IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 15 September 2024, accepted 11 October 2024, date of publication 28 October 2024, date of current version 8 November 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3486769

RESEARCH ARTICLE

The Age-of-Information Distribution in Slotted ALOHA

DIETER FIEMS[1], (Senior Member, IEEE), AND **ALEXEY VINEL**[2], (Senior Member, IEEE)

[1]Department of Telecommunications and Information Processing, Ghent University, 9000 Ghent, Belgium

[2]Institute of Applied Informatics and Formal Description Methods, Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany

Corresponding author: Dieter Fiems (Dieter.Fiems@UGent.be)

This work was supported by the framework of "Future Mobility Grants" program of the Innovation Campus Future Mobility (ICM).

ABSTRACT Age-of-Information (AoI) is a new performance metric for critical networked cyber-physical systems (CPS), where performance depends on how quickly fresh data is received in update packets. When multiple users communicate these updates over a shared channel in a random access manner, packet losses can occur due to overlapping transmissions. In this study, we analyse the AoI for slotted ALOHA, which is the classical medium access control protocol in communication networks. Our stochastic analysis provides exact results for the moments of the AoI at random slot boundaries and for the peak AoI, which is different from previous studies that rely on approximations. Additionally, we provide a recursive algorithm to calculate the probability mass function of the AoI. Finally, we explore how packet arrival and transmission probabilities impact the AoI through a numerical example. Our results, which characterise the AoI distribution in random access channels, have significant implications for the safety assessment of CPS in critical scenarios.

INDEX TERMS Age of information (AoI), probability generating function, slotted ALOHA.

I. INTRODUCTION

Cyber-Physical Systems (CPS) depend on the exchange of updates on physical processes facilitated by digital communication [1], [2], [3]. In the era of the Internet of Things, traditional networking metrics such as throughput and delay are no longer sufficient. The focus has shifted to Age-of-Information (AoI) or the freshness of available updates, as highlighted in literature [4], [5], [6], [7]. In mobile ad-hoc networks, communicating nodes normally use a dedicated random access channel for their mutual updates [8], [9]. The inherent broadcast feature of the radio channel naturally fits the communication goals since the updates are often intended for all the nodes in the immediate proximity [10], [11]. Furthermore, random access avoids the difficulty with centralised channel access coordination in highly dynamic network topologies [12], [13]. Hence, examining the AoI characterisation within ALOHA, a classical random access scheme [14], becomes significant for evaluating CPS performance [15], [16].

The associate editor coordinating the review of this manuscript and approving it for publication was Angel F. García-Fernández.

A prominent illustration of such a cyber-physical system in action is seen in cooperative autonomous vehicles, as highlighted by several studies [17], [18], [19]. In these application scenarios, a vehicular ad-hoc network is used for the exchange of periodic broadcast packets over a dedicated communication channel [20], [21], [22]. The content of these packets enhances vehicular awareness (for example awareness about the presence of vulnerable road users [23]), improves vehicular perception (e.g. via collective perception services [24]) or serves as an input for autonomous driving control loops (e.g. during cooperative manoeuvring [25]). Driving safety, which is the main argument for introducing vehicular communication, relies on the timeliness of status updates received from the surrounding road users [26], [27], [28]. It is thereby important to consider the complete distribution of the Age of Information (AoI) while evaluating the risks associated with packet losses in cooperative driving scenarios. More specifically, probabilities of the AoI exceeding certain threshold values characterise the lack of up-to-date information about the status of autonomous vehicles and, thus, the impossibility of safely accomplishing the driving manoeuvre in case of a hazardous road event [29].

© 2024 The Authors. This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 License.
For more information, see https://creativecommons.org/licenses/by-nc-nd/4.0/

The AoI tail distributions may also play a crucial role in the cooperative planning of pre-crash manoeuvring [30].

Both state-of-the-art vehicular networking technologies — broadband cellular approaches (e.g. 5G NR) and wireless local area-based systems (e.g. IEEE 802.11bd) — incorporate, at least in certain modes, random access mechanisms [31]. The industrial standards mentioned above are often approximated by the ALOHA protocol [32], [33], [34] at least via modelling back-off mechanisms by a fixed channel access probability. Such approximations date back to Bianchi's analysis of IEEE 802.11 [35].

Few ALOHA modifications have been considered in analytic AoI studies. In [36] and [37] a user transmits via slotted ALOHA only when its status update reaches a certain threshold. Moreover, a short control sequence ahead of the actual data transmission and respective carrier-sensing is considered in [38]. The AoI in frame slotted ALOHA is analysed in [39], [40], [41], and [42]. The age-aware back-off protocol is discussed in [43]. In the current work, we consider classical ALOHA with a fixed channel access probability. The paper extends the findings of [44] which studies the expected age and peak age of information in classical ALOHA. The present extension provides recursive expressions for higher order moments of the age and peak age, as well as recursive expressions for the age and peak age distribution.

There are but a few generic studies on the AoI in random access channels. Age-efficient transmission policies are proposed in [45] under the assumption that new packets can replace undelivered older ones. The relation between the retransmission policy and the peak AoI is investigated in [46]. Retransmission strategies to minimise the average AoI are proposed in [47]. In [48], a system with source nodes and relays is studied where source nodes neither buffer nor retransmit, while relay nodes do. Unfortunately, it is often practically not possible to adopt retransmission policies for broadcast (and not unicast) packets. Hence, various authors study bufferless systems without retransmissions. Allowing for packet drops naturally improves information freshness, since new packets contain the latest information [49]. The peak AoI is studied in [50] when the packet at the source can be updated if new information becomes available. A similar model is studied in [51] for Gilbert-Elliot channels with a focus on evaluating penalty functions of the peak AoI. The authors in [52] investigate how (not) replacing the information in the buffer affects the peak age of information as it may not be practically feasible to remove old packets already waiting in the queue. In this work, we made the practical assumption that the buffer size is limited to one packet which cannot be replaced and only one transmission attempt is allowed.

When studying the AoI, some authors adopt different assumptions about the random access system. For example, [53] assumes that the number of contending users is known. The ability for successive interference cancellation (allowing to decode overlapping packets) is incorporated in [54] and [55], and the capture effect is considered in [56]. Our work follows the classical random access vision: the number of users is unknown and the simultaneous transmission of any two (or more) of them makes the reception impossible. Our stochastic model includes the states of the different users, and the age of information for a particular user. We present an exact analysis which relies on probability generating function techniques to calculate various performance measures of interest.

The remainder of this paper is organised as follows. We introduce the AoI model and its analysis in the next section. We first study the packet arrival and departure processes and the probability generating function of the age of information. These probability generating functions are then used to obtain recursive expressions for the moments of the AoI, as well as for the probability mass function of the AoI. We then illustrate our results by a numerical example in section III and conclude in section IV.

II. STOCHASTIC MODEL AND ANALYSIS

To assess the age of information in slotted ALOHA, we examine a system with $M + 1$ users who communicate information over a shared channel. Each user possesses a buffer capable of storing a single piece of information or packet, if available. Each user with a packet accesses the channel with a fixed probability $0 \leq p \leq 1$, and a transmission is considered successful if no other users are transmitting simultaneously. If a transmission fails, the packet is lost and not retransmitted. New packets arrive at the users' buffers according to a Bernoulli process with a positive success probability $\lambda > 0$, in accordance with [57]. Generated packets are immediately stored in the buffer, such that the arrival time at the buffer equals the packet generation time. There may be multiple synchronous events at a slot boundary. We here assume that transmissions at a slot boundary precede arrivals: if a user transmits at a slot boundary and there is a new packet arrival, the old packet is transmitted and the new packet occupies the user's buffer. Moreover, the new packet cannot be transmitted at its arrival slot boundary.

Remark 1: Although the assumption that transmissions are only successful if no other users are transmitting is not true in general, the assumption holds for the context of our study. Cooperative autonomous vehicles in a vehicular ad-hoc network are located in a close proximity to each other and communication between them is short-range. This makes the received powers from the different users similar for the collided packets.

Remark 2: Buffering is necessary if one excludes the case of immediate transmission ($p = 1$). As the case $p = 1$ overloads the multiple access channel, the generation process of the packets is restricted by regulations. It can be argued that dropping old packets in the buffer and transmitting new ones is a better strategy from the freshness perspective [58]. However, from the perspective of the practical implementation, it is often impossible to remove the packet from the buffer.

We focus on a single user in particular. In the remainder, we refer to this user as the tagged user. For this tagged user we track the age of information at the receiver A_k at the beginning of slot k, and the age of information at the sender B_k at the beginning of slot k. In line with the literature, the age at the receiver equals the time since the arrival time of the last correctly received packet. The age at the sender is the time since the arrival of the last packet, if there is such a packet in the buffer. If the tagged user's buffer is empty, it is convenient to set $B_k = -1$. For the other users, we only track how many users have information, that is, how many users have a packet to send. Let X_k denote the number of users with a packet at the beginning of slot k.

A. THE NUMBER OF USERS WITH A PACKET

We now aim at relating the age of information of the tagged user at the sender B_k and receiver A_k at consecutive slot boundaries. To this end, we first study how the number of users with a packet evolves over time. Let X_k denote the number of users with a packet at the kth slot boundary. To explicitly express X_{k+1} in terms of X, let $\Lambda_{\ell,k}$ and $T_{\ell,k}$ denote the indicator that there is an arrival and transmission for the ℓth user, respectively. With the assumptions above, $\{\Lambda_{\ell,k}\}$ and $\{T_{\ell,k}\}$ are independent doubly indexed sequences of Bernoulli random variables with success probabilities λ and p, respectively. As the order of the users does not affect the evolution of X_k, we can assume that the users with packets have the lowest indices. This notational convenience then allows for the following explicit recursion,

$$X_{k+1} = \sum_{\ell=1}^{X_k} Q_{\ell,k} + \sum_{\ell=X_k+1}^{M} \Lambda_{\ell,k}. \tag{1}$$

Here, we introduced the random variables

$$Q_{\ell,k} \doteq 1 - T_{\ell,k} + T_{\ell,k}\Lambda_{\ell,k} \tag{2}$$

to simplify the notation. The random variable $Q_{\ell,k}$ is the indicator that the ℓth user with a packet at the kth slot boundary, has a packet at the $(k+1)$st slot boundary as well. Note that a user with a packet at the kth boundary still has one at the $k+1$st slot boundary if she did not transmit, or if she transmits and there is a new packet arrival. A user without a packet cannot transmit and has a packet at boundary $k+1$ with a new arrival. Clearly, each $Q_{\ell,k}$ is an independent Bernoulli distributed random variable, with success probability

$$q \doteq \mathsf{P}[Q_{\ell,k} = 1] = \lambda p + 1 - p.$$

Clearly, the process $\{X_k\}$ is a Markov process. This is immediate from the observation that equation (1) expresses X_{k+1} in terms of X_k and independent random variables. Let x_{mn} denote the transition probability from state $X_k = m$ to state $X_{k+1} = n$,

$$x_{mn} = \mathsf{P}[X_{k+1} = n | X_k = m],$$

for $m, n \in \mathcal{M} = \{0, 1, 2, \ldots, M\}$. By means of equation (1), and by invoking the independence of the random variables $\{Q_{\ell,k}\}$ and $\{\Lambda_{\ell,k}\}$ we find the following expression for the transmission probabilities,

$$\begin{aligned} x_{mn} &= \mathsf{P}\left[\sum_{\ell=1}^{m} Q_{\ell,k} + \sum_{\ell=m+1}^{M} \Lambda_{\ell,k} = n\right] \\ &= \sum_{i=0}^{n} \mathsf{P}\left[\sum_{\ell=1}^{m} Q_{\ell,k} = i\right] \mathsf{P}\left[\sum_{\ell=m+1}^{M} \Lambda_{\ell,k} = n - i\right]. \end{aligned}$$

Now both probabilities in the expression above concern the probabilities of fixed sums of Bernoulli random variables. As a sum of Bernoulli random variables follows a binomial distribution, we have

$$x_{mn} = \sum_{i=\max(0,n+m-M)}^{\min(n,m)} \binom{m}{i} q^i \bar{q}^{m-i} \binom{M-m}{n-i} \lambda^{n-i} \bar{\lambda}^{M-m-n+i}. \tag{3}$$

for $m, n \in \mathcal{M}$. Note that we here accounted for the range of the Binomial random variables by changing the upper and lower bounds of the summation. Moreover, we introduced the notation $\bar{q} = 1 - q$ and $\bar{\lambda} = 1 - \lambda$ for convenience.

For the evolution of the age of information, we need a more detailed description of the process $\{X_k\}$. To this end, let Y_k denote the indicator that there is a transmission at the kth slot boundary. The process $\{(X_k, Y_k)\}$ constitutes a marked Markov process, where X_k is the state of the Markov process, and Y_k is the mark. The process is now characterised by the marked transition probabilities. Let x^{t}_{mn} denote the probability that there is a transition from state m to state n with a transmission, and let x^{nt}_{mn} denote the probability that there is a transition from state m to state n without transmissions,

$$\begin{aligned} x^{\mathrm{t}}_{mn} &= \mathsf{P}[X_{k+1} = n, Y_k = 1 | X_k = m], \\ x^{\mathrm{nt}}_{mn} &= \mathsf{P}[X_{k+1} = n, Y_k = 0 | X_k = m]. \end{aligned}$$

We have no transmission ($Y_k = 0$) if and only if none of the non-tagged users transmits. Hence, we have $Y_k = 0$ if and only if $T_{\ell,k} = 0$ for all ℓ. Moreover, $T_{\ell,k} = 0$ implies $Q_{\ell,k} = \Lambda_{\ell,k}$, see equation (2). Combining these observations with equation (1), and accounting for the independence of $\Lambda_{\ell,k}$ and X_k, we can express the transition probability with no transmissions as follows,

$$x^{\mathrm{nt}}_{mn} = \mathsf{P}\left[m + \sum_{\ell=m+1}^{M} \Lambda_{\ell,k} = n, \sum_{\ell=1}^{m} T_{\ell,k} = 0\right].$$

Again we use the property that the sum of Bernoulli random variables follows a binomial distribution. Therefore, the expression above simplifies to,

$$x^{\mathrm{nt}}_{mn} = \begin{cases} \bar{p}^m \binom{M-m}{n-m} \lambda^{n-m} \bar{\lambda}^{M-n} & \text{for } m = 0, \ldots, n, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

for $m, n \in \mathcal{M}$.

To calculate the marked transition probabilities with transmissions, we can develop analogous arguments. It is however more convenient to note that there are either transmissions or no transmissions. Therefore the sum of the marked transition probabilities equals the transmission probability and we have,

$$x^{\mathrm{t}}_{mn} = x_{mn} - x^{\mathrm{nt}}_{mn}\,,$$

for $m, n \in \mathcal{M}$.

B. AGE OF INFORMATION

1) STOCHASTIC RECURSION

Having studied the evolution of the number of users that have a packet, we now turn to the evolution of the age of information of the tagged user. In line with the notation for the other users, let Λ_k denote the indicator that there is an arrival for the tagged user, and let T_k denote the indicator that the tagged user transmits, assuming there is a packet to transmit in the tagged user's buffer.

The age at the sender is the time since the arrival instant of the packet in the buffer (if any), or equivalently, the time since the packet in the buffer was generated. Hence, the age at the sender is incremented if there is a packet at the sender and there is neither arrival nor transmission. The age is reset to 0 if there is an arrival. In all other cases, the age is reset to -1, which indicates the absence of a packet. Hence, we can concisely express B_{k+1} in terms of B_k as follows,

$$B_{k+1} = \begin{cases} B_k + 1 & \text{if } T_k = \Lambda_k = 0\,, B_k \geq 0\,, \\ \Lambda_k - 1 & \text{otherwise.} \end{cases} \tag{5}$$

The age at the receiver is the time since the arrival of the last received packet. Hence, the age at the receiver is incremented as long as there is no successful transmission. If there is such a transmission, the new age is the age of the correctly received packet. More precisely, the age at the receiver is reset to B_k+1 if there is a successful transmission. Recall that one can only have a correct transmission if there is a packet ($B_k > 0$), if there is a transmission ($T_k = 1$) and if there are no other transmissions ($Y_k = 0$). Given these observations we can express the age at the $(k+1)$st slot in terms of A_k and B_k,

$$A_{k+1} = \begin{cases} B_k + 1 & \text{if } B_k \geq 0, T_k = 1, Y_k = 0\,, \\ A_k + 1 & \text{otherwise.} \end{cases} \tag{6}$$

2) JOINT GENERATING FUNCTION

In view of the stochastic recursions (1), (5) and (6), the process $\{(A_k, B_k, X_k), k \in \mathbb{N}\}$ constitutes a Markov process with state space $\mathbb{N} \times (\mathbb{N} \cup \{-1\}) \times \mathcal{M}$. We now study the limiting distribution of this Markov process. To this end, let $P_n(y,z)$ denote the following partial probability generating function,

$$\begin{aligned} P_n(y,z) &= \lim_{k\to\infty} P_n(y,z;k) \\ &= \lim_{k\to\infty} \mathsf{E}\left[y^{B_k+1} z^{A_k} 1_{\{X_k=n\}}\right]\,, \end{aligned}$$

for $|y| \leq 1$, $|z| \leq 1$ and $n \in \mathcal{M}$. Here $1_{\{\cdot\}}$ is the indicator function which evaluates to 1 if its argument is true, and to 0 otherwise. For notational convenience, we first condition on the number of users with a packet in the preceding slot and on the presence of transmissions of other users. We have,

$$P_n(y,z) = \sum_{m=0}^{M} (P^{\mathrm{t}}_{mn}(y,z) + P^{\mathrm{nt}}_{mn}(y,z))\,, \tag{7}$$

with $P^{\mathrm{t}}_{mn}(y,z) = \lim_{k\to\infty} P^{\mathrm{t}}_{mn}(y,z;k+1)$ and $P^{\mathrm{nt}}_{mn}(y,z) = \lim_{k\to\infty} P^{\mathrm{nt}}_{mn}(y,z;k+1)$, and with,

$$\begin{aligned} P^{\mathrm{t}}_{mn}(y,z;k+1) &= \mathsf{E}\left[y^{B_{k+1}+1} z^{A_{k+1}} 1_{\{X_{k+1}=n, X_k=m, Y_k=1\}}\right]\,, \\ P^{\mathrm{nt}}_{mn}(y,z;k+1) &= \mathsf{E}\left[y^{B_{k+1}+1} z^{A_{k+1}} 1_{\{X_{k+1}=n, X_k=m, Y_k=0\}}\right]\,, \end{aligned}$$

for $m, n \in \mathcal{M}$.

By conditioning on the number of arrivals, transmissions and the presence of a packet, we can express the partial joint probability generating functions at slot $k+1$ in terms of random variables at slot k. In view of the recursions (5) and (6), we find,

$$\begin{aligned} &P^{\mathrm{t}}_{mn}(y,z;k+1) \\ &\quad = \mathsf{E}\left[y^{B_k+1} z^{A_k+1} 1_{\{X_{k+1}=n, X_k=m, Y_k=1\}} F_k\right] \\ &\qquad + \mathsf{E}\left[y^{\Lambda_k} z^{A_k+1} 1_{\{X_{k+1}=n, X_k=m, Y_k=1\}} (1-F_k)\right]. \end{aligned}$$

where we introduced $F_k = 1_{\{T_k=\Lambda_k=0, B_k\geq 0\}}$ to simplify notation. A further simplification of the equation above by some standard z-transform manipulations and sending $k \to \infty$ yields,

$$\begin{aligned} P^{\mathrm{t}}_{mn}(y,z) &= yz\bar{\lambda}\bar{p}P_m(y,z)x^{\mathrm{t}}_{mn} + \bar{\lambda}\bar{p}(1-y)zP_m(0,z)x^{\mathrm{t}}_{mn} \\ &\quad + (y\lambda + \bar{\lambda}p)zP_m(1,z)x^{\mathrm{t}}_{mn}\,, \end{aligned} \tag{8}$$

for $|y| \leq 1$, $|z| \leq 1$ and $n \in \mathcal{M}$, with $\bar{p} = 1-p$. Analogously, conditioning on the number of arrivals, transmissions and the presence of a packet yields,

$$\begin{aligned} &P^{\mathrm{nt}}_{mn}(y,z;k+1) \\ &\quad = \mathsf{E}\left[y^{B_k+2} z^{A_k+1} 1_{\{X_{k+1}=n, X_k=m, Y_k=0\}} 1_{\{T_k=0, \Lambda_k=0, B_k\geq 0\}}\right] \\ &\qquad + \mathsf{E}\left[y z^{A_k+1} 1_{\{X_{k+1}=n, X_k=m, Y_k=0\}} 1_{\{T_k=0, \Lambda_k=1, B_k\geq 0\}}\right] \\ &\qquad + \mathsf{E}\left[y^{\Lambda_k} z^{B_k+1} 1_{\{X_{k+1}=n, X_k=m, Y_k=0\}} 1_{\{T_k=1, B_k\geq 0\}}\right] \\ &\qquad + \mathsf{E}\left[y^{\Lambda_k} z^{A_k+1} 1_{\{X_{k+1}=n, X_k=m, Y_k=0\}} 1_{\{B_k=-1\}}\right]. \end{aligned}$$

After some standard z-transform manipulations and sending $k \to \infty$, we have,

$$\begin{aligned} &P^{\mathrm{nt}}_{mn}(y,z) \\ &\quad = \bar{\lambda}\bar{p}yzP_m(y,z)x^{\mathrm{nt}}_{mn} \\ &\qquad + (\bar{\lambda} + \lambda y)p(P_m(z,1) - P_m(0,1))x^{\mathrm{nt}}_{mn} \\ &\qquad + yz\bar{p}\lambda P_m(1,z)x^{\mathrm{nt}}_{mn} + z(\bar{\lambda}(1-y) + py)P_m(0,z)x^{\mathrm{nt}}_{mn}\,, \end{aligned} \tag{9}$$

for $m, n \in \{0, 1, \ldots, M\}$, $|y| \leq 1$ and $|z| \leq 1$.

Plugging equations (8) and (9) in (7), yields the following system of equations,

$$\begin{aligned}
&P_n(y,z)\\
&= \sum_{m=0}^{M} \Big(yz\bar{\lambda}\bar{p}P_m(y,z)x^{\mathrm{t}}_{mn} + \bar{\lambda}\bar{p}(1-y)zP_m(0,z)x^{\mathrm{t}}_{mn}\\
&\quad + (y\lambda + \bar{\lambda}p)zP_m(1,z)x^{\mathrm{t}}_{mn} + \bar{\lambda}\bar{p}yzP_m(y,z)x^{\mathrm{nt}}_{mn}\\
&\quad + (\bar{\lambda} + \lambda y)p(P_m(z,1) - P_m(0,1))x^{\mathrm{nt}}_{mn}\\
&\quad + yz\bar{p}\lambda P_m(1,z)x^{\mathrm{nt}}_{mn} + z(\bar{\lambda}(1-y) + py)P_m(0,z)x^{\mathrm{nt}}_{mn}\Big),
\end{aligned} \tag{10}$$

for $n \in \mathcal{M}$. This system of equations contains all information that is needed for determining the partial joint generating functions $P_n(y, z)$, and therefore also for the stationary distribution of the Markov chain. The expressions however contain several unknown functions which will be determined in the next section. To this end, it is convenient to introduce vector-matrix notation for the system of equations above.

3) MATRIX REPRESENTATION

Let $\mathbf{P}(y, z)$ be the row-vector with entries $P_m(y, z)$ ($m \in \mathcal{M}$), and let $\mathcal{X}^{\mathrm{t}}$ and $\mathcal{X}^{\mathrm{nt}}$ be the $(M + 1) \times (M + 1)$ matrices with entries x^{t}_{mn} and x^{nt}_{mn}, respectively ($m, n \in \mathcal{M}$). Finally, let $\mathcal{X} = \mathcal{X}^{\mathrm{t}} + \mathcal{X}^{\mathrm{nt}}$. The matrix $\mathcal{X}$ collects the transition probabilities x_{mn}.

We then have the following vector representation of the system of equations (10),

$$\begin{aligned}
&\mathbf{P}(y,z)\left(\mathcal{I} - \bar{\lambda}\bar{p}yz\mathcal{X}\right)\\
&= \mathbf{P}(0,z)\left(\bar{\lambda}\bar{p}(1-y)z\mathcal{X}^{\mathrm{t}} + z(\bar{\lambda}(1-y) + py)\mathcal{X}^{\mathrm{nt}}\right)\\
&\quad + \mathbf{P}(1,z)\left((y\lambda + \bar{\lambda}p)z\mathcal{X}^{\mathrm{t}} + yz\bar{p}\lambda\mathcal{X}^{\mathrm{nt}}\right)\\
&\quad + (\mathbf{P}(z,1) - \mathbf{P}(0,1))(\bar{\lambda} + \lambda y)p\mathcal{X}^{\mathrm{nt}}.
\end{aligned} \tag{11}$$

Here, we collected the terms in $\mathbf{P}(y, z)$ on the left-hand side for convenience.

We now find an explicit expression for $\mathbf{P}(y, z)$. To this end, we evaluate the functional equation (11) in specific values to determine the unknown functions in the functional equation. First, substituting $y = 1$ and $z = 1$ gives

$$\mathbf{P}(1,1) = \mathbf{P}(1,1)\mathcal{X}.$$

This is not unexpected. This expression corresponds to the balance equations of the Markov process $\{X_k\}$. Indeed, we have $P_m(1, 1) = \lim_{k\to\infty} \mathsf{P}[X_k = m]$. Complementing this expression with the normalisation condition $\mathbf{P}(1, 1)\mathbf{e}' = 1$ ($\mathbf{e}$ is a row vector of ones, and $\mathbf{x}'$ is the transpose of $\mathbf{x}$), we have,

$$\mathbf{P}(1,1) = \mathbf{e}\,(\mathcal{I} - \mathcal{X} + \mathbf{e}'\mathbf{e})^{-1}. \tag{12}$$

Secondly, evaluating (11) in $y = 0$ and $z = 1$, and solving for $\mathbf{P}(0, 1)$ gives,

$$\begin{aligned}
\mathbf{P}(0,1) &= \mathbf{P}(1,1)\bar{\lambda}p(\mathcal{I} - \bar{\lambda}\bar{p}\mathcal{X})^{-1}\\
&= \mathbf{e}\,(\mathcal{I} - \mathcal{X} + \mathbf{e}'\mathbf{e})^{-1}\bar{\lambda}p(\mathcal{I} - \bar{\lambda}\bar{p}\mathcal{X})^{-1}.
\end{aligned} \tag{13}$$

Note that the inverse matrix in the expression above is well defined as $\bar{\lambda}\bar{p}\mathcal{X}$ is a sub-stochastic matrix. The mth entry of this vector is the probability of having m users with a vector when the tagged user does not have a packet: $P_m(0, 1) = \lim_{k\to\infty} \mathsf{P}[X_k = m, B_k = -1]$.

Now we set $z = 1$ and $y = z$ in equation (11), which yields an equation where only $\boldsymbol{P}(z, 1)$ is not known. Solving for $\boldsymbol{P}(z, 1)$ yields,

$$\begin{aligned}
\mathbf{P}(z,1) = &\left(\mathbf{P}(0,1)\bar{\lambda}\bar{p}(1-z) + \mathbf{P}(1,1)(z\lambda + \bar{\lambda}p)\right)\\
&\times \left(\mathcal{I} - \bar{\lambda}\bar{p}z\mathcal{X}\right)^{-1},
\end{aligned} \tag{14}$$

for $|z| \le 1$. The inverse matrix in the expression above is well defined, as the spectral norm of $\bar{\lambda}\bar{p}z\mathcal{X}$ is smaller than 1 for $|z| < \bar{\lambda}^{-1}$. The expression above particularly reveals that the stationary distribution of the age of information at the sender does not depend on the availability of the channel. Indeed, the expression does not depend on the marked transition probability matrices $\mathcal{X}^{\mathrm{t}}$ and $\mathcal{X}^{\mathrm{nt}}$, but only on the transition matrix $\mathcal{X}$. This is in line with the modelling assumptions: the age increments every slot and resets to 0 with a new arrival, and to -1 with a transmission (independent of its success).

Finally, evaluating (11) in $y = 0$ and $y = 1$ yields,

$$\begin{aligned}
&\mathbf{P}(0,z)\left(\mathcal{I} - \bar{\lambda}z\left(\bar{p}\mathcal{X}^{\mathrm{t}} + \mathcal{X}^{\mathrm{nt}}\right)\right)\\
&\quad = \mathbf{P}(1,z)\bar{\lambda}pz\mathcal{X}^{\mathrm{t}} + (\mathbf{P}(z,1) - \mathbf{P}(0,1))\bar{\lambda}p\mathcal{X}^{\mathrm{nt}},
\end{aligned}$$

and,

$$\begin{aligned}
&\mathbf{P}(1,z)\left(\mathcal{I} - z(\mathcal{X}^{\mathrm{t}} + \bar{p}\mathcal{X}^{\mathrm{nt}})\right)\\
&\quad = \mathbf{P}(0,z)pz\mathcal{X}^{\mathrm{nt}} + (\mathbf{P}(z,1) - \mathbf{P}(0,1))p\mathcal{X}^{\mathrm{nt}},
\end{aligned}$$

respectively. Solving for $\mathbf{P}(0, z)$ and $\mathbf{P}(1, z)$ leads to,

$$\mathbf{P}(0,z) = \mathbf{P}(1,z)\mathcal{A}(z), \tag{15}$$

$$\mathbf{P}(1,z) = (\mathbf{P}(z,1) - \mathbf{P}(0,1))p\mathcal{X}^{\mathrm{nt}}(\mathcal{B}(z))^{-1}, \tag{16}$$

with,

$$\mathcal{A}(z) = \bar{\lambda}(\mathcal{I} - z\bar{p}\mathcal{X})(\mathcal{I} - z\bar{p}\bar{\lambda}\mathcal{X})^{-1}, \tag{17}$$

$$\mathcal{B}(z) = \mathcal{I} - z(\mathcal{X}^{\mathrm{t}} + \bar{p}\mathcal{X}^{\mathrm{nt}}) - pz\mathcal{A}(z)\mathcal{X}^{nt}, \tag{18}$$

for $|z| < 1$. The inverse of $\mathcal{B}(z)$ in (16) is again well defined for all $|z| < 1$.

4) GENERATING FUNCTIONS OF THE AGE AND PEAK AGE

We have now determined all unknown functions on the right-hand side of (11). We now express the probability generating functions of the mean AoI at random slot boundaries and the peak AoI in terms of $\boldsymbol{P}(y, z)$. Let $A_r(z)$ denote the probability generating function of the age of information at random slot boundaries. From the definition of $\boldsymbol{P}(y, z)$, we immediately have,

$$A_r(z) = \mathbf{P}(1,z)\mathbf{e}'. \tag{19}$$

For the peak age of information, we note that there is a successful transmission provided that (i) there is a packet at the sender, (ii) there is a transmission by the sender, and (iii) there are no transmissions by other senders. We can therefore express the probability generating function of the peak age in terms of known functions as follows,

$$A_p(z) = \frac{(P(1,z) - P(0,z))\mathcal{X}^{\mathrm{nt}}\mathbf{e}'}{(P(1,1) - P(0,1))\mathcal{X}^{\mathrm{nt}}},$$

or equivalently,

$$A_p(z) = \frac{\mathbf{P}(1,z)(\mathcal{I} - \mathcal{A}(z))\mathcal{X}^{\mathrm{nt}}\mathbf{e}'}{\mathbf{P}(1,1)(\mathcal{I} - \mathcal{A}(1))\mathcal{X}^{\mathrm{nt}}\mathbf{e}'}. \tag{20}$$

These expressions will be used in the following two subsections to construct recursive expressions for the moments and probabilities of the age and peak age.

C. MOMENTS

The moment-generating property allows for retrieving moments up to any order by evaluating higher-order derivatives of the generating function in 1. As such moment expressions quickly grow in size, we here focus on recursive equations for the factorial moments. The nth factorial moment equals the nth derivative of the generating function evaluated in $z = 1$. In particular, we have

$$\mathsf{E}[(A_r)_n] = A_r^{(n)}(1).$$

where $(x)_n = x(x-1)\ldots(x-(n-1))$ denotes the falling factorial and $A_r^{(n)}(z)$ is the nth derivative of the generating function $A_r(z)$ of A_r. Factorial moments easily relate to moments and central moments. In particular, the mean and variance equal

$$\mathsf{E}[A_r] = A_r'(1), \quad \mathsf{var}[A_r] = A_r''(1) + A_r'(1) - (A_r'(1))^2.$$

More generally the nth moment can be expressed in terms of the factorial moments up to order n as follows,

$$\mathsf{E}[A_r^n] = \sum_{m=0}^{n} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \mathsf{E}[(A_r)_m]. \tag{21}$$

The curly braces in the expression above denote Stirling numbers of the second kind,

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_{\ell=0}^{m} \frac{(-1)^{m-\ell}\ell^n}{(m-\ell)!\ell!}.$$

Of course, similar expressions relate the moments and factorial moments of the peak age to the derivatives of $A_p(z)$ in $z = 1$.

To retrieve the moment recursion, we introduce the following notation for the derivatives of $\mathbf{P}(1,z)$, $\mathbf{P}(z,1)$, $\mathcal{A}(z)$, and $\mathcal{B}(z)$,

$$\boldsymbol{\alpha}_n = \left.\frac{\partial^n P(1,z)}{\partial z^n}\right|_{z=1}, \qquad \boldsymbol{\beta}_n = \left.\frac{\partial^n P(z,1)}{\partial z^n}\right|_{z=1},$$
$$\widehat{\mathcal{A}}_n = \left.\frac{\partial^n \mathcal{A}(z)}{\partial z^n}\right|_{z=1}, \qquad \widehat{\mathcal{B}}_n = \left.\frac{\partial^n \mathcal{B}(z)}{\partial z^n}\right|_{z=1}.$$

Combining the definition of $A_r(z)$, see equation (19), with the definition of $\boldsymbol{\alpha}_n$ above, we immediately have,

$$\mathsf{E}[(A_r)_n] = A_r^{(n)}(1) = \boldsymbol{\alpha}_n\mathbf{e}'. \tag{22}$$

The expressions of the factorial moments of the peak AoI A_p are somewhat more involved. To calculate the nth derivative of the vector-matrix product in the numerator on the right-hand side, we apply the general Leibniz rule. Evaluating in $z = 1$ yields,

$$\mathsf{E}[(A_p)_n] = \frac{\boldsymbol{\alpha}_n\mathcal{X}^{\mathrm{nt}}\mathbf{e}'}{\boldsymbol{\alpha}_0(\mathcal{I} - \widehat{\mathcal{A}}_0)\mathcal{X}^{\mathrm{nt}}\mathbf{e}'} - \sum_{k=0}^{n}\binom{n}{k}\frac{\boldsymbol{\alpha}_k\widehat{\mathcal{A}}_{n-k}\mathcal{X}^{\mathrm{nt}}\mathbf{e}'}{\boldsymbol{\alpha}_0(\mathcal{I} - \widehat{\mathcal{A}}_0)\mathcal{X}^{\mathrm{nt}}\mathbf{e}'}. \tag{23}$$

We now find recursive equations for the unknown vectors $\boldsymbol{\alpha}_n$ and $\boldsymbol{\beta}_n$, and for the unknown matrices $\widehat{\mathcal{A}}_n$ and $\widehat{\mathcal{B}}_n$. By evaluating the derivatives of (17) and (18) in $z = 1$, we find,

$$\begin{aligned}
\widehat{\mathcal{A}}_0 &= \bar{\lambda}(\mathcal{I} - \bar{p}\mathcal{X})(\mathcal{I} - \bar{\lambda}\bar{p}\mathcal{X})^{-1}, \\
\widehat{\mathcal{A}}_1 &= \bar{\lambda}\bar{p}(\widehat{\mathcal{A}}_0 - \mathcal{I})\mathcal{X}(\mathcal{I} - \bar{\lambda}\bar{p}\mathcal{X})^{-1}, \\
\widehat{\mathcal{A}}_n &= n\widehat{\mathcal{A}}_{n-1}\bar{p}\bar{\lambda}\mathcal{X}(\mathcal{I} - \bar{\lambda}\bar{p}\mathcal{X})^{-1},
\end{aligned} \tag{24}$$

and,

$$\begin{aligned}
\widehat{\mathcal{B}}_0 &= \mathcal{I} - (\mathcal{X}^{\mathrm{t}} + (1-p)\mathcal{X}^{\mathrm{nt}}) - p\widehat{\mathcal{A}}_0\mathcal{X}^{\mathrm{nt}}, \\
\widehat{\mathcal{B}}_1 &= -\mathcal{X}^{\mathrm{t}} - (1-p)\mathcal{X}^{\mathrm{nt}} - p\widehat{\mathcal{A}}_0\mathcal{X}^{\mathrm{nt}} - p\widehat{\mathcal{A}}_1\mathcal{X}^{\mathrm{nt}}, \\
\widehat{\mathcal{B}}_n &= -p\widehat{\mathcal{A}}_n\mathcal{X}^{\mathrm{nt}} - np\widehat{\mathcal{A}}_{n-1}\mathcal{X}^{\mathrm{nt}},
\end{aligned} \tag{25}$$

respectively.

By using the definition $\boldsymbol{\beta}_0 = \mathbf{P}(1,1)$ and equation (12), the identity $\mathbf{P}(0,1) = \boldsymbol{\beta}_0\widehat{\mathcal{A}}_0$ which follows from evaluating (15) in $z = 1$, and by evaluating the derivatives of (14) in $z = 1$, we find,

$$\begin{aligned}
\boldsymbol{\beta}_0 &= \mathbf{e}(\mathcal{I} - \mathcal{X} + \mathbf{e}'\mathbf{e})^{-1}, \\
\boldsymbol{\beta}_1 &= \boldsymbol{\beta}_0\left(\lambda\mathcal{I} + \bar{\lambda}\bar{p}(\mathcal{I} - \widehat{\mathcal{A}}_0)\right)\left(\mathcal{I} - \bar{\lambda}\bar{p}\mathcal{X}\right)^{-1}, \\
\boldsymbol{\beta}_n &= n\boldsymbol{\beta}_{n-1}\bar{\lambda}\bar{p}\mathcal{X}\left(\mathcal{I} - \bar{\lambda}\bar{p}\mathcal{X}\right)^{-1},
\end{aligned} \tag{26}$$

Finally, we can express $\boldsymbol{\alpha}_0$ in terms of known quantities by noting that $\boldsymbol{\alpha}_0 = \mathbf{P}(1,1)$ and by equation (12). Moreover, by evaluating the nth derivative of (16) in $z = 1$ and by applying the generalised Leibniz rule, we find,

$$\begin{aligned}
\boldsymbol{\alpha}_0 &= \mathbf{e}(\mathcal{I} - \mathcal{X} + \mathbf{e}'\mathbf{e})^{-1}, \\
\boldsymbol{\alpha}_n &= \left(\boldsymbol{\beta}_n p\mathcal{X}^{\mathrm{nt}} - \sum_{k=1}^{n}\binom{n}{k}\boldsymbol{\alpha}_{n-k}\widehat{\mathcal{B}}_k\right)(\widehat{\mathcal{B}}_0)^{-1}.
\end{aligned} \tag{27}$$

Summarising, in the nth step of the recursion, we first determine $\widehat{\mathcal{A}}_n$ by (24) and $\widehat{\mathcal{B}}_n$ by (25). We need these matrices to calculate the vectors $\boldsymbol{\alpha}_n$ by (27) and $\boldsymbol{\beta}_n$ by (26). Evaluating (22) and (23) then yields the nth factorial moments of the age and peak age, respectively. The corresponding nth moment then follows from (21). The calculations are indeed recursive: we only need vectors and matrices up to order $(n-1)$ in the nth step.

D. AGE OF INFORMATION DISTRIBUTION

The probability generating functions $A_r(z)$ and $A_p(z)$ of the average and peak age of information can be inverted numerically, see e.g. [59]. In the present setting, we can however also retrieve a simple recursion for the probabilities. To this end, consider the following Taylor series expansions of the vector generating functions $\mathbf{P}(1, z)$ and $\mathbf{P}(z, 1)$,

$$\mathbf{P}(1, z) = \sum_{k=0}^{\infty} \mathbf{a}_k z^k \,, \quad \mathbf{P}(z, 1) = \sum_{k=0}^{\infty} \mathbf{b}_k z^k .$$

To simplify notation, we further introduce the expansion of the matrix-generating function as well,

$$\mathcal{A}(z) = \sum_{k=0}^{\infty} \tilde{\mathcal{A}}_k z^k .$$

From equation (17), we have,

$$\mathcal{A}(z)(\mathcal{I}-z\bar{p}\bar{\lambda}\mathcal{X}) - \bar{\lambda}(\mathcal{I} - z\bar{p}\mathcal{X}) = 0.$$

Plugging the expansion of $\mathcal{A}(z)$ in this expression and isolating the terms in z^k ($k = 0, 1, \ldots$) yields,

$$\tilde{\mathcal{A}}_0 = \bar{\lambda}\mathcal{I} \,, \quad \tilde{\mathcal{A}}_1 = -\bar{p}\bar{\lambda}\mathcal{X} \,,$$

and,

$$\tilde{\mathcal{A}}_k = \bar{p}\bar{\lambda}\mathcal{A}_{k-1}\mathcal{X} \,,$$

for $k \in \{2, 3, \ldots\}$. Hence, we can recursively calculate the matrices $\tilde{\mathcal{A}}_k$ for any index k.

Analogously, using (14), we have,

$$\mathbf{P}(z, 1)\left(\mathcal{I} - \bar{\lambda}\bar{p}z\mathcal{X}\right) - \mathbf{P}(0, 1)\bar{\lambda}\bar{p}(1-z) - \mathbf{P}(1, 1)(z\lambda + \bar{\lambda}p) = 0.$$

Plugging the expansion of $P(z, 1)$ in the expression above and isolating the terms in z^k ($k = 0, 1, \ldots$) leads to,

$$\mathbf{b}_0 = \mathbf{P}(0, 1) \,, \quad \mathbf{b}_1 = \lambda\mathbf{P}(1, 1).$$

and

$$\mathbf{b}_k = (1-\lambda)(1-p)\mathbf{b}_{k-1}\mathcal{X}$$

for $k \in \{2, 3, \ldots\}$. Note that $\mathbf{P}(1, 1)$ and $\mathbf{P}(0, 1)$ are expressed in terms of the system parameters in equations (12) and (13), respectively.

Finally, from equation (14), we have

$$\mathbf{P}(z, 1)\left(\mathcal{I} - \bar{\lambda}\bar{p}z\mathcal{X}\right) - \mathbf{P}(0, 1)\bar{\lambda}\bar{p}(1-z) - \mathbf{P}(1, 1)(z\lambda + \bar{\lambda}p) = 0 \,, \tag{28}$$

We can then express $\mathbf{a}_k$ in terms of the sequences $\{\mathbf{b}_k\}$ and $\{\mathcal{A}_k\}$ by plugging the three expansions in the expression above. By isolating the terms in z^k ($k = 0, 1, \ldots$) in the resulting expression, we find,

$$\mathbf{a}_0 = 0 \,, \quad \mathbf{a}_1 = p\mathbf{b}_1\mathcal{X}^{\mathrm{nt}}$$

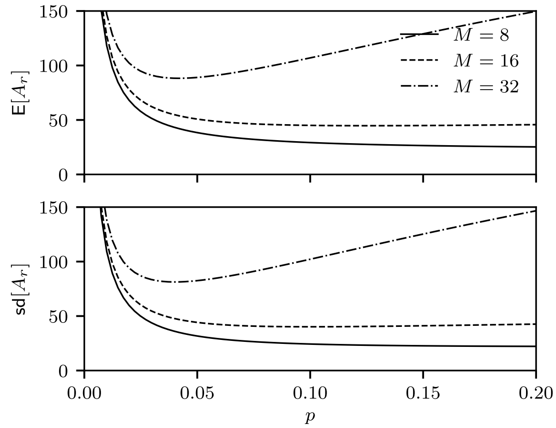

FIGURE 1. Mean and standard deviation of the AoI at random slot boundaries vs. the transmission probability p for low arrival intensity $\lambda = 0.1$ and for different numbers of users M as indicated.

and,

$$\mathbf{a}_k = \mathbf{b}_k p\mathcal{X}^{\mathrm{nt}} + \mathbf{a}_{k-1}(\mathcal{X}^{\mathrm{t}} + (1-p)\mathcal{X}^{\mathrm{nt}}) + p\sum_{\ell=1}^{k-1}\mathbf{a}_\ell\mathcal{A}_{k-\ell-1}\mathcal{X}^{nt}$$

for $k \in \{2, 3, \ldots\}$.

The recursions above allow for determining $\mathbf{a}_k$ and $\mathbf{b}_k$ for any index k. Note that the nth term in the expansion of a probability generating function equals the probability that the corresponding random variable equals n. We can therefore simply express the distribution of the age of information at random slot boundaries and of the peak age of information in terms of the vectors $\mathbf{a}_k$ and the matrices $\tilde{\mathcal{A}}_k$. Indeed, substituting $\mathbf{P}(1, z)$ and $\mathcal{A}(z)$ with their expansions in (19) and (20), and isolating the term in z^k yields,

$$\mathsf{P}[A_r = k] = \mathbf{a}_k\mathbf{e}' \,,$$

$$\mathsf{P}[A_p = k] = \frac{(\mathbf{a}_k - \sum_{\ell=0}^{k}\mathbf{a}_\ell\tilde{\mathcal{A}}_{k-\ell})\mathcal{X}^{\mathrm{nt}}\mathbf{e}'}{\boldsymbol{\alpha}_0(\mathcal{I} - \hat{\mathcal{A}}_0)\mathcal{X}^{\mathrm{nt}}\mathbf{e}'} \,,$$

for $k \in \{0, 1, 2, \ldots\}$.

III. NUMERICAL EXAMPLE

We now illustrate our results with some numerical examples. All results follow from the analytic calculations in the preceding sections. To verify the correctness of the analytic results, we also simulated the system for various parameter sets. We do not add the simulation results to the plots, as these do not provide additional information.

Figures 1 and 2 show the mean and standard deviation of the age at random slot boundaries and the peak age vs. the transmission probability p, respectively. The arrival intensity is fairly low ($\lambda = 0.05$) and the number of users varies from $M = 8$ to $M = 32$ as indicated. For the parameters at hand, increasing the transmission probabilities improves performance, that is, the mean and standard deviation are lower. The value of the transmission probability affects the age in two ways. First, information remains longer at the

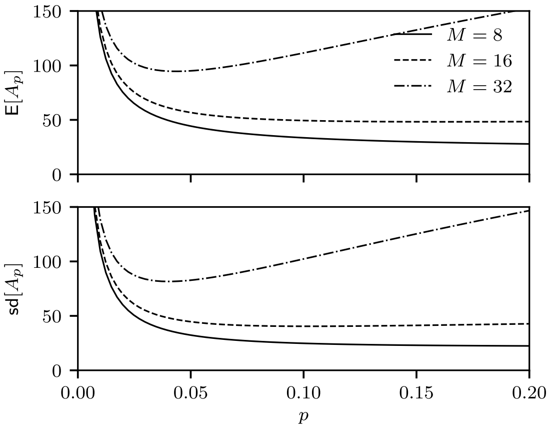

FIGURE 2. Mean and standard deviation of the peak AoI vs. the transmission probability p for low arrival intensity $\lambda = 0.1$ and for different numbers of users M as indicated.

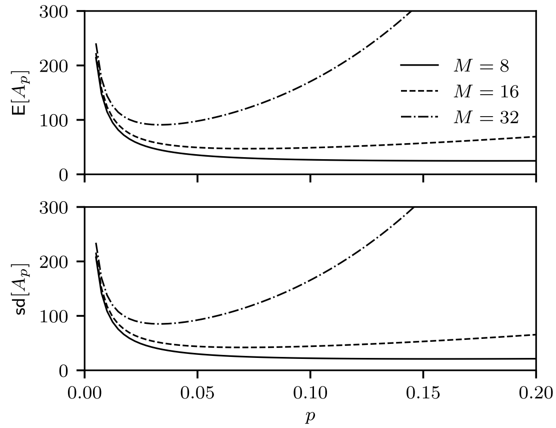

FIGURE 4. Mean and standard deviation of the peak AoI vs. the transmission probability p for high arrival intensity $\lambda = 0.25$ and for different numbers of users M as indicated.

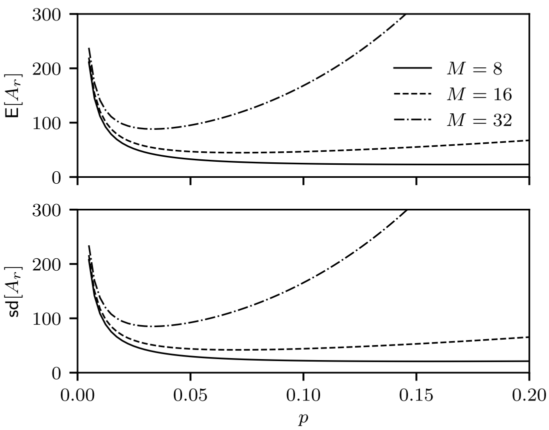

FIGURE 3. Mean and standard deviation of the AoI at random slot boundaries vs. the transmission probability p for high arrival intensity $\lambda = 0.25$ and for different numbers of users M as indicated.

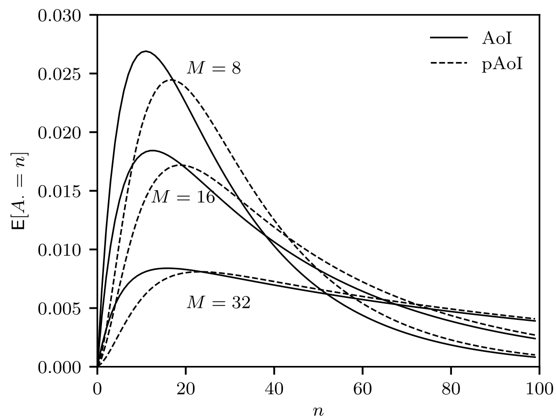

FIGURE 5. Probability mass function of the age and peak age of information for $\lambda = 0.1$, $p = 0.1$ and M as indicated.

sender if the transmission probability is lower. An increase therefore means that the information is sent more timely. On the other hand, a higher transmission probability also increases the probability that there is a collision. For low arrival intensity, one expects that a timely transmission is more important as there is not that much information to send and the collision probability is fairly low. These figures indeed confirm this. We further observe that the presence of more users negatively affects performance. This of course stems from having more collisions if there are more users. Comparing the mean age and mean peak age, the latter is higher for all p. Somewhat surprisingly, the standard deviations of the age and peak age are almost identical, the standard deviation of the peak age is somewhat larger, but the difference is limited to 3.5%.

In Figures 3 and 4 we retain the assumptions of the preceding figures, but we increase the arrival intensity to $\lambda = 0.2$. Here the adverse effects of the transmission probability on the age become apparent. The performance only improves with increasing p for low values, while for higher p a further increase affects performance negatively. This can be explained by the increased arrival rate and the corresponding increase of collisions for higher p. We again see that more users have a negative impact, and we find that the mean peak age exceeds the mean age. The somewhat surprising fact that the variance of age and peak age are almost identical is confirmed as well. The standard deviation of the peak age is again slightly larger, the difference being less than 1%.

The standard deviation is somewhat smaller than the mean (peak) age of information in the preceding plots. As the age is non-negative, one expects that the probability mass function of the age is positively skewed. This is indeed confirmed by Figures 5 and 6 which show the age and peak age probability mass functions for different values of the number of users M as indicated. The arrival rate equals $\lambda = 0.05$ in both plots, which corresponds to the rate in plots 1 and 2. The transmission probability equals $p = 0.1$ if Figure 5 and $p = 0.3$ in Figure 6. A comparison of the age and peak age probability mass functions in both plots and for all M shows that low values of the age are more likely than low values of the peak age, while for somewhat larger values we observe

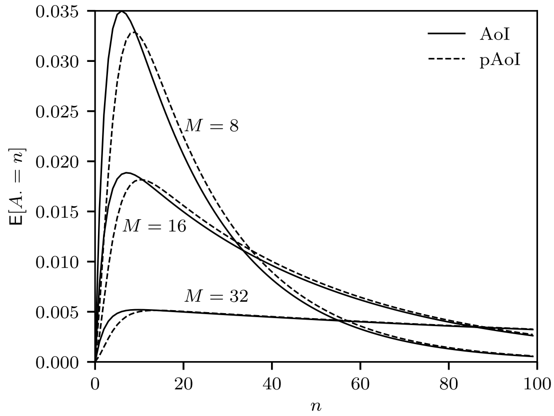

FIGURE 6. Probability mass function of the age and peak age of information for $\lambda = 0.1$, $p = 0.3$ and M as indicated.

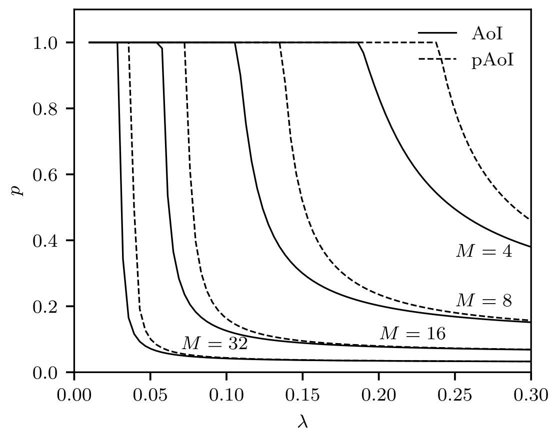

FIGURE 7. AoI-optimal and pAoI-optimal transmission probabilities vs. the arrival probability λ for different M as indicated.

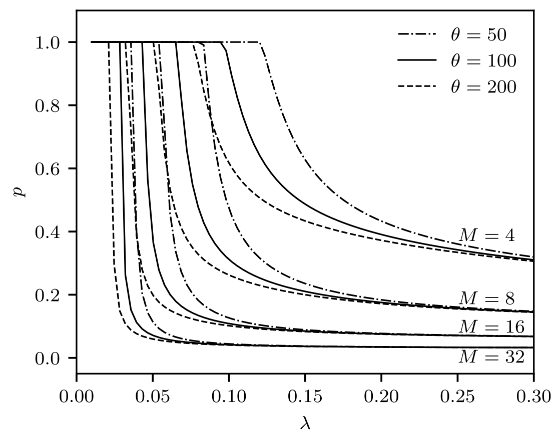

FIGURE 8. AoI-threshold-optimal transmission probabilities vs. the arrival probability λ for different M and θ as indicated.

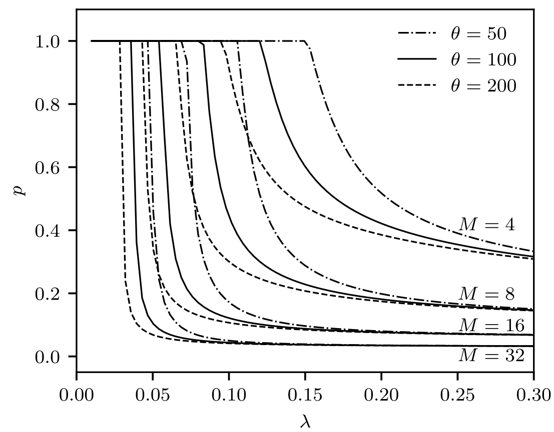

FIGURE 9. pAoI-threshold-optimal transmission probabilities vs. the arrival probability λ for different M and θ as indicated.

the opposite. The difference between both probability mass functions is most outspoken for these low to intermediate age values. For larger values of the (peak) age, the (tail) probabilities converge to the same value which does depend on M. The number of users M affects the probability mass functions as expected: higher M means more collisions and the probability shifts to the right, with tail probabilities that decay more slowly.

To investigate how the optimal p changes with the packet arrival rate λ, Figure 7 depicts the optimal p which minimises the mean AoI at slot boundaries, as well as the p which minimises the mean pAoI. When there are but a few packets, it is optimal to immediately send. When the packet arrival rate is higher, the optimal probability drops quickly. The drop is initiated for lower λ when the number of users $M+1$ is higher. This is not unexpected, as the chance of contention increases when the number of users increases. Finally, note that optimising the AoI and pAoI yields considerably different p, the optimal p for the pAoI is higher than the p optimising the AoI. Hence, there is no single optimal transmission probability p, and the choice between optimising for the AoI at slot boundaries or the peak AoI depends on the application at hand.

Depending on the application at hand, one prefers minimising the probability that the age of information exceeds some threshold θ over minimising the mean (peak) age of information. Therefore, Figure 8 shows the optimal transmission probability to minimise the probability that the age exceeds a threshold θ while Figure 9 shows the optimal transmission probability for the peak age. On both plots, different thresholds θ (in terms of slots) and different numbers of users M are assumed as indicated. When there are but a few packets, it is optimal to immediately send, in line with the p that optimises the mean (peak) age. When the packet arrival rate is higher, the optimal probability drops quickly. The drop is initiated for lower λ when the number of users $M + 1$ is higher, and when the threshold θ is higher. While the shape of the curves in Figures 7, 8 and 9 is similar, the drop is quick, and initiated at considerably different λ which complicates the optimal control.

IV. CONCLUSION

We considered the age and peak age of information for slotted ALOHA. Our analytic approach is exact and can generate various performance measures like the moments and the probability mass function of the age and peak age of information fast. While we relied on a probability generating function approach for the main performance measures, we also provided recursive algorithms to calculate the moments and the probability mass function of the age of information and the peak age of information. No numerical conversion of the generating function is needed. We then explore the impact of packet arrival and transmission probabilities on the age of information through a numerical example. In particular, we find the optimal transmission probabilities for ALOHA which either minimise the mean (peak) age of information or the tail probabilities of the age. The work paves the way towards safety assessment of networked cyber-physical systems.

REFERENCES

[1] S. K. Khaitan and J. D. McCalley, "Design techniques and applications of cyberphysical systems: A survey," *IEEE Syst. J.*, vol. 9, no. 2, pp. 350–365, Jun. 2015.

[2] D. Jia, K. Lu, J. Wang, X. Zhang, and X. Shen, "A survey on platoon-based vehicular cyber-physical systems," *IEEE Commun. Surveys Tuts.*, vol. 18, no. 1, pp. 263–284, 1st Quart., 2016.

[3] Y. Wang, S. Wu, C. Lei, J. Jiao, and Q. Zhang, "A review on wireless networked control system: The communication perspective," *IEEE Internet Things J.*, vol. 11, no. 5, pp. 7499–7524, Mar. 2024.

[4] A. Kosta, N. Pappas, and V. Angelakis, "Age of information: A new concept, metric, and tool," *Found. Trends Netw.*, vol. 12, no. 3, pp. 162–259, 2017, doi: 10.1561/1300000060.

[5] R. D. Yates, Y. Sun, D. R. Brown, S. K. Kaul, E. Modiano, and S. Ulukus, "Age of information: An introduction and survey," *IEEE J. Sel. Areas Commun.*, vol. 39, no. 5, pp. 1183–1210, May 2021.

[6] İ. Kahraman, A. Köse, M. Koca, and E. Anarim, "Age of information in Internet of Things: A survey," *IEEE Internet Things J.*, vol. 11, no. 6, pp. 9896–9914, Mar. 2024.

[7] D. Fiems, "Age of information analysis with preemptive packet management," *IEEE Commun. Lett.*, vol. 27, no. 4, pp. 1105–1109, Apr. 2023.

[8] D. Zucchetto and A. Zanella, "Uncoordinated access schemes for the IoT: Approaches, regulations, and performance," *IEEE Commun. Mag.*, vol. 55, no. 9, pp. 48–54, Sep. 2017.

[9] J. Jiao, S. Wu, R. Lu, and Q. Zhang, "Massive access in space-based Internet of Things: Challenges, opportunities, and future directions," *IEEE Wireless Commun.*, vol. 28, no. 5, pp. 118–125, Oct. 2021.

[10] D. Shumeye Lakew, U. Sa'ad, N.-N. Dao, W. Na, and S. Cho, "Routing in flying ad hoc networks: A comprehensive survey," *IEEE Commun. Surveys Tuts.*, vol. 22, no. 2, pp. 1071–1120, 2nd Quart., 2020.

[11] H. Sun, Z. Meng, D. Wang, and H. Li, "Fedab: A low-latency energy-efficient proactive neighbor discovery protocol in MLDC-WSN," *IEEE Access*, vol. 11, pp. 22843–22854, 2023.

[12] M. Derakhshani, Y. Khan, D. T. Nguyen, S. Parsaeefard, A. D. Shoaei, and T. Le-Ngoc, "Self-organizing TDMA: A distributed contention-resolution MAC protocol," *IEEE Access*, vol. 7, pp. 144845–144860, 2019.

[13] D. D. Olatinwo, A. M. Abu-Mahfouz, and G. P. Hancke, "A hybrid multi-class MAC protocol for IoT-enabled WBAN systems," *IEEE Sensors J.*, vol. 21, no. 5, pp. 6761–6774, Mar. 2021.

[14] N. Abramson, "Internet access using VSATs," *IEEE Commun. Mag.*, vol. 38, no. 7, pp. 60–68, Jul. 2000.

[15] X. Zhao and L. Dai, "Connection-based ALOHA: Modeling, optimization, and effects of connection establishment," *IEEE Trans. Wireless Commun.*, vol. 23, no. 2, pp. 1008–1023, Feb. 2024, doi: 10.1109/TWC.2023.3284907.

[16] J. M. de Souza Sant'Ana, O. D. S. Neto, A. Hoeller, J. L. Rebelatto, R. D. Souza, and H. Alves, "Asynchronous contention resolution-aided ALOHA in LR-FHSS networks," *IEEE Internet Things J.*, vol. 11, no. 9, pp. 16684–16692, May 2024.

[17] R. Hult, G. R. Campos, E. Steinmetz, L. Hammarstrand, P. Falcone, and H. Wymeersch, "Coordination of cooperative autonomous vehicles: Toward safer and more efficient road transportation," *IEEE Signal Process. Mag.*, vol. 33, no. 6, pp. 74–84, Nov. 2016.

[18] O. Sawade, M. Schulze, and I. Radusch, "Robust communication for cooperative driving maneuvers," *IEEE Intell. Transp. Syst. Mag.*, vol. 10, no. 3, pp. 159–169, Fall 2018.

[19] S. Liu, B. Yu, J. Tang, Y. Zhu, and X. Liu, "Communication challenges in infrastructure-vehicle cooperative autonomous driving: A field deployment perspective," *IEEE Wireless Commun.*, vol. 29, no. 4, pp. 126–131, Aug. 2022.

[20] M. Torrent-Moreno, J. Mittag, P. Santi, and H. Hartenstein, "Vehicle-to-vehicle communication: Fair transmit power control for safety-critical information," *IEEE Trans. Veh. Technol.*, vol. 58, no. 7, pp. 3684–3703, Sep. 2009.

[21] A. Festag, "Cooperative intelligent transport systems standards in Europe," *IEEE Commun. Mag.*, vol. 52, no. 12, pp. 166–172, Dec. 2014.

[22] H. Zhou, W. Xu, J. Chen, and W. Wang, "Evolutionary V2X technologies toward the Internet of Vehicles: Challenges and opportunities," *Proc. IEEE*, vol. 108, no. 2, pp. 308–323, Feb. 2020.

[23] S. Lobo, A. Festag, and C. Facchi, "Enhancing the safety of vulnerable road users: Messaging protocols for V2X communication," in *Proc. IEEE 96th Veh. Technol. Conf. (VTC-Fall)*, Sep. 2022, pp. 1–7.

[24] Q. Delooz, A. Willecke, K. Garlichs, A.-C. Hagau, L. Wolf, A. Vinel, and A. Festag, "Analysis and evaluation of information redundancy mitigation for V2X collective perception," *IEEE Access*, vol. 10, pp. 47076–47093, 2022.

[25] A. Vinel, L. Lan, and N. Lyamin, "Vehicle-to-vehicle communication in C-ACC/platooning scenarios," *IEEE Commun. Mag.*, vol. 53, no. 8, pp. 192–197, Aug. 2015.

[26] S. M. Loos, D. Witmer, P. Steenkiste, and A. Platzer, "Efficiency analysis of formally verified adaptive cruise controllers," in *Proc. 16th Int. IEEE Conf. Intell. Transp. Syst. (ITSC)*, Oct. 2013, pp. 1565–1570.

[27] A. Vinel, N. Lyamin, and P. Isachenkov, "Modeling of V2V communications for C-ITS safety applications: A CPS perspective," *IEEE Commun. Lett.*, vol. 22, no. 8, pp. 1600–1603, Aug. 2018.

[28] G. Sidorenko, D. Plöger, J. Thunberg, and A. Vinel, "Emergency braking with ACC: How much does V2V communication help?" *IEEE Netw. Lett.*, vol. 4, no. 3, pp. 157–161, Sep. 2022.

[29] J. Thunberg, D. Bischoff, F. A. Schiegg, T. Meuser, and A. Vinel, "Unreliable V2X communication in cooperative driving: Safety times for emergency braking," *IEEE Access*, vol. 9, pp. 148024–148036, 2021.

[30] G. Sidorenko, J. Thunberg, and A. Vinel, "Ethical V2X: Cooperative driving as the only ethical path to multi-vehicle safety," in *Proc. IEEE 98th Veh. Technol. Conf. (VTC-Fall)*, vol. 50, Oct. 2023, pp. 1–6.

[31] G. Naik, B. Choudhury, and J.-M. Park, "IEEE 802.11bd & 5G NR V2X: Evolution of radio access technologies for V2X communications," *IEEE Access*, vol. 7, pp. 70169–70184, 2019.

[32] F. Cali, M. Conti, and E. Gregori, "Dynamic tuning of the IEEE 802.11 protocol to achieve a theoretical throughput limit," *IEEE/ACM Trans. Netw.*, vol. 8, no. 6, pp. 785–799, Dec. 2000.

[33] A. Baiocchi, I. Turcanu, N. Lyamin, K. Sjöberg, and A. Vinel, "Age of information in IEEE 802.11p," in *Proc. IFIP/IEEE Int. Symp. Integr. Netw. Manage. (IM)*, May 2021, pp. 1024–1031.

[34] A. Rolich, I. Turcanu, A. Vinel, and A. Baiocchi, "Impact of persistence on the age of information in 5G NR-V2X sidelink communications," in *Proc. 21st Medit. Commun. Comput. Netw. Conf. (MedComNet)*, Jun. 2023, pp. 15–24.

[35] G. Bianchi, "Performance analysis of the IEEE 802.11 distributed coordination function," *IEEE J. Sel. Areas Commun.*, vol. 18, no. 3, pp. 535–547, Mar. 2000.

[36] O. T. Yavascan and E. Uysal, "Analysis of slotted ALOHA with an age threshold," *IEEE J. Sel. Areas Commun.*, vol. 39, no. 5, pp. 1456–1470, May 2021.

[37] H. H. Yang, N. Pappas, T. Q. S. Quek, and M. Haenggi, "Analysis of the age of information in age-threshold slotted ALOHA," in *Proc. 21st Int. Symp. Model. Optim. Mobile, Ad Hoc, Wireless Netw. (WiOpt)*, Aug. 2023, pp. 366–373.

[38] M. Ahmetoglu, O. T. Yavascan, and E. Uysal, "MiSTA: An age-optimized slotted ALOHA protocol," *IEEE Internet Things J.*, vol. 9, no. 17, pp. 15484–15496, Sep. 2022.

[39] J. Wang, J. Yu, X. Chen, L. Chen, C. Qiu, and J. An, "Age of information for frame slotted ALOHA," *IEEE Trans. Commun.*, vol. 71, no. 4, pp. 2121–2135, Apr. 2023.

[40] Q. Wang and H. Chen, "Age of information in reservation multi-access networks with stochastic arrivals: Analysis and optimization," *IEEE Trans. Commun.*, vol. 71, no. 8, pp. 4707–4720, Aug. 2023.

[41] Z. Yue, H. H. Yang, M. Zhang, and N. Pappas, "Age of information under frame slotted ALOHA in random access networks," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2023, pp. 969–974.
[42] M. Moradian, A. Dadlani, A. Khonsari, and H. Tabassum, "Age-aware dynamic frame slotted ALOHA for machine-type communications," *IEEE Trans. Commun.*, vol. 72, no. 5, pp. 2639–2654, May 2024.
[43] P. Mollahosseini, S. Asvadi, and F. Ashtiani, "Effect of variable backoff algorithms on age of information in slotted ALOHA networks," *IEEE Trans. Mobile Comput.*, vol. 23, no. 9, pp. 8620–8633, Sep. 2024.
[44] D. Fiems and A. Vinel, "Mean value analysis of the age of information in slotted ALOHA," *IEEE Commun. Lett.*, vol. 27, no. 10, pp. 2588–2592, Oct. 2023.
[45] X. Chen, K. Gatsis, H. Hassani, and S. S. Bidokhti, "Age of information in random access channels," *IEEE Trans. Inf. Theory*, vol. 68, no. 10, pp. 6548–6568, Oct. 2022.
[46] P. S. Dester, P. H. J. Nardelli, F. H. C. dos Santos Filho, P. Cardieri, and P. Popovski, "Delay and peak-age-of-information of ALOHA networks with limited retransmissions," *IEEE Wireless Commun. Lett.*, vol. 10, no. 10, pp. 2328–2332, Oct. 2021.
[47] H. Wang, X. Xie, X. Li, and J. Yang, "Scheduling schemes for age optimization in IoT systems with limited retransmission times," *IEEE Internet Things J.*, vol. 9, no. 21, pp. 21458–21468, Nov. 2022.
[48] K. Vaezi, N. Akar, and E. Karaşan, "Modeling age of information in a cooperative slotted ALOHA network," *Wireless Netw.*, vol. 29, no. 5, pp. 2405–2417, Jul. 2023.
[49] H. Pan, T.-T. Chan, J. Li, and V. C. M. Leung, "Age of information with collision-resolution random access," *IEEE Trans. Veh. Technol.*, vol. 71, no. 10, pp. 11295–11300, Oct. 2022.
[50] S. Asvadi and F. Ashtiani, "Peak age of information in slotted ALOHA networks," *IEEE Trans. Commun.*, vol. 71, no. 10, pp. 6018–6030, Oct. 2023.
[51] A. Munari and G. Liva, "Information freshness analysis of slotted ALOHA in Gilbert–Elliot channels," *IEEE Commun. Lett.*, vol. 25, no. 9, pp. 2869–2873, Sep. 2021.
[52] W. Zhan, D. Wu, X. Sun, Z. Guo, P. Liu, and J. Liu, "Peak age of information optimization of slotted aloha: FCFS versus LCFS," *IEEE Trans. Netw. Sci. Eng.*, vol. 10, no. 6, pp. 3719–3731, Feb. 2023.
[53] Y. H. Bae and J. W. Baek, "Age of information and throughput in random access-based IoT systems with periodic updating," *IEEE Wireless Commun. Lett.*, vol. 11, no. 4, pp. 821–825, Apr. 2022.
[54] A. Munari, "Modern random access: An age of information perspective on irregular repetition slotted ALOHA," *IEEE Trans. Commun.*, vol. 69, no. 6, pp. 3572–3585, Jun. 2021.
[55] J. F. Grybosi, J. L. Rebelatto, and G. L. Moritz, "Age of information of SIC-aided massive IoT networks with random access," *IEEE Internet Things J.*, vol. 9, no. 1, pp. 662–670, Jan. 2022.
[56] L. Badia, A. Zanella, and M. Zorzi, "A game of ages for slotted ALOHA with capture," *IEEE Trans. Mobile Comput.*, vol. 23, no. 5, pp. 4878–4889, May 2024.
[57] A. Vinel, Y. Koucheryavy, S. Andreev, and D. Staehle, "Estimation of a successful beacon reception probability in vehicular ad-hoc networks," in *Proc. Int. Conf. Wireless Commun. Mobile Comput., Connecting World Wirelessly*, New York, NY, USA, Jun. 2009, pp. 416–420, doi: 10.1145/1582379.1582470.
[58] M. van Eenennaam, L. Hendriks, G. Karagiannis, and G. Heijenk, "Oldest packet drop (OPD): A buffering mechanism for beaconing in IEEE 802.11p VANETs (poster)," in *Proc. IEEE Veh. Netw. Conf. (VNC)*, Nov. 2011, pp. 252–259.
[59] J. Abate and W. Whitt, "Numerical inversion of probability generating functions," *Oper. Res. Lett.*, vol. 12, no. 4, pp. 245–251, Oct. 1992.

DIETER FIEMS (Senior Member, IEEE) received the Ph.D. degree in engineering from Ghent University, in 2004. He is an Associate Professor with the Department of Telecommunications and Information Processing, Ghent University, Belgium. His current research interests include various applications of stochastic processes for the performance analysis of communication networks. Particularly, he is interested in the applications of queuing theory and branching processes in wireless networks.

ALEXEY VINEL (Senior Member, IEEE) received the Ph.D. degree from Tampere University of Technology, Finland, in 2013. He is a Professor with Karlsruhe Institute of Technology (KIT), Germany. Previously, he was a Professor with the University of Passau, Germany. Since 2015, he has been a Professor with Halmstad University, Sweden (currently part-time). His areas of interests include vehicular communications and networking, cooperative autonomous driving, and future smart mobility solutions.