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# Invariant Sets of a Nonlinear Oscillator with Delayed Restoring Force: Comparison System Approach ⋆

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**Abstract:** The subject is a continuous-time second-order damped oscillator whose restoring force is nonlinear and delayed. It is otherwise called the "sunflower equation." The goal is to find a forward invariant set containing the equilibrium, as well as to estimate its domain of attraction. The aim is for delay-dependent results. We formulate and apply the method of delay-free comparison systems. Using a reasoning similar to the classical Bendixson pocket principle, we develop an invariant set estimation algorithm based on the numerical integration of the comparison system. The approach is nonlocal: we give an example where it handles relatively large delays and yields an invariant set containing a large periodic trajectory.



*Keywords:* Time-delay systems, nonlinear systems, sunflower equation, oscillations, invariant set, stability, numerical analysis, comparison systems.

## 1. INTRODUCTION

A nonlinear oscillator with time delay serves as a simple model of many dynamical systems in science and engineering, including synchronization networks and power electronic devices. An oscillator with delayed restoring force such as $\ddot{x}(t) + k\dot{x}(t) + \sin x(t-\tau) = 0$ may occur due to delayed control action; this equation has also been named the *sunflower equation* as it models the circadian rotation of the sunflower head. Smirnova et al. (2021) give an overview of known results for the sunflower equation.

Analysis of a nonlinear oscillator near an equilibrium is often focused first of all on the stability of the equilibrium. If it is asymptotically stable, then an estimated domain of attraction is of interest. Such questions can be answered by linearizing the system. There is a large volume of research concerning linear delayed oscillators – e.g., see Scholl et al. (2019) and references therein.

If the equilibrium turns out to be unstable, then the next question to be answered is whether there exists a certain region surrounding the equilibrium such that after a small perturbation the system remains in the region. *The goal* is formalized as, firstly, finding a forward invariant set (also called a "positively invariant set") containing the equilibrium and, secondly, estimating a larger set from which the system is known to converge to the invariant set. This is the goal we aim for.

We apply the technique of *comparison systems*. It consists in designing a simpler system whose solutions bound or otherwise estimate the solutions of the original system. The comparison system may be solvable analytically or numerically, or lend itself to qualitative analysis readier than the original system. This idea was used by Serebryakova and Barbashin (1961) to study a couple of interacting massive points on a circle. A nonlinear term in the equations was replaced by its bounds – upper or lower depending on the current state. The resulting simplified system could be integrated analytically. Its trajectories, in some sense, bounded the trajectories of the original system. In a more general case, the method was extended by Belykh (1975) for the global portrait analysis of a planar time-varying system using *autonomous* comparison systems. Later the approach was generalized to $n$-dimensional systems where an $(n-2)$-dimensional part is treated as a bounded disturbance of the remaining two-dimensional subsystem – see the bottom of page *iv* in Leonov et al. (1992) for an overview. In Ponomarev et al. (2024) we applied the technique to the analysis of the periodic solutions of a phase synchronization system – the so-called *phase-locked loop* used, e.g., in power electronics.

An advantage of the comparison system method over linearization is that the former is nonlocal. Comparison-based results also often win against Lyapunov's direct method due to being less conservative. Indeed, Lyapunov function design in most practical cases is guided by a simplification of the system equations. A comparison system likely requires less simplification: often it is sufficient to solve it numerically to obtain an estimation for the system's solutions.

The *main contribution* of the present paper is an extension of the comparison system technique to time-delay systems. The idea is to estimate the delayed position of the system based on its current position. The estimation is then used to bound the right-hand side, leading to a comparison

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system. On our way to attain *delay-dependent* results, the primary challenge is to formulate conditions on the infinite-dimensional state of the system in such a way that the estimated delayed position is close to the current position when delay is small – otherwise the results would be delay-independent. We overcome the challenge, essentially, by letting the system evolve for one delay period due to its own dynamics and assessing its state after this transition time. We arrive at an implicit definition of the invariant set in the form: "a given initial state belongs to the invariant set if the solution it generates during one delay period satisfies certain constraints."

In Section 2 we describe the time-delay system that is investigated, specify the notion of a forward invariant set, and formulate the goal of the study. In Section 3, the delay-free comparison system is introduced, and a concept of *invariant pocket* is formulated to enable qualitative analysis in the spirit of the classical "Bendixson pocket" argument. Section 4 contains the main result: an algorithm for the construction of an invariant set and estimation of its domain of attraction. In Section 5 we provide numerical examples to showcase the algorithm.

## 2. PROBLEM STATEMENT

### 2.1 Nominal System

Let us begin by considering the nonlinear planar system

$$\dot{x} = y, \tag{1a}$$
$$\dot{y} = -f(x) - ky \tag{1b}$$

where $k > 0$ is a constant and $f$ is a bounded analytic function satisfying

$$f(0) = 0, \tag{2a}$$
$$xf(x) > 0 \text{ in a neighborhood of } x = 0. \tag{2b}$$

A typical example of (1) is the mathematical pendulum with friction where the restoring force $f(x) = \sin x$.

### 2.2 Assumption of Oscillatory Behavior

The qualitative behavior of system (1) near the equilibrium $(0, 0)$ depends on the eigenvalues of the linearization at the equilibrium. The system can be underdamped (oscillatory) or overdamped. For the lack of space, we limit our attention to the underdamped case. It is characterized by the condition

$$k^2 < 4f'(0) \tag{3}$$

which means that the origin of (1) is a stable focus (spiral).

### 2.3 System with Delay

By adding a time delay $\tau > 0$ to the restoring force $f$ in (1) we create the modified system

$$\dot{x}(t) = y(t), \tag{4a}$$
$$\dot{y}(t) = -f\big(x(t-\tau)\big) - ky(t), \quad t \ge 0. \tag{4b}$$

The *state* of the system at time $t$ is the history of the $x$ coordinate over the past $\tau$ seconds, together with the current value of the $y$ coordinate: the tuple

$$\big(x_t, y(t)\big) \tag{5}$$

where

$$x_t \colon [-\tau, 0] \to \mathbb{R}, \quad \theta \mapsto x(t+\theta). \tag{6}$$

The *state space* $\mathcal{S}$ is the space of tuples (5).

In what follows we shall assume that the initial state at $t = 0$ generates a unique smooth solution for all $t \ge 0$. This is ensured, e.g., by assuming that the initial history $x_0$ is a continuous function. Accordingly, we postulate

$$\mathcal{S} = C^0\big([-\tau, 0], \mathbb{R}\big) \times \mathbb{R} \tag{7}$$

where $C^0$ is the space of continuous functions.

### 2.4 Invariant Sets

Let us first mention two concepts of a forward invariant set found in the literature. The first one is common for abstract infinite-dimensional systems.

*Definition 1.* A set $\widetilde{\mathcal{D}} \subset \mathcal{S}$ is forward invariant if every solution that starts in this set at $t = 0$ remains there for all $t > 0$.

Definition 1 is in line with the finite-dimensional theory. However, it specifies an infinite-dimensional set which may be difficult to describe practically. A typical estimation of it by a normed ball is sensitive to the choice of the norm, as explained in Scholl et al. (2020).

For the special class of time-delay systems, the following definition is often assumed.

*Definition 2.* A set $\mathcal{D} \subset \mathbb{R}^2$ is forward invariant for system (4) if for every initial state that is *pointwise* in $\mathcal{D}$, i.e.,

$$\big(x(s), y(0)\big) \in \mathcal{D} \text{ for all } s \in [-\tau, 0] \tag{8}$$

the corresponding future trajectory remains in $\mathcal{D}$:

$$\big(x(t), y(t)\big) \in \mathcal{D} \text{ for all } t > 0. \tag{9}$$

Definition 2 is used, e.g., in Dórea et al. (2022) and explored in detail by Laraba et al. (2016) in the discrete-time setting. It describes a set $\mathcal{D}$ which is two-dimensional and thus certainly convenient in practice but is much narrower than its infinite-dimensional counterpart.

In the forthcoming discussion, another notion of a forward invariant set is adopted. It combines the above definitions in the following manner.

*Definition 3.* We say that $\widetilde{\mathcal{D}} \subset \mathcal{S}$ is a *forward invariant set associated with* a set $\mathcal{D} \subset \mathbb{R}^2$ if for every initial state

$$\big(x_0, y(0)\big) \in \widetilde{\mathcal{D}} \tag{10}$$

the corresponding trajectory satisfies

$$\big(x(t), y(t)\big) \in \mathcal{D} \text{ for all } t \ge \tau. \tag{11}$$

Note that the system is required to reach $\mathcal{D}$ and stay there only after one delay period.

Definition 3 describes an infinite-dimensional set $\widetilde{\mathcal{D}}$. However, as will be seen later, our construction of $\widetilde{\mathcal{D}}$ is such that condition (10) can be verified in practice through numerical integration of (4) over the finite time $[0, \tau]$. This test can be further simplified via various conservative estimations of the solutions of (4) which would result in a fully analytic description of $\widetilde{\mathcal{D}}$.

*Remark 4.* Definition 3 "implies" Definition 1 and "is implied" by Definition 2 in the following sense:

- If $\widetilde{\mathcal{D}} \subset \mathcal{S}$ is the *maximal* forward invariant set associated with $\mathcal{D} \subset \mathbb{R}^2$ per Definition 3 then it is forward invariant by Definition 1. Indeed, if (10) implies (11), then every future state $(x_t, y(t))$, by the semigroup property of (4), generates the same rest of trajectory as $(x_0, y(0))$ and thus also implies (11). If $\widetilde{\mathcal{D}}$ is maximal, this verifies $(x_t, y(t)) \in \widetilde{\mathcal{D}}$ and shows that $\widetilde{\mathcal{D}}$ is forward invariant by Definition 1.
- If $\mathcal{D} \subset \mathbb{R}^2$ is forward invariant by Definition 2 then the set $\widetilde{\mathcal{D}} \subset \mathcal{S}$ described by (8) is a forward invariant set associated with $\mathcal{D}$ according to Definition 3, even with "$t \geq 0$" in (11) instead of "$t \geq \tau$."

### *2.5 Main Goal*

Our goal is to localize the solutions of the system (4) in a neighborhood of the equilibrium $(0,0)$ by tackling the following problems:

1. For system (4), construct a *forward invariant set* $\widetilde{\mathcal{D}} \subset \mathcal{S}$ associated with a set $\mathcal{D} \subset \mathbb{R}^2$ according to Definition 3 such that $(0,0) \in \mathcal{D}$.
2. Given an invariant set $\widetilde{\mathcal{D}}$ associated with $\mathcal{D}$, estimate its *domain of attraction*. Specifically, find a larger invariant set such that all solutions it produces converge into $\mathcal{D}$ after a finite time.

## 3. PRELIMINARIES

### *3.1 Comparison System*

Before we formulate the comparison system for (4) it is necessary to specify a restriction of the state space for which the intended comparison will be valid. This is done in the following definition.

*Definition 5.* Given a set $\mathcal{D} \in \mathbb{R}^2$, we say that the tuple $(\xi(\cdot), \eta) \in \mathcal{S}$ is a *$\mathcal{D}$-self-consistent state* of system (4) if there exists an initial state $(x_0, y(0)) \in \mathcal{S}$ such that the corresponding solution $(x_t, y(t))$ satisfies two conditions:

- $(x(t), y(t)) \in \mathcal{D}$ for all $t \in [0, \tau]$;
- $(x_\tau, y(\tau)) = (\xi(\cdot), \eta)$.

In other words, a $\mathcal{D}$-self-consistent state is the one produced by the system itself, and in such a way that the trajectory via which it is produced remains in the set $\mathcal{D}$. This concept enables the following statement.

*Lemma 6.* Consider a box

$$\mathcal{B} = \begin{Bmatrix} x_{\min} \leq x \leq x_{\max} \\ y_{\min} \leq y \leq y_{\max} \end{Bmatrix} \subset \mathbb{R}^2 \tag{12}$$

and a $\mathcal{B}$-self-consistent state $(x_t, y(t))$ of (4). It holds that

$$x(s) \in [x_{\min}, x_{\max}] \cap [x(t) - \tau y_{\max}, x(t) - \tau y_{\min}] \tag{13}$$

for all $s \in [t-\tau, t]$.

**Proof.** Since $(x_t, y(t))$ is a $\mathcal{B}$-self-consistent state, we have $x(s) \in [x_{\min}, x_{\max}]$ and

$$\frac{\mathrm{d}}{\mathrm{d}s} x(s) \in [y_{\min}, y_{\max}] \tag{14}$$

for all $s \in [t-\tau, t]$. The estimation follows trivially.

Lemma 6 provides an estimation of $x(t-\tau)$ via $x(t)$. This allows an estimation of the right-hand side of (4) via $x(t)$, leading to the following definition of a delay-free comparison system.

*Definition 7.* The *delay-free comparison system* for (4) in the box (12) is the system

$$\dot{x} = y, \tag{15a}$$

$$\dot{y} = \begin{cases} -f_{\min}(x) - ky, & y \geq 0, \\ -f_{\max}(x) - ky, & \text{otherwise} \end{cases} \tag{15b}$$

where

$$f_{\min}(x) = \min_{\xi \in \mathcal{X}(x)} f(\xi), \quad f_{\max}(x) = \max_{\xi \in \mathcal{X}(x)} f(\xi) \tag{16}$$

and

$$\mathcal{X}(x) = [x_{\min}, x_{\max}] \cap [x - \tau y_{\max}, x - \tau y_{\min}]. \tag{17}$$

Note that (15) is a system of ordinary differential equations with, generally, a discontinuous right-hand side. However, at least for small $\tau$ it permits safe numerical integration starting from every point in a neighborhood of $(0,0)$, except for a small segment of the $x$ axis containing 0. We claim it for the following reason.

*Proposition 8.* System (15) has only piecewise smooth Carathéodory solutions starting from almost every initial point – specifically, excluded are initial points $(x, 0)$ such that the interval $[x - \tau y_{\max}, x - \tau y_{\min}]$ contains a zero of the function $f$.

**Proof.** The set $\mathcal{X}(x)$ is compact and changes smoothly with $x$. Function $f$ is assumed to be analytic. Therefore, the extrema $f_{\min}$ and $f_{\max}$ are smooth with respect to $x$ by Theorem 6.2 from Fiacco and Ishizuka (1990). Thus, the only switching surface in (15) is the line $y = 0$. On both sides of the line the velocity vector points either in the same direction or in the opposite directions away from the line. The points with opposing velocities are excluded by the Proposition. Starting from every other point, the solution of (15) proceeds piecewise smoothly in the Carathéodory sense – it does not reach the excluded points and thus does not go into the sliding mode on the switching surface, see Cortés (2008). The proof is thus complete.

*Remark 9.* Delay-free comparison system (15) is an analogue of the autonomous comparison systems $(A^+)$ and $(A^-)$ of Belykh (1975) – it is similar to $(A^+)$ for $y \geq 0$ and to $(A^-)$ for $y < 0$.

The geometric meaning of (15) is explained by the following lemma.

*Lemma 10.* Given a box $\mathcal{B}$ and a point $(x, y) \in \mathcal{B}$, suppose $(\xi(\cdot), y)$ is a $\mathcal{B}$-self-consistent state of the time-delay system (4) with $\xi(0) = x$. Then the trajectory of (4), as it originates from $(\xi(\cdot), y)$, crosses the trajectory of the comparison system (15) starting from $(x, y)$ in the direction "left to right," which is to say that the shortest rotation from the velocity of (15) to the velocity of (4) at $(x, y)$ is clockwise, unless the velocities are collinear.

**Proof.** The proof is illustrated by Fig. 1. Owing to the $\mathcal{B}$-self-consistency of $(\xi(\cdot), y)$, the historical value $\xi(-\tau)$ can be estimated from $\xi(0) = x$ by Lemma 6 which yields $\xi(-\tau) \in \mathcal{X}(x)$. The initial velocity of (4) starting from the state $(\xi(\cdot), y)$ is the vector

$$\begin{bmatrix} y \\ -f\big(\xi(-\tau)\big) - ky \end{bmatrix} \tag{18}$$

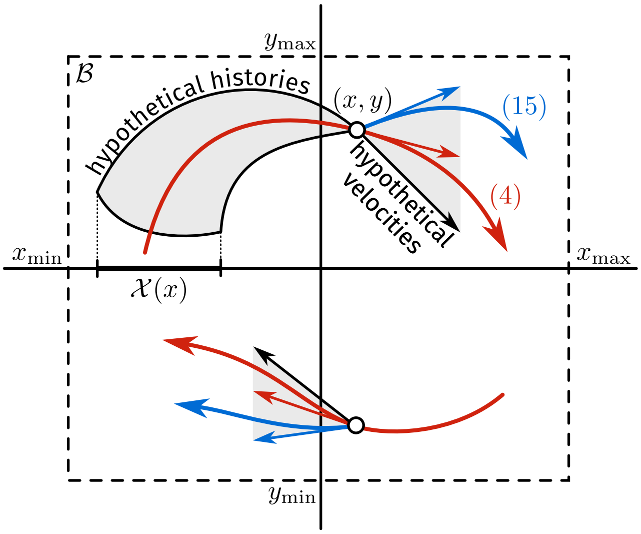


Fig. 1. Illustration for the proof of Lemma 10.

where the term $f(\xi(-\tau))$ is estimated by $[f_{\min}(x), f_{\max}(x)]$ giving rise to the cone of "hypothetical velocities" of (4), shown in the figure. If $y \geq 0$, the comparison system (15) assumes $f = f_{\min}(x)$ and follows the upper direction of the cone; otherwise, the lower one. Since both (4) and (15) move rightward if $y > 0$, vertically if $y = 0$, and leftward if $y < 0$, this finishes the proof.

### *3.2 Invariant Pocket*

We construct an invariant set and its estimated domain of attraction using regions formed by the trajectories of the comparison system (15). Let us define the basic building block of this construction, called an invariant pocket.

The following explanation is illustrated by Fig. 2. Recall that the delay-free system (1) is assumed to be oscillatory by virtue of condition (3). Consider a box $\mathcal{B}$ given by (12) which contains the origin. Due to continuity, at least for small $\tau$, there exists a trajectory $\Gamma$ of the comparison system (15) which starts at a point $(x^1, 0)$ with $x^1 > 0$, makes a turn around the origin and returns to the positive $x$ axis at $x = x^2 < x^1$. Together with the segment $[x^2, x^1]$, trajectory $\Gamma$ bounds a region which we name $\mathcal{P}$. It is sometimes referred to as a *Bendixson pocket* – see p. 170 in Petrovski (1966).

Lemma 10 states that every trajectory of the time-delay system (4) starting from a $\mathcal{B}$-self-consistent initial state $(\xi(\cdot), \eta)$ whose head $(\xi(0), \eta)$ is on $\Gamma$, must certainly enter the pocket $\mathcal{P}$. This fact is symbolized in Fig. 2 by arrows crossing $\Gamma$ into $\mathcal{P}$. As for the arrow that crosses the segment $[x^2, x^1]$ downward, it represents another condition which makes the pocket $\mathcal{P}$ "invariant."

*Definition 11.* Consider a Bendixson pocket $\mathcal{P}$ constructed as described above inside a box $\mathcal{B}$ defined by (12). We say that the pocket $\mathcal{P}$ is *invariant relative to* the box $\mathcal{B}$ if $f(\cdot) > 0$ on the interval $[x^2 - \tau y_{\max}, x^1 - \tau y_{\min}]$.

Such a pocket is invariant in the following sense.

*Lemma 12.* Suppose that a pocket $\mathcal{P}$ is invariant relative to a box $\mathcal{B}$. Every trajectory of the time-delay system (4) starting from a $\mathcal{B}$-self-consistent state $(\xi(\cdot), \eta)$ with $(\xi(0), \eta) \in \mathcal{P}$ remains in $\mathcal{P}$ in the future.

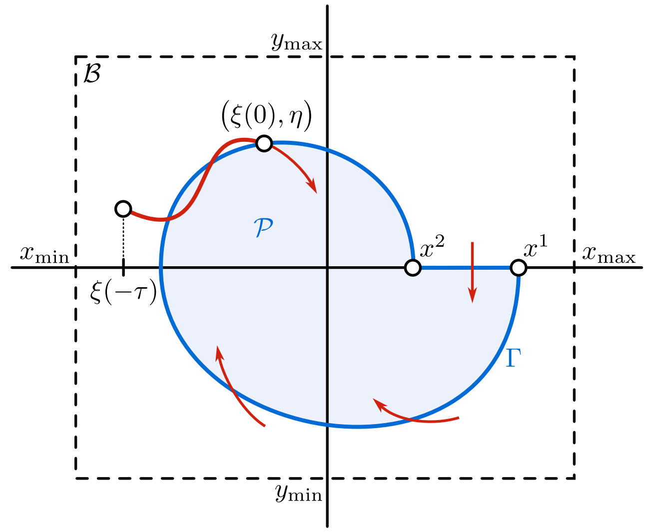


Fig. 2. Invariant pocket $\mathcal{P}$ relative to the box $\mathcal{B}$.

**Proof.** Consider the pocket $\mathcal{P}$ as pictured in Fig. 2 and suppose that the head $(\xi(0), \eta)$ of the initial state lies on the segment $[x^2, x^1]$ of the $x$ axis. Under the condition prescribed by Definition 11, it holds that $f_{\min}(x) > 0$ for all $x \in [x^2, x^1]$ where $f_{\min}(x)$ comes from Definition 7. This verifies that (4) crosses the $x$-axis segment of the pocket's boundary downward and into $\mathcal{P}$. As explained previously, the curve $\Gamma$ is also crossed inward which completes the invariance proof.

## 4. MAIN RESULT

Using the concept of an invariant pocket, we propose the following numerical algorithm to construct an invariant set of the time-delay system (4) and estimate its domain of attraction. The algorithm is illustrated by Fig. 3:

1. Begin by constructing an invariant pocket $\mathcal{P}_1$ in a box $\mathcal{B}_1$. This can be done as follows:
   - choose an initial point $(x^1, 0)$ and generate a Bendixson pocket $\mathcal{P}_{1,0}$ via numerical integration of the *delay-free* system (1) starting from $(x^1, 0)$;
   - define the smallest box $\mathcal{B}_{1,0}$ that contains $\mathcal{P}_{1,0}$;
   - based on the box $\mathcal{B}_{1,0}$, generate a new (larger) pocket $\mathcal{P}_{1,1}$ from the same initial point $(x^1, 0)$ by integration of the *comparison* system (15);
   - contain $\mathcal{P}_{1,1}$ in the smallest box $\mathcal{B}_{1,1}$;
   - based on $\mathcal{B}_{1,1}$, generate $\mathcal{P}_{1,2}$ by integration of the comparison system;
   - contain $\mathcal{P}_{1,2}$ in the smallest box $\mathcal{B}_{1,2}$;
   - repeat until the process practically converges to a pocket $\mathcal{P}_{1,\infty} =: \mathcal{P}_1$ and box $\mathcal{B}_{1,\infty} =: \mathcal{B}_1$.

   If the process does not converge (e.g., the trajectory of (15) may fail to turn around the origin or its endpoint may fall further from the origin than $x^1$), then the initial point $(x^1, 0)$ must be placed closer to the origin.

   Denote $(x^2, 0)$ the endpoint of the trajectory that forms the pocket $\mathcal{P}_1$.
2. Using the same box $\mathcal{B}_1$, generate another invariant pocket $\mathcal{P}_2$ bounded by the trajectory of (15) starting from $(x^2, 0)$ and ending at some point $(x^3, 0)$. Define $\mathcal{B}_2$ as the smallest box containing $\mathcal{P}_2$.
3. Using $\mathcal{B}_2$, generate $\mathcal{P}_3$ and define $\mathcal{B}_3$, etc.

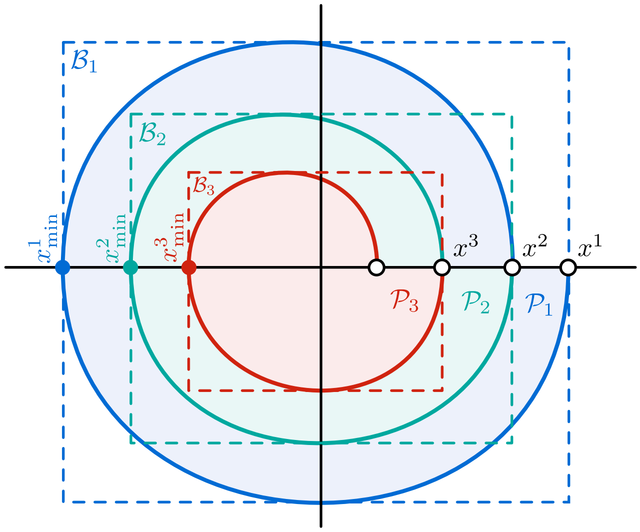


Fig. 3. A sequence of three invariant pockets and corresponding boxes produced by the numerical algorithm.

It follows from the previous section that the algorithm is feasible, at least for small $\tau$ and in a neighborhood of the origin. More precisely, see the following remark.

*Remark 13.* The success of the algorithm is decided at its first step where the trajectory of (15) starting at $(x^1, 0)$ should encircle the origin and form the pocket $\mathcal{P}_1$. It is guaranteed to happen for small $\tau$ and $x^1$. An initial guess for $\tau$ and $x^1$ that are "small enough" can be obtained, e.g., by linearization at the origin. These values can then be gradually increased until the first step of the algorithm starts failing. If the first step passes then the output of the algorithm is valid.

The algorithm outputs a sequence of nested invariant pockets $\mathcal{P}_i$ and boxes $\mathcal{B}_i$. The structure of the sequence and its relation to our goals can be summarized as follows:

- The first pocket $\mathcal{P}_1$ is invariant relative to the box $\mathcal{B}_1$.
- Each of the following pockets $\mathcal{P}_i$ ($i \geq 2$) is invariant relative to $\mathcal{B}_{i-1}$ by construction, and is also invariant relative to $\mathcal{B}_i$ since $\mathcal{B}_i \subset \mathcal{B}_{i-1}$.
- Ideally, we would like to end up with the largest possible pocket $\mathcal{P}_1$ and the smallest possible $\mathcal{P}_N$ after a large enough number of steps $N$.
- The last pocket $\mathcal{P}_N$ implicitly defines a forward invariant set $\widetilde{\mathcal{P}}_N \subset \mathcal{S}$ of (4). The first pocket $\mathcal{P}_1$ defines an estimation $\widetilde{\mathcal{P}}_1 \subset \mathcal{S}$ of the domain of attraction of $\widetilde{\mathcal{P}}_N$. The exact formulation is contained in the following main Theorem.

*Theorem 14.* Consider a sequence of pockets $\mathcal{P}_i$ and boxes

$$\mathcal{B}_i = \begin{Bmatrix} x^i_{\min} \leq x \leq x^i_{\max} \\ y^i_{\min} \leq y \leq y^i_{\max} \end{Bmatrix} \tag{19}$$

constructed by the above algorithm ($i = 1, 2, \dots, N$). Assume that

$$f(x) < 0 \text{ for all } x \in [x^1_{\min} - \tau y^1_{\max}, x^N_{\min} - \tau y^N_{\min}]. \tag{20}$$

Let $\widetilde{\mathcal{P}}_i \subset S$ be the set of initial states $(x_0, y(0))$ of the time-delay system (4) such that the corresponding solution satisfies

$$\big(x(s), y(s)\big) \in \mathcal{B}_i \text{ for all } s \in [0, \tau] \tag{21}$$

and

$$\big(x(\tau), y(\tau)\big) \in \mathcal{P}_i. \tag{22}$$

Then $\widetilde{\mathcal{P}}_N$ is a forward invariant set associated with $\mathcal{P}_N$ in the sense of Definition 3. Furthermore, $\widetilde{\mathcal{P}}_1$ is an estimation of the domain of attraction of $\widetilde{\mathcal{P}}_N$ in the sense of Section 2.5.

**Proof.** By definition of $\widetilde{\mathcal{P}}_i$, for every $(x_0, y(0)) \in \widetilde{\mathcal{P}}_i$ the corresponding state $(x_\tau, y(\tau))$ of (4) is $\mathcal{B}_i$-self-consistent in the sense of Definition 5. Furthermore, $(x(\tau), y(\tau)) \in \mathcal{P}_i$. By Lemma 12 the trajectory of (4) is guaranteed to stay in $\mathcal{P}_i$ after time $t = \tau$. Therefore, $\widetilde{\mathcal{P}}_i$ is a forward invariant set associated with $\mathcal{P}_i$ according to Definition 3.

Let us now prove that $\widetilde{\mathcal{P}}_1$ is an estimation of the domain of attraction of $\widetilde{\mathcal{P}}_N$. Take a solution of (4) starting from $\widetilde{\mathcal{P}}_1$ that has entered $\mathcal{P}_1$ and suppose that it never enters $\mathcal{P}_2$, so it remains in the gap $\mathcal{P}_1 \backslash \mathcal{P}_2$ (see Fig. 3). Consider a small $\varepsilon > 0$. The solution cannot remain forever in the part of the gap where $y < -\varepsilon$ since then it moves leftward with non-zero velocity, and in the same way it cannot remain in the part where $y > \varepsilon$ where it moves rightward. It remains to show that the solution cannot remain forever in the $\varepsilon$-neighborhood of the segments $[x^2, x^1]$ and $[x^1_{\min}, x^2_{\min}]$ of the $x$ axis. From the proof of Lemma 12 the velocity of (4) points downward on the segment $[x^2, x^1]$. By continuity, the velocity is non-zero in the $\varepsilon$-neighborhood of $[x^2, x^1]$. Similarly, from the condition (20) it follows that in the $\varepsilon$-neighborhood of $[x^1_{\min}, x^2_{\min}]$ the velocity is non-zero as well. Therefore, we conclude that the solution must enter $\mathcal{P}_2$ after a finite time. In the same way, it must enter all further pockets and eventually $\mathcal{P}_N$. The proof is finished.

## 5. EXAMPLE

Consider (4) with $f(x) = \sin x$, $k = 1$, and two cases of $\tau$.

The first case (Fig. 4) is $\tau = 0.8$. We implemented the algorithm of the previous section and observed that the sequence of pockets $\mathcal{P}_i$ shrinks completely down to the origin as $i \to \infty$. Linearization at the origin shows local asymptotic stability, see Cooke and Grossman (1982). Thus, we conclude that the set $\widetilde{\mathcal{P}}_1$ specified by Theorem 14

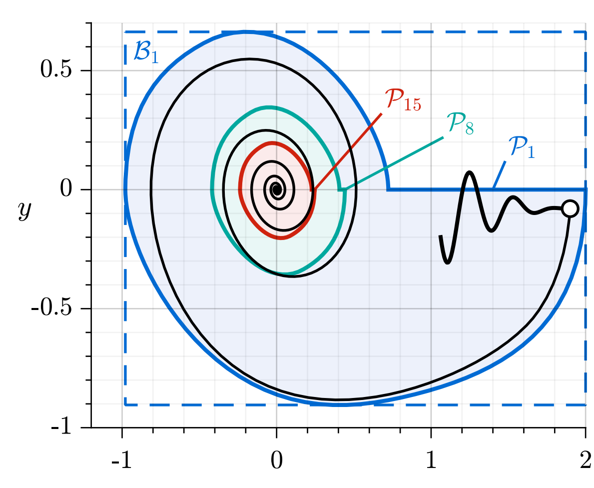


Fig. 4. Example ($\tau = 0.8$).

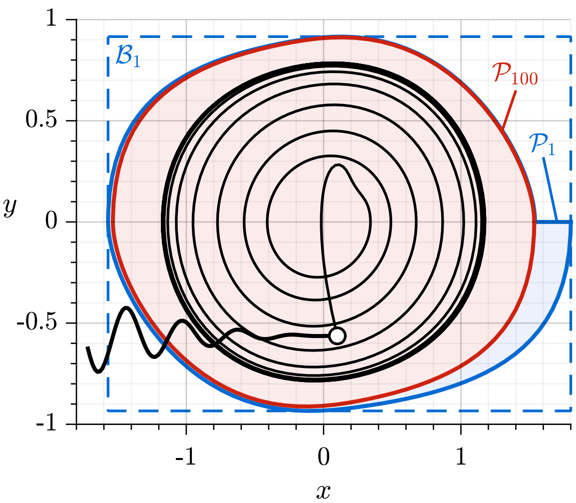


Fig. 5. Example ($\tau = 1.4$).

is, in fact, an estimation of the domain of attraction *of the origin*. To reiterate: the trajectory $(x_t, y(t))$ of (4) starting from any initial state $(x_0, y(0))$ such that $(x(s), y(s)) \in \mathcal{B}_1$ for all $s \in [0, \tau]$ and $(x(\tau), y(\tau)) \in \mathcal{P}_1$ stays in $\mathcal{P}_1$ for $t \geq \tau$ and converges to $(0, 0)$. The black curve in the figure is a sample trajectory of (4) starting from a sinusoidal initial state shown by the bold curve. As expected, it goes through all the pockets and approaches the origin.

The second case (Fig. 5) is $\tau = 1.4$. The pockets do not shrink to the origin but quickly approach an oval shape. As we increase $\tau$ from 0.8 to 1.4, one pair of eigenvalues at the origin moves into the right half of the complex plane which indicates a supercritical Hopf bifurcation, i.e., the birth of a stable limit cycle near the origin, see Somolinos (1978). The limit cycle is contained in our pockets which is confirmed by the sample trajectory. In practice, pocket $\mathcal{P}_{100}$ is an estimation of the largest oscillation resulting from a slight disturbance at the origin. The set $\widetilde{\mathcal{P}}_1$ defined by Theorem 14 estimates the domain of attraction of $\mathcal{P}_{100}$.

## 6. CONCLUSION

We introduce the concept of a delay-free comparison system. Based on that, we propose a numerical algorithm to find a forward invariant set and estimate its domain of attraction for a nonlinear time-delay system.

Although the algorithm is only proven to be feasible for small delays, the example shows that it works for sensible delays and in a moderately large region around the origin. As a nonlocal technique, it can even generate an invariant set with large limit cycles inside. In practical terms, this result brings about, e.g., an estimation of the amplitude of the possible oscillations near the origin. If the origin is asymptotically stable, then the method can enlarge the linearization-based domain of attraction.

In the future, the method can be adapted to distributed delays and to time-varying systems by incorporating the classical autonomous comparison system idea of Belykh (1975). It can be extended to systems of a more general structure and enhanced by the *left* and *right* comparison systems in the sense of Ponomarev et al. (2024). Furthermore, the method is applicable to the analysis of *cycle slipping*, see Smirnova and Proskurnikov (2019).

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