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ORIGINAL PAPER

A Note on Asymmetric Hypergraphs

Dominik Bohnert[1] · **Christian Winter**[1]

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Abstract

A k-graph $\mathcal{G}$ is asymmetric if there does not exist an automorphism on $\mathcal{G}$ other than the identity, and $\mathcal{G}$ is called minimal asymmetric if it is asymmetric but every non-trivial induced sub-hypergraph of $\mathcal{G}$ is non-asymmetric. Extending a result of Jiang and Nešetřil (J Comb Theory Ser B 164: 105–118, 2024), we show that for every $k \geq 3$, there exist infinitely many minimal asymmetric k-graphs which have maximum degree 2 and are linear. Further, we show that there are infinitely many 2-regular asymmetric k-graphs for $k \geq 3$.

1 Introduction

For $k \geq 2$, a *k-uniform hypergraph*, or *k-graph*, is a pair $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ such that the edge set $\mathcal{E}(\mathcal{G})$ consists of k-element subsets of the vertex set $\mathcal{V}(\mathcal{G})$. Note that 2-graphs are commonly known as graphs. An *automorphism* on a k-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a bijection $\phi\colon \mathcal{V} \to \mathcal{V}$ such that for every $E \in \mathcal{E}$, $\{\phi(v) : v \in E\} \in \mathcal{E}$. An automorphism which is not the identity is called *non-trivial*. We say that a k-graph $\mathcal{G}$ is *symmetric* if there exists a non-trivial automorphism on $\mathcal{G}$ and *asymmetric* otherwise. $\mathcal{G}$ is *minimal asymmetric* if it is asymmetric and every induced sub-hypergraph $\mathcal{H}$ of $\mathcal{G}$ with $2 \leq |\mathcal{V}(\mathcal{H})| < |\mathcal{V}(G)|$ is symmetric.

Asymmetry of graphs was first considered by Frucht [6] in 1949. It was famously observed by Erdős and Rényi [5] that random graphs are asymmetric with high probability. In 1988, Nešetřil conjectured that the number of minimal asymmetric graphs is finite, see [1]. After several partial results [10–12], this conjecture was recently confirmed by Schweitzer and Schweitzer [13] who showed that there are exactly 18 minimal asymmetric graphs. In the hypergraph setting, Ellingham and Schroeder [4] studied a connection between asymmetric hypergraphs and color-preserving vertex partitions. Jiang and Nešetřil showed in [9] (also published as an extended abstract in [8]) that the natural generalization of Nešetřil's conjecture to k-graphs does not hold.

✉ Christian Winter
christian.winter@kit.edu

1 Karlsruhe Institute of Technology, Karlsruhe, Germany

Theorem 1 (Jiang, Nešetřil [9]) *Let $k \geq 3$ be a positive integer. Then there exist infinitely many minimal asymmetric k-graphs.*

They provided an explicit construction where each k-graph has maximum degree 3, i.e. every vertex is contained in at most three edges. It is a natural follow-up question to study how *sparse* a minimal asymmetric k-graphs can be. In this paper, we consider *sparsity* with respect to maximum degree and maximum codegree. In a k-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for any two distinct vertices $u, v \in \mathcal{V}$ the *codegree* of u and v is the number of edges in $\mathcal{E}$ which contain both u and v. The *maximum codegree* of $\mathcal{G}$ is the maximum over the codegrees of all vertex pairs $u, v \in \mathcal{V}$, $u \neq v$. Here we prove the following strengthening of Theorem 1.

Theorem 2 *Let $k \geq 3$ be a positive integer. There exist infinitely many minimal asymmetric k-graphs which have maximum degree 2 and maximum codegree 1.*

We remark that a k-graph with maximum codegree 1 is commonly referred to as *linear*. Equivalently, a k-graph is linear if any two edges intersect in at most one vertex. Note that every k-graph with maximum degree 1 or maximum codegree 0 is symmetric, so our result is best possible with respect to both parameters. In our construction for Theorem 2, most vertices have degree 2, but crucially some vertices have degree 1. This raises the question whether there exist (minimal) asymmetric k-graphs where every vertex has the same degree. We say that a k-graph $\mathcal{G}$ is *r-regular* if every vertex has degree r. Based on a result by Izbicki [7] we obtain the following.

Theorem 3 *There are infinitely many 2-regular, asymmetric k-graphs for every $k \geq 3$.*

It remains open if this result extends to *minimal* asymmetric k-graphs. We raise the following question.

Questions 4 *For $k \geq 3$ and $r \geq 2$, is there an r-regular, minimal asymmetric k-graph?*

Note that this question can be answered negatively for $k = 2$ and arbitrary r: None of the 18 minimal asymmetric graphs characterized by Schweitzer and Schweitzer [13] is regular.

In this paper we use standard graph theoretic notions; for formal definitions we refer the reader to Diestel [3]. We denote by $[n]$ the set of the first n integers $\{1, \ldots, n\}$. For consistency, let $[0] = \varnothing$. Given a function $\phi : \mathcal{V} \to \mathcal{V}$ and a subset $W \subseteq \mathcal{V}$, we denote the image of W by $\phi(W) := \{\phi(v) : v \in W\}$.

The organization of this paper is as follows. In Sect. 2.1 we present the constructions needed for the proof of Theorem 2, in Sects. 2.2 and 2.3 we show some properties of these constructions. Subsequently, in Sect. 2.4 we prove Theorem 2 and in Sect. 3 we give a proof of Theorem 3.

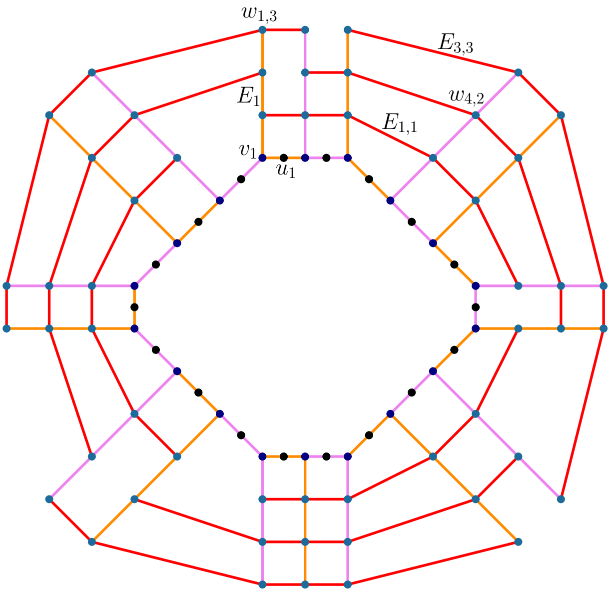

Fig. 1 The 6-graph $\mathcal{G}_{6,3}$

2 Sparse Minimal Asymmetric *k*-Graphs

2.1 Constructions

Our asymmetric hypergraph is constructed in two steps. The basic framework is the following (symmetric) construction given by Jiang and Nešetřil [9]. Throughout this section, all indices in $[tk]$ are considered modulo tk. In particular, $tk \equiv 0$.

Construction 5 (Jiang, Nešetřil [9]) Let $k \geq 3$ and $t \geq 2$. Let $\mathcal{G}_{k,t}$ be the k-graph with vertices

$$\mathcal{V}(\mathcal{G}_{k,t}) = \left\{u_i \,:\, i \in [tk]\right\} \cup \left\{v_i \,:\, i \in [tk]\right\} \cup \left\{w_{i,j} \,:\, i \in [tk],\, j \in [k-3]\right\}$$

and edges $\mathcal{E}(\mathcal{G}_{k,t}) = \mathcal{E}_L \cup \mathcal{E}_{cyc}$. Here $\mathcal{E}_L = \left\{E_i \,:\, i \in [tk]\right\}$ is the set of *L-edges*

$$E_i = \left\{v_i, u_i, v_{i+1}, w_{i,1}, \ldots, w_{i,k-3}\right\}.$$

Furthermore, the set of *cyclic edges* is $\mathcal{E}_{cyc} = \left\{E_{i,j} \,:\, j \in [k-3], i = j + sk, s \in [t]\right\}$ where

$$E_{i,j} = \left\{w_{i,j}, \ldots, w_{i+k-1,j}\right\}.$$

An illustration of this construction is given in Fig. 1.

Jiang and Nešetřil [9] proved Theorem 1 by adding a single edge to $\mathcal{G}_{k,t}$. In this paper, we extend Construction 5 as follows.

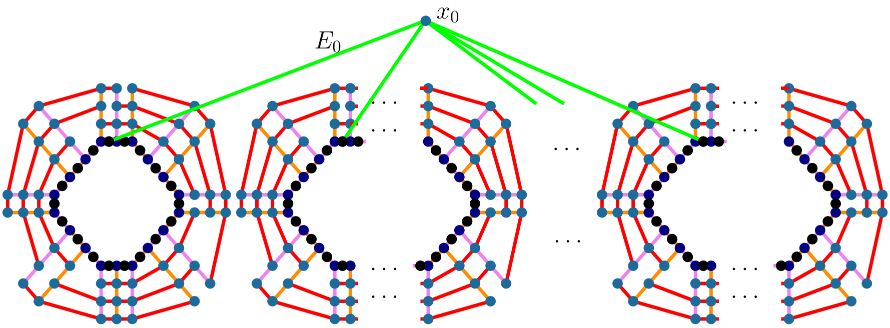

Fig. 2 The k-graph $\mathcal{H}(t_1, \ldots, t_{k-1})$

Construction 6 Let $k \geq 3$ and let $t_i \in \mathbb{N}$ for $i \in [k-1]$ such that $2 \leq t_1 < t_2 < \cdots < t_{k-1}$. We denote by $\mathcal{G}^\ell = (\mathcal{V}^\ell, \mathcal{E}^\ell)$ a copy of $\mathcal{G}_{k,t_\ell}$ as introduced in Construction 5, such that the vertex sets $\mathcal{V}^\ell$ are pairwise disjoint. For every $\ell \in [k-1]$, we write u_i^ℓ when referring to the vertex of $\mathcal{G}^\ell$ corresponding to u_i in $\mathcal{G}_{k,t_\ell}$ and similarly for v_i^ℓ, $w_{i,j}^\ell$, E_i^ℓ and $E_{i,j}^\ell$.

Now let x_0 be an additional vertex which is not contained in any $\mathcal{V}^\ell$, $\ell \in [k-1]$. We define $\mathcal{H}(t_1, \ldots, t_{k-1}) = (\mathcal{V}, \mathcal{E})$ such that

$$\mathcal{V} = \mathcal{V}^1 \cup \cdots \cup \mathcal{V}^{k-1} \cup \{x_0\} \quad \text{and} \quad \mathcal{E} = \mathcal{E}^1 \cup \cdots \cup \mathcal{E}^{k-1} \cup \{E_0\},$$

where $E_0 = \{x_0, u_1^1, u_1^2, \ldots, u_1^{k-1}\}$. See Fig. 2 for an illustration of $\mathcal{H}(t_1, \ldots, t_{k-1})$.

The k-graph $\mathcal{H}(t_1, \ldots, t_{k-1})$ is non-asymmetric if $k = 3$ or $k = 5$, because Lemma 10 does not hold for such k, see also Fig. 4. Therefore, we provide two additional constructions covering those cases.

Construction 7 Let $k \in \{3, 5\}$ and $2 \leq t < t'$. Let $\mathcal{G}$ and $\mathcal{G}'$ be vertex-disjoint copies of $\mathcal{G}_{k,t}$ and $\mathcal{G}_{k,t'}$, respectively. We denote by u_i' the vertex corresponding to u_i in $\mathcal{G}_{k,t'}$ and similarly for v_i', $w_{i,j}'$, E_i' and $E_{i,j}'$. For the vertices in $\mathcal{G}$ we use the same labels as defined for $\mathcal{G}_{k,t}$, e.g. u_i refers to the vertex in $\mathcal{G}$ corresponding to u_i in $\mathcal{G}_{k,t}$. Let x_0, y and y' be three distinct vertices, disjoint from $\mathcal{V}(\mathcal{G}) \cup \mathcal{V}(\mathcal{G}')$.
For $k = 3$, let $E_0 = \{x_0, u_1, u_1'\}$, $E_y = \{y, u_2, u_3\}$ and $E_y' = \{y', u_2', u_3'\}$. We define the 3-graph

$$\mathcal{H}^3(t, t') = \big(\mathcal{V}(\mathcal{G}) \cup \mathcal{V}(\mathcal{G}') \cup \{x_0, y, y'\}, \mathcal{E}(\mathcal{G}) \cup \mathcal{E}(\mathcal{G}') \cup \{E_0, E_y, E_y'\}\big).$$

For $k = 5$, let $E_0 = \{x_0, u_1, u_2, u_1', u_2'\}$ and define the 5-graph

$$\mathcal{H}^5(t, t') = \big(\mathcal{V}(\mathcal{G}) \cup \mathcal{V}(\mathcal{G}') \cup \{x_0\}, \mathcal{E}(\mathcal{G}) \cup \mathcal{E}(\mathcal{G}') \cup \{E_0\}\big).$$

Both constructions are illustrated in Fig. 3.

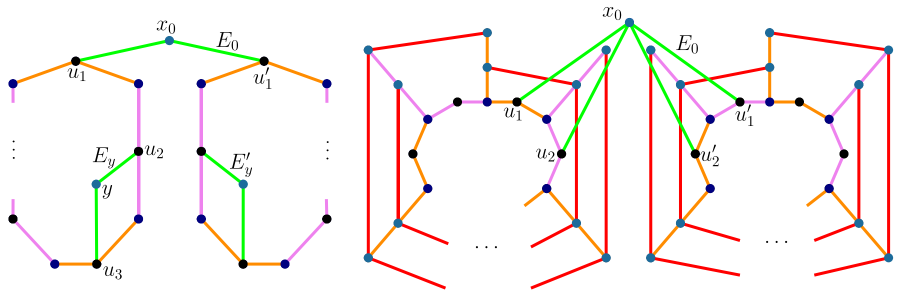

Fig. 3 The hypergraphs $\mathcal{H}^3(t, t')$ and $\mathcal{H}^5(t, t')$

2.2 Properties of $\mathcal{G}_{k,t}$

First we state slight reformulations of two properties shown by Jiang and Nešetřil [8].

Lemma 8 (Jiang, Nešetřil [8]) *Let $k \geq 3$ and $t \geq 2$. Let $\mathcal{G}'$ be an induced sub-hypergraph of $\mathcal{G}_{k,t}$ on at least two vertices.*

(i) *There is a non-trivial automorphism on $\mathcal{G}'$, i.e. $\mathcal{G}'$ is symmetric.*
(ii) *If $E_1 \in \mathcal{E}(\mathcal{G}')$ and $|\mathcal{V}(\mathcal{G}')| < |\mathcal{V}(\mathcal{G}_{k,t})|$, then there is a non-trivial automorphism ϕ on $\mathcal{G}'$ such that $\phi(E_1) = E_1$ and $\phi(u_1) = u_1$.*

A stronger version of Lemma 8(i) is given in Lemma 4(2) of [8], where *weak* sub-hypergraphs are considered. Lemma 8(ii) follows from the proof of Lemma 3(3) of [8].

Lemma 9 *Let $k \geq 3$ and $t \geq 2$. Let ϕ be an automorphism on $\mathcal{G}_{k,t}$.*

(i) *Then $\{\phi(u_i) : i \in [tk]\} = \{u_i : i \in [tk]\}$. Furthermore, $\phi(E) \in \mathcal{E}_L$ for every $E \in \mathcal{E}_L$ and $\{\phi(v_i) : i \in [tk]\} = \{v_i : i \in [tk]\}$.*
(ii) *There is a $j \in [tk]$ such that either $\phi(E_i) = E_{i+j-1}$ for every $i \in [tk]$ or $\phi(E_i) = E_{j-i+1}$ for every $i \in [tk]$, where the indices are considered modulo tk.*

We remark that the statement of Lemma 9(i) is given implicitly in [8]. A statement similar to Lemma 9(ii) appears as Lemma 9(1) in [9].

Proof of Lemma 9 Note that the u_i's are exactly the vertices of degree 1 in $\mathcal{G}_{k,t}$. Observe that, since ϕ is an automorphism, v and $\phi(v)$ have the same degree for every vertex v, thus $\phi(\{u_i : i \in [tk]\}) = \{u_i : i \in [tk]\}$. This implies (i).

A consequence of (i) is that $\phi(E_1) = E_j$ for some $j \in [tk]$, so $\{\phi(v_1), \phi(v_2)\} = \{v_j, v_{j+1}\}$. If $\phi(v_1) = v_j$ and $\phi(v_2) = v_{j+1}$, then $\phi(E_2) = E_{j+1}$ and thus $\phi(v_3) = v_{j+2}$. Iteratively, we find that $\phi(E_i) = E_{i+j-1}$ for every $i \in [tk]$. Now suppose that $\phi(v_1) = v_{j+1}$ and $\phi(v_2) = v_j$. Then $\phi(E_2) = E_{j-1}$ and iteratively $\phi(E_i) = E_{j-i+1}$ for every $i \in [tk]$. This completes the proof of (ii). □

Lemma 10 *Let $k = 4$ or $k \geq 6$ and $t \geq 2$. Let ϕ be an automorphism on $\mathcal{G}_{k,t}$. If $\phi(E_1) = E_1$, then ϕ is the identity.*

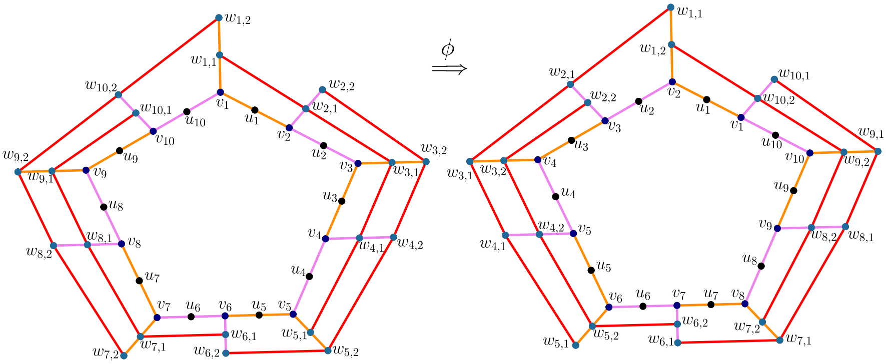

Fig. 4 Non-trivial automorphism ϕ on $\mathcal{G}_{5,2}$ with $\phi(E_1) = E_1$

A result closely related to Lemma 10 is given in Lemma 9(2) of [9] without a proof. Note that Lemma 3(2) of [8] almost corresponds to our Lemma 10, but it does not hold for $k = 3$ and $k = 5$, see for example the non-trivial automorphism illustrated in Fig. 4.

Proof of Lemma 10 Assume that ϕ is not the identity. Then Lemma 9(ii) provides that $\phi(E_i) = E_{2-i}$ for every $i \in [tk]$.

If $k = 4$, then Lemma 9(i) implies that $\{\phi(w_{i,1}) : i \in [tk]\} = \{w_{i,1} : i \in [tk]\}$. Since $\phi(E_1) = E_1$ and $\phi(E_2) = E_{tk}$, we find that $\phi(w_{1,1}) = w_{1,1}$ and $\phi(w_{2,1}) = w_{tk,1}$, respectively. Observe that the edge $E_{1,1}$ contains vertices $w_{1,1}$ and $w_{2,1}$. However, there is no edge in $\mathcal{G}_{k,t}$ containing $\phi(w_{1,1}) = w_{1,1}$ and $\phi(w_{2,1}) = w_{tk,1}$, a contradiction.

Now suppose that $k \geq 6$. Then consider the edge $E_{3,3} = \{w_{3,3}, \ldots, w_{k+2,3}\}$. It intersects each of the edges $E_3, \ldots, E_{k+2}$. Because ϕ is an automorphism, $\phi(E_{3,3})$ has a non-empty intersection with each of the edges $\{\phi(E_3), \ldots, \phi(E_{2+k})\} = \{E_{(t-1)k}, \ldots, E_{tk-1}\}$. However, it is easy to see from our construction that in $\mathcal{G}_{k,t}$ such an edge does not exist. □

2.3 Connectivity

Next we introduce a notion of *connectivity* between two vertices of a k-graph. In a k-graph $\mathcal{G}$, a $v_1 v_{r+1}$-*path* is an alternating sequence $(v_1, E_1, v_2, E_2, \ldots, E_r, v_{r+1})$ of $r + 1$ distinct vertices $v_i \in \mathcal{V}(\mathcal{G})$ and r distinct edges $E_i \in \mathcal{E}(\mathcal{G})$ such that both v_i and v_{i+1} are contained in E_i for any $i \in [r]$. Such paths are commonly known as *Berge paths*. Two xy-paths are *edge-disjoint* if the underlying edge sets of the paths are disjoint. We say that x and y are t-*connected* if there are t pairwise edge-disjoint xy-paths. It is a simple observation that an automorphism leaves the connectivity invariant:

Proposition 11 *Let $\mathcal{G}$ be a k-graph and $x, y \in \mathcal{V}(\mathcal{G})$. Let ϕ be an automorphism on $\mathcal{G}$. Then x and y are t-connected if and only if $\phi(x)$ and $\phi(y)$ are t-connected.*

Lemma 12 *Let $k \geq 3$ and $t \geq 2$ and consider $\mathcal{G} := \mathcal{G}_{k,t}$. Let $E \in \mathcal{E}(\mathcal{G})$ and let $x, y \in E$ be distinct vertices of degree 2, i.e. $x, y \notin \{u_1, \dots, u_{tk}\}$. Then x and y are 2-connected.*

Proof Consider $\mathcal{G}'$, the k-graph obtained from $\mathcal{G}$ by deleting the edge E. We shall show that x and y are 1-connected in $\mathcal{G}'$. Recall that the edge set $\mathcal{E}(\mathcal{G}) = \mathcal{E}_L \cup \mathcal{E}_{cyc}$ consists of L-edges E_i and cyclic edges $E_{i,j}$.

If $E \in \mathcal{E}_{cyc}$, then x and y are contained in distinct L-edges of $\mathcal{G}$, say without loss of generality $x \in E_1$ and $y \in E_j$. Then $(x, E_1, v_2, E_2, v_3, \dots, E_j, y)$ is a xy-path in $\mathcal{G}'$.

If $E \in \mathcal{E}_L$, assume that $E = E_i$ for some i. Note that the L-edges of $\mathcal{G}'$ form a $v_{i+1}v_i$-path containing all v_j, $j \in [tk]$. Therefore, if there is a xv_j-path and a $yv_{j'}$-path in $\mathcal{G}'$ for any $j, j' \in [tk]$, then we also find a xy-path in $\mathcal{G}'$. If $x \in \{v_1, v_2\}$, there is a trivial xv_i-path. Otherwise, $x \in E'$ for some cyclic edge $E' \in \mathcal{E}_{cyc}$. Let z be an arbitrary vertex in $E' \setminus \{x\}$. Then z is also contained in some L-edge $E_j \in \mathcal{E}_L$ where $j \in [tk]$. Then (x, E', z, E_j, v_j) is a xv_j-path. Similarly, we find a $yv_{j'}$-path, which completes the proof. □

2.4 Proof of the main result

Proof of Theorem 2 For the first part of the proof, let $k = 4$ or $k \geq 6$. Let $t_i \in \mathbb{N}$ for $i \in [k-1]$ be an integer such that $2 \leq t_1 < t_2 < \cdots < t_{k-1}$. We shall show that $\mathcal{H} := \mathcal{H}(t_1, \dots, t_{k-1})$ is minimal asymmetric. In order to verify that $\mathcal{H}$ is asymmetric, let ϕ be an arbitrary automorphism on $\mathcal{H}$. Recall that E_0 is an edge of $\mathcal{H}$ which connects otherwise disjoint copies of $\mathcal{G}_{k,t_i}$, $i \in [k-1]$.

First, we show that $\phi(E_0) = E_0$. We know that $\phi(E_0) \in \mathcal{E}(\mathcal{H})$, so assume that $\phi(E_0) = E$ for some $E \neq E_0$. Then $E \in \mathcal{E}(\mathcal{G}^\ell)$ for some $\ell \in [k-1]$. Consider two distinct vertices $u, v \in E_0 \setminus \{x_0\}$. Both vertices have degree 2 in $\mathcal{H}$. Since ϕ is an automorphism, $\phi(u)$ and $\phi(v)$ are distinct vertices in E with degree 2. By Lemma 12, $\phi(u)$ and $\phi(v)$ are 2-connected in $\mathcal{G}^\ell$, so in particular 2-connected in $\mathcal{H}$. However, in our construction the vertices u and v are not 2-connected, because the only path between them is (u, E_0, v). This contradicts Proposition 11. Therefore, we conclude $\phi(E_0) = E_0$, so in particular $\phi(x_0) = x_0$.

Since $\phi(E_0) = E_0$, ϕ is also an automorphism on the k-graph $\mathcal{H} - E_0$, which is the disjoint union of an isolated vertex x_0 and k-graphs $\mathcal{G}^\ell$. Recall that the t_ℓ's are pairwise distinct, therefore the $\mathcal{G}^\ell$'s are pairwise non-isomorphic. This implies that $\{\phi(E) : E \in \mathcal{E}(\mathcal{G}^\ell)\} = \mathcal{E}(\mathcal{G}^\ell)$ for every $\ell \in [k-1]$, i.e. ϕ maps each $\mathcal{G}^\ell$ to itself.

Now we show that ϕ is the identity, thus $\mathcal{H}$ is asymmetric. Fix an arbitrary $\ell \in [k-1]$. Note that $\phi(u_1^\ell) \in \phi(E_0) \cap \phi(E_1^\ell)$, thus $\phi(u_1^\ell) \in E_0 \cap \mathcal{V}(\mathcal{G}^\ell)$. This implies that $\phi(u_1^\ell) = u_1^\ell$ and therefore $\phi(E_1^\ell) = E_1^\ell$. Now Lemma 10 provides that ϕ restricted to $\mathcal{G}^\ell$ is the identity. Since ℓ was chosen arbitrarily, the entire automorphism $\phi \colon \mathcal{H} \to \mathcal{H}$ is an identity function. We conclude that $\mathcal{H}$ is asymmetric.

Now let $\mathcal{H}'$ be an arbitrary induced sub-hypergraph of $\mathcal{H}$ with $2 \leq |\mathcal{V}(\mathcal{H}')| < |\mathcal{V}(\mathcal{H})|$. We shall show that that $\mathcal{H}'$ is symmetric. We can assume that $|\mathcal{E}(\mathcal{H}')| \geq 2$, otherwise $\mathcal{H}'$ is trivally symmetric.

Case 1: $E_0 \notin \mathcal{E}(\mathcal{H}')$.

Let $E \in \mathcal{E}(\mathcal{H}')$ with $E \neq E_0$, i.e. $E \in \mathcal{E}(\mathcal{G}^\ell)$ for some fixed $\ell \in [k-1]$. Let $\mathcal{H}''$ be the sub-hypergraph of $\mathcal{H}'$ induced by the vertex set $\mathcal{V}(\mathcal{G}^\ell)$. Because $E \in \mathcal{E}(\mathcal{H}'')$, $\mathcal{H}''$ has at least two vertices. Now Lemma 8(i) provides a non-trivial automorphism ψ on $\mathcal{H}''$. We extend this automorphism to $\mathcal{H}'$ as follows. Let $\phi : \mathcal{V}(\mathcal{H}') \to \mathcal{V}(\mathcal{H}')$ with $\phi(w) = \psi(w)$ for every $w \in \mathcal{V}(\mathcal{H}'')$ and $\phi(w) = w$ for every $w \notin \mathcal{V}(\mathcal{H}'')$. Then ϕ is a non-trivial automorphism on $\mathcal{H}'$, so $\mathcal{H}'$ is symmetric.

Case 2: $E_0 \in \mathcal{E}(\mathcal{H}')$.

If there is some $\ell \in [k-1]$ such that $E_1^\ell \notin \mathcal{E}(\mathcal{H}')$, then consider the function $\phi : \mathcal{V}(\mathcal{H}') \to \mathcal{V}(\mathcal{H}')$ with $\phi(u_1^\ell) = x_0$, $\phi(x_0) = u_1^\ell$ and $\phi(w) = w$ for every $w \in \mathcal{V}(\mathcal{H}') \setminus \{u_1^\ell, x_0\}$. Observe that this is a non-trivial automorphism on $\mathcal{H}'$.

If $E_1^{\ell'} \in \mathcal{E}(\mathcal{H}')$ for every $\ell' \in [k-1]$, fix ℓ such that there is a vertex $v \in \mathcal{V}(\mathcal{G}^\ell) \setminus \mathcal{V}(\mathcal{H}')$ and let $\mathcal{H}''$ be the sub-hypergraph of $\mathcal{H}'$ induced by $\mathcal{V}(\mathcal{G}^\ell)$. Then $2 < |\mathcal{V}(\mathcal{H}'')| < |\mathcal{V}(\mathcal{G}^\ell)|$, thus Lemma 8(ii) yields a non-trivial automorphism ψ on $\mathcal{H}''$ with $\psi(u_1^\ell) = u_1^\ell$. Similarly to Case 1, we extend ψ to a non-trivial automorphism on $\mathcal{H}'$.

This completes the proof for $k = 4$ and $k \geq 6$. For $k = 3$ and $k = 5$, let t and t' be arbitrary integers with $2 \leq t < t'$. We show that $\mathcal{H}^3 := \mathcal{H}^3(t, t')$ and $\mathcal{H}^5 := \mathcal{H}^5(t, t')$ are minimal asymmetric. The proof is similar to the argumentation presented above, so we only provide a sketch. A detailed proof is given in the first author's thesis [2].

If $k = 3$, let ϕ be an arbitrary automorphism on $\mathcal{H}^3$. Then $\phi(E_0) = E_0$, and thus $\{\phi(E) : E \in \mathcal{E}(\mathcal{G})\} = \mathcal{E}(\mathcal{G})$ as well as $\{\phi(E) : E \in \mathcal{E}(\mathcal{G}')\} = \mathcal{E}(\mathcal{G}')$. Therefore, $\phi(u_1) = u_1$ and $\phi(u_1') = u_1'$. There are six edges in which every vertex has degree 2, namely E_i and E_i' for $i \in [3]$. It is easy to see that each of them is invariant under ϕ, which then implies that ϕ is the identity. Thus, $\mathcal{H}^3$ is asymmetric.

Now consider an induced sub-hypergraph $\mathcal{H}'$ of $\mathcal{H}^3$ with $2 \leq |\mathcal{V}(\mathcal{H}')| < |\mathcal{V}(\mathcal{H}^3)|$. We can suppose that there is no edge which contains two vertices of degree 1, otherwise $\mathcal{H}'$ is clearly symmetric. If $E_0 \notin \mathcal{E}(\mathcal{H}')$, then there is a non-trivial automorphism ϕ on $\mathcal{H}'$ with $\phi(u_2) = u_3$ and $\phi(u_3) = u_2$. If $E_0 \in \mathcal{E}(\mathcal{H}')$, then $E_1, E_1' \in \mathcal{E}(\mathcal{H}')$. Let $E \in \mathcal{E}(\mathcal{H}^3) \setminus \mathcal{E}(\mathcal{H}')$ and say that $E \in \mathcal{E}(\mathcal{G})$. Now if $E_y \notin \mathcal{E}(\mathcal{H}')$, we can apply Lemma 8(ii) as in Case 2, so suppose that $E_y \in \mathcal{E}(\mathcal{H}')$. Since no edge contains two vertices of degree 1, we find that $\mathcal{E}(\mathcal{H}') \cap \mathcal{E}(\mathcal{G}) = \{E_1, E_2, E_3, E_y\}$. Then there is an automorphism ϕ on $\mathcal{G}$ with $\phi(E_y) = E_3$ and $\phi(E_3) = E_y$.

If $k = 5$, given an automorphism ϕ on $\mathcal{H}^5$, we see that $\phi(E_0) = E_0$, thus $\{\phi(u_1), \phi(u_2)\} = \{u_1, u_2\}$. If $\phi(u_1) = u_2$ and $\phi(u_2) = u_1$, then $\phi(E_1) = E_2$, $\phi(E_2) = E_1$ and $\phi(E_3) = E_{tk}$. Note that the edge $E_{1,1}$ intersects each of E_1, E_2 and E_3, but there does not exist an edge in $\mathcal{E}(\mathcal{H}^5)$ which intersects $\phi(E_1) = E_2$, $\phi(E_2) = E_1$, $\phi(E_3) = E_{tk}$, a contradiction. Thus $\phi(u_1) = u_1$ and $\phi(u_2) = u_2$, and similarly, $\phi(u_1') = u_1'$ and $\phi(u_2') = u_2'$. By Lemma 9(ii), ϕ is the identity, so $\mathcal{H}^5$ is asymmetric.

Let $\mathcal{H}'$ be an induced sub-hypergraph of $\mathcal{H}^5$ such that $2 \leq |\mathcal{V}(\mathcal{H}')| < |\mathcal{V}(\mathcal{H}^5)|$. If $E_0 \notin \mathcal{E}(\mathcal{H}')$, we proceed as in Case 1. Otherwise, we can suppose that $E_1, E_2, E_1', E_2' \in \mathcal{E}(\mathcal{H}')$ by a similar argument as in Case 2. Then a variant of Lemma 8(ii), see Lemma 2.9(i) of [2], provides a non-trivial automorphism on $\mathcal{H}'$. □

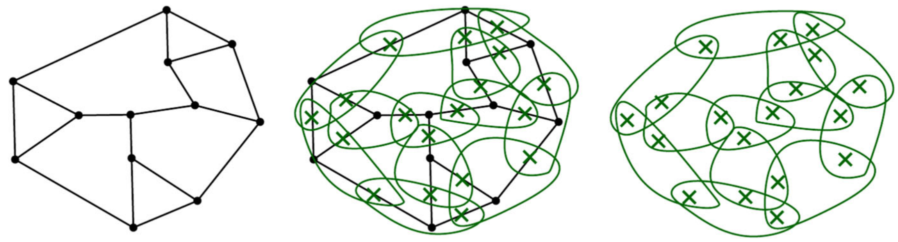

Fig. 5 The 3-regular Frucht graph (left) and its hypergraph dual (right)

3 Regular Asymmetric *k*-Graphs

In order to prove Theorem 3, we need a result by Izbicki [7].

Theorem 13 (Izbicki [7]) *For every $k \geq 3$, there exist infinitely many k-regular asymmetric 2-graphs.*

Given a k-regular 2-graph $G = (\mathcal{V}, \mathcal{E})$, the *(hypergraph) dual* of G is the k-graph $\mathcal{H} = (\mathcal{E}, \{A(v) : v \in \mathcal{V}\})$ where $A(v) = \{E \in \mathcal{E} : v \in E\}$ is the *adjacency set* of v. Note that $|A(v)| = k$ for every v, thus $\mathcal{H}$ is a well-defined k-graph. An example for a hypergraph dual is provided in Fig. 5. Observe that adjacency sets are unique in regular 2-graphs:

Proposition 14 *Let G be an r-regular 2-graph, $r \geq 2$. Let $u, v \in \mathcal{V}(G)$ be two distinct vertices of G. Then $A(u) \neq A(v)$.*

Lemma 15 *Let G be an r-regular asymmetric graph for some $r \geq 3$. Then the hypergraph dual of G is also asymmetric.*

Proof Let $G = (\mathcal{V}, \mathcal{E})$. Let $\phi_{\mathcal{H}}$ be an arbitrary automorphism on the dual $\mathcal{H}$ of G. By the definition of a dual we know that $\phi_{\mathcal{H}} \colon \mathcal{E} \to \mathcal{E}$ is a bijection such that for any $v \in \mathcal{V}$, the edges in its adjacency set $A(v)$ are mapped to $\{\phi_{\mathcal{H}}(E) : E \in A(v)\} = A(w_v)$ for some vertex $w_v \in \mathcal{V}$. By Proposition 14, w_v is uniquely determined.

We define the function $\phi_G \colon \mathcal{V} \to \mathcal{V}$, $\phi_G(v) = w_v$. Observe that ϕ_G is a bijection. Now we show that ϕ_G is an automorphism on G. Consider an arbitrary edge $E_{uv} = \{u, v\} \in \mathcal{E}$. Then $E_{uv} \in A(v)$, thus $\phi_{\mathcal{H}}(E_{uv}) \in \{\phi_{\mathcal{H}}(E) : E \in A(v)\} = A(w_v) = A(\phi_G(v))$. Similarly, we obtain $\phi_{\mathcal{H}}(E_{uv}) \in A(\phi_G(u))$. Therefore,

$$\phi_{\mathcal{H}}(E_{uv}) = \{\phi_G(u), \phi_G(v)\}. \tag{1}$$

This implies that $\{\phi_G(u), \phi_G(v)\} \in \mathcal{E}$, so ϕ_G is an automorphism on G. Since G is asymmetric, ϕ_G is the identity. Then for every edge $E_{uv} = \{u, v\} \in \mathcal{E}$, (1) implies that $\phi_{\mathcal{H}}(E_{uv}) = \{u, v\} = E_{uv}$, i.e. $\phi_{\mathcal{H}}$ is the identity. Consequently, $\mathcal{H}$ is asymmetric.

Proof of Theorem 3 Let $k \geq 3$. Let $G_1, G_2, \ldots$ be an infinite family of pairwise distinct k-regular, asymmetric graphs as provided by Theorem 13. Let $\mathcal{H}_i$ be the hypergraph dual of G_i, $i \in \mathbb{N}$. Observe that the $\mathcal{H}_i$'s are pairwise distinct 2-regular k-graphs. Lemma 15 provides that the $\mathcal{H}_i$'s are asymmetric. □

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Declarations

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