Available online at www.sciencedirect.com

ScienceDirect

IFAC Papers Online CONFERENCE PAPER ARCHIVE

IFAC PapersOnLine 58-27 (2024) 219–224

Lyapunov-Krasovskii Functionals of Robust Type and Their Legendre-Tau-Based Approximation

Tessina H. Scholl * **Veit Hagenmeyer** * **Lutz Gröll** *

* *Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany (e-mail: {tessina.scholl, veit.hagenmeyer, lutz.groell}@kit.edu).*

Abstract: Lyapunov-Krasovskii functionals of robust type have recently been introduced for the stability analysis in time-delay systems. They are inspired by the wide-spread Lyapunov-Krasovskii functionals of complete type, but aim at improved robustness results. The present paper recaps the most important aspects and proposes a numerical method to solve the defining equation of these functionals numerically. The numerical method relies on a Legendre-tau-based ODE approximation of the time-delay system and has recently been shown to yield convincing results for the finite-dimensional approximation of Lyapunov-Krasovskii functionals of complete type. In the latter case, a Lyapunov equation has to be solved. In the case of Lyapunov-Krasovskii functionals of robust type, an algebraic Riccati equation has to be solved instead.

Keywords: Lyapunov-Krasovskii functional, spectral methods, robustness, absolute stability

1. INTRODUCTION

An important application of complete-type Lyapunov-Krasovskii (LK) functionals (Kharitonov and Zhabko, 2003) and of related concepts (Alexandrova, 2020) is the question of robustness. The LK functional is constructed for an asymptotically stable nominal linear system $\dot{x}(t) = A_0x(t) + A_1x(t-h)$. It aims at characterizing admissible added terms in

$$\dot{x}(t) = \underbrace{A_0x(t) + A_1x(t-h)}_{f(x_t)} + \underbrace{\tilde{g}(x(t), x(t-h))}_{g(x_t)} \quad (1)$$

($A_0, A_1 \in \mathbb{R}^{n\times n}$, $\tilde{g}$ is locally Lipschitz, $\tilde{g}(0_n, 0_n) = 0_n$). For instance, $\tilde{g}(x(t), x(t-h))$ might address higher order remainder terms from a linearization, a saturation nonlinearity, or additional linear terms that amount to uncertainties in the coefficient matrices A_0, A_1. The robustness statement derivable from complete-type LK functionals provides a linear norm bound on such terms

$$\|\tilde{g}(x(t), x(t-h))\|_2 \leq \gamma \left\| \begin{bmatrix} x(t-h) \\ x(t) \end{bmatrix} \right\|_2, \quad \gamma < \gamma_{\max} \quad (2)$$

with some $\gamma_{\max} > 0$, see Melchor-Aguilar and Niculescu (2007). If this perturbation restriction is satisfied, the stability of the zero equilibrium of the nominal system is not compromised by the perturbation $\tilde{g}(x(t), x(t-h))$.

A main advantage of LK-functional-based methods compared to mere frequency domain methods (e.g., Bliman (2000)) is the ability to cope with nonlinearities that reside only locally within the perturbation restriction. In that case, a regional stability statement can be derived in further steps; Melchor-Aguilar and Niculescu (2007); Villafuerte and Mondié (2007); Alexandrova (2020). In order to obtain such an estimation of the domain of attraction, the existence statement of the LK functional and the resulting perturbation restriction is not enough. Such applications also require the explicit knowledge of the LK functional (to make sense of a sublevel set) as well as non-conservative bounds on that functional. The common formula for the computation of complete-type LK functionals is based on an a priori calculation of the so-called delay Lyapunov matrix function (Kharitonov, 2013). As an alternative, we have recently proposed an approach to derive LK functionals of complete type numerically without such an intermediate step (Scholl et al., 2024). The approach also comes along with improved bounds on the functional. It relies on an ODE approximation of the time-delay system. Only a finite-dimensional Lyapunov equation has to be solved to obtain a finite-dimensional approximation of the LK functional of complete type. This numerical approach however can also be applied to more general defining equations of LK functionals and thus paves the way to a more adapted construction of the LK functional.

The prevalent problem of LK functionals of complete type is that the obtainable robustness statement is very restrictive, i.e., $\gamma_{\max}$ in (2) is small. This to improve has been the main objective of the introduction of LK functionals of robust type in Scholl (2023). So far, however, only the definition of LK functionals of robust type, their existence, the achievable robustness statements, and properties of the functional have been presented. The question of how to compute the functional remained unanswered, despite of being crucial for the above described application.

To answer this question is the objective of the present paper. We are going to use the above mentioned ODE-based approach that has proven to yield convincing results for LK functionals of complete type in Scholl et al. (2024). The defining equation of LK functionals of robust type is more involved, but in terms of the matrix equation that arises in the numerical approach, in the end, the Lyapunov equation

10.1016/j.ifacol.2024.10.327

will simply be replaced by an algebraic Riccati equation. As a result, the implementation effort remains similarly low, while, at the same time, the obtained functional comes along with a significantly improved robustness statement.

The paper is organized as follows. First, the most important aspects of LK functionals of robust type are discussed in Section 2. Then, Section 3 presents the proposed numerical approach. Its applicability is finally confirmed by an example in Section 4, before Section 5 concludes the paper.

2. LK FUNCTIONALS OF ROBUST TYPE

LK functionals of robust type have the structure

$$\begin{aligned} V(\phi) = \phi^\top(0)\, P_{\mathrm{xx}}\, \phi(0) + 2\int_{-h}^{0} \phi^\top(0)\, P_{\mathrm{xz}}(\eta)\phi(\eta)\,\mathrm{d}\eta \\ + \int_{-h}^{0}\int_{-h}^{0} \phi^\top(\xi) P_{\mathrm{zz}}(\xi,\eta)\phi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \\ + \int_{-h}^{0} \phi^\top(\eta) P_{\mathrm{zz,diag}}\phi(\eta)\,\mathrm{d}\eta, \end{aligned} \tag{3}$$

where $P_{\mathrm{xx}}, P_{\mathrm{zz,diag}} \in \mathbb{R}^{n\times n}$, $P_{\mathrm{xz}} \in L_2([-h,0],\mathbb{R}^{n\times n})$, and $P_{\mathrm{zz}} \in L_2([-h,0]\times[-h,0],\mathbb{R}^{n\times n})$. Thus, the structure is the one known from complete-type LK functionals, respectively, it is slightly simpler since $P_{\mathrm{zz,diag}}(\eta) \equiv P_{\mathrm{zz,diag}} \in \mathbb{R}^{n\times n}$ is constant. Note that complete-type LK functionals (Kharitonov, 2013, Def. 2.7) are defined via their derivative D_f^+V along solutions of the unperturbed linear nominal system $\dot{x}(t) = A_0x(t) + A_1x(t-h) =: f(x_t)$. In order to be a complete-type LK functional, that derivative must have a very special structure. It depends on three arbitrarily chosen positive definite matrices that have to be chosen a priori when constructing such a functional. LK functionals of robust type are also defined in terms of the derivative D_f^+V. However, the derivative does not show the special structure required for being a complete-type LK functional. Moreover, contrary to complete-type LK functionals, the derivative is not prescribed by a fixed ansatz with arbitrarily chosen matrices.

Instead, D_f^+V in the defining equation of LK functionals of robust type is tailored to the class of perturbations $g(x_t)$ in (1) that shall be tackled in the robustness analysis. To this end, it incorporates a characterization of the perturbation $g(x_t)$ in terms of the perturbation structure and in terms of a compatible sector-shaped perturbation restriction. We are going to take a closer look at these ingredients before the defining equation is finally stated in Sec. 2.3.

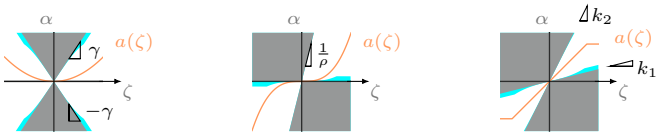
2.1 Perturbation Structure

The perturbation structure is described via a tuple $(B,\mathcal{C})$, referring to the decomposition of the perturbation as

$$g(x_t) = -Ba(\mathcal{C}x_t), \tag{4}$$

where $x_t\colon [-h,0]\to\mathbb{R}^n; x_t(\theta) = x(t+\theta)$ denotes the state. The matrix $B\in\mathbb{R}^{n\times m}$ encodes which components of the system equation are affected by the injection of the perturbation, whereas the operator $\mathcal{C}\colon C([-h,0],\mathbb{R}^n)\to\mathbb{R}^p$ generates the argument of the core nonlinearity $a\colon \mathbb{R}^p\to\mathbb{R}^m$. Since the perturbation only depends on $x(t) = x_t(0)$ and $x(t-h) = x_t(-h)$, this operator $\mathcal{C}$ takes the form

$$\mathcal{C}\phi = \begin{bmatrix} C_1\phi(-h) \\ C_0\phi(0)\end{bmatrix}, \text{ i.e., } \mathcal{C}x_t = \begin{bmatrix} C_1\, x(t-h) \\ C_0\, x(t)\end{bmatrix}, \tag{5}$$



(a) $\Pi_{\zeta\zeta} = \gamma^2$, $\Pi_{\zeta a} = 0$, $\Pi_{aa} = -1$

(b) $\Pi_{\zeta\zeta} = 0$, $\Pi_{\zeta a} = \frac{1}{2}$, $\Pi_{aa} = -\rho$

(c) $\Pi_{\zeta\zeta} = -k_1k_2$, $\Pi_{\zeta a} = \frac{k_1+k_2}{2}$, $\Pi_{aa} = -1$

Fig. 1. If $p=m=1$, then $w(\zeta,\alpha)\geq 0$ with w from (7) describes a sector in the (ζ,α) plane (non-gray region).

$C_0\in\mathbb{R}^{p_0\times n}, C_1\in\mathbb{R}^{p_1\times n}$. The result of (5) has the overall number of rows $p = p_0+p_1$. A vanishing C_0 or C_1, i.e., $p_0 = 0$ or $p_1 = 0$, is also possible. Still, to establish some desired properties in the later defined LK functional (e.g., certain lower bounds on $V(\phi)$ or upper bounds on $D^+_{(f+g)}V(\phi)$), full rank matrices for C_0 and/or C_1 might be desirable. Consider a simple example (Scholl, 2023).

Example 2.1. In the described framework,

$$g(x_t) = \begin{bmatrix} 0 \\ -x_1^3(t-h)\end{bmatrix}$$

can be expressed via $B = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $C_1 = [1\ 0]$, and $p_0 = 0$, with the core nonlinearity $a(\zeta) = \zeta^3$ in (4). An alternative choice is $B = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, full-rank matrices $C_1 = \left[\begin{smallmatrix}1&0\\0&\varepsilon\end{smallmatrix}\right]$, $C_0 = \varepsilon I_2$, $\varepsilon\in\mathbb{R}$, and $a([\zeta_1,\zeta_2,\zeta_3,\zeta_4]^\top) = \zeta_1^3$.

As a consequence, the perturbation restriction, which will be discussed in the sequel, only needs to refer to the specific function $\zeta\mapsto a(\zeta)$ from (4). Choosing

$$B = C_0 = C_1 = I_n \tag{6}$$

leads again to an unstructured consideration of the overall $\tilde{g}(x(t),x(t-h)) \overset{(4),(1)}{=} -Ba\left(\left[\begin{smallmatrix}C_1x(t-h)\\C_0x(t)\end{smallmatrix}\right]\right) \overset{(6)}{=} -a\left(\left[\begin{smallmatrix}x(t-h)\\x(t)\end{smallmatrix}\right]\right)$ as in the result (2) from complete-type LK functionals.

2.2 Perturbation Restriction

The concept allows to choose a type of perturbation restriction that fits best to the nonlinearity $a(\cdot)$.

Example 2.2. For simplicity, consider $p = m = 1$, i.e. $\zeta\mapsto\alpha = a(\zeta)$ in (4) is a scalar map. A possible type of perturbation restriction is a linear norm bound $|a(\zeta)|\leq\gamma|\zeta|$, see Fig. 1a. The robustness statement will then describe the maximum slope γ similarly to the statement known from complete-type LK functionals in (2). However, for a nonlinearity like $a(\zeta) = \zeta^3$, the sector in Fig. 1b fits much better. The robustness statement then describes the admissible upper slope $\frac{1}{\rho}$ which might be considerably larger than γ. As a result, the range of ζ, for which $a(\zeta) = \zeta^3$ resides within the sector, will also be larger. For a saturation nonlinearity, we rather choose Fig. 1c with a fixed k_2, and determine the admissible lower slope k_1.

The perturbation restriction will be specified via three matrices $(\Pi_{\zeta\zeta},\Pi_{\zeta a},\Pi_{aa})$, where the last one Π_{aa} is required to be negative definite ($\Pi_{\zeta\zeta} = \Pi_{\zeta\zeta}^\top\in\mathbb{R}^{p\times p}, \Pi_{aa} = \Pi_{aa}^\top\in\mathbb{R}^{m\times m}, \Pi_{\zeta a}\in\mathbb{R}^{p\times m}$). These matrices define an indefinite quadratic form in $[\zeta^\top,\alpha^\top]^\top$

$$w(\zeta,\alpha) := \zeta^\top\Pi_{\zeta\zeta}\zeta + 2\zeta^\top\Pi_{\zeta a}\alpha + \alpha^\top\Pi_{aa}\alpha. \tag{7}$$

If $p = m = 1$, then $w(\zeta, \alpha) \geq 0$ describes a sector in the (ζ, α) plane shown in Fig. 1. The nonlinearity $a(\zeta)$ might satisfy the accordingly defined perturbation restriction

$$w(\zeta, a(\zeta)) \geq 0 \tag{8}$$

either locally for some ζ or even globally for all $\zeta \in \mathbb{R}^p$.

The simplest perturbation restriction is described by

$$\Pi_{\zeta\zeta} = \gamma^2 I_p, \quad \Pi_{\zeta a} = 0_{p\times m}, \quad \Pi_{aa} = -I_m \tag{9}$$

with some parameter $\gamma > 0$. It amounts by (7) and (8) to

$$w(\zeta, a(\zeta)) = \gamma^2 \zeta^\top \zeta - a^\top(\zeta) a(\zeta) \geq 0. \tag{10}$$

Thus, (9) stands for $a^\top(\zeta) a(\zeta) \leq \gamma^2 \zeta^\top \zeta$, which is nothing else than the linear norm bound

$$\|a(\zeta)\|_2 \leq \gamma \|\zeta\|_2. \tag{11}$$

In the unstructured case (6), the latter describes the same type of restriction like (2) from complete-type LK functionals, but γ can become significantly larger. These admissible values of γ depend on the robustness of the nominal system under the given perturbation structure $(B, \mathcal{C})$. Namely, all

$$\gamma < \gamma_{\max} := \frac{1}{\sup_{\omega\in\mathbb{R}} \|G(\mathrm{i}\omega)\|_2} = \frac{1}{\|G\|_{H_\infty}} \tag{12}$$

are admissible, based on the transfer function

$$G(s) = \begin{bmatrix} C_1 \mathrm{e}^{-sh} \\ C_0 \end{bmatrix} (sI - A_0 - \mathrm{e}^{-sh} A_1)^{-1} B. \tag{13}$$

If $\gamma < \gamma_{\max}$, then an LK functional of robust type exists, i.e., the defining equation that will be described below is solvable. If, however, $\gamma > \gamma_{\max}$, then no LK functional of robust type exists. In Scholl (2023), both statements are proven by means of an operator-valued version of the Kalman-Yakubovich-Popov lemma (Likhtarnikov and Yakubovich, 1977, Thm. 3). For admissible parameter values (cf. ρ and k_1 in Fig. 1) arising in other types of perturbation restrictions, see Scholl (2023).

2.3 Defining Equation

A final term that will also be encountered is

$$v(\phi) := P_{\mathrm{xx}}\, \phi(0) + \int_{-h}^{0} P_{\mathrm{xz}}(\eta) \phi(\eta)\, \mathrm{d}\eta, \tag{14}$$

which relies on a part of the searched functional (3). Altogether, we have the following definition (Scholl, 2023).

Definition 2.1. (LK functional of robust type).
A functional $V\colon C([-h,0],\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ that has the structure (3) is called a *L*yapunov-Krasovskii functional of robust type with respect to

- the nominal linear system $\dot{x}(t) = f(x_t)$,
- the perturbation structure $(B, \mathcal{C})$, and
- the perturbation restriction matrices $(\Pi_{\zeta\zeta}, \Pi_{\zeta a}, \Pi_{aa})$

if for all $\phi \in C([-h,0],\mathbb{R}^n)$ it holds

$$\begin{aligned} D^+_f V(\phi) = &-(\mathcal{C}\phi)^\top \Pi_{\zeta\zeta} \mathcal{C}\phi - \Big[v^\top(\phi) B \\ &- (\mathcal{C}\phi)^\top \Pi_{\zeta a}\Big] (-\Pi_{aa})^{-1} \Big[B^\top v(\phi) - \Pi_{\zeta a}^\top \mathcal{C}\phi\Big], \end{aligned} \tag{15}$$

where $v\colon C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n$ is given by (14).

Using the linear-norm-bound perturbation restriction (9), the defining equation (15) simplifies to

$$D^+_f V(\phi) = -\gamma^2 (\mathcal{C}\phi)^\top \mathcal{C}\phi - v^\top(\phi) B B^\top v(\phi), \tag{16}$$

where $(\mathcal{C}\phi)^\top \mathcal{C}\phi = \phi^\top(0) C_0^\top C_0 \phi(0) + \phi^\top(-h) C_1^\top C_1 \phi(-h)$.

2.4 What is Achieved by the Defining Equation

For complete-type LK functionals, the restrictive bound on the perturbation in (2) shall ensure that the perturbation cannot turn the nonpositive chosen derivative $D^+_f V(\phi)$ into a $D^+_{(f+g)} V(\phi)$ that is no longer nonpositive. In contrast, for LK functionals of robust type, nonpositivity of $D^+_{(f+g)} V(\phi)$ is – by construction – ensured for all ϕ for which

$$w(\mathcal{C}\phi, a(\mathcal{C}\phi)) \geq 0, \tag{17}$$

i.e., only the perturbation restriction (8) must be satisfied. This result emerges from $\ell(\zeta) \equiv 0$ in the following theorem.

Theorem 2.1. (Scholl (2023)). Let V be an LK functional of robust type (Definition 2.1). For any $\phi \in C([-h,0],\mathbb{R}^n)$ for which the perturbation restriction is exceeded by a given offset function $\ell\colon \mathbb{R}^p \to \mathbb{R}$ in the sense of

$$w(\mathcal{C}\phi, a(\mathcal{C}\phi)) \geq \ell(\mathcal{C}\phi), \tag{18}$$

where w is defined in (7), the derivative of V along solutions of the perturbed equation satisfies

$$D^+_{(f+g)} V(\phi) \leq -\ell(\mathcal{C}\phi). \tag{19}$$

To make the result more plausible we shortly recap the proof for the special case of (16), i.e., a linear norm bound.

Proof. The effect of the perturbation on the derivative can (analogously to a known result from complete-type LK functionals, cf. Kharitonov (2013)) shown to be $D^+_{(f+g)} V(\phi) = D^+_f V(\phi) + 2v^\top(\phi) g(\phi)$. By (4), we have

$$\begin{aligned} D^+_{(f+g)} V(\phi) &= D^+_f V(\phi) - 2v^\top(\phi) B a(\mathcal{C}\phi) \\ &\overset{(16)}{=} -\gamma^2 (\mathcal{C}\phi)^\top \mathcal{C}\phi - v^\top(\phi) B B^\top v(\phi) - 2v^\top(\phi) B a(\mathcal{C}\phi) \\ &\overset{(10)}{=} -w(\mathcal{C}\phi, a(\mathcal{C}\phi)) - \|B^\top v(\phi) + a(\mathcal{C}\phi)\|_2^2 \overset{(18)}{\leq} -\ell(\mathcal{C}\phi). \end{aligned}$$

□

The effect of a small positive definite function $\ell(\zeta)$ in (18) is indicated by the turquoise shading in Fig. 1. In terms of the linear norm bound, choosing a quadratic $\ell(\zeta) = k_3 \|\zeta\|_2^2$ with some small $k_3 > 0$ means that the reduced bound

$$\|a(\mathcal{C}x_t)\|_2 \leq \sqrt{\gamma^2 - k_3}\, \|\mathcal{C}x_t\|_2 \tag{20}$$

(which is (18) combined with (10) for $\ell(\zeta) = k_3 \zeta^\top \zeta$) already implies

$$D^+_{(f+g)} V(x_t) \leq -k_3 \|\mathcal{C}x_t\|_2^2. \tag{21}$$

Besides of an upper bound on $D^+_{(f+g)} V(\phi)$, also a lower bound on $V(\phi)$ is of interest for the usability of the LK functional. See Scholl (2023) for respective results.

Altogether, for the linear norm bound and with full rank matrices C_0 and C_1, LK functionals of robust type are very similar to LK functionals of complete type. They share all the advantages but overcome the problem of a usually much too restrictive robustness statement.

3. THE NUMERICAL APPROACH

The LK functional V maps the state function $x_t = \phi \in C([-h,0],\mathbb{R}^n)$ to a scalar value $V(x_t) \in \mathbb{R}$. The idea of the numerical approach applied in the present paper is to map instead a polynomial approximation of the state function to a scalar value. Since a polynomial is

uniquely represented by a finite number of coefficients (e.g., Legendre coordinates, interpolation coordinates, or monomial coefficients), which can be collected in a vector c, such an approximation of the quadratic LK functional simplifies to a quadratic form

$$V(\phi) \approx c^\top P_c c \tag{22}$$

in the polynomial coordinates c. Thus, it is fully described by a finite dimensional matrix P_c. The coordinates c will be created from $\phi = x_t$ via a projection that relies on taking integrals and on $\phi(0)$. Therefore, the resulting $c^\top P_c c$, when written out in terms of the original function $\phi = x_t$, even stands for an expression having the form (3) with some explicitly known kernel functions. However, there is no need to write it out since the finite-dimensional quadratic form $c^\top P_c c$ is much simpler to handle.

The numerical approach, which thus only has to find an appropriate matrix P_c in (22), directly tackles the defining equation of the LK functional. The defining equation (15) refers to the derivative of $V(x_t)$ along the solution of the nominal linear time-delay system, i.e., the directional derivative in the direction in which the state x_t evolves. Thus, in view of $c^\top P_c\, c$, we have to clarify how coordinates $c(t)$ that provide an approximation of x_t evolve with time t. Spectral methods like the Legendre tau method, which has been applied to time-delay systems by Ito and Teglas (1986), exactly address that task. See Hesthaven et al. (2007) for a general introduction. The result is an ODE

$$\dot c = A_c\, c \tag{23}$$

that describes the evolution of such polynomial coordinates. Consequently, the left-hand side of the defining equation (15) becomes handleable in terms of matrices

$$D_f^+ V(\phi) \approx D^+_{(\dot c = A_c c)} c^\top P_c c = c^\top P_c A_c c + c^\top A_c^\top P_c c. \tag{24}$$

In contrast to complete-type LK functionals, the right-hand side of the defining equation (15) also depends on the unknown functional. In total, an algebraic Riccati equation will arise in (50), from which P_c can be computed. Regarding the underlying operator-valued equation, cf. Scholl (2023), the result is along the lines of Ito and Teglas (1987), who approximate operator-valued algebraic Riccati equations that arise in optimal control problems.

3.1 Polynomial Approximation of the State x_t

Let us make the approach sketched above more explicit. The state function $\theta \mapsto x_t(\theta)$ at each time $t \geq 0$ shall be approximated by a polynomial on $\theta \in [-h, 0]$. A polynomial of degree N is uniquely determined by $N+1$ coordinates which represent the coefficients of some basis polynomials. Legendre basis polynomials $p_k \colon [-1,1] \to \mathbb{R}$ are originally defined on $[-1,1]$ and therefore first have to be scaled and shifted via a composition with

$$\vartheta \colon [-h,0] \to [-1,1]; \qquad \theta \mapsto \vartheta(\theta) := \tfrac{2}{h}\theta + 1 \tag{25}$$

to cope with the domain $[-h,0]$. Then, the approximating polynomial for x_t at time $t \geq 0$ can be written as

$$x_t(\theta) \approx \sum_{k=0}^{N} c^k(t)\, p_k(\vartheta(\theta)). \tag{26}$$

Since $x_t(\theta)$ is $\mathbb{R}^n$-valued, any Legendre coordinate $c^k(t)$ is also $\mathbb{R}^n$-valued, referring to a vector-valued component of

$$c(t) = [(c^0(t))^\top, \dots, (c^N(t))^\top]^\top \in \mathbb{R}^{n(N+1)}. \tag{27}$$

3.2 Exact Evolution of the State x_t

In order to derive the time evolution of the coordinates $t \mapsto c(t) \in \mathbb{R}^{n(N+1)}$ that uniquely describe the approximated state, first the time evolution of the exact state $t \mapsto x_t \in C([-h,0],\mathbb{R}^n) =: C$ must be clarified. It obeys an abstract differential equation $\frac{\mathrm{d}}{\mathrm{d}t} x_t = \mathcal{A} x_t$ on the state space C, respectively, noting that $\phi \in C$ gives rise to

$$\left[\begin{smallmatrix} \phi(\cdot) \\ \phi(0) \end{smallmatrix}\right] \in C \times \mathbb{R}^n \subset L_2 \times \mathbb{R}^n =: M_2, \tag{28}$$

an abstract differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\begin{smallmatrix} x_t(\cdot) \\ x_t(0) \end{smallmatrix}\right] = \mathscr{A} \left[\begin{smallmatrix} x_t(\cdot) \\ x_t(0) \end{smallmatrix}\right] \tag{29}$$

for the embedding in the larger space M_2. So far, however, it suffices to recognize the underlying PDE that arises if not only the time t but both t and θ are understood as independent variables in $(t,\theta) \mapsto x_t(\theta) \in \mathbb{R}^n$. Note that $x_t(\theta) := x(t+\theta)$ inevitably has equal derivatives w.r.t. both t and θ and that $\dot x(t) = A_0 x(t) + A_1 x(t-h)$ relates $\dot x(t) = \frac{\partial}{\partial t} x_t(\theta)|_{\theta=0}$ with $x(t) = x_t(0)$ and $x(t-h) = x_t(-h)$. Therefore, $(t,\theta) \mapsto x_t(\theta)$ obeys

$$\frac{\partial}{\partial t} x_t(\theta) = \frac{\partial}{\partial \theta} x_t(\theta), \qquad \theta \in [-h,0), t \geq 0, \tag{30a}$$

$$\frac{\partial}{\partial t} x_t(0) = A_0 x_t(0) + A_1 x_t(-h), \qquad t \geq 0. \tag{30b}$$

3.3 A_c in the Legendre-Tau-Based ODE Approximation

We are going to apply the Legendre tau method to (30) in order to obtain A_c in (23), see also (Ito and Teglas, 1986).

Assume for a moment that $\phi : [-h,0] \to \mathbb{R}; \theta \mapsto \phi(\theta)$ is a scalar polynomial of degree N, described by its Legendre coordinates $c \in \mathbb{R}^{N+1}$. Its derivative $\theta \mapsto \phi'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\phi(\theta)$ is a polynomial of degree $N-1$ and thus can still be represented in the same basis. In fact, ϕ' is exactly represented by the Legendre coordinates $D_c c$, where the differentiation matrix $D_c \in \mathbb{R}^{(N+1)\times(N+1)}$ is given by

$$D_c = \vartheta' \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 3 & 0 & \dots & 3 \\ 0 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & \dots & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 7 & 0 & \dots & 7 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 9 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & \dots & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 15 \\ & & & & & & & & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{with } \vartheta' = \frac{2}{h}, \tag{31}$$

cf. (Hesthaven et al., 2007, eq. (5.8)), with ϑ' from (25); exemplarily, (31) is shown for N even, otherwise a last column $[1,0,5,0,9,\dots,0,(2N-1),0]^\top$ has to be appended.

Thus, if $\theta \mapsto x_t(\theta)$ was a scalar polynomial of degree at most N, which is uniquely represented by its Legendre coordinates $c(t)$, then (30a) is represented by $\dot c = D_c c$. If it was $\mathbb{R}^n$-valued, with c stacked as in (27), then (30a) was exactly described by the coordinate representation

$$\dot c = (D_c \otimes I_n) c. \tag{32}$$

However, the boundary condition (30b) must somehow be incorporated. This is where the various spectral methods differ (in fact, the differentiation matrices for different coordinate choices, e.g., D_y for interpolation coordinates used in Breda et al. (2016) and D_c for Legendre coordinates, are related by a similarity transform). The Legendre tau method replaces the last row of (32) by an equation

that addresses the boundary condition (30b). For the ansatz (26), (30b) becomes

$$\sum_{k=0}^{N} \dot{c}_k(t) \underbrace{p_k(\vartheta(0))}_{1} = A_0 \sum_{k=0}^{N} c^k(t) \overbrace{p_k(\vartheta(0))}^{1} + A_1 \sum_{k=0}^{N} c^k(t) \overbrace{p_k(\vartheta(-h))}^{(-1)^k} \quad (33)$$

The first part of $\sum_{k=0}^{N} \dot{c}^k(t) = \sum_{k=0}^{N-1} \dot{c}^k(t) + \dot{c}^N(t)$ on the left-hand side is the sum of the rows in (32), which yield

$$\sum_{k=0}^{N-1} \dot{c}^k(t) = \vartheta' \sum_{k=0}^{N} \frac{k(k+1)}{2} c^k(t). \quad (34)$$

Altogether, the ODE approximation from the Legendre tau method (Ito and Teglas, 1986) is (23) with

$$A_c = D_c \otimes I_n + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ A_0+A_1 & A_0-A_1 & A_0+A_1 & \cdots & A_0+(-1)^N A_1 \end{bmatrix} + \frac{2}{h} \begin{bmatrix} 0 & & & & & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -3 & -6 & -10 & \cdots & -\frac{N(N+1)}{2} \end{bmatrix} \otimes I_n. \quad (35)$$

3.4 Deriving c for a Given Function ϕ

Let an argument $\phi \in C([-h,0],\mathbb{R}^n)$ of $V(\phi)$ in (22) or an initial function $\phi = x_0$ be given. The function ϕ might already be a polynomial of degree at most N. Then the coordinates to be chosen are the Legendre coordinates that represent ϕ exactly. Clearly, ϕ can also be represented by any other coordinate choice, which might be easier to derive. For instance, interpolation coordinates are simply the pointwise evaluations

$$y^k = \phi(\tilde{\theta}_k) \quad (36)$$

on some grid $\{\tilde{\theta}_k\}_{k\in\{0,\dots,N\}}$ (preferably Chebyshev nodes). The coordinates $y = (y^k)_{k\in\{0,\dots,N\}}$ can be converted to

$$c = T_{cy} y \quad (37)$$

via a transformation matrix T_{cy} (see Scholl et al. (2024) for implementation hints) which establishes that

$$\phi(\theta) = \sum_{k=0}^{N} y^k \ell_k(\vartheta(\theta)) = \sum_{k=0}^{N} c^k p_k(\vartheta(\theta)) \quad (38)$$

is a change from Lagrange interpolation basis polynomials $\{\ell_k(\vartheta(\cdot))\}_k$ to Legendre basis polynomials $\{p_k(\vartheta(\cdot))\}_k$. Even if ϕ is not yet a polynomial, the interpolating polynomial (38) that is built from the evaluations (36), using Chebyshev nodes for $\tilde{\theta}_k$, gives usually a very good approximation of ϕ. Therefore, (37) with (36) can in practice also be used for non-polynomial functions.

Strictly speaking, however, if ϕ is not yet a polynomial of degree at most N, the projection that gives the approximating polynomial (respectively its Legendre coordinates c) the Legendre tau method relies upon is as follows. First note that ϕ can be written as a Legendre series

$$\phi(\theta) = \sum_{k=0}^{\infty} \tilde{c}^k p_k(\vartheta(\theta)), \quad (39)$$

$$\text{with } \tilde{c}^k = \frac{2k+1}{2} \int_{-1}^{1} \phi(\vartheta^{-1}(\tilde{\vartheta})) p_k(\tilde{\vartheta}) \, \mathrm{d}\tilde{\vartheta}. \quad (40)$$

However, the $N+1$ coordinates $(c^k)_{k\in\{0,\dots,N\}}$ that define

$$\phi^{[N]}(\theta) := \sum_{k=0}^{N} c^k p_k(\vartheta(\theta)) \approx \phi(\theta) \quad (41)$$

are not straightforwardly taken from truncating that series (which would be the L_2-optimal polynomial approximation). Instead, analogously to the treatment of the residual in the Legendre tau method described above, the last coordinate $\tilde{c}^N$ is replaced and chosen such that the boundary value at $\theta = 0$, which is $\sum_{k=0}^{N} c^k p_k(\vartheta(0)) = \sum_{k=0}^{N} c^k$, coincides with $\phi(0)$. Consequently, cf. Ito and Teglas (1986),

$$c^k = \begin{cases} \tilde{c}^k, & \text{if } k < N, \\ \hat{x} - \sum_{k=0}^{N-1} \tilde{c}^k, & \text{if } k = N, \end{cases} \qquad \text{with } \hat{x} = \phi(0) \quad (42)$$

are the searched $N+1$ Legendre coordinates c in (41).

3.5 How to Determine P_c

Consider the system with in- and output

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \quad (43)$$

$$\zeta(t) = \mathcal{C} x_t = \begin{bmatrix} C_1 x(t-h) \\ C_0 x(t) \end{bmatrix}. \quad (44)$$

Combined with the feedback $u(t) = -a(\zeta(t))$, the latter describes the perturbed system $\dot{x}(t) = A_0 x(t) + A_1 x(t-h) - Ba(\left[\begin{smallmatrix} C_1 x(t-h) \\ C_0 x(t) \end{smallmatrix}\right])$. Its Legendre-tau-based discretization

$$\begin{aligned} \dot{c}(t) &= A_c c(t) + B_c u(t) \\ \zeta(t) &= C_c c(t) \end{aligned} \quad (45)$$

relies on A_c from (35) and on an input matrix

$$B_c = \left[\begin{smallmatrix} 0_{nN\times m} \\ B \end{smallmatrix}\right] \quad (46)$$

that reflects that (1) only affects the last row of the ODE approximation. Moreover, the output is

$$\mathcal{C}\phi \approx \mathcal{C}\phi^{[N]} = C_c c, \quad (47)$$

where, analogously to (33), $\phi^{[N]}(0) = \sum_{k=0}^{N} c^k$ and $\phi^{[N]}(-h) = \sum_{k=0}^{N} (-1)^k c^k$, and therefore

$$C_c = \begin{bmatrix} C_1 & -C_1 & C_1 & \cdots & (-1)^N C_1 \\ C_0 & C_0 & C_0 & \cdots & C_0 \end{bmatrix}. \quad (48)$$

Altogether, with $D_f^+ V(\phi) \approx c^\top (P_c A_c + A_c^\top P_c) c$ from (24), $\mathcal{C}\phi \approx C_c c$ from (47), and

$$v^\top(\phi)\, B \approx c^\top P_c B_c, \quad (49)$$

the finite-dimensional approximation of the defining equation (15) is the algebraic Riccati equation (ARE)

$$P_c A_c + A_c^\top P_c = -C_c^\top \Pi_{\zeta\zeta} C_c - \Big[P_c B_c - C_c^\top \Pi_{\zeta a} \Big] (-\Pi_{aa})^{-1} \Big[B_c^\top P_c - \Pi_{\zeta a}^\top C_c \Big] \quad (50)$$

for the searched matrix $P_c = P_c^\top$. In view of the convergence proof (see outlook), we are only interested in the so-called stabilizing solution of the ARE, which is a unique solution. For the special case (16), (50) simplifies to

$$P_c A_c + A_c^\top P_c = -\gamma^2 C_c^\top C_c - P_c B_c B_c^\top P_c. \quad (51)$$

3.6 Change of Coordinates

When using (37), it might be more convenient to transform the quadratic form once by

$$V(\phi) \approx c^\top P_c c = y^\top T_{cy}^\top P_c T_{cy} y = y^\top P_y y. \quad (52)$$

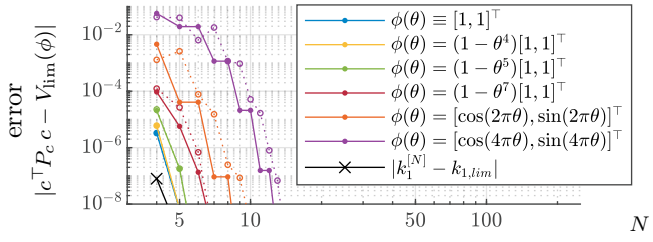


Fig. 2. Error between $c^\top P_c c$ (equivalently $\chi^\top P_\chi \chi$) and its converged value for some exemplary $\phi(\theta)$. See Sec. 4 for the considered problem. Solid lines rely on the correct c from (42), dotted lines on c from (36), (37).

An alternative coordinate choice are the mixed coordinates

$$\chi = [(c^0)^\top, \dots, (c^{N-1})^\top, \hat{x}^\top]^\top \tag{53}$$

where $\hat{x} = \sum_{k=0}^{N} c^k$, cf. (42). In particular,

$$P_\chi = T_{c\chi}^\top P_c T_{c\chi} \quad \text{with } T_{c\chi} = T_{\chi c}^{-1} = \begin{bmatrix} I_{nN} & 0_{nN\times n} \\ -1_N^\top \otimes I_n & I_n \end{bmatrix} \tag{54}$$

contains the Legendre coefficients of the kernel functions [1] in the approximation of $V(\phi)$, taking the special role of $\hat{x} = \phi(0)$ in (3) into account. From a numerical point of view, it is preferable to state the overall ARE (50) in these coordinates, replacing (A_c, B_c, C_c) by

$$(A_\chi, B_\chi, C_\chi) = (T_{\chi c} A_c T_{c\chi},\ T_{\chi c} B_c,\ C_c T_{c\chi}) \tag{55}$$

and replacing P_c by P_χ. In the end, $c^\top P_c c = \chi^\top P_\chi \chi$.

4. EXAMPLE

We consider the example

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t-1) + g(x_t), \tag{56}$$

which in (Kharitonov and Zhabko, 2003, Example 1) supplements the introduction of complete-type LK functionals. As already shown in Scholl (2023), $\gamma_{\max} = 0.0227$ from a LK functional of complete type in (2) improves to $\gamma_{\max} = \frac{1}{\|G\|_\infty} = 0.1059$ for an LK functional of robust type with $B = C_0 = C_1 = I_2$. If (56) stems from a second order system in x_1, it can be expected that $g(x_t) = \begin{bmatrix} 0 \\ g_2(x_t) \end{bmatrix}$. Choosing thus $B = [0,1]^\top$, even $\gamma_{\max} = \frac{1}{\|G\|_\infty} = 0.2462$ is obtained. A first observation is that $\gamma_{\max}^{[N]} = \frac{1}{\|G^{[N]}\|_\infty}$ from $G^{[N]}(s) = C_c(sI_n - A_c)^{-1}B_c$ converges rapidly, with $|\gamma_{\max}^{[N]} - \gamma_{\max}| < 10^{-8}$ for all tested $N \geq 4$. We choose $\gamma = (1-10^{-5})\gamma_{\max}$. The ARE (51) must be solved for P_c or, preferably, its counterpart with (55) must be solved for P_χ. We use the standard Matlab implementation `icare`. The convergence of the approximation of $V(\phi)$ for some exemplary $\phi \in C$ is shown in Fig. 2. The vector c is derived from the argument ϕ according to (42). As an alternative, we obtain c from the interpolating polynomial by applying (36) and (37). Besides of the possibility to evaluate $V(\phi)$, we are also interested in the lower bound on the functional. To this end, we compute the coefficient in $k_1^{[N]}\|\hat{x}\|_2^2 \leq c^\top P_c c = \chi^\top P_\chi \chi$, for which a formula is given in Scholl et al. (2024). This coefficient also converges rapidly as indicated by the black lower line in Fig. 2.

[1] If $p_1 \neq 0$ in (5), a splitting approach in Scholl (2023) establishes the separation between $P_{\mathrm{zz}}(\xi,\eta)$ and $P_{\mathrm{zz,diag}}$.

5. CONCLUSION AND OUTLOOK

The paper proposes a numerical approach for the recently introduced LK functionals of robust type $V(\phi)$. Only an algebraic Riccati equation (50) has to be solved for a matrix P_c. Then $c^\top P_c c$ approximates $V(\phi)$ based on coordinates c that represent a polynomial approximation of the argument ϕ. An example already indicates a rapid convergence as the approximation order N increases.

The proof that $c^\top P_c c$ indeed converges to $V(\phi)$ for any given $\phi \in C$ will, due to space reasons, be given separately. Using an operator-based description and a splitting approach from Scholl (2023), this proof benefits from a known convergence analysis by Ito and Teglas (1987) on Legendre-tau-based results for optimal control problems. Still, due to the different type of equation, aspects like the existence of solutions and uniform boundedness are proven differently.

REFERENCES

Alexandrova, I.V. (2020). On the robustness and estimation of the attraction region for a class of nonlinear time delay systems. *Appl. Math. Lett.*, 106, 106374.

Bliman, P.A. (2000). Extension of Popov absolute stability criterion to non-autonomous systems with delays. *Int. J. Control*, 73(15), 1349–1361.

Breda, D., Diekmann, O., Gyllenberg, M., Scarabel, F., and Vermiglio, R. (2016). Pseudospectral discretization of nonlinear delay equations: New prospects for numerical bifurcation analysis. *SIAM J. Appl. Dyn. Syst.*, 15(1), 1–23.

Hesthaven, J.S., Gottlieb, S., and Gottlieb, D. (2007). *Spectral methods for time-dependent problems*. Cambridge University Press, Cambridge.

Ito, K. and Teglas, R. (1986). Legendre–tau approximations for functional-differential equations. *SIAM J. Control Optim.*, 24(4), 737–759.

Ito, K. and Teglas, R. (1987). Legendre-tau approximation for functional differential equations part II: The linear quadratic optimal control problem. *SIAM J. Control Optim.*, 25(6), 1379–1408.

Kharitonov, V.L. and Zhabko, A.P. (2003). Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39(1), 15–20.

Kharitonov, V.L. (2013). *Time-delay systems: Lyapunov functionals and matrices*. Birkhäuser Springer, NY.

Likhtarnikov, A.L. and Yakubovich, V.A. (1977). The frequency theorem for continuous one-parameter semigroups. *Mathematics of the USSR-Izvestiya*, 11(4), 849.

Melchor-Aguilar, D. and Niculescu, S.I. (2007). Estimates of the attraction region for a class of nonlinear time-delay systems. *IMA J. Math. Control Inf.*, 24(4), 523–550.

Scholl, T.H. (2023). Lyapunov-Krasovskii functionals of robust type for the stability analysis in time-delay systems. *submitted to Automatica*. arXiv:2312.16738.

Scholl, T.H., Hagenmeyer, V., and Gröll, L. (2024). What ODE-approximation schemes of time-delay systems reveal about Lyapunov–Krasovskii functionals. *IEEE Trans. Automat. Contr.*, 69(7), 4614–4629.

Villafuerte, R. and Mondié, S. (2007). On improving estimate of the region of attraction of a class of nonlinear time delay system. *IFAC Proc. Vol.*, 40(23), 227–232.