Compressible and immiscible fluids with arbitrary density ratio

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For the water-air system, the density ratio is as high as about 1000; no model can fully tackle such a high density ratio system. Here, we present an alternative theory for the density evolution equations in computational fluid dynamics, differing from the concept of Navier-Stokes and Euler equations. Our derivation is built upon the physical principle of energy minimization from the aspect of thermodynamics. The present results are consistent with Landau's theory of sound speed, and, most importantly, provide a generalization of Bernoulli's principle for energy conservation. The present model can be applied for immiscible fluids with arbitrary density ratios, thereby, opening a new window for computational fluid dynamics both for compressible and incompressible fluids.

Introduction. Immiscible fluids with a high density ratio exist everywhere in our daily lives [1]. A typical example is the water-air system, where the density ratio is about 1000. In the classic fluid dynamics, for the waterair system, the density is either assumed to be a constant according to the Boussinesq approximation [2, 3] or interpolated over the composition [4–6]. The latter consideration leads to the paradox of density interpolation. In these considerations, the variation of the density at the water-air interface has not yet been resolved, preventing us from a quantitative understanding on the fundamentals of interfacial fluid dynamics for immiscible fluids. Some recent attempts [7–9] have been made to shed light on the density ratio problem.

Another daily example for the density variation is evaporation of water and condensation of vapour. The effect of diffusion and convection in the evaporation and condensation processes cannot be coped by the Navier-Stokes equations [8]. Even for pure fluids, like gas phase, when the fluid velocity is close to the speed of sound, the density varies in space with time (see Landau's theory of sound wave [10]). The density evolution equation for high speed fluids is beyond the scope of Euler and Navier-Stokes equations.

In this Letter, we will propose a generalized theory for the evolution equations of fluids. Our concept is based on the physical principle of energy minimization. We will derive a novel kinetic equation for the density at the water-air interface. In special cases, the present results are shown to be well consistent with Landau's theory of sound speed for inviscid fluids. The derived density evolution equation also provides a generalization of Bernoulli's law for the conservation of potential energy plus kinetic energy. We show that for inviscid fluids, the classic Bernoulli's equation loses its validity when there is a change in the density at the droplet-air interface.

Setup. We consider an isothermal closed system consisting with two immiscible phases, say α and β . These two phases consist of $K \in \mathbb{Z}$ components in a domain Ω . The volume concentration of component i $(i = 1, 2, \dots, K)$ is denoted by $\phi_i(\mathbf{r}), \forall \mathbf{r} \in \Omega$, leading to a concentration vector $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_K)$. The boundary of the domain Ω is denoted by Γ . For a binary system, we use the notation $\phi_1 = \phi, \phi_2 = 1 - \phi$. The density of the mixture at any position $\mathbf{r} \in \Omega$ is denoted by $\rho(\mathbf{r})$.

Paradox of the density interpolation. For a binary sys-



FIG. 1. Origin of the pressure energy for uniform and nonuniform systems. For a uniform system (left), the pressure at any position is the same, which is taken as a reference value. In a non-uniform system with density variation (right), the pressure is generally non-uniform due to the asymmetric collisions of different sized molecules. Green and red spheres depict two molecules or atoms with distinct sizes, leading to a variation of the density in space. The arrows sketch the thermal motion of the molecules/atoms.

tem, the density of a mixture is often considered to be a function of the composition ϕ , namely $\rho = \rho(\phi)$. In such a consideration, it implies that the time evolution of the density is defined when the time evolution of the composition is given; the system is over-defined if both ρ and ϕ are considered to be unknowns. As an example, we discuss the linear interpolation of the density over the composition for a binary system as

$$\rho(\phi) = \rho_1 \phi + \rho_2 (1 - \phi), \tag{1}$$

where ρ_1 and ρ_2 are respectively the density of the species 1 and 2, which are constants but generally $\rho_1 \neq \rho_2$. According to the definition of density, we have the following expression for the mixture density

$$\rho = \frac{\rho_1 v_1 + \rho_2 v_2}{V} = \rho_1 \frac{v_1}{V} + \rho_2 \frac{v_2}{V}, \qquad (2)$$

where $V \subset \Omega$ is a volume element and expressed as $V = v_1 + v_2 + v_e$; v_e depicts the excess volume and v_i (i = 1, 2) is the volume of the *i* species. The volume element includes sufficient amount of fluid particles for statistic thermodynamics. By assuming a zero excess volume, i.e., $v_e = 0$, we have $V = v_1 + v_2$. Substituting this expression into Eq. (2) and using the definition of volume concentration $\phi_i = v_i/V$, Eq. (1) is replicated. Suffice to say, the linear interpolation in Eq. (1) indicates a zero excess volume of mixing throughout the system. Any other kinds of interpolations define a certain excess volume, which is uniform everywhere in the domain Ω . The assumption of uniform excess volume (either zero or non-zero) deviates from the physical fact that the local excess volume v_e is affected by the local pressure, which should be space dependent; the excess volume at each position in the domain should have its own characteristics. An additional paradox in the interpolation concept is the conflict of incompressibility with mass diffusion. The incompressibility requires $\dot{\rho} = 0$. When diffusion takes place, i.e., $\phi \neq 0$, the density evolution based on the linear interpolation is $\dot{\rho} = (\rho_1 - \rho_2)\phi \neq 0$. In the following, we will shed light on this density paradox from the physical principle of energy minimization.

System energy. We propose that the total energy of the system consists of potential free energy \mathcal{F} , potential pressure energy \mathcal{P} , macroscopic kinetic energy \mathcal{K} , and wall free energy \mathcal{W} , reading

$$\mathcal{E} = \mathcal{F} + \mathcal{P} + \mathcal{K} + \mathcal{W}.$$
 (3)

As derived by Cahn [11] and van der Waals [12], the potential free energy $\mathcal{F}(\phi, \nabla \phi)$ is written as the integration over the non-local potential energy density $e(\phi, \nabla \phi)$ as [13]

$$\mathcal{F}(\boldsymbol{\phi}, \nabla \boldsymbol{\phi}) = \int_{\Omega} e(\boldsymbol{\phi}, \nabla \boldsymbol{\phi}) d\Omega, \tag{4}$$

$$e(\boldsymbol{\phi}, \nabla \boldsymbol{\phi}) = f(\boldsymbol{\phi}) + \sum_{i=1}^{K} \frac{1}{2} \kappa_i (\nabla \phi_i)^2, \qquad (5)$$

where $f(\phi)$ depicts the bulk free energy density and κ_i represents the gradient energy coefficient which is related to the interfacial tension of the α - β interface, $\sigma_{\alpha\beta}$.

The $\mathcal{P} = \int_{\Omega} p d\Omega$ term, often overlooked previously and introduced here, describes the potential pressure energy. We formulate the pressure energy density as

$$-p = e - \sum_{i=1}^{K} \mu_i \phi_i, \tag{6}$$

where μ_i is the chemical potential of species *i*, the formulation of which will be derived in the following via the energy dissipation principle. Herein, an essential consideration is that the pressure p is always associated with the potential energy \mathcal{F} . A physical interpretation of the pressure is shown in Fig. 1. A homogenous mixture of two distinct species (Fig. 1a) leads to a uniform pressure $p_0(\mathbf{r}) = \text{const}, \ \forall \mathbf{r} \in \Omega$, which can be taken as a reference value. However, when the two species are not homogeneously mixed (Fig. 1b), the pressure is nonuniform in space; the non-uniform pressure is caused, for example, by the collisions of different molecules with distinct sizes due to the asymmetry in space. As considered in Dalton's law, the partial pressure from different phases is additive, namely $p = p_{\alpha}\omega_{\alpha} + p_{\beta}\omega_{\beta}$, where ω_{θ} $(\theta = \alpha, \beta)$ depicts the volume fraction of phase θ . In the special case of a homogenous mixture, the pressure defined in Eq. (6) is consistent with the Landau potential, $-p = f(\phi) - \sum_{i=1}^{K} (\partial f / \partial \phi_i) \phi_i$. The difference in the Landau potential of different phases is the driving force for the movement of the α - β interface, e.g., phase transformation. A mathematical derivation of Eq. (6)is similar to that for Landau potential; the derivation is based on the fact that the internal energy of the system is an extensive variable.

The \mathcal{K} term denotes the macroscopic kinetic energy, which is formulated as

$$\mathcal{K} = \int_{\Omega} \boldsymbol{\zeta} \cdot \mathbf{u} d\Omega, \tag{7}$$

where $\boldsymbol{\zeta}$ is the linear momentum density expressed in the classic physics as $\boldsymbol{\zeta} = \rho \mathbf{u}$. We present three remarks for the kinetic energy: (I) One should not write the kinetic energy density as $\frac{1}{2}\rho \mathbf{u}^2$; the reason will be elucidated in the following. (II) The macroscopic kinetic energy should not be confused with the microscopic kinetic energy, which has already been considered in the potential energies, \mathcal{F} and \mathcal{P} . (III) One should not interpolate the velocity via $\mathbf{u} = \mathbf{u}_1\phi_1 + \mathbf{u}_2\phi_2$; this interpolation leads to an artificial contribution to the kinetic energy $\rho \mathbf{u}_1 \cdot \mathbf{u}_2$ [14]. In other words, the fluid velocity is not a thermodynamically extensive variable.

The \mathcal{W} term in Eq. (3) represents the wall free energy, which is integrated over the wall free energy density f_w on the boundary Γ as $\mathcal{W} = \int_{\Gamma} f_w d\Gamma$. The formulation of the wall free energy density f_w has been comprehensively discussed elsewhere [14–16] and is not the scope of this work.

Energy dissipation. We derive the equilibrium condition and the kinetic equations based on the principle of energy minimization, namely

$$\frac{d\mathcal{E}}{dt} = \dot{\mathcal{E}} \le 0.$$

Next, we evaluate the total time derivative of the free energy \mathcal{F} , the pressure energy \mathcal{P} , and the kinetic energy \mathcal{K} . By using the chain rule and noting that the functional \mathcal{F} has two arguments, ϕ and $\nabla \phi$, the total time derivative of the potential free energy \mathcal{F} reads

$$\dot{\mathcal{F}}(\boldsymbol{\phi}, \nabla \boldsymbol{\phi}) = \int_{\Omega} \sum_{i=1}^{K} \frac{\partial e}{\partial \phi_i} \frac{d\phi_i}{dt} + \frac{\partial e}{\partial \nabla \phi_i} \cdot \frac{d\nabla \phi_i}{dt} d\Omega.$$
(8)

By using the vector calculus $\nabla \cdot (\nabla \phi_i \otimes \nabla \phi_i) = \nabla^2 \phi_i \nabla \phi_i + (\nabla \phi_i \cdot \nabla) \nabla \phi_i$, the definition of materials derivative $\nabla \phi_i = \partial_t \nabla \phi_i + \mathbf{u} \cdot \nabla \nabla \phi_i$, and integration by parts with no-flux boundary condition for a closed system (more details can be found in Refs. [14, 17]), we obtain the following equations

$$\dot{\mathcal{F}} = \int_{\Omega} \sum_{i=1}^{K} \mu_i \dot{\phi}_i + (\nabla \cdot \underline{\underline{\Theta}}) \cdot \mathbf{u} d\Omega; \tag{9}$$

$$\mu_i = \frac{\partial e}{\partial \phi_i} - \nabla \cdot \frac{\partial e}{\partial \nabla \phi_i}; \tag{10}$$

$$\underline{\underline{\Theta}} = \sum_{i} \frac{\partial e}{\partial \nabla \phi_i} \otimes \nabla \phi_i.$$
(11)

The first part of Eq. (9) defines a generalized formulation for the chemical potential, as stated in Eq. (10). At thermodynamic equilibrium, the chemical potential is a constant value throughout the system. In a special case of a zero chemical potential, i.e., $\mu_i = 0$, Eq. (10) is equivalent to the Euler-Lagrange equation; Eq. (10) also provides a physical interpretation for the variational derivative in mathematics. The second part of Eq. (9) defines a stress tensor $\underline{\Theta}$, which is responsible for the transformation of the potential free energy into the macroscopic kinetic energy [18]. This stress tensor is consistent with the Korteweg stress by evaluating the derivative via considering Eq. (5), namely $\partial_{\nabla \phi_i} e = \kappa_i \nabla \phi_i$.

Noteworthily, a uniform chemical potential throughout the system cannot well define the thermodynamic equilibrium; the pressure evolution should be considered as well. An evaluation of the time derivative for the pressure energy and the kinetic energy leads to

$$\dot{\mathcal{P}} = \int_{\Omega} \frac{dp(\mathbf{r})}{dt} d\Omega = \int_{\Omega} \nabla p \cdot \dot{\mathbf{r}} d\Omega = \int_{\Omega} \nabla \cdot (p\mathbf{I}) \cdot \mathbf{u} d\Omega,$$
(12)

$$\dot{\mathcal{K}} = \int_{\Omega} (\dot{\boldsymbol{\zeta}} \cdot \mathbf{u} + \boldsymbol{\zeta} \cdot \dot{\mathbf{u}}) d\Omega, \tag{13}$$

where **I** is an identity tensor. For the derivation of Eq. (12), we have used the chain rule and the definition of the velocity, $\dot{\mathbf{r}} = \mathbf{u}$; for the derivation of Eq. (13), the chain rule of total derivative has been applied. Summarizing Eq. (9), Eq. (10), Eq. (12), and Eq. (13) and collecting all the $\cdot \mathbf{u}$ terms, we obtain the total time derivative of the system energy \mathcal{E} as

$$\dot{\mathcal{E}} = \int_{\Omega} \sum_{i=1}^{K} \underbrace{\mu_i \dot{\phi_i}}_{\mathrm{I}} + \underbrace{\left[\nabla \cdot (\underline{\underline{\Theta}} + p\mathbf{I}) + \dot{\zeta}\right]}_{\mathrm{II}} \cdot \mathbf{u} + \underbrace{\zeta \cdot \dot{\mathbf{u}}}_{\mathrm{III}} d\Omega.$$
(14)

The physical interpretation for the three parts of the energy evolution in Eq. (14) is sketched in Fig. 2.

Before further analyzing the energy dissipation, we state the following dissipation-conservation theorem which connects the first law and the second law of thermodynamics.

I: Diffusion dissipation due to exchange of atoms and vacancies



FIG. 2. The energy dissipation mechanisms in Eq. (14). I: The diffusion via the exchange of the molecules and vacancies leads to the dissipation of the free energy \mathcal{F} . II: The gradient of the free energy \mathcal{F} and the associated pressure \mathcal{P} result in an increase in the momentum $\boldsymbol{\zeta} = \rho \mathbf{u}$. III: The difference in the velocity, density, and viscosity in distinct subdomains (dashed rectangle) gives rise to the dissipation of the velocity \mathbf{u} caused by the "spring" drag effect; the spring symbol denote the interaction between the atoms/molecules in different subdomains.

Dissipation-conservation theorem. In a closed system, for a conserved scalar field ϕ and the conjugate potential μ , the energy dissipation $\int_{\Omega} \mu \dot{\phi} d\Omega = -\int_{\Omega} M (\nabla \mu)^2 \leq 0$ is equivalent to the conservation equation $\dot{\phi} = \nabla \cdot M \nabla \mu$. Here, M is a positive mobility. The proof is straightforward by using integration by parts $\int_{\Omega} \mu \dot{\phi} d\Omega = \int_{\Omega} \mu \nabla \cdot M \nabla \mu d\Omega = \int_{\Gamma} \mu M \nabla \mu \cdot \mathbf{n} d\Gamma - \int_{\Omega} M (\nabla \mu)^2 d\Omega$ and the noflux boundary condition, $\nabla \mu \cdot \mathbf{n} = 0$. The dissipationconservation theorem shows that the conservation and dissipation are not independent, but always associated.

According to the dissipation-conservation theorem, the first part in Eq. (14), I, defines the generalized diffusion-convection equation as

$$\dot{\phi}_i = \nabla \cdot \varsigma_i \nabla \mu_i. \tag{15}$$

The coefficient ς_i is the mobility subjected to the Gibbs-Duhem constraint; see Ref. [19] for the derivation of the mobility. For a binary case, this equation returns to the Cahn-Hilliard equation.

The second part in Eq. (14), II, reveals the momentum conservation

$$\dot{\boldsymbol{\zeta}} = \frac{d(\rho \mathbf{u})}{dt} = -\nabla \cdot (\underline{\boldsymbol{\Theta}} + p\mathbf{I}). \tag{16}$$

The conservative force at the right hand side is responsible for the exchange of the *microscopic* potential energy $\mathcal{F} + \mathcal{P}$ with the *macroscopic* kinetic energy \mathcal{K} . This result is nothing but Newton's law of momentum balance. Here, a more clear physical meaning of the force is presented; the force term consists of the pressure p and the stress tensor $\underline{\Theta}$. The stress tensor is due to the gradient term in the potential energy \mathcal{F} , while the pressure is contributed by the associated free energy.

The only term left in the time evolution of system energy, III, is $\int_{\Omega} \boldsymbol{\zeta} \cdot \dot{\mathbf{u}} d\Omega$. By applying the same dissipation principle as done for the scalar field in Eq. (15), we obtain the dissipation equation for the velocity as

$$\dot{\mathbf{u}} = \nabla \cdot \tau_1 (\nabla \boldsymbol{\zeta} + \nabla \boldsymbol{\zeta}^T) + \nabla \cdot \tau_2 (\nabla \cdot \boldsymbol{\zeta}) \mathbf{I}.$$
(17)

Unlike the dissipation rule for the scalar variable ϕ_i with the conjugate potential $\nabla \mu_i$ (see Eq. (15)), three terms $\nabla \boldsymbol{\zeta}, \nabla \boldsymbol{\zeta}^T$, and $(\nabla \cdot \boldsymbol{\zeta})\mathbf{I}$ have to be taken into account for the dissipation of a vector \mathbf{u} [10] (for the dissipation of a tensor field, like strain tensor, one has to consider Oldroyd derivative [20]).

The right hand side of Eq. (17) is equivalent to Newton viscous stress tensor if we define the mobilities τ_1 and τ_2 as

$$\tau_1 = \eta/\rho^2, \ \tau_2 = \lambda/\rho^2, \tag{18}$$

where η and λ are the shear and bulk viscosities [22], respectively.

Another formulation for the kinetic energy is $\mathcal{K}' = \int_{\Omega} \frac{1}{2} \rho \mathbf{u}^2 d\Omega$. In this case, the time derivative of the kinetic energy reads

$$\frac{d\mathcal{K}'}{dt} = \int_{\Omega} \frac{1}{2} \frac{d\rho}{dt} \mathbf{u}^2 + \rho \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} d\Omega.$$
(19)

The second part leads to the classic Navier-Stokes equation by applying the dissipation-conservation theorem to the momentum vector $\rho \mathbf{u}$. A noteworthy shortcoming for the formulation of Eq. (19) is the missing information of the momentum balance as well as the transformation between the potential and kinetic energies. Moreover, the first part in Eq. (19) requires a prior knowledge about the time evolution of the density, which is unknown for two immiscible phases. The density evolution can be avoided if the kinetic energy is written as $\rho \mathbf{u} \cdot \mathbf{u} = \boldsymbol{\zeta} \cdot \mathbf{u}$; in this case, the time evolution of the momentum $\boldsymbol{\zeta}$ as a whole term is sufficient, subjected to the momentum conservation. Noteworthily, the so-called continuity equation $\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$ cannot be applied here for density evolution. The reason is as follows: (I) The classic continuity equation is derived for pure fluid, where there is no density variation in space. (II) When considering two immiscible fluids, the classic continuity equation cannot be straightforwardly adopted, since the density varies across the interface. In this case, the mass conservation and energy dissipation are not independent. Suffice to say, one cannot directly apply a mass conservation without considering the energy dissipation across the interface. (III) At the interface of two immiscible fluids, one cannot use the condition $\nabla \cdot \mathbf{u} = 0$ which is used for pure fluids. The divergence free condition $\nabla \cdot \mathbf{u} = 0$ implies a uniform excess volume everywhere in the whole domain Ω and that the mass is uniformly distributed in the speed of sound. In other words, when the fluid velocity is much less than the speed of sound, the density almost has no variation in space; only in this case, the condition $\nabla \cdot \mathbf{u} = 0$ can be applied. For pure homogenous fluids, the condition $\nabla \cdot \mathbf{u} = 0$ is used to solve the unknown pressure p_0 , which is homogenous and can be considered as a reference value (Fig. 1a). However, for two immiscible fluids, the pressure is inhomogeneous across the interface, which cannot be solved by $\nabla \cdot \mathbf{u} = 0$ due to the density variation.

Density evolution equation and generalization of Bernoulli's law. Eq. (16) subtracting Eq. (17) with the relation $\dot{\boldsymbol{\zeta}} = \dot{\rho} \mathbf{u} + \rho \dot{\mathbf{u}}$, we obtain the temporal equation for the density as

$$\mathbf{u}^{2}\dot{\rho} = \{\nabla \cdot [(p\mathbf{I} + \underline{\underline{\Theta}}) + \tau_{1}(\nabla \boldsymbol{\zeta} + \nabla \boldsymbol{\zeta}^{T}) + \tau_{2}(\nabla \cdot \boldsymbol{\zeta})\mathbf{I}]\} \cdot \mathbf{u}.$$
(20)

As demonstrated in the following, Eq. (20) is consistent with Landau's theory of sound speed and provides a generalization of Bernoulli's law.

For inviscid flow, there is no energy dissipation for kinetic energy, i.e., $\tau_1 = \tau_2 \approx 0$. In this case, Eq. (20) is rewritten as

$$\mathbf{u}^{2}\frac{d\rho}{dt} = \nabla \cdot (p\mathbf{I} + \underline{\Theta}) \cdot \mathbf{u}.$$
 (21)

By using the definition of the velocity and pressure gradient in one dimension, u = dr/dt, and $\nabla p = dp/dr$, we obtain the relation between velocity, pressure, and density as

$$u^2 = \left(\frac{dp}{d\rho}\right)_s,\tag{22}$$

where the stress tensor has been overlooked. The result stated by Eq. (22) is fully consistent with the definition of sound speed. The condition for this definition is that there is no energy dissipation, namely, isentropic, as indicated by the subscription s denoting the isentropic condition. This result coincides with Landau's theory of sound speed [10].

Considering inviscid fluids via setting $\tau_1 = \tau_2 \approx 0$, and the steady state, $\dot{\rho} = \mathbf{u} \cdot \nabla \rho$ where we set $\partial_t \rho = 0$, we rewrite Eq. (20) as

$$\mathbf{u}^{2}(\mathbf{u}\cdot\nabla\rho) = \left[\nabla\cdot\left(p\mathbf{I}+\underline{\Theta}\right)\right]\cdot\mathbf{u}.$$
 (23)

By considering an one-dimensional setup and using the relation $d(\rho u^2) = u^2 d\rho + 2\rho u du$, Eq. (23) is further simplified as

$$d(p + \rho u^2) = 2ud\rho. \tag{24}$$

This result is nothing but a generalized Bernoulli's equation. When there is no density variation, namely, $d(p + \rho u^2) = 0$, we have $p + \rho u^2 = \text{constant}$, which is Bernoulli's principle [23, 24]. However, when density varies, $d\rho \neq 0$, the associated energy is changed as well; in this case, the classic Bernoulli's equation loses its validity.

Discussion. The key point of the present work is the energy minimization based on an alternative formulation for the macroscopic kinetic energy, $\rho \mathbf{u} \cdot \mathbf{u}$, rather than $\frac{1}{2}\rho \mathbf{u}^2$. In our derivation, the dissipation of the kinetic energy is divided into two parts. The first part is the momentum conservation, $d(\rho \mathbf{u})/dt$ which changes not only the fluid velocity \mathbf{u} but also the density ρ . This derivation differs from Euler and Navier-Stokes equations where the momentum variation is only caused by the evolution of the fluid velocity. The second part of the kinetic energy dissipation is the velocity dissipation due to the difference in the density, viscosity, and velocity, as explained in Fig. 2III.

Another vital point of the current work is the pressure energy, the formulation of which is consistent with many pioneers [13]. The physical meaning of the pressure has been elucidated in Fig. 1. The minimization of the pressure energy leads to the so-called grand potential model [25]. In the lattice model, the free energy and the pressure can be explicitly formulated according to the theory of thermodynamics, such as Landau potential. However, for non-lattice models, the pressure cannot be explicitly identified. This case is the consideration in Euler and Navier-Stokes equations, where the unknown pressure is solved by the divergence free of the velocity for incompressible flow. For liquid and solid phases, the lattice model can be applied because of the regularized order of the structure within the short range. For gas phases, the lattice model loses its validity due to the disorder of the structure; one has to use the equation of state to formulate the partial pressure of the gas phase [7, 26].

Additional remark is the dissipation of the velocity. Following the dissipation-conservation principle, the dissipation of the velocity equals to $\nabla \cdot \tau_1 \nabla \boldsymbol{\zeta} = \nabla \cdot \tau_1 \nabla (\partial \mathcal{K}/\partial \mathbf{u})$, where $\boldsymbol{\zeta} = \partial \mathcal{K}/\partial \mathbf{u}$ is analogous to the chemical potential $\mu = \partial \mathcal{F}/\partial \phi$ with the associated dissipation term $\nabla \cdot \varsigma \nabla \mu = \nabla \cdot \varsigma \nabla (\partial \mathcal{F}/\partial \phi)$. Within the context of the dissipation-conservation theorem, the Newton viscous stress tensor is obtained when defining $\tau_1 = \eta/\rho^2$. The question is that if the velocity is a conserved variable or not. If not, the dissipation has to be modified by the non-conserved form as

$$\dot{\mathbf{u}} = -\tau \boldsymbol{\zeta},\tag{25}$$

where τ is a positive mobility. The non-conserved form is known as the Allen-Cahn type dissipation following the gradient descendent path, being consistent with the Langevin equation. In the current work, we consider the linear momentum $\boldsymbol{\zeta} = \rho \mathbf{u}$. Non-linear terms in terms of fluid velocity may be accounted for the dissipation if we consider non-linear momentum, for example, $\boldsymbol{\zeta} \propto \mathbf{u}^n$ or $\boldsymbol{\zeta} \propto (\nabla \mathbf{u})^n$. The non-linear momentum not only modifies the velocity dissipation but also changes the momentum balance, which may be applied for understanding highly nonlinear effects, such as turbulence. These specialized cases are out of the scope of the present discussion.

In conclusion, we have proposed an alternative theory differing from Navier-Stokes and Euler equations to address the high density ratio problem in fluid dynamics. Our consideration is based on the basic principle of energy minimization. A noteworthy outcome of the present work is a novel evolution equation for the high density ratio system, such as water-air. For inviscid fluids, our derivation of the density evolution is fully consistent with Landau's theory of sound speed. Our derivation also leads to a generalization of Bernoulli's law for steadystate flow when there is a density variation in space. The current work reshapes the understanding of momentum conservation in Newton's law. We expect that the present model can be used for immiscible fluids with arbitrary density ratios as well as for pure fluids when the fluid velocity is close to the speed of sound.

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