Efficient Recognition of Subgraphs of Planar Cubic Bridgeless Graphs

Miriam Goetze ✉ 🄔
Karlsruhe Institute of Technology, Germany

Paul Jungeblut ✉ 🄔
Karlsruhe Institute of Technology, Germany

Torsten Ueckerdt ✉
Karlsruhe Institute of Technology, Germany

───── **Abstract** ─────

It follows from the work of Tait and the Four-Color-Theorem that a planar cubic graph is 3-edge-colorable if and only if it contains no bridge. We consider the question of which planar graphs are subgraphs of planar cubic bridgeless graphs, and hence 3-edge-colorable. We provide an efficient recognition algorithm that given an n-vertex planar graph, augments this graph in $\mathcal{O}(n^2)$ steps to a planar cubic bridgeless supergraph, or decides that no such augmentation is possible. The main tools involve the Generalized (Anti)factor-problem for the fixed embedding case, and SPQR-trees for the variable embedding case.

1 Introduction

Whether or not the 3-Edge Colorability-problem is solvable in polynomial time for planar graphs is one of the most fundamental open problems in algorithmic graph theory:

▶ **Question 1.** *Can we decide in polynomial time, whether the edges of a given planar graph can be colored in three colors such that any two adjacent edges receive distinct colors?*

In other words, can we decide for a planar graph G in polynomial time whether $\chi'(G) \leq 3$, where $\chi'(G)$ denotes the chromatic index of G? Clearly, it is enough to consider planar graphs G of maximum degree $\Delta(G) = 3$. If G is planar and 3-regular, then by the Four-Color-Theorem [1, 2] and the work of Tait [26] we know that G is 3-edge-colorable if and only if G is bridgeless. An edge is a bridge if its removal increases the number of connected components (note that this definition also applies to disconnected graphs). As we can check the existence of bridges in linear time [28], we hence can decide in polynomial time whether a given 3-regular planar graph is 3-edge-colorable.

In particular, subgraphs of bridgeless 3-regular planar graphs are 3-edge-colorable. However, this does not answer Question 1 yet (as sometimes wrongly claimed, e.g., in [7]), because it is for example not clear which planar graphs of maximum degree 3 are subgraphs of bridgeless 3-regular planar graphs, and whether these can be recognized efficiently.

In this paper we consider the corresponding decision problem: Given a graph G, is there a bridgeless 3-regular planar graph H, such that $G \subseteq H$? In other words, can G be augmented, by adding edges and (possibly) vertices, to a supergraph H of G that is planar, 3-regular, and contains no bridge? For brevity we call such a supergraph H a 3-*augmentation* of G and denote the above decision problem as 3-Augmentation. Our main result is that 3-Augmentation is in P.

(a) A planar graph G_1 with a 3-augmentation H_1.

(b) A different planar embedding of G_1 that does not allow a 3-augmentation.

(c) A graph G_2 with a 3-regular planar supergraph H_2. However, there is no 3-augmentation of G_2.
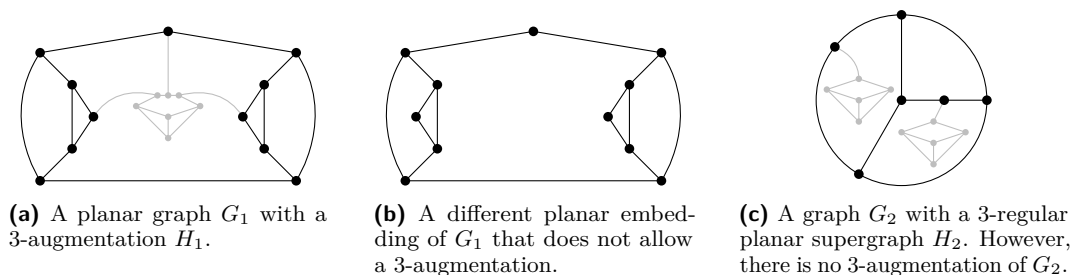
Figure 1 Example instances for the 3-AUGMENTATION problem.

▶ **Theorem 2.** *For a given n-vertex graph G we can construct in $\mathcal{O}(n^2)$ time a 3-regular bridgeless planar supergraph H of G, or conclude that no such exists.*

Theorem 2 is the main result of the present paper and we emphasize that this does not answer Question 1 yet. In fact, admitting a 3-augmentation is a sufficient condition for 3-edge colorability; but it is in general not necessary. For example, $K_{2,3}$ admits a proper 3-edge coloring but no 3-augmentation. Question 1 remains open and we discuss it and its connection to 3-augmentations in more detail in Section 3.

In order to decide the existence of a 3-augmentation (i.e., proving Theorem 2), we may of course assume that the graph G itself is planar and of maximum degree at most 3. Observe that in this case it is always possible to find a 3-regular planar supergraph of G, for example by adding the small gadget $K_4^{(1)}$ consisting of K_4 with one subdivided edge to each vertex that has not degree 3 yet, see Figure 1c. The difficult part is to prevent bridges in the resulting graph, even if the input graph G is already bridgeless. In fact, our task boils down to finding a suitable planar embedding of G such that for each vertex v of G and each missing edge at v, we can assign an incident face at v that should contain the new edge. We avoid the creation of bridges by assigning each face either no or at least two such new edges. Having assigned k new edges to a face f, we insert the small gadget $K_4^{(k)}$ consisting of K_4 with one edge subdivided k times into f. See Figure 1a for an example. Let us note that this might only work for some planar embeddings of G. See Figure 1b for a negative example.

We show Theorem 2 in three steps. First, we show that G admits a 3-augmentation if and only if each inclusion-maximal 2-connected component, called a *block*, of G admits a 3-augmentation. As all blocks can be found in linear time [27], we may restrict to the 2-connected case henceforth. Second, we consider a 2-connected G with a fixed planar embedding $\mathcal{E}$ and use the GENERALIZED ANTIFACTOR-problem to test whether G admits a 3-augmentation $H \supseteq G$ with a planar embedding whose restriction to G equals $\mathcal{E}$. Finally, for a 2-connected G with variable embedding, we use an SPQR-tree of G to efficiently go through the possible planar embeddings of G with a dynamic program and to identify one such embedding that allows for a 3-augmentation, or conclude that no such exists.

Outline.   After discussing related work below, we give necessary definitions in Section 1.1, including the GENERALIZED ANTIFACTOR-problem and SPQR-trees. In Section 2 we develop our algorithm for the 3-AUGMENTATION-problem, where we reduce to the 2-connected case in Section 2.1, and handle the fixed embedding in Section 2.2, and variable embedding in Section 2.3. Finally, in Section 3 we complete the loop back to the 3-EDGE COLORABILITY-problem for planar graphs. Lemmas marked with * are proven in the full version [13].

Related work.   Hartmann, Rollin and Rutter [16] studied a similar augmentation problem for planar graphs, where we are only allowed to add edges (but no vertices) to the graph. In particular, for given $c, k \in \{1, \dots, 5\}$ they define the c-Connected Planar k-Regular Augmentation-problem where one seeks to add edges to a given planar graph G, so that the resulting supergraph H of G is planar, c-connected, and k-regular. Observe that the 2-Connected Planar 3-Regular Augmentation-problem is more restrictive than the 3-Augmentation-problem: The former forbids to add new vertices, therefore refuses all input graphs with an odd number of vertices, and requires the result to be connected, therefore refusing all input graphs that are 3-regular and disconnected. In fact, reducing from Planar 3Sat, they show that 2-Connected Planar 3-Regular Augmentation is NP-complete [16, Theorem 3], while we show that 3-Augmentation lies in P.

Let us mention a few more examples from the rich and diverse area of augmentation problems. Eswaran and Tarjan [12] pioneered the systematic investigation of augmentation problems. They presented algorithms to find in $\mathcal{O}(|V| + |E|)$ a smallest number of edges whose addition to a given (not necessarily planar) graph $G = (V, E)$ results in a 2-connected respectively 2-edge-connected graph (a connected graph with no bridge), while the weighted versions of either problem is NP-complete. If we additionally require the result to be planar, already both unweighted problems are NP-complete [18,22]. Other problems of augmenting to a planar graph consider augmenting to a grid graph [3], or triangulating while minimizing the maximum degree [9,19], avoiding separating triangles [4], creating a Hamiltonian cycle [11], or resulting in a chordal graph [20], just to name a few.

1.1   Preliminaries

All graphs considered here are finite, undirected, and contain no loops but possibly multiedges. We write $\|G\|$ for the *size* of G (its number of edges) and denote the degree of a vertex v by $\deg(v)$, the minimum degree in G by $\delta(G)$, and the maximum degree in G by $\Delta(G)$. A graph G is *d-regular*, for some non-negative integer d, if we have $\delta(G) = \Delta(G) = d$. A 3-regular graph is also called *cubic*, while a graph G is *subcubic* if $\Delta(G) \leq 3$.

A *bridge* in a graph G is an edge e whose removal increases the number of connected components, i.e., $G - e$ has strictly more components than G. Equivalently, e is a bridge if e is not contained in any cycle of G. A *bridgeless* graph is one that contains no bridge. Note that a bridgeless graph may be disconnected. On the other hand, for a positive integer k, a graph $G = (V, E)$ is *k-connected* if $|V| \geq k + 1$ and for any set U of $k - 1$ vertices in G the graph $G - U$ is connected. In particular, a graph G of maximum degree $\Delta(G) \leq 3$ is 2-connected if and only if G is connected and bridgeless. A 2-connected graph is sometimes also called *biconnected*, while a 3-connected graph is sometimes also called *triconnected*.

A *planar embedding* $\mathcal{E}$ of a (planar) graph G is (in a sense that we need not make precise here) an equivalence class of crossing-free drawings of G in the plane. In particular, a planar embedding determines the set F of all *faces*, the distinguished *outer face* $f_0 \in F$, the clockwise ordering of incident edges around each vertex and the *boundary* of each face as a set of *facial walks*, each being a clockwise ordering of vertices and edges (with repetitions allowed). The edges and vertices incident to the outer face are called *outer edges* and *outer vertices*, while all others are *inner edges* and *inner vertices*. For every embedding $\mathcal{E}$ of G we define the *flipped embedding* $\mathcal{E}'$ to be the embedding obtained from $\mathcal{E}$ by reversing the clockwise order of incident edges at each vertex. This operation changes neither the set of faces nor the outer face. Whitney's Theorem [30] states that a 3-connected planar graph G has a unique embedding (up to the choice of the outer face and flipping).

Generalized (Anti)factors.   If G is a subgraph of H, denoted $G \subseteq H$, and v is a vertex of G, then we denote the degree of v in G by $\deg_G(v)$. If $V(G) = V(H)$, then G is called a *spanning* subgraph of H. If each vertex v of H is assigned a set $B(v) \subseteq \{0, \ldots, \deg_H(v)\}$, then a spanning subgraph G of H is called a *B-factor* of H if and only if $\deg_G(v) \in B(v)$ for every vertex v. Lovász [21] introduced B-factors and the GENERALIZED FACTOR-problem that, given graph H and for each vertex v in H a set $B(v)$, asks whether H admits some B-factor. A set $B(v)$ is said to have a *gap of length* $\ell \geq 1$ if there is an integer $i \in B(v)$ such that $i + 1, \ldots, i + \ell \notin B(v)$, and $i + \ell + 1 \in B(v)$. While the GENERALIZED FACTOR-problem is NP-complete in general [21], it can be solved in polynomial time if all gaps of each $B(v)$ have length one [8].

Now let $\overline{B}(v) \subseteq \{0, \ldots, \deg_H(v)\}$ be another set assigned to each vertex v. A spanning subgraph G of H is called a *$\overline{B}$-antifactor*, if and only if $\deg_G(v) \notin \overline{B}(v)$. One can think of $\overline{B}(v)$ as forbidden degrees for v in G. The GENERALIZED ANTIFACTOR-problem asks whether H admits a $\overline{B}$-antifactor. Note that the set $\{0, \ldots, \deg_H(v)\} \setminus \overline{B}(v)$ is finite, so the GENERALIZED ANTIFACTOR-problem is indeed a special case of the GENERALIZED FACTOR-problem. Therefore, an instance of the GENERALIZED ANTIFACTOR-problem with no two consecutive integers in any $\overline{B}(v)$ corresponds to an instance of the GENERALIZED FACTOR-problem with gaps of length at most one[1] and can be solved in polynomial time [8].

In Section 2.2 we use a theorem by Sebö [24], giving an efficient algorithm to compute generalized antifactors without two consecutive forbidden degrees.

▶ **Theorem 3** (Sebö [24])**.** *Let $H = (V, E)$ be a graph and for each vertex $v \in V$ let $\overline{B}(v) \subseteq \{0, \ldots, \deg_H(v)\}$ be a set containing no two consecutive integers. Then we can compute a $\overline{B}$-antifactor in time $\mathcal{O}(|V| \cdot |E|)$, or conclude that no such exists.*

SPQR-Tree.   The *SPQR-tree* is a tree-like data structure that compactly encodes all planar embeddings of a biconnected planar graph. It was introduced by Di Battista and Tamassia [10] and can be computed in linear time [15]. Its precise definition includes quite a number of technical terms, of which we define the crucial ones below. This makes our exposition self-contained, while also ensuring the established terminology for experienced readers. We give an illustrating example in Figure 2.

The SPQR-tree of a biconnected planar graph G is a rooted tree T, where each vertex μ of T is associated to a multigraph $\mathrm{skel}(\mu)$ that is called the *skeleton of μ*. This multigraph $\mathrm{skel}(\mu)$ must be of one of four types determining whether μ is an S-, a P-, a Q- or an R-vertex:

- S-vertex: $\mathrm{skel}(\mu)$ is a simple cycle.
- P-vertex: $\mathrm{skel}(\mu)$ consists of two vertices and at least three parallel edges.
- Q-vertex: $\mathrm{skel}(\mu)$ consists of two vertices with two parallel edges.
- R-vertex: $\mathrm{skel}(\mu)$ is triconnected.

Some of the edges of the skeletons can be marked as *virtual* edges. An edge $e = \mu\nu$ of the SPQR-tree T corresponds to two virtual edges, exactly one in $\mathrm{skel}(\mu)$ and one in $\mathrm{skel}(\nu)$. Conversely, each virtual edge corresponds to exactly one tree edge of T in this way. We refer again to Figure 2 for an example.

Under above conditions, the defining property of the SPQR-tree T is that G can be obtained by *gluing* along the virtual edges: For each tree edge $e = \mu\nu$, the skeletons $\mathrm{skel}(\mu)$ and $\mathrm{skel}(\nu)$ are identified at the corresponding endpoints of the two virtual edges associated to e and then the virtual edges are removed.

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[1] Let us point out a subtlety here illustrating that this correspondence is not one-to-one. Requiring that $\overline{B}(v)$ does not contain two consecutive integers is stronger than requiring gaps of length 1 in $B(v) := \{0, \ldots, \deg_H(v)\} \setminus \overline{B}(v)$. For example, consider a vertex v with $\deg_H(v) = 5$ and $\overline{B}(v) = \{1, 3, 4, 5\}$. Then $B(v) = \{0, 2\}$ has a gap of length 1, even though $\overline{B}(v)$ contained consecutive integers.
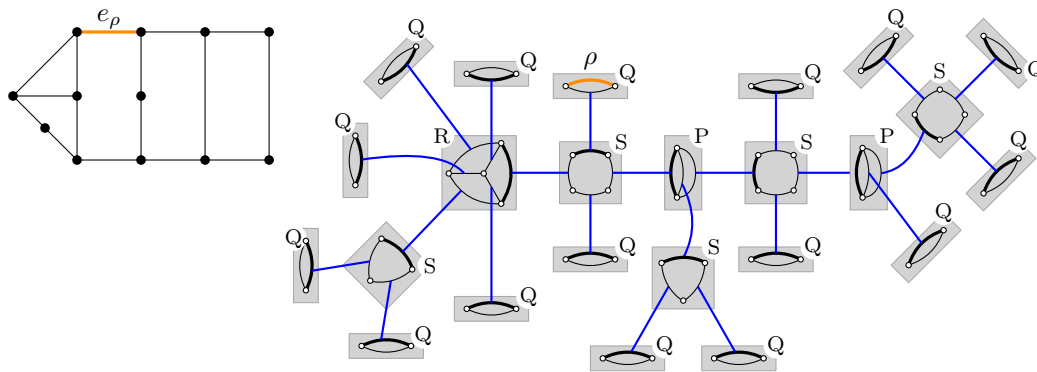
Figure 2 A graph with an edge e_ρ (left) and its SPQR-tree rooted at the Q-vertex ρ corresponding to e_ρ (right). Each tree node μ shows the skeleton $\mathrm{skel}(\mu)$ in which the virtual edge to its parent is shown thicker. The (blue) tree edges indicate the associated pairs of virtual edges.

We additionally require that no two S-vertices and no two P-vertices are adjacent in T, as otherwise the skeletons of two such vertices can be merged into the skeleton of a new vertex of the same type. Further, exactly one of the two parallel edges in a Q-vertex is a virtual edge while S-, P- and R-vertices contain only virtual edges. Under these conditions the SPQR-tree of G is unique. There is exactly one Q-vertex per edge in G and these form the leaves of the SPQR-tree. The inner S-, P- and R-vertices correspond more or less[2] to the separation pairs (that is, pairs of vertices forming a cut set) of G [10].

Assume that an arbitrary vertex ρ of T is fixed as the root. For some vertex μ in T let π be its parent. Further, let u, v be the endpoints of the virtual edge in $\mathrm{skel}(\mu)$ associated with the tree edge $\mu\pi$ in T. Then the graph obtained by gluing $\mathrm{skel}(\mu)$ with all skeletons in its subtree and without the virtual edge uv is called the *pertinent graph* of μ and denoted by $\mathrm{pert}(\mu)$. Note that $\mathrm{pert}(\mu)$ is always connected.

SPQR-Tree and Planar Embeddings. If the SPQR-tree T is rooted at a Q-vertex ρ corresponding to an edge e_ρ of G, then T represents all planar embeddings of G in which e_ρ is an outer edge [10]. When G is constructed by gluing corresponding virtual edges, one has the following choices on the planar embedding:

- Whenever the corresponding virtual edges of an S-, P- or R-vertex μ and its parent are glued together, this leaves two choices for the planar embedding: Having decided for an embedding $\mathcal{E}_\mu$ of $\mathrm{pert}(\mu)$ already, we can insert $\mathcal{E}_\mu$ or the flipped embedding $\mathcal{E}'_\mu$.
- The parallel virtual edges of a P-vertex μ associated to virtual edges of children can be permuted arbitrarily. Every permutation leads to a different planar embedding of $\mathrm{skel}(\mu)$.
- Gluing at the virtual edge of a Q-vertex μ replaces the virtual edge uv by the "real" edge uv in G. This has no effect on the embedding.

Let $\mathcal{E}$ be a planar embedding of G having e_ρ as an outer edge. Further, let μ be an inner vertex of the SPQR-tree and u_μ, v_μ be the endpoints of the virtual edge in $\mathrm{skel}(\mu)$ corresponding to the parent edge of μ in T. Lastly, let $\mathcal{E}_\mu$ be the restriction of $\mathcal{E}$ to $\mathrm{pert}(\mu)$ and let f_μ^o be the outer face of $\mathcal{E}_\mu$. As e_ρ is an outer edge of $\mathcal{E}$, it follows that u_μ and v_μ are outer vertices in $\mathcal{E}_\mu$. The $u_\mu v_\mu$-path in $\mathrm{pert}(\mu)$ having f_μ^o to its left (right) is the *left (right) outer path* of $\mathcal{E}_\mu$. Lastly, we define the *left (right) outer face* of $\mathcal{E}_\mu$ inside $\mathcal{E}$ to be the face of $\mathcal{E}$ left (right) of the left (right) outer path of $\mathcal{E}_\mu$.

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[2] In fact they correspond to so-called *split pairs*. However, we omit their formal discussion, as it is not needed here.

2 The 3-Augmentation-Problem

2.1 Reduction to the 2-Connected Case

▶ **Proposition 4.** *For a disconnected graph G with connected components $G_1, \ldots, G_k$, $k \geq 2$, we have that*

(i) *G has a 3-augmentation if and only if each G_i has a 3-augmentation, $i = 1, \ldots, k$, and*

(ii) *G has a 2-connected 3-augmentation if and only if each G_i has a 3-augmentation and no G_i is 3-regular, $i = 1, \ldots, k$.*

Proof.

(i) Any 3-augmentation of G is also a 3-augmentation of each G_i, showing already necessity. For sufficiency, observe that the 3-augmentations of different G_i are vertex-disjoint and hence their union is a 3-augmentation of G.

(ii) Like above, a 2-connected 3-augmentation H of G is also a 3-augmentation of each G_i, $i = 1, \ldots, k$. Moreover, as H is connected, each G_i has a vertex with at least one incident edge in $E(H) - E(G)$, showing that G_i is not 3-regular.

On the other hand, for $i = 1, \ldots, k$ let H_i be a 3-augmentation of G_i. Without loss of generality each H_i is connected (hence 2-connected since 3-augmentations are bridgeless). As G_i is not 3-regular, we can pick an edge e_i from $E(H_i) - E(G_i)$, $i = 1, \ldots, k$. Next, choose a planar embedding $\mathcal{E}$ of the disjoint union $H_1 \cup \cdots \cup H_k$ where each of $e_1, \ldots, e_k$ is an outer edge. Finally, add a copy of $K_4^{(2k)}$ into the outer face of $\mathcal{E}$, delete $e_1, \ldots, e_k$, and connect the $2k$ degree-2 vertices of $H_1 \cup \cdots \cup H_k$ with the $2k$ degree-2 vertices of $K_4^{(2k)}$ by a non-crossing matching. The result is a 2-connected 3-augmentation of G (by definition a 3-augmentation is bridgeless, so connectivity implies 2-connectivity). ◀

▶ **Proposition 5.** *A graph G admits a 3-augmentation if and only if $\Delta(G) \leq 3$ and each block of G admits a 3-augmentation.*

Proof. If G is bridgeless, then each connected component is a single block and thus admits a 3-augmentation by assumption. The disjoint union of these is a 3-augmentation of G.

Otherwise, consider G with a bridge $e = uv$. Let G_1 be the connected component of $G - e$ containing u, and let the remaining graph be $G_2 = G - G_1$. It is enough to show that if G_1 and G_2 have 3-augmentations H_1 respectively H_2, then G has a 3-augmentation, too. To this end, consider an edge $e_1 \in E(H_1) - E(G_1)$ incident to u and an edge $e_2 \in E(H_2) - E(G_2)$ incident to v. These edges exist as $\deg_{G_1}(u), \deg_{G_2}(v) \leq \Delta(G) - 1 \leq 2$ but $\deg_{H_1}(u) = \deg_{H_2}(v) = 3$. Choose a planar embedding of $H_1 \cup H_2$ with e_1 and e_2 being outer edges. Denoting by a, b the endpoints of e_1, e_2 different from u, v, we see that $(H_1 - e_1) \cup (H_2 - e_2) \cup \{uv, ab\}$ is a 3-augmentation of G, as desired. ◀

2.2 The Fixed Embedding Setting

As usual for embedding-dependent problems for planar graphs, it makes sense to distinguish between the planar graph G being given with a fixed embedding that shall not be altered, and the setting with variable embedding where we solely have G as the input and shall find a suitable embedding for G or decide that no such exists. The 3-augmentation problem is formulated in the variable embedding setting. However, let us treat the variant with a fixed embedding first, as this will be a crucial subroutine for the variable embedding setting later.

▶ **Proposition 6.** *Let G be an n-vertex 2-connected planar multigraph of maximum degree $\Delta(G) \leq 3$ with a fixed planar embedding $\mathcal{E}$. Then we can compute in time $\mathcal{O}(n^2)$ a 3-augmentation H of G with a planar embedding $\mathcal{E}_H$ whose restriction to G equals $\mathcal{E}$, or conclude that no such exists.*

Proof. Let V' denote the subset of the vertices of G with $\deg_G(v) \leq 2$ and let F denote the set of faces of $\mathcal{E}$. Note that since G is 2-connected, all $v \in V'$ have $\deg_G(v) = 2$. We consider the bipartite vertex-face incidence graph $I = (V' \uplus F, E(I))$ with vertex-set $V' \uplus F$ and edge-set $E(I) := \{vf \mid v \in V', f \in F, v \text{ is incident to } f\}$. Note that I has $O(n)$ vertices and at most $2n$ edges, since $\Delta(G) \leq 3$. We define an instance of the Generalized Antifactor-problem by assigning each vertex x of I (corresponding to a vertex in G or a face in F) a set $\overline{B}(x) \subseteq \{0, \ldots, \deg_I(x)\}$:

$$\overline{B}(x) := \begin{cases} \{0, 2\} & \text{for } x \in V' \\ \{1\} & \text{for } x \in F \end{cases}$$

Note that no $\overline{B}(x)$ contains two consecutive integers.

▷ **Claim 7.** Graph G admits a 3-augmentation H extending the embedding $\mathcal{E}$ if and only if I admits a $\overline{B}$-antifactor.

Proof. First assume H is a 3-augmentation of G with a planar embedding $\mathcal{E}_H$ that extends $\mathcal{E}$. Hence every edge $e \in E(H) - E(G)$ lies in a unique face of $\mathcal{E}$. We construct a $\overline{B}$-antifactor of I as follows. For each degree-2 vertex v of G, let f_v be the face of $\mathcal{E}$ that contains the unique edge in $E(H) - E(G)$ incident to v. We claim that $J = (V' \uplus F, \{vf_v \mid v \in V'\})$ is a $\overline{B}$-antifactor of I. In fact, $\deg_J(v) = 1$ for each degree-2 vertex $v \in V$. Now if we would have $\deg_J(f) = 1$ for some face $f \in F$, then exactly one vertex $v \in V'$ has exactly one incident edge e lying in face f. In particular, the other endpoint of e is not a vertex of G. But then e is a bridge and H is not a 3-augmentation. Hence $\deg_J(f) \neq 1$ for each $f \in F$ and I indeed admits a $\overline{B}$-antifactor.

Conversely assume now that I has some $\overline{B}$-antifactor J. Then we construct the desired 3-augmentation H of G as follows. Inside each face f of $\mathcal{E}$ with $\deg_J(f) > 0$ place a copy K_f of $K_4^{(\deg_J(f))}$. Connect the $\deg_J(f)$ degree-2 vertices $v \in V'$ with $vf \in E(J)$ by a non-crossing matching with the $\deg_J(f)$ degree-2 vertices of K_f. Call the resulting graph H and its resulting planar embedding $\mathcal{E}_H$. Then H is 2-connected (in particular bridgeless) as G is 2-connected and $\deg_J(f) \neq 1$ for each $f \in F$. Moreover, H is 3-regular. In fact, for each vertex $v \in V'$ we have $\deg_H(v) = \deg_G(v) + 1 = 3$, as J is a $\overline{B}$-antifactor. Finally, restricting $\mathcal{E}_H$ to G gives back embedding $\mathcal{E}$. ◁

Now Claim 7 immediately finishes the proof because no $\overline{B}(x)$ contains two consecutive integers. Hence, by Sebö's algorithm [24] (cf. Theorem 3) we can compute a $\overline{B}$-antifactor of I in $\mathcal{O}(n^2)$ time, or conclude that no such exists. ◀

2.3　The Variable Embedding Setting

Even an unlabeled 2-connected subcubic planar graph G can have exponentially many different planar embeddings (e.g., the $(2 \times n)$-grid graph). Thus, iterating over all embeddings of G and applying the algorithm from Proposition 6 to each of them is not a polynomial-time algorithm and hence no feasible approach for us. In this section we describe how to use the SPQR-tree of G to efficiently find a planar embedding $\mathcal{E}$ of G such that there is a 3-augmentation H of G extending $\mathcal{E}$, or conclude that no such embedding exists. The algorithm from Proposition 6 will be an important subroutine.

▶ **Proposition 8.** *Let G be an n-vertex 2-connected planar graph of maximum degree $\Delta(G) \leq 3$. Then we can compute in $\mathcal{O}(n^2)$ time a 3-augmentation H of G or conclude that no such exists.*

Overview.   The proof of Proposition 8 uses a bottom-up dynamic programming approach on the SPQR-tree T of G rooted at a Q-vertex ρ corresponding to some edge e_ρ in G. Consider a vertex $\mu \neq \rho$ in T. Let uv be the virtual edge in skel(μ) that is associated to the parent edge of μ. Recall that each embedding $\mathcal{E}$ of G with e_ρ on the outer face, when restricted to the pertinent graph pert(μ), gives an embedding $\mathcal{E}_\mu$ of pert(μ) whose inner faces are also inner faces of $\mathcal{E}$, and with u and v being outer vertices of $\mathcal{E}_\mu$. The outer face of $\mathcal{E}_\mu$ is composed of two (not necessarily edge-disjoint) u-v-paths; the left and right outer path of $\mathcal{E}_\mu$, which are contained in the left and right outer face of $\mathcal{E}_\mu$ inside $\mathcal{E}$, respectively. We seek to partition the (possibly exponentially many) planar embeddings of pert(μ) with u, v on its outer face into a constant number of equivalence classes based on how many edges in a 3-augmentation of G could possibly "connect" pert(μ) with the rest of the graph G inside the left or right outer face of $\mathcal{E}_\mu$ inside $\mathcal{E}$. This corresponds[3] to the number of degree-2 vertices on the left and right side in so-called *inner augmentations* of $\mathcal{E}_\mu$. Loosely speaking, it will be enough for us to distinguish three cases for the left side (0, 1, or at least 2 connections), the symmetric three cases for the right side, and to record which of the nine resulting combinations are possible. Note that this grouping of embeddings of $\mathcal{E}_\mu$ into constantly many classes is the key insight that allows an efficient dynamic program.

Whether a particular equivalence class is realizable by some planar embedding $\mathcal{E}_\mu$ of pert(μ) will depend on the vertex type of μ (S-, P- or R-vertex) and the realizable equivalence classes of its children $\mu_1, \ldots, \mu_k$. In the end, we shall conclude that the whole graph G has a 3-augmentation if and only if for the unique child μ of the root ρ of T the equivalence class of embeddings of pert(μ) for which neither the left nor the right side has any connections is non-empty.

Most of our arguments are independent of SPQR-trees and we instead consider so-called uv-graphs, which are slightly more general than pertinent graphs. We shall introduce inner augmentations of uv-graphs, which then give rise to label sets for uv-graphs, both in a fixed and variable embedding setting. These label sets encode the aforementioned number of connections between the uv-graph as a subgraph of G and the rest of G in a potential 3-augmentation. After showing that we can compute even variable label sets by resorting to the fixed embedding case and Proposition 6, we then present the final dynamic program along the rooted SPQR-tree T of G.

uv-Graphs and Labels.   A *uv-graph* is a connected multigraph G_{uv} with $\Delta(G_{uv}) \leq 3$, two distinguished vertices u, v of degree at most 2, together with a planar embedding $\mathcal{E}_{uv}$ such that u and v are outer vertices. A connected multigraph $H_{uv} \supseteq G_{uv}$ with planar embedding $\mathcal{E}_H$ is an *inner augmentation* of G_{uv} if

-   $\mathcal{E}_H$ extends $\mathcal{E}_{uv}$ and has u, v on its outer face,
-   each of u, v has the same degree in H_{uv} as in G_{uv},
-   every vertex of H_{uv} except for u, v has degree 1 or 3,
-   every degree-1 vertex of H_{uv} lies in the outer face of $\mathcal{E}_H$ and
-   every bridge of H_{uv} that is not a bridge of G_{uv} is incident to a degree-1 vertex.

Because u, v are outer vertices in $\mathcal{E}_H$, one could add another edge e_{uv} (oriented from u to v) into the outer face of $\mathcal{E}_H$ preserving planarity (this edge is not part of the inner augmentation). Then e_{uv} splits the outer face into two faces f_A, f_B left and right of e_{uv}, respectively. Each degree-1 vertex of H_{uv} now lies either inside f_A or f_B.

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[3]   up to the fact that left and right outer path may share degree-2 vertices, each of which sends however its third edge into only one of the left and right outer face

We are interested in the number of degree-1 vertices in each of these faces of $\mathcal{E}_H$ and write $d(H_{uv}, \mathcal{E}_H) = (a, b)$ if an inner augmentation H_{uv} of G_{uv} has exactly a degree-1 vertices inside f_A and exactly b degree-1 vertices inside f_B.

▶ **Lemma 9.** *Let H_{uv} be an inner augmentation of G_{uv} with $d(H_{uv}, \mathcal{E}_H) = (a, b)$. If $a \geq 2$, then G_{uv} has an inner augmentation H_{uv}^0 with $d(H_{uv}^0, \mathcal{E}_H^0) = (0, b)$ and an inner augmentation H_{uv}^1 with $d(H_{uv}^1, \mathcal{E}_H^1) = (1, b)$. A symmetric statement holds when $b \geq 2$.*

Proof. Add edge uv to the inner augmentation H_{uv} such that it has a degree-1 vertices in f_A. We add a copy of $K_4^{(a)}$ into f_A and identify the a degree-2 vertices of $K_4^{(a)}$ with the a degree-1 vertices in f_A in a non-crossing way. Ignoring edge uv, the obtained graph is the desired inner augmentation H_{uv}^0 with $d(H_{uv}^0, \mathcal{E}_H^0) = (0, b)$. We obtain H_{uv}^1 by additionally subdividing an edge of $K_4^{(a)}$ that is incident to f_A once and by attaching a degree-1 vertex to it into f_A. ◀

Motivated by Lemma 9, we focus on inner augmentations H_{uv} with $d(H_{uv}, \mathcal{E}_H) = (a, b)$ where $a, b \in \{0, 1\}$, and assign to H_{uv} in this case the *label* `ab` with `a, b` $\in \{0, 1\}$.

The *embedded label set* $L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$ contains all labels `ab` such that there is an inner augmentation H_{uv} of G_{uv} with label `ab`. Allowing other planar embeddings of G_{uv}, we further define the *variable label set* as $L_{\mathrm{var}}(G_{uv}) = \bigcup_{\mathcal{E}} L_{\mathrm{emb}}(G_{uv}, \mathcal{E})$, where $\mathcal{E}$ runs over all planar embeddings of G_{uv} where u and v are outer vertices. As this in particular includes for each embedding $\mathcal{E}$ of G_{uv} also the flipped embedding $\mathcal{E}'$ of G_{uv}, it follows that `ab` $\in L_{\mathrm{var}}(G_{uv})$ if and only if `ba` $\in L_{\mathrm{var}}(G_{uv})$. Whenever this property holds for a (variable or embedded) label set, we call the label set *symmetric*. Hence, all variable label sets are symmetric, but embedded label sets may or may not be symmetric.

For brevity, let us use $\star$ as a wildcard character, in the sense that if $\{$`x0, x1`$\}$ is in an embedded or variable label set for some `x` $\in \{0, 1\}$, then we shorten the notation and replace them by a label `x`$\star$. Symmetrically, we use the notation $\star$`x` and in particular define $\{\star\star\} := \{$`00, 01, 10, 11`$\}$. Using this notation, the eight possible symmetric label sets are:

$$\emptyset, \{\texttt{00}\}, \{\texttt{01}, \texttt{10}\}, \{\texttt{11}\}, \{\texttt{0}\star, \star\texttt{0}\}, \{\texttt{00}, \texttt{11}\}, \{\texttt{1}\star, \star\texttt{1}\}, \{\star\star\} \tag{1}$$
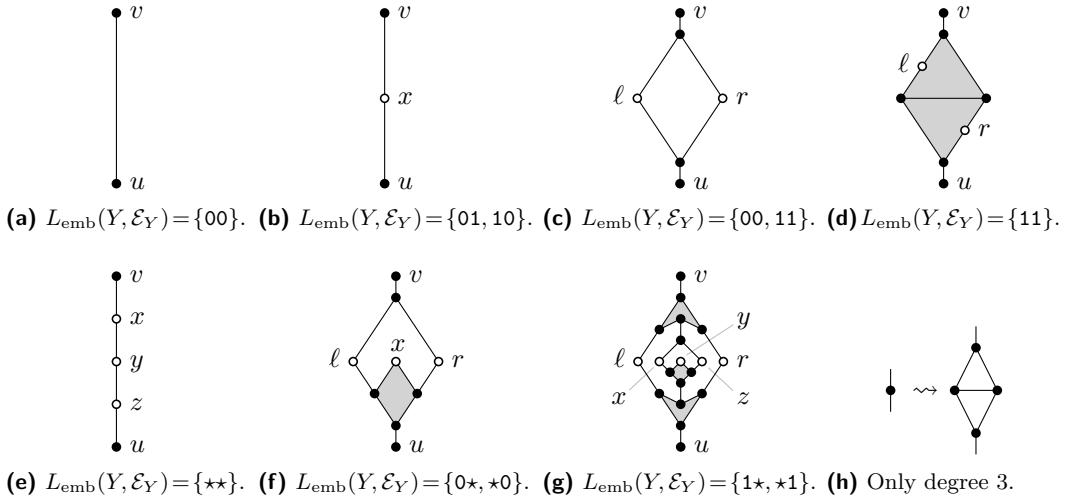
The following lemma reveals the significance of inner augmentations and label sets.

▶ **Lemma 10.** *Let G be a 2-connected graph with $\Delta(G) \leq 3$ and $\mathcal{E}$ be an embedding of G with some outer edge $e = xy$. Further, let G_{uv} be the uv-graph obtained from G by deleting e and adding two new vertices u, v with edges ux and vy into the outer face of $\mathcal{E}$. Then G has a 3-augmentation if and only if $\texttt{00} \in L_{\mathrm{var}}(G_{uv})$.*

Proof. First let $H \supseteq G$ be a 3-augmentation of G and let $\mathcal{E}_H$ be an embedding of H with $e = xy$ being an outer edge. Then deleting e and adding two new vertices u, v with edges ux and vy into the outer face of $\mathcal{E}_H$ results in an inner augmentation H_{uv} of G_{uv} with respect to the embedding of G_{uv} inherited from $\mathcal{E}_H$. As adding an edge e_{uv} from u to v into H_{uv} gives a graph with no degree-1 vertices, we have $\texttt{00} \in L_{\mathrm{var}}(G_{uv})$.

Conversely, assume that $\texttt{00} \in L_{\mathrm{var}}(G_{uv})$. Then there is an embedding $\mathcal{E}_{uv}$ of G_{uv} that allows for some inner augmentation H_{uv} with embedding $\mathcal{E}_H$ for which $H_{uv} + e_{uv}$ has no degree-1 vertices, where $e_{uv} = uv$ denotes a new edge between u and v. Thus, in H_{uv} the vertices u and v have degree 1 (as in G_{uv}), every vertex of H_{uv} except u, v has degree 3, and the only bridges of H_{uv} are the edges ux and vy. Then we obtain a 3-augmentation H of G by removing ux, vy from H_{uv} and adding the edge xy into the outer face of $\mathcal{E}_H$. In case, H_{uv} already contains the edge xy, this is replaced by a copy of $K_4^{(2)}$ with two non-crossing edges between x, y and the two degree-2 vertices of $K_4^{(2)}$. ◀

(a) $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{00\}$. **(b)** $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{01, 10\}$. **(c)** $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{00, 11\}$. **(d)** $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{11\}$.

(e) $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{\star\star\}$. **(f)** $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{0\star, \star0\}$. **(g)** $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{1\star, \star1\}$. **(h)** Only degree 3.

Figure 3 a–g The seven gadgets for the seven non-empty variable label sets $L_{\mathrm{var}}(G_{uv})$ of a uv-graph G_{uv}. h Modification to locally replace a degree-2 vertex by four degree-3-vertices.

Gadgets. In our algorithm below, we aim to replace certain uv-graphs X (with variable embedding) by uv-graphs Y with fixed embedding $\mathcal{E}_Y$, such that the variable label set $L_{\mathrm{var}}(X)$ equals the embedded label set $L_{\mathrm{emb}}(Y, \mathcal{E}_Y)$. This will allow us to use Proposition 6 from the fixed embedding setting as a subroutine.

The following lemma describes seven uv-graphs, each with a fixed embedding, corresponding to the seven different non-empty variable label sets as given in (1). For this purpose, each such *gadget* is itself a uv-graph Y with a fixed embedding $\mathcal{E}_Y$.

▶ **Lemma* 11.** *For every uv-graph G_{uv} with $L_{\mathrm{var}}(G_{uv}) \neq \emptyset$ there exists a gadget Y with an embedding $\mathcal{E}_Y$ such that u, v are outer vertices and $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = L_{\mathrm{var}}(G_{uv})$.*

Lemma 11 is proven in the full version [13], but the claimed gadgets are shown in Figure 3.

Computing a Label Set. In our algorithm below we want to compute the variable label sets of $\mathrm{pert}(\mu)$ for vertices μ of the rooted SPQR-tree T of G. As we will see, we can reduce this to a constant number of computations of embedded label sets of certain uv-graphs that are specifically crafted to encode all the possible embeddings of $\mathrm{pert}(\mu)$. The following lemma describes how to do this.

▶ **Lemma* 12.** *Let G_{uv} be an n-vertex uv-graph and $\mathcal{E}_{uv}$ a planar embedding where u and v are outer vertices. Then we can check each of the following in time $\mathcal{O}(n^2)$:*
- *Whether $00 \in L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$.*
- *Whether $01 \in L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$ or $10 \in L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$.*
- *Whether $11 \in L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$.*
In particular, if $L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$ is symmetric, then this is sufficient to determine the exact embedded label set $L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$.

The idea for Lemma 12 is similar to Lemma 10: For each check, we do some small local modifications to G_{uv} in order to obtain an embedded planar graph G_{uv}^+ that has a 3-augmentation extending its embedding if and only if $L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$ contains the specific label. Then the result follows from Proposition 6. A full proof is given in the full version [13].

Algorithm for Variable Embedding. In order to decide whether a given biconnected planar graph G admits some planar embedding which admits a 3-augmentation, we use the SPQR-tree T of G. Rooting T at some Q-vertex ρ, the pertinent graph $\text{pert}(\mu)$ of a vertex μ in T is a subgraph of G. Moreover, if $u_\mu v_\mu$ is the virtual edge in $\text{skel}(\mu)$ associated to the parent edge of μ, then $\text{pert}(\mu)$ is a uv-graph (with u_μ, v_μ taking the roles of u, v in the uv-graph). Now the variable label set $L_{\text{var}}(\text{pert}(\mu))$ is a constant-size representation of all possible labels that any possible embedding of an inner augmentation of $\text{pert}(\mu)$ can have (having u_μ and v_μ on its outer face). The remainder of this section describes how the variable label sets of all vertices in the SPQR-tree can be computed by a bottom-up dynamic program.

▶ **Lemma* 13.** *Let μ be an inner R-, S-, or P-vertex of the SPQR-tree and $\mu_1, \ldots, \mu_k$ be its children. Further assume that the variable label sets $L_{\text{var}}(\text{pert}(\mu_i))$, for $i = 1, \ldots, k$, are non-empty and known. Then the variable label set $L_{\text{var}}(\text{pert}(\mu))$ can be computed in time $\mathcal{O}(\|\text{skel}(\mu)\|^2)$.*

The idea to prove Lemma 13 is to consider $\text{skel}(\mu)$ with its essentially unique embedding and to replace for each μ_i the associated virtual edge by the embedded gadget Y from Lemma 11 (cf. Figure 3) with $L_{\text{emb}}(Y, \mathcal{E}_Y) = L_{\text{var}}(\text{pert}(\mu_i))$. Then the virtual edge uv associated to the parent of μ gets removed to obtain an embedded uv-graph on $\mathcal{O}(\|\text{skel}(\mu)\|)$ vertices, whose embedded label set can then be computed with Lemma 12. A full proof is given in the full version [13].

Lemma 13 computes the variable label set of an inner vertex of the SPQR-tree, requiring that the variable label sets of its children are non-empty. If this condition is not satisfied, i.e., at least one vertex μ has $L_{\text{var}}(\text{pert}(\mu)) = \emptyset$, then the following lemma applies:

▶ **Lemma 14.** *If $L_{\text{var}}(\text{pert}(\mu)) = \emptyset$ for some vertex μ of the SPQR-tree T of G, then G has no 3-augmentation.*

Proof. Assuming that G has a 3-augmentation H, we shall show that $L_{\text{var}}(\text{pert}(\mu)) \neq \emptyset$ for every vertex μ of T. If μ is the root, let u, v be the two unique vertices in $\text{skel}(\mu)$ (because $\mu = \rho$ is a Q-vertex). If μ is not the root, let u, v be the endpoints of the virtual edge associated to the parent edge of μ.

By the definition of labels, $L_{\text{var}}(\text{pert}(\mu)) \neq \emptyset$ if there is some inner augmentation of $\text{pert}(\mu)$ for at least one of its planar embeddings with u, v on its outer face. But the 3-augmentation H of G induces an inner augmentation of $\text{pert}(\mu)$ as follows: Let $\mathcal{E}_H$ be a planar embedding of H with outer edge e_ρ and $\mathcal{E}_G$ its restriction to G. Recall that then u, v are outer vertices of $\text{pert}(\mu)$ in $\mathcal{E}_G$. Consider the embedded subgraph of H consisting of $\text{pert}(\mu)$ and all vertices and edges of H inside inner faces of $\text{pert}(\mu)$ in $\mathcal{E}_G$. For each vertex $w \neq u, v$ on the outer face of $\text{pert}(\mu)$ in $\mathcal{E}_G$ incident to an edge of H in the outer face of $\text{pert}(\mu)$, we add a new pendant edge at w into the outer face of $\text{pert}(\mu)$ in $\mathcal{E}_G$. The resulting graph is an inner augmentation of $\text{pert}(\mu)$ and hence $L_{\text{var}}(\text{pert}(\mu)) \neq \emptyset$. ◀

Now that we considered S-, P- and R-vertices, we are finally set up to prove Proposition 8. There we claim that we can decide in polynomial time whether a biconnected planar graph G with $\Delta(G) \leq 3$ has a 3-augmentation.

Proof of Proposition 8. As mentioned above, we use bottom-up dynamic programming on the SPQR-tree T of G rooted at an arbitrary Q-vertex ρ corresponding to an edge e_ρ in G.

The base cases are the leaves of T, all of which are Q-vertices. The variable label set of a leaf μ is $L_{\text{var}}(\text{pert}(\mu)) = \{00\}$: $\text{pert}(\mu)$ is just a single edge and the only inner augmentation of $\text{pert}(\mu)$ is $\text{pert}(\mu)$ itself, and as such has label 00.

Now let μ be an inner vertex of T and thus be either an S-, a P- or an R-vertex. All its children $\mu_1, \ldots, \mu_k$ have already been processed and their variable label sets $L_{\mathrm{var}}(\mathrm{pert}(\mu_i))$ are known. Then the variable label set $L_{\mathrm{var}}(\mathrm{pert}(\mu))$ can be computed in time $\mathcal{O}(\| \mathrm{skel}(\mu)\|^2)$ (which is actually $\mathcal{O}(1)$ in case of a P-vertex) by Lemma 13. To apply this lemma, we need to guarantee that the variable label sets $L_{\mathrm{var}}(\mathrm{pert}(\mu_i))$ of the children are non-empty. If this is not the case, then by Lemma 14 graph G has no 3-augmentation and we can stop immediately.

It remains to consider the root ρ of the SPQR-tree. Recall that $\mathrm{pert}(\rho) = G$. Following the setup of Lemma 10, let x, y be the two unique vertices of $\mathrm{skel}(\rho)$ and xy be the unique non-virtual edge, i.e., the edge $e_\rho = xy$ of G. Let G_{uv} be the uv-graph obtained from $G = \mathrm{pert}(\rho)$ by deleting $e_\rho = xy$ and adding two new pendant edges ux, vy. Note that x and y have the same degree in G_{uv} as in G. By Lemma 10, G has a 3-augmentation if and only if $00 \in L_{\mathrm{var}}(G_{uv})$.

To check whether $00 \in L_{\mathrm{var}}(G_{uv})$, let μ be the unique child of ρ. Thus we have $\mathrm{pert}(\mu) = G - e_\rho$. We have already computed $L_{\mathrm{var}}(\mathrm{pert}(\mu))$ and can assume by Lemma 14 that it is non-empty. Consider the gadget Y with embedding $\mathcal{E}_Y$ from Lemma 11 such that $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = L_{\mathrm{var}}(\mathrm{pert}(\mu))$. Let u' and v' denote the two degree-1 vertices in Y. If both x and y have degree 3 in G (hence also in G_{uv}), then $L_{\mathrm{var}}(G_{uv}) = L_{\mathrm{var}}(\mathrm{pert}(\mu)) = L_{\mathrm{emb}}(Y, \mathcal{E}_Y)$ and we already know whether or not 00 is contained in these label sets.

If x has degree 2 in G (hence also degree 2 in G_{uv}, while degree 1 in $\mathrm{pert}(\mu)$), then x receives a new edge in inner augmentations of G_{uv} but not in inner augmentations of $\mathrm{pert}(\mu)$. For Y to model $L_{\mathrm{var}}(G_{uv})$ instead of $L_{\mathrm{var}}(\mathrm{pert}(\mu))$, we subdivide in Y the edge at u' by a new vertex x'. Similarly, if y has degree 2 in G, we subdivide in Y the edge at v'. For the resulting graph Y' with embedding $\mathcal{E}_{Y'}$ it follows that $L_{\mathrm{var}}(G_{uv}) = L_{\mathrm{emb}}(Y', \mathcal{E}_{Y'})$ and we can check whether 00 is contained in these label sets by calling Lemma 12 on Y' with embedding $\mathcal{E}_{Y'}$. This takes constant time, as Y' has constant size.

The overall runtime is the time needed to construct the SPQR-tree plus the time spent processing each of its vertices. Gutwenger and Mutzel [15] show how to construct the SPQR-tree in time $\mathcal{O}(n)$. The time for the dynamic program traversing the SPQR-tree T is

$$\mathcal{O}\Big(\sum_{\mu \in V(T)} \| \mathrm{skel}(\mu)\|^2 \Big) \subseteq \mathcal{O}\Big(\big(\sum_{\mu \in V(T)} \| \mathrm{skel}(\mu)\| \big)^2 \Big) \subseteq \mathcal{O}(n^2),$$

where the first step uses that for a set of positive integers the sum of their squares is at most the square of their sum, and the second step uses that the SPQR-tree has linear size.   ◀

3   Discussion and Open Problems

In this paper we showed how to test in polynomial time whether a planar graph G is a subgraph of some bridgeless cubic planar graph H. (We call such H a 3-augmentation of G.) Our motivation was to test whether G admits a proper 3-edge-coloring, because admitting a 3-augmentation is sufficient to conclude that $\chi'(G) \leq 3$. (This follows from the Four-Color-Theorem [1, 2] and the work of Tait [26].) However, there are 3-edge-colorable planar graphs with no 3-augmentation; $K_{2,3}$ is an easy example. For another class of examples, consider for instance any 3-connected 3-regular plane graph G (that is, the dual of a plane triangulation) and subdivide (with a new degree-2 vertex each) any set of at least two edges, where no two of these are incident to the same face of G (so their dual edges form a matching in the triangulation). The resulting graph G' has only one embedding (up to the choice of the outer face) and clearly no 3-augmentation. On the other hand, Conjecture 15 below predicts that G' is 3-edge colorable.

The computational complexity of the 3-ᴇᴅɢᴇ Cᴏʟᴏʀᴀʙɪʟɪᴛʏ-problem for planar graphs remains open, while it is known to be NP-complete already for 3-regular, but not necessarily planar, graphs [17]. Similarly to our methods in Section 2.1, one can easily show that a planar subcubic graph is 3-edge-colorable if and only if all of its blocks (inclusion-maximal biconnected subgraphs) are 3-edge-colorable, i.e., 3-ᴇᴅɢᴇ Cᴏʟᴏʀᴀʙɪʟɪᴛʏ reduces to the 2-connected case. A simple counting argument shows that a 2-connected subcubic graph G with exactly one degree-2 vertex is not 3-edge-colorable (independent of whether G is planar or not). The following conjecture, attributed to Grötzsch by Seymour [25], states that in the case of planar graphs, this is the only obstruction.

▶ **Conjecture 15** (Grötzsch, cf. [25]). *If G is a 2-connected planar graph of maximum degree $\Delta(G) \leq 3$, then G is 3-edge-colorable, unless it has exactly one vertex of degree 2.*

If Conjecture 15 is true, 3-ᴇᴅɢᴇ Cᴏʟᴏʀᴀʙɪʟɪᴛʏ would be in P, as its condition is easy to check in linear time.

Finally, let us also briefly discuss planar graphs of maximum degree larger than 3. Vizing conjectured in 1965 that all planar graphs of maximum degree $\Delta \geq 6$ are Δ-edge-colorable, proving it only for $\Delta \geq 8$ [29]. As of today, it is known that all planar graphs of maximum degree $\Delta \geq 7$ are Δ-edge-colorable [14, 23, 31], and optimal edge colorings can be computed efficiently in these cases. The case $\Delta = 6$ is still open, while for $\Delta = 3, 4, 5$ there are planar graphs of maximum degree Δ that are not Δ-edge-colorable [29], and at least for $\Delta = 4, 5$ the Δ-ᴇᴅɢᴇ Cᴏʟᴏʀᴀʙɪʟɪᴛʏ-problem is suspected to be NP-complete for planar graphs [6].

Generalizing Conjecture 15, Seymour's Exact Conjecture [25] states that every planar graph G is $\lceil \eta'(G) \rceil$-edge-colorable, where $\eta'(G)$ denotes the fractional chromatic index of G. It is worth noting that Seymour's Exact Conjecture implies Vizing's Conjecture, as well as the Four-Color-Theorem; see e.g., the recent survey [5].

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