On Comparable Box Dimension

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Abstract

Two boxes in $\mathbb{R}^d$ are *comparable* if one of them is a subset of a translation of the other one. The *comparable box dimension* of a graph G is the minimum integer d such that G can be represented as a touching graph of comparable axis-aligned boxes in $\mathbb{R}^d$. We show that proper minor-closed classes have bounded comparable box dimension and explore further properties of this notion.

1 Introduction

Given a system $\mathcal{O}$ of subsets of $\mathbb{R}^d$, we say that a graph G is a *touching graph of objects from $\mathcal{O}$* if there exists a function $f : V(G) \to \mathcal{O}$ (called a *touching representation by objects from $\mathcal{O}$*) such that the interiors of $f(u)$ and $f(v)$ are disjoint for all distinct $u, v \in V(G)$, and $f(u) \cap f(v) \neq \emptyset$ if and only if $uv \in E(G)$. Famously, Koebe [13] proved that a graph is planar if and only if it is a touching graph of balls in $\mathbb{R}^2$. This result has motivated numerous strengthenings and variations (see [14, 19] for some classical examples); most

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relevantly for us, Felsner and Francis [11] showed that every planar graph is a touching graph of cubes in $\mathbb{R}^3$.

An attractive feature of touching representations is that it is possible to represent graph classes that are sparse (e.g., planar graphs, or more generally, graph classes with bounded expansion [15]). This is in contrast to general intersection representations where the represented class always includes arbitrarily large cliques. Of course, whether the class of touching graphs of objects from $\mathcal{O}$ is sparse or not depends on the system $\mathcal{O}$. For example, all complete bipartite graphs $K_{n,m}$ are touching graphs of boxes in $\mathbb{R}^3$, where the vertices in one part are represented by $m \times 1 \times 1$ boxes and the vertices of the other part are represented by $1 \times n \times 1$ boxes (throughout the paper, by a *box* we mean an axis-aligned one, i.e., the Cartesian product of closed intervals of non-zero length). Dvořák, McCarty and Norin [6] noticed that this issue disappears if we forbid such a combination of long and wide boxes, a condition which can be expressed as follows. For two boxes B_1 and B_2, we write $B_1 \sqsubseteq B_2$ if B_2 contains a translate of B_1. We say that B_1 and B_2 are *comparable* if $B_1 \sqsubseteq B_2$ or $B_2 \sqsubseteq B_1$. A *touching representation by comparable boxes* of a graph G is a touching representation f by boxes such that for every $u, v \in V(G)$, the boxes $f(u)$ and $f(v)$ are comparable. Let the *comparable box dimension* $\dim_{cb}(G)$ of a graph G be the smallest integer d such that G has a touching representation by comparable boxes in $\mathbb{R}^d$. Let us remark that the comparable box dimension of every graph G is at most $|V(G)|$, see Section 3.1 for details. Then for a class $\mathcal{G}$ of graphs, let $\dim_{cb}(\mathcal{G}) := \sup\{\dim_{cb}(G) : G \in \mathcal{G}\}$. Note that $\dim_{cb}(\mathcal{G}) = \infty$ if the comparable box dimension of graphs in $\mathcal{G}$ is not bounded.

Dvořák, McCarty and Norin [6] proved some basic properties of this notion. In particular, they showed that if a class $\mathcal{G}$ has finite comparable box dimension, then it has polynomial strong coloring numbers, which implies that $\mathcal{G}$ has strongly sublinear separators. They also provided an example showing that for many functions h, the class of graphs with strong coloring numbers bounded by h has infinite comparable box dimension[1]. Dvořák et al. [9] proved that graphs of comparable box dimension 3 have exponential weak coloring numbers, giving the first natural graph class with polynomial strong coloring numbers and superpolynomial weak coloring numbers (the previous example is obtained by subdividing edges of every graph suitably many times [12]).

We show that the comparable box dimension behaves well under the operations of addition of apex vertices, clique-sums, and taking subgraphs. Together with known results on product structure [4], this implies the main result of this paper.

Theorem 1. *The comparable box dimension of every proper minor-closed class of graphs is finite.*

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[1] In their construction $h(r)$ has to be at least 3, and has to tend to $+\infty$.

Additionally, we show that classes of graphs with finite comparable box dimension are fractionally treewidth-fragile. This gives arbitrarily precise approximation algorithms for all monotone maximization problems that are expressible in terms of distances between the solution vertices and tractable on graphs of bounded treewidth [8] or expressible in the first-order logic [7].

2 Parameters

In this section we bound some basic graph parameters in terms of comparable box dimension. The first result bounds the clique number $\omega(G)$ in terms of $\dim_{cb}(G)$.

Lemma 1. *For any graph G, we have $\omega(G) \leq 2^{\dim_{cb}(G)}$.*

Proof. We may assume that G has bounded comparable box dimension witnessed by a representation f. To represent any clique $A = \{a_1, \ldots, a_w\}$ in G, the corresponding boxes $f(a_1), \ldots, f(a_w)$ have pairwise non-empty intersections. Since axis-aligned boxes have the Helly property, there is a point $p \in \mathbb{R}^d$ contained in $f(a_1) \cap \cdots \cap f(a_w)$. As each box is full-dimensional, its interior intersects at least one of the 2^d orthants at p. At the same time, it follows from the definition of a touching representation that $f(a_1), \ldots, f(a_d)$ have pairwise disjoint interiors, and hence $w \leq 2^d$. $\square$

Note that a clique with 2^d vertices has a touching representation by comparable boxes in $\mathbb{R}^d$, where each vertex is a hypercube defined as the Cartesian product of intervals of form $[-1, 0]$ or $[0, 1]$. Together with Lemma 1, it follows that $\dim_{cb}(K_{2^d}) = d$.

In the following we consider the chromatic number $\chi(G)$, and two of its variants. An *acyclic coloring* (resp. *star coloring*) of a graph G is a proper coloring such that any two color classes induce a forest (resp. star forest, i.e., a forest in which each component is a star). The *acyclic chromatic number* $\chi_a(G)$ (resp. *star chromatic number* $\chi_s(G)$) of G is the minimum number of colors in an acyclic (resp. star) coloring of G. We will need the fact that all the variants of the chromatic number are at most exponential in the comparable box dimension; this follows from [6], although we include an argument to make the dependence clear.

Lemma 2. *For any graph G we have $\chi(G) \leq 3^{\dim_{cb}(G)}$, $\chi_a(G) \leq 5^{\dim_{cb}(G)}$ and $\chi_s(G) \leq 2 \cdot 9^{\dim_{cb}(G)}$.*

Proof. We focus on the star chromatic number and note that the chromatic number and the acyclic chromatic number may be bounded similarly. Suppose that G has comparable box dimension d witnessed by a representation f, and let $v_1, \ldots, v_n$ be the vertices of G written so that $\text{vol}(f(v_1)) \geq \ldots \geq \text{vol}(f(v_n))$. Equivalently, we have $f(v_i) \sqsubseteq f(v_j)$ whenever $i > j$. Now define a greedy coloring c so that $c(v_i)$ is the smallest

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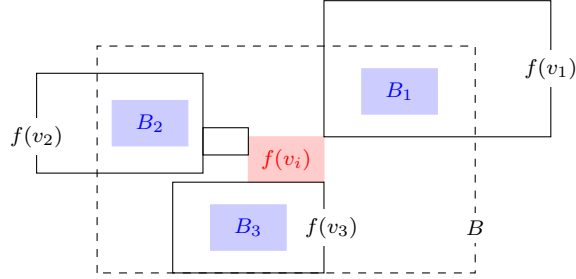Figure 1: Nearby boxes obstructing colors at v_i.

color such that $c(v_i) \neq c(v_j)$ for any $j < i$ for which either $v_j v_i \in E(G)$ or there exists $m > j$ such that $v_j v_m, v_m v_i \in E(G)$. Note that this gives a star coloring, since a path on four vertices always contains a 3-vertex subpath of the form $v_{i_1} v_{i_2} v_{i_3}$ such that $i_1 < i_2, i_3$ and our coloring procedure gives distinct colors to vertices forming such a path.

It remains to bound the number of colors used. Suppose we are coloring v_i. We shall bound the number of vertices v_j such that $j < i$ and such that there exists $m > i$ for which $v_j v_m, v_m v_i \in E(G)$. Let B be the box obtained by scaling up $f(v_i)$ by a factor of 5 while keeping the same center. Since $f(v_m) \sqsubseteq f(v_i) \sqsubseteq f(v_j)$, there exists a translation B_j of $f(v_i)$ contained in $f(v_j) \cap B$ (see Figure 1). Two boxes B_j and $B_{j'}$ for $j \neq j'$ have disjoint interiors since their intersection is contained in the intersection of the touching boxes $f(v_j)$ and $f(v_{j'})$, and their interiors are also disjoint from $f(v_i) \subset B$. Thus the number of such indices j is at most $\mathrm{vol}(B)/\mathrm{vol}(f(v_i)) - 1 = 5^d - 1$.

A similar argument shows that the number of indices m such that $m < i$ and $v_m v_i \in E(G)$ is at most $3^d - 1$. Consequently, the number of indices $j < i$ for which there exists m such that $j < m < i$ and $v_j v_m, v_m v_i \in E(G)$ is at most $(3^d - 1)^2$. This means that when choosing the color of v_i greedily, we only need to avoid colors of at most $(5^d - 1) + (3^d - 1) + (3^d - 1)^2$ vertices, so $2 \cdot 9^d$ colors suffice. □

3 Operations

It is clear that given a touching representation of a graph G, one can easily obtains a touching representation by boxes of an induced subgraph H of G by simply deleting the boxes corresponding to the vertices in $V(G) \setminus V(H)$. In this section we are going to consider other basic operations on graphs. In the following, to describe the boxes, we are going to use the Cartesian product $\times$ defined among boxes ($A \times B$ is the box whose projection on the first dimensions gives the box A, while the projection on the remaining dimensions gives the box B) or we are going to provide its projections for

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every dimension ($A[i]$ is the interval obtained from projecting A on its ith dimension).

3.1 Vertex addition

Let us start with a simple lemma saying that the addition of a vertex increases the comparable box dimension by at most one. In particular, this implies that $\dim_{cb}(G) \leq |V(G)|$.

Lemma 3. *For any graph G and $v \in V(G)$, we have $\dim_{cb}(G) \leq \dim_{cb}(G - v) + 1$.*

Proof. Let f be a touching representation of $G - v$ by comparable boxes in $\mathbb{R}^d$, where $d = \dim_{cb}(G - v)$. We define a representation h of G as follows. For each $u \in V(G) \setminus \{v\}$, let $h(u) = [0, 1] \times f(u)$ if $uv \in E(G)$ and $h(u) = [1/2, 3/2] \times f(u)$ if $uv \notin E(G)$. Let $h(v) = [-1, 0] \times [-M, M] \times \cdots \times [-M, M]$, where M is chosen large enough so that $f(u) \subseteq [-M, M] \times \cdots \times [-M, M]$ for every $u \in V(G) \setminus \{v\}$. Then h is a touching representation of G by comparable boxes in $\mathbb{R}^{d+1}$. $\square$

3.2 Strong product

Let $G \boxtimes H$ denote the *strong product* of the graphs G and H, i.e., the graph with vertex set $V(G) \times V(H)$ and with distinct vertices (u_1, v_1) and (u_2, v_2) adjacent if and only if u_1 is equal to or adjacent to u_2 in G and v_1 is equal to or adjacent to v_2 in H. To obtain a touching representation of $G \boxtimes H$ it suffices to take a product of representations of G and H, but the resulting representation may contain incomparable boxes. Indeed, $\dim_{cb}(G \boxtimes H)$ in general is not bounded by a function of $\dim_{cb}(G)$ and $\dim_{cb}(H)$; for example, every star has comparable box dimension at most two, but the strong product of the star $K_{1,n}$ with itself contains $K_{n,n}$ as an induced subgraph, and thus its comparable box dimension is at least $\Omega(\log n)$. However, as shown in the following lemma, this issue does not arise if the representation of H consists of translates of a single box; by scaling, we can without loss of generality assume this box is a unit hypercube.

Lemma 4. *Consider a graph H having a touching representation h in $\mathbb{R}^{d_H}$ by axis-aligned hypercubes of unit size. Then for any graph G, the strong product $G \boxtimes H$ of these graphs has comparable box dimension at most $\dim_{cb}(G) + d_H$.*

Proof. The proof simply consists in taking a product of the two representations. Indeed, consider a touching respresentation g of G by comparable boxes in $\mathbb{R}^{d_G}$, with $d_G = \dim_{cb}(G)$, and the representation h of H. Let us define a representation f of $G \boxtimes H$ in $\mathbb{R}^{d_G + d_H}$ as follows.

$$f((u, v))[i] = \begin{cases} g(u)[i] & \text{if } i \leq d_G \\ h(v)[i - d_G] & \text{if } i > d_G \end{cases}$$

Consider distinct vertices (u, v) and (u', v') of $G \boxtimes H$. The boxes $g(u)$ and $g(u')$ are comparable, say $g(u) \sqsubseteq g(u')$. Since $h(v')$ is a translation of $h(v)$, this implies that $f((u, v)) \sqsubseteq f((u', v'))$. Hence, the boxes of the representation f are pairwise comparable.

The boxes of the representations g and h have pairwise disjoint interiors. Hence, if $u \neq u'$, then there exists $i \leq d_G$ such that the interiors of the intervals $f((u, v))[i] = g(u)[i]$ and $f((u', v'))[i] = g(u')[i]$ are disjoint; and if $v \neq v'$, then there exists $i \leq d_H$ such that the interiors of the intervals $f((u, v))[i + d_G] = h(v)[i]$ and $f((u', v'))[i + d_G] = h(v')[i]$ are disjoint. Consequently, the interiors of boxes $f((u, v))$ and $f((u', v'))$ are pairwise disjoint. Moreover, if $u \neq u'$ and $uu' \notin E(G)$, or if $v \neq v'$ and $vv' \notin E(G)$, then the intervals discussed above (not just their interiors) are disjoint for some i; hence, if (u, v) and (u', v') are not adjacent in $G \boxtimes H$, then $f((u, v)) \cap f((u', v')) = \emptyset$. Therefore, f is a touching representation of a subgraph of $G \boxtimes H$.

Finally, suppose that (u, v) and (u', v') are adjacent in $G \boxtimes H$. Then there exists a point p_G in the intersection of $g(u)$ and $g(u')$, since $u = u'$ or $uu' \in E(G)$ and g is a touching representation of G; and similarly, there exists a point p_H in the intersection of $h(v)$ and $h(v')$. Then $p_G \times p_H$ is a point in the intersection of $f((u, v))$ and $f((u', v'))$. Hence, f is indeed a touching representation of $G \boxtimes H$. $\qquad\square$

3.3 Taking a subgraph

The comparable box dimension of a subgraph of a graph G may be larger than $\dim_{cb}(G)$, see the end of this section for an example. However, we show that the comparable box dimension of a subgraph is at most exponential in the comparable box dimension of the whole graph. This is essentially Corollary 25 in [6], but since the setting is somewhat different and the construction of [6] uses rotated boxes, we provide details of the argument.

Lemma 5. *If G is a subgraph of a graph G', then $\dim_{cb}(G) \leq \dim_{cb}(G') + \frac{1}{2}\chi_s^2(G')$.*

Proof. As we can remove the boxes that represent the vertices, we can assume $V(G') = V(G)$. Let f be a touching representation of G' by comparable boxes in $\mathbb{R}^d$, where $d = \dim_{cb}(G')$. Let φ be a star coloring of G' using colors $\{1, \ldots, c\}$, where $c = \chi_s(G')$.

For any distinct colors $i, j \in \{1, \ldots, c\}$, let $A_{i,j} \subseteq V(G)$ be the set of vertices u of color i such that there exists a vertex v of color j such that $uv \in E(G') \setminus E(G)$. For each $u \in A_{i,j}$, let $a_j(u)$ denote such a vertex v chosen arbitrarily.

Let us define a representation h by boxes in $\mathbb{R}^{d+\binom{c}{2}}$ by starting from the representation f and, for each pair $i < j$ of colors, adding a dimension $d_{i,j}$

and setting

$$h(v)[d_{i,j}] = \begin{cases} [1/3, 4/3] & \text{if } v \in A_{i,j} \\ [-4/3, -1/3] & \text{if } v \in A_{j,i} \\ [-1/2, 1/2] & \text{otherwise.} \end{cases}$$

Note that the boxes in this extended representation are comparable, as in the added dimensions, all the boxes have size 1.

Suppose $uv \in E(G)$, where $\varphi(u) = i$ and $\varphi(v) = j$ and say $i < j$. We cannot have $u \in A_{i,j}$ and $v \in A_{j,i}$, as then $a_j(u)uva_i(v)$ would be a 4-vertex path in G' in colors i and j. Hence, in any added dimension d', we have $h(u)[d'] = [-1/2, 1/2]$ or $h(v)[d'] = [-1/2, 1/2]$, and thus $h(u)[d'] \cap h(v)[d'] \neq \emptyset$. Since the boxes $f(u)$ and $f(v)$ touch, it follows that the boxes $h(u)$ and $h(v)$ touch as well.

Suppose now that $uv \notin E(G)$. If $uv \notin E(G')$, then $f(u)$ is disjoint from $f(v)$, and thus $h(u)$ is disjoint from $h(v)$. Hence, we can assume $uv \in E(G') \setminus E(G)$, $\varphi(u) = i$, $\varphi(v) = j$ and $i < j$. Then $u \in A_{i,j}$, $v \in A_{j,i}$, $h(u)[d_{i,j}] = [1/3, 4/3]$, $h(v)[d_{j,i}] = [-4/3, -1/3]$, and $h(u) \cap h(v) = \emptyset$.

Consequently, h is a touching representation of G by comparable boxes in dimension $d + \binom{c}{2} \leq d + c^2/2$. $\square$

Let us now combine Lemmas 2 and 5.

Corollary 1. *If G is a subgraph of a graph G', then $\dim_{cb}(G) \leq \dim_{cb}(G') + 2 \cdot 81^{\dim_{cb}(G')} \leq 3 \cdot 81^{\dim_{cb}(G')}$.*

Let us remark that an exponential increase in the dimension is unavoidable: We have $\dim_{cb}(K_{2^d}) = d$, but the graph obtained from K_{2^d} by deleting a perfect matching has comparable box dimension 2^{d-1}. Indeed, for every pair u, v of non-adjacent vertices there is a specific dimension i such that their boxes span intervals $[a, b]$ and $[c, d]$ with $b < c$, while for every other box in the representation their i^{th} interval contains $[b, c]$.

3.4 Clique-sums

A *clique-sum* of two graphs G_1 and G_2 is obtained from their disjoint union by identifying vertices of a clique in G_1 and a clique of the same size in G_2 and possibly deleting some of the edges of the resulting clique. A *full clique-sum* is a clique-sum in which we keep all the edges of the resulting clique. The main issue to overcome in obtaining a representation for a (full) clique-sum is that the representations of G_1 and G_2 can be "degenerate". Consider e.g. the case that G_1 is represented by unit squares arranged in a grid; in this case, there is no space to attach G_2 at the cliques formed by four squares intersecting in a single corner. This can be avoided by increasing the dimension, but we need to be careful so that the dimension stays bounded even after an arbitrary number of clique-sums. We thus introduce the notion of *clique-sum extendable* representations.

Definition 1. *Consider a graph G with a distinguished clique $C^\star$, called the* root clique *of G. A touching representation h of G by (not necessarily comparable) boxes in $\mathbb{R}^d$ is called $C^\star$-clique-sum extendable if the following conditions hold for every sufficiently small $\varepsilon > 0$.*

(vertices) *For each $u \in V(C^\star)$, there exists a dimension d_u, such that:*

(v0) $d_u \neq d_{u'}$ *for distinct $u, u' \in V(C^\star)$,*

(v1) *each vertex $u \in V(C^\star)$ satisfies $h(u)[d_u] = [-1, 0]$ and $h(u)[i] = [0, 1]$ for any dimension $i \neq d_u$, and*

(v2) *each vertex $v \notin V(C^\star)$ satisfies $h(v) \subset [0, 1)^d$.*

(cliques) *For every clique C of G, there exists a point $p(C) \in [0, 1)^d \cap \left(\bigcap_{v \in V(C)} h(v) \right)$ such that, defining the clique box $h^\varepsilon(C)$ by setting $h^\varepsilon(C)[i] = [p(C)[i], p(C)[i] + \varepsilon]$ for every dimension i, the following conditions are satisfied:*

(c1) *For any two cliques $C_1 \neq C_2$, $h^\varepsilon(C_1) \cap h^\varepsilon(C_2) = \emptyset$ (equivalently, $p(C_1) \neq p(C_2)$).*

(c2) *A box $h(v)$ intersects $h^\varepsilon(C)$ if and only if $v \in V(C)$, and in that case their intersection is a facet of $h^\varepsilon(C)$ incident to $p(C)$. That is, there exists a dimension $i_{C,v}$ such that for each dimension j,*

$$h(v)[j] \cap h^\varepsilon(C)[j] = \begin{cases} \{p(C)[i_{C,v}]\} & \text{if } j = i_{C,v} \\ [p(C)[j], p(C)[j] + \varepsilon] & \text{otherwise.} \end{cases}$$

Note that the root clique can be empty, that is the empty subgraph with no vertices. In that case the clique is denoted $\emptyset$. Let $\dim_{cb}^{ext}(G)$ be the minimum dimension such that G has an $\emptyset$-clique-sum extendable touching representation by comparable boxes.

Let us remark that a clique-sum extendable representation in dimension d implies such a representation in higher dimensions as well.

Lemma 6. *Let G be a graph with a root clique $C^\star$ and let h be a $C^\star$-clique-sum extendable touching representation of G by comparable boxes in $\mathbb{R}^d$. Then G has such a representation in $\mathbb{R}^{d'}$ for every $d' \geq d$.*

Proof. It clearly suffices to consider the case that $d' = d + 1$. Note that the **(vertices)** conditions imply that $h(v') \sqsubseteq h(v)$ for every $v' \in V(G) \setminus V(C^\star)$ and $v \in V(C^\star)$. We extend the representation h by setting $h(v)[d+1] = [0, 1]$ for $v \in V(C^\star)$ and $h(v)[d + 1] = [0, \frac{1}{2}]$ for $v \in V(G) \setminus V(C^\star)$. The clique point $p(C)$ of each clique C is extended by setting $p(C)[d+1] = \frac{1}{4}$. It is easy to verify that the resulting representation is $C^\star$-clique-sum extendable. $\square$

The following lemma ensures that clique-sum extendable representations behave well with respect to full clique-sums.

Lemma 7. *Consider two graphs G_1 and G_2, given with a $C_1^\star$- and a $C_2^\star$-clique-sum extendable representations h_1 and h_2 by comparable boxes in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Let G be the graph obtained by performing a full clique-sum of these two graphs on any clique C_1 of G_1, and on the root clique $C_2^\star$ of G_2. Then G admits a $C_1^\star$-clique sum extendable representation h by comparable boxes in $\mathbb{R}^{\max(d_1,d_2)}$.*

Proof. By Lemma 6, we can assume that $d_1 = d_2$; let $d = d_1$. The idea is to translate (allowing also exchanges of dimensions) and scale h_2 to fit in $h_1^\varepsilon(C_1)$. Consider an $\varepsilon > 0$ sufficiently small so that $h_1^\varepsilon(C_1)$ satisfies all the **(cliques)** conditions, and such that $h_1^\varepsilon(C_1) \sqsubseteq h_1(v)$ for any vertex $v \in V(G_1)$. Let $V(C_1) = \{v_1, \ldots, v_k\}$; without loss of generality, we can assume $i_{C_1, v_i} = i$ for $i \in \{1, \ldots, k\}$, and thus

$$h_1(v_i)[j] \cap h_1^\varepsilon(C_1)[j] = \begin{cases} \{p_1(C_1)[i]\} & \text{if } j = i \\ [p_1(C_1)[j], p_1(C_1)[j] + \varepsilon] & \text{otherwise.} \end{cases}$$

Now let us consider G_2 and its representation h_2. Here the vertices of $C_2^\star$ are also denoted $v_1, \ldots, v_k$, and without loss of generality, the **(vertices)** conditions are satisfied by setting $d_{v_i} = i$ for $i \in \{1, \ldots, k\}$

We are now ready to define h. For $v \in V(G_1)$, we set $h(v) = h_1(v)$. We now scale and translate h_2 to fit inside $h_1^\varepsilon(C_1)$. That is, we fix $\varepsilon > 0$ small enough so that

- the conditions **(cliques)** hold for h_1,

- $h_1^\varepsilon(C_1) \subset [0,1)^d$, and

- $h_1^\varepsilon(C_1) \sqsubseteq h_1(u)$ for every $u \in V(G_1)$,

and for each $v \in V(G_2) \setminus V(C_2^\star)$, we set $h(v)[i] = p_1(C_1)[i] + \varepsilon h_2(v)[i]$ for $i \in \{1, \ldots, d\}$. Note that the condition (v2) for h_2 implies $h(v) \subset h_1^\varepsilon(C_1)$. Each clique C of H is a clique of G_1 or G_2. If C is a clique of G_2, we set $p(C) = p_1(C_1) + \varepsilon p_2(C)$, otherwise we set $p(C) = p_1(C)$. In particular, for subcliques of $C_1 = C_2^\star$, we use the former choice.

Let us now check that h is a $C_1^\star$-clique sum extendable representation by comparable boxes. The fact that the boxes are comparable follows from the fact that those of h_1 and h_2 are comparable and from the scaling of h_2: By construction both $h_1(v) \sqsubseteq h_1(u)$ and $h_2(v) \sqsubseteq h_2(u)$ imply $h(v) \sqsubseteq h(u)$, and for any vertex $u \in V(G_1)$ and any vertex $v \in V(G_2) \setminus V(C_2^\star)$, we have $h(v) \subset h_1^\varepsilon(C_1) \sqsubseteq h(u)$.

We now check that h is a contact representation of G. For $u, v \in V(G_1)$ (resp. $u, v \in V(G_2) \setminus V(C_2^\star)$) it is clear that $h(u)$ and $h(v)$ have disjoint

interiors, and that they intersect if and only if $h_1(u)$ and $h_1(v)$ intersect (resp. if $h_2(u)$ and $h_2(v)$ intersect). Consider now a vertex $u \in V(G_1)$ and a vertex $v \in V(G_2) \setminus V(C_2^\star)$. As $h(v) \subset h^\varepsilon(C_1)$, the condition (v2) for h_1 implies that $h(u)$ and $h(v)$ have disjoint interiors.

Furthermore, if $uv \in E(G)$, then $u \in V(C_1) = V(C_2^\star)$, say $u = v_1$. Since $uv \in E(G_2)$, the intervals $h_2(u)[1]$ and $h_2(v)[1]$ intersect, and by (v1) and (v2) for h_2, we conclude that $h_2(v)[1] = [0, \alpha]$ for some positive $\alpha < 1$. Therefore, $p_1(C_1)[1] \in h(v)[1]$. Since $p_1(C_1) \in \bigcap_{x \in V(C_1)} h_1(x)$, we have $p_1(C_1) \in h(u)$, and thus $p_1(C_1)[1] \in h(u)[1] \cap h(v)[1]$. For $i \in \{2, \dots, d\}$, note that $i \neq 1 = i_{C_1,u}$, and thus by (c2) for h_1, we have $h_1^\varepsilon(C_1)[i] \subseteq h_1(u)[i] = h(u)[i]$. Since $h(v)[i] \subseteq h_1^\varepsilon(C_1)[i]$, it follows that $h(u)$ intersects $h(v)$.

Finally, let us consider the $C_1^\star$-clique-sum extendability. The **(vertices)** conditions hold, since (v0) and (v1) are inherited from h_1, and (v2) is inherited from h_1 for $v \in V(G_1) \setminus V(C_1^\star)$ and follows from the fact that $h(v) \subseteq h_1^\varepsilon(C_1) \subset [0,1)^d$ for $v \in V(G_2) \setminus V(C_2^\star)$. For the **(cliques)** condition (c1), the mapping p inherits injectivity when restricted to cliques of G_2, or to cliques of G_1 not contained in C_1. For any clique C of G_2, the point $p(C)$ is contained in $h_1^\varepsilon(C_1)$, since $p_2(C) \in [0,1)^d$. On the other hand, if C' is a clique of G_1 not contained in C_1, then there exists $v \in V(C') \setminus V(C_1)$, we have $p(C') = p_1(C') \in h_1(v)$, and $h_1(v) \cap h_1^\varepsilon(C_1) = \emptyset$ by (c2) for h_1. Therefore, the mapping p is injective, and thus for sufficiently small $\varepsilon' > 0$, we have $h^{\varepsilon'}(C) \cap h^{\varepsilon'}(C') = \emptyset$ for any distinct cliques C and C' of G.

The condition (c2) of h is (for sufficiently small $\varepsilon' > 0$) inherited from the property (c2) of h_1 and h_2 when C is a clique of G_2 and $v \in V(G_2) \setminus V(C_2^\star)$, or when C is a clique of G_1 not contained in C_1 and $v \in V(G_1)$. If C is a clique of G_1 not contained in C_1 and $v \in V(G_2) \setminus V(C_2^\star)$, then by (c1) for h_1 we have $h_1^\varepsilon(C) \cap h_1^\varepsilon(C_1) = \emptyset$, and since $h^{\varepsilon'}(C) \subseteq h_1^\varepsilon(C)$ and $h(v) \subseteq h_1^\varepsilon(C_1)$, we conclude that $h(v) \cap h^{\varepsilon'}(C) = \emptyset$. It remains to consider the case that C is a clique of G_2 and $v \in V(G_1)$. Note that $h^{\varepsilon'}(C) \subseteq h_1^\varepsilon(C_1)$.

- If $v \notin V(C_1)$, then by the property (c2) of h_1, the box $h(v) = h_1(v)$ is disjoint from $h_1^\varepsilon(C_1)$, and thus $h(v) \cap h^{\varepsilon'}(C) = \emptyset$.

- Otherwise $v \in V(C_1) = V(C_2^\star)$, say $v = v_1$. Note that by (v1), we have $h_2(v) = [-1, 0] \times [0, 1]^{d-1}$.

  - If $v \notin V(C)$, then by the property (c2) of h_2, the box $h_2(v)$ is disjoint from $h_2^\varepsilon(C)$. Since $h_2^\varepsilon(C)[i] \subseteq [0,1] = h_2(v)[i]$ for $i \in \{2, \dots, d\}$, it follows that $h_2^\varepsilon(C)[1] \subseteq (0,1)$, and thus $h^{\varepsilon'}(C)[1] \subseteq h_1^\varepsilon(C_1)[1] \setminus \{p(C_1)[1]\}$. By (c2) for h_1, we have $h(v)[1] \cap h_1^\varepsilon(C_1)[1] = h_1(v)[1] \cap h_1^\varepsilon(C_1)[1] = p(C_1)[1]$, and thus $h(v) \cap h^{\varepsilon'}(C) = \emptyset$.

  - If $v \in V(C)$, then by the property (c2) of h_2, the intersection of $h_2(v)[1] = [-1, 0]$ and $h_2^\varepsilon(C)[1] \subseteq [0,1)$ is the single point $p_2(C)[1] = 0$, and thus $p(C)[1] = p_1(C_1)[1]$ and $h^{\varepsilon'}(C)[1] =$

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$[p_1(C_1)[1], p_1(C_1)[1] + \varepsilon']$. Recall that the property (c2) of h_1 implies $h(v)[1] \cap h_1^\varepsilon(C_1)[1] = \{p(C_1)[1]\}$, and thus $h(v)[1] \cap h^{\varepsilon'}(C)[1] = \{p(C)[1]\}$. For $i \in \{2, \ldots, d\}$, the property (c2) of h_1 implies $h_1^\varepsilon(C_1)[i] \subseteq h_1(v)[i] = h(v)[i]$, and since $h^{\varepsilon'}(C)[i] \subseteq h_1^\varepsilon(C_1)[i]$, it follows that $h^{\varepsilon'}(C)[i] \subseteq h(v)[i]$.

$\square$

The following lemma enables us to pick the root clique at the expense of increasing the dimension by $\omega(G)$.

Lemma 8. *For any graph G and any clique $C^\star$, the graph G admits a $C^\star$-clique-sum extendable touching representation by comparable boxes in $\mathbb{R}^d$, for $d = |V(C^\star)| + \dim_{cb}^{ext}(G \setminus V(C^\star))$.*

Proof. The proof is essentially the same as the one of Lemma 3. Consider a $\emptyset$-clique-sum extendable touching representation h' of $G \setminus V(C^\star)$ by comparable boxes in $\mathbb{R}^{d'}$, with $d' = \dim_{cb}(G \setminus V(C^\star))$, and let $V(C^\star) = \{v_1, \ldots, v_k\}$. We now construct the desired representation h of G as follows. For each vertex $v_i \in V(C^\star)$, let $h(v_i)$ be the box in $\mathbb{R}^d$ uniquely determined by the condition (v1) with $d_{v_i} = i$. For each vertex $u \in V(G) \setminus V(C^\star)$, if $i \leq k$ then let $h(u)[i] = [0, 1/2]$ if $uv_i \in E(G)$, and $h(u)[i] = [1/4, 3/4]$ if $uv_i \notin E(G)$. For $i > k$ we have $h(u)[i] = \alpha h'(u)[i - k]$, for some $\alpha > 0$. The value $\alpha > 0$ is chosen sufficiently small so that $h(u)[i] \subset [0, 1)$ whenever $u \notin V(C^\star)$. We proceed similarly for the clique points. For any clique C of G, if $i \leq k$ then let $p(C)[i] = 0$ if $v_i \in V(C)$, and $p(C)[i] = 1/4$ if $v_i \notin V(C)$. For $i > k$ we refer to the clique point $p'(C')$ of $C' = C \setminus \{v_1, \ldots, v_k\}$, and we set $p(C)[i] = \alpha p'(C')[i - k]$.

By the construction, it is clear that h is a touching representation of G. As $h'(u) \sqsubset h'(v)$ implies that $h(u) \sqsubset h(v)$, and as $h(u) \sqsubset h(v_i)$ for every $u \in V(G) \setminus V(C^\star)$ and every $v_i \in V(C^\star)$, we have that h is a representation by comparable boxes.

For the $C^\star$-clique-sum extendability, the **(vertices)** conditions hold by the construction. For the **(cliques)** condition (c1), let us consider distinct cliques C_1 and C_2 of G such that $|V(C_1)| \geq |V(C_2)|$, and let $C_i' = C_i \setminus V(C^\star)$. If $C_1' = C_2'$, there is a vertex $v_i \in V(C_1) \setminus V(C_2)$, and $p(C_1)[i] = 0 \neq 1/4 = p(C_2)[i]$. Otherwise, if $C_1' \neq C_2'$, then $p'(C_1') \neq p'(C_2')$, which implies $p(C_1) \neq p(C_2)$ by construction.

For the **(cliques)** condition (c2), let us first consider a vertex $v \in V(G) \setminus V(C^\star)$ and a clique C of G containing v. In the dimensions $i \in \{1, \ldots, k\}$, we always have $h^\varepsilon(C)[i] \subseteq h(v)[i]$. Indeed, if $v_i \in V(C)$, then $h^\varepsilon(C)[i] \subseteq [0, 1/2] = h(v)[i]$, as in this case v and v_i are adjacent; and if $v_i \notin V(C)$, then $h^\varepsilon(C)[i] \subseteq [1/4, 1/2] \subseteq h(v)[i]$. By the property (c2) of h', we have $h^\varepsilon(C)[i] \subseteq h(v)[i]$ for every $i > k$, except one, for which $h^\varepsilon(C)[i] \cap h(v)[i] = \{p(C)[i]\}$.

Next, let us consider a vertex $v \in V(G) \setminus V(C^\star)$ and a clique C of G not containing v. As $v \notin V(C')$, the condition (c2) for h' implies that $p'(C')$ is disjoint from $h'(v)$, and thus $p(C)$ is disjoint from $h(v)$.

Finally, we consider a vertex $v_i \in V(C^\star)$. Note that for any clique C containing v_i, we have that $h^\varepsilon(C)[i] \cap h(v_i)[i] = [0, \varepsilon] \cap [-1, 0] = \{0\}$, and $h^\varepsilon(C)[j] \subseteq [0, 1] = h(v_i)[j]$ for any $j \neq i$. For a clique C that does not contain v_i we have that $h^\varepsilon(C)[i] \cap h(v_i)[i] \subset (0, 1) \cap [-1, 0] = \emptyset$. Condition (c2) is thus fulfilled and this completes the proof of the lemma. $\square$

The following lemma provides an upper bound on $\dim_{cb}^{ext}(G)$ in terms of $\dim_{cb}(G)$ and $\chi(G)$.

Lemma 9. *For any graph G, $\dim_{cb}^{ext}(G) \leq \dim_{cb}(G) + \chi(G)$.*

Proof. Let h be a touching representation of G by comparable boxes in $\mathbb{R}^d$, with $d = \dim_{cb}(G)$, and let c be a $\chi(G)$-coloring of G. We start with a slightly modified version of h. We first scale h to fit in $(0, 1)^d$, and for a sufficiently small real $\alpha > 0$ we increase each box in h by 2α in every dimension, that is we replace $h(v)[i] = [a, b]$ by $[a - \alpha, b + \alpha]$ for each vertex v and dimension i. We choose α sufficiently small so that the boxes representing non-adjacent vertices remain disjoint, and thus the resulting representation h_1 is an intersection representation of the same graph G. Moreover, observe that for every clique C of G, the intersection $I_C = \bigcap_{v \in V(C)} h_1(v)$ is a box with non-zero edge lengths. For any clique C of G, let $p_1(C)$ be a point in the interior of I_C different from the points chosen for all other cliques.

Now we add $\chi(G)$ dimensions to make the representation touching again, and to ensure some space for the clique boxes $h^\varepsilon(C)$. Formally we define h_2 as follows.

$$h_2(u)[i] = \begin{cases} h_1(u)[i] & \text{if } i \leq d \\ [1/5, 3/5] & \text{if } i > d \text{ and } c(u) < i - d \\ [0, 2/5] & \text{if } i > d \text{ and } c(u) = i - d \\ [2/5, 4/5] & \text{otherwise (if } c(u) > i - d > 0) \end{cases}$$

For any clique C of G, let $c(C)$ denote the color set $\{c(u) \mid u \in V(C)\}$. We now set

$$p_2(C)[i] = \begin{cases} p_1(C)[i] & \text{if } i \leq d \\ 2/5 & \text{if } i > d \text{ and } i - d \in c(C) \\ 1/2 & \text{otherwise} \end{cases}$$

As h_2 is an extension of h_1, and as in each dimension $j > d$, $h_2(v)[j]$ is an interval of length $2/5$ containing the point $2/5$ for every vertex v, we have that h_2 is an intersection representation of G by comparable boxes. To prove that it is touching consider two adjacent vertices u and v such that

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$c(u) < c(v)$, and let us note that $h_2(u)[d + c(u)] = [0, 2/5]$ and $h_2(v)[d + c(u)] = [2/5, 4/5]$.

For the $\emptyset$-clique-sum extendability, the **(vertices)** conditions are void. For the **(cliques)** conditions, since p_1 is chosen to be injective, the mapping p_2 is injective as well, implying that (c1) holds.

Consider now a clique C in G and a vertex $v \in V(G)$. If $c(v) \notin c(C)$, then $h_2(v)[c(v) + d] = [0, 2/5]$ and $p_2(C)[c(v) + d] = 1/2$, implying that $h_2^\varepsilon(C) \cap h_2(v) = \emptyset$. If $c(v) \in c(C)$ but $v \notin V(C)$, then letting $v' \in V(C)$ be the vertex of color $c(v)$, we have $vv' \notin E(G)$, and thus $h_1(v)$ is disjoint from $h_1(v')$. Since $p_1(C)$ is contained in the interior of $h_1(v')$, it follows that $h_2^\varepsilon(C) \cap h_2(v) = \emptyset$. Finally, suppose that $v \in C$. Since $p_1(C)$ is contained in the interior of $h_1(v)$, we have $h_2^\varepsilon(C)[i] \subset h_2(v)[i]$ for every $i \leq d$. For $i > d$ distinct from $d + c(v)$, we have $p_2^\varepsilon(C)[i] \in \{2/5, 1/2\}$ and $[2/5, 3/5] \subseteq h_2(v)[i]$, and thus $h_2^\varepsilon(C)[i] \subset h_2(v)[i]$. For $i = d + c(v)$, we have $p_2^\varepsilon(C)[i] = 2/5$ and $h_2(v)[i] = [0, 2/5]$, and thus $h_2^\varepsilon(C)[i] \cap h_2(v)[i] = \{p_2^\varepsilon(C)[i]\}$. Therefore, (c2) holds. $\square$

A touching representation of axis-aligned boxes in $\mathbb{R}^d$ is said *fully touching* if any two intersecting boxes intersect on a $(d-1)$-dimensional box. Note that the construction above is fully touching. Indeed, two intersecting boxes corresponding to vertices u, v of colors $c(u) < c(v)$, only touch at coordinate $2/5$ in the $(d + c(u))^{\text{th}}$ dimension, while they fully intersect in every other dimension. This observation with Lemma 2 lead to the following.

Corollary 2. *Any graph G has a fully touching representation of comparable axis-aligned boxes in $\mathbb{R}^d$, where $d = \dim_{cb}(G) + 3^{\dim_{cb}(G)}$.*

Together, the lemmas from this section show that comparable box dimension is almost preserved by full clique-sums.

Corollary 3. *Let $\mathcal{G}$ be a class of graphs of chromatic number at most k. If $\mathcal{G}'$ is the class of graphs obtained from $\mathcal{G}$ by repeatedly performing full clique-sums, then*
$$\dim_{cb}(\mathcal{G}') \leq \dim_{cb}(\mathcal{G}) + 2k.$$

Proof. Suppose a graph G is obtained from $G_1, \ldots, G_m \in \mathcal{G}$ by performing full clique-sums. Without loss of generality, the labelling of the graphs is chosen so that we first perform the full clique-sum on G_1 and G_2, then on the resulting graph and G_3, and so on. Let $C_1^\star = \emptyset$ and for $i = 2, \ldots, m$, let $C_i^\star$ be the root clique of G_i on which it is glued in the full clique-sum operation. By Lemmas 9 and 8, G_i has a $C_i^\star$-clique-sum extendable touching representation by comparable boxes in $\mathbb{R}^d$, where $d = \dim_{cb}(\mathcal{G}) + 2k$. Repeatedly applying Lemma 7, we conclude that $\dim_{cb}(G) \leq d$. $\square$

By Lemmas 2 and 5, this gives the following bounds.

Corollary 4. *Let $\mathcal{G}$ be a class of graphs of comparable box dimension at most d.*

- *The class $\mathcal{G}'$ of graphs obtained from $\mathcal{G}$ by repeatedly performing full clique-sums has comparable box dimension at most $d + 2 \cdot 3^d$.*

- *The closure of $\mathcal{G}'$ by taking subgraphs has comparable box dimension at most 1250^d.*

Proof. The former bound directly follows from Corollary 3 and the bound on the chromatic number from Lemma 2. For the latter one, we need to bound the star chromatic number of $\mathcal{G}'$. Suppose a graph G is obtained from $G_1, \ldots, G_m \in \mathcal{G}$ by performing full clique-sums. For $i = 1, \ldots, m$, suppose G_i has an acyclic coloring φ_i by at most k colors. Note that the vertices of any clique get pairwise different colors, and thus by permuting the colors, we can ensure that when we perform the full clique-sum, the vertices that are identified have the same color. Hence, we can define a coloring φ of G such that for each i, the restriction of φ to $V(G_i)$ is equal to φ_i. Let C be the union of any two color classes of φ. Then for each i, $G_i[C \cap V(G_i)]$ is a forest, and since $G[C]$ is obtained from these graphs by full clique-sums, $G[C]$ is also a forest. Hence, φ is an acyclic coloring of G by at most k colors. By [1], G has a star coloring by at most $2k^2 - k$ colors. Hence, Lemma 2 implies that $\mathcal{G}'$ has star chromatic number at most $2 \cdot 25^d - 5^d$. The bound on the comparable box dimension of subgraphs of graphs from $\mathcal{G}'$ then follows from Lemma 5. $\qquad\square$

4 The strong product structure and minor-closed classes

A *k-tree* is any graph obtained by repeated full clique-sums on cliques of size k from cliques of size at most $k + 1$. A *k-tree-grid* is a strong product of a k-tree and a path. An *extended k-tree-grid* is a graph obtained from a k-tree-grid by adding at most k apex vertices. Dujmović et al. [4] proved the following result.

Theorem 2. *Any graph G is a subgraph of the strong product of a k-tree-grid and K_m, where*

- *$k = 3$ and $m = 3$ if G is planar, and*

- *$k = 4$ and $m = \max(2g, 3)$ if G has Euler genus at most g.*

Moreover, for every t, there exists an integer k such that any K_t-minor-free graph G is a subgraph of a graph obtained by repeated clique-sums from extended k-tree-grids.

Let us first bound the comparable box dimension of a graph in terms of its Euler genus. As paths and m-cliques admit touching representations with hypercubes of unit size in $\mathbb{R}^1$ and in $\mathbb{R}^{\lceil \log_2 m \rceil}$ respectively, by Lemma 4 it suffices to bound the comparable box dimension of k-trees.

Theorem 3. *For any k-tree G, $\dim_{cb}(G) \leq \dim_{cb}^{ext}(G) \leq k + 1$.*

Proof. Let H be a complete graph with $k + 1$ vertices and let $C^\star$ be a clique of size k in H. By Lemma 7, it suffices to show that H has a $C^\star$-clique-sum extendable touching representation by hypercubes in $\mathbb{R}^{k+1}$. Let $V(C^\star) = \{v_1, \ldots, v_k\}$. We construct the representation h so that (v1) holds with $d_{v_i} = i$ for each i; this uniquely determines the hypercubes $h(v_1)$, ..., $h(v_k)$. For the vertex $v_{k+1} \in V(H) \setminus V(C^\star)$, we set $h(v_{k+1}) = [0, 1/2]^{k+1}$. This ensures that the **(vertices)** conditions holds.

For the **(cliques)** conditions, let us set the point $p(C)$ for every clique C as follows:

- $p(C)[i] = 0$ for every $i \leq k$ such that $v_i \in C$

- $p(C)[i] = \frac{1}{4}$ for every $i \leq k$ such that $v_i \notin C$

- $p(C)[k + 1] = \frac{1}{2}$ if $v_{k+1} \in C$

- $p(C)[k + 1] = \frac{3}{4}$ if $v_{k+1} \notin C$

By construction, it is clear that for each vertex $v \in V(H)$, $p(C) \in h(v)$ if and only if $v \in V(C)$.

For any two distinct cliques C_1 and C_2, the points $p(C_1)$ and $p(C_2)$ are distinct. Indeed, by symmetry we can assume that for some i we have $v_i \in V(C_1) \setminus V(C_2)$, and this implies that $p(C_1)[i] < p(C_2)[i]$. Hence, the condition (c1) holds.

Consider now a vertex v_i and a clique C. As we observed before, if $v_i \notin V(C)$, then $p(C) \notin h(v_i)$, and thus $h^\varepsilon(C)$ and $h(v_i)$ are disjoint (for sufficiently small $\varepsilon > 0$). If $v_i \in C$, then the definitions ensure that $p(C)[i]$ is equal to the maximum of $h(v_i)[i]$, and that for $j \neq i$, $p(C)[j]$ is in the interior of $h(v_i)[j]$, implying $h(v_i)[j] \cap h^\varepsilon(C)[j] = [p(C)[j], p(C)[j] + \varepsilon]$ for sufficiently small $\varepsilon > 0$. $\square$

The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum k such that G is a subgraph of a k-tree. Note that actually the bound on the comparable box dimension of Theorem 3 extends to graphs of treewidth at most k.

Corollary 5. *Every graph G satisfies $\dim_{cb}(G) \leq \mathrm{tw}(G) + 1$.*

Proof. Let $k = \mathrm{tw}(G)$. Observe that there exists a k-tree T with the root clique $C^\star$ such that $G \subseteq T - V(C^\star)$. Inspection of the proof of Theorem 3 (and Lemma 7) shows that we obtain a representation h of $T - V(C^\star)$ in $\mathbb{R}^{k+1}$ such that

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- the vertices are represented by hypercubes of pairwise different sizes,

- if $uv \in E(T - V(C^\star))$ and $h(u) \sqsubseteq h(v)$, then $h(u) \cap h(v)$ is a facet of $h(u)$ incident with its point with minimum coordinates, and

If for some $u, v \in V(G)$, we have $uv \in E(T) \setminus E(G)$, where without loss of generality $h(u) \sqsubseteq h(v)$, we now alter the representation by shrinking $h(u)$ slightly away from $h(v)$ (so that all other touchings are preserved). Since the hypercubes of h have pairwise different sizes, the resulting touching representation of G is by comparable boxes. $\square$

As every planar graph G has a touching representation by cubes in $\mathbb{R}^3$ [11], we have that $\dim_{cb}(G) \leq 3$. For the graphs with higher Euler genus we can also derive upper bounds. Indeed, combining the previous observation on the representations of paths and K_m, with Theorem 3, Lemma 4, and Corollary 1 we obtain:

Corollary 6. *For every graph G of Euler genus g, there exists a supergraph G' of G such that $\dim_{cb}(G') \leq 6 + \lceil \log_2 \max(2g, 3) \rceil$. Consequently,*

$$\dim_{cb}(G) \leq 3 \cdot 81^7 \cdot \max(2g, 3)^{\log_2 81}.$$

Similarly, we can deal with proper minor-closed classes.

Proof of Theorem 1. Let $\mathcal{G}$ be a proper minor-closed class. Since $\mathcal{G}$ is proper, there exists t such that $K_t \notin \mathcal{G}$. By Theorem 2, there exists k such that every graph in $\mathcal{G}$ is a subgraph of a graph obtained by repeated clique-sums from extended k-tree-grids. As we have seen, k-tree-grids have comparable box dimension at most $k + 2$, and by Lemma 3, extended k-tree-grids have comparable box dimension at most $2k + 2$. By Corollary 4, it follows that $\dim_{cb}(\mathcal{G}) \leq 1250^{2k+2}$. $\square$

Note that the graph obtained from K_{2n} by deleting a perfect matching has Euler genus $\Theta(n^2)$ and comparable box dimension n. It follows that the dependence of the comparable box dimension on the Euler genus cannot be subpolynomial (though the degree $\log_2 81$ of the polynomial established in Corollary 6 certainly can be improved). The dependence of the comparable box dimension on the size of the forbidden minor that we established is not explicit, as Theorem 2 is based on the structure theorem of Robertson and Seymour [17]. It would be interesting to prove Theorem 1 without using the structure theorem.

5 Fractional treewidth-fragility

Suppose G is a connected planar graph and v is a vertex of G. For an integer $k \geq 2$, give each vertex at distance d from v the color $d \mod k$. Then deleting

the vertices of any of the k colors results in a graph of treewidth at most $3k$. This fact (which follows from the result of Robertson and Seymour [18] on treewidth of planar graphs of bounded radius) is (in the modern terms) the basis of Baker's technique [2] for design of approximation algorithms. However, even quite simple graph classes (e.g., strong products of three paths [3]) do not admit such a coloring (where the removal of any color class results in a graph of bounded treewidth). However, a fractional version of this coloring concept is still very useful in the design of approximation algorithms [8] and applies to much more general graph classes, including all graph classes with strongly sublinear separators and bounded maximum degree [5].

We say that a class of graphs $\mathcal{G}$ is *fractionally treewidth-fragile* if there exists a function f such that for every graph $G \in \mathcal{G}$ and integer $k \geq 2$, there exist sets $X_1, \ldots, X_m \subseteq V(G)$ such that each vertex belongs to at most m/k of them and $\mathrm{tw}(G - X_i) \leq f(k)$ for every i (equivalently, there exists a probability distribution on the set $\{X \subseteq V(G) : \mathrm{tw}(G - X) \leq f(k)\}$ such that $\Pr[v \in X] \leq 1/k$ for each $v \in V(G)$). For example, the class of planar graphs is (fractionally) treewidth-fragile, since we can let X_i consist of the vertices of color $i-1$ in the coloring described at the beginning of the section.

Before going further, let us recall some notions about treewidth. A *tree decomposition* of a graph G is a pair (T, β), where T is a rooted tree and $\beta : V(T) \to 2^{V(G)}$ assigns a *bag* to each of its nodes, such that

- for each $uv \in E(G)$, there exists $x \in V(T)$ such that $u, v \in \beta(x)$, and

- for each $v \in V(G)$, the set $\{x \in V(T) : v \in \beta(x)\}$ is non-empty and induces a connected subtree of T.

For nodes $x, y \in V(T)$, we write $x \preceq y$ if $x = y$ or x is a descendant of y in T. The *width* of the tree decomposition is the maximum of the sizes of the bags minus 1. The *treewidth* of a graph is the minimum of the widths of its tree decompositions. Let us remark that the value of treewidth obtained via this definition coincides with the one via k-trees which we used in the previous section.

Our main result is that all graph classes of bounded comparable box dimension are fractionally treewidth-fragile. We will show the result in a more general setting, motivated by concepts from [6] and by applications to related representations. The argument is motivated by the idea used in the approximation algorithms for disk graphs by Erlebach et al. [10]. Before introducing this more general setting, and as a warm-up, let us outline how to prove that disk graphs of thickness t are fractionally treewidth-fragile. Consider first unit disk graphs. By partitionning the plane with a random grid $\mathcal{H}$, having squared cells of side-length $2k$, any unit disk has probability $1/2k$ to intersect a vertical (resp. horizontal) line of the grid.

By union bound, any disk has probability at most $1/k$ to intersect the grid. Considering this probability distribution, let us now show that removing the disks intersected by the grid leads to a unit disk graph of bounded treewidth. Indeed, in such a graph any connected component corresponds to unit disks contained in the same cell of the grid. Such cell having area bounded by $4k^2$, there are at most $16tk^2/\pi$ disks contained in a cell. The size of the connected components being bounded, so is the treewidth. Note that this distribution also works if we are given disks whose diameter lie in a certain range. If any diameter δ is such that $1/c \leq \delta \leq 1$, then the same process with a random grid of $2k \times 2k$ cells, ensures that any disk is deleted with probability at most $1/k$, while now the connected components have size at most $4tc^2k^2/\pi$. Dealing with arbitrary disk graphs (with any diameter δ being in the range $0 < \delta \leq 1$) requires to delete more disks. This is why each $(2k \times 2k)$-cell is now partitionned in a quadtree-like manner. Now a disk with diameter between $\ell/2$ and ℓ (with $\ell = 1/2^i$ for some integer $i \geq 0$) is deleted if it is not contained in a $(2k\ell \times 2k\ell)$-cell of a quadtree. It is not hard to see that a disk is deleted with probability at most $1/k$. To prove that the remaining graph has bounded treewidth one should consider the following tree decomposition (T, β). The tree T is obtained by linking the roots of the quadtrees we used (as trees) to a new common root. Then for a $(2k\ell \times 2k\ell)$-cell C, $\beta(C)$ contains all the disks of diameter at least $\ell/2$ intersecting C. To see that such bag is bounded consider the $((2k+1)\ell \times (2k+1)\ell)$ square C' centered on C, and note that any disk in $\beta(C)$ intersects C' on an area at least $\pi\ell^2/16$. This implies that $|\beta(C)| \leq 16t(2k+1)^2/\pi$.

Let us now give a detailed proof in a more general setting. For a measurable set $A \subseteq \mathbb{R}^d$, let $\mathrm{vol}(A)$ denote the Lebesgue measure of A. For two measurable subsets A and B of $\mathbb{R}^d$ and a positive integer s, we write $A \sqsubseteq_s B$ if for every $x \in B$, there exists a translation A' of A such that $x \in A'$ and $\mathrm{vol}(A' \cap B) \geq \frac{1}{s}\mathrm{vol}(A)$. Note that for two boxes A and B, we have $A \sqsubseteq_1 B$ if and only if $A \sqsubseteq B$. An *s-comparable envelope representation* (ι, ω) of a graph G in $\mathbb{R}^d$ consists of two functions $\iota, \omega : V(G) \to 2^{\mathbb{R}^d}$ such that for some ordering $v_1, \ldots, v_n$ of vertices of G,

- for each i, $\omega(v_i)$ is a box, $\iota(v_i)$ is a measurable set, and $\iota(v_i) \subseteq \omega(v_i)$,

- if $i < j$, then $\omega(v_j) \sqsubseteq_s \iota(v_i)$, and

- if $i < j$ and $v_i v_j \in E(G)$, then $\omega(v_j) \cap \iota(v_i) \neq \emptyset$.

We say that the representation has *thickness at most t* if for every point $x \in \mathbb{R}^d$, there exist at most t vertices $v \in V(G)$ such that $x \in \iota(v)$.

For example:

- If f is a touching representation of G by comparable boxes in $\mathbb{R}^d$, then (f, f) is a 1-comparable envelope representation of G in $\mathbb{R}^d$ of thickness at most 2^d.

- If f is a touching representation of G by balls in $\mathbb{R}^d$ and letting $\omega(v)$ be the smallest axis-aligned hypercube containing $f(v)$, then there exists a positive integer s_d depending only on d such that (f, ω) is an s_d-comparable envelope representation of G in $\mathbb{R}^d$ of thickness at most 2.

Theorem 4. *For positive integers t, s, and d, the class of graphs with an s-comparable envelope representation in $\mathbb{R}^d$ of thickness at most t is fractionally treewidth-fragile, with a function $f(k) = O_{t,s,d}(k^d)$.*

Proof. For a positive integer k, let $f(k) = (2ksd + 2)^d st$. Let (ι, ω) be an s-comparable envelope representation of a graph G in $\mathbb{R}^d$ of thickness at most t, and let $v_1, \ldots, v_n$ be the corresponding ordering of the vertices of G. Let us define $\ell_{i,j} \in \mathbb{R}^+$ for $i = 1, \ldots, n$ and $j \in \{1, \ldots, d\}$ as an approximation of $ksd|\omega(v_i)[j]|$ such that $\ell_{i-1,j}/\ell_{i,j}$ is a positive integer. Formally it is defined as follows.

- Let $\ell_{1,j} = ksd|\omega(v_1)[j]|$.

- For $i = 2, \ldots, n$, let $\ell_{i,j} = \ell_{i-1,j}$, if $\ell_{i-1,j} < ksd|\omega(v_i)[j]|$, and otherwise let $\ell_{i,j}$ be lowest fraction of $\ell_{i-1,j}$ that is greater than $ksd|\omega(v_i)[j]|$, formally $\ell_{i,j} = \min\{\ell_{i-1,j}/b \mid b \in \mathbb{N}^+ \text{ and } \ell_{i-1,j}/b \geq ksd|\omega(v_i)[j]|\}$.

Let the real $x_j \in [0, \ell_{1,j}]$ be chosen uniformly at random, and let $\mathcal{H}_j^i$ be the set of hyperplanes in $\mathbb{R}^d$ consisting of the points whose j-th coordinate is equal to $x_j + m\ell_{i,j}$ for some $m \in \mathbb{Z}$. As $\ell_{i,j}$ is a multiple of $\ell_{i',j}$ whenever $i \leq i'$, we have that $\mathcal{H}_j^i \subseteq \mathcal{H}_j^{i'}$ whenever $i \leq i'$. For $i \in \{1, \ldots, n\}$, the i-grid is $\mathcal{H}^i = \bigcup_{j=1}^d \mathcal{H}_j^i$, and we let the 0-grid $\mathcal{H}^0 = \emptyset$. Similarly as above we have that $\mathcal{H}^i \subseteq \mathcal{H}^{i'}$ whenever $i \leq i'$.

We let $X \subseteq V(G)$ consist of the vertices $v_a \in V(G)$ such that the box $\omega(v_a)$ intersects some hyperplane $H \in \mathcal{H}^a$, that is such that $x_j + m\ell_{a,j} \in \omega(v_a)[j]$, for some $j \in \{1, \ldots, d\}$ and some $m \in \mathbb{Z}$. First, let us argue that $\Pr[v_a \in X] \leq 1/k$. Indeed, the set $[0, \ell_{1,j}] \cap \bigcup_{m \in \mathbb{Z}}(\omega(v_a)[j] - m\ell_{a,j})$ has measure $\frac{\ell_{1,j}}{\ell_{a,j}} \cdot |\omega(v_a)[j]|$, implying that for fixed j, this happens with probability $|\omega(v_a)[j]|/\ell_{a,j}$. Let a' be the largest integer such that $a' \leq a$ and $\ell_{a',j} < \ell_{a'-1,j}$ if such an index exists, and $a' = 1$ otherwise; note that $\ell_{a,j} = \ell_{a',j} \geq ksd|\omega(v_{a'})[j]|$. Moreover, since $\omega(v_a) \sqsubseteq_s \iota(v_{a'}) \subseteq \omega(v_{a'})$, we have $\omega(v_a)[j] \leq s\omega(v_{a'})[j]$. Combining these inequalities,

$$\frac{|\omega(v_a)[j]|}{\ell_{a,j}} \leq \frac{s\omega(v_{a'})[j]}{ksd|\omega(v_{a'})[j]|} = \frac{1}{kd}.$$

By the union bound, we conclude that $\Pr[v_a \in X] \leq 1/k$.

Let us now bound the treewidth of $G - X$. For $a \geq 0$, an *a-cell* is a maximal connected subset of $\mathbb{R}^d \setminus \left(\bigcup_{H \in \mathcal{H}^a} H\right)$. A set $C \subseteq \mathbb{R}^d$ is a *cell* if it is an a-cell for some $a \geq 0$. A cell C is *non-empty* if there exists $v \in V(G - X)$

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such that $\iota(v) \subseteq C$. Note that there exists a rooted tree T whose vertices are the non-empty cells and such that for $x, y \in V(T)$, we have $x \preceq y$ if and only if $x \subseteq y$. For each non-empty cell C, let us define $\beta(C)$ as the set of vertices $v_i \in V(G - X)$ such that $\iota(v) \cap C \neq \emptyset$ and C is an a-cell for some $a \geq i$.

Let us show that (T, β) is a tree decomposition of $G - X$. For each $v_j \in V(G - X)$, the j-grid is disjoint from $\omega(v_j)$, and thus $\iota(v_j) \subseteq \omega(v_j) \subset C$ for some j-cell $C \in V(T)$ and $v_j \in \beta(C)$. Consider now an edge $v_i v_j \in E(G - X)$, where $i < j$. We have $\omega(v_j) \cap \iota(v_i) \neq \emptyset$, and thus $\iota(v_i) \cap C \neq \emptyset$ and $v_i \in \beta(C)$. Finally, suppose that $v_j \in C'$ for some $C' \in V(T)$. Then C' is an a-cell for some $a \geq j$, and since $\iota(v_j) \cap C' \neq \emptyset$ and $\iota(v_j) \subset C$, we conclude that $C' \subseteq C$, and consequently $C' \preceq C$. Moreover, any cell C'' such that $C' \preceq C'' \preceq C$ (and thus $C' \subseteq C'' \subseteq C$) is an a'-cell for some $a' \geq j$ and $\iota(v_j) \cap C'' \supseteq \iota(v_j) \cap C' \neq \emptyset$, implying $v_j \in \beta(C'')$. It follows that $\{C' : v_j \in \beta(C')\}$ induces a connected subtree of T.

Finally, let us bound the width of the decomposition (T, β). Let C be a non-empty cell and let a be maximum such that C is an a-cell. Then C is an open box with sides of lengths $\ell_{a,1}, \ldots, \ell_{a,d}$. Consider $j \in \{1, \ldots, d\}$:

- If $a = 1$, then $\ell_{a,j} = ksd|\omega(v_a)[j]|$.

- If $a > 1$ and $\ell_{a,j} = \ell_{a-1,j}$, then $\ell_{a,j} = \ell_{a-1,j} < 2ksd|\omega(v_a)[j]|$ (otherwise $\ell_{a,j} = \ell_{a-1,j}/b$ for some integer $b \geq 2$).

- If $a > 1$ and $\ell_{a,j} < \ell_{a-1,j}$, then $\ell_{a-1,j} \geq b \times ksd|\omega(v_a)[j]|$ for some integer $b \geq 2$. Now let b be the greatest such integer (that is such that $\ell_{a-1,j} < (b+1) \times ksd|\omega(v_a)[j]|$) and note that

$$\ell_{a,j} = \frac{\ell_{a-1,j}}{b} < \frac{b+1}{b} ksd|\omega(v_a)[j]| < \tfrac{3}{2} ksd|\omega(v_a)[j]|.$$

Hence, $\ell_{a,j} < 2ksd|\omega(v_a)[j]|$. Let C' be the box with the same center as C and with $|C'[j]| = (2ksd + 2)|\omega(v_a)[j]|$. For any $v_i \in \beta(C) \setminus \{v_a\}$, we have $i \leq a$ and $\iota(v_i) \cap C \neq \emptyset$, and since $\omega(v_a) \sqsubseteq_s \iota(v_i)$, there exists a translation B_i of $\omega(v_a)$ that intersects $C \cap \iota(v_i)$ and such that $\mathrm{vol}(B_i \cap \iota(v_i)) \geq \frac{1}{s} \mathrm{vol}(\omega(v_a))$. Note that as B_i intersects C, we have that $B_i \subseteq C'$. Since the representation

has thickness at most t,

$$\text{vol}(C') \geq \text{vol}\left(C' \cap \bigcup_{v_i \in \beta(C)\setminus\{v_a\}} \iota(v_i)\right)$$

$$\geq \text{vol}\left(\bigcup_{v_i \in \beta(C)\setminus\{v_a\}} B_i \cap \iota(v_i)\right)$$

$$\geq \frac{1}{t} \sum_{v_i \in \beta(C)\setminus\{v_a\}} \text{vol}(B_i \cap \iota(v_i))$$

$$\geq \frac{\text{vol}(\omega(v_a))(|\beta(C)| - 1)}{st}.$$

Since $\text{vol}(C') = (2ksd + 2)^d \text{vol}(\omega(v_a))$, it follows that

$$|\beta(C)| - 1 \leq (2ksd + 2)^d st = f(k),$$

as required. $\qquad\square$

The proof that (generalizations of) graphs with bounded comparable box dimensions have sublinear separators in [6] is indirect; it is established that these graphs have polynomial coloring numbers, which in turn implies they have polynomial expansion, which then gives sublinear separators using the algorithm of Plotkin, Rao, and Smith [16]. The existence of sublinear separators is known to follow more directly from fractional treewidth-fragility. Indeed, since $\Pr[v \in X] \leq 1/k$, there exists $X \subseteq V(G)$ such that $\text{tw}(G - X) \leq f(k)$ and $|X| \leq |V(G)|/k$. The graph $G - X$ has a balanced separator of size at most $\text{tw}(G - X) + 1$, which combines with X to a balanced separator of size at most $V(G)|/k + f(k) + 1$ in G. Optimizing the value of k (choosing it so that $V(G)|/k = f(k)$), we obtain the following corollary of Theorem 4.

Corollary 7. *For positive integers t, s, and d, every graph G with an s-comparable envelope representation in $\mathbb{R}^d$ of thickness at most t has a sublinear separator of size $O_{t,s,d}\left(|V(G)|^{\frac{d}{d+1}}\right)$.*

6   Acknowledgement

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