Combinatorial Designs Meet Hypercliques: Higher Lower Bounds for Klee's Measure Problem and Related Problems in Dimensions $d \geq 4$

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Abstract

Klee's measure problem (computing the volume of the union of n axis-parallel boxes in $\mathbb{R}^d$) is well known to have $n^{\frac{d}{2} \pm o(1)}$-time algorithms (Overmars, Yap, SICOMP'91; Chan FOCS'13). Only recently, a conditional lower bound (without any restriction to "combinatorial" algorithms) could be shown for $d = 3$ (Künnemann, FOCS'22). Can this result be extended to a tight lower bound for dimensions $d \geq 4$?

In this paper, we formalize the technique of the tight lower bound for $d = 3$ using a combinatorial object we call *prefix covering design*. We show that these designs, which are related in spirit to combinatorial designs, directly translate to conditional lower bounds for Klee's measure problem and various related problems. By devising good prefix covering designs, we give the following lower bounds for Klee's measure problem in $\mathbb{R}^d$, the depth problem for axis-parallel boxes in $\mathbb{R}^d$, the largest-volume/max-perimeter empty (anchored) box problem in $\mathbb{R}^{2d}$, and related problems:

- $\Omega(n^{1.90476})$ for $d = 4$,
- $\Omega(n^{2.22222})$ for $d = 5$,
- $\Omega(n^{d/3+2\sqrt{d}/9-o(\sqrt{d})})$ for general d,

assuming the 3-uniform hyperclique hypothesis. For Klee's measure problem and the depth problem, these bounds improve previous lower bounds of $\Omega(n^{1.777\ldots}), \Omega(n^{2.0833\ldots})$ and $\Omega(n^{d/3+1/3+\Theta(1/d)})$ respectively.

Our improved prefix covering designs were obtained by (1) exploiting a computer-aided search using problem-specific insights as well as SAT solvers, and (2) showing how to transform combinatorial *covering designs* known in the literature to strong prefix covering designs. In contrast, we show that our lower bounds are close to best possible using this proof technique.

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1 Introduction

For various problems in computational geometry, the best known algorithms display a running time of the form $n^{\Theta(d)}$ where d denotes the number of dimensions: Klee's measure problem and the depth problem for axis-parallel boxes in $\mathbb{R}^d$ can be solved in time $n^{d/2 \pm o(1)}$ [31, 12, 13], a recent algorithm [15] computes the largest-volume empty axis-parallel box among a given set of points in time $\widetilde{\mathcal{O}}(n^{(5d+2)/6})$, the star discrepancy can be computed in time $O(n^{d/2+1})$ [17], the maximum-weight rectangle problem can be solved in time $O(n^d)$ [6], to name few examples. Indeed, for all listed problems, it can be shown [12, 20, 6] that an $n^{o(d)}$-time algorithm would refute the Exponential Time Hypothesis (ETH). Thus, the subsequent challenge is to determine running times $n^{f(d)}$ with $f(d) = \Theta(d)$ that are optimal under fine-grained complexity assumptions. By the nature of these running times (which quickly increase with d), it is particularly interesting to determine optimal time bounds for small dimensions such as $d \in \{2, 3, 4, 5\}$.

For some of these problems, strong conditional lower bounds are known: For Klee's measure problem and the depth problem, Chan [12] gives a tight conditional lower bound of $n^{d/2-o(1)}$ for *combinatorial* algorithms – roughly speaking, algorithms that avoid the algebraic techniques underlying fast matrix multiplication algorithms. When considering general algorithms (not only combinatorial ones), tight lower bounds are only known for *weighted* problems or small dimensions: For the weighted depth problem and the maximum-weight rectangle problem, tight lower bounds of $n^{d/2-o(1)}$ and $n^{d-o(1)}$, respectively, can be shown under the Weighted k-Clique Hypothesis [5]. Showing strong lower bounds for the simpler, *unweighted* problems appears to be more difficult, however. For Klee's measure problem and the unweighted depth problem, a recent result shows an $n^{d/(3-3/d)-o(1)}$ conditional lower bound under the 3-uniform hyperclique hypothesis [27], which yields a tight bound for $d = 3$, but not for $d \geq 4$.

Thus, the motivating question of this paper is the following:

Can we prove conditional optimality of known algorithms for Klee's measure problem, the depth problem and related problems for small dimensions $d \geq 4$, such as $d \in \{4, 5, 6\}$?

1.1 Our Results

As a starting point of this work, we formalize the approach used in [27] to obtain tight hardness for $d = 3$. To this end, we define the following combinatorial object, which we term *prefix covering designs* (due to its conceptual similarity to certain combinatorial designs[1]).

In the following definition, let $\binom{S}{t}$ denote the set of t-element subsets of S.

▶ **Definition 1.** *Let $d, K, \alpha \in \mathbb{N}$ with $d \geq 3$ and $K \geq 4$. A (d, K, α)-*prefix covering design *consists of d sequences $s_1, \ldots, s_d$ over $[K]$ with the following properties.*

- ***Triplet condition:*** *For every $\{a, b, c\} \in \binom{[K]}{3}$, there are $i, i', i'' \in [d]$ and $\ell, \ell', \ell'' \in \mathbb{N}_0$ such that*
  - *each element of $\{a, b, c\}$ is contained in $s_i[..\ell]$, $s_{i'}[..\ell']$, or $s_{i''}[..\ell'']$. (Here, $s[..\ell]$ denotes the prefix of the first ℓ elements of s.)*
  - *$\ell + \ell' + \ell'' \leq \alpha$.*
- ***Singleton condition:*** *For every $x \in [K]$ occurring more than once in $s_1, \ldots, s_d$, define $\ell_{\min}(x)$ ($\ell_{\max}(x)$) as the minimal (maximal) ℓ such that there is some i with $s_i[\ell] = x$. Then we have*
  - *$\ell_{\min}(x) + \ell_{\max}(x) \leq \alpha + 1$.*

As an example, it is straightforward to see that for any d, the sequences $s_1 = (1, d+1), s_2 = (2, d+1), \ldots, s_d = (d, d+1)$ constitute a $(d, d+1, 3)$ prefix covering design.[2]

[1] In fact, we will later establish a formal connection between these concepts.

[2] For the triplet condition, note that the triplet $\{a, b, c\} \in \binom{[d]}{3}$ is contained in the prefixes $s_a[..1], s_b[..1], s_c[..1]$ of total length $\alpha = 3$ and that any triplet $\{a, b, d+1\}$ with $\{a, b\} \in \binom{[d]}{2}$ is

Prefix covering designs constitute the core of the proof technique used in [27]. Specifically, we show that the existence of good prefix covering designs directly leads to strong lower bounds for several problems (these reductions are implicit in [27] or adapted to prefix covering designs from [20]).

▶ **Proposition 2.** *Let $d, K, \alpha \in \mathbb{N}$ such that there exists a (d, K, α) prefix covering design. Unless the 3-uniform Hyperclique Hypothesis fails, there is no $\varepsilon > 0$ such that there exists an $O(n^{\frac{K}{\alpha}-\varepsilon})$-time algorithm for any of the following problems:*

- *Klee's Measure problem in $\mathbb{R}^d$,*
- *Depth problem in $\mathbb{R}^d$,*
- *Largest-Volume Empty Anchored Box problem in $\mathbb{R}^{2d}$,*
- *Maximum-Perimeter Empty Anchored Box problem in $\mathbb{R}^{2d}$.*

Beyond these problems, similar reductions are also possible for related problems such as the Bichromatic Box problem in $\mathbb{R}^{2d}$ (given sets of red and blue points, find the axis-parallel box containing the maximum number of blue points while avoiding any red point) and various related discrepancy problems such as the Star Discrepancy, see [20]. Note that there is a blow-up in the dimension for the Empty Anchored Box problems, which turns out to be unavoidable assuming the 3-uniform hyperclique hypothesis, as there are $O(n^{(1/2-\varepsilon)d})$-algorithms for these problems (see below). At this point, we only give a rough sketch of the reduction, with the full proof deferred to the full version of this paper [22], where we also formally define all listed problems and discuss the 3-uniform hyperclique hypothesis.

Proof sketch for Proposition 2. For each problem, we give a reduction from the 3-uniform hyperclique problem: Given a 3-uniform hypergraph $G = (V, E)$ with $V = V^{(1)} \cup \cdots \cup V^{(K)}$ and $|V^{(1)}| = \cdots = |V^{(K)}| = n$, determine whether there are $v^{(1)} \in V^{(1)}, \ldots, v^{(K)} \in V^{(K)}$ that form a clique in G. The 3-uniform hyperclique hypothesis states that this problem requires running time $n^{K-o(1)}$.

Intuitively, a special case of each of the problems listed above is to find an axis-parallel box Q satisfying certain properties. More specifically, any candidate box Q is given by choosing some value $v_i \in \{0, \ldots, U-1\}$ for each dimension $i \in [d]$. We use a (d, K, α) prefix covering design $s_1, \ldots, s_d$ to interpret the values $v_1, \ldots, v_d$ as choices of vertices in $V^{(1)}, \ldots, V^{(K)}$: Namely, with $s_i = (s_i[1], \ldots, s_i[L])$, we think of any number $v_i \in \{0, \ldots, U-1\}$ with $U = n^L$ as a base-n number $v_i = (v_i[1], \ldots, v_i[L])$. We interpret $(v_i[1], \ldots, v_i[L]) \in \{0, \ldots, n-1\}^L$ as choosing the $(v_i[\ell] + 1)$-st vertex in $V^{(s_i[\ell])}$ for all $1 \le \ell \le L$.

With this encoding fixed, it remains to ensure that the only true solutions Q encode a *clique* in G. This consists of two tasks: (1) ensuring that the candidate box Q chooses vertices consistently, i.e., for each $V^{(x)}$ such that x occurs in more than one s_i, we need to make sure that the same vertex is chosen in each occurrence, and (2) ensuring that the chosen vertices form a clique. Crucially, for both tasks, our geometric problems allow us to exclude candidate boxes Q where the v_i have certain prefixes. Specifically, due to the singleton condition, we only need to construct $O(n^\alpha)$ boxes to ensure consistency of the remaining candidate solutions Q. Likewise, the triplet condition is used to ensure that all candidate boxes Q that encode a non-clique (for which one of the triplets $\{v^{(a)}, v^{(b)}, v^{(c)}\}$ is not an edge in G) are excluded, using only $O(n^\alpha)$ additional boxes. In total, this creates an instance of size $O(n^\alpha)$ for the target problem, which yields an $n^{\frac{K}{\alpha}-o(1)}$ lower bound under the 3-uniform hyperclique hypothesis. ◀

contained in the prefixes $s_a[..2], s_b[..1]$ of total length $\alpha = 3$. The singleton condition only needs to be checked for $x = d+1$, for which we note that $\ell_{\min}(d+1) = \ell_{\max}(d+1) = 2$ and thus $\ell_{\min}(d+1) + \ell_{\max}(d+1) = 4 \le \alpha + 1$ for $\alpha = 3$.

From Proposition 2, we obtain the following direct corollary.

▶ **Corollary 3.** *For any $d \geq 3$, let $\gamma_d := \sup\{\frac{K}{\alpha} \mid$ there is a (d, K, α) prefix covering design$\}$. Then for no $\varepsilon > 0$ there exists an $O(n^{\gamma_d - \varepsilon})$-algorithm for any of the problems listed in Proposition 2, unless the 3-uniform hyperclique hypothesis fails.*

The tight conditional lower bound [27] for Klee's Measure problem and the depth problem in $\mathbb{R}^3$ follows from the following construction: For any $g \in \mathbb{N}$, we set $K = 3g$, write $[K] = \{a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_g\}$ and observe that

$$s_1 = (a_1, \ldots, a_g, b_g, \ldots, b_1), s_2 = (b_1, \ldots, b_g, c_g, \ldots, c_1), s_3 = (c_1, \ldots, c_g, a_g, \ldots, a_1)$$

provide a $(3, 3g, 2g+1)$ prefix covering design. Thus, we obtain $\gamma_3 \geq \lim_{g \to \infty} \frac{3g}{2g+1} = \frac{3}{2}$, establishing an $n^{\frac{3}{2} - o(1)}$ conditional lower bound for KMP in $\mathbb{R}^3$ and related problems.[3]

Given the direct applicability of prefix covering designs to Klee's measure problem, the depth problem and many related problems, it is only natural to ask what the highest obtainable lower bounds are using this technique. For one, designing better prefix covering designs gives stronger lower bounds. On the other hand, establishing limits for prefix covering designs may indicate potential for improved algorithms for KMP and related problems (such a phenomenon has been observed in other contexts, e.g., [11]).

Our first result is that prefix covering designs cannot establish a higher lower bound than $n^{\frac{d}{3} + O(\sqrt{d})}$. The following bound will be proved in Section 3.

▶ **Proposition 4.** *We have that $\gamma_d \leq \frac{d}{3(1 - \sqrt{\frac{2}{d}})} = \frac{d}{3} + \sqrt{\frac{2}{9}} \cdot \sqrt{d} + o(\sqrt{d})$.*

However, as $\frac{d}{3(1-\sqrt{2/d})} \geq \frac{d}{2}$ for $d \leq 18$, this result does not rule out tight lower bounds for small dimensions. In fact, combining a computer-aided search with problem-specific insights, we give improved constructions for $d \in \{4, 5\}$, which give lower bounds that are surprisingly close to $\frac{d}{2}$.

▶ **Theorem 5.** *There is a $(4, 40, 21)$ prefix covering design, which yields $\gamma_4 \geq \frac{40}{21} > 1.90476$. There is a $(5, 40, 18)$ prefix covering design, which yields $\gamma_5 \geq \frac{40}{18} > 2.22222$.*

Proof. The following sequences give a $(4, 40, 21)$ prefix covering design:

$$s_1 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 40, 19, 28, 37, 26),$$
$$s_2 = (11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 30, 9, 38, 27, 36),$$
$$s_3 = (21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 20, 39, 8, 7, 37),$$
$$s_4 = (31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 10, 29, 18, 17, 27).$$

The following sequences give a $(5, 40, 18)$ prefix covering design:

$$s_1 = (1, 2, 3, 4, 5, 6, 7, 8, 24, 31, 38, 30, 14),$$
$$s_2 = (9, 10, 11, 12, 13, 14, 15, 16, 32, 40, 6, 31, 22),$$
$$s_3 = (17, 18, 19, 20, 21, 22, 23, 24, 8, 7, 39, 15, 30),$$
$$s_4 = (25, 26, 27, 28, 29, 30, 31, 32, 40, 16, 23, 39, 6),$$
$$s_5 = (33, 34, 35, 36, 37, 38, 39, 40, 16, 32, 15, 23).$$

For the readers' convenience, we provide checker programs to verify the singleton and triplet conditions in [21] (see the full version of this paper [22] for details). ◀

[3] It is not hard to prove that $\gamma_3 \leq \frac{3}{2}$, resulting in $\gamma_3 = \frac{3}{2}$. This raises the question whether we can find exact values of γ_d for $d \geq 4$.

For Klee's measure problem and the depth problem in $\mathbb{R}^4$ and $\mathbb{R}^5$, the gap between the resulting conditional lower bound and the known upper bound is thus at most $O(n^{0.09524})$ and $O(n^{0.27778})$, respectively. This improves over previous hyperclique-based lower bounds of $\Omega(n^{1.777})$ and $\Omega(n^{2.0833})$, respectively.

These results may (re-)ignite hope that it might be possible to find prefix covering designs that establish tight lower bounds for $d = 4$ and $d = 5$. Alas, by a careful investigation of the limits of prefix covering designs, we refute this hope.

▶ **Theorem 6.** *We have $\gamma_4 < 2$.*

This result is proven via a careful analysis of the structure of prefix covering designs with quality $\frac{K}{\alpha}$ approaching 2: We show that certain *levels* (i.e., $s_1[\ell], \dots, s_4[\ell]$ for certain values of ℓ) must have a very rigid structure. Essentially, every element on such a level must have exactly a single copy on a corresponding other level. A detailed analysis of all possibilities displays a contradiction; we cannot get a quality $\frac{K}{\alpha}$ that is arbitrarily close to 2. The proof is in the full version of this paper [22]. It remains an interesting question to determine the precise value of γ_4; our results yield $1.90476 \leq \gamma_4 < 2$.

Connection to covering designs

Our previous results give evidence of the intricacy of designing good prefix covering designs. Unfortunately, designing optimized designs for small dimensions like $d = 4$ and $d = 5$ offers little insights into the asymptotics in d as well as the general structure of good prefix designs for larger dimensions.

We address this by providing general constructions that are applicable for all d and make use of the extensive literature on *combinatorial designs*. Specifically, we observe an interesting connection between so-called *covering designs* (see, e.g., the surveys [29, 24, 25, 23] and [14] for an algorithmic application in computational geometry) and prefix covering designs. A (v, k, t) covering design is a collection of k-sized subsets $B_1, \dots, B_b$ – called *blocks* – of $[v]$ such that every t-element subset of $[v]$ is fully contained in some block B_i. These covering designs constitute a relaxation of balanced incomplete block designs.

Note that a (d, K, α) prefix covering design $s_1, \dots, s_d$ where each s_i has length at most L is superficially similar to a (v, k, t)-covering design with $v = K$ elements, block size $k = L$, parameter $t = 3$ and d blocks: in both designs, we cover triplets among $v = K$ elements using d sequences/blocks. However, there are two key differences. (1) In covering designs, we cover each triplet in a single block, while in prefix covering designs, we may use prefixes from up to three sequences. (2) The sequences of prefix covering designs are inherently ordered (due to the *prefix* nature of the singleton and triplet conditions), while covering designs have unordered blocks. A priori, it is unclear whether there is a general way to use good covering designs to obtain good prefix covering designs or vice versa. Maybe surprisingly, we show how to use good (v, k, t) covering designs with $t = 2$ (rather than $t = 3$, which might appear as the more natural correspondence) to obtain strong prefix covering designs.

Specifically, for any such covering design satisfying a mild matching-like condition (which is satisfied by many constructions known in the literature), we obtain high-quality prefix covering designs. We will see below that by plugging in known constructions, we get prefix covering designs that are close to optimal when $d \to \infty$.

▶ **Theorem 7.** *Let $d \geq 3, k \in \mathbb{N}$ and v be a multiple of d such that there is a $(v, k, 2)$ covering design with d blocks with the following property: For every block B_i, there exists $U_i \subseteq B_i$ of size $\frac{v}{d}$ such that $U_1, \dots, U_d$ partition $[v]$. Then $\gamma_d \geq \frac{d}{3 - 2\frac{v}{kd}}$.*

Let us give an example application of this theorem (see Sections 1.2 and the full version of this paper [22] for stronger consequences). It is well known that the projective plane of order q (where q is a prime power) yields a set of $v = q^2 + q + 1$ points, $d = q^2 + q + 1$ lines, with $k = q + 1$ points on each line, such that every pair of points is connected by a line. This yields a $(v, k, 2)$-design with $d = v = q^2 + q + 1$ and $k = q + 1$. One can show that this design satisfies the matching-like condition (see the full version of this paper [22]). Thus, for infinitely many d, we obtain a lower bound of $\gamma_d \geq \frac{d}{3-\frac{2}{q+1}}$. Since $q = O(\sqrt{d})$, we obtain $\gamma_d \geq \frac{d}{3-\Omega(1/\sqrt{d})} = \frac{d}{3} + \Omega(\sqrt{d})$ for infinitely many d, improving over the lower bound of $\gamma_d \geq \frac{d}{3} + \frac{1}{3} + \frac{1}{3(d-1)}$ that is implicit in [27].

1.2 Consequences: Improved conditional lower bounds

Using Theorem 7, we may take any $(v, k, 2)$ covering design with d blocks that is known in the literature, check whether it satisfies the matching-like condition, and obtain the corresponding lower bound on γ_d. In Table 1, we list lower bounds on $\gamma_d, d \leq 10$ obtained this way, specifically, by using covering designs listed in the La Jolla Covering Repository [23] (see Section 2 for details). Notably, the resulting lower bounds improve over the constructions in [27] for $d \geq 4$.

We also provide a lower bound for *all* γ_d that is close to optimal when $d \to \infty$.

► **Theorem 8.** *There is some function $f(d) = d/3 + 2\sqrt{d}/9 - o(\sqrt{d})$ such that $\gamma_d \geq f(d)$ for all $d \geq 3$.*

This lower bound is obtained by showing how to extend the projective planes covering designs (in a suitable way) to obtain strong prefix covering designs for all values of d.

By the above theorem, we obtain a $n^{d/3+2/9\sqrt{d}-o(\sqrt{d})}$ conditional lower bound for Klee's measure problem and related problems. Note that Chan's reduction from K-clique [12] can be interpreted as a lower bound of $n^{(\omega/6)d-o(1)}$ assuming that current K-clique algorithms are optimal. If $\omega = 2$, this cannot give any higher lower bound than $n^{d/3-o(1)}$.

Table 1 also lists the corresponding upper bound of $O(n^{d/2})$ for Klee's measure problem and the depth problem for comparison. The gaps for the Largest-Volume/Maximum-Perimeter Empty (Anchored) Box problem in $\mathbb{R}^d$ are a bit larger: Chan [15] obtains an upper bound[4] for the anchored version of $\widetilde{\mathcal{O}}(n^{d/3+\lfloor d/2 \rfloor/6}) \leq \widetilde{\mathcal{O}}(n^{5d/12})$ for $d \geq 4$. In particular, this yields upper bounds of $\widetilde{\mathcal{O}}(n^{2.5})$, $\widetilde{\mathcal{O}}(n^{3.3334})$, and $\widetilde{\mathcal{O}}(n^{4.1667})$ for $d = 6$, $d = 8$ and $d = 10$, respectively, while we supply a conditional lower bound of $n^{\gamma_{d/2}-o(1)}$ for even $d \geq 6$, which yields lower bounds of $n^{1.5-o(1)}$, $n^{1.9047-o(1)}$ and $n^{2.2222-o(1)}$ for $d = 6$, $d = 8$ and $d = 10$, respectively. It is an interesting question whether we can prove a higher lower bound than $n^{d/4-o(1)}$ for any d or whether Chan's algorithms can be improved further.

Related Work. Klee's measure problem has been well-studied since the 1970s [26, 7, 19, 32, 31, 12, 13, 27], including algorithms beating $n^{d/2\pm o(1)}$ for various special cases, e.g., [2, 8, 1, 9, 33, 10].

The depth problem for axis-parallel boxes is closely related to Klee's measure problem and often admits similar algorithmic ideas, see particularly [13].

Finding a largest-volume empty axis-parallel box has initially been mostly studied in two dimensions (see, e.g., [30, 16, 3]). In higher dimensions, Backer and Keil [4] give a $\widetilde{\mathcal{O}}(n^d)$ algorithm, which was recently improved to $\widetilde{\mathcal{O}}(n^{(5d+2)/6})$ by Chan [15]. Note that

[4] While Chan focuses on the Largest-Volume Empty Box problem, he states that his algorithms for $d \geq 4$ also work for the Maximum-Perimeter version, see [15, Section 5].

Table 1 The exponents of the upper and conditional lower bounds for Klee's measure problem and the depth problem in $\mathbb{R}^d$ for $d \le 10$. The upper bound column is due to the $n^{d/2 \pm o(1)}$-time algorithms [31, 12, 13], the conditional lower bounds are based on the 3-uniform hyperclique hypothesis and result from [27] (3rd column), Theorem 5 (4th column) and from combining Theorem 7 with covering designs found in the La Jolla Covering Repository maintained by D. Gordon [23] (5th column).

d	Upper bound	Previously known lower bound	SAT-solver lower bound	Covering designs lower bound
3	1.5	1.5		1.5
4	2	1.7777	1.9047	1.8461
5	2.5	2.0833	2.2222	2.1929
6	3	2.4		2.5714
7	3.5	2.7222		3
8	4	3.0476		3.3333
9	4.5	3.375		3.6818
10	5	3.7037		4.0540

our lower bounds are most interesting for the anchored version of the problem, which is solvable in faster running time $\widetilde{\mathcal{O}}(n^{5d/12})$ [15]. Approximation algorithms have been given in [18]. Giannopoulos et al. [20] give a reduction from d-clique, which can be understood as an $n^{(\omega/12)d-o(1)}$ lower bound assuming that current clique algorithms are optimal.

2 Constructions

In this section, we prove our general result transforming covering designs to prefix covering designs (Theorem 7). All remaining proofs and details on constructing prefix covering designs can be found in the full version of this paper [22].

For a (d, K, α) prefix covering design (PCD) with sequences $s_1, s_2, \ldots, s_d$ we call elements $s_1[i], s_2[i], \ldots, s_d[i]$ the i-th level of the PCD.

When analyzing such prefix covering designs, it is helpful to distinguish between the "first" occurrence of some element, which we call the *primary* element, and all other occurrences, which we call *copies*. We call a pair (i, ℓ) a position if $1 \le i \le d$, $1 \le \ell$, and there exists ℓ-th element in s_i.

▶ **Definition 9.** *For any prefix covering design $s_1, \ldots, s_d$, we call a position (i, ℓ) the* primary position *of value x ($1 \le x \le K$) if and only if $s_i[\ell] = x$ and $s_{i'}[\ell'] \ne x$ for every other position (i', ℓ') such that (ℓ', i') precedes (ℓ, i) in the lexicographic ordering.*

Every other occurrence (i', ℓ') with $s_{i'}[\ell'] = x$ is called a copy *of x.*

Note that if (i, ℓ) is a primary position of value x, then $\ell = \ell_{\min}(x)$.

▶ **Definition 10.** *A (v, k, t) covering design where $v \ge 2$, $k \ge t \ge 1$ is a collection of k-element subsets (called blocks) of $[v]$ such that any t-element subset is contained in at least one block.*

In the following proof, we will be extensively using (v, k, t) covering designs for $t = 2$. So, every pair of elements is contained in at least one block.

Proof of Theorem 7. Consider some $(v, k, 2)$ covering design consisting of d blocks where v is divisible by d and set $v' := \frac{v}{d} \in \mathbb{N}$. Define $B_1, B_2, \ldots, B_d$ as the blocks of this covering design. Assume there exist sets $U_1 \subseteq B_1, U_2 \subseteq B_2, \ldots, U_d \subseteq B_d$ such that

s_1	8	9	10	1	2	3
s_2	11	12	13	4	1	5
s_3	14	15	16	7	1	6
s_4	17	18	19	6	2	4
s_5	20	21	22	2	5	7
s_6	23	24	25	3	4	7
s_7	26	27	28	5	3	6

Figure 1 Example of a (7, 28, 10) PCD construction from a $(7, 3, 2)$ covering design with 7 blocks.

$|U_1| = |U_2| = \ldots = |U_d| = v'$ and $U_1, U_2, \ldots, U_d$ partition $[v]$. Then we will prove that for every $\varepsilon > 0$, there exist K and α such that $\frac{K}{\alpha} \geq \frac{d}{3-2\frac{v}{kd}} - \varepsilon$ and (d, K, α) PCD exists. From this we automatically get that $\gamma_d \geq \frac{d}{3-2\frac{v}{kd}}$ by going to the limit.

First, we present a slightly worse construction.

Order elements inside blocks of a given covering design in such a way that elements of U_i are located in the first v' positions of B_i, i.e., $\{B_i[j] \mid 1 \leq j \leq v'\} = U_i$. To construct sequences of our PCD, we take these blocks of the covering design and put kd new different elements in front of them by prepending k elements in each sequence. In other words, the resulting PCD has sequences s_1, s_2, …, s_d each of length $2k$ such that $s_i[j] = v + (i-1) \cdot k + j$ for $j \leq k$ and $s_i[j] = b_i[j-k]$ for $j > k$. An example for $d = 7$ is given in Figure 1. We will prove that this gives a (d, K, α) PCD with $K = (v' + k)d$ and $\alpha \leq 3k + v'$.

There are $v'd$ elements from a covering design and kd more unique elements that we added, so $K = (v' + k)d$. It remains to check that $\alpha \leq 3k + v'$.

First, we check the singleton condition. Due to our ordering of the covering design blocks, all primary positions of all elements are located in the first $k + v'$ levels, so $\ell_{\min}(x) \leq k + v'$ for every element x. At the same time, there are $2k$ elements in each sequence in total, so $\ell_{\max}(x) \leq 2k$. Thus, $\ell_{\min}(x) + \ell_{\max}(x) \leq (k + v') + 2k = 3k + v'$ for each $x \in [K]$.

Second, we check the triplet condition. Assume we chose three elements a, b and c. Define their primary positions as (i_a, ℓ_a), (i_b, ℓ_b) and (i_c, ℓ_c) respectively. Without loss of generality, assume that $\ell_a \leq \ell_b \leq \ell_c$. Consider two cases.

1. If there is at most one element from the covering design among these three, then $\ell_a \leq k$, $\ell_b \leq k$ and $\ell_c \leq k + v'$, so we can cover them with prefixes $s_{i_a}[..\ell_a]$, $s_{i_b}[..\ell_b]$ and $s_{i_c}[..\ell_c]$ of total size $\ell_a + \ell_b + \ell_c \leq k + k + (k + v') = 3k + v'$.
2. If there are at least two elements from the covering design among these three, then b and c are in the covering design. By the definition of a covering design, there should be a sequence s_i that contains both b and c. Thus we can cover all three elements with two prefixes: $s_i[..2k]$ (whole sequence) and $s_{i_a}[..\ell_a]$ of total size $2k+\ell_a \leq 2k+(k+v') = 3k+v'$.

This concludes the proof that $\alpha \leq 3k + v'$ and already gives a bound $\gamma_d \geq \frac{K}{\alpha} \geq \frac{(k+v')d}{3k+v'} = \frac{d}{3} \cdot \frac{3k+3v'}{3k+v'} = \frac{d}{3} \cdot \left(1 + \frac{2v'}{3k+v'}\right) = \frac{d}{3} \cdot \left(1 + \frac{2v}{3dk+v}\right)$.

To improve this construction we will replicate the covering design n times for some positive integer n. Define B_i^j for $1 \leq i \leq d$ and $1 \leq j \leq n$ as the i-th block of the j-th copy of the covering design. We want different copies of the covering design to be over different elements, so the v elements of B^j are $\{(j-1)v + 1, \ldots, jv\}$. Define U_i^j as v'-element subsets of B_i^j such that U_1^j, U_2^j, …, U_d^j partition $\{(j-1)v+1, \ldots, jv\}$. Define $R_i^j := B_i^j \setminus U_i^j$ as the remaining $k - v'$ elements of each block. Also, for every sequence of our PCD, we define

s_1	22	23	24	25	26	27	28	1	8	15	19	20	12	13	5	6
s_2	29	30	31	32	33	34	35	2	9	16	15	21	8	14	1	7
s_3	36	37	38	39	40	41	42	3	10	17	15	18	8	11	1	4
s_4	43	44	45	46	47	48	49	4	11	18	19	16	12	9	5	2
s_5	50	51	52	53	54	55	56	5	12	19	21	17	14	10	7	3
s_6	57	58	59	60	61	62	63	6	13	20	16	17	9	10	2	3
s_7	64	65	66	67	68	69	70	7	14	21	20	18	13	11	6	4

Figure 2 Example of a $(7, 70, 24)$ prefix covering design obtained by a scaled construction with $v = 7$ ($v' = 1$), $k = 3$, $d = 7$ and $n = 3$.

$m \coloneqq nk - (n-1)v'$ unique elements that are put at the beginning of this sequence. Let these unique elements for sequence i be called A_i ($A_i = \{nv + (i-1)m + 1, \dots, nv + im\}$). Now we are ready to construct the sequences $s_1, \dots, s_d$ of our prefix covering design by

$$s_i = (A_i, U_i^1, U_i^2, \dots, U_i^n, R_i^n, R_i^{n-1}, \dots, R_i^1).$$

An example of such a construction is given in Figure 2.

We will prove that such a PCD has $K = (nk + v')d$ and $\alpha \leq 3nk - (2n-3)v'$, similarly to the proof for the simpler construction. First, there are $v'd$ elements from every covering design, and there are n designs, so overall, there are $nv'd$ elements from covering designs. Additionally, there are $md = nkd - (n-1)v'd$ more unique elements that we added, so $K = (nk + v')d$ indeed. It remains to check that $\alpha \leq T$ where $T \coloneqq 3nk - (2n-3)v'$. We will use that $T = 2m + nk + v' = 3m + nv'$.

First, we check the singleton condition. Due to our ordering of the covering design blocks, all primary positions of all elements are located in the first $m+nv'$ levels, so $\ell_{\min}(x) \leq m+nv'$ for every element x. If $\ell_{\min}(x) \leq m$, this element has only one occurrence, and we do not need to check the singleton condition for it. If $\ell_{\min}(x) = m + (n-i)v' + j$ for some $1 \leq i \leq n$ and $1 \leq j \leq v'$, then it means that element x belongs to the $(n-i+1)$-st covering design, and its other occurrences are located in the levels from $m + nv' + (i-1)(k-v') + 1$ to $m + nv' + i(k-v')$. So $\ell_{\max}(x) \leq m + nv' + i(k-v')$. Consequently, $\ell_{\min}(x) + \ell_{\max}(x) \leq (m+(n-i)v'+j)+(m+nv'+i(k-v')) = 2m+2nv'+i(k-2v')+j \leq 2m+2nv'+n(k-2v')+v' = 2m + nk + v' = T < T+1$ where we used the fact that $k - 2v' = k - 2\frac{v}{d} \geq 0$ due to the lemma below. We have even proved a slightly stronger inequality:

$$\ell_{\min}(x) + \ell_{\max}(x) \leq T. \tag{1}$$

▶ **Lemma 11.** *For every $(v, k, 2)$ covering design with $d \geq 2$ blocks, $k \geq 2v/d$ holds.*

Proof of Lemma. If $k < v$, then every element $x \in [v]$ should be located in at least two sets: otherwise, we would cover only $k - 1 < v - 1$ pairs involving x, which contradicts the fact that it is a covering design. But if every element is located in at least two sets, then the sum of all set sizes kd is at least $2v$. Dividing both numbers by d, we get the desired inequality.

If $k \geq v$, then $k \geq v \geq 2\frac{v}{d}$ because $d \geq 2$. ◀

Second, we check the triplet condition. Consider any three elements a, b and c. Define their primary positions as (i_a, ℓ_a), (i_b, ℓ_b) and (i_c, ℓ_c) respectively. Without loss of generality, assume that $\ell_a \leq \ell_b \leq \ell_c$. Consider two cases.

1. If at most one element out of these three is from covering designs, we know that $\ell_a \leq m$, $\ell_b \leq m$ and $\ell_c \leq m+nv'$, so we can cover them with prefixes $s_{i_a}[..\ell_a]$, $s_{i_b}[..\ell_b]$ and $s_{i_c}[..\ell_c]$ with total size $\ell_a + \ell_b + \ell_c \leq m + m + (m + nv') = 3m + nv' = T$.

2. If at least two elements out of these three are from covering designs, then b and c are in the covering designs. By the definition of a covering design there should be a sequence s_i that contains both b and c.[5] Then we can cover all three elements with two prefixes: $s_i[..\max(\ell_b^i, \ell_c^i)]$ and $s_{i_a}[..\ell_a]$ where ℓ_b^i and ℓ_c^i are positions of elements b and c, respectively, in the sequence i. We already know that elements b and c satisfy (1). It follows that $\ell_a + \ell_b^i \leq \ell_b + \ell_b^i \leq T$ and $\ell_a + \ell_c^i \leq \ell_c + \ell_c^i \leq T$. From this we can conclude that $\ell_a + \max(\ell_b^i, \ell_c^i) \leq T$, as desired.

This concludes the proof that $\alpha \leq T = 3nk - (2n-3)v'$. This construction gives us a bound $\gamma_d \geq \frac{K}{\alpha} \geq \frac{(nk+v')d}{3nk-(2n-3)v'} = \frac{(k+\frac{v'}{n})d}{3k-(2-\frac{3}{n})v'}$ where $n \in \mathbb{N}$ can be chosen arbitrarily. When n approaches infinity, this value approaches $\frac{kd}{3k-2v'} = \frac{d}{3-\frac{2v'}{k}} = \frac{d}{3-2\frac{v}{kd}}$. Thus, for every $\varepsilon > 0$ there exists n such that such a construction gives $\frac{K}{\alpha} \geq \frac{d}{3-2\frac{v}{kd}} - \varepsilon$, as desired. ◀

We say that a $(v, k, 2)$ covering design with d blocks *admits a multi-matching* if for every block B_i we can choose a subset U_i of size v/d such that $U_1, U_2, \ldots, U_d$ partition $[v]$. The following observation shows that in Theorem 7 it is not a restriction to assume that v is divisible by d, since we can always suitably scale covering designs:

▶ **Observation 12.** *Every $(v, k, 2)$ covering design with d blocks can be transformed into a $(vd, kd, 2)$ covering design with d blocks by replacing each of v elements with d distinct elements. If this scaled covering design admits a multi-matching, we get a lower bound for γ_d of $\frac{d}{3-2\frac{(vd)}{(kd)d}} = \frac{d}{3-2\frac{v}{kd}}$.*

The bound we give depends on the existence of specific covering designs admitting multi-matchings. This lower bound can be transformed into a general lower bound depending only on d (see the full version of this paper [22] for details); one can also obtain lower bounds for specific values of d: for a fixed value of d, the lower bound can be obtained by finding a covering design that minimizes the value $\mathrm{freq} := \frac{kd}{v}$ which we call *frequency* (for a fixed covering design, the frequency is the average number of occurrences of elements). We searched for covering designs in the La Jolla Covering Repository [23], fixing the number of blocks to d and choosing the ones with the smallest frequencies. Then we multiplied the number of elements and set sizes in these covering designs by d using Observation 12 (because Theorem 7 works only for covering designs with v divisible by d) and checked whether they admit multi-matching. Perhaps surprisingly, for all specific values of d that we checked, the obtained covering designs indeed admit a multi-matching. The covering designs used and their multi-matchings can be found in [21] along with a computer program that checks that provided constructions are indeed covering designs, and they indeed admit multi-matchings.

The question remains whether the frequency in some dimension d could be minimized by a covering design that does *not* admit a multi-matching. Indeed, one can construct covering designs that do not admit a multi-matching. However, since we aim to minimize the frequencies, we are considering covering designs that should have a relatively small degree of redundancy – otherwise, they probably could be improved. In the full version of this paper [22], we formulate the corresponding conjecture that "sufficiently good" covering designs always admit a multi-matching and discuss some evidence. We also provide weaker bounds obtained from covering designs not admitting multi-matchings.

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[5] b and c may belong to different copies of our covering design, but all copies are identical, so equivalent elements from all covering designs occur in the same sequences, so there indeed should exist such s_i.

3 Limits

In this section, we prove limits of prefix covering designs, i.e., upper bounds on $\gamma_d = \sup\{\frac{K}{\alpha} \mid$ there exists a (d, K, α) prefix covering design$\}$. The proof that $\gamma_4 < 2$ is in the full version of this paper [22]. The following lemma formalizes the intuition that increasing the value of K can only lead to better (more precisely, not worse) prefix covering designs.

▶ **Lemma 13** (Scaling Lemma). *For every (d, K, α) PCD and positive integer $\lambda \in \mathbb{N}$, there also exists a $(d, \lambda \cdot K, \lambda \cdot \alpha)$ PCD.*

The proof of this fact is in the full version of this paper [22].

Proof of Proposition 4. For a fixed (d, K, α) PCD define $g := \lceil \frac{K}{d} \rceil$. If $\alpha \geq 3g$ then $\frac{K}{\alpha} \leq \frac{K}{3g} \leq \frac{K}{3K/d} = \frac{d}{3}$ and the proposition statement holds. Otherwise define $a := \lceil g - \frac{\alpha}{3} \rceil \geq 1$, i.e., $3(g-a) \leq \alpha < 3(g-a+1)$. We will prove that $a < \sqrt{\frac{2}{d}} + 2$. If $a = 1$, it is correct, so from now on we assume that $a \geq 2$.

Define B as the set of all elements x that have $\ell_{\min}(x) > g-a$. We claim that $|B| \geq d(a-1)$: The number of (not necessarily distinct) elements in the first $(g-a)$ positions (over all s_i) is $d \cdot (g-a) = dg - da < d \cdot (\frac{K}{d} + 1) - da = K - d(a-1)$. Since there are K distinct numbers in total, the claim follows.

To prove the proposition, we will define a graph G_B with vertex set B. We connect two elements $x, y \in B$ by an edge if and only if there is some sequence s_i containing both x and y. We obtain our result by proving both an upper and a lower bound on the number of edges.

For a lower bound on the number of edges, consider how triplets $\{a, b, c\} \in \binom{B}{3}$ are covered by prefixes: For any such triplet $\{a, b, c\}$, there are prefixes $s_i[..\ell]$, $s_{i'}[..\ell']$ and $s_{i''}[..\ell'']$ which contain a, b and c and satisfy $\ell + \ell' + \ell'' \leq \alpha$.

▷ Claim 14. Without loss of generality, we may assume that at least one of ℓ, ℓ' and ℓ'' is zero.

Proof. If all of them are at least $g-a+1$, then $\ell + \ell' + \ell'' \geq 3(g-a+1) > \alpha$, which yields a contradiction. Otherwise, if at least one of them is at most $g-a$, then this prefix cannot contain any of a, b and c as $\ell_{\min}(a), \ell_{\min}(b), \ell_{\min}(c) > g-a$. We can set this prefix to the empty prefix without loss of generality. ◁

So indeed, we can imagine that triplets of elements in B must be covered by using only two prefixes, not three. In particular, for every triplet of elements from B, at least two of them must occur in the same sequence, i.e., they must have an edge in G_B. Put differently, the complement graph of G_B is triangle-free and thus contains at most $|B|^2/4$ edges by Mantel's Theorem [28] (a special case of Turan's Theorem). We conclude that G_B has at least $\binom{|B|}{2} - \frac{|B|^2}{4} = \frac{|B|^2 - 2|B|}{4} \geq \frac{(|B|-2)^2}{4}$ edges because $|B| \geq 2$.

We now show that either the number of edges is at most $dg^2/2$ or $|B| \leq 2g$. We ask on which positions elements from B can be located in the sequences. We know that $\ell_{\min}(x) \geq g-a+1$ for any $x \in B$. At the same time, if some element from B is located in position $\geq 2(g-a)+3$ (in some sequence i), then this must be its only occurrence since otherwise, it would violate the singleton condition. Furthermore, any covering of a triplet with such an element cannot contain elements from B in other sequences because it would take a prefix of length at least $2(g-a)+3$ in sequence i and a prefix of length at least $g-a+1$ in some other sequence, which would violate the triplet condition. From this, we can conclude that if every triplet with this element and other elements in B is covered, all elements from B have to occur in sequence i. We can assume that all elements have indices at most α (otherwise, they are useless for coverings), so there are at most $\alpha - (g-a) \leq 2g$

elements from B in this sequence. This yields $|B| \leq 2g$. In the remaining case all $x \in B$ satisfy $\ell_{\max}(x) \leq 2(g-a)+2$ and $\ell_{\min}(x) > (g-a)$, so there are at most $g-a+2 \leq g$ elements from B in each sequence. Thus, there are at most $d \cdot \binom{g}{2}$ pairs of elements from B that occur in the same sequence.

From the above lower and upper bounds on the number of edges, we derive that

$$\frac{(|B|-2)^2}{4} \leq d \cdot \binom{g}{2} < \frac{dg^2}{2}.$$

Combining this with the fact that $d(a-1) \leq |B|$, we deduce that $d(a-1)-2 \leq |B|-2 < \sqrt{2d}g$. (Note that in the case $|B| \leq 2g$, the upper bound is trivially satsified since $d \geq 2$.) Consequently,

$$a < \frac{\sqrt{2d}g+2}{d} + 1 \leq \sqrt{\frac{2}{d}}g + 2$$

for $d \geq 2$. We plug this inequality into our initial inequality on α:

$$\alpha \geq 3(g-a) > 3g\left(1 - \sqrt{\frac{2}{d}} - \frac{2}{g}\right) \geq \frac{3K}{d}\left(1 - \sqrt{\frac{2}{d}} - \frac{2}{g}\right).$$

It follows that

$$\frac{K}{\alpha} \leq \frac{K}{\frac{3K}{d}\left(1-\sqrt{\frac{2}{d}}-\frac{2}{g}\right)} = \frac{d}{3\cdot\left(1-\sqrt{\frac{2}{d}}-\frac{2}{g}\right)}.$$

Due to Scaling Lemma 13 we know that if there exists a (d, K, α) PCD then there also exists a $(d, K \cdot \lambda, \alpha \cdot \lambda)$ PCD for every positive integer λ. If we plug this covering design into the inequality above, we will get that

$$\frac{K}{\alpha} = \frac{\lambda \cdot K}{\lambda \cdot \alpha} \leq \frac{d}{3\cdot\left(1-\sqrt{\frac{2}{d}}-\frac{2}{g'}\right)}$$

where $g' := \left\lceil \frac{K\cdot\lambda}{d} \right\rceil$. If we take $\lambda \to +\infty$ then $\frac{2}{g'} \to 0$ and in the limit, we get the desired upper bound on $\frac{K}{\alpha}$:

$$\frac{K}{\alpha} \leq \frac{d}{3\cdot\left(1-\sqrt{\frac{2}{d}}\right)} = \frac{d}{3}\cdot\left(1+\frac{\sqrt{\frac{2}{d}}}{1-\sqrt{\frac{2}{d}}}\right) = \frac{d}{3} + \frac{\sqrt{2d}}{3\left(1-\sqrt{\frac{2}{d}}\right)} = \frac{d}{3} + \sqrt{\frac{2}{9}}\cdot\sqrt{d} + o(\sqrt{d}). \blacktriangleleft$$

4 Conclusion and Outlook

In this work, we make progress on obtaining tight conditional lower bounds for Klee's measure problem and related problems for $d \geq 4$. We give improved lower bounds that leave gaps of only $O(n^{0.09524})$, $O(n^{0.27778})$ and $O(n^{0.4286})$ for $d = 4$, $d = 5$ and $d = 6$, respectively. On the negative side, we prove that the proof technique via prefix covering designs and Proposition 2 – despite yielding a tight lower bound for $d = 3$ – cannot give tight lower bounds for $d \geq 4$, so that a novel reduction approach is needed for this task. Of course, it remains a tantalizing possibility that the $n^{d/2 \pm o(1)}$ running time for Klee's measure problem for large dimensions $d \geq 4$ can be broken.

We feel that the prefix covering designs formalized in this work are interesting in their own right. We establish a connection to the well-studied covering designs, by giving a framework that turns 2-covering designs into prefix covering designs. This connection leads to the asymptotic bound $\gamma_d = \frac{d}{3} + \Theta(\sqrt{d})$, leading to an $n^{d/3+\Theta(\sqrt{d})}$ conditional lower bound for Klee's measure problem and related problems, improving over a previous bound of $n^{d/3+1/3+\Omega(1/d)}$.

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