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RAINER MANDEL

We prove interpolation inequalities of Gagliardo–Nirenberg type involving Fourier symbols that vanish on hypersurfaces in $\mathbb{R}^d$.

1. Introduction

In a recent paper by Fernández, Jeanjean, Mariş and the author the following inequality of Gagliardo–Nirenberg-type was proved:

$$\|u\|_q \lesssim \|(|D|^s - 1)u\|_2^{1-\kappa} \|u\|_2^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d). \tag{1}$$

Here, $(|D|^s - 1)u = \mathcal{F}^{-1}((|\cdot|^s - 1)\hat{u})$, the symbol $\lesssim$ stands for $\le C$ for some positive number C independent of u and the parameters are supposed to satisfy

$$s > 0, \quad \kappa \ge \frac{1}{2}, \quad 2 \le q < \infty, \quad d \in \mathbb{N}, \quad d \ge 2 \quad \text{and} \quad \frac{2(1-\kappa)}{d+1} \le \frac{1}{2} - \frac{1}{q} \le \frac{(1-\kappa)s}{d}; \tag{2}$$

see [Fernández et al. 2022, Theorem 2.6]. In this paper we investigate such inequalities in greater generality both by extending the analysis to a larger class of exponents, but also by allowing for more general Fourier symbols. We expect applications in the context of normalized solutions of elliptic PDEs and orbital stability [Cazenave and Lions 1982; Bartsch et al. 2016; Noris et al. 2014] or long-time behaviour [Weinstein 1982/1983] of time-dependent PDEs just as in the case of the classical Gagliardo–Nirenberg inequality [Nirenberg 1959]. In [Fernández et al. 2022; Lenzmann and Weth 2024] applications of (1) to variational existence results and symmetry-breaking phenomena for biharmonic nonlinear Schrödinger equations are given. For the existence and qualitative properties of maximizers in classical Gagliardo–Nirenberg inequalities we refer to [Weinstein 1982/1983; Del Pino and Dolbeault 2002; Bellazzini et al. 2014; Lenzmann and Sok 2021; Zhang 2021]. Interpolation inequalities in different spaces like Lorentz spaces, Besov spaces, BMO or weighted Lebesgue spaces can be found in [Brezis et al. 2021; Hajaiej et al. 2011; Brezis and Mironescu 2019; Dao et al. 2022; Caffarelli et al. 1984; McCormick et al. 2013].

We shall be concerned with inequalities of the form

$$\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa}, \tag{3}$$

where $q, r_1, r_2 \in [1, \infty]$, $\kappa \in [0, 1]$ and $P_1, P_2 : \mathbb{R}^d \to \mathbb{R}$ are Fourier symbols that may vanish on a given smooth compact hypersurface $S \subset \mathbb{R}^d$, $d \ge 2$, with at least $k \in \{1, \dots, d-1\}$ nonvanishing principal

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curvatures in each point. In the case $d=1$ the symbols are allowed to have a finite set of zeros $S\subset\mathbb{R}$. We will assume that P_i vanishes of order α_i on S and behaves like $|\cdot|^{s_i}$ at infinity; see assumptions (A1), (A2) below for a precise statement. This covers (1) as a special case, where $d\ge 2$, $(\alpha_1,\alpha_2,s_1,s_2)=(1,0,s,0)$ and S is the unit sphere in $\mathbb{R}^d$, so $k=d-1$. As an application of our results for (3) we obtain the following generalization of [Fernández et al. 2022, Theorem 2.6].

Theorem 1. *Assume $d\in\mathbb{N}$, $d\ge 2$, $\kappa\in[0,1]$, $s>0$. Then*

$$\|u\|_q \lesssim \|(|D|^s-1)u\|_r^{1-\kappa}\,\|u\|_r^{\kappa},\qquad u\in\mathcal{S}(\mathbb{R}^d),$$

holds provided that the exponents $r\in[1,2]$, $q\in[2,\infty]$ satisfy

$$\frac{2(1-\kappa)}{d+1}\le\frac{1}{r}-\frac{1}{q}\le\frac{(1-\kappa)s}{d}\qquad\text{and}\qquad \min\left\{\frac{1}{r},\frac{1}{q'}\right\}\begin{cases}\ge\frac{d+1-2\kappa}{2d} & \text{if } \kappa>0,\\ >\frac{d+1}{2d} & \text{if } \kappa=0.\end{cases}$$

So our result from [Fernández et al. 2022] is recovered, as (2) is nothing but the special case $r=2$ in the above theorem. We even obtain sufficient conditions for general $q,r_1,r_2\in[1,\infty]$. In the one-dimensional case we obtain the following generalization of [Fernández et al. 2022, Theorem 2.3].

Theorem 2. *Assume $\kappa\in[0,1]$, $s>0$. Then*

$$\|u\|_q \lesssim \|(|D|^s-1)u\|_{r_1}^{1-\kappa}\,\|u\|_{r_2}^{\kappa},\qquad u\in\mathcal{S}(\mathbb{R}),$$

holds provided that $q,r_1,r_2\in[1,\infty]$ satisfy

$$1-\kappa\le\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}\le(1-\kappa)s.$$

Both our main results arise as special cases of Theorems 18 and 19 where interpolation inequalities of the form (3) are proved for symbols $P_1,P_2:\mathbb{R}^d\to\mathbb{R}$ that satisfy the following abstract conditions:

(A1) There is a compact hypersurface $S=\{\xi\in\mathbb{R}^d:F(\xi)=0\}$, with $F\in C^\infty(\mathbb{R}^d)$, $|\nabla F|\ne 0$ on S and at least $k\in\{1,\dots,d-1\}$ nonvanishing principal curvatures at each point such that $\{\xi\in\mathbb{R}^d:P_i(\xi)=0\}\subset S$. For ξ near S we have $P_i(\xi)=a_{i+}(\xi)F(\xi)_+^{\alpha_i}+a_{i-}(\xi)F(\xi)_-^{\alpha_i}$ for smooth nonvanishing functions a_{i+},a_{i-} and $\alpha_i>-1$. In the case $\alpha_i=1$, additionally assume $a_{i-}=-a_{i-}$, and in the case $\alpha_i=0$, additionally assume $a_{i-}=a_{i+}$.

(A2) There are $s_1,s_2\in\mathbb{R}$, $\delta>0$ such that for $\operatorname{dist}(\xi,S)\ge\delta>0$ the functions $Q_i(\xi):=\langle\xi\rangle^{s_i}/P_i(\xi)$ satisfy for some $\varepsilon>0$

$$\begin{aligned}|\partial^\gamma Q_i(\xi)|&\lesssim\langle\xi\rangle^{-|\gamma|}&&\text{if } \gamma\in\mathbb{N}_0^d,\ 0\le|\gamma|\le\lfloor d/2\rfloor,\\ |\partial^\gamma Q_i(\xi)|&\lesssim\langle\xi\rangle^{-\varepsilon-|\gamma|}&&\text{if } \gamma\in\mathbb{N}_0^d,\ |\gamma|=\lfloor d/2\rfloor+1.\end{aligned}$$

Here and in the following we set $\langle\xi\rangle:=(1+|\xi|^2)^{1/2}$ and $|\gamma|:=|(\gamma_1,\dots,\gamma_d)|:=\gamma_1+\dots+\gamma_d$ for multi-indices $\gamma\in\mathbb{N}_0^d$, $F(\xi)_+:=\max\{F(\xi),0\}$ and $F(\xi)_-=:-\min\{F(\xi),0\}$. In the case $d=1$ assumption (A1) is supposed to mean $S=\{\xi\in\mathbb{R}:F(\xi)=0\}=\{\xi_1^*,\dots,\xi_L^*\}$, with F,P_i,a_{i+},a_{i-}

as above. Given the importance of the fractional Laplacian $(-\Delta)^{s/2} = |D|^s$ we mention that one may generalize this further by allowing the symbols P_1, P_2 to vanish at some finite set of points in $\mathbb{R}^d \setminus S$; see Remark 10. The choice $P_1 = P_2$ or $\kappa \in \{0, 1\}$ leads to Sobolev inequalities. In the elliptic case $-\Delta - 1 = |D|^2 - 1$ such results are due to Kenig, Ruiz and Sogge [Kenig et al. 1987, Theorem 2.3], Gutiérrez [2004, Theorem 6] and Evequoz [2017]. Our most general result from Theorem 19 contains these results as a special case $(k, s_1, \alpha_1, \kappa) = (d-1, 2, 1, 0)$. Sharp results for special nonelliptic symbols with unbounded characteristic set S are due to Kenig, Ruiz and Sogge [Kenig et al. 1987, Theorem 2.1], Koch and Tataru [2005] and Jeong, Kwon and Lee [Jeong et al. 2016, Theorem 1.1].

Remark 3. (a) In the case $S = \varnothing$ the main results of this paper hold without any assumptions on α_1, α_2. Similarly, if the Fourier support of the given functions is contained in a fixed compact subset of $\mathbb{R}^d$, then all conditions involving s_1, s_2 can be ignored.

(b) Theorems 1 and 2 equally hold for symbols $P_i(|D|)$, where P_i are polynomials of degree s with simple zeros only or no zeros at all.

(c) Our analysis may be extended to vectorial differential operators with constant coefficients $P_1(D)$, $P_2(D)$, where, according to Cramer's rule, the characteristic set S is then supposed to satisfy $\{\det(P_i(\xi)) = 0\} \subset S$ for $i = 1, 2$. Such a situation occurs in the context of Maxwell's equations, Dirac equations or Lamé equations with constant coefficients.

(d) The Gagliardo–Nirenberg inequalities from this paper hold for functions with Fourier support in bounded smooth pieces of more general sets $S \subset \mathbb{R}^d$. In this way, unbounded characteristic sets S or characteristic sets with singularities as in [Mandel and Schippa 2022, Section 3] may be partially analyzed, but a full analysis remains to be done. In the special case of the wave and Schrödinger operator one may nevertheless implement the strategy from [Fernández et al. 2022] to get such inequalities at least for $r = 2$; see Section 7.

(e) The admissible set of exponents for Gagliardo–Nirenberg inequalities may become larger by imposing symmetries. For instance, the Stein–Tomas theorem for $O(d-k) \times O(k)$-symmetric functions from [Mandel and Oliveira e Silva 2023] may substitute the classical Stein–Tomas theorem in Lemma 13 to prove better dyadic estimates. The latter yield larger values for $A_\varepsilon(p, q)$ in (17), which allows one to deduce Gagliardo–Nirenberg inequalities for a wider range of exponents.

Our strategy is as follows. We decompose the pseudodifferential operators $P_1(D)$, $P_2(D)$ dyadically, both for frequencies close to the critical surface S and at infinity. Assumption (A1) allows us to analyze the first-mentioned part with the aid of Bochner–Riesz estimates from [Mandel and Schippa 2022; Cho et al. 2005]. Here, only the parameters α_1, α_2 will play a role. Assumption (A2) will be used to estimate the second-mentioned part that only involves s_1, s_2. Interpolating the bounds for the dyadic operators in both frequency regimes then allows us to conclude. We stress that the proof from [Fernández et al. 2022] does not carry over from the $L^2(\mathbb{R}^d)$-setting since Plancherel's theorem does not have a counterpart in $L^r(\mathbb{R}^d)$ with $r \neq 2$.

2. Preliminaries

In the following we decompose a given Schwartz function $u \in \mathcal{S}(\mathbb{R}^d)$ in frequency space. We start by separating the frequencies close to the critical surface from the others by defining

$$u_1 := \mathcal{F}^{-1}(\tau\hat{u}), \quad u_2 := \mathcal{F}^{-1}((1-\tau)\hat{u}), \quad \text{where } \tau \in C_0^\infty(\mathbb{R}^d),\ \tau = 1 \text{ near } S. \tag{4}$$

More precisely, τ is chosen in such a way that S admits local parametrizations in Euclidean coordinates within $\operatorname{supp}(\tau)$, that a_{i+}, a_{i-} from (A1) are uniformly positive near S and that the functions Q_i from (A2) behave as required for $\xi \in \mathbb{R}^d \setminus \operatorname{supp}(\tau)$. The function τ is considered as fixed from now on. For both u_1 and u_2 we will introduce a dyadic decomposition into infinitely many annular regions in order to prove our estimates mostly via Bourgain's summation argument [1985]. We will need the following abstract version of this result from [Carbery et al. 1999, p. 604].

Lemma 4. *Let $\beta_1, \beta_2 \in \mathbb{R}$, $\theta \in (0,1)$, and let (X_1, X_2) and (Y_1, Y_2) be real interpolation pairs of Banach spaces. For $j \in \mathbb{N}$ let $\mathcal{T}_j$ be linear operators satisfying*

$$\|\mathcal{T}_j f\|_{Y_1} \le M_1\, 2^{\beta_1 j}\|f\|_{X_1}, \quad \|\mathcal{T}_j f\|_{Y_2} \le M_2\, 2^{\beta_2 j}\|f\|_{X_2}.$$

Then we have

$$\Big\|\sum_{j\in\mathbb{N}} \mathcal{T}_j f\Big\|_{(Y_1,Y_2)_{\theta,\infty}} \le C(\beta_1,\beta_2) M_1^{1-\theta} M_2^{\theta}\|f\|_{(X_1,X_2)_{\theta,1}} \tag{5}$$

provided that $(1-\theta)\beta_1 + \theta\beta_2 = 0$, with $\beta_1, \beta_2 \neq 0$. In the case $(1-\theta)\beta_1 + \theta\beta_2 < 0$ we have for all $r \in [1,\infty]$

$$\Big\|\sum_{j\in\mathbb{N}} \mathcal{T}_j f\Big\|_{(Y_1,Y_2)_{\theta,r}} \le C M_1^{1-\theta} M_2^{\theta}\|f\|_{(X_1,X_2)_{\theta,r}}. \tag{6}$$

The whole point of this result is (5); the estimate (6) is a rather trivial consequence of the summability of the interpolated bounds

$$\|\mathcal{T}_j f\|_{(Y_1,Y_2)_{\theta,r}} \lesssim 2^{j((1-\theta)\beta_1+\theta\beta_2)}\|f\|_{(X_1,X_2)_{\theta,r}} \quad \text{for all } r \in [1,\infty].$$

Here, $(Y_1,Y_2)_{\theta,r}, (X_1,X_2)_{\theta,r}$ denote real interpolation spaces [Bergh and Löfström 1976]. The choice $Y_1 = L^{q_1}(\mathbb{R}^d)$, $Y_2 = L^{q_2}(\mathbb{R}^d)$, with

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad q_1 \neq q_2,$$

yields the Lorentz space $(Y_1,Y_2)_{\theta,r} = L^{q,r}(\mathbb{R}^d)$, whereas $q_1 = q_2 = q$ leads to $(Y_1,Y_2)_{\theta,r} = L^q(\mathbb{R}^d)$. In our context, the spaces X_i are defined as the completion of $\{u \in \mathcal{S}(\mathbb{R}^d) : P_i(D)u \in L^r(\mathbb{R}^d)\}$ with respect to the norm $\|u\|_{X_i} := \|P_i(D)u\|_r$. Exploiting assumptions (A1), (A2) we find that for any given $u \in \mathcal{S}(\mathbb{R}^d)$ the function $P_i(D)u$ is a priori well-defined as a function in $L^\infty(\mathbb{R}^d)$ because $\xi \mapsto P_i(\xi)\hat{u}(\xi)$ is integrable due to $\alpha_i > -1$. (Choosing the completion of a smaller set one may extend the analysis to $\alpha_i \le -1$.) The link to Gagliardo–Nirenberg-type inequalities is provided by the general interpolation property [Bergh and Löfström 1976, Theorem 3.1.2], namely

$$\|f\|_{(X_1,X_2)_{\kappa,r}} \le \|f\|_{X_1}^{1-\kappa}\|f\|_{X_2}^{\kappa}, \quad 0<\kappa<1,\ 1\le r\le\infty.$$

In fact, choosing X_1, X_2 as above we obtain for $u \in \mathcal{S}(\mathbb{R}^d)$

$$\|u\|_{(X_1,X_2)_{\kappa,r}} \le \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad 0<\kappa<1, 1\le r\le\infty. \tag{7}$$

The same estimate holds for $(X_1, X_2)_{\kappa,r}$ replaced by the complex interpolation space $[X_1, X_2]_\kappa$. This can be deduced from (7) and $[X_1, X_2]_\kappa \subset (X_1, X_2)_{\kappa,\infty}$; see [Bergh and Löfström 1976, Theorem 4.7.1].

3. Large frequency analysis

We start with our analysis for large frequencies or, more precisely, for those frequencies with uniformly positive distance to the critical surface S given by our assumption (A1). To this end we first choose a function η such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}(\eta) \subset \left[-2,-\tfrac{1}{2}\right]\cup\left[\tfrac{1}{2},2\right], \quad \sum_{j\in\mathbb{Z}} \eta(2^j\,\cdot) = 1 \text{ almost everywhere on } \mathbb{R};$$

see [Bergh and Löfström 1976, Lemma 6.1.7]. For $\xi_0 \in \mathbb{R}^d$ define

$$\begin{aligned} T_j f &:= \mathcal{F}^{-1}(\eta(2^j|\xi-\xi_0|)\hat{f}) = K_j * f, \quad \text{where} \\ K_j(x) &:= \mathcal{F}^{-1}(\eta(2^j|\xi-\xi_0|))(x) = 2^{-jd}\mathcal{F}^{-1}(\eta(|\cdot|))(2^{-j}x)e^{ix\cdot\xi_0}. \end{aligned} \tag{8}$$

Later on, we will choose $\xi_0 \in S$ in order to have $T_j u_2 = 0$ for $j \ge j_0$, where $j_0 \in \mathbb{Z}$ only depends on ξ_0 and τ. Indeed, (4) implies that $\hat{u}_2(\xi) = (1-\tau(\xi))\hat{u}(\xi)$ vanishes for frequencies ξ close to S. As a consequence, only the bounds for $j \searrow -\infty$ will be of importance.

Lemma 5. *Assume $d \in \mathbb{N}$ and let $\eta \in C_0^\infty(\mathbb{R})$, $\xi_0 \in \mathbb{R}^d$. Then we have for $j \in \mathbb{Z}$*

$$\|T_j\|_{p\to q} \lesssim 2^{-jd(\frac{1}{p}-\frac{1}{q})} \quad \textit{for } 1\le p\le q\le\infty.$$

Proof. For all $r \in [1,\infty]$ we have $\|K_j\|_r = 2^{-jd}\|\mathcal{F}^{-1}(\eta(|\cdot|))(2^{-j}\cdot)\|_r \lesssim 2^{-jd/r'}$. Hence, for $1 \le p \le q \le \infty$ and $\frac{1}{r} := 1 + \frac{1}{q} - \frac{1}{p}$ we get from Young's convolution inequality

$$\|T_j f\|_q \lesssim \|K_j\|_r\|f\|_p \lesssim 2^{-j\frac{d}{r'}}\|f\|_p \lesssim 2^{-jd(\frac{1}{p}-\frac{1}{q})}\|f\|_p. \qquad \square$$

In the following, we will need a multiplier theorem in $L^\mu(\mathbb{R}^d)$ for arbitrary $\mu \in [1,\infty]$. The natural candidate — Mikhlin's multiplier theorem [Bergh and Löfström 1976, Theorem 6.1.6] — is only available for $\mu \in (1,\infty)$. In order to avoid tiresome separate discussions we first provide a simple sufficient condition for a given function $m : \mathbb{R}^d \to \mathbb{R}$ to be an L^μ-multiplier for all $\mu \in [1,\infty]$. The following result essentially says that a function m serves our purpose provided that its derivatives grow a bit slower near zero and decay a bit faster near infinity compared to the requirements of Mikhlin's multiplier theorem.

Proposition 6. *Let $d \in \mathbb{N}$, $k := \lfloor d/2 \rfloor + 1$ and $m \in C^k(\mathbb{R}^d \setminus \{0\})$. Then m is an L^μ multiplier for all $\mu \in [1,\infty]$ provided that there is $\varepsilon > 0$ such that*

$$|\partial^\alpha m(\xi)| \lesssim \langle \xi \rangle^{-2\varepsilon} |\xi|^{-k+\varepsilon} \quad \textit{for all } \alpha \in \mathbb{N}_0^d \textit{ such that } |\alpha| = k.$$

Proof. We show that the assumptions imply that $\rho := \mathcal{F}^{-1}m$ is integrable. Once this is shown, the result follows from Young's convolution inequality because of

$$\|\mathcal{F}^{-1}(m\hat{f})\|_\mu = \|\rho * f\|_\mu \le \|\rho\|_1 \|f\|_\mu.$$

We may without loss of generality assume $0 < \varepsilon \le 2k - d$. For all $\alpha \in \mathbb{N}_0^d$, $|\alpha| = k$ we have

$$|\mathcal{F}((-ix)^\alpha \rho)(\xi)| = |\partial^\alpha \hat{\rho}(\xi)| = |\partial^\alpha m(\xi)| \lesssim \langle \xi \rangle^{-2\varepsilon} |\xi|^{-k+\varepsilon}.$$

Hence, $\mathcal{F}(x^\alpha \rho)$ belongs to the space $L^{\sigma_1}(\mathbb{R}^d) \cap L^{\sigma_2}(\mathbb{R}^d)$, where

$$\sigma_1 := \frac{d}{k + \varepsilon/2}, \quad \sigma_2 := \frac{d}{k - \varepsilon/2}.$$

Our choice for ε implies $1 \le \sigma_1 \le \sigma_2 \le 2$, so the Hausdorff–Young inequality gives

$$|x|^k \rho \in L^{\sigma_1'}(\mathbb{R}^d) \cap L^{\sigma_2'}(\mathbb{R}^d).$$

To conclude $\rho \in L^1(\mathbb{R}^d)$ with Hölder's inequality it remains to check

$$|x|^{-k} \in L^{\sigma_1}(\mathbb{R}^d) + L^{\sigma_2}(\mathbb{R}^d).$$

But this follows from $|x|^{-k}\mathbb{1}_{|x|\le 1} \in L^{\sigma_1}(\mathbb{R}^d)$ and $|x|^{-k}\mathbb{1}_{|x|>1} \in L^{\sigma_2}(\mathbb{R}^d)$ due to $k\sigma_1 < d < k\sigma_2$, which finishes the proof. ☐

Next we provide our estimates in the large-frequency regime. To this end we analyze the mapping properties of $\mathcal{T}_j u := T_j(u_2)$, where T_j and $u_2 = \mathcal{F}^{-1}((1-\tau)\hat{u})$ were defined in (8), (4), respectively.

Proposition 7. *Assume $d \in \mathbb{N}$ and (A2) with $s_1, s_2 \in \mathbb{R}$. Then, for $i = 1, 2$,*

$$\|\mathcal{T}_j u\|_q \lesssim 2^{j(s_i - d(\frac{1}{p} - \frac{1}{q}))} \|P_i(D)u\|_p \quad \textit{for } 1 \le p \le q \le \infty,\ j \in \mathbb{Z}.$$

Proof. In order to use Lemma 5 for $\xi_0 \in S$ we set $\eta_i(z) := \eta(z)|z|^{-s_i}$ for $z \in \mathbb{R}$. Then $\eta \in C_0^\infty(\mathbb{R})$, $0 \notin \operatorname{supp}(\eta)$ implies $\eta_i \in C_0^\infty(\mathbb{R})$ for $i = 1, 2$. Moreover, we have for $i = 1, 2$ and $j \in \mathbb{Z}$

$$\begin{aligned}
\mathcal{T}_j u &= \mathcal{F}^{-1}\big(\eta(2^j|\xi - \xi_0|)\hat{u}_2(\xi)\big) \\
&= \mathcal{F}^{-1}\big(\eta_i(2^j|\xi - \xi_0|)\,(2^j|\xi - \xi_0|)^{s_i}\,\hat{u}_2(\xi)\big) \\
&= 2^{js_i}\mathcal{F}^{-1}\big(\eta_i(2^j|\xi - \xi_0|) m_i(\xi) P_i(\xi)\hat{u}(\xi)\big),
\end{aligned}$$

where $m_i(\xi) := (1 - \tau(\xi))|\xi - \xi_0|^{s_i}/P_i(\xi)$. Since τ is smooth and identically 1 near $\xi_0 \in S$, a calculation shows that our assumptions on P_i from (A2) imply that m_i satisfies the assumptions of Proposition 6. In fact, for $|\alpha| = k := \lfloor d/2 \rfloor + 1$ and $Q_i, \varepsilon > 0$ as in assumption (A2),

$$\begin{aligned}|\partial^\alpha m_i(\xi)| &\lesssim \sum_{0\le\gamma\le\alpha} \binom{\alpha}{\gamma} \big|\partial^{\alpha-\gamma}\big((1-\tau(\xi))|\xi-\xi_0|^{s_i}\langle\xi\rangle^{-s_i}\big)\big|\,|\partial^\gamma Q_i(\xi)| \\ &\lesssim 1\cdot|\partial^\alpha Q_i(\xi)| + \sum_{0\le\gamma<\alpha} \langle\xi\rangle^{-|\alpha-\gamma|-1}|\partial^\gamma Q_i(\xi)| \\ &\lesssim \langle\xi\rangle^{-\varepsilon-|\gamma|} + \langle\xi\rangle^{-|\alpha-\gamma|-1}\langle\xi\rangle^{-|\gamma|} \lesssim \langle\xi\rangle^{-\min\{1,\varepsilon\}-|\alpha|}.\end{aligned}$$

Here we used the Leibniz rule. So, by Proposition 6, m_i is an L^μ-multiplier for all $\mu \in [1,\infty]$. Hence, Lemma 5 yields for all $q \in [p,\infty]$

$$\begin{aligned}\|\mathcal{T}_j u\|_q &\lesssim 2^{js_i}\|\mathcal{F}^{-1}(\eta_i(2^j|\xi-\xi_0|)m_i(\xi)\widehat{P_i(D)u}(\xi))\|_q \\ &\lesssim 2^{j(s_i-d(\frac{1}{p}-\frac{1}{q}))}\|\mathcal{F}^{-1}(m_i(\xi)\widehat{P_i(D)u}(\xi))\|_p \\ &\lesssim 2^{j(s_i-d(\frac{1}{p}-\frac{1}{q}))}\|P_i(D)u\|_p. \qquad \square\end{aligned}$$

Next we use these dyadic estimates to prove estimates of Gagliardo–Nirenberg type. We deduce our results from a detailed analysis of the special case $P_i(D) = \langle D\rangle^{s_i}$ for $s_1, s_2 \in \mathbb{R}$. This is possible due to

$$\|\langle D\rangle^{s_i}u_2\|_p \lesssim \|P_i(D)u\|_p, \quad 1 \le p \le \infty, \tag{9}$$

for symbols P_1, P_2 as in (A2) thanks to Proposition 6. So we collect some mapping properties of the Bessel potential operators $\langle D\rangle^{-s}$, where $s > 0$.

Proposition 8. *Assume $d \in \mathbb{N}$, $s > 0$ and $p, q, r \in [1,\infty]$, $u \in \mathcal{S}(\mathbb{R}^d)$.*

(i) *If $0 \le \frac{1}{p} - \frac{1}{q} < \frac{s}{d}$ then $\|u\|_q \lesssim \|\langle D\rangle^s u\|_p$.*
(ii) *If $0 \le \frac{1}{p} - \frac{1}{q} = \frac{s}{d}$ and $1 < p, q < \infty$ then $\|u\|_{q,r} \lesssim \|\langle D\rangle^s u\|_{p,r}$ and $\|u\|_q \lesssim \|\langle D\rangle^s u\|_p$.*
(iii) *If $0 \le \frac{1}{p} - \frac{1}{q} = \frac{s}{d}$ and $s = d = 1$ then $\|u\|_\infty \lesssim \|\langle D\rangle u\|_1$.*
(iv) *If $0 \le \frac{1}{p} - \frac{1}{q} = \frac{s}{d}$ and $1 = p < q < \infty$ then $\|u\|_{q,\infty} \lesssim \|\langle D\rangle^s u\|_1$.*

Proof. The parts (i), (iv) and the second part of (ii) are given in [Grafakos 2014, Corollary 1.2.6]; the Lorentz space mapping properties from (ii) follow from real interpolation. The estimate (iii) follows from

$$\|u\|_\infty \lesssim \|u'\|_1 = \|m(D)(\langle D\rangle u)\|_1 \lesssim \|\langle D\rangle u\|_1, \quad u \in \mathcal{S}(\mathbb{R}).$$

Here we used that $m(\xi) := \xi(1+|\xi|^2)^{-1/2}$ satisfies the assumptions of Proposition 6. $\square$

We finally use these estimates to prove Gagliardo–Nirenberg inequalities for large frequencies.

Proposition 9. *Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A2) for $s_1, s_2 \in \mathbb{R}$. Then*

$$\|u_2\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{10}$$

holds provided that the exponents $q, r_1, r_2 \in [1,\infty]$ satisfy $0 \le \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \le \frac{\bar{s}}{d}$, as well as the following conditions in the endpoint case $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d}$:

(i) *If* $q=\infty$ *then* $\frac{1}{r_1}-\frac{s_1}{d}\neq 0\neq \frac{1}{r_2}-\frac{s_2}{d}$ *or* $r_1=r_2=\infty,\ s_1=s_2=0$ *or* $d=1, (r_1,r_2)=(\frac{1}{s_1},\frac{1}{s_2})$, $s_1,s_2\in\{0,1\}$.

(ii) *If* $1<q<\infty$ *and* $\frac{1}{r_1}-\frac{s_1}{d}=\frac{1}{q}=\frac{1}{r_2}-\frac{s_2}{d}$ *and additionally, if* $r_1=1,\kappa<1$, *then* $1<r_2<q,\ \kappa\geq\frac{r_2}{q}$ *or* $r_2=\infty,\ \frac{1}{q}\leq\kappa\leq\frac{1}{q'}$.

(iii) *If* $1<q<\infty$ *and* $\frac{1}{r_1}-\frac{s_1}{d}=\frac{1}{q}=\frac{1}{r_2}-\frac{s_2}{d}$ *and additionally, if* $r_2=1,\ \kappa>0$, *then* $1<r_1<q$, $1-\kappa\geq\frac{r_1}{q}$ *or* $r_1=\infty,\ \frac{1}{q}\leq 1-\kappa\leq\frac{1}{q'}$.

Proof. As mentioned before, it is sufficient to prove the estimates in the prototypical case $P_i(D)=\langle D\rangle^{s_i}$. The case $\kappa\in\{0,1\}$ is covered by Proposition 8(i), (ii), (iii). So we may concentrate on $\kappa\in(0,1)$ in the following. We combine Proposition 7 and Lemma 4 for the Bessel potential spaces $X_i := P_i(D)^{-1}L^{r_i}(\mathbb{R}^d)=\langle D\rangle^{-s_i}L^{r_i}(\mathbb{R}^d)$ and $i=1,2$. Here we use the identity

$$u_2=\sum_{j=-\infty}^{j_0}\mathcal{T}_j u,\quad \text{where } \|\mathcal{T}_j u\|_{q_i}\lesssim 2^{j(s_i-d(\frac{1}{r_i}-\frac{1}{q_i}))}\|u\|_{X_i}\quad (j\in\mathbb{Z},\ 1\leq r_i\leq q_i\leq\infty);$$

see Proposition 7. Our strategy is as follows. We first prove apply Lemma 4 to get strong bounds. This will cover all nonendpoint cases $0\leq\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}<\frac{\bar{s}}{d}$, as well as the endpoint cases involving $q\in\{1,\infty\}$. The remaining discussion for $1<q<\infty$ and $1<r_1,r_2<\infty$ can be taken from the literature, but the analysis for $\{r_1,r_2\}\cap\{1,\infty\}\neq\emptyset$ is more delicate. We will first address the case $\frac{1}{r_1}-\frac{1}{r_2}=\frac{s_1-s_2}{d}$, where we prove our claim using complex and real interpolation theory. Finally, in the case $\frac{1}{r_1}-\frac{1}{r_2}\neq\frac{s_1-s_2}{d}$ we will first deduce restricted weak-type bounds from Lemma 4 and upgrade them to strong bounds by interpolating the restricted weak-type bounds with each other. We will need in the following that our assumptions imply $\bar{s}\geq 0$.

<u>Step 1</u>: We start the interpolation procedure with (nonendpoint) exponents satisfying

$$0\leq\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}<\frac{\bar{s}}{d}. \tag{11}$$

In that case the interpolation estimate (6) with $(Y_1,Y_2,\theta,r):=(L^{q_1}(\mathbb{R}^d),L^{q_2}(\mathbb{R}^d),\kappa,q)$ gives the bound

$$\|u_2\|_q=\left\|\sum_{j=-\infty}^{j_0}\mathcal{T}_j u\right\|_q\overset{(6)}{\lesssim}\|u\|_{(X_1,X_2)_{\kappa,q}}\overset{(7)}{\lesssim}\|\langle D\rangle^{s_1}u\|_{r_1}^{1-\kappa}\|\langle D\rangle^{s_2}u\|_{r_2}^{\kappa}.$$

Here, (6) applies because (11) allows us to find $q_i\in[r_i,\infty]$ such that

$$(1-\kappa)\left(s_1-d\left(\frac{1}{r_1}-\frac{1}{q_1}\right)\right)+\kappa\left(s_2-d\left(\frac{1}{r_2}-\frac{1}{q_2}\right)\right)>0,\quad \frac{1}{q}=\frac{1-\kappa}{q_1}+\frac{\kappa}{q_2}.$$

So the claim is proved for all nonendpoint exponents given by (11).

It remains to discuss the endpoint case $0\leq\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\frac{\bar{s}}{d}$. Using (5) for $Y_1=Y_2=L^q(\mathbb{R}^d)$ we get the claim for all exponents satisfying

$$0\leq\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\frac{\bar{s}}{d}\quad\text{and}\quad q\geq\max\{r_1,r_2\},\ \frac{1}{r_1}-\frac{s_1}{d}\neq\frac{1}{q}\neq\frac{1}{r_2}-\frac{s_2}{d}. \tag{12}$$

Here the latter two inequalities correspond to $\beta_1, \beta_2 \neq 0$ in Lemma 4. From this we infer that the claimed endpoint estimates hold for $q \in \{1, \infty\}$ via the following cases:

- Case $q = 1$: $r_1 = r_2 = 1,\ s_1 = s_2 = 0$ is trivial.
- Case $q = 1$: $r_1 = r_2 = 1,\ \bar{s} = 0,\ s_1 \neq 0 \neq s_2$ is covered by (12).
- Case $q = \infty$: $r_1 = r_2 = \infty,\ s_1 = s_2 = 0$ is trivial.
- Case $q = \infty$: $\frac{1}{r_1} - \frac{s_1}{d} \neq 0 \neq \frac{1}{r_2} - \frac{s_2}{d}$ is covered by (12).
- Case $q = \infty$: $(d, r_1, r_2) = \left(1, \frac{1}{s_1}, \frac{1}{s_2}\right),\ s_1, s_2 \in \{0, 1\}$ is covered by Proposition 8(iii).

These are all cases involving $q \in \{1, \infty\}$ and in particular claim (i) is proved. So we are left with those endpoint estimates for $1 < q < \infty$ that are not covered by (12).

Step 2: The claim holds for $1 < r_1, r_2 < \infty$ due to

$$\|u\|_q \lesssim \|\langle D\rangle^{\bar{s}} u\|_{\bar{r}} \lesssim \|\langle D\rangle^{s_1} u\|_{r_1}^{1-\kappa} \|\langle D\rangle^{s_2} u\|_{r_2}^{\kappa},$$

where $\frac{1}{\bar{r}} := \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2}$. This is a consequence of Sobolev's embedding theorem [Bergh and Löfström 1976, Theorem 6.5.1] and the complex interpolation result from [loc. cit., Theorem 6.4.5(7)]. So we may in the following assume $\{r_1, r_2\} \cap \{1, \infty\} \neq \varnothing$. As announced earlier, we first deal with $\frac{1}{r_1} - \frac{1}{r_2} = \frac{s_1 - s_2}{d}$.

Step 3: Assume we are in the endpoint case with $1 < q < \infty$, $\frac{1}{r_1} - \frac{1}{r_2} = \frac{s_1 - s_2}{d}$, $r_1 \leq r_2$ (without loss of generality) and $\{r_1, r_2\} \cap \{1, \infty\} \neq \varnothing$. Then $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d}$ implies $\frac{1}{r_1} - \frac{s_1}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{s_2}{d}$. We distinguish the following cases:

- Case $r_1 = 1,\ r_2 = 1$: This case is excluded, so there is nothing to prove.
- Case $r_1 = 1,\ 1 < r_2 < q$: By Proposition 8(ii), (iv), we have $\|u\|_{q,\infty} \lesssim \|\langle D\rangle^{s_1} u\|_1$, as well as $\|u\|_{q,r_2} \lesssim \|\langle D\rangle^{s_2} u\|_{r_2}$. Applying the interpolation identity [loc. cit., Theorem 5.3.1]

$$L^q(\mathbb{R}^d) = (L^{q,\infty}(\mathbb{R}^d), L^{q,\kappa q}(\mathbb{R}^d))_{\kappa,q}, \quad \kappa \in (0, 1], \tag{13}$$

we infer for all $\kappa \in \left[\frac{r_2}{q}, 1\right]$

$$\|u\|_q \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,\kappa q}^{\kappa} \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,r_2}^{\kappa} \lesssim \|\langle D\rangle^{s_1} u\|_1^{1-\kappa} \|\langle D\rangle^{s_2} u\|_{r_2}^{\kappa}.$$

- Case $r_1 = 1, r_2 = \infty$: We have to prove (10) for $\frac{1}{q} \leq \kappa \leq \frac{1}{q'}$. It is sufficient to prove the claim first for $\kappa = \frac{1}{q}$ and then for $\kappa = \frac{1}{q'}$. We use $\|u\|_{q,\infty} \lesssim \|\langle D\rangle^{s_1} u\|_1$ and

$$\|u\|_{q,2}^2 \lesssim \|\langle D\rangle^{\frac{d}{2} - \frac{d}{q}} u\|_2^2 = \int_{\mathbb{R}^d} \langle D\rangle^{\frac{d}{q'}} u \cdot \langle D\rangle^{-\frac{d}{q}} u \, dx \leq \|\langle D\rangle^{s_1} u\|_1 \|\langle D\rangle^{s_2} u\|_\infty. \tag{14}$$

In (14) we subsequently used Proposition 8(ii) and the L^2-isometry property of the Fourier transform, as well as $s_1 = \frac{d}{q'}$, $s_2 = -\frac{d}{q}$. Real interpolation of these two estimates and $L^q(\mathbb{R}^d) = (L^{q,\infty}(\mathbb{R}^d), L^{q,2}(\mathbb{R}^d))_{2/q,q}$, which is (13) for $\kappa = \frac{2}{q}$, gives

$$\|u\|_q \lesssim \|u\|_{q,\infty}^{1-\frac{2}{q}} \|u\|_{q,2}^{\frac{2}{q}} \lesssim \|\langle D\rangle^{s_1} u\|_1^{\frac{1}{q'}} \|\langle D\rangle^{s_2} u\|_\infty^{\frac{1}{q}}. \tag{15}$$

So the claim holds for $\kappa = \frac{1}{q}$ and we now consider $\kappa = \frac{1}{q'}$. Here we use Stein's interpolation theorem [1956] in a more general setting [Voigt 1992, Theorem 2.1] for the family of linear operators $\mathcal{T}^s u := e^{s^2}\langle D\rangle^{s/2-d/q}u$, with $s \in \mathbb{C},\ 0 \le \mathrm{Re}(s) \le 1$. We have

$$\|\mathcal{T}^{it}u\|_{\mathrm{BMO}(\mathbb{R}^d)} = e^{-t^2}\|\langle D\rangle^{it}(\langle D\rangle^{-\frac{d}{q}}u)\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|\langle D\rangle^{-\frac{d}{q}}u\|_\infty,$$
$$\|\mathcal{T}^{1+it}u\|_2 = e^{1-t^2}\|\langle D\rangle^{\frac{d}{2}-\frac{d}{q}}u\|_2 \overset{(14)}{\lesssim} \|\langle D\rangle^{\frac{d}{q'}}u\|_1^{\frac{1}{2}}\|\langle D\rangle^{-\frac{d}{q}}u\|_\infty^{\frac{1}{2}}.$$

Here we used the validity of Mikhlin's multiplier theorem in $\mathrm{BMO}(\mathbb{R}^d)$ to deduce that the operator norm $\langle D\rangle^{it} : L^\infty(\mathbb{R}^d) \to \mathrm{BMO}(\mathbb{R}^d)$ is polynomially bounded with respect to t and thus compensated for by the mitigating factor e^{-t^2} as $|t| \to \infty$. We refer to Proposition 3.4, Theorem 4.4 and the comments on pages 20-21 in Tao's lecture notes [2018], where such an application in the context of Stein's interpolation theorem is explicitly mentioned. In view of $[\mathrm{BMO}(\mathbb{R}^d), L^2(\mathbb{R}^d)]_\theta = L^{2/\theta}(\mathbb{R}^d)$ for $0 < \theta \le 1$ we may plug in $\theta = \frac{2}{q}$ and get in view of $s_1 = \frac{d}{q'}, s_2 = -\frac{d}{q}$

$$\|u\|_q = \|\mathcal{T}^{\frac{2}{q}}u\|_q \lesssim \|\langle D\rangle^{-\frac{d}{q}}u\|_\infty^{1-\theta}(\|\langle D\rangle^{\frac{d}{q'}}u\|_1^{\frac{1}{2}}\|\langle D\rangle^{-\frac{d}{q}}u\|_\infty^{\frac{1}{2}})^\theta = \|\langle D\rangle^{s_1}u\|_1^{\frac{1}{q}}\|\langle D\rangle^{s_2}u\|_\infty^{\frac{1}{q'}}.$$

- Case $1 < r_1 < r_2 = \infty$: We have to prove (10) for $1 < q < r_1,\ \kappa \ge \frac{r_1}{q}$. We consider $\mathcal{T}^s u := e^{s^2}\langle D\rangle^{s_2+s(s_1-s_2)}u$ and obtain as before

$$\|\mathcal{T}^{it}u\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|\langle D\rangle^{s_2}u\|_\infty, \quad \|\mathcal{T}^{1+it}u\|_{r_1} \lesssim \|\langle D\rangle^{s_1}u\|_{r_1}.$$

So we conclude for $\kappa := \frac{r_1}{q} = \frac{s_2}{s_2-s_1}$

$$\|u\|_q = \|\mathcal{T}^\kappa u\|_{\frac{r_1}{\kappa}} \lesssim \|\langle D\rangle^{s_2}u\|_\infty^{1-\kappa}\|\langle D\rangle^{s_1}u\|_{r_1}^\kappa.$$

This proves the claim for $\kappa = \frac{r_1}{q}$. Since the desired bound for $\kappa = 1$ follows from Proposition 8(ii), we get the claim for $\kappa \in \left[\frac{r_1}{q}, 1\right]$.

- Case $1 < r_1 = r_2 = \infty$: This case does not occur because $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = -\frac{1}{q} < 0$.

Step 4: To prove the remaining estimates we first prove restricted weak-type estimates $\|u_2\|_{q,\infty} \lesssim \|u\|_{(X_1,X_2)_{\kappa,1}}$ for all exponents satisfying

$$0 \le \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d} \quad \text{and} \quad 1 < q < \infty \quad \text{and} \quad \frac{1}{r_1} - \frac{1}{r_2} \ne \frac{s_1 - s_2}{d}. \tag{16}$$

For $s_1 = s_2 = 0$ this is implied by Hölder's inequality, so we may assume $\bar{s} > 0$ or $\bar{s} = 0,\ (s_1, s_2) \ne (0,0)$. For $\bar{s} = 0,\ (s_1, s_2) \ne (0,0),\ q = r_1 = r_2$ this is implied by the strong estimates in the case (12), so we may even assume $\bar{s} > 0$ or $\bar{s} = 0,\ (s_1, s_2) \ne (0,0),\ (r_1, r_2) \ne (q,q)$. For the remaining exponents the weak estimate is a consequence of (6) because one can find $q_i \in [r_i, \infty]$ such that

$$(1-\kappa)\left(s_1 - d\left(\frac{1}{r_1} - \frac{1}{q_1}\right)\right) + \kappa\left(s_2 - d\left(\frac{1}{r_2} - \frac{1}{q_2}\right)\right) = 0,$$
$$\frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2}, \quad s_i - d\left(\frac{1}{r_i} - \frac{1}{q_i}\right) \ne 0, \quad q_1 \ne q_2.$$

Indeed, this condition is equivalent to $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\frac{\bar{s}}{d}$ and finding q_2 such that

$$\frac{1}{q}-\frac{1-\kappa}{r_1}\le\frac{\kappa}{q_2}\le\frac{\kappa}{r_2},\quad q_2\ne q,\quad \frac{1}{q}-(1-\kappa)\left(\frac{1}{r_1}-\frac{s_1}{d}\right)\ne\frac{\kappa}{q_2}\ne\kappa\left(\frac{1}{r_2}-\frac{s_2}{d}\right),$$

and such a choice is possible due to our assumptions. (In the case $\bar{s}=0$, $(s_1,s_2)\ne(0,0)$, $(r_1,r_2)\ne(q,q)$ choose $q_2=r_2$, $q_1=r_1$.) In this way we obtain $\|u_2\|_{q,\infty}\lesssim\|u\|_{(X_1,X_2)_{\kappa,1}}$ for all exponents satisfying (16). We finally interpolate these restricted weak-type estimates with each other to prove strong estimates for exponents as in (16). To this end let $\delta>0$ be sufficiently small (but fixed) and $\varepsilon:=\delta\big(\frac{s_1-s_2}{d}-\frac{1}{r_1}+\frac{1}{r_2}\big)\ne 0$ and define $\tilde{q},q^*,\tilde{\kappa},\kappa^*$ via $\frac{1}{\tilde{q}}-\varepsilon=\frac{1}{q}=\frac{1}{q^*}+\varepsilon$ and $\tilde{\kappa}-\delta=\kappa=\kappa^*+\delta$. Then $(\tilde{q},r_1,r_2,\tilde{\kappa})$, (q^*,r_1,r_2,κ^*) satisfies (16) and the reiteration property of real interpolation [Bergh and Löfström 1976, Theorem 3.5.3] gives

$$\begin{aligned}\|u_1\|_q&\lesssim\|u_1\|_{(L^{q^*}(\mathbb{R}^d),L^{\tilde{q}}(\mathbb{R}^d))_{\frac12,q}}\\&\lesssim\|u\|_{((X_1,X_2)_{\kappa^*,1},(X_1,X_2)_{\tilde{\kappa},1})_{\frac12,q}}\\&\lesssim\|u\|_{(X_1,X_2)_{\kappa,q}}\overset{(7)}{\lesssim}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.\end{aligned}$$

Here the first bound uses $\frac{1}{q}=\frac{1}{2}\big(\frac{1}{q^*}+\frac{1}{\tilde{q}}\big)$ and the third uses $\kappa=\frac{1}{2}(\tilde{\kappa}+\kappa^*)$. □

We have thus proved that the Gagliardo–Nirenberg inequality (3) holds for noncritical frequencies whenever the exponents belong to the set

$$\mathcal{B}(\kappa):=\{(q,r_1,r_2)\in[1,\infty]^3:\ (q,r_1,r_2)\text{ as in Proposition 9}\}.$$

Remark 10. (a) The original Gagliardo–Nirenberg inequality $\|\nabla^j v\|_q\lesssim\|\nabla^m v\|_{r_1}^{1-\kappa}\|v\|_{r_2}^{\kappa}$ from [Nirenberg 1959, p. 125] holds for $j,m\in\mathbb{N}$ provided that $\frac{1}{q}-\frac{j}{d}=(1-\kappa)\big(\frac{1}{r_1}-\frac{m}{d}\big)+\frac{\kappa}{r_2}$ and $\frac{j}{m}\le 1-\kappa<1$. Our result shows that "in most cases" the large-frequency part of this estimate holds provided that $\frac{j}{m}\le 1-\kappa<1$ holds and $\frac{1}{q}-\frac{j}{d}\ge(1-\kappa)\big(\frac{1}{r_1}-\frac{m}{d}\big)+\frac{\kappa}{r_2}$. The exceptions are due to the fact that, in $L^1(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$, the term $\langle D\rangle^j u$ does not control $D^j u$, i.e., not every single partial derivative of order j. This is a consequence of the unboundedness of the Riesz transform on these spaces.

(b) Our proof indicates which function spaces to choose in order to get some endpoint estimates in the exceptional cases as well. Roughly speaking, one may replace $L^q(\mathbb{R}^d)$ by $L^{q,r}(\mathbb{R}^d)$ for suitable $r>q$ and $L^\infty(\mathbb{R}^d)$ by $\mathrm{BMO}(\mathbb{R}^d)$ on the left-hand side. On the right-hand side the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$ may replace $L^1(\mathbb{R}^d)$.

(c) One may as well consider symbols $P_i(D)$ that vanish at some finite set of points in $\mathbb{R}^d\setminus S$. If for instance one has $P_i(\xi)=b_i(\xi)|\xi-\xi^*|^{t_i}$ near $\xi^*\in\mathbb{R}^d\setminus S$ for $t_1,t_2>-d$ and nonvanishing $b_i\in C^\infty(\mathbb{R}^d)$, then one finds as in Proposition 9 that the interpolation estimate holds in this frequency regime whenever $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}>\frac{\bar{t}}{d}$, where $\bar{t}:=(1-\kappa)t_1+\kappa t_2$. Under suitable extra conditions similar to the ones above, this may be extended to the endpoint case $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\frac{\bar{t}}{d}$.

(d) The proof in the important special case $1<r_1,r_2,q<\infty$ is much shorter than the complete analysis; see the beginning of Step 2.

4. Critical frequency analysis

We introduce a real number $A_\varepsilon(p,q)$ such that $\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-jA_\varepsilon(p,q)}$ holds for suitably defined dyadic operators $\widetilde{T}_j$ that play the role of the T_j in the previous section. Unfortunately, the definition of $A_\varepsilon(p,q)$ is rather complicated for $d \geq 2$. It involves the number

$$A(p,q) := \min\{A_0, A_1, A_2, A_2', A_3, A_3', A_4, A_4'\},$$

where $A_i = A_i(p,q)$ and $A_i' = A_i(q',p')$ are given by

$$A_0 = 1, \quad A_1 = \frac{k+2}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \quad A_2 = \frac{k+2}{2} - \frac{k+1}{q},$$

as well as

$$A_3 = \frac{2d-k}{2} - \frac{2d-k-1}{q}, \quad A_4 = \frac{k+2}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{2d-k-2}{2} - \frac{2d-k-2}{q}.$$

The values $A_0, A_1, A_1', A_2, A_2'$ will be important for $1 \leq p \leq 2 \leq q \leq \infty$, whereas all other exponents satisfying $1 \leq p \leq q \leq \infty$ come with A_3, A_3', A_4, A_4'. Then we define for $\varepsilon > 0$

$$A_\varepsilon(p,q) := \frac{1}{p} - \frac{1}{q} \quad \text{if } d = 1, \qquad A_\varepsilon(p,q) := A(p,q) - \varepsilon \cdot \mathbb{1}_{(p,q)\in\mathcal{E}} \quad \text{if } d \geq 2. \tag{17}$$

Here, $\mathcal{E}$ denotes a set of exceptional points where we do not have strong bounds, but only weak bounds or restricted weak-type bounds. It is given by

$$\mathcal{E} := \left\{(p,q) \in [1,\infty]^2 : \frac{1}{p} = \frac{k+2}{2(k+1)}, \ \frac{1}{q} \leq \frac{k^2}{2(k+1)(k+2)} \quad \text{or } \frac{1}{q} = \frac{k}{2(k+1)}, \ \frac{1}{p} \geq \frac{k^2+6k+4}{2(k+1)(k+2)}\right\}$$

and coincides with the red points in Figure 1.

We first prove dyadic estimates in the frequency regime close to the critical surface S. The latter can be locally parametrized as a graph $\xi_d = \psi(\xi')$ after some permutation of coordinates, where $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \simeq \mathbb{R}^d$. In view of (A1) we study operators of the form

$$\begin{aligned} \widetilde{T}_j f &:= \mathcal{F}^{-1}\big(\eta(2^j(\xi_d - \psi(\xi')))\chi(\xi')\hat{f}(\xi)\big) = \widetilde{K}_j * f, \quad \text{where} \\ \widetilde{K}_j &:= \mathcal{F}^{-1}\big(\eta(2^j(\xi_d - \psi(\xi')))\chi(\xi')\big) \end{aligned} \tag{18}$$

and

$$\psi \in C^\infty(\mathbb{R}^{d-1}), \ \chi \in C_0^\infty(\mathbb{R}^{d-1}) \text{ and at least } k \in \{1,\dots,d-1\} \text{ eigenvalues of the Hessian } D^2\psi \text{ are nonzero on } \operatorname{supp}(\chi). \tag{19}$$

In the degenerate case $d = 1$ we interpret $\eta(2^j(\xi_d - \psi(\xi')))\chi(\xi')$ as $\eta(2^j(\xi - c))$ for some constant $c \in \mathbb{R}$. Our analysis of the mapping properties of $\widetilde{T}_j$ follows [Mandel and Schippa 2022, Section 4]. Contrary to the situation for T_j, only the bounds for $j \nearrow +\infty$ will be of importance. Repeating the proof of Lemma 5 gives the following result in the one-dimensional case.

Lemma 11. *Assume $d = 1$ and $\eta \in C_0^\infty(\mathbb{R})$. Then we have*

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-j(\frac{1}{p}-\frac{1}{q})} \quad \text{for } 1 \leq p \leq q \leq \infty, \ j \in \mathbb{Z}.$$

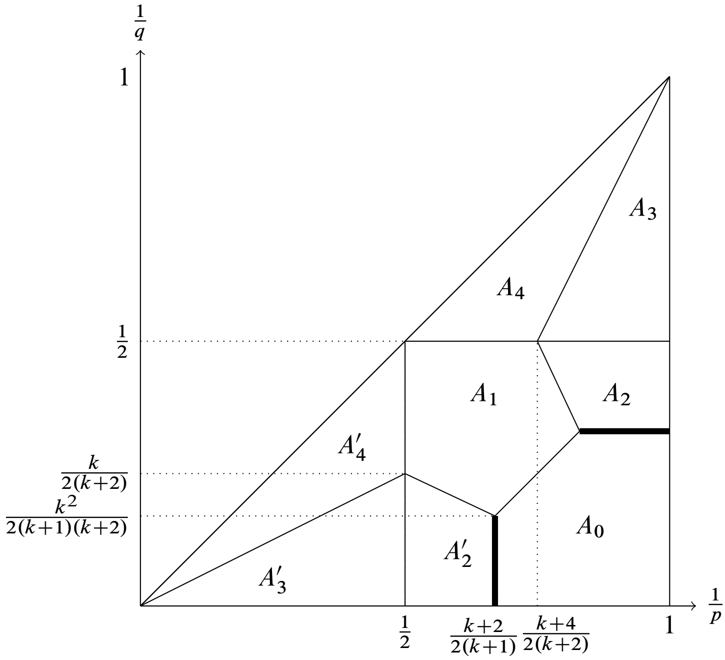

Figure 1. Riesz diagram with the bounds for the mapping constant of $\widetilde{T}_j$ from Lemma 13. The exceptional points from $\mathcal{E}$ are in bold.

The bounds in higher dimensions are more complicated and depend on the number $k \in \{1, \dots, d-1\}$ of nonvanishing principal curvatures of S. We first analyze the kernel function $\widetilde{K}_j$ following [Mandel and Schippa 2022, Lemma 4.4].

Proposition 12. *Assume $d \in \mathbb{N}$, $d \geq 2$, let χ, ψ, k be as in (19) and $\eta \in C_0^\infty(\mathbb{R})$. Then the kernel function $\widetilde{K}_j$ satisfies for $j \in \mathbb{Z}$, $j \geq j_0$*

$$\|\widetilde{K}_j\|_r \lesssim 2^{-j\left(\frac{2d-k}{2} - \frac{2d-k-1}{r}\right)} \quad \text{if } 1 \leq r \leq 2, \qquad \|\widetilde{K}_j\|_\infty \lesssim 2^{-j}. \tag{20}$$

Proof. The bound $\|\widetilde{K}_j\|_2 \lesssim 2^{-j/2}$ follows from Plancherel's identity and (18). Indeed,

$$\begin{aligned}\|\widetilde{K}_j\|_2^2 &= \int_{\mathbb{R}^d} \eta(2^j(\xi_d - \psi(\xi')))^2 \chi(\xi')^2 \, d(\xi', \xi_d) \\ &= \int_{\mathbb{R}^{d-1}} \chi(\xi')^2 \left(\int_{\mathbb{R}} \eta(2^j t)^2 \, dt \right) d\xi' \\ &= 2^{-j} \|\chi\|_2^2 \|\eta\|_2^2.\end{aligned}$$

To prove (20) it thus suffices to show $\|\widetilde{K}_j\|_1 \lesssim 2^{-j((k+2)/2-d)}$, as well as $\|\widetilde{K}_j\|_\infty \lesssim 2^{-j}$, and to apply the Riesz–Thorin interpolation theorem. These two norm bounds for the kernel function are consequences of the pointwise bounds for arbitrary $N, M \in \mathbb{N}_0$

$$\begin{aligned} |\widetilde{K}_j(x)| &\lesssim_{N,M} 2^{-j}(1 + 2^{-j}|x_d|)^{-M}(1 + |x'|)^{-N} \quad &&\text{if } |x'| \geq c|x_d|, \\ |\widetilde{K}_j(x)| &\lesssim_M 2^{-j}(1 + 2^{-j}|x_d|)^{-M}(1 + |x_d|)^{-\frac{k}{2}} \quad &&\text{if } |x'| \leq c|x_d|, \end{aligned} \tag{21}$$

where $c>0$ is suitably chosen. Indeed, choosing M,N sufficiently large we get

$$\begin{aligned}\|\widetilde{K}_j\|_1 &\lesssim_{N,M} \int_{\mathbb{R}}\biggl(\int_{|x'|\le cx_d} 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x_d|)^{-\frac{k}{2}}\,dx'\biggr)\,dx_d \\ &\qquad\qquad + \int_{\mathbb{R}}\biggl(\int_{|x'|\ge cx_d} 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x'|)^{-N}\,dx'\biggr)\,dx_d \\ &\lesssim_{M,N} 2^{-j}\int_{\mathbb{R}}(1+2^{-j}|x_d|)^{-M}|x_d|^{d-1}(1+|x_d|)^{-\frac{k}{2}}\,dx_d \\ &\qquad\qquad + 2^{-j}\int_{\mathbb{R}}(1+2^{-j}|x_d|)^{-M}(1+|x_d|)^{d-N}\,dx_d \\ &\lesssim_{M,N} 2^{-j}\int_0^{2^j}|x_d|^{d-1}(1+|x_d|)^{-\frac{k}{2}}\,dx_d + 2^{(M-1)j}\int_{2^j}^{\infty}|x_d|^{d-\frac{k}{2}-1-M}\,dx_d \\ &\lesssim_{M,N} 2^{-j(\frac{k+2}{2}-d)}.\end{aligned}$$

Here we used $2^j\ge 2^{j_0}>0$. So it remains to prove the pointwise bounds by adapting the arguments from [Mandel and Schippa 2022]. We have

$$\widetilde{K}_j(x) = c_d\, 2^{-j}(\mathcal{F}^{-1}\eta)(2^{-j}x_d)\int_{\mathbb{R}^{d-1}} e^{i(x'\cdot\xi'+x_d\psi(\xi'))}\chi(\xi')\,d\xi'$$

for some dimensional constant $c_d>0$. We choose $c>0$ so large that the smooth phase function $\Phi(\xi')=x'\cdot\xi'+x_d\psi(\xi')$ satisfies $|\nabla\Phi(\xi')|\ge c^{-1}|x'|$ for all $\xi'\in\mathbb{R}^{d-1}$ whenever $|x'|\ge c|x_d|$. In view of $\chi\in C_0^\infty(\mathbb{R}^{d-1})$ the method of nonstationary phase gives

$$\begin{aligned}|\widetilde{K}_j(x)| &\lesssim_N 2^{-j}|(\mathcal{F}^{-1}\eta)(2^{-j}x_d)|(1+|x'|)^{-N} \\ &\lesssim_{N,M} 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x'|)^{-N} \quad \text{for } |x'|\ge c|x_d|.\end{aligned}$$

In the second estimate we used that $\mathcal{F}^{-1}\eta$ is a Schwartz function. On the other hand, the theory of oscillatory integrals gives (see [Stein 1993, p. 361])

$$|\widetilde{K}_j(x)| \lesssim_M 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x_d|)^{-\frac{k}{2}} \quad \text{for } |x'|\le c|x_d|. \qquad \square$$

Next we use Proposition 12 to find upper bounds for the operator norms of $\widetilde{T}_j$ as maps from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, where $1\le p\le q\le\infty$. The latter condition is mandatory since $\widetilde{T}_j$ is a translation-invariant operator covered by Hörmander's result from [Hörmander 1960, Theorem 1.1].

Lemma 13. *Assume $d\in\mathbb{N}$, $d\ge 2$ and let χ, ψ, k be as in (19) and $\eta\in C_0^\infty(\mathbb{R})$. Then, for any fixed $\varepsilon>0$,*

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-jA_\varepsilon(p,q)} \quad \textit{for } 1\le p\le q\le\infty,\ j\in\mathbb{Z},\ j\ge j_0.$$

Proof. We first analyze the range $1\le p\le 2\le q\le\infty$. Plancherel's theorem gives

$$\|\widetilde{T}_j f\|_2 = \|\eta(2^j(\xi_d-\psi(\xi')))\chi(\xi')\hat{f}\|_2 \lesssim \|\hat{f}\|_2 = \|f\|_2$$

due to $\eta, \chi \in L^\infty(\mathbb{R}^d)$. The Stein–Tomas theorem for surfaces with k nonvanishing principal curvatures [Stein 1993, p. 365] yields as in [Mandel and Schippa 2022, Lemma 4.3]

$$\|\widetilde{T}_j f\|_q \lesssim 2^{-\frac{j}{2}} \|f\|_2, \quad \|\widetilde{T}_j f\|_2 \lesssim 2^{-\frac{j}{2}} \|f\|_{q'} \quad \text{if } \frac{1}{q} \le \frac{k}{2(k+2)}.$$

The Restriction-Extension operator $f \mapsto \mathcal{F}^{-1}(\hat{f}\, d\sigma_M)$ for compact pieces M of hypersurfaces with k nonvanishing principal curvatures has the mapping properties from [Mandel and Schippa 2022, Corollary 5.1], so it is bounded for (p,q) belonging to the pentagonal region

$$\frac{1}{p} > \frac{k+2}{2(k+1)}, \quad \frac{1}{q} < \frac{k}{2(k+1)}, \quad \frac{1}{p} - \frac{1}{q} \ge \frac{2}{k+2}. \tag{22}$$

So for these exponents and $M_t := \{\xi = (\xi', \xi_d) \in \operatorname{supp}(\chi) \times \mathbb{R} : \xi_d - \psi(\xi') = t\}$ with induced surface measure $d\sigma_{M_t} = (1 + |\nabla\psi(\xi')|^2)^{1/2}\, d\xi'$ we have for $\hat{g}(\xi) := \chi(\xi')\hat{f}(\xi)(1 + |\nabla\psi(\xi')|^2)^{-1/2}$

$$\|\widetilde{T}_j f\|_q \lesssim \int_{\mathbb{R}} |\eta(2^j t)|\, \|\mathcal{F}^{-1}(\hat{g}\, d\sigma_{M_t})\|_q \, dt \lesssim \int_{\mathbb{R}} |\eta(2^j t)|\, \|g\|_p \, dt \lesssim 2^{-j} \|f\|_p.$$

Moreover, [Mandel and Schippa 2022, Corollary 5.1] yields restricted weak-type bounds from $L^{p,1}(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$ for all (p,q) belonging to the closure of the above-mentioned pentagon, which implies $\|\widetilde{T}_j f\|_{q,\infty} \lesssim 2^{-j} \|f\|_{p,1}$ in the same manner. Interpolating all these bounds gives

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-j(\min\{A_0, A_1, A_2, A_2'\} - \varepsilon \cdot \mathbb{1}_{(p,q)\in\mathcal{E}})} = 2^{-jA_\varepsilon(p,q)} \quad \text{for } 1 \le p \le 2 \le q \le \infty,\ \varepsilon > 0.$$

This finishes the analysis in the case $1 \le p \le 2 \le q \le \infty$. For $2 \le p \le q \le \infty$ or $1 \le p \le q \le 2$ we get from Proposition 12

$$\|\widetilde{T}_j\|_{1\to 1} + \|\widetilde{T}_j\|_{\infty\to\infty} \lesssim \|\widetilde{K}_j\|_1 \lesssim 2^{-j(\frac{k+2}{2} - d)}.$$

Interpolating the estimates for $(p,q) = (\infty,\infty)$ with the ones for $p = 2,\ q \ge 2$ from above yields the estimates in the region A_3', A_4'; the dual ones follow analogously. So we get

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-j \min\{A_3, A_3', A_4, A_4'\}} = 2^{-jA_\varepsilon(p,q)},$$

which proves the claim. □

The optimality of our constants is open. It would be interesting to see whether recent results and techniques for oscillatory integral operators by Guth, Hickman and Iliopolou [Guth et al. 2019] or Kwon and Lee [2020] (Proposition 2.4, Proposition 2.5) can be adapted to prove better bounds, especially in the range $1 \le p \le q < 2$ or $2 < p \le q \le \infty$. Any theorem leading to a larger value of $A_\varepsilon(p,q)$ will automatically provide a larger range of exponents q, r_1, r_2 for which our Gagliardo–Nirenberg inequalities hold. Candidates for such values $\ge A_\varepsilon(p,q)$ are given in [Cho et al. 2005, Lemma 2.2] and [Mandel and Schippa 2022, Lemma 4.4], but it seems nontrivial to make use of those in our setting. Next we use the estimates for $\widetilde{T}_j$ to discuss the relevant operators at distance 2^{-j} from the critical surface where $j \nearrow +\infty$.

Proposition 14. *Assume $d \in \mathbb{N}$ and (A1) with $\alpha_1, \alpha_2 > -1$. Then there are bounded linear operators $\mathcal{T}_j : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ and $j_0 \in \mathbb{Z}$ with $\sum_{j=j_0}^{\infty} \mathcal{T}_j u = u_1$ such that, for $i = 1, 2$ and any given $\varepsilon > 0$, we have, for all $u \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|\mathcal{T}_j u\|_q \lesssim 2^{j(\alpha_i - A_\varepsilon(p,q))} \|P_i(D)u\|_p \quad \textit{for } 1 \le p \le q \le \infty,\ j \in \mathbb{Z},\ j \ge j_0.$$

Proof. Recall $u_1 = \mathcal{F}^{-1}(\tau \hat{u})$, where τ was chosen in (4); we first consider the case $d \ge 2$. According to assumption (A1) there are $\tau_1, \dots, \tau_L \in C_0^\infty(\mathbb{R}^d)$ such that $\tau_1 + \cdots + \tau_L = \tau$ holds and $S \cap \operatorname{supp}(\tau_l) = \{\xi \in \operatorname{supp}(\tau_l) : \tilde{\xi}_d = \psi_l(\tilde{\xi}'), \text{where } \tilde{\xi} = \Pi_l \xi\}$. Here, Π_l denotes some permutation of coordinates in $\mathbb{R}^d$. Since P_i vanishes of order α_i near the surface in the sense of assumption (A1), we may write

$$P_i(\xi)^{-1}\tau_l(\xi) = [\tau_{l+}(\xi)(\tilde{\xi}_d - \psi_l(\tilde{\xi}'))_+^{-\alpha_i} + \tau_{l-}(\xi)(\tilde{\xi}_d - \psi_l(\tilde{\xi}'))_-^{-\alpha_i}]\chi_l(\tilde{\xi}'),$$
$$\text{with } \tau_{l+}, \tau_{l-} \in C_0^\infty(\mathbb{R}^d),\ \chi_l \in C_0^\infty(\mathbb{R}^{d-1}),\ \tilde{\xi} := \Pi_l \xi, \tag{23}$$

for suitable functions χ_l, ψ_l that satisfy (19). In view of this we define

$$\mathcal{T}_j := \sum_{l=1}^{L} \mathcal{T}_j^l, \quad \text{where } \mathcal{T}_j^l u := \mathcal{F}^{-1}\big(\tau_l(\xi)\hat{u}(\xi)\,\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\big) \quad (\tilde{\xi} = \Pi_l \xi).$$

Since 0 does not belong to the support of η, there is $j_0 \in \mathbb{Z}$ such that $u_1 = \sum_{j=j_0}^{\infty} \mathcal{T}_j u$ in the sense of distributions. We introduce the smooth function $\eta_i(z) := \eta(z)|z|^{-\alpha_i}$. Then Lemma 13 yields

$$\begin{aligned}
\|\mathcal{T}_j u\|_q &\lesssim \sum_{l=1}^{L} \|\mathcal{T}_j^l u\|_q \\
&= \sum_{l=1}^{L} \big\|\mathcal{F}^{-1}\big(\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\,\tau_l(\xi)\hat{u}(\xi)\big)\big\|_q \\
&= \sum_{l=1}^{L} \big\|\mathcal{F}^{-1}\big(\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\,P_i(\xi)^{-1}\tau_l(\xi)\widehat{P_i(D)u}(\xi)\big)\big\|_q \\
&\overset{(23)}{=} \sum_{l=1}^{L} 2^{j\alpha_i} \big\|\mathcal{F}^{-1}\big(\eta_i(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')(\tau_{li+}(\xi) + \tau_{li-}(\xi))\widehat{P_i(D)u}(\xi)\big)\big\|_q \\
&\lesssim \sum_{l=1}^{L} 2^{j(\alpha_i - A_\varepsilon(p,q))} \big\|\mathcal{F}^{-1}\big((\tau_{li+}(\xi) + \tau_{li-}(\xi))\widehat{P_i(D)u}(\xi)\big)\big\|_p \\
&\lesssim 2^{j(\alpha_i - A_\varepsilon(p,q))} \|P_i(D)u\|_p.
\end{aligned}$$

In the last inequality we used that τ_{li+}, τ_{li-} are L^p-multipliers since their Fourier transforms are integrable. □

In the forthcoming analysis we shall need the following auxiliary result. The proof mainly follows Stein's analysis of oscillatory integrals [1993, p. 380–386].

Proposition 15. *Assume $0 \le \alpha < \frac{1}{2}$ and that χ, ψ are as in (19), $\tau \in C_0^\infty(\mathbb{R}^d)$; set*

$$L_\alpha u := \mathcal{F}^{-1}((\xi_d - \psi(\xi'))_+^{-\alpha}\chi(\xi')\tau(\xi)u).$$

Then $L_\alpha : L^2(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded linear operator for $q := \frac{2(k+2)}{k+2-4\alpha}$.

Proof. Define the family of distributions γ_s as in [Stein 1993, p. 381] (called α_s in this book) via

$$\gamma_s(y) = \frac{e^{s^2}}{\Gamma(s)} y^{s-1}\zeta(y)1_{y>0} \quad \text{if } \Re(s) > 0,$$

where ζ is smooth with compact support and $\zeta(y) = 1$ for $|y| \le y_0$, where y_0 is chosen so large that $\zeta(\xi_d - \psi(\xi')) = 1$ holds whenever $\chi(\xi')\tau(\xi) \neq 0$. The family (γ_s) is extended to all $s \in \mathbb{C}$ via analytic continuation. Then introduce the family of linear operators

$$M_s f := \mathcal{F}^{-1}(\chi(\xi')^2\gamma_s(\xi_d - \psi(\xi'))\hat{f}).$$

Plancherel's identity gives

$$\|M_s f\|_2 \lesssim \|f\|_2 \quad \text{if } \Re(s) = 1.$$

On the other hand

$$M_s f = \Phi * f, \quad \Phi(z) := \hat{\gamma}_s(-z_d)\cdot \int_{\mathbb{R}^{d-1}} \chi(\xi')^2 e^{iz\cdot(\xi',\psi(\xi'))}\, d\xi'.$$

From equation (15) in [Stein 1993] and equation (32) in [Mandel and Schippa 2022] we infer

$$|\Phi(z)| \lesssim (1+|z_d|)^{-\Re(s)}(1+|z_d|)^{-\frac{k}{2}} \lesssim 1 \quad \text{if } \Re(s) = -\frac{k}{2}.$$

We conclude

$$\|M_s f\|_\infty \lesssim \|f\|_1 \quad \text{if } \Re(s) = -\frac{k}{2}.$$

Furthermore, for any given Schwartz functions f, g the function $s \mapsto \int_{\mathbb{R}^d}(M_s f)g$ is holomorphic in the open strip $-\frac{k}{2} < \Re(s) < 1$ with continuous extension to the boundary. So the family M_s is admissible for Stein's interpolation theorem [1956, Theorem 1] and we obtain

$$\|M_{1-2\alpha} f\|_q \lesssim \|f\|_{q'} \quad \text{if } \theta \in [0,1],\ 1-2\alpha = (1-\theta)\cdot\left(-\frac{k}{2}\right) + \theta\cdot 1,\ \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$

This leads to $\theta = \frac{2(k+2-4\alpha)}{2(k+2)}$ and $q = \frac{2(k+2)}{k+2-4\alpha}$. In view of $0 < 2\alpha < 1$ this implies

$$\big\|\mathcal{F}^{-1}\big(\chi(\xi')^2(\xi_d - \psi(\xi'))_+^{-2\alpha}\zeta(\xi_d - \psi(\xi'))\hat{f}\big)\big\|_q \lesssim \|f\|_{q'}.$$

Now we consider functions $\hat{f} = \tau^2\hat{g}$. By choice of ζ and of y_0 we then have

$$\big\|\mathcal{F}^{-1}\big(\chi(\xi')^2(\xi_d - \psi(\xi'))_+^{-2\alpha}\tau(\xi)^2\hat{g}\big)\big\|_q \lesssim \|\mathcal{F}^{-1}(\tau^2\hat{g})\|_{q'} \lesssim \|g\|_{q'}.$$

This implies the claim given that this operator coincides with $L_\alpha L_\alpha^*$. □

We now use the dyadic estimates from Proposition 14 to prove Gagliardo–Nirenberg inequalities in the special case $P_1(D) = P_2(D)$ where the exponents satisfy $A_\varepsilon(p,q) = \alpha \in [0,1]$. This result plays the same role in the critical frequency regime as Proposition 8 does in the noncritical regime. For $d \ge 2$ we concentrate on exponents with $1 \le p \le 2 \le q \le \infty$.

Lemma 16. *Assume $d \in \mathbb{N}$ and let $P := P_1 = P_2$ satisfy (A1) for $\alpha := \alpha_1 = \alpha_2 \in [0,1]$. Then $\|u_1\|_q \lesssim \|P(D)u\|_p$ holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ provided that*

(i) $d=1$ *and* $1\le p,q\le\infty$ *satisfy* $\frac{1}{p}-\frac{1}{q}=\alpha$ *and, if* $0<\alpha<1$, $(p,q)\notin\{(1,\frac{1}{1-\alpha}),(\frac{1}{\alpha},\infty)\}$,

(ii) $d\ge 2$ *and* $1\le p\le 2\le q\le\infty$ *satisfy* $\frac{1}{p}-\frac{1}{q}=\frac{2\alpha}{k+2}$ *and* $\min\{\frac{1}{p},\frac{1}{q'}\}>\frac{k+2\alpha}{2(k+1)}$.

The estimate $\|u_1\|_{q,\infty}\lesssim\|P(D)u\|_p$ *holds for exponents as in* (i), (ii) *or*

(iii) $d=1,\ p=1,\ q=\frac{1}{1-\alpha}$ *if* $\alpha\in(0,1)$,

(iv) $d\ge 2,\ 1\le p<\frac{2(k+1)}{k+2\alpha},\ q=\frac{2(k+1)}{k+2-2\alpha}$ *if* $\alpha\in(\frac{1}{2},1]$.

Proof. With the same notation as before we have

$$P(\xi)^{-1}\tau_l(\xi)=[\tau_{l+}(\xi)(\tilde{\xi}_d-\psi_l(\tilde{\xi}'))_+^{-\alpha}+\tau_{l-}(\xi)(\tilde{\xi}_d-\psi_l(\tilde{\xi}'))_-^{-\alpha}]\chi_l(\tilde{\xi}'),$$
$$\text{with } \tau_{l+},\tau_{l-}\in C_0^\infty(\mathbb{R}^d),\ \chi_l\in C_0^\infty(\mathbb{R}^{d-1}),\ \tilde{\xi}:=\Pi_l\xi,$$

for functions χ_l,ψ_l that satisfy (19). So $u_1=\sum_{j=j_0}^\infty\mathcal{T}_j u$. Assuming $1\le p\le 2\le q\le\infty$ are chosen as above we obtain (ii), (iv) as follows:

- Case $d\ge 2,\ \alpha=0$. Our assumptions give that $A_\varepsilon(p,q)=\alpha=0$ only occurs for $p=q=2$. Here the estimate $\|u_1\|_2\lesssim\|P(D)u\|_2$ follows from Plancherel's theorem.

- Case $d\ge 2,\ \alpha\in(0,1)$. We first consider the case $\alpha<\frac{1}{2}$. By assumption, $(\frac{1}{p},\frac{1}{q})$ lies on the green diagonal line in Figure 2. By Proposition 15, the claimed inequality holds for the endpoints of that line given by $p=2,\ q=\frac{2(k+2)}{k+2-4\alpha}$ and its dual $p=\frac{2(k+2)}{k+2+4\alpha},\ q=2$. Interpolating these two estimates with each other provides the desired inequality for all tuples on the green line in Figure 2 and thus proves the claim for $\alpha<\frac{1}{2}$.

  Now consider the case $\alpha\ge\frac{1}{2}$. Our assumptions imply that $(\frac{1}{p},\frac{1}{q})$ lies on the blue line in Figure 2 with endpoints excluded. In particular, $(\frac{1}{p},\frac{1}{q})$ is in the interior of the A_1-region, so $A(\tilde{p},\tilde{q})=\frac{k+2}{2}(\frac{1}{\tilde{p}}-\frac{1}{\tilde{q}})$ for all $(\tilde{p},\tilde{q})$ close to (p,q). For small $\delta>0$ we choose $\frac{1}{q_1}=\frac{1}{q}+\delta,\ \frac{1}{q_2}=\frac{1}{q}-\delta$. Interpolating the estimates for (p,q_1) and (p,q_2) with interpolation parameter $\theta=\frac{1}{2}$ gives, due to $(1-\theta)A_\varepsilon(p,q_1)+\theta A_\varepsilon(p,q_2)=\alpha$, the weak estimate $\|u\|_{q,\infty}\lesssim\|P(D)u\|_p$. Here we used $u_1=\sum_{j=j_0}^\infty\mathcal{T}_j u$, the dyadic estimates from Proposition 14 and the interpolation lemma, Lemma 4. These weak estimates hold for all $(\frac{1}{p},\frac{1}{q})$ on the blue line with endpoints excluded. Interpolating these inequalities with each other gives $\|u\|_q\lesssim\|P(D)u\|_p$ for the same set of exponents, which proves (ii) for $\alpha\in(0,1)$.

  To prove the weak estimate from (iv) assume $\alpha\in(\frac{1}{2},1)$. For any given $(\frac{1}{p},\frac{1}{q})$ on the dashed horizontal blue line in Figure 2 with left endpoint excluded we can choose q_1,q_2 as above and the same argument gives $\|u\|_{q,\infty}\lesssim\|P(D)u\|_p$. Since these exponents are given by $1\le p<\frac{2(k+1)}{k+2\alpha}$ and $q=\frac{2(k+1)}{k+2-2\alpha}$, we are done.

- Case $d\ge 2,\ \alpha=1$. It was shown in [Mandel and Schippa 2022, Section 5] that the linear operators $(P(D)+i\delta)^{-1}:L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)$ are uniformly bounded with respect to small $|\delta|>0$ given that our additional regularity assumptions on P from (A1) imply that $S=\{\xi\in\mathbb{R}^d:P(\xi)=0\}$ is a smooth compact manifold with $|\nabla P|\ne 0$ on S. This implies $\|u_1\|_q\lesssim\|P(D)u\|_p$ and analogous arguments yield the weak bounds claimed in (iv).

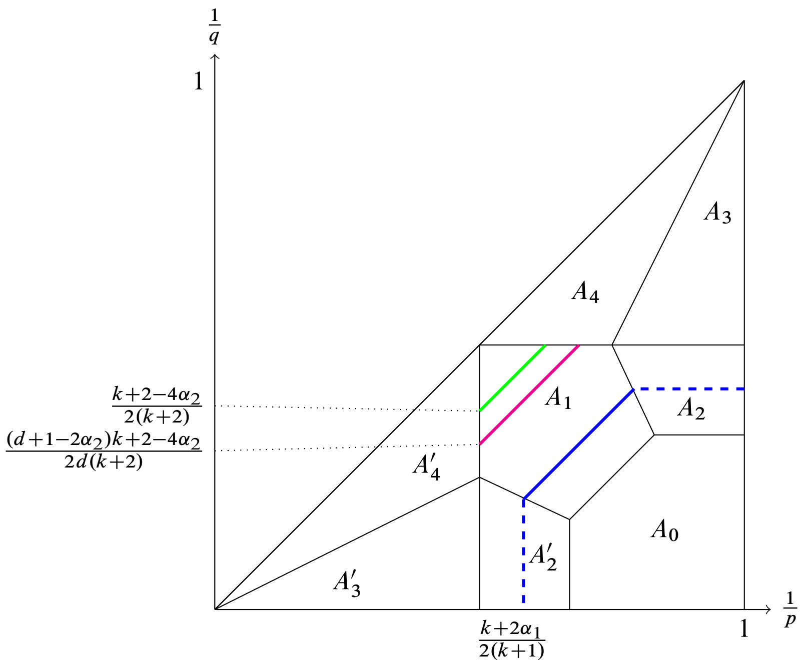

Figure 2. Riesz diagram showing the exponents $1 \le p \le 2 \le q \le \infty$ satisfying $A_\varepsilon(p,q) = \alpha$ in the case $\alpha = \alpha_1 \in \left(\frac{1}{2}, 1\right)$ (blue) and for $\alpha = \alpha_2 \in \left(0, \frac{1}{2}\right)$ (green). For the green resp. nondashed blue, exponent pairs Lemma 16 (i), (ii) give $\|u\|_q \le \|P(D)u\|_p$. In the case $\alpha = \alpha_2$ the corresponding estimates from [Mandel and Schippa 2022, Theorem 1.4(ii)] only hold for exponents on the magenta line. The picture was produced with parameter values $(d, k, \alpha_1, \alpha_2) = \left(4, 2, \frac{3}{4}, \frac{1}{4}\right)$.

Next we turn to the one-dimensional case $d = 1$. The representation formula then reads

$$u_1 = \sum_{l=1}^{L} \mathcal{F}^{-1}\left([\tau_{l+}(\xi)(\xi - \xi_l^*)_+^{-\alpha} + \tau_{l-}(\xi)(\xi - \xi_l^*)_-^{-\alpha}]\widehat{P(D)u}\right), \tag{24}$$

where $\{P(\xi) = 0\} = \{\xi_1^*, \dots, \xi_L^*\}$. Using our assumption $\frac{1}{p} - \frac{1}{q} = \alpha$ we obtain the claims (i), (iii) from the following arguments:

- Case $d = 1$, $\alpha = 0$. We then have $p = q$ and we first analyze $1 < p = q < \infty$. In this case the Hilbert transform $f \mapsto \mathcal{F}^{-1}(\operatorname{sign}(\xi)\hat{f})$ is bounded on $L^p(\mathbb{R})$, and so is $f \mapsto \mathcal{F}^{-1}(\operatorname{sign}(\xi - \xi_l^*)\hat{f})$ for $l = 1, \dots, L$. So the representation formula (24) implies $\|u_1\|_p \lesssim \|P(D)u\|_p$. In the case $p = q \in \{1, \infty\}$ we make use of our additional regularity assumption $\tau_l := \tau_{l+} = \tau_{l-}$ from (A1), so

$$\|u_1\|_p \le \sum_{l=1}^{L} \|\mathcal{F}^{-1}(\tau_l \widehat{P(D)u})\|_p \lesssim \sum_{l=1}^{L} \|\mathcal{F}^{-1}(\tau_l) * (P(D)u)\|_p \lesssim \|P(D)u\|_p.$$

Here we used that $\mathcal{F}^{-1}(\tau_l)$ is a Schwartz function for $l = 1, \dots, L$.

- Case $d = 1$, $\alpha \in (0, 1)$. If $1 < p < q < \infty$ we deduce the claimed estimate from the boundedness of the Hilbert transform on $L^q(\mathbb{R})$ and the Riesz potential estimate $\|\mathcal{F}^{-1}(|\cdot|^{-\alpha}\hat{f})\|_q \lesssim \|f\|_p$. For $p = 1$,

$0<\alpha<1$ we have a weak estimate $\|\mathcal{F}^{-1}(|\cdot|^{-\alpha}\hat{f})\|_{q,\infty}\lesssim\|f\|_1$; see [Grafakos 2014, Theorem 1.2.3]. Note that the Hilbert transform is bounded on $L^{q,\infty}(\mathbb{R})$ as well by real interpolation.

• Case $d=1$, $\alpha=1$. We now have $\frac{1}{p}-\frac{1}{q}=1$, so $p=1, q=\infty$. We exploit the additional smoothness assumption $\tau_{l+}=-\tau_{l-}$ from (A1). Then $P\in C^\infty(\mathbb{R})$ is a smooth function with simple zeros $\xi_1^*,\dots,\xi_L^*$. To prove the claimed inequality we start with the trivial estimate $\|v\|_\infty\lesssim\|v'\|_1=\|\mathcal{F}^{-1}(i\xi\hat{v})\|_1$ for all $v\in\mathcal{S}(\mathbb{R})$. Translation in Fourier space gives $\|v\|_\infty\lesssim\|\mathcal{F}^{-1}(i(\xi-\xi_l^*)\hat{v})\|_1$ for all $u\in\mathcal{S}(\mathbb{R}), l=1,\dots,L$. So (24) implies as above

$$\|u_1\|_\infty\lesssim\sum_{l=1}^{L}\|\mathcal{F}^{-1}((\xi-\xi_l^*)^{-1}\tau_l\widehat{P(D)u})\|_\infty\lesssim\sum_{l=1}^{L}\|\mathcal{F}^{-1}(\tau_l\widehat{P(D)u})\|_1\lesssim\|P(D)u\|_1. \qquad \square$$

As remarked in Figure 2, claim (ii) of the previous lemma improves upon the corresponding bounds from [Mandel and Schippa 2022, Theorem 1.4] in the case $0<\alpha<\frac{1}{2}$. We finally combine all these estimates to prove Gagliardo–Nirenberg inequalities in the critical frequency regime. Given the rather complicated definition of $A_\varepsilon(p,q)$, an explicit characterization of the admissible exponents is possible in principle, but extremely laborious. We prefer to avoid most of the computations. Instead, we describe the set of admissible exponents in an abstract way and provide the required computations in the reasonably simple special case $1\le p\le 2\le q\le\infty$ that allows us to prove our main results. Proceeding in this way it becomes clear how eventual improvements of Lemma 13 affect the final range of exponents. Once more we exploit Bourgain's summation argument, which allows us to argue almost as in the large-frequency regime. On a formal level, comparing Lemma 5 (large frequencies) with Lemma 13 (critical frequencies), we essentially have to replace $s_i-d\big(\frac{1}{r_i}-\frac{1}{q_i}\big)$ by $A_\varepsilon(r_i,q_i)-\alpha_i$ because the summation index now ranges from some $j=j_0$ to $+\infty$ and not from $j=j_0$ to $-\infty$. It will be convenient to formulate our sufficient conditions in terms of $\bar{\alpha}:=(1-\kappa)\alpha_1+\kappa\alpha_2$.

We provide a definition of the set $\mathcal{A}(\kappa)$ of exponents (q,r_1,r_2) that are admissible for

$$\|u_1\|_q\lesssim\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa},\quad u\in\mathcal{S}(\mathbb{R}^d). \tag{25}$$

Lemma 16 provides the definition for $\kappa\in\{0,1\}$, namely

$$\begin{aligned}\mathcal{A}(0)&:=\{(q,r_1,r_2)\in[1,\infty]^3:(q,r_1,\alpha_1)\text{ as in Lemma 16(i),(ii)}\},\\ \mathcal{A}(1)&:=\{(q,r_1,r_2)\in[1,\infty]^3:(q,r_2,\alpha_2)\text{ as in Lemma 16(i),(ii)}\}.\end{aligned} \tag{26}$$

In the case $0<\kappa<1$ the definition is more involved and relies on the interpolation lemma (Lemma 4) and the dyadic estimates for critical frequencies from Proposition 14. Combining the latter with (6) we obtain $\|u_1\|_q\lesssim\|u\|_{(X_1,X_2)_{\kappa,q}}$ and deduce (25) for exponents (q,r_1,r_2) belonging to the set

$$\mathcal{A}_1(\kappa):=\Big\{(q,r_1,r_2)\in[1,\infty]^3:\text{ there are }\varepsilon>0,\ q_1\in[r_1,\infty],\ q_2\in[r_2,\infty]\text{ such that } \frac{1}{q}=\frac{1-\kappa}{q_1}+\frac{\kappa}{q_2}\text{ and }(1-\kappa)A_\varepsilon(r_1,q_1)+\kappa A_\varepsilon(r_2,q_2)>\bar{\alpha}\Big\}.$$

This result covers all nonendpoint cases in our considerations further below. Using (5) with $Y_1 = Y_2 = L^q(\mathbb{R}^d)$ we obtain $\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$ for exponents in

$$\mathcal{A}_2(\kappa) := \big\{(q,r_1,r_2) \in [1,\infty]^3 : q \geq \max\{r_1,r_2\} \text{ and there is } \varepsilon > 0 \text{ such that } (1-\kappa)A_\varepsilon(r_1,q) + \kappa A_\varepsilon(r_2,q) = \bar{\alpha},\ A_\varepsilon(r_i,q) \neq \alpha_i,\ i = 1,2\big\}.$$

Next we use $\|u\|_q = \|u\|_q^{1-\kappa}\|u\|_q^{\kappa}$ to deduce further estimates from Lemma 16 for exponents in

$$\mathcal{A}_3(\kappa) := \{(q,r_1,r_2) \in [1,\infty]^3 : (q,r_1,\alpha_1),\ (q,r_2,\alpha_2) \text{ as in Lemma 16(i), (ii)}\}.$$

Using (5) with $Y_1 = L^{q_1}(\mathbb{R}^d),\ Y_2 = L^{q_2}(\mathbb{R}^d)$, we get the weak bound $\|u_1\|_{q,\infty} \lesssim \|u\|_{(X_1,X_2)_{\kappa,1}}$ for exponents belonging to

$$\mathcal{A}_4^w(\kappa) := \Big\{(q,r_1,r_2) \in [1,\infty]^3 : \text{there are } \varepsilon > 0,\ q_1 \in [r_1,\infty],\ q_2 \in [r_2,\infty] \text{ such that } (1-\kappa)A_\varepsilon(r_1,q_1) + \kappa A_\varepsilon(r_2,q_2) = \bar{\alpha},\ \frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2},\ \alpha_i \neq A_\varepsilon(r_i,q_i),\ q_1 \neq q_2\Big\}.$$

Interpolating the (weak or strong) endpoint estimates for $\mathcal{A}_2(\kappa) \cup \mathcal{A}_3(\kappa) \cup \mathcal{A}_4^w(\kappa)$ with each other exactly as in the final step of the proof of Proposition 9 we deduce $\|u_1\|_q \lesssim \|u\|_{(X_1,X_2)_{\kappa,q}} \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$ for exponents from

$$\mathcal{A}_4(\kappa) := \Big\{(q,r_1,r_2) \in [1,\infty]^3 : \text{there are } \varepsilon \neq 0, \delta > 0,\ \tilde{q}, q^* \in [1,\infty],\ \tilde{\kappa}, \kappa^* \in (0,1) \text{ with } \frac{1}{\tilde{q}} - \varepsilon = \frac{1}{q} = \frac{1}{q^*} + \varepsilon,\ \tilde{\kappa} - \delta = \kappa = \kappa^* + \delta \text{ and } (\tilde{q},r_1,r_2) \in \mathcal{A}_4^w(\tilde{\kappa}) \cup \mathcal{A}_3(\tilde{\kappa}) \cup \mathcal{A}_2(\tilde{\kappa}),\ (q^*,r_1,r_2) \in \mathcal{A}_4^w(\kappa^*) \cup \mathcal{A}_3(\kappa^*) \cup \mathcal{A}_2(\kappa^*)\Big\}.$$

Summarizing these interpolation results we obtain the following interpolation inequality in the critical frequency regime.

Proposition 17. *Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A1) for $\alpha_1, \alpha_2 > -1$. Then*

$$\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

holds provided that $(q,r_1,r_2) \in \mathcal{A}(\kappa) := \mathcal{A}_1(\kappa) \cup \mathcal{A}_2(\kappa) \cup \mathcal{A}_3(\kappa) \cup \mathcal{A}_4(\kappa)$.

5. Gagliardo–Nirenberg inequalities and proofs of Theorems 1 and 2

We first discuss the one-dimensional case. As before, we use the notation

$$\bar{\alpha} := (1-\kappa)\alpha_1 + \kappa\alpha_2 \quad \text{and} \quad \bar{s} := (1-\kappa)s_1 + \kappa s_2.$$

Theorem 18. *Assume $d = 1$, $\kappa \in [0,1]$ and that (A1), (A2) hold for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$ such that $0 < \bar{\alpha} \leq \bar{s}$. Then*

$$\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}),$$

holds provided that $q, r_1, r_2 \in [1,\infty]$ satisfy $\bar{\alpha} \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \leq \bar{s}$, as well as the conditions (i), (ii), (iii) *and* (iv), (v), (vi) *in the endpoint cases $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{s}$ and $\bar{\alpha} = \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q}$, respectively:*

(i) *If* $q=\infty$ *then* $\frac{1}{r_1}-s_1\neq 0\neq \frac{1}{r_2}-s_2$ *or* $(r_1,r_2)=\big(\frac{1}{s_1},\frac{1}{s_2}\big)$, $s_1,s_2\in\{0,1\}$.
(ii) *If* $1<q<\infty$, $\frac{1}{r_1}-\frac{s_1}{d}=\frac{1}{q}=\frac{1}{r_2}-\frac{s_2}{d}$ *and* $r_1=1$ *then* $1<r_2<q$, $\kappa\geq\frac{r_2}{q}$ *or* $r_2=\infty$, $\frac{1}{q}\leq\kappa\leq\frac{1}{q'}$.
(iii) *If* $1<q<\infty$ *and* $\frac{1}{r_1}-\frac{s_1}{d}=\frac{1}{q}=\frac{1}{r_2}-\frac{s_2}{d}$ *and* $r_2=1$ *then* $1<r_1<q$, $1-\kappa\geq\frac{r_1}{q}$ *or* $r_1=\infty$, $\frac{1}{q}\leq 1-\kappa\leq\frac{1}{q'}$.
(iv) *If* $q=\infty$ *then* $\frac{1}{r_1}-\alpha_1\neq 0\neq\frac{1}{r_2}-\alpha_2$ *or* $(r_1,r_2)=\big(\frac{1}{\alpha_1},\frac{1}{\alpha_2}\big)$, $\alpha_1,\alpha_2\in\{0,1\}$.
(v) *If* $1<q<\infty$, $\frac{1}{r_1}-\alpha_1=\frac{1}{q}=\frac{1}{r_2}-\alpha_2$ *then* $\alpha_1,\alpha_2\in[0,1]$ *and* $r_1=1,\kappa<1$ *only if* $1<r_2<q$, $\kappa\geq\frac{r_2}{q}$.
(vi) *If* $1<q<\infty$, $\frac{1}{r_1}-\alpha_1=\frac{1}{q}=\frac{1}{r_2}-\alpha_2$ *then* $\alpha_1,\alpha_2\in[0,1]$ *and* $r_2=1,\kappa>0$ *only if* $1<r_1<q$, $1-\kappa\geq\frac{r_1}{q}$.

Proof. Proposition 9 shows that the large-frequency part of the inequality (involving s_1,s_2 and thus (i), (ii) and (iii)) holds. In view of Proposition 17 it remains to show that all exponents satisfying $\bar\alpha\leq\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}$ with (iv), (v) and (vi) in the endpoint case $\bar\alpha=\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}$ are covered by $\mathcal{A}(\kappa)$. In the case $\kappa=0$ this holds by definition of $\mathcal{A}(0)$ from (26) because the requirement $(r_1,q)\notin\big\{1,\frac{1}{1-\alpha},\frac{1}{\alpha},\infty\big\}$ if $0<\alpha<1$ from Lemma 16 (i) is met by (iv), (v) and (vi). The discussion for $\kappa=1$ is analogous. So from now on consider the case $0<\kappa<1$.

We now retrieve some information about $\mathcal{A}(\kappa)$ by exploiting the formula $A_\varepsilon(p,q)=\frac{1}{p}-\frac{1}{q}$ for $1\leq p\leq q\leq\infty$; see (17). Going back to the definition of the sets $\mathcal{A}_i(\kappa)$ we find

$$\mathcal{A}_1(\kappa)=\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}>\bar\alpha\right\},$$
$$\mathcal{A}_2(\kappa)\supset\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha,\ 0\leq\frac{1}{r_i}-\frac{1}{q}\neq\alpha_i\text{ for }i=1,2\right\},$$
$$\mathcal{A}_3(\kappa)\supset\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha,\ \frac{1}{r_i}-\frac{1}{q}=\alpha_i\in[0,1]\text{ and }\right.$$
$$\left.(r_i,q)\notin\left\{\left(1,\frac{1}{1-\alpha_i}\right),\left(\frac{1}{\alpha_i},\infty\right)\right\}\text{ if }\alpha_i\in(0,1)\text{ for }i=1,2\right\}.$$

Since the interpolation inequality holds for these exponents, our claim is proved in the following cases:

- $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}>\bar\alpha$: see $\mathcal{A}_1(\kappa)$.
- $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha$ and $q=1$: we necessarily have $\bar\alpha=0$, $r_1=r_2=1$, which is covered by $\mathcal{A}_2(\kappa)$ for $\alpha_1,\alpha_2\neq 0$ or $\mathcal{A}_3(\kappa)$ for $\alpha_1=\alpha_2=0$, respectively.
- $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha$ and $q=\infty$: $\frac{1}{r_1}-\alpha_1\neq 0\neq\frac{1}{r_2}-\alpha_2$ is covered by $\mathcal{A}_2(\kappa)$ and $\frac{1}{r_1}-\alpha_1=0=\frac{1}{r_2}-\alpha_2$ with $\alpha_1,\alpha_2\in\{0,1\}$ is covered by $\mathcal{A}_3(\kappa)$.

So it remains to show the remaining endpoint estimates dealing with $1<q<\infty$. By the definition of $\mathcal{A}_4^w(\kappa)$ we have restricted weak-type estimates for exponents from

$$\mathcal{A}_4^w(\kappa)=\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha\text{ and there are }q_1\in[r_1,\infty],\ q_2\in[r_2,\infty]\right.$$
$$\left.\text{such that }q_1\neq q_2,\ \frac{1}{r_i}-\frac{1}{q_i}\neq\alpha_i\ (i=1,2),\ \frac{1-\kappa}{q_1}+\frac{\kappa}{q_2}=\frac{1}{q}\right\}$$
$$=\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha,\ 1<q<\infty\right\}.$$

(Indeed, thanks to $\bar{\alpha} > 0$ we may choose $\frac{1}{q_1} := \frac{1}{r_1} - \varepsilon$ and $\frac{\kappa}{q_2} := \frac{1}{q} - \frac{1-\kappa}{q_1}$ for small $\varepsilon > 0$ provided that $1 \le r_1 < \infty$, analogously for $r_2 < \infty$.) This implies

$$\mathcal{A}_4(\kappa) \supset \left\{ (q, r_1, r_2) \in [1,\infty]^3 : \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{\alpha},\ 1 < q < \infty,\ \frac{1}{r_1} - \frac{1}{r_2} \neq \alpha_1 - \alpha_2 \right\}.$$

This yields the claim for the following exponents:

- $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{\alpha},\ 1 < q < \infty$ and $\frac{1}{r_1} - \frac{1}{r_2} \neq \alpha_1 - \alpha_2$, which is covered by $\mathcal{A}_4(\kappa)$,
- $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{\alpha},\ 1 < q < \infty$ and $\frac{1}{r_i} - \frac{1}{q} = \alpha_i \in [0,1]$ with $(r_i, q) \neq \left(1, \frac{1}{1-\alpha_i}\right)$ if $\alpha_i \in (0,1)$, which is covered by $\mathcal{A}_3(\kappa)$.

So it remains to prove the claim for

$$1 < q < \infty, \quad \frac{1}{r_1} - \alpha_1 = \frac{1}{q} = \frac{1}{r_2} - \alpha_2 \quad \text{and}$$

$$\left[r_1 = 1 < r_2 < q, \quad 1 > \kappa \ge \frac{r_2}{q} \qquad \text{or} \qquad r_2 = 1 < r_1 < q, \quad 1 > 1-\kappa \ge \frac{r_1}{q} \right].$$

By symmetry we may concentrate on $r_1 = 1 < r_2 < q,\ 1 > \kappa \ge \frac{r_2}{q}$, where the estimate follows from

$$\|u\|_q \overset{(13)}{\lesssim} \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,\kappa q}^{\kappa} \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,r_2}^{\kappa} \lesssim \|P_1(D)u\|_1^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa}.$$

Here we used Proposition 8(iv) and (ii) (for $r = r_2$). ☐

Proof of Theorem 2. We apply Theorem 18 to the symbols $P_1(D) = |D|^s - 1, s > 0$ and $P_2(D) = I$ that satisfy the hypotheses of the theorem for $(\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0)$. Then $\bar{\alpha} = 1 - \kappa$, $\bar{s} = (1-\kappa)s$, so Theorem 18 implies that the Gagliardo–Nirenberg inequality holds provided that $1 - \kappa \le \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \le (1-\kappa)s$. The latter restriction comes from Theorem 18(i) and one checks that (ii)–(vi) are not restrictive for our choice of parameters $(\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0),\ s > 0$. ☐

We continue with the higher-dimensional case where a computation of $\mathcal{A}(\kappa) \cap \mathcal{B}(\kappa)$ is rather cumbersome. To simplify the discussion we concentrate on the special case $r_1 = r_2 = r \in [1,2]$ and $q \in [2,\infty]$ and only consider the special ansatz $q_1 = q_2 = q$ in the definition of the sets $\mathcal{A}_i(\kappa)$.

Theorem 19. *Assume $d \in \mathbb{N},\ d \ge 2, \kappa \in [0,1]$ and that (A1), (A2) hold for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$ such that $0 \le \bar{\alpha} \le 1$. Then*

$$\|u\|_q \lesssim \|P_1(D)u\|_r^{1-\kappa} \|P_2(D)u\|_r^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

holds provided that $\bar{\alpha} < 1,\ \alpha_1 \neq \alpha_2,\ 0 < \kappa < 1$ and the exponents $r \in [1,2],\ q \in [2,\infty]$ satisfy

$$\frac{2\bar{\alpha}}{k+2} \le \frac{1}{r} - \frac{1}{q} \le \frac{\bar{s}}{d} \quad \textit{and} \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \ge \frac{k+2\bar{\alpha}}{2(k+1)}, \tag{27}$$

as well as $(q,r) \neq \left(\infty, \frac{d}{\bar{s}}\right)$ if $s_1 = s_2 = \bar{s} \in (0,d]$. In the case $\bar{\alpha} = 1$ or $\alpha_1 = \alpha_2$ or $\kappa \in \{0,1\}$ the same is true provided that the last condition in (27) is replaced by $\min\left\{\frac{1}{r}, \frac{1}{q'}\right\} > \frac{k+2\bar{\alpha}}{2(k+1)}$.

Proof. The conditions for large frequencies (involving s_1, s_2) were shown to be sufficient in Proposition 9. So we concentrate on the critical frequency part involving α_1, α_2. The following computations are based on the formula $A_\varepsilon(r,q) = A(r,q) - \varepsilon \cdot \mathbb{1}_{(p,q)\in\mathcal{E}}$, where

$$A(r,q) = \min\left\{1, \frac{k+2}{2}\left(\frac{1}{r}-\frac{1}{q}\right), \frac{k+2}{2} - \frac{k+1}{q}, -\frac{k}{2} + \frac{k+1}{r}\right\}$$

for $1 \le r \le 2 \le q \le \infty$; see (17) and Figure 1. Our definitions of $\mathcal{A}_1(\kappa), \mathcal{A}_2(\kappa), \mathcal{A}_3(\kappa)$ yield in the case $0 < \kappa < 1$

$$\mathcal{A}_1(\kappa) \supset \{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) > \bar{\alpha} \text{ for some } \varepsilon > 0\},$$
$$\mathcal{A}_2(\kappa) \supset \{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) = \bar{\alpha} \text{ for some } \varepsilon > 0,\ \alpha_1 \neq \bar{\alpha} \neq \alpha_2\},$$
$$\mathcal{A}_3(\kappa) \supset \left\{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) = \bar{\alpha} \text{ for some } \varepsilon > 0,\ \alpha_1 = \bar{\alpha} = \alpha_2 \in [0,1] \text{ and } \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} > \frac{k+2\bar{\alpha}}{2(k+1)}\right\}.$$

From $\mathcal{A}(\kappa) \supset \mathcal{A}_1(\kappa) \cup \mathcal{A}_2(\kappa) \cup \mathcal{A}_3(\kappa)$ we thus get

$$\mathcal{A}(\kappa) \supset \left\{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) \ge \bar{\alpha} \text{ for some } \varepsilon > 0 \text{ and if } A_\varepsilon(r,q) = \bar{\alpha} = \alpha_1 = \alpha_2 \in [0,1] \text{ then } \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} > \frac{k+2\bar{\alpha}}{2(k+1)}\right\}.$$

Since $A_\varepsilon(r,q) \ge \bar{\alpha}$ for some $\varepsilon > 0$ is equivalent to

$$\frac{1}{r} - \frac{1}{q} \ge \frac{2\bar{\alpha}}{k+2} \quad \text{and} \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \begin{cases} \ge \frac{k+2\bar{\alpha}}{2(k+1)} & \text{if } \bar{\alpha} < 1 \text{ and } \alpha_1 \neq \alpha_2, \\ > \frac{k+2\bar{\alpha}}{2(k+1)} & \text{if } \bar{\alpha} = 1 \text{ or } \alpha_1 = \alpha_2. \end{cases}$$

This proves the claim for $0 < \kappa < 1$. When $\kappa \in \{0,1\}$ the claim follows from (26) and Lemma 16(i), (ii). □

Proof of Theorem 1. We apply Theorem 19 to $P_1(D) = |D|^s - 1$, $P_2(D) = I$. Again, the hypotheses of the theorem hold for $(\alpha_1, \alpha_2, s_1, s_2, k) = (1, 0, s, 0, d-1)$ because S is the unit sphere with $d-1$ nonvanishing principal curvatures. □

6. Local Gagliardo–Nirenberg inequalities

In [Fernández et al. 2022] it was shown that a "local" version of Gagliardo–Nirenberg inequalities is of interest, too. Here one looks for a larger set of exponents where (3) holds under the additional hypothesis $\|P_1(D)u\|_{r_1} \le R\|P_2(D)u\|_{r_2}$, where $R > 0$ is fixed; see Corollary 2.10 in that paper. A simple consequence of our estimates above is the following.

Corollary 20. *Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A1), (A2) for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$. Then the inequality*

$$\|u\|_q \lesssim (R^{\kappa-\kappa_1} + R^{\kappa-\kappa_2}) \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa}$$

holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ and satisfying $\|P_1(D)u\|_{r_1} \le R\|P_2(D)u\|_{r_2}$ provided that for some $\kappa_1, \kappa_2 \in [0,\kappa]$ we have $(q, r_1, r_2) \in \mathcal{A}(\kappa_1) \cap \mathcal{B}(\kappa_2)$.

Proof. Choose κ_1, κ_2 as required. Then Proposition 17 gives

$$\begin{aligned}\|u_1\|_q &\lesssim \|P_1(D)u\|_{r_1}^{1-\kappa_1}\|P_2(D)u\|_{r_2}^{\kappa_1} \\ &= (\|P_1(D)u\|_{r_1}\|P_2(D)u\|_{r_2}^{-1})^{\kappa-\kappa_1}\cdot\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa} \\ &\lesssim R^{\kappa-\kappa_1}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.\end{aligned}$$

Similarly, Proposition 9 implies

$$\|u_2\|_q \lesssim R^{\kappa-\kappa_2}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.$$

Summing up these inequalities gives the claim. □

In the context of our particular example $P_1(D) = |D|^s - 1$, $s > 0$, and $P_2(D) = I$ this gives the following generalization of [Fernández et al. 2022, Corollary 2.10].

Corollary 21. *Assume $d \in \mathbb{N}$, $d \geq 2$, $\kappa \in (0,1)$, $s > 0$. Then*

$$\|u\|_q \lesssim (R^\kappa + 1)\|(|D|^s - 1)u\|_r^{1-\kappa}\|u\|_r^{\kappa}$$

holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\|(|D|^s - 1)u\|_r \leq R\|u\|_r$ provided that $(q,r) \neq \left(\infty, \frac{d}{s}\right)$ if $0 < s \leq d$ and

(i) *$d = 1$, $1 \leq r, q \leq \infty$ and $1-\kappa \leq \frac{1}{r} - \frac{1}{q} \leq s$ or*

(ii) *$d \geq 2$, $1 \leq r \leq 2 \leq q \leq \infty$ and $\frac{2(1-\kappa)}{k+2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{s}{d}$, $\min\{\frac{1}{r}, \frac{1}{q'}\} \geq \frac{k+2-2\kappa}{2(k+1)}$.*

Proof. This corresponds to the special case

$$(\kappa_1, \kappa_2) = (\kappa, 0) \quad \text{and} \quad (\alpha_1, \alpha_2, s_1, s_2, k, r_1, r_2) = (1, 0, s, 0, d-1, r, r)$$

in Corollary 20. The computation of $\mathcal{A}(\kappa)$ and $\mathcal{B}(0)$ can be done as in the proof of Theorem 19. Note that the assumptions imply $\bar{\alpha} = 1-\kappa \in (0,1)$, $\alpha_1 \neq \alpha_2$ and $0 < \kappa < 1$. □

7. Gagliardo–Nirenberg inequalities with unbounded characteristic sets

In the previous sections we provided a systematic study of Gagliardo–Nirenberg inequalities, where the characteristic set S of the symbols is smooth and compact. In the case of unbounded characteristic sets our analysis works for Schwartz functions whose Fourier transform is supported in a given bounded set, but an argument for general Schwartz functions is lacking so far, even in the case of simple differentiable operators with suitable scaling behaviour like the wave operator or the Schrödinger operator. In the L^2-setting, a less technical approach based on Plancherel's identity can be used. We follow the ideas presented in [Fernández et al. 2022] to prove Gagliardo–Nirenberg inequalities of the form

$$\|u\|_q \lesssim \|\partial_{tt}u - \Delta u\|_r^{1-\kappa}\|u\|_r^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{28}$$

$$\|v\|_q \lesssim \|i\partial_t v - \Delta v\|_r^{1-\kappa}\|v\|_r^{\kappa}, \quad v \in \mathcal{S}(\mathbb{R}^d), \tag{29}$$

where $r = 2$. We denote the space-time variable by $z = (x,t) \in \mathbb{R}^{d-1} \times \mathbb{R} = \mathbb{R}^d$.

Theorem 22. *Let $d \in \mathbb{N}$. Then (28) holds provided that $r = 2$, $q = \frac{2d}{d-4+4\kappa}$, where $\frac{1}{2} \le \kappa \le 1$ if $d \ge 3$ and $\frac{1}{2} < \kappa \le 1$ if $d = 2$.*

Proof. We first consider the case $d \ge 3$, and define $\mathcal{C}_t := \{\xi = (\xi', \xi_d) \in \mathbb{R}^d : \xi_d^2 - |\xi'|^2 = t\}$ and the induced surface measure σ_t. Then we have the representation formula

$$u(z) = c_d \int_{\mathbb{R}^d} \hat{u}(\xi) e^{iz\cdot\xi}\, d\xi = \frac{c_d}{2} \int_{\mathbb{R}} \int_{\mathcal{C}_t} \hat{u}(\xi)|\xi|^{-1} e^{iz\cdot\xi}\, d\sigma_t(\xi)\, dt,$$

where $c_d = (2\pi)^{-d/2}$. Strichartz' inequality [1977, Theorem I, case III(b)] implies that we have for $\frac{2(d+1)}{d-1} \le q \le \frac{2d}{d-2}$

$$\begin{aligned}
\|u\|_q &\lesssim \int_{\mathbb{R}} \|\mathcal{F}^{-1}(\hat{u}|\cdot|^{-1}\, d\sigma_t)\|_q\, dt \\
&\lesssim \int_{\mathbb{R}} |t|^{\frac{d-1}{4} - \frac{d}{2q}} \|\hat{u}|\cdot|^{-1}\|_{L^2(\mathcal{C}_t, d\sigma_t)}\, dt \\
&\lesssim \int_{\mathbb{R}} |t|^{\frac{d-2}{4} - \frac{d}{2q}} \|\hat{u}|\cdot|^{-\frac{1}{2}}\|_{L^2(\mathcal{C}_t, d\sigma_t)}\, dt.
\end{aligned}$$

Here, the factor $|t|^{(d-1)/4-d/(2q)}$ is obtained via scaling and in the last estimate we used $|\xi| \ge \sqrt{|t|}$ for $\xi \in \mathcal{C}_t$. On the other hand, Plancherel's theorem gives

$$\begin{aligned}
\|\partial_{tt}u - \Delta u\|_2^2 &= \int_{\mathbb{R}^d} |\xi_d^2 - |\xi'|^2|^2 |\hat{u}(\xi)|^2\, d\xi \\
&= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathcal{C}_t} |t|^2 |\hat{u}(\xi)|^2 |\xi|^{-1}\, d\sigma_t(\xi)\, dt \\
&= \frac{1}{2} \int_{\mathbb{R}} t^2 \|\hat{u}|\cdot|^{-\frac{1}{2}}\|_{L^2(\mathcal{C}_t, d\sigma_t)}^2\, dt
\end{aligned}$$

and

$$\|u\|_2^2 = \frac{1}{2} \int_{\mathbb{R}} \|\hat{u}|\cdot|^{-\frac{1}{2}}\|_{L^2(\mathcal{C}_t, d\sigma_t)}^2\, dt.$$

Writing $\varphi(t) := \|\hat{u}|\cdot|^{-1/2}\|_{L^2(\mathcal{C}_t, d\sigma_t)}$ it remains to prove that the quotient

$$\frac{\int_{\mathbb{R}} |t|^{\frac{d-2}{4} - \frac{d}{2q}} \varphi(t)\, dt}{\left(\int_{\mathbb{R}} t^2 \varphi(t)^2\, dt\right)^{\frac{1-\kappa}{2}} \left(\int_{\mathbb{R}} \varphi(t)^2\, dt\right)^{\frac{\kappa}{2}}}$$

is bounded independently of φ. According to [Fernández et al. 2022, Lemma 2.1], with $w(t) = |t|^{(d-2)/4-d/(2q)}$, $w_1(t) = 1$ and $w_2(t) = t$, this is the case if and only if the following quantity is finite:

$$\begin{aligned}
\sup_{s>0} s^{\frac{1-\kappa}{2}} \left\| \frac{w}{(w_1^2 + s w_2^2)^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R})} &= \sup_{s>0} s^{\frac{1-\kappa}{2}} \left(\int_{\mathbb{R}} \frac{|t|^{\frac{d-2}{2} - \frac{d}{q}}}{1 + st^2}\, dt \right)^{\frac{1}{2}} \\
&= \sup_{s>0} s^{\frac{1-\kappa}{2} - \frac{1}{4}\left(\frac{d}{2} - \frac{d}{q}\right)} \left(\int_{\mathbb{R}} \frac{|\rho|^{\frac{d-2}{2} - \frac{d}{q}}}{1 + \rho^2}\, d\rho \right)^{\frac{1}{2}}.
\end{aligned}$$

This leads to $q = \frac{2d}{d-4+4\kappa}$. In view of $\frac{2(d+1)}{d-1} \le q \le \frac{2d}{d-2}$ this requires $\frac{1}{2} \le \kappa \le \frac{d+2}{2(d+1)}$, but the upper bound for κ may be removed just as in [Fernández et al. 2022, p. 20–21] by combining the already established inequality for $\frac{2(d+1)}{d-1}$ with

$$\|u\|_q \le \|u\|_2^{1-\theta}\|u\|_{\frac{2(d+1)}{d-1}}^{\theta}, \quad 2 \le q \le \frac{2(d+1)}{d-1}, \quad \frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{\frac{2(d+1)}{d-1}}.$$

In the case $d = 2$ the analogous reasoning based on [Strichartz 1977, Theorem I, case III(c)]. It is also shown in that work that the above estimates are valid for $6 = \frac{2(d+1)}{d-1} \le q < \frac{2d}{d-2} = \infty$ and thus $\frac{1}{2} < \kappa \le \frac{d+2}{2(d+1)}$. The same interpolation trick then allows to extend this to the whole range $\kappa > \frac{1}{2}$. □

We now apply this method to the Schrödinger operator.

Theorem 23. *Let $d \in \mathbb{N}, d \ge 2$. Then (29) holds provided that $r = 2, q = \frac{2(d+1)}{d-3+4\kappa}$ and $\frac{1}{2} \le \kappa \le 1$.*

Proof. Define $\mathcal{P}_t := \{\xi = (\xi', \xi_d) \in \mathbb{R}^d : \xi_d - |\xi'|^2 = t\}$ and the induced surface measure σ_t. Plancherel's identity gives

$$\begin{aligned}\|v\|_2^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} |\hat{v}(\xi', t + |\xi'|^2)|^2 \, d\xi' \, dt \\ &= \int_{\mathbb{R}} |t|^{\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} |\hat{v}(\sqrt{t}\xi', t(1 + |\xi'|^2))|^2 \, d\xi' \, dt \\ &= \int_{\mathbb{R}} |t|^{\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} |\hat{v}_t|^2 \sqrt{1 + 4|\xi'|^2} \, d\xi' \, dt \\ &= \int_{\mathbb{R}} |t|^{\frac{d-1}{2}} \|\hat{v}_t\|^2_{L^2(\mathcal{P}_1, d\sigma_1)} \, dt,\end{aligned}$$

where $\hat{v}_t(\xi) := \hat{v}(\sqrt{t}\xi', t\xi_d)(1 + 4|\xi'|^2)^{-1/4}$. Similarly,

$$\|i\partial_t v - \Delta v\|_2^2 = \int_{\mathbb{R}} t^{2+\frac{d-1}{2}} \|\hat{v}_t\|^2_{L^2(\mathcal{P}_1, d\sigma_1)} \, dt.$$

Strichartz' inequality from [Strichartz 1977, Theorem I, case I] implies for $q = \frac{2(d+1)}{d-1}$

$$\begin{aligned}\|v\|_q &= \left\| c_d \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} \hat{v}(\xi', t + |\xi'|^2) e^{iz\cdot(\xi', t+|\xi'|^2)} \, d\xi' \, dt \right\|_q \\ &\lesssim \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^{d-1}} \hat{v}(\xi', t + |\xi'|^2) e^{iz\cdot(\xi', t+|\xi'|^2)} \, d\xi' \right\|_q dt \\ &\lesssim \int_{\mathbb{R}} |t|^{\frac{d-1}{2}} \left\| \int_{\mathbb{R}^{d-1}} \hat{v}(\sqrt{t}\xi', t(1 + |\xi'|^2)) e^{iz\cdot(\sqrt{t}\xi', t(1+|\xi'|^2))} \, d\xi' \right\|_q dt \\ &\lesssim \int_{\mathbb{R}} |t|^{\frac{d-1}{2}} \left\| \mathcal{F}^{-1}\big(\hat{v}(\sqrt{t}\xi', t\xi_d)(1 + 4|\xi'|^2)^{-\frac{1}{2}} \, d\sigma_1\big)(\sqrt{t}z', tz_1) \right\|_q dt \\ &= \int_{\mathbb{R}} |t|^{\frac{d-1}{2} - \frac{d+1}{2q}} \left\| \mathcal{F}^{-1}\big(\hat{v}(\sqrt{t}\xi', t\xi_d)(1 + 4|\xi'|^2)^{-\frac{1}{2}} \, d\sigma_1\big) \right\|_q dt \\ &\lesssim \int_{\mathbb{R}} |t|^{\frac{d-1}{2} - \frac{d+1}{2q}} \|\hat{v}(\sqrt{t}\xi', t\xi_d)(1 + 4|\xi'|^2)^{-\frac{1}{2}}\|_{L^2(\mathcal{P}_1, d\sigma_1)} \, dt \\ &\lesssim \int_{\mathbb{R}} |t|^{\frac{d-1}{2} - \frac{d+1}{2q}} \|\hat{v}_t\|_{L^2(\mathcal{P}_1, d\sigma_1)} \, dt.\end{aligned}$$

We set $\varphi(t) := |t|^{(d-1)/4}\|\hat{v}_t\|_{L^2(\mathcal{P}_1, d\sigma_1)}$ and it remains to show that the quotient

$$\frac{\int_{\mathbb{R}} |t|^{\frac{d-1}{4}-\frac{d+1}{2q}} \varphi(t)\, dt}{\left(\int_{\mathbb{R}} t^2\varphi(t)^2\, dt\right)^{\frac{1-\kappa}{2}} \left(\int_{\mathbb{R}} \varphi(t)^2\, dt\right)^{\frac{\kappa}{2}}}$$

is bounded independently of φ. We apply [Fernández et al. 2022, Lemma 2.1] once more:

$$\begin{aligned}\sup_{s>0} s^{\frac{1-\kappa}{2}} \left(\int_{\mathbb{R}} \frac{|t|^{\frac{d-1}{2}-\frac{d+1}{q}}}{1+st^2}\, dt\right)^{\frac{1}{2}} &= \sup_{s>0} s^{\frac{1-\kappa}{2}} \left(\left(\frac{1}{\sqrt{s}}\right)^{\frac{d+1}{2}-\frac{d+1}{q}} \int_{\mathbb{R}} \frac{|\rho|^{\frac{d-1}{2}-\frac{d+1}{q}}}{1+\rho^2}\, d\rho\right)^{\frac{1}{2}} \\ &= \sup_{s>0} s^{\frac{1-\kappa}{2}-\frac{d+1}{8}+\frac{d+1}{4q}} \left(\int_{\mathbb{R}} \frac{|\rho|^{\frac{d-1}{2}-\frac{d+1}{q}}}{1+\rho^2}\, d\rho\right)^{\frac{1}{2}}.\end{aligned}$$

This term is indeed finite for $q = \frac{2(d+1)}{d-1}$ and $\kappa = \frac{1}{2}$, which proves the claim in this special case. The claim for general $\kappa \geq \frac{1}{2}$ follows as above by interpolation. □

We conjecture that at least for $1 < r \leq 2 \leq q < \infty$ and $0 < \kappa < 1$ the inequality (28) actually holds for exponents

$$\frac{1}{r} - \frac{1}{q} = \frac{2(1-\kappa)}{d}, \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \frac{d-2\kappa}{2(d-1)}, \tag{30}$$

whereas the corresponding inequality involving the Schrödinger operator holds whenever

$$\frac{1}{r} - \frac{1}{q} = \frac{2(1-\kappa)}{d+1}, \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \frac{d+1-2\kappa}{2d}.$$

Note that the Sobolev inequalities [Jeong et al. 2016, Theorem 1.1] then take the form of the endpoint estimate $\kappa = 0$ in (30).

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