International Journal of Mechanical Sciences 275 (2024) 109294

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

Dislocations in nonlocal simplified strain gradient elasticity: Eringen meets Aifantis

Markus Lazar

Karlsruhe Institute of Technology (KIT), Institute of Engineering Mechanics, D-76131 Karlsruhe, Germany

ARTICLE INFO

Keywords:
Nonlocal gradient elasticity
Gradient elasticity
Nonlocal elasticity
Dislocations
Peach-Koehler force

ABSTRACT

The nonlocal strain gradient elasticity theory is used to address mechanical problems at small scales where size effects and regularization cannot be neglected. In this work, dislocations are investigated in the framework of nonlocal simplified first strain gradient elasticity. It is shown that nonlocal simplified strain gradient elasticity is the unification of the theories of Eringen's nonlocal elasticity of Helmholtz type and simplified first strain gradient elasticity. Nonlocal simplified strain gradient elasticity contains two characteristic lengths, namely the characteristic length of nonlocal elasticity of Helmholtz type and the characteristic length of simplified first strain gradient elasticity. The advantage of nonlocal simplified first strain gradient elasticity is that the displacement, elastic distortion, plastic distortion, total stress, Cauchy stress and double stress fields of screw and edge dislocations which are calculated here are nonsingular and finite everywhere. Moreover, the Peach-Koehler force of two screw dislocations and two edge dislocations is derived and it is shown that the Peach-Koehler force is also nonsingular. Numerical examples for all dislocation fields of screw and edge dislocations in aluminum are given.

1. Introduction

A dislocation is the most important crystal defect causing plasticity and breaking the translation symmetry of a crystal within the dislocation core. Generalized continuum field theories (nonlocal and gradient theories) of dislocations are a challenging field of research in order to find nonsingular dislocation fields and to describe the mechanics at small scales. Classical continuum theories, which are scale-free continuum theories, are not valid at small scales and lead for dislocations to unphysical singularities in the dislocation core region. Therefore, a nonsingular dislocation field theory based on generalized continuum field theories is of high relevance in order to describe the physical behavior of dislocations at small scales without singularities. There exist two main classes of generalized continuum theories, namely nonlocal elasticity and strain gradient elasticity (see, e.g., [1–4]). In the framework of nonlocal elasticity, effects of nonlocal long–range interactions are expressed by the integral convolution form of Hooke's law using a nonlocal kernel. In first strain gradient elasticity, the first gradient of the elastic strain tensor is also taken into account in the strain energy density in addition to the elastic strain tensor.

The first success was achieved by Eringen [1,5,6,7,8] using nonlocal elasticity to find nonsingular stress fields of straight dislocations (see also [9,10]). Solutions of straight screw and edge dislocations within nonlocal elasticity based on Gaussian kernels have been given by Eringen [5,6]. Using a scalar nonlocal kernel, which is the Green function of the Helmholtz equation, nonlocal elasticity has been studied by Eringen [1,7]. Such a framework can be called nonlocal elasticity of Helmholtz type [10], where one characteristic length scale parameter, namely the characteristic length of nonlocal elasticity enters the Helmholtz operator. Solutions of straight screw and edge dislocations within nonlocal elasticity of Helmholtz type have been given by Eringen [1,7], Lazar [9] and Lazar et al. [10]. The main feature of these solutions is the regularization of the stress field singularities at the dislocation line towards nonsingular stress fields. The main advantage of nonlocal elasticity of Helmholtz type is that the integral equation for the stress tensor is reduced to a Helmholtz equation for the stress tensor where the inhomogeneous part is given by the singular Cauchy stress tensor of classical elasticity. Therefore, if the scalar nonlocal kernel function is a Green function, then the integral constitutive relation can be reduced to a partial differential equation and in this way strong nonlocal elasticity is reduced to weak nonlocal elasticity (see [11]). It has been shown that nonlocal elasticity of Helmholtz type is valid down to the Ångström-scale (see [1,4,7]). However, in Eringen's nonlocal elasticity, the displacement, elastic strain and plastic distortion of straight dislocations are still the classical singular fields.

Later, in the framework of simplified strain gradient elasticity proposed by Aifantis [12] (see also [13]), Gutkin and Aifantis [14,15] found nonsingular fields for the displacement and elastic strain of straight dislocations. Lazar and Maugin [16], Lazar et al. [10], Lazar

E-mail address: markus.lazar@kit.edu.

https://doi.org/10.1016/j.ijmecsci.2024.109294
Received 27 October 2023; Received in revised form 10 April 2024; Accepted 12 April 2024
Available online 19 April 2024

[17,18] completed the framework of simplified strain gradient elasticity including double stresses and showed that even the Cauchy stress tensor and the plastic distortion tensor are nonsingular and that the dislocation density is regularized and possesses a weaker singularity than in classical dislocation theory. Gutkin and Aifantis [14,15] mentioned that the stress tensor appearing in the simplified gradient elasticity in a "generalized Hooke law" is still singular and identical to the classical stress tensor. Lazar and Po [19] have proven that this singular stress tensor has the physical meaning of the so-called total stress tensor in simplified strain gradient elasticity. In addition, double stresses of dislocations are singular in the framework of simplified strain gradient elasticity [16]. It is important to note that the nonsingular stress fields of screw and edge dislocations obtained in the framework of simplified strain gradient elasticity coincide with the stress fields calculated in nonlocal elasticity of Helmholtz type. Moreover, simplified first strain gradient elasticity theory [13,16] is a particular version of Mindlin's first strain gradient elasticity theory [2,20] (see also [18,21]) with one length scale parameter, namely the characteristic length of simplified strain gradient elasticity in addition to the two Lamé parameters. It allows eliminating singularities and discontinuities and to interpret size effects. Aspects of nonsingular cracks in the framework of simplified first strain gradient elasticity can be found in [22]. Since a dislocation is the building block of a crack, nonsingular crack fields can be computed in the framework of dislocation based fracture mechanics using nonlocal and gradient elasticity theories. Using the nonsingular stress fields of screw and edge dislocations, nonsingular stress fields of cracks of mode III, mode II and mode I were given by Mousavi and Lazar [23] in the framework of nonlocal elasticity of Helmholtz type and by Mousavi and Aifantis [24,25] in the framework of simplified strain gradient elasticity. In nonlocal elasticity of Helmholtz type, the stress fields of cracks are nonsingular and in simplified strain gradient elasticity, the stress, elastic strain and plastic distortion fields of cracks are nonsingular and finite. An overview on simplified gradient theory and its extensions to fractional/fractal media and applications to various disciplines of science and engineering can be found in Aifantis [26,27,28]. Aspects of gradient electrodynamics can be found in Lazar [17,29]. A gradient generalization of the Newton law was first suggested by Aifantis [27]. The field theoretical framework of the gradient modification of Newtonian gravity including nonsingular gravitational fields (nonsingular Newtonian potential and nonsingular gravitational force) and its relation to quantum gravity have been given in Lazar [30].

In order to eliminate the singularities in both the elastic strain and stress fields of dislocations, Gutkin and Aifantis [31] postulated a generalized Hooke law with two different Helmholtz operators and two different length scales, one Helmholtz operator acts on the stress and the other Helmholtz operator acts on the elastic strain. No variational derivation of such a generalized Hooke law was used by Gutkin and Aifantis [31] and it was a postulated law aimed to regularize both the elastic strain and stress fields. Using the generalized Hooke law with two different Helmholtz operators and two different length scales, Gutkin and Aifantis [31] and Gutkin [32,33] found nonsingular stress and elastic strain fields of screw and edge dislocations in terms of the two length scales, namely one length scale for the stress fields and the other length scale for the elastic strain fields. Later, Aifantis [34] connected the generalized Hooke law with the framework of implicit constitutive equations. Furthermore, Aifantis [35] derived the generalized Hooke law from a particular form of second strain gradient elasticity using a generalized Hu–Washizu variational principle.

More than 10 years later, Fafalis et al. [36] (see also [37]) have proposed the theory of nonlocal simplified strain gradient elasticity, where nonlocal elasticity is combined with simplified strain gradient elasticity. Fafalis et al. [36] combined in a unique theory both the nonlocal elasticity theory of Eringen and the simplified strain gradient elasticity. The resulting approach, called nonlocal simplified strain gradient elasticity, contains two characteristic length scales (the length scale of nonlocal elasticity and the length scale of simplified strain gradient elasticity). In this way, nonlocal simplified strain gradient elasticity is nothing but the nonlocal version or extension of simplified strain gradient elasticity. In a popular way, one may say:

nonlocal simplified strain gradient elasticity = Eringen meets Aifantis.

In this sense, nonlocal simplified strain gradient elasticity represents the unification of the theories of Eringen's nonlocal elasticity [1,4,7] and Aifantis' simplified strain gradient elasticity [13,16,38].

Although, nonlocal strain gradient elasticity is an active research field (see, e.g., [36,37,39–41]), so far, no single work exists for dislocations in nonlocal simplified strain gradient elasticity in the literature. For that reason, the aim of the present work is to study screw and edge dislocations in the framework of nonlocal simplified strain gradient elasticity. The results demonstrate the elimination of singularities.

The paper is organized as follows. In Section 2, the theoretical framework of nonlocal simplified strain gradient elasticity is presented. In Section 3, we derive for screw and edge dislocations exact analytical solutions for the displacement, elastic distortion, plastic distortion, total stress, Cauchy stress and double stress fields within this framework. The Peach-Koehler force of screw and edge dislocations is computed in Section 4. Conclusions are given in Section 5. In Appendix, the boundary conditions of nonlocal first strain gradient elasticity are given.

2. Nonlocal simplified strain gradient elasticity

In this section, we present the theoretical framework and the material parameters of nonlocal simplified strain gradient elasticity.

2.1. Theoretical framework

In nonlocal simplified strain gradient elasticity (NSSGE), the strain energy density for isotropic materials is given by (see [36])

$$\mathcal{W}(\boldsymbol{e}, \nabla\boldsymbol{e}, \alpha) = \frac{1}{2}\,\mathbb{C}_{ijkl} e_{ij} e_{kl} * \alpha + \frac{1}{2}\,\ell_G^2\,\mathbb{C}_{ijkl} \partial_m e_{ij} \partial_m e_{kl} * \alpha, \tag{1}$$

where the isotropic constitutive tensor of rank four reads

$$\mathbb{C}_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big). \tag{2}$$

Here λ and μ are the Lamé moduli (elastic constants), ℓ_G is the characteristic length of simplified strain gradient elasticity (or gradient elasticity of Helmholtz type), δ_{ij} denotes the Kronecker symbol and $*$ denotes the spatial convolution.[1] α is the so-called nonlocal kernel function. Therefore, Eq. (1) gives the nonlocal form of simplified strain gradient elasticity due to the convolution with the nonlocal kernel function α. Note that we use the following abbreviation for the partial derivative: $\partial_m = \partial/\partial x_m$. Like in Eringen's nonlocal micropolar elasticity [1,43] and in nonlocal micromorphic elasticity [44], only one nonlocal kernel function appears in nonlocal simplified strain gradient elasticity. This ensures a mathematical simplicity of a nonlocal microcontinuum field theory. On the other hand, Lim et al. [39] have proposed nonlocal gradient elasticity with two different nonlocal kernel functions leading to more complicated field equations (see also [37]). Furthermore, as in Eringen's nonlocal elasticity [1,4,7], the scalar nonlocal kernel function α is the Green function (fundamental solution) of the linear differential operator L_N

$$L_N \alpha = \delta(\boldsymbol{x}). \tag{3}$$

Here, we choose L_N as Helmholtz operator according to

$$L_N = 1 - \ell_N^2 \Delta, \tag{4}$$

[1] The following notation for the spatial convolution is used: $A(\boldsymbol{x}) * B(\boldsymbol{x}) = \int_V A(\boldsymbol{x}-\boldsymbol{y})B(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ (see, e.g., [42]).

where Δ denotes the Laplace operator. If $\ell_N^2 > 0$, then the nonlocal kernel α is positive definite that means: $\alpha > 0$. Therefore, ℓ_N is the characteristic length of nonlocal elasticity of Helmholtz type. Strictly speaking, using the strain energy density (1) together with the nonlocal kernel α as Green function of the Helmholtz operator, Eqs. (3) and (4), we deal with nonlocal simplified strain gradient elasticity of Helmholtz type.

The conditions for positive definiteness (for reasons of uniqueness and stability) of the strain energy density (1) are given by

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \ell_G^2 > 0, \qquad \ell_N^2 > 0. \tag{5}$$

The incompatible elastic strain tensor e_{ij} reads

$$e_{ij} = \frac{1}{2}\left(\beta_{ij} + \beta_{ji}\right), \tag{6}$$

which is given by the symmetric part of the incompatible elastic distortion tensor

$$\beta_{ij} = \partial_j u_i - \beta^{\mathrm{P}}_{ij}. \tag{7}$$

The incompatible elastic distortion tensor (7) is nothing but the gradient of the displacement vector u_i minus the plastic distortion (or eigendistortion) tensor β^{P}_{ij}.

In dislocation theory, the dislocation density tensor is defined in terms of the incompatible plastic distortion tensor (see, e.g., [17,45])

$$\alpha_{ij} = -\epsilon_{jkl}\partial_k \beta^{\mathrm{P}}_{il} \tag{8}$$

and can also be expressed in terms of the incompatible elastic distortion tensor

$$\alpha_{ij} = \epsilon_{jkl}\partial_k \beta_{il}, \tag{9}$$

where ϵ_{jkl} indicates the Levi-Civita tensor.

The Cauchy stress tensor σ_{ij} and the double stress tensor τ_{ijm} are defined as response fields to the elastic strain tensor and the gradient of the elastic strain tensor

$$\sigma_{ij} = \frac{\partial \mathcal{W}}{\partial e_{ij}} = \mathbb{C}_{ijkl} e_{kl} * \alpha, \tag{10}$$

$$\tau_{ijm} = \frac{\partial \mathcal{W}}{\partial(\partial_m e_{ij})} = \ell_G^2 \mathbb{C}_{ijkl}\partial_m e_{kl} * \alpha, \tag{11}$$

respectively. Eq. (10) is the constitutive equation between the Cauchy stress σ_{ij} and the elastic strain e_{kl} and Eq. (11) is the constitutive equation between the double stress τ_{ijm} and the gradient of the elastic strain $\partial_m e_{kl}$. Due to the convolution with the nonlocal kernel function α, the constitutive Eqs. (10) and (11) have a nonlocal form. Using Eq. (2), the stress tensors read

$$\sigma_{ij} = (\lambda\,\delta_{ij} e_{ll} + 2\mu\, e_{ij}) * \alpha, \tag{12}$$

$$\tau_{ijm} = \ell_G^2(\lambda\,\delta_{ij}\partial_m e_{ll} + 2\mu\,\partial_m e_{ij}) * \alpha. \tag{13}$$

As in simplified strain gradient elasticity [16], the double stress tensor can be written as gradient of the Cauchy stress tensor (10)

$$\tau_{ijm} = \ell_G^2 \partial_m \sigma_{ij}. \tag{14}$$

Now, using the constitutive Eqs. (10) and (11), the strain energy density (1) can be rewritten in the form

$$\mathcal{W} = \frac{1}{2}\sigma_{ij} e_{ij} + \frac{1}{2}\tau_{ijm}\partial_m e_{ij}. \tag{15}$$

Applying the differential operator L_N to the nonlocal constitutive Eqs. (10) and (11) and using Eq. (3), we get the constitutive equations in differential form as inhomogeneous Helmholtz equations

$$L_N \sigma_{ij} = \mathbb{C}_{ijkl} e_{kl}, \tag{16}$$

$$L_N \tau_{ijm} = \ell_G^2 \mathbb{C}_{ijkl}\partial_m e_{kl}. \tag{17}$$

The equilibrium condition is given by the Euler–Lagrange equation of the strain energy density (1)

$$\frac{\delta \mathcal{W}}{\delta u_i} := \frac{\partial \mathcal{W}}{\partial u_i} - \partial_j \frac{\partial \mathcal{W}}{\partial(\partial_j u_i)} + \partial_m \partial_j \frac{\partial \mathcal{W}}{\partial(\partial_m \partial_j u_i)} = 0, \tag{18}$$

where the left hand side is the functional or variational derivative (see, e.g., [46]). In terms of the Cauchy stress tensor (10) and double stress tensor (11), the equation of equilibrium (18) takes the following form

$$\partial_j\left(\sigma_{ij} - \partial_m \tau_{ijm}\right) = 0, \tag{19}$$

which is the same as in first strain gradient elasticity (see also [2]).

Using the relation (14), the equation of equilibrium (19) reduces to

$$\partial_j L_G \sigma_{ij} = 0 \tag{20}$$

with the Helmholtz operator L_G to be given in terms of the characteristic length of simplified gradient elasticity as

$$L_G = 1 - \ell_G^2 \Delta \tag{21}$$

with $\ell_G^2 > 0$ and

$$L_G G^{L_G} = \delta(\boldsymbol{x}), \tag{22}$$

where G^{L_G} denotes the Green function of the Helmholtz operator L_G. It is noted for the interested reader that the boundary conditions of nonlocal first strain gradient elasticity, which are not needed in this work, were given by Lim et al. [39] (see also [40]) and can be found in Appendix.

If we substitute Eq. (12) into Eq. (20), then the equation of equilibrium reads in terms of the elastic strain tensor

$$\mathbb{C}_{ijkl}\partial_j L_G e_{kl} * \alpha = 0\,. \tag{23}$$

Furthermore, using Eqs. (6) and (7), the equation of equilibrium reads in terms of the displacement vector and the plastic distortion tensor

$$L_{ik} L_G u_k * \alpha = \mathbb{C}_{ijkl}\partial_j L_G \beta^{\mathrm{P}}_{kl} * \alpha, \tag{24}$$

where the Navier operator L_{ik} is defined by

$$L_{ik} = \mathbb{C}_{ijkl}\partial_j \partial_l. \tag{25}$$

Applying the differential operator L_N to Eq. (24), it gives

$$L_N L_{ik} L_G u_k * \alpha = \mathbb{C}_{ijkl}\partial_j L_N L_G \beta^{\mathrm{P}}_{kl} * \alpha\,. \tag{26}$$

Using the property (3) that the nonlocal kernel function α is the Green function of L_N, Eq. (26) simplifies to

$$L_{ik} L_G u_k = \mathbb{C}_{ijkl}\partial_j L_G \beta^{\mathrm{P}}_{kl}\,. \tag{27}$$

Eq. (27) is the field equation for u_k in simplified first strain gradient elasticity, namely an inhomogeneous Helmholtz-Navier equation, which is a partial differential equation of fourth order (see, e.g., [17, 47]).

Thus, all the analysis and techniques of simplified strain gradient elasticity like operator split, bifield ansatz and “Ru-Aifantis technique” (see, e.g., [17,38,47]) can be taken over in nonlocal simplified strain gradient elasticity. For that reason, the plastic distortion tensor in NSSGE fulfills the inhomogeneous Helmholtz equation with the operator L_G

$$L_G \beta^{\mathrm{P}}_{kl} = \beta^{\mathrm{P},0}_{kl}, \tag{28}$$

where the right hand side is given by the classical plastic distortion tensor $\beta^{\mathrm{P},0}_{kl}$. Then the displacement vector satisfies the following Helmholtz-Navier equation

$$L_{ik} L_G u_k = \mathbb{C}_{ijkl}\partial_j \beta^{\mathrm{P},0}_{kl}, \tag{29}$$

where the right hand side is given by the gradient of the classical plastic distortion tensor. Moreover, the displacement vector fulfills the inhomogeneous Helmholtz equation

$$L_G u_k = u^0_k, \tag{30}$$

where the right hand side is given by the classical displacement vector u_k^0. The elastic distortion tensor fulfills the inhomogeneous Helmholtz equation

$$L_G \beta_{kl} = \beta_{kl}^0, \tag{31}$$

where the right hand side is given by the classical elastic distortion tensor. The elastic strain tensor fulfills the inhomogeneous Helmholtz equation

$$L_G e_{kl} = e_{kl}^0, \tag{32}$$

where the right hand side is given by the classical elastic strain tensor e_{kl}^0. The dislocation density tensor fulfills the inhomogeneous Helmholtz equation

$$L_G \alpha_{kl} = \alpha_{kl}^0, \tag{33}$$

where the right hand side is given by the classical dislocation density tensor α_{kl}^0.

It can be seen in the Eqs. (30)–(33) that the kinetic quantities such as the displacement vector u_k, the plastic distortion tensor β_{kl}^{P}, the elastic distortion tensor β_{kl}, the elastic strain tensor e_{kl} and the dislocation density tensor α_{kl} are determined by the Helmholtz operator L_G including the gradient length scale ℓ_G like in simplified strain gradient elasticity (see also [15–17,47]).

Moreover, the total stress tensor t_{ij} can be defined as the functional or variational derivative of the strain energy density $\mathcal{W}$ with respect to the elastic strain tensor e_{ij} (see, e.g., [19,48])

$$t_{ij} := \frac{\delta \mathcal{W}}{\delta e_{ij}} = \frac{\partial \mathcal{W}}{\partial e_{ij}} - \partial_k \frac{\partial \mathcal{W}}{\partial(\partial_k e_{ij})}. \tag{34}$$

Using Eqs. (10) and (11), Eq. (34) reads (see also [19,21,49,50])

$$t_{ij} = \sigma_{ij} - \partial_m \tau_{ijm}. \tag{35}$$

If we substitute the relation (14) into Eq. (35), then it reduces to

$$t_{ij} = L_G \sigma_{ij} \tag{36}$$

and the equilibrium equation (19) simplifies to

$$\partial_j t_{ij} = 0\,. \tag{37}$$

Moreover, substituting the constitutive relation (10) into the total stress tensor (36), it becomes

$$t_{ij} = \mathbb{C}_{ijkl} L_G e_{kl} * \alpha. \tag{38}$$

Note that the form (38) of the total stress tensor t_{ij} is an integro-partial differential equation due to the differential operator L_G and the convolution with the nonlocal kernel α. Using the convolution of the total stress tensor (38) with the Green function G^{L_G} and the property (22), we obtain the relation between the total stress tensor and the Cauchy stress tensor

$$\sigma_{ij} = t_{ij} * G^{L_G}, \tag{39}$$

which means that the Cauchy stress tensor σ_{ij} is nothing but the convolution of the total stress tensor t_{ij} with G^{L_G} and is smoother than the total stress tensor due to this convolution.

In general, constitutive equations involve a set of constitutive parameters and a set of constitutive operators. The constitutive operators may be linear and integro-differential operators (see, e.g., [51]). In this sense, the integro-partial differential equation (38) is a "constitutive equation" between the total stress tensor t_{ij} and the elastic strain tensor e_{kl} in form of an integro-partial differential equation involving the constitutive tensor $\mathbb{C}_{ijkl}$, the differential operator L_G and the convolution with the nonlocal kernel α.

In order to connect Eq. (38) with the "generalized Hooke law" postulated by Gutkin and Aifantis [31], we apply the differential operator L_N together with the property (3) to the total stress tensor (38) and obtain from the integro-partial differential equation (38) the following equation in differential form

$$L_N t_{ij} = \mathbb{C}_{ijkl} L_G e_{kl}, \tag{40}$$

where on the left hand side the Helmholtz operator L_N acts on the total stress tensor and on the right hand side the Helmholtz operator L_G acts on the elastic strain tensor. In this way, Eq. (40) represents the "constitutive equation" between the total stress and the elastic strain in differential form. If we substitute the Helmholtz operators (4) and (21) into Eq. (40), we recover Aifantis' postulated "generalized Hooke law" [31,52] with the Helmholtz operators L_N and L_G acting on the stress tensor and the elastic strain tensor,[2] respectively, namely

$$(1 - \ell_N^2 \Delta) t_{ij} = \mathbb{C}_{ijkl} (1 - \ell_G^2 \Delta) e_{kl}. \tag{41}$$

One can say that Eq. (41) has the physical meaning of a "generalized Hooke law" between the total stress and the elastic strain having the form of a partial differential equation in terms of the Helmholtz operators L_N and L_G. If we substitute the constitutive tensor (2) into Eq. (41), we obtain the explicit form for an isotropic material

$$(1 - \ell_N^2 \Delta) t_{ij} = (1 - \ell_G^2 \Delta)[\lambda \delta_{ij} e_{kk} + 2\mu e_{ij}], \tag{42}$$

originally postulated by Gutkin and Aifantis [31] (see also [32,52]). However, since it was just postulated without to be derived from a proper theoretical framework, as part of a theory, the physical meaning of the stress tensor, t_{ij}, was lost in [31].

From arguments of physically realistic dispersion relations with normal dispersion in NSSGE [53], the nonlocal length scale ℓ_N and gradient length scale ℓ_G have to fulfill the constraint

$$\ell_N^2 > \ell_G^2. \tag{43}$$

Substituting Eq. (32) into Eq. (40), it can be observed that in NSSGE the total stress tensor satisfies the inhomogeneous Helmholtz equation with the operator L_N

$$L_N t_{ij} = \sigma_{ij}^0, \tag{44}$$

where the right hand side is given by the classical Cauchy stress tensor σ_{ij}^0 of classical elasticity

$$\sigma_{ij}^0 = \mathbb{C}_{ijkl} e_{kl}^0. \tag{45}$$

The form of Eq. (44) is known from the stress tensor in Eringen's nonlocal elasticity of Helmholtz type (see [1,4,7,10]). Therefore, the total stress tensor t_{ij} is determined by the Helmholtz operator L_N including the nonlocal length ℓ_N. The solution of Eq. (44) reads as

$$t_{ij} = \sigma_{ij}^0 * \alpha. \tag{46}$$

The total stress tensor (46) is nothing but the convolution of the classical stress tensor σ_{ij}^0 and the nonlocal kernel α like in nonlocal elasticity of Helmholtz type (see [1,4,7]).

Furthermore, applying the differential operator L_G to Eq. (16) and using Eq. (32), in NSSGE the Cauchy stress tensor satisfies the inhomogeneous bi-Helmholtz equation with the operators L_N and L_G

$$L_G L_N \sigma_{ij} = \sigma_{ij}^0, \tag{47}$$

where the right hand side is given by the classical Cauchy stress tensor σ_{ij}^0 of classical elasticity. The form of Eq. (47) is known from the Cauchy

[2] In the notation of Gutkin and Aifantis [31]: $c_1 = \ell_N^2$ and $c_2 = \ell_G^2$. Then Eq. (41) reads

$$(1 - c_1 \Delta) t_{ij} = \mathbb{C}_{ijkl} (1 - c_2 \Delta) e_{kl}.$$

It was pointed out by Gutkin and Aifantis [31] that in order to solve the "generalized Hooke law", one can solve separately the equations:

$$(1 - c_1 \Delta) t_{ij} = \sigma_{ij}^0, \qquad (1 - c_2 \Delta) e_{ij} = e_{ij}^0$$

by utilizing the solutions σ_{ij}^0 and e_{ij}^0 of classical elasticity.

Table 1
Lamé moduli, characteristic lengths, equilibrium lattice parameter and Poisson ratio for aluminum (Al).

λ (eV/Å^3)	μ (eV/Å^3)	ℓ_G (Å)	ℓ_N (Å)	ℓ_N/ℓ_G	a (Å)	ν
0.38649	0.19704	1.1300	2.3638	2.0919	4.0495	0.3312

stress tensor in gradient elasticity of bi-Helmholtz type (see [16,54]). The solution of Eq. (47) reads as

$$\sigma_{ij} = \sigma^0_{ij} * \alpha * G^{L_G}. \tag{48}$$

The Cauchy stress tensor (48) is nothing but the double convolution of the classical stress tensor σ^0_{ij} and the nonlocal kernel α and the Green function G^{L_G} similar to gradient elasticity of bi-Helmholtz type (see [10]).

Applying the differential operator L_G to Eq. (17) and using Eq. (32), in NSSGE the double stress tensor satisfies the inhomogeneous bi-Helmholtz equation with the operators L_N and L_G

$$L_G L_N \tau_{ijm} = \ell_G^2 \partial_m \sigma^0_{ij}. \tag{49}$$

Therefore, the Cauchy stress tensor σ_{ij} and the double stress tensor τ_{ijm} are determined by both the Helmholtz operators L_G and L_N including the gradient length scale ℓ_G and the nonlocal length scale ℓ_N.

2.2. Material parameters of NSSGE

Like the Lamé moduli, the characteristic length scales ℓ_N and ℓ_G are also material parameters. Similar to nonlocal elasticity of Helmholtz type [7], in nonlocal simplified strain gradient elasticity, the dispersion curve of nonlocal simplified strain gradient elasticity can be matched with the Born-von Kármán lattice model to get a relation between the two characteristic lengths ℓ_N and ℓ_G. The match of the dispersion relation of nonlocal simplified strain gradient elasticity with the dispersion relation of the Born-von Karmán lattice dynamics at the end of the first Brillouin zone $k = \pi/a$ gives the relation between ℓ_N^2 and ℓ_G^2 (see [53])

$$\ell_N^2 = \frac{a^2}{4}\left(1 + \frac{\ell_G^2 \pi^2}{a^2}\right) - \frac{a^2}{\pi^2}. \tag{50}$$

In first strain gradient elasticity, the gradient length scale ℓ_G and the Lamé moduli μ and λ can be computed from a second nearest-neighbor modified embedded-atom method (2NN MEAM) interatomic potential as done in [55,56]. Since aluminum is nearly an isotropic material, we will use aluminum in the numerical study of NSSGE. For aluminum, the gradient length was computed as $\ell_G = 0.279\,a$ (see [55]). Using Eq. (50), the nonlocal length is obtained as $\ell_N = 0.584\,a$. The Lamé moduli, gradient length and nonlocal length of Al used in this work for the numerics of dislocations in NSSGE are given in Table 1. Note that the length scales ℓ_N and ℓ_G satisfy the constraint (43).

3. Straight dislocations in nonlocal simplified strain gradient elasticity

In this section, we investigate screw and edge dislocations in the framework of NSSGE. For the numerical study of the dislocation fields, the elastic constants and the characteristic length scales of aluminum given in Table 1 are used.

3.1. Screw dislocation

The screw dislocation is located at the position $(x, y) = (0, 0)$ with Burgers vector b_z and dislocation line in the z-direction of a Cartesian coordinate system.

The classical plastic distortion of a screw dislocation given by deWit [57] (see also [58]) reads

$$\beta^{P,0}_{zy} = b_z \delta(y) H(-x) = b_z \delta(y) \int_x^\infty \delta(X)\,\mathrm{d}X, \tag{51}$$

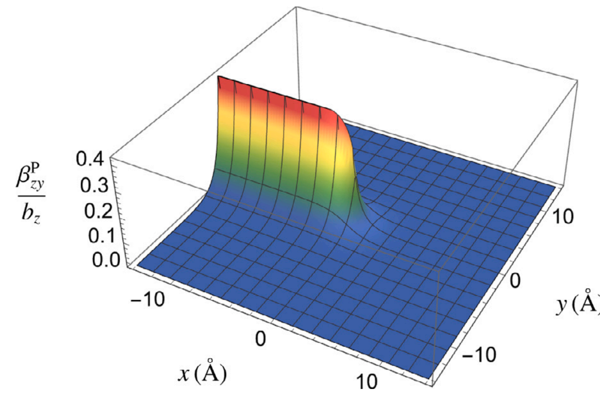

Fig. 1. Plastic distortion β^P_{zy} of a screw dislocation.

which possesses a discontinuity at $y = 0$ for $x < 0$. Here $H(.)$ denotes the Heaviside step function and $\delta(.)$ denotes the Dirac function. The classical dislocation density of a screw dislocation reads (see [57])

$$\alpha^0_{zz} = b_z \delta(x)\delta(y). \tag{52}$$

The classical elastic distortion and Cauchy stress fields read (see [57])

$$\beta^0_{zx} = \frac{\sigma^0_{zx}}{\mu} = -\frac{b_z}{2\pi}\frac{y}{r^2}, \tag{53}$$

$$\beta^0_{zy} = \frac{\sigma^0_{zy}}{\mu} = \frac{b_z}{2\pi}\frac{x}{r^2}, \tag{54}$$

where $r = \sqrt{x^2 + y^2}$.

Using Eq. (51), the solution of Eq. (28) gives the plastic distortion of a screw dislocation in NSSGE

$$\beta^P_{zy} = \frac{b_z}{2\pi\ell_G^2}\int_x^\infty K_0\left(\sqrt{X^2 + y^2}/\ell_G\right)\mathrm{d}X, \tag{55}$$

which is nonsingular, smooth and finite as it can be seen in Fig. 1. Here $K_n(.)$ denotes the modified Bessel function of second kind and $n = 0, 1, 2$. The plastic distortion of a screw dislocation in NSSGE is in agreement with the plastic distortion of a screw dislocation in simplified strain gradient elasticity given in [18,55].

In NSSGE, the dislocation density of a screw dislocation is obtained as solution of Eq. (33)

$$\alpha_{zz} = \frac{b_z}{2\pi\ell_G^2} K_0(r/\ell_G). \tag{56}$$

The dislocation density (56) is plotted in Fig. 2 and gives the shape and size of the dislocation core of a screw dislocation. The dislocation density of a screw dislocation in NSSGE is in agreement with the dislocation density of a screw dislocation in simplified strain gradient elasticity given in [18,55,59].

Substituting Eq. (51) into Eq. (29), the displacement field u_z is calculated as

$$u_z = \frac{b_z}{2\pi}\left(\arctan\frac{y}{x} + \pi H(-x)\,\mathrm{sgn}(y) + \partial_y \int_x^\infty K_0\left(\sqrt{X^2 + y^2}/\ell_G\right)\mathrm{d}X\right). \tag{57}$$

The displacement field (57) is plotted in Fig. 3. The displacement field (57) is nonsingular and has a smooth form due to the superposition of the classical jump discontinuity (first term) and the gradient term (second term). Here, sgn denotes the signum function. The displacement field of a screw dislocation in NSSGE is in agreement with the displacement field of a screw dislocation in simplified strain gradient elasticity given in [18,55].

In NSSGE, the two non-vanishing components of the elastic distortion are calculated as solution of Eq. (31)

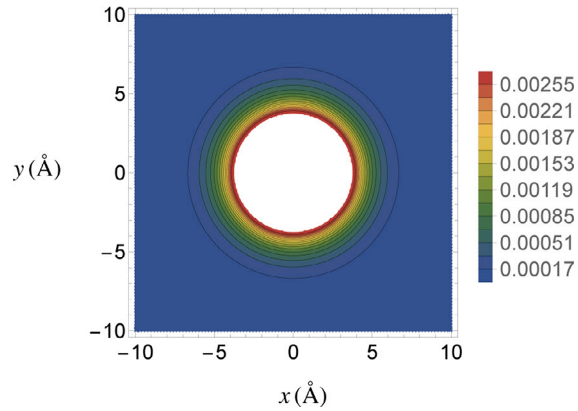

Fig. 2. Contour plot of the dislocation density of a screw dislocation α_{zz} (normalized by the Burgers vector b_z).

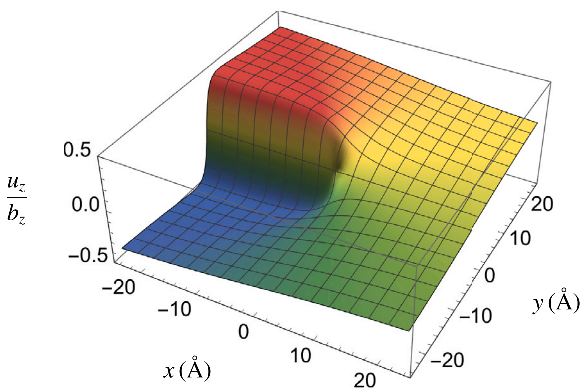

Fig. 3. Displacement field u_z of a screw dislocation.

$$\beta_{zx} = -\frac{b_z}{2\pi}\,\frac{y}{r^2}\left[1 - \frac{r}{\ell_G}\,K_1(r/\ell_G)\right], \tag{58}$$

$$\beta_{zy} = \frac{b_z}{2\pi}\,\frac{x}{r^2}\left[1 - \frac{r}{\ell_G}\,K_1(r/\ell_G)\right]. \tag{59}$$

The two components of the elastic distortion tensor, Eqs. (58) and (59), are plotted in Figs. 4(a) and (b). It can be seen that they are nonsingular. The elastic distortion of a screw dislocation in NSSGE is in agreement with the elastic distortion of a screw dislocation in simplified strain gradient elasticity given in [14,47,59].

In NSSGE for the total stress tensor of a screw dislocation, solution of Eq. (44) gives

$$t_{zx} = -\frac{\mu b_z}{2\pi}\,\frac{y}{r^2}\left[1 - \frac{r}{\ell_N}\,K_1(r/\ell_N)\right], \tag{60}$$

$$t_{zy} = \frac{\mu b_z}{2\pi}\,\frac{x}{r^2}\left[1 - \frac{r}{\ell_N}\,K_1(r/\ell_N)\right]. \tag{61}$$

The two components of the total stress tensor, Eqs. (60) and (61), are plotted in Figs. 5(a) and (b). It can be seen that they are nonsingular. The total stress of a screw dislocation in NSSGE is in agreement with the stress of a screw dislocation in nonlocal elasticity of Helmholtz type given in [1,7,9,10]. Moreover, the total stress of a screw dislocation in NSSGE is in agreement with the stress of a screw dislocation obtained by Gutkin and Aifantis [31] using the postulated "generalized Hooke law" with two different Helmholtz operators.

In NSSGE for the Cauchy stress tensor of a screw dislocation, solution of Eq. (47) gives

$$\sigma_{zx} = -\frac{\mu b_z}{2\pi}\,\frac{y}{r^2}\left[1 - \frac{1}{\ell_G^2-\ell_N^2}\left[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\right]\right], \tag{62}$$

$$\sigma_{zy} = \frac{\mu b_z}{2\pi}\,\frac{x}{r^2}\left[1 - \frac{1}{\ell_G^2-\ell_N^2}\left[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\right]\right]. \tag{63}$$

The two components of the Cauchy stress tensor, Eqs. (62) and (63), are plotted in Figs. 6(a) and (b). It can be seen that they are nonsingular. The Cauchy stress of a screw dislocation in NSSGE is in agreement with the Cauchy stress of a screw dislocation in gradient elasticity of bi-Helmholtz type given in [54]. It can be seen in Figs. 5 and 6 that the Cauchy stress tensor is slightly smoother than the total stress tensor.

In NSSGE, using Eq. (14), the double stress tensor of a screw dislocation reads as

$$\tau_{(zy)x} = -\frac{\mu\ell_G^2 b_z}{2\pi}\left[\frac{x^2-y^2}{r^4}\left(1 - \frac{1}{\ell_G^2-\ell_N^2}\left[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\right]\right) - \frac{x^2}{r^2}\,\frac{1}{\ell_G^2-\ell_N^2}\left[K_0(r/\ell_G) - K_0(r/\ell_N)\right]\right], \tag{64}$$

$$\tau_{(zx)y} = -\frac{\mu\ell_G^2 b_z}{2\pi}\left[\frac{x^2-y^2}{r^4}\left(1 - \frac{1}{\ell_G^2-\ell_N^2}\left[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\right]\right) + \frac{y^2}{r^2}\,\frac{1}{\ell_G^2-\ell_N^2}\left[K_0(r/\ell_G) - K_0(r/\ell_N)\right]\right], \tag{65}$$

$$\tau_{(zy)y} = -\frac{\mu\ell_G^2 b_z}{2\pi}\,\frac{xy}{r^4}\left[2\left(1 - \frac{1}{\ell_G^2-\ell_N^2}\left[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\right]\right) - r^2\,\frac{1}{\ell_G^2-\ell_N^2}\left[K_0(r/\ell_G) - K_0(r/\ell_N)\right]\right], \tag{66}$$

$$\tau_{(zx)x} = -\tau_{(zy)y}. \tag{67}$$

The components of the double stress tensor, Eqs. (64)–(67), are plotted in Figs. 7(a)–(d). It can be seen that the stresses are nonsingular unlike the double stresses calculated within the simplified first strain gradient elasticity [16]. The double stress of a screw dislocation in NSSGE is in agreement with the double stress of a screw dislocation in gradient elasticity of bi-Helmholtz type given in [54].

3.2. Edge dislocation

The edge dislocation of glide-mode is located at the position $(x, y) = (0, 0)$ with Burgers vector b_x. The dislocation line coincides with the z-axis of a Cartesian coordinate system.

The classical plastic distortion of an edge dislocation of glide-mode given by deWit [57] (see also [58]) reads

$$\beta^{\mathrm{P},0}_{xy} = b_x\delta(y)H(-x) = b_x\delta(y)\int_x^\infty \delta(X)\,\mathrm{d}X. \tag{68}$$

The classical dislocation density of an edge dislocation reads as (see [57])

$$\alpha^0_{xz} = b_x\delta(x)\delta(y). \tag{69}$$

The classical elastic distortion components are [57]

$$\beta^0_{xx} = -\frac{b_x}{4\pi(1-\nu)}\,\frac{y}{r^2}\left[(1-2\nu) + \frac{2x^2}{r^2}\right], \tag{70}$$

$$\beta^0_{yy} = -\frac{b_x}{4\pi(1-\nu)}\,\frac{y}{r^2}\left[(1-2\nu) - \frac{2x^2}{r^2}\right], \tag{71}$$

$$\beta^0_{xy} = \frac{b_x}{4\pi(1-\nu)}\,\frac{x}{r^2}\left[(3-2\nu) - \frac{2y^2}{r^2}\right], \tag{72}$$

$$\beta^0_{yx} = -\frac{b_x}{4\pi(1-\nu)}\,\frac{x}{r^2}\left[(1-2\nu) + \frac{2y^2}{r^2}\right] \tag{73}$$

and the classical stress components are [57]

$$\sigma^0_{xx} = -\frac{\mu b_x}{2\pi(1-\nu)}\,\frac{y}{r^4}\left(3x^2+y^2\right), \tag{74}$$

$$\sigma^0_{yy} = \frac{\mu b_x}{2\pi(1-\nu)}\,\frac{y}{r^4}\left(x^2-y^2\right), \tag{75}$$

$$\sigma^0_{xy} = \frac{\mu b_x}{2\pi(1-\nu)}\,\frac{x}{r^4}\left(x^2-y^2\right), \tag{76}$$

$$\sigma^0_{zz} = -\frac{\mu\nu b_x}{\pi(1-\nu)}\,\frac{y}{r^2}. \tag{77}$$

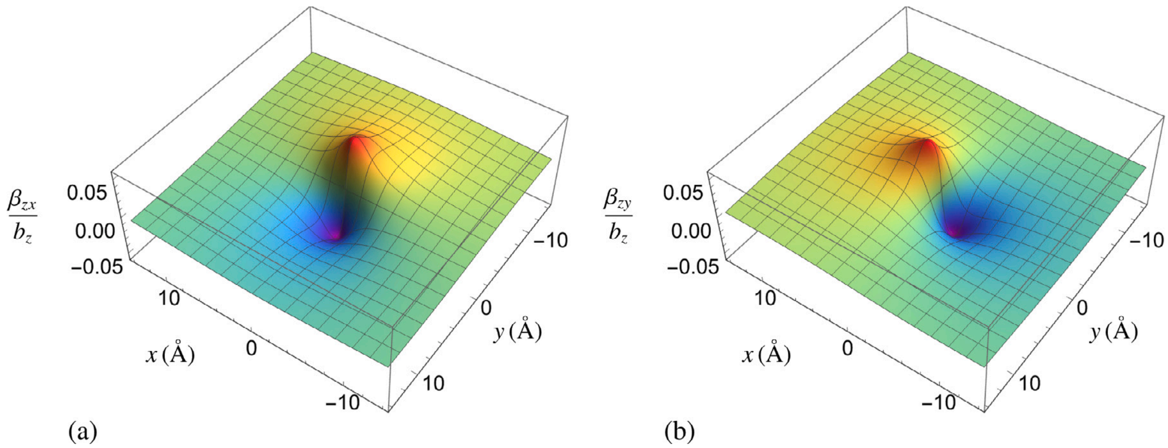

Fig. 4. Elastic distortion components of a screw dislocation: (a) β_{zx} and (b) β_{zy}.

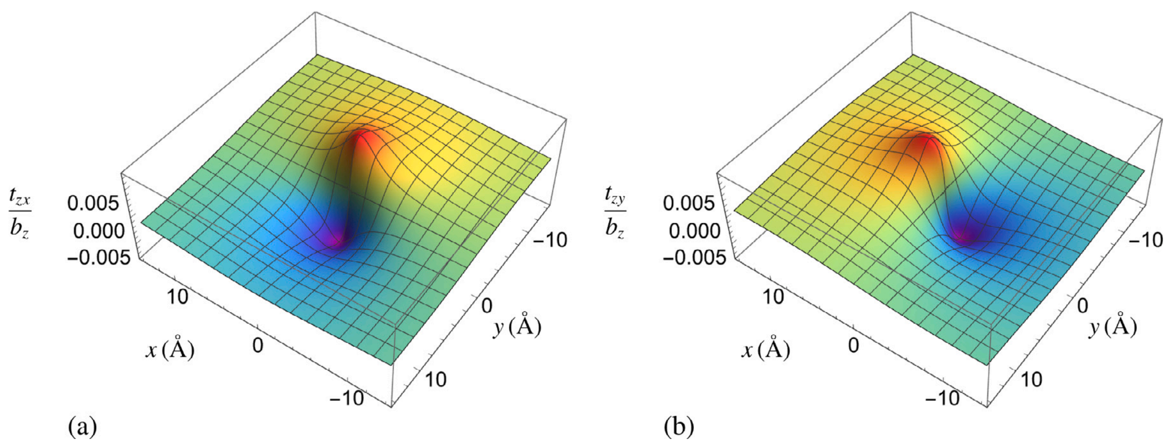

Fig. 5. Total stress components of a screw dislocation: (a) t_{zx} and (b) t_{zy}.

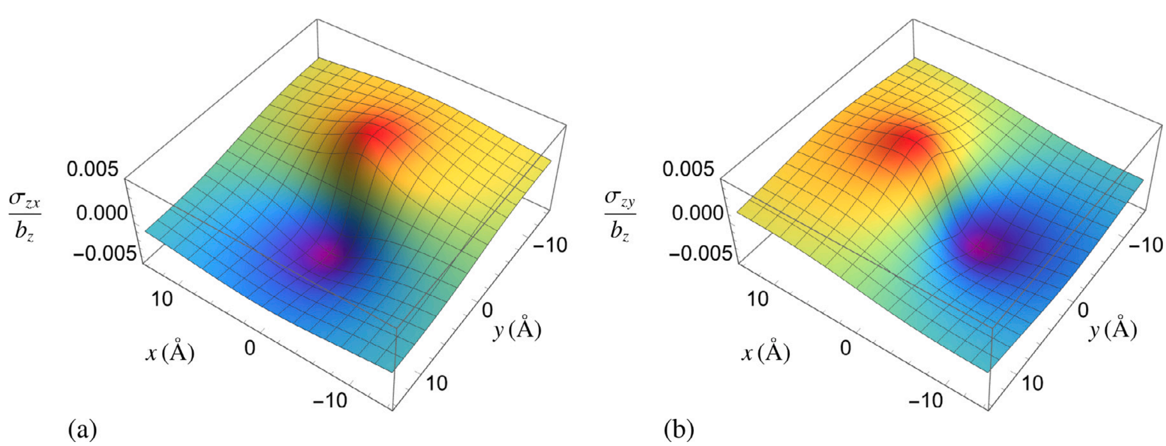

Fig. 6. Cauchy stress components of a screw dislocation: (a) σ_{zx} and (b) σ_{zy}.

Substituting Eq. (68) into Eq. (28), the plastic distortion of an edge dislocation is calculated as

$$\beta^{\mathrm{P}}_{xy} = \frac{b_x}{2\pi\ell_G^2}\int_x^\infty K_0\big(\sqrt{X^2+y^2}/\ell_G\big)\,\mathrm{d}X, \tag{78}$$

which is nonsingular, smooth and finite as it can be seen in Fig. 8. The plastic distortion of an edge dislocation in NSSGE is in agreement with the plastic distortion of an edge dislocation in simplified strain gradient elasticity given in [18,55].

In NSSGE, the dislocation density of a screw dislocation is obtained as solution of Eq. (33)

$$\alpha_{xz} = \frac{b_x}{2\pi\ell_G^2}\,K_0\big(r/\ell_G\big). \tag{79}$$

The dislocation density (79) is plotted in Fig. 9 and gives the shape and size of the dislocation core of an edge dislocation. The dislocation density of an edge dislocation in NSSGE is in agreement with the

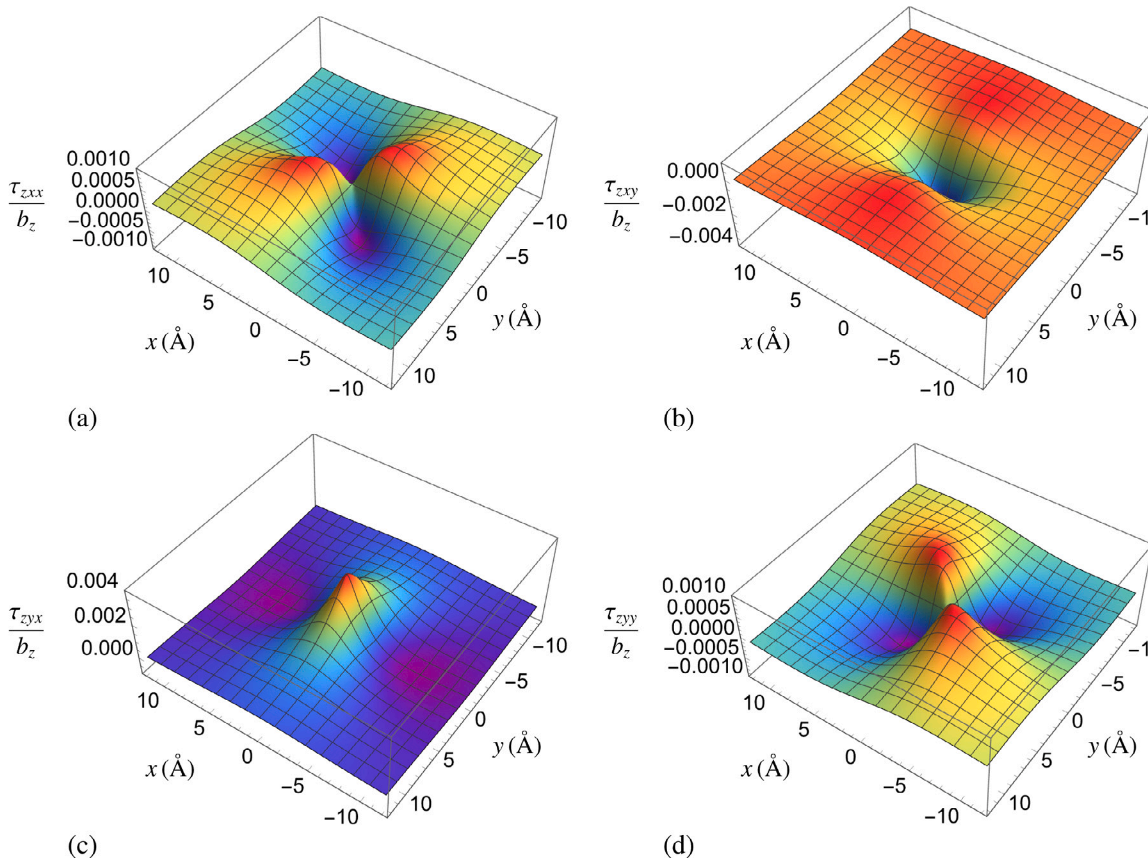

Fig. 7. Double stress components of a screw dislocation: (a) τ_{zxx}, (b) τ_{zxy}, (c) τ_{zyx} and (c) τ_{zyy}.

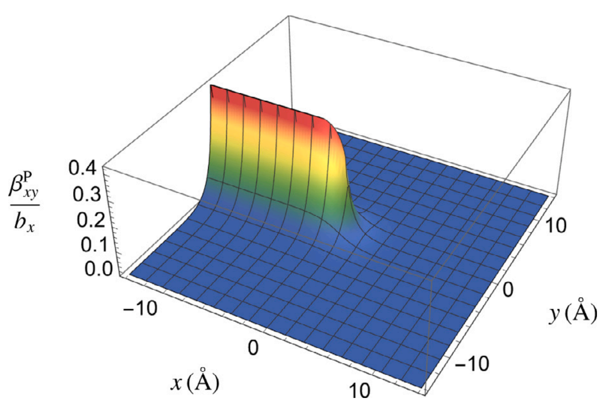

Fig. 8. Plastic distortion β^{P}_{xy} of an edge dislocation.

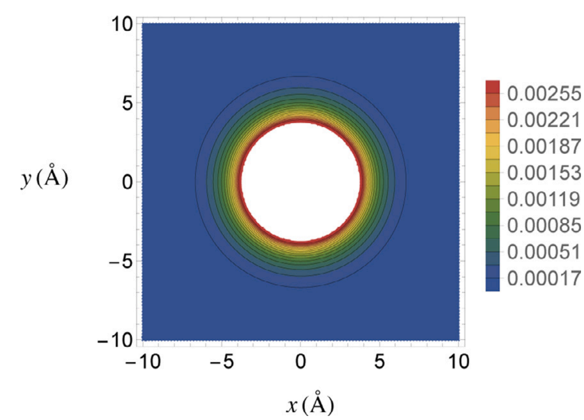

Fig. 9. Contour plot of the dislocation density of an edge dislocation α_{xz} (normalized by the Burgers vector b_x).

dislocation density of an edge dislocation in simplified strain gradient elasticity given in [18,55,59].

Substituting Eq. (68) into Eq. (29), the displacement fields u_x and u_y of an edge dislocation in NSSGE are calculated as

$$u_x = \frac{b_x}{4\pi(1-\nu)}\Bigg[2(1-\nu)\bigg(\arctan\frac{y}{x} + \pi H(-x)\,\mathrm{sgn}(y) + \partial_y \int_x^\infty K_0\big(\sqrt{X^2+y^2}/\ell_G\big)\,\mathrm{d}X\bigg) + \frac{xy}{r^2}\bigg(1 - \frac{4\ell_G^2}{r^2} + 2K_2(r/\ell_G)\bigg)\Bigg], \tag{80}$$

$$u_y = -\frac{b_x}{4\pi(1-\nu)}\Bigg[(1-2\nu)\Big(\ln r + K_0(r/\ell_G)\Big) + \frac{x^2-y^2}{2r^2}\bigg(1 - \frac{4\ell_G^2}{r^2} + 2K_2(r/\ell_G)\bigg)\Bigg]. \tag{81}$$

The displacement fields (80) and (81) are plotted in Figs. 10(a) and (b). The displacement fields (80) and (81) are nonsingular and have a smooth form. The displacement fields of an edge dislocation in NSSGE are in agreement with the displacement fields of an edge dislocation in simplified strain gradient elasticity given in [18,55].

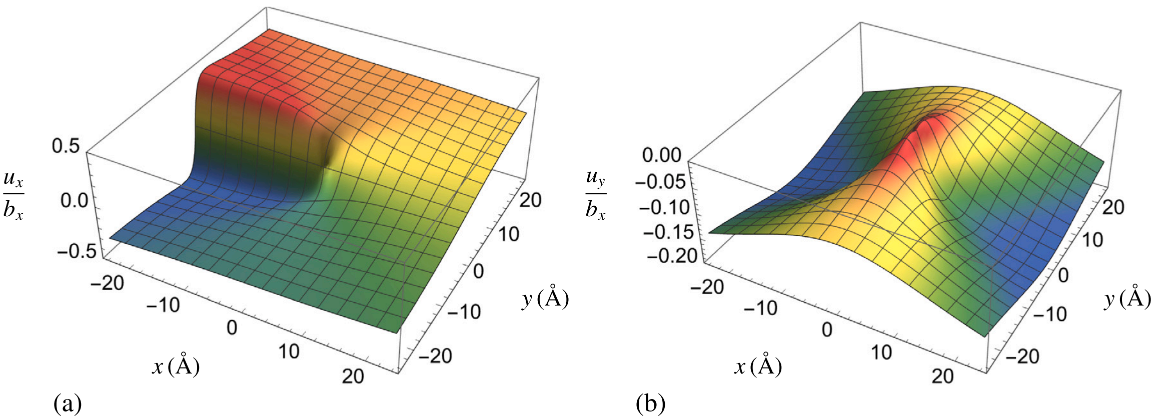

Fig. 10. Displacement fields of an edge dislocation: (a) u_x and (b) u_y.

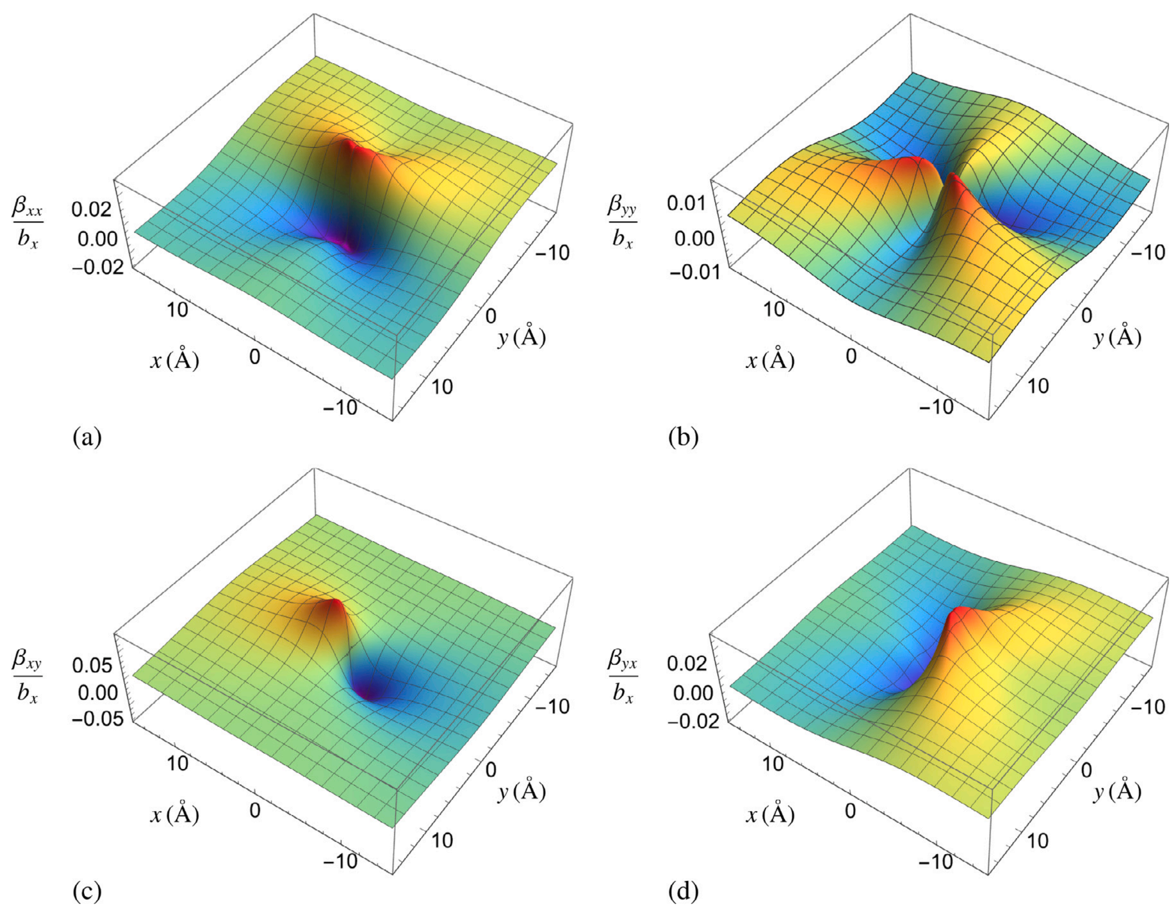

Fig. 11. Elastic distortion components of an edge dislocation: (a) β_{xx}, (b) β_{yy}, (c) β_{xy} and (d) β_{yx}.

In NSSGE, the elastic distortion tensor components of an edge dislocation are calculated as solution of Eq. (31)

$$\beta_{xx} = -\frac{b_x}{4\pi(1-\nu)}\,\frac{y}{r^2}\,\bigg[(1-2\nu) + \frac{2x^2}{r^2} - \frac{3x^2-y^2}{r^2}\bigg(\frac{4\ell_G^2}{r^2} - 2K_2(r/\ell_G)\bigg) - \frac{2(y^2-\nu r^2)}{\ell_G r}\,K_1(r/\ell_G)\bigg], \tag{82}$$

$$\beta_{yy} = -\frac{b_x}{4\pi(1-\nu)}\,\frac{y}{r^2}\,\bigg[(1-2\nu) - \frac{2x^2}{r^2} + \frac{3x^2-y^2}{r^2}\bigg(\frac{4\ell_G^2}{r^2} - 2K_2(r/\ell_G)\bigg) - \frac{2(x^2-\nu r^2)}{\ell_G r}\,K_1(r/\ell_G)\bigg], \tag{83}$$

$$\beta_{xy} = \frac{b_x}{4\pi(1-\nu)}\,\frac{x}{r^2}\,\bigg[(3-2\nu) - \frac{2y^2}{r^2} - \frac{x^2-3y^2}{r^2}\bigg(\frac{4\ell_G^2}{r^2} - 2K_2(r/\ell_G)\bigg) - \frac{2\big(y^2+(1-\nu)r^2\big)}{\ell_G r}\,K_1(r/\ell_G)\bigg], \tag{84}$$

$$\beta_{yx} = -\frac{b_x}{4\pi(1-\nu)}\,\frac{x}{r^2}\,\bigg[(1-2\nu) + \frac{2y^2}{r^2} + \frac{x^2-3y^2}{r^2}\bigg(\frac{4\ell_G^2}{r^2} - 2K_2(r/\ell_G)\bigg) - \frac{2\big(x^2-\nu r^2\big)}{\ell_G r}\,K_1(r/\ell_G)\bigg]. \tag{85}$$

The components of the elastic distortion tensor, Eqs. (82)–(84), are plotted in Figs. 11(a)–(d). It can be seen that they are nonsingular. The elastic distortion of an edge dislocation in NSSGE is in agreement with

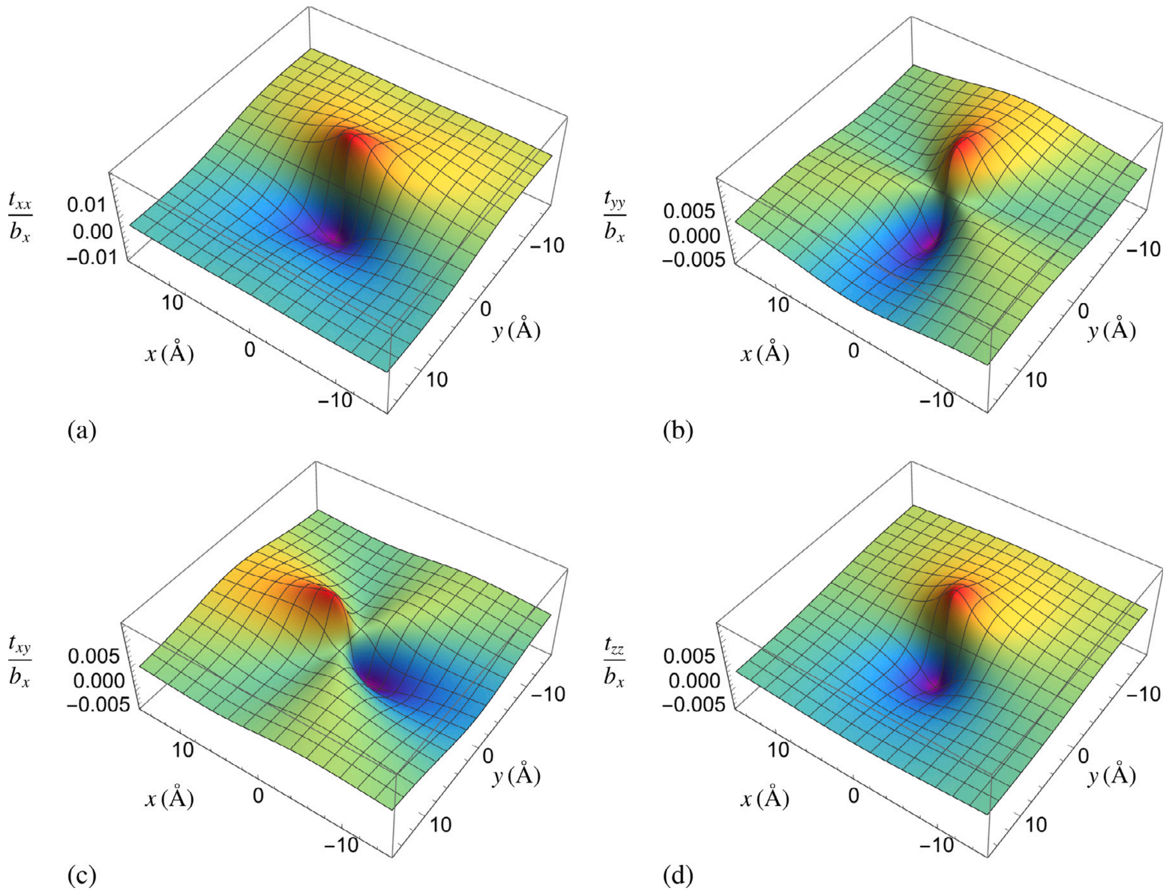

Fig. 12. Total stress components of an edge dislocation: (a) t_{xx}, (b) t_{yy}, (c) t_{xy} and (d) t_{zz}.

the elastic distortion of an edge dislocation in simplified strain gradient elasticity given in [18,47,59].

For the total stress tensor produced by an edge dislocation, solution of Eq. (44) gives

$$t_{xx} = -\frac{\mu b_x}{2\pi(1-\nu)}\,\frac{y}{r^4}\left[(3x^2+y^2)-(3x^2-y^2)\left(\frac{4\ell_N^2}{r^2}-2K_2(r/\ell_N)\right) - \frac{2y^2 r}{\ell_N}\,K_1(r/\ell_N)\right], \tag{86}$$

$$t_{yy} = \frac{\mu b_x}{2\pi(1-\nu)}\,\frac{y}{r^4}\left[(x^2-y^2)-(3x^2-y^2)\left(\frac{4\ell_N^2}{r^2}-2K_2(r/\ell_N)\right) + \frac{2x^2 r}{\ell_N}\,K_1(r/\ell_N)\right], \tag{87}$$

$$t_{xy} = \frac{\mu b_x}{2\pi(1-\nu)}\,\frac{x}{r^4}\left[(x^2-y^2)-(x^2-3y^2)\left(\frac{4\ell_N^2}{r^2}-2K_2(r/\ell_N)\right) - \frac{2y^2 r}{\ell_N}\,K_1(r/\ell_N)\right], \tag{88}$$

$$t_{zz} = -\frac{\mu\nu b_x}{\pi(1-\nu)}\,\frac{y}{r^2}\left[1-\frac{r}{\ell_N}\,K_1(r/\ell_N)\right]. \tag{89}$$

The components of the total stress tensor, Eqs. (86)–(89), are plotted in Figs. 12(a)–(d). They are nonsingular. The total stress of an edge dislocation in NSSGE is in agreement with the stress of an edge dislocation in nonlocal elasticity of Helmholtz type given in [9,10]. Additionally, the total stress of an edge dislocation in NSSGE is in agreement with the stress of an edge dislocation obtained by Gutkin and Aifantis [31] using the postulated "generalized Hooke law" with two different Helmholtz operators.

In NSSGE, the Cauchy stress tensor produced by an edge dislocation is given by the solution of Eq. (47)

$$\sigma_{xx} = -\frac{\mu b_x}{2\pi(1-\nu)}\,\frac{y}{r^4}\left[(3x^2+y^2)-\frac{4(\ell_G^2+\ell_N^2)}{r^2}(3x^2-y^2) - \frac{2y^2}{\ell_G^2-\ell_N^2}\big[\ell_G r\,K_1(r/\ell_G)-\ell_N r\,K_1(r/\ell_N)\big] + \frac{2(3x^2-y^2)}{\ell_G^2-\ell_N^2}\big[\ell_G^2\,K_2(r/\ell_G)-\ell_N^2\,K_2(r/\ell_N)\big]\right], \tag{90}$$

$$\sigma_{yy} = \frac{\mu b_x}{2\pi(1-\nu)}\,\frac{y}{r^4}\left[(x^2-y^2)-\frac{4(\ell_G^2+\ell_N^2)}{r^2}(3x^2-y^2) + \frac{2x^2}{\ell_G^2-\ell_N^2}\big[\ell_G r\,K_1(r/\ell_G)-\ell_N r\,K_1(r/\ell_N)\big] + \frac{2(3x^2-y^2)}{\ell_G^2-\ell_N^2}\big[\ell_G^2\,K_2(r/\ell_G)-\ell_N^2\,K_2(r/\ell_N)\big]\right], \tag{91}$$

$$\sigma_{xy} = \frac{\mu b_x}{2\pi(1-\nu)}\,\frac{x}{r^4}\left[(x^2-y^2)-\frac{4(\ell_G^2+\ell_N^2)}{r^2}(x^2-3y^2) - \frac{2y^2}{\ell_G^2-\ell_N^2}\big[\ell_G r\,K_1(r/\ell_G)-\ell_N r\,K_1(r/\ell_N)\big] + \frac{2(x^2-3y^2)}{\ell_G^2-\ell_N^2}\big[\ell_G^2\,K_2(r/\ell_G)-\ell_N^2\,K_2(r/\ell_N)\big]\right], \tag{92}$$

$$\sigma_{zz} = -\frac{\mu\nu b_x}{\pi(1-\nu)}\,\frac{y}{r^2}\left[1-\frac{1}{\ell_G^2-\ell_N^2}\big[\ell_G r\,K_1(r/\ell_G)-\ell_N r\,K_1(r/\ell_N)\big]\right]. \tag{93}$$

The components of the Cauchy stress tensor, Eqs. (90)–(93), are plotted in Figs. 13(a)–(b). It can be seen that they are nonsingular. The Cauchy stress of an edge dislocation in NSSGE is in agreement with the Cauchy

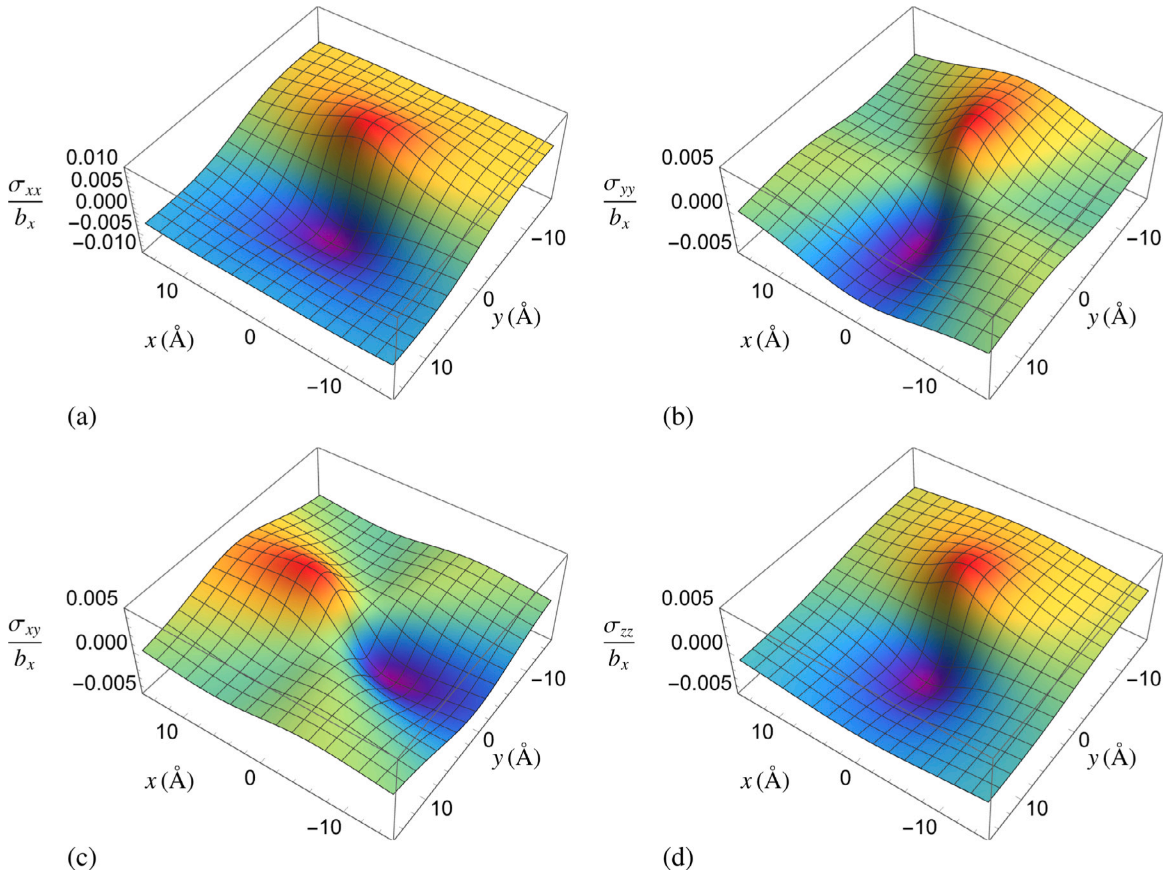

Fig. 13. Cauchy stress components of an edge dislocation: (a) σ_{xx}, (b) σ_{yy}, (c) σ_{xy} and (d) σ_{zz}.

stress of an edge dislocation in gradient elasticity of bi-Helmholtz type given in [54]. Again, it can be seen in Figs. 12 and 13 that the Cauchy stress tensor is slightly smoother than the total stress tensor.

In NSSGE, using Eq. (14), the double stress tensor of an edge dislocation is given by

$$
\begin{aligned}
\tau_{(xx)x} = \frac{\mu \ell_G^2 b_x}{2\pi(1-\nu)} \frac{2xy}{r^6} \bigg[& (3x^2-y^2) - 24\,\frac{\ell_G^2+\ell_N^2}{r^2}(x^2-y^2) \\
& + \frac{3(x^4-y^4)}{(\ell_G^2-\ell_N^2)r}\big[\ell_G K_1(r/\ell_G) - \ell_N K_1(r/\ell_N)\big] \\
& + \frac{12(x^2-y^2)}{\ell_G^2-\ell_N^2}\big[\ell_G^2 K_2(r/\ell_G) - \ell_N^2 K_2(r/\ell_N)\big] \\
& - \frac{y^2 r^2}{\ell_G^2-\ell_N^2}\big[K_2(r/\ell_G) - K_2(r/\ell_N)\big] \bigg],
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
\tau_{(xx)y} = -\frac{\mu \ell_G^2 b_x}{2\pi(1-\nu)} \frac{1}{r^6} \bigg[& (3x^4-6x^2y^2-y^4) - 12\,\frac{\ell_G^2+\ell_N^2}{r^2}(x^4-6x^2y^2+y^4) \\
& + \frac{2y^4 r^2}{\ell_G^2-\ell_N^2}\big[K_2(r/\ell_G) - K_2(r/\ell_N)\big] \\
& + \frac{6(x^4-6x^2y^2+y^4)}{\ell_G^2-\ell_N^2}\big[\ell_G^2 K_2(r/\ell_G) - \ell_N^2 K_2(r/\ell_N)\big] \\
& - \frac{12x^2y^2}{\ell_G^2-\ell_N^2}\big[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\big] \bigg],
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
\tau_{(yy)x} = -\frac{\mu \ell_G^2 b_x}{2\pi(1-\nu)} \frac{2xy}{r^6} \bigg[& (x^2-3y^2) - 24\,\frac{\ell_G^2+\ell_N^2}{r^2}(x^2-y^2) \\
& + \frac{3(x^4-y^4)}{(\ell_G^2-\ell_N^2)r}\big[\ell_G K_1(r/\ell_G) - \ell_N K_1(r/\ell_N)\big] \\
& + \frac{12(x^2-y^2)}{\ell_G^2-\ell_N^2}\big[\ell_G^2 K_2(r/\ell_G) - \ell_N^2 K_2(r/\ell_N)\big] \\
& + \frac{x^2 r^2}{\ell_G^2-\ell_N^2}\big[K_2(r/\ell_G) - K_2(r/\ell_N)\big] \bigg],
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
\tau_{(yy)y} = \frac{\mu \ell_G^2 b_x}{2\pi(1-\nu)} \frac{1}{r^6} \bigg[& (x^4-6x^2y^2+y^4) - 12\,\frac{\ell_G^2+\ell_N^2}{r^2}(x^4-6x^2y^2+y^4) \\
& - \frac{2x^2y^2 r^2}{\ell_G^2-\ell_N^2}\big[K_2(r/\ell_G) - K_2(r/\ell_N)\big] \\
& + \frac{6(x^4-6x^2y^2+y^4)}{\ell_G^2-\ell_N^2}\big[\ell_G^2 K_2(r/\ell_G) - \ell_N^2 K_2(r/\ell_N)\big] \\
& + \frac{2(x^6-3x^4y^2-3x^2y^4+y^6)}{(\ell_G^2-\ell_N^2)r}\big[\ell_G K_1(r/\ell_G) - \ell_N K_1(r/\ell_N)\big] \bigg],
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
\tau_{(zz)x} = \frac{\mu \nu \ell_G^2 b_x}{\pi(1-\nu)} \frac{xy}{r^4} \bigg[& 2\Big(1 - \frac{1}{\ell_G^2-\ell_N^2}\big[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\big]\Big) \\
& - r^2\,\frac{1}{\ell_G^2-\ell_N^2}\big[K_0(r/\ell_G) - K_0(r/\ell_N)\big] \bigg],
\end{aligned} \tag{98}
$$

$$
\begin{aligned}
\tau_{(zz)y} = -\frac{\mu \nu \ell_G^2 b_x}{\pi(1-\nu)} \bigg[& \frac{x^2-y^2}{r^4}\Big(1 - \frac{1}{\ell_G^2-\ell_N^2}\big[\ell_G r K_1(r/\ell_G) - \ell_N r K_1(r/\ell_N)\big]\Big) \\
& + \frac{y^2}{r^2}\,\frac{1}{\ell_G^2-\ell_N^2}\big[K_0(r/\ell_G) - K_0(r/\ell_N)\big] \bigg]
\end{aligned} \tag{99}
$$

and $\tau_{(xy)x} = -\tau_{(yy)y}$, $\tau_{(xy)y} = -\tau_{(xx)x}$. The components of the double stress tensor, Eqs. (94)–(99), are plotted in Figs. 14(a)–(f). It can be seen that the stresses are nonsingular unlike the double stresses calculated within the simplified first strain gradient elasticity [16]. The double stress of an edge dislocation in NSSGE is in formal agreement with the double stress of an edge dislocation in gradient elasticity of bi-Helmholtz type given in [54].

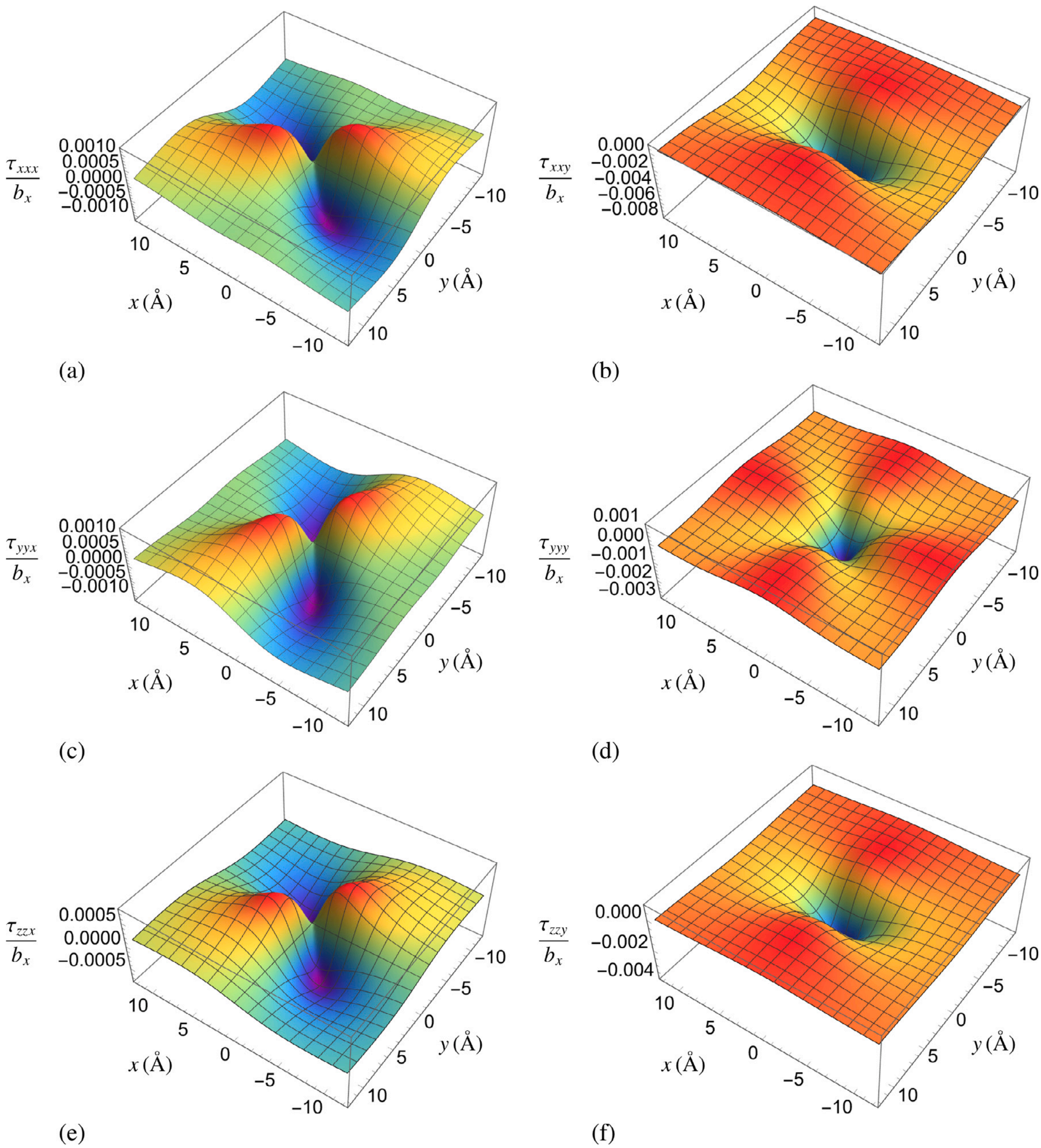

Fig. 14. Double stress components of an edge dislocation: (a) τ_{xxx}, (b) τ_{xxy}, (c) τ_{yyx}, (d) τ_{yyy}, (e) τ_{zzx} and (c) τ_{zzy}.

4. Peach-Koehler force

Now, we examine the Peach-Koehler force in the framework of nonlocal simplified strain gradient elasticity. The Eshelby stress tensor in nonlocal strain gradient elasticity is given by

$$P_{kj} = \mathcal{W}\delta_{jk} - \big(\sigma_{ij} - \partial_l \tau_{ijl}\big)\beta_{ik} - \tau_{ilj}\partial_l \beta_{ik}, \tag{100}$$

which is similar to the Eshelby stress tensor in strain gradient elasticity given in Lazar and Kirchner [21] (see also [17]) and the Eshelby stress tensor in nonlocal elasticity given in Lazar and Kirchner [60]. The corresponding Peach-Koehler force is obtained as

$$\int_V \partial_j P_{kj}\, \mathrm{d}V = \mathcal{F}_k^{\mathrm{PK}}. \tag{101}$$

The Peach-Koehler force in nonlocal simplified strain gradient elasticity reads

$$\begin{aligned}
\mathcal{F}_k^{\mathrm{PK}} &= \int_V \epsilon_{kjl}\big(\sigma_{ij}\alpha_{il} + \tau_{ijm}\,\partial_m \alpha_{il}\big)\,\mathrm{d}V \\
&= \int_V \epsilon_{kjl}\big((\sigma_{ij} - \partial_m \tau_{ijm})\alpha_{il} + \partial_m(\tau_{ijm}\alpha_{il})\big)\,\mathrm{d}V \\
&= \int_V \epsilon_{kjl} t_{ij}\alpha_{il}\,\mathrm{d}V \\
&= \int_V \epsilon_{kjl}\sigma_{ij}\alpha^0_{il}\,\mathrm{d}V.
\end{aligned} \tag{102}$$

From the second to the third line, the div-term (surface term) can be neglected at infinity.

If we substitute the classical dislocation density of a straight dislocation (see Eqs. (52) and (69)) into Eq. (102), we find for the Peach-Koehler force per unit length of a straight dislocation

$$\mathcal{F}_k^{\mathrm{PK}} = \epsilon_{kjl} b'_i \sigma_{ij}\xi_l. \tag{103}$$

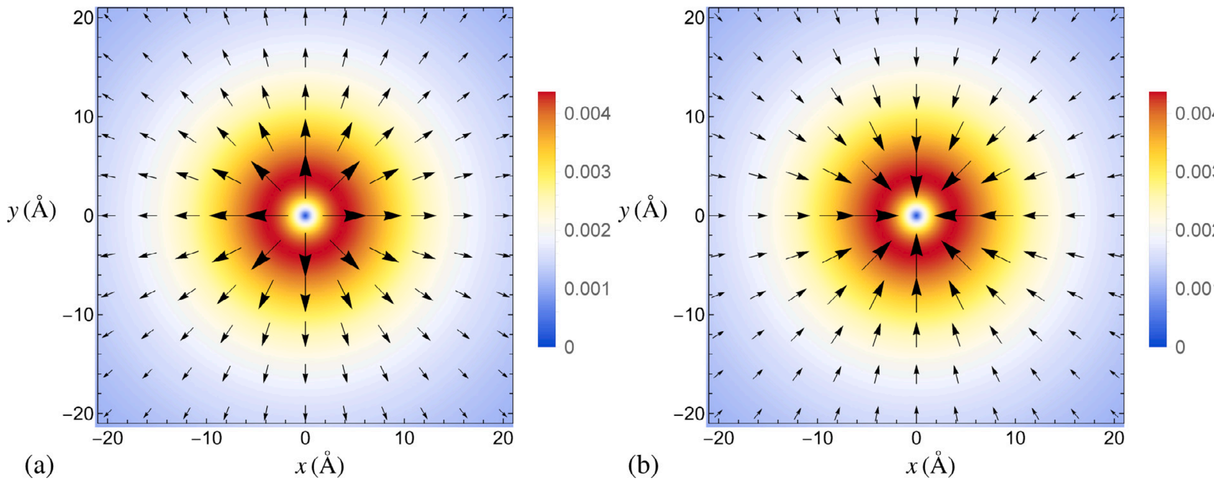

Fig. 15. Peach-Koehler force $\mathcal{F}^{\rm PK}$ of two parallel screw dislocations with: (a) $b'_z b_z > 0$, (b) $b'_z b_z < 0$.

The Peach-Koehler force (103) is the interaction force of a straight dislocation with Burgers vector b'_i and the dislocation line direction ξ_l in the stress field σ_{ij}. The Peach-Koehler force (103) is nonsingular due to the nonsingular Cauchy stress in nonlocal simplified strain gradient elasticity. Eq. (103) is important for the interaction between dislocations.

For two parallel screw dislocations with Burgers vector in z-direction and $\xi_z = 1$, the Peach-Koehler force per unit length reads

$$\mathcal{F}^{\rm PK}_x = b'_z \sigma_{zy}, \qquad \mathcal{F}^{\rm PK}_y = -b'_z \sigma_{zx}, \tag{104}$$

where the corresponding Cauchy stress fields of a screw dislocation are given in Eqs. (62) and (63). The Peach-Koehler force (104) is plotted in Figs. 15(a) and 15(b). It can be seen that the Peach-Koehler force (104) is nonsingular. The Peach-Koehler force (104) is repulsive if $b'_z b_z > 0$ (see Fig. 15(a)) and is attractive if $b'_z b_z < 0$ (see Fig. 15(b)). It is obvious that the Peach-Koehler force of two parallel screw dislocations (104) is a central force.

For two parallel edge dislocations with Burger vector in x-direction and $\xi_z = 1$, the Peach-Koehler force per unit length reads

$$\mathcal{F}^{\rm PK}_x = b'_x \sigma_{xy}, \qquad \mathcal{F}^{\rm PK}_y = -b'_x \sigma_{xx}, \tag{105}$$

where the corresponding Cauchy stress fields of an edge dislocation are given in Eqs. (90) and (92). The Peach-Koehler force (105) is plotted in Figs. 16(a) and 16(b). It can be seen that the Peach-Koehler force (105) is nonsingular. The Peach-Koehler force (105) is repulsive if $b'_x b_x > 0$ (see Fig. 16(a)) and is attractive if $b'_x b_x < 0$ (see Fig. 16(b)). It is obvious that the Peach-Koehler force of two parallel edge dislocations (105) is not a central force. The glide and climb components of two parallel edge dislocations are plotted in Figs. 17 and 18, respectively. Of course, they are nonsingular (see also the discussion in [9] for the Peach-Koehler force in nonlocal elasticity). Also, it can be observed that the climb component is greater than the glide component in the dislocation core region (or near the dislocation line): $\mathcal{F}^{\rm PK}_y > \mathcal{F}^{\rm PK}_x$ and this is because the extremum of the Cauchy stress component σ_{xx} is greater than the extremum of the Cauchy stress component σ_{xy} in the dislocation core region (see Fig. 13 and also [9]).

5. Conclusions

In this paper, we have investigated nonlocal simplified strain gradient elasticity. Nonlocal simplified strain gradient elasticity is a model of generalized continuum theory involving two internal characteristic lengths in addition to the two Lamé parameters. It allows to eliminate elastic singularities and discontinuities and to interpret size effects.

In order to show the main advantages of the nonlocal simplified strain gradient elasticity model, it has been employed to investigate straight dislocations. Exact analytical solutions for the displacement fields, elastic distortions, Cauchy stresses, double stresses, total stresses, plastic distortions and dislocation densities of screw and edge dislocations have been derived which demonstrate the elimination of any singularity in the elastic and plastic fields at the dislocation line, except the dislocation density field possessing a logarithmic singularity at the dislocation line. The fields of straight dislocations depend on the characteristic gradient length ℓ_G and the characteristic nonlocal length ℓ_N in the following way:

- displacement vector: $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, \ell_G)$
- plastic distortion tensor: $\boldsymbol{\beta}^{\rm P} = \boldsymbol{\beta}^{\rm P}(\boldsymbol{r}, \ell_G)$
- dislocation density tensor: $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{r}, \ell_G)$
- elastic distortion tensor: $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{r}, \ell_G)$
- Cauchy stress tensor: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{r}, \ell_G, \ell_N)$
- double stress tensor: $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{r}, \ell_G, \ell_N)$
- total stress tensor: $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, \ell_N)$.

Therefore, the displacement, plastic distortion, elastic distortion and dislocation density fields incorporate the nonlocality of strain gradient elasticity, the total stress incorporates the nonlocality of nonlocal elasticity, and the Cauchy stress and double stress fields incorporate both the nonlocality of strain gradient elasticity and the nonlocality of nonlocal elasticity. It is important to note that the main feature of the obtained solutions of screw and edge dislocations is the absence of any singularity in the displacement, elastic distortion, plastic distortion, Cauchy stress, double stress and total stress fields due to the regularization in the framework of nonlocal simplified strain gradient elasticity. We have proven that the solutions for the stress and elastic strain fields of screw and edge dislocations given by Gutkin and Aifantis [31] are actually the solutions of the total stress and elastic strain fields in the framework of nonlocal simplified strain gradient elasticity.

Moreover, we have calculated the Peach-Koehler force in the framework of nonlocal simplified strain gradient elasticity. The Peach-Koehler force (per unit length) reads for a straight dislocation with Burgers vector $\boldsymbol{b}'$ and dislocation line direction $\boldsymbol{\xi}$ in the stress field $\boldsymbol{\sigma}$:

$$\mathcal{F}^{\rm PK} = (\boldsymbol{b}' \cdot \boldsymbol{\sigma}) \times \boldsymbol{\xi}. \tag{106}$$

It is important to note that the Cauchy stress $\boldsymbol{\sigma}$, which incorporates the nonlocality of strain gradient elasticity and the nonlocality of nonlocal elasticity, is the physical stress entering the Peach-Koehler force, which is felt by the dislocation with Burgers vector $\boldsymbol{b}'$. The Peach-Koehler force (106) has the same formal mathematical form as the Peach-Koehler force in classical elasticity (see [61]). The only difference is

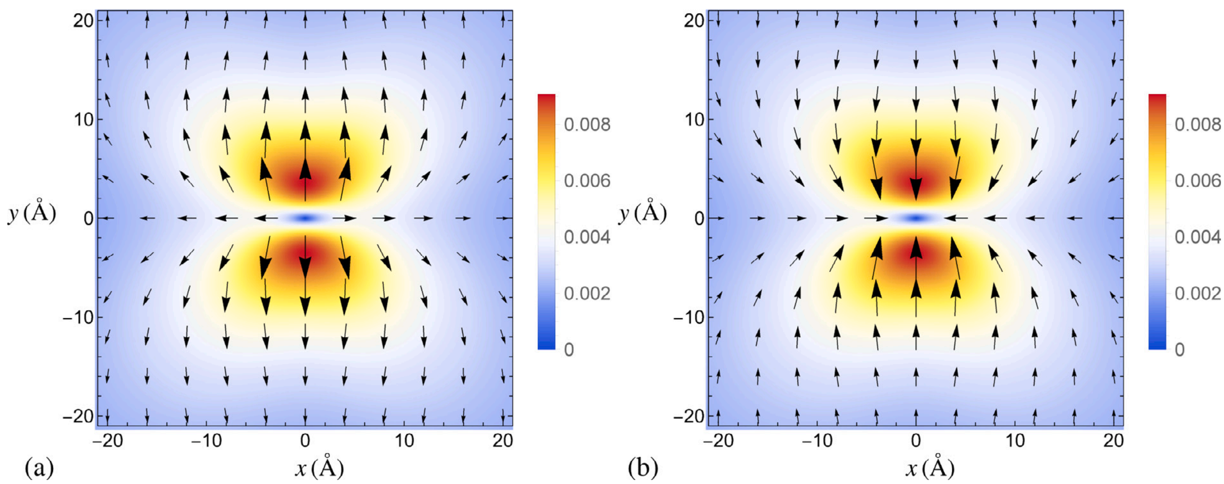

Fig. 16. Peach-Koehler force $\mathcal{F}^{\rm PK}$ of two parallel edge dislocations with: (a) $b'_x b_x > 0$, (b) $b'_x b_x < 0$.

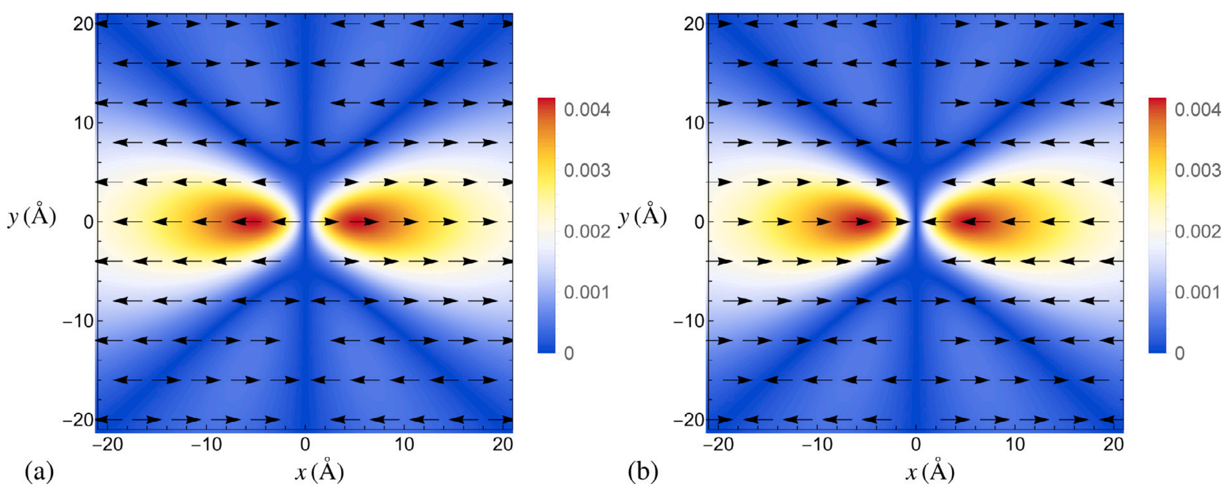

Fig. 17. Glide component (Peach-Koehler force component $\mathcal{F}^{\rm PK}_x$) of two parallel edge dislocations with: (a) $b'_x b_x > 0$, (b) $b'_x b_x < 0$.

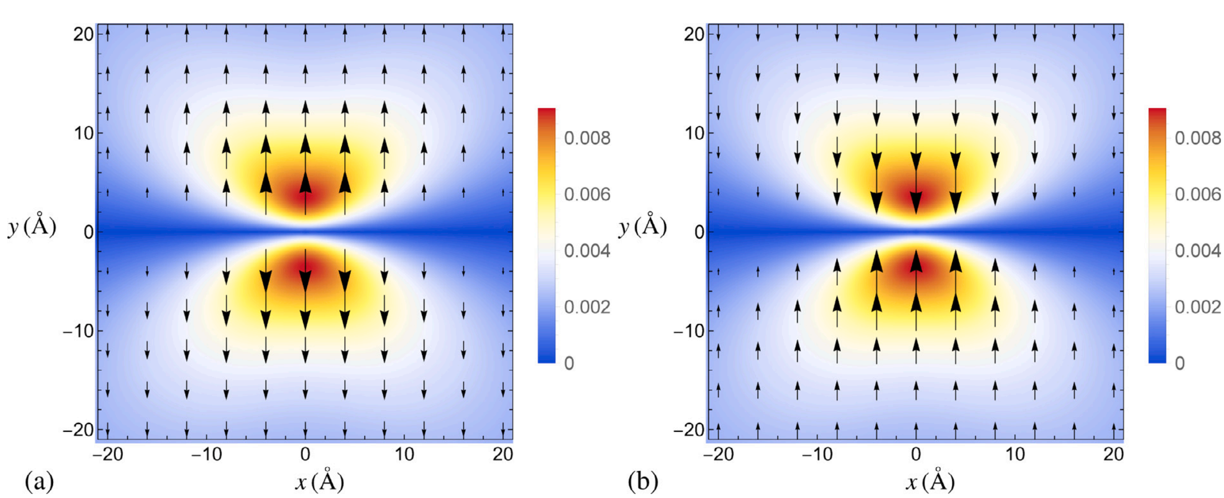

Fig. 18. Climb component (Peach-Koehler force component $\mathcal{F}^{\rm PK}_y$) of two parallel edge dislocations with: (a) $b'_x b_x > 0$, (b) $b'_x b_x < 0$.

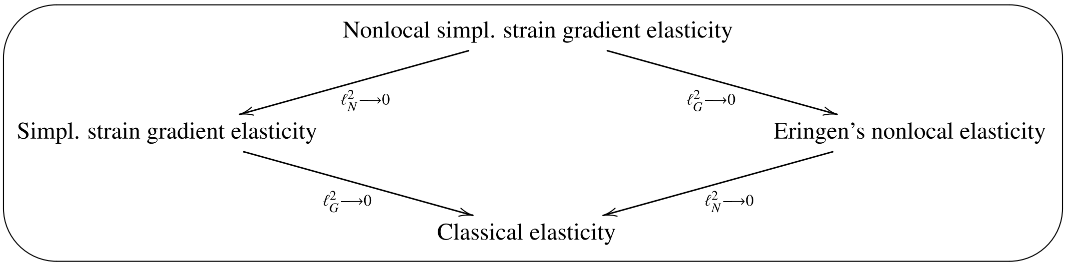

Fig. 19. Limits of nonlocal simplified strain gradient elasticity.

Table 2
Behavior of dislocation fields in simplified strain gradient elasticity, Eringen's nonlocal elasticity and nonlocal simplified strain gradient elasticity.

	Simplified gradient elasticity	Nonlocal elasticity	Nonlocal simplified gradient elasticity
$\boldsymbol{u}$	Smooth and nonsingular	Discontinuous and singular	Smooth and nonsingular
$\boldsymbol{\beta}^{\mathrm{P}}$	Nonsingular	Singular	Nonsingular
$\boldsymbol{\beta}$	Nonsingular	Singular	Nonsingular
$\boldsymbol{\sigma}$	Nonsingular	–	Nonsingular
$\boldsymbol{\tau}$	Singular	–	Nonsingular
$\boldsymbol{t}$	Singular	Nonsingular	Nonsingular

Table 3
Framework of nonlocal simplified strain gradient elasticity.

Geometric fields	Stresses	Constitutive relations
$L_G \boldsymbol{u} = \boldsymbol{u}^0$	$L_G L_N \boldsymbol{\sigma} = \boldsymbol{\sigma}^0$	$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{e} * \alpha$
$L_G \boldsymbol{\beta}^{\mathrm{P}} = \boldsymbol{\beta}^{\mathrm{P},0}$	$L_G L_N \boldsymbol{\tau} = \ell_G^2 \nabla \boldsymbol{\sigma}^0$	$\boldsymbol{\tau} = \ell_G^2 \nabla \boldsymbol{\sigma}$
$L_G \boldsymbol{\beta} = \boldsymbol{\beta}^0$	$L_N \boldsymbol{t} = \boldsymbol{\sigma}^0$	$\boldsymbol{t} = \mathbb{C} : (L_G \boldsymbol{e}) * \alpha$
$L_G \boldsymbol{\alpha} = \boldsymbol{\alpha}^0$		

Table 4
Framework of simplified strain gradient elasticity.

Geometric fields	Stresses	Constitutive relations
$L_G \boldsymbol{u} = \boldsymbol{u}^0$	$L_G \boldsymbol{\sigma} = \boldsymbol{\sigma}^0$	$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{e}$
$L_G \boldsymbol{\beta}^{\mathrm{P}} = \boldsymbol{\beta}^{\mathrm{P},0}$	$L_G \boldsymbol{\tau} = \ell_G^2 \nabla \boldsymbol{\sigma}^0$	$\boldsymbol{\tau} = \ell_G^2 \nabla \boldsymbol{\sigma}$
$L_G \boldsymbol{\beta} = \boldsymbol{\beta}^0$	$\boldsymbol{t} = \boldsymbol{\sigma}^0$	$\boldsymbol{t} = \mathbb{C} : (L_G \boldsymbol{e})$
$L_G \boldsymbol{\alpha} = \boldsymbol{\alpha}^0$		

Table 5
Framework of Eringen's nonlocal elasticity.

Geometric fields	Stresses	Constitutive relation
$\boldsymbol{u} = \boldsymbol{u}^0$	$L_N \boldsymbol{t} = \boldsymbol{\sigma}^0$	$\boldsymbol{t} = \mathbb{C} : \boldsymbol{e} * \alpha$
$\boldsymbol{\beta}^{\mathrm{P}} = \boldsymbol{\beta}^{\mathrm{P},0}$	$\boldsymbol{t} = \boldsymbol{\sigma}$	
$\boldsymbol{\beta} = \boldsymbol{\beta}^0$		
$\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$		

that the Cauchy stress $\boldsymbol{\sigma}$ is nonsingular in nonlocal simplified strain gradient elasticity unlike the singular Cauchy stress in classical elasticity. The Peach-Koehler force depends on the characteristic gradient length ℓ_G and the characteristic nonlocal length ℓ_N in the following way:

- Peach-Koehler force: $\boldsymbol{\mathcal{F}}^{\mathrm{PK}} = \boldsymbol{\mathcal{F}}^{\mathrm{PK}}(\boldsymbol{r}, \ell_G, \ell_N)$.

Nonlocal simplified strain gradient elasticity is the unification of the theories of Eringen's nonlocal elasticity and simplified strain gradient elasticity. It combines the advantages of the theories of Eringen's nonlocal elasticity and simplified strain gradient elasticity. The advantage of nonlocal simplified first strain gradient elasticity in comparison to Eringen's nonlocal elasticity and simplified first strain gradient elasticity is that all relevant dislocation fields are nonsingular (see Table 2). In this way, nonlocal simplified first strain gradient elasticity provides a new way to remove/regularize the singularities in classical dislocation fields.

Nonlocal simplified strain gradient elasticity contains three important limits (see Fig. 19):

$$\ell_N^2 \to 0 : \text{simplified strain gradient elasticity}$$
$$\ell_G^2 \to 0 : \text{nonlocal elasticity}$$
$$\ell_N^2 \to 0,\ \ell_G^2 \to 0 : \text{classical elasticity.}$$

The frameworks of nonlocal simplified strain gradient elasticity, simplified strain gradient elasticity and Eringen's nonlocal elasticity are summarized in Tables 3–5, respectively. The fields with superscript 0 are the singular fields in the framework of classical elasticity.

CRediT authorship contribution statement

Markus Lazar: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Conceptualization.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Data availability

No data was used for the research described in the article.

Acknowledgments

Markus Lazar is grateful to Eleni Agiasofitou for many fruitful discussions. Markus Lazar gratefully acknowledges the grant obtained from the Deutsche Forschungsgemeinschaft (Grant number LA1974/4-2).

Appendix. Boundary conditions in nonlocal first strain gradient elasticity

The non-standard boundary conditions are an important aspect of nonlocal first strain gradient elasticity theory. The boundary conditions (BCs) in nonlocal first strain gradient elasticity were originally given by Lim et al. [39]. The BCs corresponding to Eq. (19) read in the index notation

$$\left.\begin{aligned} t_{ij}n_j - \partial_j\big(\tau_{ijk}n_k\big) + n_j\partial_l\big(\tau_{ijk}n_kn_l\big) &= \bar{t}_i \\ \tau_{ijk}n_jn_k &= \bar{q}_i \end{aligned}\right\} \quad \text{on} \quad \partial\Omega, \tag{A.1}$$

where t_i and q_i are the Cauchy traction vector and the double stress traction vector, respectively. Moreover, $\partial\Omega$ is the smooth boundary surface of the domain Ω occupied by the body satisfying the Euler–Lagrange equation (19), n_i denotes the unit outward-directed vector normal to the boundary $\partial\Omega$, and the overhead bar represents the prescribed value. Using the constitutive equation (14), the BCs (A.1) reduce to

$$\left.\begin{aligned} t_{ij}n_j - \ell_G^2\partial_j\big(n_k\partial_k\sigma_{ij}\big) + \ell_G^2 n_j\partial_l\big(n_ln_k\partial_k\sigma_{ij}\big) &= \bar{t}_i \\ \ell_G^2 n_jn_k\partial_k\sigma_{ij} &= \bar{q}_i \end{aligned}\right\} \quad \text{on} \quad \partial\Omega. \tag{A.2}$$

In the limit $\ell_N^2 \to 0$ that means $\alpha \to \delta(\boldsymbol{x})$, the BCs (A.1) reduce to the BCs in first strain gradient elasticity (see [25,62,63]). In the limit $\ell_G^2 \to 0$, the BCs (A.2) reduce to the BC in Eringen's nonlocal elasticity: $t_{ij}n_j = \bar{t}_i$ (see [1]).

References

[1] Eringen AC. Nonlocal continuum field theories. New York: Springer; 2002.
[2] Mindlin RD. Micro-structure in linear elasticity. Arch Ration Mech Anal 1964;16:51–78.
[3] Askes H, Aifantis EC. Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results. Int J Solids Struct 2011;48:1962–90.
[4] Lazar M, Agiasofitou E, Po G. Three-dimensional nonlocal anisotropic elasticity: a generalized continuum theory of Ångström-mechanics. Acta Mech 2020;231:743–81.
[5] Eringen AC. Screw dislocation in nonlocal elasticity. J Phys D: Appl Phys 1977;10:671–8.
[6] Eringen AC. Edge dislocation in nonlocal elasticity. Internat J Engrg Sci 1977;15:177–83.
[7] Eringen AC. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J Appl Phys 1983;54:4703–10.
[8] Eringen AC. Theory of nonlocal elasticity and some applications. Res Mech 1987;21:313–42.
[9] Lazar M. Peach-Koehler forces within the theory of nonlocal elasticity. In: Steinmann P, Maugin GA, editors. Mechanics of material forces. Berlin: Springer; 2005, p. 149–58.
[10] Lazar M, Maugin GA, Aifantis EC. On dislocations in a special class of generalized elasticity. Phys Status Solidi (b) 2005;242:2365–90.
[11] Lazar M, Agiasofitou E. Nonlocal elasticity of Klein–Gordon type: Fundamentals and wave propagation. Wave Motion 2022;114:103038.
[12] Aifantis EC. On the role of gradients in the localization of deformation and fracture. Internat J Engrg Sci 1992;30:1279–99.
[13] Altan BC, Aifantis EC. On some aspects in the special theory of gradient elasticity. J Mech Behav Mater 1997;8:231–82.
[14] Gutkin MYu, Aifantis EC. Screw dislocation in gradient elasticity. Scr Mater 1996;35:1353–8.
[15] Gutkin MYu, Aifantis EC. Edge dislocation in gradient elasticity. Scr Mater 1997;36:129–35.
[16] Lazar M, Maugin GA. Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity. Internat J Engrg Sci 2005;43:1157–84.
[17] Lazar M. On gradient field theories: gradient magnetostatics and gradient elasticity. Phil Mag 2014;94:2840–74.
[18] Lazar M. Incompatible strain gradient elasticity of Mindlin type: screw and edge dislocations. Acta Mech 2021;232:3471–94.
[19] Lazar M, Po G. On Mindlin's isotropic strain gradient elasticity: Green tensors, regularization, and operator-split. J Micromech Mol Phys 2018;3(3 & 4):1840008.
[20] Mindlin RD, Eshel NN. On first strain gradient theory in linear elasticity. Int J Solids Struct 1968;4:109–24.
[21] Lazar M, Kirchner HOK. The Eshelby stress tensor, angular momentum tensor and dilatation flux in gradient elasticity. Int J Solids Struct 2007;44:2477–86.
[22] Aifantis EC. A note on gradient elasticity and nonsingular crack fields. J Mech Behav Mater 2011;20:103–5.
[23] Mousavi SM, Lazar M. Distributed dislocation technique for cracks based on non-singular dislocations in nonlocal elasticity of Helmholtz type. Eng Fract Mech 2015;136:79–95.
[24] Mousavi SM, Aifantis EC. A note on dislocation-based mode III gradient elastic fracture mechanics. J Mech Behav Mater 2015;24:115–9.
[25] Mousavi SM, Aifantis EC. Dislocation-based gradient elastic fracture mechanics for in-plane analysis of cracks. Int J Fract 2016;202:93–110.
[26] Aifantis EC. Internal length gradient (ILG) material mechanics across scales and disciplines. Adv Appl Mech 2016;49:1–110.
[27] Aifantis EC. A concise review of gradient models in mechanics and physics. Front Phys 2020;7:239.
[28] Aifantis EC. Gradient extension of classical material models: from nuclear & condensed matter scales to earth & cosmological scales. In: Springer tracts in mechanical engineering. 2021, p. 417–52.
[29] Lazar M. Green functions and propagation in the Bopp-Podolsky electrodynamics. Wave Motion 2019;91:102388.
[30] Lazar M. Gradient modification of Newtonian gravity. Phys Rev D 2020;102:096002.
[31] Gutkin MYu, Aifantis EC. Dislocations in gradient elasticity. Scr Mater 1999;40:559–66.
[32] Gutkin MYu. Nanoscopics of dislocations and disclinations in gradient elasticity. Rev Adv Mater Sci 2000;1:27–60.
[33] Gutkin MYu. Elastic behaviour of defects in nanomaterials I. Rev Adv Mater Sci 2006;13:125–61.
[34] Aifantis EC. On the gradient approach–relation to Eringen's nonlocal theory. Internat J Engrg Sci 2011;49:1367–77.
[35] Aifantis EC. On non-singular GRADELA crack fields. Theor Appl Mech Lett 2014;4:051005.
[36] Fafalis DA, Filopoulos SP, Tsamasphyros GJ. On the capability of generalized continuum theories to capture dispersion characteristics at the atomic scale. Eur J Mech A Solids 2012;36:25–37.
[37] Tsamasphyros GJ, Koutsoumaris CC. Mixed nonlocal-gradient elastic materials with applications in wave propagation of beams. AIP Conf Proc 2016;1790:150031.
[38] Ru CQ, Aifantis EC. A simple approach to solve boundary-value problems in gradient elasticity. Acta Mech 1993;101:59–68.
[39] Lim CW, Zhang G, Reddy JN. A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation. J Mech Phys Solids 2015;78:298–313.
[40] Zaera R, Serrano Ó, Fernández-Sáez J. On the consistency of the nonlocal strain gradient elasticity. Internat J Engrg Sci 2019;138:65–81.
[41] Jiang Y, Li L, Hu Y. A physically-based nonlocal strain gradient theory for crosslinked polymers. Int J Mech Sci 2023;245:108094.
[42] Gel'fand IM, Shilov GE. Generalized functions, Vol. I, New York: Academic; 1964.
[43] Eringen AC. Plane waves in nonlocal micropolar elasticity. Internat J Engrg Sci 1984;22:1113–21.
[44] Zeng X, Chen Y, Lee JD. Determining material constants in nonlocal micromorphic theory through phonon dispersion relations. Internat J Engrg Sci 2006;44:1334–45.
[45] Kröner E. Kontinuumstheorie der Versetzungen und Eigenspannungen. Berlin: Springer; 1958.
[46] Frankel T. The geometry of physics – An introduction. 3rd ed. Cambridge: Cambridge University Press; 2012.
[47] Lazar M. The fundamentals of non-singular dislocations in the theory of gradient elasticity: Dislocation loops and straight dislocations. Int J Solids Struct 2013;50:352–62.
[48] Maugin GA. Non-classical continuum mechanics: a dictionary. Singapore: Springer; 2017.
[49] Toupin RA, Grazis DC. Surface effects and initial stress in continuum and lattice models of elastic crystals. In: Wallis RF, editor. Proceedings of the international conference on lattice dynamics. Copenhagen, Pergamon Press; 1964, p. 597–602.
[50] Maugin GA. Material inhomogeneities in elasticity. London: Chapman and Hall; 1993.
[51] Rothwell EJ, Cloud MJ. Electromagnetics. 2nd ed. Boca Raton: CRC Press; 2009.
[52] Aifantis EC. Update on a class of gradient theories. Mech Mater 2003;35:259–80.
[53] Lazar M, Agiasofitou E. On wave dispersion in nonlocal simplified strain gradient elasticity. 2024, submitted.
[54] Lazar M, Maugin GA, Aifantis EC. Dislocations in second strain gradient elasticity. Int J Solids Struct 2006;43:1787–817.
[55] Lazar M. Reduced strain gradient elasticity model with two characteristic lengths: fundamentals and application to straight dislocations. Contin Mech Thermodyn 2022;34:1433–54.
[56] Lazar M, Agiasofitou E, Böhlke T. Mathematical modeling of the elastic properties of cubic crystals at small scales based on the Toupin-Mindlin anisotropic first strain gradient elasticity. Contin Mech Thermodyn 2022;34:107–36.
[57] deWit R. Theory of disclinations IV. J Res Natl Bur Stand 1973;77A:607–58.
[58] Mura T. Micromechanics of defects in solids. 2nd ed. Dordrecht: Martinus Nijhoff; 1987.
[59] Lazar M, Maugin GA. Dislocations in gradient elasticity revisited. Proc R Soc Lond Ser A Math Phys Eng Sci 2006;462:3465–80.
[60] Lazar M, Kirchner HOK. The Eshelby tensor in nonlocal elasticity and in nonlocal micropolar elasticity. J Mech Mater Struct 2006;1:325–37.
[61] Hirth JP. Theory of dislocations. 2nd ed. New York: Wiley; 1982.
[62] Gao X-L, Park SK. Variational formulation of a simplified strain gradient elasticity theory and its application to a pressurized thick-walled cylinder problem. Int J Solids Struct 2007;44:7486–99.
[63] Jaunzemis W. Continuum mechanics. New York: The Macmillan Company; 1967.