satisfied:

5) $X^{+} \cap X^{-} = (\partial^{\infty} X)^{+} = (\partial^{\infty} X)^{-}$.

Now we show that the limit spaces fulfill the two topological properties:

Proof of Theorem 1.1 (Part I) There is a sequence $(X_n)_{n\in\mathbb{N}}$ of length spaces that are homeomorphic to a fixed compact surface such that $X_n \to X$. If X_n is a closed surface, then X is at most 2-dimensional and locally simply connected by Proposition 2.4. Hence we may assume that the boundary of X_n is non-empty.

(1) We note that $2X_n$ is a closed surface. Hence $2X$ is at most 2-dimensional. Since we have $X^{+} \subset 2X$, we derive that X^{+} is also at most 2-dimensional (cf. [17, p. 266]). Because X and X^{+} are isometric, we deduce that X is at most 2-dimensional.
(2) The statements listed in Notation 3.1 imply the following: Let $V \subset 2X$ be an open and simply connected subset. Since V is open, the same applies to $\tau^{+}(V)$. Moreover every loop in $\tau^{+}(V)$ is the composition of τ^{+} with some loop in V. The map τ^{+} is continuous. Due to the fact that V is simply connected, we hence get that $\tau^{+}(V)$ is simply connected. We further note that the diameter of $\tau^{+}(V)$ does not exceed that of V.

Because $2X_n$ is a closed surface, the space $2X$ is locally simply connected. It follows that X^{+} is locally simply connected. Therefore the same applies to X. □

3.2 Regular convergence

In this subsection we consider sequences in $\mathcal{S}(c, b)$ with additional topological control:

Definition 3.2 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c, b)$ where $b > 0$. Then the sequence is called *regular* provided $\inf\{diam(J_n) \colon n \in \mathbb{N}\}$ is positive for every sequence $(J_n)_{n\in\mathbb{N}}$ such that J_n is a non-contractible Jordan curve in $2X_n$.

In other words, the sequence $(X_n)_{n\in\mathbb{N}}$ is regular if and only if the sequence $(2X_n)_{n\in\mathbb{N}}$ is regular. Provided the sequence $(X_n)_{n\in\mathbb{N}}$ converges to some $X \in \mathcal{M}$, the definition directly implies that $\partial^{\infty} X$ has b connected components.

The next result states a property of non-regular sequences. We remark that Notation 3.1 can also be applied to the constant sequence $(X_n, X_n, \ldots)$. In the upcoming proof we use this notation and denote the corresponding maps by $\tau_n^{\pm}$.

Lemma 3.3 *Let $(X_n)_{n\in\mathbb{N}}$ be a non-regular sequence in $\mathcal{S}(c, b)$ where $b > 0$. After passing to a subsequence, we may assume that one of the following cases applies:*

(1) *There is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that γ_n is a non-separating simple arc in X_n. Moreover we have $diam(\gamma_n) \to 0$.*
(2) *There is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that γ_n is a separating simple arc in X_n which does not form a contractible Jordan curve together with a subarc of some boundary component. Moreover we have $diam(\gamma_n) \to 0$.*
(3) *There is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that γ_n is a non-contractible simple Jordan curve in X_n. Moreover we have $diam(\gamma_n) \to 0$.*

Proof We consider the case that the first two statements of the lemma do not apply. By non-regularity we may assume the existence of a sequence $(J_n)_{n\in\mathbb{N}}$ such that J_n is a non-contractible Jordan curve in $2X_n$ and $diam(J_n) \to 0$. Since the first statement of the lemma does not apply, we may assume that J_n intersects exactly one boundary component $b_n \subset X_n^+$ and the intersection is non-degenerate.

There is a homeomorphism $f_n : 2X_n \to Y_n$ such that Y_n is a Riemannian manifold and $f_n(b_n)$ is a piecewise geodesic Jordan curve. We note that the Jordan curve $f_n(J_n)$ can be obtained as the Hausdorff limit of piecewise geodesic Jordan curves that are homotopic to $f_n(J_n)$ (cf. [18, p. 1794], [19, pp. 413–415]). Since Y_n is a compact Riemannian manifold, there is some $\varepsilon_n > 0$ such that every pair of points in Y_n with distance less than ε_n can be connected by a unique geodesic. It follows that two distinct geodesic Jordan curves in Y_n intersect at most finitely many times.

By the observations above we may assume that J_n and b_n intersect only finitely many times. Then there is a finite subdivision of J_n into simple arcs in X_n^+ and X_n^- and arcs in b_n. Because the second statement of the lemma does not apply, we may assume that each of the arcs is homotopic to some arc in b_n. This yields that J_n is homotopic to some loop in b_n.

We derive that $\gamma_n := \tau_n^+(J_n)$ is a non-contractible loop in X_n^+ and $diam(\gamma_n) \to 0$. In particular, we may assume that γ_n does not intersect ∂X_n^+. Moreover we find a Jordan curve in γ_n which is non-contractible in X_n^+ (cf. [9, p. 626]). Therefore we finally may assume that γ_n is a Jordan curve. □

3.2.1 Limits of boundary components

As a first step we investigate the limits of boundary components for regular sequences. In the next three results we extend Whyburn's proof ideas regarding the limits of discs (cf. [19, pp. 421–424]):

Proposition 3.4 *Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c, q)$, where $q > 0$, and $X \in \mathcal{M}$ with $X_n \to X$. If b is a connected component of $\partial^\infty X$, then b is a 1-cactoid.*

Proof By regularity there is a sequence $(b_n)_{n\in\mathbb{N}}$ such that b_n is a boundary component of X_n and $b_n \to b$. Since b can be obtained as the Hausdorff limit of continua, it is also a continuum.

For the sake of contradiction, we assume that b is not a 1-cactoid. From Proposition 2.12 it follows the existence of conjugate points $x, y \in b$ such that $b \setminus \{x, y\}$ is connected. There are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with $x_n, y_n \in b_n$ and $x_n \neq y_n$ such that $x_n \to x$ and $y_n \to y$. We denote the subarcs of b_n connecting x_n and y_n by α_n and β_n. Moreover we may assume that there are $\alpha, \beta \subset b$ such that $\alpha_n \to \alpha$ and $\beta_n \to \beta$.

Then we have $\alpha \cup \beta = b$ and there exists $z \in \alpha \cap \beta \setminus \{x, y\}$. Further we choose sequences $(z_n)_{n\in\mathbb{N}}$ and $(\tilde{z}_n)_{n\in\mathbb{N}}$ with $z_n \in \alpha_n$ and $\tilde{z}_n \in \beta_n$ such that $z_n \to z$ and $\tilde{z}_n \to z$.

Let $\gamma_n \subset X_n$ be a geodesic between z_n and $\tilde{z}_n$. After passing to a subsequence and subarcs of the geodesics, we may assume γ_n to be a simple arc. By regularity and $diam(\gamma_n) \to 0$ we also may assume that γ_n is separating.

Now there are compact surfaces $U_n, V_n \subset X_n$ such that $x_n \in U_n$, $y_n \in V_n$, $U_n \cup V_n = X_n$ and $U_n \cap V_n = \gamma_n$. We may assume the corresponding sequences to be convergent with limits U and V. This leads to $U \cup V = X$ and $U \cap V = \{z\}$ (cf. [19, p. 412]). Finally we derive that z separates x and y in X and therefore also in b. A contradiction. □

The proof above also demonstrates the following lemma:

Lemma 3.5 *Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c,q)$, where $q>0$, and $X\in\mathcal{M}$ with $X_n\to X$. If b is a connected component of $\partial^\infty X$ and $x,y,z\in b$ are such that z separates x and y in b, then z separates x and y in X.*

In the next two results we study the intersections with maximal cyclic subsets of $2X$:

Lemma 3.6 *Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c,q)$, where $q>0$, and $X\in\mathcal{M}$ with $X_n\to X$. If b is a connected component of $\partial^\infty X$ and T is a maximal cyclic subset of $2X$ with $\left|T\cap b^+\right|>1$, then the intersection is homeomorphic to the 1-sphere.*

Proof Let $x,y\in T\cap b^+$ with $x\neq y$. Since $x,y\in T$, the points lie on some Jordan curve in $2X$. If $\gamma\subset 2X$ is a path between x and y which does not contain a certain point of b^+, then $\tau^+\circ\gamma$ is also such a path. Therefore Lemma 3.5 implies that x and y are conjugate in b^+.

By Proposition 3.4 the subset b^+ is a 1-cactoid. From Lemma 2.10 we derive that x and y are contained in some maximal cyclic subset $S\subset b^+$. We note that S is homeomorphic to the 1-sphere. Using Lemmas 2.9 and 2.10, we get $T\cap b^+\subset S$ and $S\subset T$. Especially we have $S\subset T\cap b^+$ and therefore $S=T\cap b^+$. □

Lemma 3.7 *Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c,q)$, where $q>1$, and $X\in\mathcal{M}$ with $X_n\to X$. If b is a connected component of $\partial^\infty X$, then there is a maximal cyclic subset of $2X$ which intersects b^+ and a further connected component of $(\partial^\infty X)^+$.*

Proof By regularity there is a sequence $(b_n)_{n\in\mathbb{N}}$ such that b_n is a boundary component of X_n and $b_n\to b$. We choose a geodesic $\gamma_n\subset X_n$ between some point of b_n and some point of $\partial X_n\setminus b_n$. Then we may assume the existence of a geodesic $\gamma\subset X$ such that $\gamma_n\to\gamma$.

Due to regularity γ connects some point of b with some point of $\partial^\infty X\setminus b$. After passing to a subarc, we may assume that the interior of γ does not intersect $\partial^\infty X$. It follows that $J{:=}\gamma^+\cup\gamma^-$ is a non-degenerate Jordan curve in $2X$. By Lemma 2.10 there is a maximal cyclic subset $T\subset 2X$ containing J. We conclude that T intersects b^+ and a further connected component of $(\partial^\infty X)^+$. □

3.2.2 Regular limit spaces

Now we describe the limits of regular sequences:

Lemma 3.8 *Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c,b)$, where $b>0$, and $X\in\mathcal{M}$ with $X_n\to X$. Further let T be a maximal cyclic subset of X^+. Then one of the following cases applies:*

(1) *T is a maximal cyclic subset of $2X$ and is a closed surface. Moreover we have $\left|T\cap(\partial^\infty X)^+\right|\leq 1$.*
(2) *$T\cup\tau^-(T)$ is a maximal cyclic subset of $2X$ and T is a compact surface with non-empty boundary. Moreover we have $\partial T=T\cap(\partial^\infty X)^+$.*

Also the following statement applies: Every maximal cyclic subset of $2X$ which is not a maximal cyclic subset of X^+ or X^- can be obtained as in the second case.

Proof Since the sequence is regular, Proposition 2.6 implies that every maximal cyclic subset of $2X$ is a closed surface.

First we consider the case that T is a maximal cyclic subset of $2X$: For the sake of contradiction, we assume $\left|T\cap(\partial^\infty X)^+\right|>1$. Then Proposition 2.11 implies that $T\cup\tau^-(T)$

is cyclicly connected. Further we have $T \neq \tau^-(T)$ by Proposition 3.4. Hence T is a proper subset of some cyclicly connected subset in $2X$. A contradiction.

Now we consider the case that T is not a maximal cyclic subset of $2X$: By Lemma 2.10 there is a maximal cyclic subset $S \subset 2X$ containing T.

Let V be the closure of a connected component of $S \setminus (\partial^\infty X)^+$. Then we may assume $V \subset X^+$. As a consequence of Lemma 3.6, the subset V is a compact surface. It also follows that $V \cap (\partial^\infty X)^+$ is a disjoint union of ∂V and k points. In particular, ∂V is non-empty since S is a closed surface. Due to the fact that $W := V \cup \tau^-(V)$ is cyclicly connected, Lemmas 2.9 and 2.10 yield $W \subset S$. Because S is a closed surface, we derive $k = 0$ and $S = W$. This implies $V = X^+ \cap S$ and therefore $T \subset V$. We note that V is cyclicly connected. Hence we get $T = V$.

Using Lemmas 2.9 and 2.10, the paragraph above also implies the last statement of our result. □

It follows the main result of this subsection:

Proposition 3.9 *Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c, b)$ where $b > 0$ and $c + b > 1$. Further let $X \in \mathcal{M}$ with $X_n \to X$. Then X is a compact length space satisfying the following properties:*

(1) *All but one maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc and one maximal cyclic subset is homeomorphic to X_n for all but finitely many $n \in \mathbb{N}$.*
(2) *X is a generalized cactoid with b boundary components and $\partial^\infty X \subset \partial X$.*

Proof From Proposition 2.6 and Lemma 3.8 we get the following: All but one maximal cyclic subsets of X are homeomorphic to the 2-sphere or the 2-disc. Moreover one maximal cyclic subset $T \subset X$ is a compact surface with non-empty boundary whose connectivity number is equal to $c + b$. We also have that T is orientable if and only if X_n is orientable for all but finitely many $n \in \mathbb{N}$.

By regularity $\partial^\infty X$ has b connected components. Combining Lemmas 3.7 and 3.8, we derive that T has b boundary components. Hence the reduced connectivity number of T is equal to c. This yields that T is homeomorphic to X_n for all but finitely many $n \in \mathbb{N}$.

Moreover the connected components of $\partial^\infty X$ are disjoint subcontinua of X. Due to Lemmas 3.6 and 3.8 the subcontinua are admissible and they cover the maximal cyclic subsets of X. Therefore X is a generalized cactoid.

Since T has b boundary components, the pre-boundary $\partial^\infty X$ is minimal. We conclude that X has b boundary components and $\partial^\infty X \subset \partial X$. □

3.3 The general case

We already described the limits of closed length surfaces and regular sequences. Now we investigate non-regular sequences: Let $(X_n)_{n\in\mathbb{N}}$ be a non-regular sequence in $\mathcal{S}(c, b)$ where $b > 0$. Further let $X \in \mathcal{M}$ with $X_n \to X$.

By non-regularity we may assume that there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of simple arcs or simple Jordan curves as in Lemma 3.3. Since the diameters of the curves vanish, the metric quotients X_n/γ_n converge to X. From Proposition 2.17, Proposition 2.18 and Proposition 2.19 we get a topological description of these quotient spaces. Using this description, we derive the following two results:

Lemma 3.10 *If the curves of the sequence $(\gamma_n)_{n\in\mathbb{N}}$ are simple arcs, then one of the following cases applies:*

(1) *There are $c_1, c_2 \in \mathbb{N}_{\geq 2}$ with $c_1+c_2 = c+1$ and a sequence $(Y_{i,n})_{n\in\mathbb{N}}$ in $\mathcal{S}(c_i)$ converging to some $Y_i \in \mathcal{M}$ such that X is isometric to a metric wedge sum of Y_1 and Y_2. Furthermore the wedge point lies in $\partial^\infty Y_1 \cap \partial^\infty Y_2$ and we find a corresponding isometry p such that $p(\partial^\infty Y_1 \cup \partial^\infty Y_2) = \partial^\infty X$.*
Provided X_n is non-orientable for infinitely many $n \in \mathbb{N}$, the surfaces of at least one of the sequences may be chosen to be non-orientable.
(2) *There is a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c-1)$ converging to some $Y \in \mathcal{M}$ such that X is isometric to Y or a metric 2-point identification of it. Furthermore the glued points lie in $\partial^\infty Y$ and we find a corresponding isometry or projection map p such that $p(\partial^\infty Y) = \partial^\infty X$.*

If X_n is orientable for infinitely many $n \in \mathbb{N}$, then the surfaces of the sequences above may be chosen to be orientable.

Lemma 3.11 *If the curves of the sequence $(\gamma_n)_{n\in\mathbb{N}}$ are simple Jordan curves, then one of the following cases applies:*

(1) *There are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$ and a sequence $(Y_{i,n})_{n\in\mathbb{N}}$ in $\mathcal{S}(c_i)$ converging to some $Y_i \in \mathcal{M}$ such that X is isometric to a metric wedge sum of Y_1 and Y_2. Furthermore we find a corresponding isometry p such that $p(\partial^\infty Y_1 \cup \partial^\infty Y_2) = \partial^\infty X$.*
Provided X_n is non-orientable for infinitely many $n \in \mathbb{N}$, the surfaces of at least one of the sequences may be chosen to be non-orientable.
(2) *There is a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c-2)$ converging to some $Y \in \mathcal{M}$ such that X is isometric to Y or a metric 2-point identification of it. Furthermore we find a corresponding isometry or projection map p such that $p(\partial^\infty Y) = \partial^\infty X$.*
(3) *There is a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c-1)$ converging to some $Y \in \mathcal{M}$ such that X is isometric to Y. Furthermore we find a corresponding isometry p such that $p(\partial^\infty Y) = \partial^\infty X$.*

If X_n is orientable for infinitely many $n \in \mathbb{N}$, then always one of the first two cases applies and the surfaces of the corresponding sequences may be chosen to be orientable.

Now we prove that the first statement of the Main Theorem implies the second. In particular, we describe what happens if we restrict ourselves to orientable or non-orientable surfaces:

Theorem 3.12 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \to X$. Then there are $k, k_0 \in \mathbb{N}_0$ and a space $Y \in \mathcal{G}(c_0)$, where $c_0 - k_0 + 2k \leq c$, such that the following statements apply:*

(1) *X can be obtained by a successive application of k metric 2-point identifications to Y such that k_0 of them are boundary identifications.*
(2) *If X_n is orientable for infinitely many $n \in \mathbb{N}$, then the maximal cyclic subsets of Y are orientable.*
(3) *If X_n is non-orientable for infinitely many $n \in \mathbb{N}$ and the maximal cyclic subsets of Y are orientable, then $c_0 < c$.*

Proof First we add a statement to the claim: There is a choice $p_1, \ldots, p_k$ of the corresponding projection maps such that $\partial^\infty X \subset (p_k \circ \ldots \circ p_0)(\partial Y)$ where $p_0 := id_Y$.

We show the claim using an induction over the connectivity number:

The case $c = 0$ is a direct consequence of Theorem 2.5.

Now let $c > 0$. For the sake of induction, we assume that the claim is true for connectivity numbers less than c. If the sequence $(X_n)_{n\in\mathbb{N}}$ contains infinitely many closed surfaces or is regular, then the claim follows from Theorem 2.3, Theorem 2.5 or Proposition 3.9. Therefore we may assume that one of the cases in Lemmas 3.10 or 3.11 applies.

The surfaces of the sequences appearing there have a connectivity number less than c. Hence we can apply the induction hypothesis and derive the claim. □

Finally we are able to prove the local description of the limit spaces:

Proof of Theorem 1.2 Due to Theorem 3.12 there is some generalized cactoid Y such that X is homeomorphic to a topological quotient of Y whose underlying equivalence relation identifies only finitely many points. We denote the number of maximal cyclic subsets in Y that are not homeomorphic to the 2-sphere or the 2-disc by k.

First we show the following claim: Every $y \in Y$ admits an open neighborhood that is homeomorphic to an open subset of some Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc.

In the case $k = 0$ the claim follows directly.

If $k = 1$, then we may assume that y lies in the maximal cyclic subset $T \subset Y$ that is not homeomorphic to the 2-sphere or the 2-disc. There is a neighborhood D of y in T that is homeomorphic to the 2-disc. We denote the union of the connected components of $Y \setminus T$ whose closures intersect D by A. It follows that $Z := D \cup A$ is a Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc.

Moreover there is an open neighborhood V of y in T that is contained in D. We denote the union of the connected components of $Y \setminus T$ whose closures intersect V by B. Then $V \cup B$ is an open neighborhood of y in Y and Z. This yields the claim.

If $k \geq 2$, then Y is a wedge sum of Peano spaces satisfying the following property: All maximal cyclic subsets are compact surfaces and less than k of them are not homeomorphic to the 2-sphere or the 2-disc.

Provided both spaces locally look like Peano spaces whose maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc, the same applies to Y. Hence the claim follows by induction.

By the claim X locally looks like a successive wedge sum of Peano spaces whose maximal cyclic subsets are homeomorphic to the 2-sphere or the 2-disc. We note that such a successive wedge sum is then also a Peano space of this kind. □

4 Approximation of generalized cactoids

In this chapter we show that the second statement of the Main Theorem implies the first. For this we successively reduce the complexity of the problem. We also prove the third statement of Theorem 1.1.

4.1 Approximation by surface gluings

The goal of this subsection is to approximate generalized cactoids by suitable spaces in $\mathcal{W}_0$. Our construction extends over the next three results:

Lemma 4.1 *Let $X \in \mathcal{G}(c, b)$ and $(T_k)_{k\in\mathbb{N}}$ be an enumeration of its maximal cyclic subsets. Then X can be obtained as the limit of compact length spaces $(X_n)_{n\in\mathbb{N}}$ satisfying the following properties:*

(1) *X_n has only finitely many maximal cyclic subsets. Moreover the maximal cyclic subsets of X_n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^{n}$.*

(2) *After passing to a subsequence, we may assume the existence of a pre-boundary $\partial^* X_n \subset X_n$ with b connected components such that $\partial^* X_n \to \partial X$.*

Proof We extend the author's proof in [4, p. 9]: First we define an equivalence relation on X by: $x \sim y$ if and only if x and y lie in the same connected component of $\cup_{m=n+1}^{n+k} T_m$. Moreover we denote the corresponding metric quotient by $X_{n,k}$ and the corresponding projection map by $p_{n,k}$. As shown in the reference, after passing to a subsequence, we may assume that there is a space $X_n \in \mathcal{M}$ and a map $p_n : X \to X_n$ such that $X_{n,k} \to X_n$ and $p_{n,k} \to p_n$ uniformly.

From the reference we also already know the following: The maximal cyclic subsets of X_n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^n$ via the map p_n. Further we have $X_n \to X$.

Now we show that $p_n(\partial X)$ is a pre-boundary for infinitely many $n \in \mathbb{N}$: Let C be a connected component of ∂X. Since p_n is continuous, $p_n(C)$ is a subcontinuum of X_n.

Further let $m \in \{1, \dots, n\}$ and $x_1, x_2 \in p_n(C) \cap p_n(T_m)$ with $x_1 \neq x_2$. Then there are $c_i \in C$ and $t_i \in T$ such that $p_n(c_i) = p_n(t_i) = x_i$. Moreover we choose a geodesic $\gamma_i \subset X$ between c_i and t_i and derive $p_n(\gamma_i) = \{x_i\}$. Since $x_1 \neq x_2$, the geodesics do not intersect. From Lemma 2.13 it follows that C is arcwise connected and we find a non-degenerate arc $\alpha \subset C$ connecting c_1 and c_2. Because γ_1 and γ_2 do not intersect, we may assume that α intersects $\gamma_1 \cup \gamma_2$ only twice. We derive that $\gamma := \gamma_1 \cup \alpha \cup \gamma_2$ is an arc and Lemma 2.9 implies $\gamma \subset T$. Finally we conclude $x_i \in p_n(T \cap C)$.

The observation above implies the following: If the subset $p_n(C) \cap p_n(T_m)$ is non-degenerate, then it equals $p_n(C \cap T_m)$. Because C is admissible and p_n is an isometry on T_m, we deduce that $p_n(C)$ is admissible.

Due to the fact that C contains a boundary component of some maximal cyclic subset, we may assume that the same applies to $p_n(C)$. Moreover $p_n(\partial X)$ covers the boundary components of the maximal cyclic subsets as ∂X does.

The map p_n is 1-lipschitz. After passing to a subsequence, we hence may assume $(p_n)_{n\in\mathbb{N}}$ to be convergent. We denote its limit by p and it follows that p is an isometry. Further we may assume the sequence $(p_n(\partial X))_{n\in\mathbb{N}}$ to be convergent. Since $p(\partial X) = \partial X$, we have $p_n(\partial X) \to \partial X$. Therefore we may assume that ∂X has as many connected components as $p_n(\partial X)$. We finally deduce that $p_n(\partial X)$ is a pre-boundary with b connected components. □

Lemma 4.2 *Let X be a geodesic generalized cactoid having only finitely many maximal cyclic subsets. Further let $\partial^* X \subset X$ be a pre-boundary with b connected components. Then X can be obtained as the Hausdorff limit of compact subsets $(X_n)_{n\in\mathbb{N}}$ satisfying the following properties:*

(1) *We have $X_n \in \mathcal{W}$ and the maximal cyclic subsets of X_n are equal to those of X.*
(2) *There is a pre-boundary $\partial^* X_n \subset X_n$ with b connected components.*
(3) *The sequence $(\partial^* X_n)_{n\in\mathbb{N}}$ Hausdorff converges to $\partial^* X$.*

Proof We extend the author's proof in [4, p. 9]: First we define ε as the minimum of the diameters of the maximal cyclic subsets in X. For every maximal cyclic subset $T \subset X$ we remove the connected components of $X \setminus T$ whose diameters are less than ε/n. We denote the constructed subset by Y_n.

The subset Y_n is a compact length space. Furthermore it is a successive metric wedge sum of its maximal cyclic subsets and compact metric trees $D_1, \dots, D_k$. For every metric tree D_i there is a finite metric tree $F_i \subset D_i$ whose Hausdorff distance to D_i is less than ε/n (cf. [2, p. 267]). In particular, we may assume that F_i intersects the same maximal cyclic subsets as

D_i. Next we define X_n as the union of the maximal cyclic subsets of Y_n and the finite metric trees. Moreover we set $\partial^* X_n := \partial^* X \cap X_n$.

By construction we have $X_n \in \mathcal{W}$. Further the maximal cyclic subsets of X_n are equal to those of X and $\partial^* X_n$ is a pre-boundary of X_n with b connected components. Finally the sequences $(X_n)_{n\in\mathbb{N}}$ and $(\partial^* X_n)_{n\in\mathbb{N}}$ Hausdorff converge to X and $\partial^* X$. □

Lemma 4.3 *Let X be a generalized cactoid in $\mathcal{W}$. Further let $\partial^* X \subset X$ be a pre-boundary with b connected components. Then there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{W}_0$ satisfying the following properties:*

(1) *The maximal cyclic subsets of X_n are in isometric one-to-one correspondence with the maximal cyclic subsets of X and finitely many length spaces that are homeomorphic to the 2-sphere or the 2-disc.*
(2) *X_n has b boundary components.*
(3) *There is an ε_n-isometry $f_n : X_n \to X$ such that $f_n(\partial X_n) = \partial^* X$ and $\varepsilon_n \to 0$.*

Proof The space X is a successive metric wedge sum of its maximal cyclic subsets and compact intervals. In particular, we may assume that the wedge points do not lie in the interior of the intervals and that every interval whose interior intersects $\partial^* X$ lies in $\partial^* X$.

We consider the following construction: Let e be one of the wedged intervals. Then there is a $1/n$-isometry $f : D \to e$ satisfying the following properties: The preimages of the endpoints of e contain exactly one point. If e lies in $\partial^* X$, then D is a length space that is homeomorphic to the 2-disc and $f(\partial D) = e$. Otherwise D is a length space that is homeomorphic to the 2-sphere.

Now we remove the interior of e from X and paste D along f. This yields a compact length space Y. Especially the maximal cyclic subsets of Y are in isometric one-to-one correspondence with the set consisting of D and the maximal cyclic subsets of X.

If e lies in $\partial^* X$, then we set $\partial^* Y := (\partial^* X \backslash e) \cup \partial D$. Otherwise we set $\partial^* Y := \partial^* X$. We note that $\partial^* Y$ is a pre-boundary of Y with b connected components. Provided Y is a successive metric wedge sum of its maximal cyclic subsets, we have $\partial^* Y = \partial Y$.

Furthermore the map f naturally induces a $1/n$-isometry $g : Y \to X_n$ with $g(\partial^* Y) = \partial^* X$.

We successively repeat this construction until there is no wedged interval left and denote the constructed space by X_n.

The space X_n is a successive metric wedge sum of its maximal cyclic subsets. For every wedge point $p \in X_n$ which lies in more than one maximal cyclic subset there is a $1/n$-isometry $f : D \to \{p\}$ satisfying the following properties: If p lies in ∂X_n, then D is a length space that is homeomorphic to the 2-disc. Otherwise D is a length space that is homeomorphic to the 2-sphere.

Using a similar construction as above, we finally may assume that $X_n \in \mathcal{W}_0$. □

We note that a sequence of ε_n-isometries between converging spaces has a convergent subsequence provided $\varepsilon_n \to 0$. In particular, its limit is an isometry between the limit spaces. Morever the boundary of a generalized cactoid is invariant under self-isometries. Combining Proposition 2.1 and the last three results, we hence get the desired approximating sequence:

Corollary 4.4 *Let $X \in \mathcal{G}(c, b)$ and $(T_k)_{k\in\mathbb{N}}$ be an enumeration of its maximal cyclic subsets. Then X can be obtained as the limit of spaces $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{W}_0$ satisfying the following properties:*

(1) *The maximal cyclic subsets of X_n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^n$ and finitely many length spaces that are homeomorphic to the 2-sphere or the 2-disc.*

(2) *After passing to a subsequence, we may assume that X_n has b boundary components and $\partial X_n \to \partial X$.*

4.2 Elementary surface gluings

Now we provide useful tools concerning the approximation of elementary surface gluings. We start with wedge sums:

Lemma 4.5 *Let $S_1 \in \mathcal{S}(c_1, b_1)$, $S_2 \in \mathcal{S}(c_2, b_2)$ and X be a metric wedge sum of S_1 and S_2. Then the following statements apply:*

(1) *There is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c_1 + c_2, b_1 + b_2)$ and an ε_n-isometry $f_n : X_n \to X$ such that $f_n(\partial X_n) = \partial X$ and $\varepsilon_n \to 0$.*
(2) *If the wedge point is contained in $\partial S_1 \cap \partial S_2$, then there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c_1 + c_2, b_1 + b_2 - 1)$ and an ε_n-isometry $f_n : X_n \to X$ such that $f_n(\partial X_n) = \partial X$ and $\varepsilon_n \to 0$.*

If S_1 and S_2 are orientable, then the surfaces of the sequence may be chosen to be orientable. Provided at least one of the wedged surfaces is non-orientable, the surfaces of the sequence may be chosen to be non-orientable.

Proof (1) We may assume the wedge points not to lie in $\partial S_1 \cup \partial S_2$. In [4, p. 14] the author already showed the statement for closed surfaces. The corresponding proof does not depend on the fact that the wedged surfaces are closed and it gives rise to a proof for the general case.

(2) Let a_1 and a_2 be the intersecting boundary components of the surfaces and p be the wedge point. We choose an arc $\gamma_{i,n} \subset a_i$ containing p in its interior. In particular, we may assume that the arc is a geodesic of length $1/n$ such that p is its midpoint. Next we define X_n as the metric gluing of X along $\gamma_{1,n}$ and $\gamma_{2,n}$. Moreover we denote the corresponding projection map by g_n and find a map $f_n : X_n \to X$ such that $f_n \circ g_n$ is the identity map on $X_n \backslash (\gamma_{1,n} \cup \gamma_{2,n})$ and $(f_n \circ g_n)(\gamma_{1,n}) = \gamma_{1,n} \cup \gamma_{2,n}$. Finally we deduce that the space X_n and the map f_n satisfy the desired properties.

□

Using similar arguments as above, we derive the following result concerning metric 2-point identifications:

Lemma 4.6 *Let S be a space in $\mathcal{S}(c)$ and X be a metric 2-point identification of S. Further let p be a corresponding projection map. Then the following statements apply:*

(1) *There is sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c+2)$ and an ε_n-isometry $f_n : X_n \to X$ with $f_n(\partial X_n) = p(\partial S)$ and $\varepsilon_n \to 0$.*
(2) *If p is a boundary identification, then there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c + 1)$ and an ε_n-isometry $f_n : X_n \to X$ with $f_n(\partial X_n) = p(\partial S)$ and $\varepsilon_n \to 0$.*

The surfaces of the sequence may be chosen to be non-orientable. If S is orientable, then the surfaces of the sequence may be chosen to be orientable.

4.3 Gluings of generalized cactoids

In this subsection we approximate spaces that can be obtained by a successive application of k metric 2-point identifications to some generalized cactoid.

Using an induction, Corollary 4.4 and Lemma 4.5 yield the case $k = 0$:

Corollary 4.7 *Let $X \in \mathcal{G}(c, b)$. Then X can be obtained as the limit of spaces $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c, b)$. Moreover the following statements apply:*

(1) *The sequence may be chosen such that $\partial X_n \to \partial X$.*
(2) *If the maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.*
(3) *If there is a non-orientable maximal cyclic subset in X, then the surfaces of the sequence may be chosen to be non-orientable.*

Now we show the general case:

Lemma 4.8 *Let $k, k_0 \in \mathbb{N}_0$ and $X \in \mathcal{G}(c)$. Further let Y be a space that can be obtained by a successive application of k metric 2-point identifications to X such that k_0 of them are boundary identifications. Then Y can be obtained as the limit of spaces $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c - k_0 + 2k)$. Moreover the following statements apply:*

(1) *If the maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.*
(2) *If there is a non-orientable maximal cyclic subset in X or $k > 0$, then the surfaces of the sequence may be chosen to be non-orientable.*

Proof First we add a statement to the claim: There is a choice $p_1, \ldots, p_k$ of the corresponding projection maps such that $\partial Y_n \to (p_k \circ \ldots \circ p_0)(\partial X)$ where $p_0 := id_X$.

We show the claim using an induction over k:

The case $k = 0$ is a direct consequence of Corollary 4.7.

Now let $k > 0$. For the sake of induction, we assume that the claim is true if the number of identifications is less than k. Let $p_1, \ldots, p_k$ be a choice of the corresponding projection maps. We set $Z := (p_{k-1} \circ \ldots \circ p_0)(X)$ and denote the number of boundary identifications in $\{p_{k-1}, \ldots, p_1\}$ by $\tilde{k}_0$. Then Z is a space that can be obtained by a successive application of $k - 1$ metric 2-point identifications to X such that $\tilde{k}_0$ of them are boundary identifications. Hence we can apply the induction hypothesis and derive a corresponding sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c - \tilde{k}_0 + 2(k - 1))$. In particular, we may assume that $\partial Z_n \to (p_{k-1} \circ \ldots \circ p_0)(\partial X)$.

Let $z_1, z_2 \in Z$ be distinct points with $p_k(z_1) = p_k(z_2)$. Then there is a sequence $(z_{i,n})_{n\in\mathbb{N}}$ with $z_{i,n} \in Z_n$ and $z_{1,n} \neq z_{2,n}$ such that $z_{i,n} \to z_i$. Provided p_k is a boundary identification, we may assume $z_{i,n} \in \partial Z_n$. Further we define W_n as the metric gluing of Z_n along $z_{1,n}$ and $z_{2,n}$. We denote the corresponding projection map by $p_{k,n}$.

By construction $p_{k,n}$ is a boundary identification if p_k is. Moreover it follows $W_n \to Y$ and we may assume that $(p_{k,n})_{n\in\mathbb{N}}$ converges to some map q_k. We note that $q_k, p_{k-1}, \ldots, p_1$ is also a possible choice of the projection maps corresponding to the construction of Y. Hence we may assume that $p_{k,n}(\partial Z_n) \to (p_k \circ \ldots \circ p_0)(\partial X)$.

Finally we apply Lemma 4.6 to W_n and denote the corresponding sequence of surfaces by $(Y_{n,m})_{m\in\mathbb{N}}$ and the corresponding sequence of almost isometries by $(f_{n,m})_{m\in\mathbb{N}}$. Then we have $f_{n,m}(\partial Y_{n,m}) = p_{k,n}(\partial Z_n)$. Choosing a diagonal sequence, we may assume that $Y_n := Y_{n,n} \to Y$ and that $(f_{n,n})_{n\in\mathbb{N}}$ converges to an isometry f. We note that $(f^{-1} \circ p_k), p_{k-1}, \ldots, p_1$ is also a possible choice of the projection maps corresponding to the construction of Y. Hence we finally may assume that $\partial Y_n \to (p_k \circ \ldots \circ p_0)(\partial X)$. □

Using sequences of 2-discs or real projective planes whose diameters tend to zero, we get the following corollary of Lemma 4.5:

Corollary 4.9 *Let $X \in \mathcal{S}(c)$. Then X can be obtained as the limit of spaces in $\mathcal{S}(c + 1)$. Moreover the following statements apply:*

(1) *If S is orientable, then the surfaces of the sequence may be chosen to be orientable.*
(2) *The surfaces of the sequence may be chosen to be non-orientable.*

As a direct consequence of the last two results, we derive that the second statement of the Main Theorem implies the first. In particular, we are able to describe under which conditions the approximating surfaces may be chosen to be orientable or non-orientable:

Theorem 4.10 *Let $c, k, k_0 \in \mathbb{N}_0$ and $X \in \mathcal{G}(c_0)$ where $c_0 - k_0 + 2k \leq c$. Further let Y be a space that can be obtained by a successive application of k metric 2-point identifications to X such that k_0 of them are boundary identifications. Then Y can be obtained as the limit of spaces $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c)$. Moreover the following statements apply:*

(1) *If the maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.*
(2) *If there is a non-orientable maximal cyclic subset in X or $c_0 < c$, then the surfaces of the sequence may be chosen to be non-orientable.*

Finally we show the third statement of Theorem 1.1:

Proof of Theorem 1.1 (Part II) If Y is a space that can be obtained by a successive application of metric 2-point identifications to some geodesic generalized cactoid, then the second statement of Theorem 1.1 and Theorem 4.10 imply that Y is locally simply connected. Due to Theorem 3.12 the space X can be obtained in this way. Hence Proposition 2.20 and Proposition 2.21 finally yield the desired fundamental group formula for X.

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