Linear power corrections to single top production and decay at the LHC in the narrow width approximation

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ABSTRACT: We consider top quark production and decay in the narrow width approximation and study if the polarisation effects, that manifest themselves in correlations of angular distributions of particles from top quark decays and final state jets in the production sub-process, are affected by linear power corrections. We find that, in general, the answer to this question is affirmative. We also discuss how these non-perturbative corrections affect polarisation observables used to study single top production at the LHC. Finally, we point out that generic kinematic distributions of leptons from top quark decays are affected by linear power corrections, which may have implications for proposals to extract the top quark mass from such leptonic observables. On the other hand, we demonstrate that the distribution of the "out-of-collision-plane" component of the positron momentum is free from linear power corrections, making it an interesting candidate for the top quark mass measurement.

Contents

1 Introduction

In a recent series of papers [1, 2] we have studied $\mathcal{O}(\Lambda_{\mathrm{QCD}})$ power corrections to top quark production in hadron collisions using the approach based on infra-red renormalons.[1] In this paper we extend these analyses by accounting for top quark decays. Such an extension is non-trivial. Indeed, since the top quark width Γ_t serves as an infra-red regulator, its interplay with the non-perturbative QCD parameter Λ_{QCD} and its proxy in the context of renormalon calculus — the gluon mass λ — is important. Since $\Gamma_t \gg \Lambda_{\mathrm{QCD}} \sim \lambda$, top quarks decay before hadronisation. A renormalon-based analysis of power corrections in such a case is technically very challenging, because the produced top quarks are off-shell, and diagrams where real or virtual gluons connect production and decay stages of off-shell top quarks have to be considered. Although this problem represents an interesting challenge for future work, we believe that it makes sense to start by considering the *opposite* case $\Gamma_t \ll \Lambda_{\mathrm{QCD}} \sim \lambda$, which can be studied in the narrow width approximation. Even if the phenomenological relevance of such an analysis is limited, it provides, for the very first time, an estimate of the non-perturbative corrections to a full physical process with unstable particles at a hadron collider and, as such, might be quite valuable for modelling the non-perturbative effects.

The narrow width approximation leads to important technical simplifications since, in this case, QCD radiative corrections cannot connect top quark production and top quark decay sub-processes [5, 6]. In fact, the QCD corrections to top quark production and decay process are known through next-to-next-to-leading order in this approximation [7, 8].

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[1]An in-depth discussion of infra-red renormalons in QCD can be found in review [3]. For a detailed description of how these methods can be used in the context of hadron collider applications, see ref. [4].
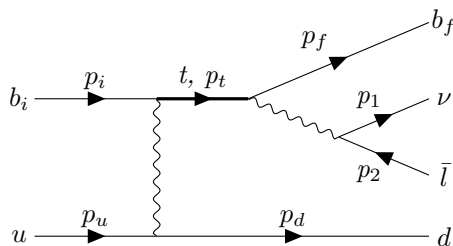
Figure 1. The Born amplitude for single top production and decay.

However, even when the narrow width approximation is used, the production and decay sub-processes are not independent; the communication between them occurs because of momentum conservation and *polarisation effects*.[2] For the case of single top production, that we study in this paper, this implies that e.g. the direction of the outgoing light-quark jet and the direction of the positron in the top quark decay are not independent in spite of the fact that they originate from widely separated sub-processes. As we will show below, linear power corrections affect angular distributions in single top production. In fact, we find that they vanish when positrons from top decays and light jets from the production process are collinear to each other, but are non-trivial functions of momenta of final state particles otherwise. We will also show that the $\mathcal{O}(\Lambda_{\text{QCD}})$ effects that we discuss in this paper impact the various observables designed to study top quark polarisation [14–16] differently, so that for each of them a dedicated study is required.

The paper is organised as follows. In the next section we describe the narrow width approximation and present the result for the differential cross section of single top production followed by top decay in a way that is useful for the subsequent analysis. In sections 3 and 4 we discuss the calculation of $\mathcal{O}(\lambda)$ corrections to the production and decay sub-processes. In section 5, $\mathcal{O}(\lambda)$ corrections due to mass-parameter redefinition are studied. In section 6, corrections to observables are discussed and the final formulas are derived. We conclude in section 7. In the appendix we illustrate an alternative approach to the calculation of power corrections where, at variance with the method used in the previous publications [1, 2], we deal directly with the amplitude of the process rather than with its square.

2 The narrow width approximation

In this paper we consider the following partonic process

$$
\begin{aligned}
u(p_u) + b(p_i) \to d(p_d) + t(p_t) \\
\searrow b(p_f) + \nu(p_1) + e^+(p_2).
\end{aligned}
\tag{2.1}
$$

It is shown in figure 1. The amplitude for this process can be written as

$$
\mathcal{M} = A_{\text{dec}}^i \frac{i\delta^{ij}(\not{p}_t + m_t)}{p_t^2 - m_t^2 + im_t\Gamma_t} A_{\text{prod}}^j,
\tag{2.2}
$$

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[2]Theoretical studies of spin correlations in top quark pair production and polarisation effects in single top production have a long history, see e.g. refs. [9–13] for original and more recent work.

where i, j are colour indices. In the on-shell $p_t^2 \to m_t^2$ limit, the quantities A_{dec} and A_{prod} correspond to on-shell amplitudes for the respective sub-processes from which the top quark spinors are removed.

The expression for the cross section becomes

$$\mathrm{d}\sigma_{PD} = \frac{1}{\mathcal{N}} \frac{\mathrm{d}\Phi(p_u, p_i; p_d, \{p_{\text{dec}}\})}{(p_t^2 - m_t^2)^2 + m_t^2 \Gamma_t^2} \sum_{\text{spins}} A_{\text{dec}}^i (\slashed{p}_t + m_t) A_{\text{prod}}^i \bar{A}_{\text{prod}}^j (\slashed{p}_t + m_t) \bar{A}_{\text{dec}}^j, \qquad (2.3)$$

where the suffix PD indicates production and decay, $\mathrm{d}\Phi$ is the phase space of the process in eq. (2.1), the sum over spins includes external particles only and $\{p_{\text{dec}}\}$ describes the momenta $p_{f,1,2}$ that refer to particles originating from the "decay" of the virtual top quark. The normalisation factor $\mathcal{N}$ includes all the averaging factors needed to compute the cross section of the process in eq. (2.1) as well as the relevant flux factor.

To simplify eq. (2.3), we note that since in the narrow width approximation no colour transfer between production and decay amplitudes occurs, the following equation holds

$$A_{\text{dec}}^i \dots \bar{A}_{\text{dec}}^j = \frac{\delta^{ij}}{N_c} A_{\text{dec}}^k \dots \bar{A}_{\text{dec}}^k, \qquad (2.4)$$

where $N_c = 3$ is the number of colours. Hence,

$$\mathrm{d}\sigma_{PD} = \frac{1}{\mathcal{N} \, N_c} \frac{\mathrm{d}\Phi(p_u, p_i; p_d, \{p_{\text{dec}}\})}{(p_t^2 - m_t^2)^2 + m_t^2 \Gamma_t^2} \sum_{\text{spins}} A_{\text{dec}}^i (\slashed{p}_t + m_t) A_{\text{prod}}^j \bar{A}_{\text{prod}}^j (\slashed{p}_t + m_t) \bar{A}_{\text{dec}}^i. \qquad (2.5)$$

To proceed further, we factorise the phase space

$$\mathrm{d}\Phi(p_u, p_i; p_d, \{p_{\text{dec}}\}) = \frac{\mathrm{d}p_t^2}{2\pi} \, \mathrm{d}\Phi(p_u, p_i; p_d, p_t) \, \mathrm{d}\Phi(p_t; \{p_{\text{dec}}\}), \qquad (2.6)$$

and make use of the fact that we work in the narrow width approximation which implies that the following equation holds

$$\left. \frac{1}{(p_t^2 - m_t^2)^2 + m_t^2 \Gamma_t^2} \right|_{\Gamma_t/m_t \to 0} = \frac{2\pi}{2 m_t \Gamma_t} \delta(p_t^2 - m_t^2). \qquad (2.7)$$

We obtain

$$\mathrm{d}\sigma_{PD} = \frac{\mathrm{d}\Phi(p_u, p_i; p_d, p_t) \, \mathrm{d}\Phi(p_t; \{p_{\text{dec}}\})}{\mathcal{N} \, N_c \, 2 m_t \Gamma_t} \sum_{\text{spins}} A_{\text{dec}}^i (\slashed{p}_t + m_t) A_{\text{prod}}^j \bar{A}_{\text{prod}}^j (\slashed{p}_t + m_t) \bar{A}_{\text{dec}}^i. \quad (2.8)$$

It is easy to see that the following (matrix) equation holds[3]

$$\sum_{\text{spins}} (\slashed{p}_t + m_t) \bar{A}_{\text{dec}}^i A_{\text{dec}}^i (\slashed{p}_t + m_t) = X \slashed{p}_{t,D}, \qquad (2.9)$$

where the sum extends over the polarisations of the top quark decay products and where we have defined the top quark spin density matrix

$$\slashed{p}_{t,D} = (\slashed{p}_t + m_t) \frac{(1 + \gamma_5 \slashed{s}_D)}{2}. \qquad (2.10)$$

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[3]This is due to the fact that helicities of all other external particles are fixed, so that the top quark must be in a pure spin state. See appendix for more details.

In eq. (2.9), X is a function of scalar products constructed out of the top quark momentum p_t and the momenta of its decay products, and s_D^μ is a space-like unit vector that also depends on p_t and the momenta of the particles in top decay. To find X, we compute the trace of both sides of eq. (2.9) and obtain

$$\sum_{\text{spins}} 2m_t \text{Tr}\left[(\slashed{p}_t + m_t)\bar{A}^i_{\text{dec}}A^i_{\text{dec}}\right] = 2m_t\, X. \tag{2.11}$$

We can write this equation in a better way by using the formula for the differential decay width of the unpolarised top quark

$$\mathrm{d}\Gamma_t = \frac{\mathrm{d}\Phi(p_t;\{p_{\text{dec}}\})}{4m_t N_c} \sum_{\text{spins}} \text{Tr}\left[(\slashed{p}_t + m_t)\bar{A}^i_{\text{dec}}A^i_{\text{dec}}\right]. \tag{2.12}$$

It follows that

$$X = \sum_{\text{spins}} \text{Tr}\left[(\slashed{p}_t + m_t)\bar{A}^i_{\text{dec}}A^i_{\text{dec}}\right] = 4m_t N_c\, \frac{\mathrm{d}\Gamma_t}{\mathrm{d}\Phi(p_t;\{p_{\text{dec}}\})}. \tag{2.13}$$

Finally, using this result as well as eq. (2.9), we re-write eq. (2.8) as

$$\mathrm{d}\sigma_{PD} = 2\frac{\mathrm{d}\Gamma_t}{\Gamma_t} \times \mathrm{d}\sigma_t(s_D). \tag{2.14}$$

In the above equation, $\mathrm{d}\sigma_t(s_D)$ is the cross section for producing a single top quark whose spin is aligned with the axis s_D; it is defined as

$$\mathrm{d}\sigma_t(s_D) = \frac{\mathrm{d}\Phi(p_u, p_i; p_d, p_t)}{\mathcal{N}} \sum_{\text{spins}} \text{Tr}\left[\slashed{p}_{t,D} A^j_{\text{prod}}\bar{A}^j_{\text{prod}}\right]. \tag{2.15}$$

We note that we will refer to the spin axis s_D as the top quark spin vector below. It is important to emphasise that for the process in eq. (2.1) this vector depends on the momenta of the top quark decay products. It is computed in the appendix and given in eq. (A.15). The calculation of the top quark polarised cross section in eq. (2.14) proceeds in the standard way. The only difference with the unpolarised case is that instead of the density matrix $(\slashed{p}_t + m_t)$ one has to use $\slashed{p}_{t,D}$.

The formula for the cross section shown in eq. (2.14) is suitable for computing QCD corrections to the production process followed by tree-level decay. When tree-level production is followed by the QCD-corrected decay, it is more convenient to deal with the decay of the polarised top quark. Following the discussion above, we find

$$\mathrm{d}\sigma_{PD} = 2\frac{\mathrm{d}\Gamma_t(s_P)}{\Gamma_t} \times \mathrm{d}\sigma_t, \tag{2.16}$$

where this time $\mathrm{d}\sigma_t$ is the unpolarised top quark production cross section and $\mathrm{d}\Gamma_t(s_P)$ is the decay rate of a polarised top quark

$$\mathrm{d}\Gamma_t(s_P) = \frac{\mathrm{d}\Phi(p_t;\{p_{\text{dec}}\})}{4m_t N_c} \sum_{\text{spins}} \text{Tr}\left[\slashed{p}_{t,P}\bar{A}^i_{\text{dec}}A^i_{\text{dec}}\right]. \tag{2.17}$$

The corresponding spin density matrix reads

$$\rho_{t,P} = (\not{p}_t + m_t)\frac{(1 + \gamma_5 \not{s}_P)}{2}.\tag{2.18}$$

The polarisation vector of the top quark s_P is given in eq. (A.16). We note that at leading order the two representations of the full differential cross section, given in eqs. (2.14) and (2.16), are equivalent, so that the following equation holds

$$\mathrm{d}\sigma_t\,\mathrm{d}\Gamma_t(s_P) = \mathrm{d}\sigma_t(s_D)\,\mathrm{d}\Gamma_t.\tag{2.19}$$

This equation allows us to use one representation to study radiative corrections to the production process and another one to study radiative corrections to top decay. Furthermore, eq. (2.19) will be useful for studying effects related to the redefinition of the top quark mass. Such a redefinition is needed to remove $\mathcal{O}(\lambda)$ corrections caused by the fact that the pole mass of the top quark is used in perturbative computations and that this mass parameter itself receives $\mathcal{O}(\lambda)$ corrections when expressed through a short-distance mass [17, 18].

Eq. (2.19) will also be useful for simplifying the final result. In particular, for single top production, the above equation assumes a particularly simple form[4]

$$\mathrm{d}\sigma_t\,\mathrm{d}\Gamma_t(s_P) = \mathrm{d}\sigma_t(s_D)\,\mathrm{d}\Gamma_t = \frac{1}{2}\mathrm{d}\sigma_t\mathrm{d}\Gamma_t\,(1 - s_D \cdot s_P).\tag{2.20}$$

Hence, by choosing the explicit form of one of the two spin vectors and pretending that the other one is general, one can put the polarisation information either to the production or to the decay sub-process of the full process in eq. (2.1).

3 QCD corrections to the production sub-process

We consider the production sub-process

$$u(p_u) + b(p_i) \to d(p_d) + t(p_t),\tag{3.1}$$

with the assumption that the top quark is polarised. As explained in ref. [19], corrections to the light-quark line do not produce linear $\mathcal{O}(\lambda)$ contributions. For this reason, we only need to consider the QCD corrections to the heavy-quark line. To describe this process, we can use eq. (2.14) and compute the standard perturbative contributions to the polarised cross section. In our discussion, we will assume that the reader is familiar with the computation for the unpolarised case reported in ref. [1], and we will mostly emphasise the differences between the two cases in what follows.

3.1 Real emission contributions

We begin with the real emission corrections to the process in eq. (3.1)

$$u(p_u) + b(p_i) \to d(q_d) + t(q_t) + g(k).\tag{3.2}$$

The gluon $g(k)$ is massive, $k^2 = \lambda^2$. We remark that we have used slightly different notations for the four-momenta of the top quark and the down quark in eqs. (3.1) and (3.2). This is

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[4]For the derivation of this formula, see eq. (A.10) and the discussion before it.

done for future convenience since, as we will see, momenta q_t and q_d will absorb the recoil due to the emitted soft gluon. We will also consider the top quark to be polarised, and we will denote its spin vector with s_D^μ.

The calculation proceeds along the lines described in ref. [1] where the case of stable top quark was discussed. The main element of that discussion was the Low-Burnett-Kroll theorem [20, 21] that can be used to describe soft radiation in QCD with next-to-leading-power accuracy.[5] This theorem was derived from the transversality of the amplitude with respect to the gluon momentum which allowed us to relate soft gluon emission by the external particles and the structure-dependent radiation. The fact that the top quark spinor represents a state with a particular polarisation plays no role in this argument. Hence, upon writing

$$\mathcal{A}_{\text{prod}} = g_s T_{ij}^a \epsilon_\mu \mathcal{M}^\mu, \tag{3.3}$$

and repeating all the steps described in ref. [1], we arrive at the following result

$$
\begin{aligned}
\mathcal{M}^\mu = {}& J_t^\mu \bar{u}_t \mathbf{N}(q_t + k, p_i, q_d, \ldots) u_i + J_i^\mu \bar{u}_t \mathbf{N}(q_t, p_i - k, q_d, \ldots) u_i \\
& + \bar{u}_t \left[\mathbf{S}_t^\mu \mathbf{N}(q_t, p_i, q_d, \ldots) + \mathbf{N}(q_t, p_i, q_d, \ldots) \mathbf{S}_i^\mu \right] u_i \\
& - \bar{u}_t \left[\frac{\partial \mathbf{N}(q_t, p_i, q_d, \ldots)}{\partial q_{t,\mu}} + \frac{\partial \mathbf{N}(q_t, p_i, q_d, \ldots)}{\partial p_{i,\mu}} \right] u_i.
\end{aligned}
\tag{3.4}
$$

In the above expression, $\mathbf{N}$ is the tree-level amplitude for single top production from which top and bottom spinors have been removed, and

$$
\begin{aligned}
J_t^\mu &= \frac{2q_t^\mu + k^\mu}{d_t}, \quad & \mathbf{S}_t^\mu &= \frac{\sigma^{\mu\nu} k_\nu}{d_t}, \\
J_i^\mu &= \frac{2p_i^\mu - k^\mu}{d_i}, \quad & \mathbf{S}_i^\mu &= \frac{\sigma^{\mu\nu} k_\nu}{d_i},
\end{aligned}
\tag{3.5}
$$

are top (bottom) currents and spin operators, respectively. The two quantities in the above equation, $d_t = (q_t + k)^2 - m_t^2$ and $d_i = (p_i - k)^2$, are the denominators of top and bottom propagators. Furthermore, $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/2$.

We can further simplify the expression in eq. (3.4) by combining the first two terms, expanded to first subleading power in k, with the last two terms. We obtain

$$
\begin{aligned}
\mathcal{M}^\mu = {}& J^\mu \bar{u}_t \mathbf{N}(q_t, p_i, q_d, \ldots) u_i + \bar{u}_t (L^\mu \mathbf{N}(q_t, p_i, q_d, \ldots)) u_i \\
& + \bar{u}_t \left[\mathbf{S}_t^\mu \mathbf{N}(q_t, p_i, q_d, \ldots) + \mathbf{N}(q_t, p_i, q_d, \ldots) \mathbf{S}_i^\mu \right] u_i,
\end{aligned}
\tag{3.6}
$$

where we have introduced the notation

$$J^\mu = J_t^\mu + J_i^\mu, \quad L^\mu = L_t^\mu - L_i^\mu, \tag{3.7}$$

with

$$L_t^\mu = \frac{2}{d_t} \left(p_t^\mu k^\nu \frac{\partial}{\partial q_t^\nu} - (p_t k) \frac{\partial}{\partial q_{t,\mu}} \right), \qquad L_i^\mu = \frac{2}{d_i} \left(p_i^\mu k^\nu \frac{\partial}{\partial p_i^\nu} - (p_i k) \frac{\partial}{\partial p_{i,\mu}} \right). \tag{3.8}$$

---

[5]An extension of this theorem to processes with polarised particles can be found in ref. [22].

– 6 –

Eq. (3.6) expresses the amplitude with the emission of a soft gluon through the elastic amplitude and its derivatives. However, we observe further simplifications if we square the amplitude and sum over the polarisations of the external particles, or consider external states with definite helicities. We find

$$
\begin{aligned}
|\mathcal{M}|^2 = -g_{\mu\nu}\mathcal{M}^\mu\mathcal{M}^{\nu,+} = &- J^\mu J_\mu \mathrm{Tr}\left[\slashed{p}_{t,D}\mathbf{N}\slashed{p}_i\bar{\mathbf{N}}\right] - J^\mu \mathrm{Tr}\left[\slashed{p}_{t,D}\mathbf{N}\slashed{p}_i L_\mu\bar{\mathbf{N}}\right] \\
&- J_\mu \mathrm{Tr}\left[\slashed{p}_{t,D}(L^\mu\mathbf{N})\slashed{p}_i\bar{\mathbf{N}}\right] + J_\mu \mathrm{Tr}\left[[\mathbf{S}_t^\mu,\slashed{p}_{t,D}]\mathbf{N}\slashed{p}_i\bar{\mathbf{N}}\right] \\
&+ J_\mu \mathrm{Tr}\left[\slashed{p}_{t,D}\mathbf{N}[\slashed{p}_i,\mathbf{S}_i^\mu]\bar{\mathbf{N}}\right].
\end{aligned}
\tag{3.9}
$$

Note that the only difference between eq. (3.9) and a similar expression for $|\mathcal{M}|^2$ in ref. [1] is that the density matrix $\slashed{p}_{t,D}$ defined in eq. (2.10) appears in eq. (3.9) instead of $(\slashed{p}_t + m_t)$. Since,

$$
[\mathbf{S}_t^\mu,\slashed{p}_{t,D}] = (-L_t^\mu - S_t^\mu)\slashed{p}_{t,D} = -(L^\mu + S_t^\mu)\slashed{p}_{t,D}, \qquad [\mathbf{S}_i^\mu,\slashed{p}_i] = -L_i^\mu\slashed{p}_i = (L^\mu + S_t^\mu)\slashed{p}_i, \tag{3.10}
$$

where

$$
S_t^\mu = \frac{2}{d_t}\left(s_D^\mu k^\nu \frac{\partial}{\partial s_D^\nu} - (s_D k)\frac{\partial}{\partial s_{D,\mu}}\right), \tag{3.11}
$$

we obtain

$$
\begin{aligned}
|\mathcal{M}|^2 = &- J^\mu J_\mu \mathrm{Tr}\left[\slashed{p}_{t,D}\mathbf{N}\slashed{p}_i\bar{\mathbf{N}}\right] - J^\mu \mathrm{Tr}\left[\slashed{p}_{t,D}\mathbf{N}\slashed{p}_i L_\mu\bar{\mathbf{N}}\right] - J^\mu \mathrm{Tr}\left[\slashed{p}_{t,D}(L_\mu\mathbf{N})\slashed{p}_i\bar{\mathbf{N}}\right] \\
&- J^\mu \mathrm{Tr}\left[((L_\mu + S_{t,\mu})\slashed{p}_{t,D})\mathbf{N}\slashed{p}_i\bar{\mathbf{N}}\right] - J^\mu \mathrm{Tr}\left[\slashed{p}_{t,D}\mathbf{N}((L_\mu + S_{t,\mu})\slashed{p}_i)\bar{\mathbf{N}}\right].
\end{aligned}
\tag{3.12}
$$

We emphasise that in the above formulas, starting from eq. (3.10), derivatives with respect to the four-momenta of partons that appear in the operators $L_{t/i}$ do not act on the polarisation vector s_D^μ.

Making use of the fact that L^μ is a linear differential operator, and that the only dependence on s_D^μ is in the density matrix $\slashed{p}_{t,D}$, we combine the last four terms to obtain a derivative of the leading order polarised amplitude. The final result reads

$$
|\mathcal{M}|^2 = -(J^\mu J_\mu + J_\mu(L^\mu + S_t^\mu))\, F_p(p_u, p_i, q_t, q_d, s_D), \tag{3.13}
$$

where

$$
F_p(p_u, p_i, q_t, q_d, s_D) = \mathrm{Tr}\left[\slashed{p}_{t,D}\mathbf{N}\slashed{p}_i\bar{\mathbf{N}}\right], \tag{3.14}
$$

is the matrix element squared for the production process with polarised top quark and where, by construction, derivatives with respect to momenta do not act on $\slashed{s}_D$. We also note that one can obtain the unpolarised result by simply setting $s_D \to 0$ in eq. (3.14) and multiplying the result by a factor 2. By a slight abuse of notation, we will write

$$
2F_p(p_u, \ldots, q_d, s_D \to 0) \equiv F_p(p_u, \ldots, q_d), \tag{3.15}
$$

in what follows.

Similarly to the stable-top case, the first term on the right hand side in eq. (3.13) contributes to the top quark production cross section starting at order $\mathcal{O}(\lambda^0)$, whereas the second and the third ones contribute at order $\mathcal{O}(\lambda)$. To extract $\mathcal{O}(\lambda)$ contributions from the first term in eq. (3.13), we redefine the momenta of various particles to remove the momentum

k from the energy-momentum conserving delta-function. However, an important difference between stable and unstable top cases occurs at this point because the top quark momentum appears both in the production and in the decay phase spaces. *Hence, redefinition of the top quark momentum in the production process leads to the redefinition of the top quark momentum in the decay and one needs to understand how to deal with it.*

To adhere as much as possible to the discussion of single top production and top decay that were studied separately in ref. [1], we employ the momenta redefinitions used there. We write

$$q_t = p_t - k + \frac{(p_t k)}{(p_t p_d)} p_d, \quad q_d = p_d - \frac{(p_t k)}{(p_t p_d)} p_d. \tag{3.16}$$

Repeating the steps described in ref. [1], we find

$$\mathrm{d}\Phi_P(p_u, p_i; q_d, q_t, k) = \mathrm{d}\Phi_P(p_u, p_i; p_d, p_t) \, [\mathrm{d}k]_\lambda \, \left(1 + \frac{(kp_d)}{(p_t p_d)} - \frac{(kp_t)}{(p_t p_d)} + \mathcal{O}(\lambda^2)\right), \tag{3.17}$$

where

$$[\mathrm{d}k]_\lambda = \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta_+(k^2 - \lambda^2). \tag{3.18}$$

In the stable-top case, once the above transformation is performed and relevant matrix elements squared are expanded in k, integration over k becomes possible. The same happens when decay is considered except that we need to account for the change in the differential decay width and the decay phase space introduced by momenta redefinitions in eq. (3.16).

Momenta redefinitions are only relevant for the leading $\mathcal{O}(\lambda^{-2})$ term in eq. (3.13). Its contribution to the cross section is proportional to

$$\mathrm{d}\Phi_P(\{q_{\mathrm{in}}\}; p_d, p_t) \, \mathrm{d}\Gamma_t(q_t, \{q_{\mathrm{dec}}\}) \, J_\mu^{(0)} J^{(0),\mu} \, F_p(\ldots, q_t, q_d, s_D(q_t, q_2))|_{q_t \to p_t + \ldots, q_d \to p_d + \ldots}, \tag{3.19}$$

where $\mathrm{d}\Phi_P$ is the phase space of the production subprocess after momenta redefinition and expansion in k, and $J_\mu^{(0)}$ is the leading power contribution to the eikonal current given in eq. (3.7). The required momenta shifts are shown in eq. (3.16). We note that the differential width that appears in the above expression depends on the *original* top quark momentum q_t. Furthermore, momenta redefinitions also affect the spin vector of the top quark s_D as it depends on the top quark momentum q_t.

To understand how to expand eq. (3.19) to first sub-leading order in the gluon momentum, we note that the top quark momentum redefinition in eq. (3.16) can be interpreted as a Lorentz transformation. Indeed, we can write eq. (3.16) as follows

$$q_t^\mu = \Lambda^{\mu\nu} p_{t,\nu}, \tag{3.20}$$

where

$$\Lambda^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu p_d^\nu - p_d^\mu k^\nu}{(p_t p_d)} = g^{\mu\nu} + \delta\Lambda^{\mu\nu}. \tag{3.21}$$

Since $\delta\Lambda^{\mu\nu}$ is an anti-symmetric traceless matrix, it can be interpreted as an infinitesimal Lorentz transformation.

This observation has important implications for the calculation of the differential decay width $\mathrm{d}\Gamma_t(q_t, \{q_{\mathrm{dec}}\})$ that appears in eq. (3.19). Indeed, after momentum transformation, it becomes

$$\mathrm{d}\Gamma_t(q_t, \{q_{\mathrm{dec}}\}) = \mathrm{d}\Gamma_t(\Lambda p_t, \{q_{\mathrm{dec}}\}) = \mathrm{d}\Gamma_t(\Lambda p_t, \{\Lambda p_{\mathrm{dec}}\}) = \mathrm{d}\Gamma_t(p_t, \{p_{\mathrm{dec}}\}), \tag{3.22}$$

where we also transformed momenta of the final state particles in the decay and made use of the fact that the differential width is invariant under Lorentz transformations.

Although the above result implies that momenta redefinitions in the production do not generate $\mathcal{O}(k)$ corrections in the unpolarised decay width that appears in eq. (3.19), the need to redefine momenta of final-state particles in the decay has implications for the spin vector s_D. Indeed, s_D depends on q_t and q_2 and, therefore, changes when the above momenta transformations are performed. It is easy to see that this change is described by a Lorentz boost

$$s_D^\mu(q_t, q_2) = \Lambda^{\mu\nu} s_{D,\nu}(p_t, p_2), \tag{3.23}$$

where $\Lambda^{\mu\nu}$ is given in eq. (3.21).

We are now in a position to write the result for the $\mathcal{O}(\lambda)$ contribution to the differential cross section of the process in eq. (2.1) due to an emission of a soft gluon in the production sub-process. It arises as a sum of the correction to the production matrix element described in eq. (3.13), correction to the production phase-space shown in eq. (3.17) and corrections to the leading order term shown in eq. (3.19) and discussed afterwards. Many of these contributions do not involve modifications of the spin vector s_D and are *identical* to contributions studied in ref. [1]. New spin-dependent contributions arise because of the spin-operator S_t in eq. (3.13) and because of the Lorentz boost of the spin-vector in eq. (3.23) that has to be inserted into the function F_p in eq. (3.19) and expanded in k.

Hence, we write

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD}^{R,\mathrm{prod}} \right] = 2 \frac{\mathrm{d}\Gamma_t}{\Gamma_t} \mathcal{T}_\lambda \left[\mathrm{d}\sigma_t^R(s_D) \right]_k + \mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\,\mathrm{new}}^{R,\mathrm{prod}} \right]_k, \tag{3.24}$$

where subscript k indicates that the integration over gluon momentum is still to be performed. Furthermore, the first term on the right hand side in eq. (3.24) is *the same as in the no-decay case* [1] except that one employs the polarised cross section for single top production, and the second term is new. It reads

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\mathrm{new}}^{R,\mathrm{prod}} \right]_k = -\frac{\mathrm{d}\Phi_P(\ldots; p_t, p_d)}{\mathcal{N}} [\mathrm{d}k]_\lambda \, 2 \frac{\mathrm{d}\Gamma_t}{\Gamma_t} J_\alpha^{(0)} \mathcal{O}^{(s),\alpha} F_p(\ldots, p_t, p_d, s_D(p_t, p_2)), \tag{3.25}$$

where

$$\mathcal{O}^{(s),\alpha} = S_t^\alpha - J^{(0),\alpha} s_{D,\mu} \, \delta\Lambda^{\mu\nu} \frac{\partial}{\partial s_D^\nu}. \tag{3.26}$$

To complete the computation of the real-emission contributions we need to integrate over the gluon momentum. This is straightforward since in both old and new contributions in eq. (3.24) the dependence on the gluon momentum k is exposed and one can use the formulas in appendix A of ref. [1] to calculate the relevant integrals over k. Hence, we find

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD}^{R,\mathrm{prod}} \right] = 2 \frac{\mathrm{d}\Gamma_t}{\Gamma_t} \mathcal{T}_\lambda \left[\mathrm{d}\sigma_t^R(s_D) \right] + \mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\mathrm{new}}^{R,\mathrm{prod}} \right]. \tag{3.27}$$

The first term on the right hand side of eq. (3.27) can be found in eq. (2.31) of ref. [1], where F_{LO} should be replaced with $F_p(\ldots, s_D)$, and the second (new) term reads

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\ \mathrm{new}}^{R,\mathrm{prod}} \right] = \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \, 2 \, \frac{\mathrm{d}\Gamma_t}{\Gamma_t} \, s_D^\mu \, W_{\mu\nu} \, \frac{\partial}{\partial s_{D,\nu}} \, \mathrm{d}\sigma_t(s_D). \tag{3.28}$$

The rank-two tensor in eq. (3.28) can be written as

$$W^{\mu\nu} = \omega_{dt}^{\mu\nu} - \omega_{it}^{\mu\nu} + \frac{2m_t^2(p_i p_d)}{(p_t p_i)(p_t p_d)} \omega_{id}^{\mu\nu}, \tag{3.29}$$

where the quantity

$$\omega_{xy}^{\mu\nu} = \frac{p_x^\mu p_y^\nu - p_y^\mu p_x^\nu}{(p_x p_y)}, \tag{3.30}$$

describes an infinitesimal boost in the $(p_x - p_y)$ plane. Eq. (3.28) provides an additional real emission contribution to the $\mathcal{O}(\lambda)$ terms when single top production process is combined with top quark decay in the narrow width approximation.

3.2 Virtual corrections and renormalisation in the production sub-process

As explained in ref. [1], the $\mathcal{O}(\lambda)$ contributions from the virtual corrections can be treated in the same way as the real emission ones. In this section we study the virtual corrections to the production sub-process of the process in eq. (2.1) and use the polarised top-quark spinor to describe the influence of the top quark decay.

Similarly to the real emission case, the $\mathcal{O}(\lambda)$ contributions to the virtual corrections can only arise from the region of soft $k \sim \lambda$ loop momenta. Our goal, therefore, is to establish a similar soft expansion of one-loop virtual corrections to the single top production process $u(p_u) + b(p_i) \to d(p_d) + t(p_t)$. We again consider corrections to the heavy-quark line since corrections to the light-quark line do not produce $\mathcal{O}(\lambda)$ contributions [19]. We write

$$\mathcal{A}_{\mathrm{virt}} = g_s^2 C_F \delta_{ij} \mathcal{M}_{\mathrm{virt}}, \tag{3.31}$$

where i, j are the colour indices of the top quark and the bottom quark. Proceeding as in ref. [1], we find

$$\mathcal{M}_{\mathrm{virt}} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2} \bigg[J_t^\alpha J_{i,\alpha} \, \bar{u}_t \left(\mathbf{N}(p_t, p_i, \ldots) + k^\mu D_{p,\mu} \mathbf{N}(p_t, p_i, \ldots) \right) u_i$$
$$- J_t^\alpha \bar{u}_t \mathbf{N}(p_t, p_i, \ldots) \mathbf{S}_{i,\alpha} u_i + J_i^\alpha \bar{u}_t \mathbf{S}_{t,\alpha} \mathbf{N}(p_t, p_i, \ldots) u_i - (J_t^\alpha + J_i^\alpha) \bar{u}_t D_{p,\alpha} \mathbf{N} u_i \bigg], \tag{3.32}$$

where $d_t = (p_t + k)^2 - m_t^2$ and $d_i = (p_i + k)^2$,

$$D_p^\mu = \frac{\partial}{\partial p_{t,\mu}} + \frac{\partial}{\partial p_{i,\mu}}, \tag{3.33}$$

and

$$J_t^\alpha = \frac{2p_t^\alpha + k^\alpha}{d_t}, \qquad \mathbf{S}_t^\alpha = \frac{\sigma^{\alpha\beta} k_\beta}{d_t},$$
$$J_i^\alpha = \frac{2p_i^\alpha + k^\alpha}{d_i}, \qquad \mathbf{S}_i^\alpha = \frac{\sigma^{\alpha\beta} k_\beta}{d_i}. \tag{3.34}$$

Similar to the real emission case, the dependence on the loop momentum has been made explicit so that the integration over k becomes straightforward. However, it is better to square the matrix element before integrating over the loop momentum k. We do this following ref. [1] and accounting for the fact that the top quark is polarised. We find

$$
\begin{aligned}
\delta_{\text{virt}}[\mathcal{M}\mathcal{M}^+] = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2} \bigg[& 2 J_t^\alpha J_{i,\alpha} \mathrm{Tr}\left[\slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i \bar{\mathbf{N}} \right] \\
& + J_t^\alpha J_{i,\alpha} k^\mu \, \mathrm{Tr}\left[\slashed{\phi}_{t,D} (D_{p,\mu}\mathbf{N}) \slashed{p}_i \bar{\mathbf{N}} + \slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i (D_{p,\mu}\bar{\mathbf{N}}) \right] \\
& - (J_t^\alpha + J_i^\alpha) \mathrm{Tr}\left[\slashed{\phi}_{t,D} (D_{p,\alpha}\mathbf{N}) \slashed{p}_i \bar{\mathbf{N}} + \slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i (D_{p,\alpha}\bar{\mathbf{N}}) \right] \\
& + J_i^\alpha \mathrm{Tr}\left[[\slashed{\phi}_{t,D}, \mathbf{S}_{t,\alpha}] \mathbf{N} \slashed{p}_i \bar{\mathbf{N}} \right] - J_t^\alpha \mathrm{Tr}\left[\slashed{\phi}_{t,D} \mathbf{N} [\mathbf{S}_{i,\alpha}, \slashed{p}_i] \bar{\mathbf{N}} \right] \bigg],
\end{aligned}
\tag{3.35}
$$

where $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_{\text{virt}}$. The above equation contains all $\mathcal{O}(\lambda)$ corrections to $\mathcal{M}\mathcal{M}^+$.

We can further simplify eq. (3.35) following steps already discussed in the previous section where the real emission contribution was considered. Indeed, using

$$
\begin{aligned}
[\slashed{\phi}_{t,D}, \mathbf{S}_t^\alpha] &= (L_t^\alpha + S_t^\alpha)\slashed{\phi}_{t,D}, \\
[\slashed{p}_i, \mathbf{S}_i^\alpha] &= L_i^\alpha \slashed{p}_i,
\end{aligned}
\tag{3.36}
$$

we arrive at

$$
\begin{aligned}
\delta[\mathcal{M}\mathcal{M}^+]_{\text{virt}} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2} \bigg[& 2 J_t^\alpha J_{i,\alpha} \mathrm{Tr}\left[\slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i \bar{\mathbf{N}} \right] \\
& + J_t^\alpha J_{i,\alpha} k^\mu \, \mathrm{Tr}\left[\slashed{\phi}_{t,D} (D_{p,\mu}\mathbf{N}) \slashed{p}_i \bar{\mathbf{N}} + \slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i (D_{p,\mu}\bar{\mathbf{N}}) \right] \\
& - (J_t^\alpha + J_i^\alpha) \mathrm{Tr}\left[\slashed{\phi}_{t,D} (D_{p,\alpha}\mathbf{N}) \slashed{p}_i \bar{\mathbf{N}} + \slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i (D_{p,\alpha}\bar{\mathbf{N}}) \right] \\
& + J_i^\alpha \mathrm{Tr}\left[((L_{t,\alpha} + S_{t,\alpha})\slashed{\phi}_{t,D}) \mathbf{N} \slashed{p}_i \bar{\mathbf{N}} \right] + J_t^\alpha \mathrm{Tr}\left[\slashed{\phi}_{t,D} \mathbf{N} (L_{i,\alpha}\slashed{p}_i) \bar{\mathbf{N}} \right] \bigg].
\end{aligned}
\tag{3.37}
$$

We stress that, similar to the real emission case, derivatives with respect to momenta do not act on the spin vector s_D that appears in the density matrix $\slashed{\phi}_{t,D}$.

We note that the difference between the result in eq. (3.37) and a similar result computed for the unpolarised case in ref. [1], is the appearance of the spin-dependent density matrix $\slashed{\phi}_{t,D}$ everywhere in eq. (3.37) and the presence of an additional term that contains the operator $S_{t,\alpha}$. This term evaluates to

$$
\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2} J_i^\alpha S_{t,\alpha} \mathrm{Tr}\left[\slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i \bar{\mathbf{N}} \right] = -\frac{1}{8\pi^2} \frac{\pi\lambda}{m_t} \frac{(p_i s_D)}{(p_i p_t)} p_t^\nu \frac{\partial}{\partial s_D^\nu} \mathrm{Tr}\left[\slashed{\phi}_{t,D} \mathbf{N} \slashed{p}_i \bar{\mathbf{N}} \right],
\tag{3.38}
$$

as follows from the integrals collected in appendix A of ref. [1]. Hence, we can write the virtual contribution as

$$
\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD}^{V,\text{prod}} \right] = 2\frac{\mathrm{d}\Gamma_t}{\Gamma_t} \mathcal{T}_\lambda \left[\mathrm{d}\sigma_t^V(s_D) \right] + \mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\text{new}}^{V,\text{prod}} \right],
\tag{3.39}
$$

where the first term on the right hand side can be found in eq. (3.16) of ref. [1] and the second term is new. We note that in the first term we again need to replace F_{LO} with

$F_p(\ldots, s_D)$ and $(\not{p}_t + m_t)$ with $\not{\rho}_{t,D}$ to account for the polarisation effects. The second term in eq. (3.39) is a new contribution. It reads

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\mathrm{new}}^{V,\mathrm{prod}} \right] = -\frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \, 2 \, \frac{\mathrm{d}\Gamma_t}{\Gamma_t} \, s_{D,\mu} \, \omega_{it}^{\mu\nu} \, \frac{\partial}{\partial s_D^\nu} \, \mathrm{d}\sigma_t(s_D). \tag{3.40}$$

Finally, contributions due to wave function and (explicit) mass renormalisation are not affected by the fact that the top quark is polarised. Hence, we conclude that these contributions can be borrowed from ref. [1] without any modification

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD}^{\mathrm{Ren,prod}} \right] = 2 \frac{\mathrm{d}\Gamma_t}{\Gamma_t} \, \mathcal{T}_\lambda \left[\mathrm{d}\sigma_t^{\mathrm{Ren}}(s_D) \right]. \tag{3.41}$$

To summarise, $\mathcal{O}(\lambda)$ contributions to the cross section of the process in eq. (2.1) caused by the radiation of real and virtual gluons and the renormalisation in the *production* sub-process are obtained as the sum of the contributions given in eqs. (3.27), (3.39) and (3.41). Each of these contributions is written as the sum of two terms: the "old one" that are identical to the stable-top production case discussed in ref. [1], except for the fact that one has to employ there the polarised leading order cross section, and the "new one" which is entirely due to the fact that there are spin correlations between production and decay processes. When single top production was considered in isolation, "old corrections" were cancelling against the redefinition of the top quark mass parameter; a similar cancellation also exists in the current case. However, before discussing this point, we need to compute the $\mathcal{O}(\lambda)$ power correction to the decay sub-process. We do this in the next section.

4 Corrections to the decay sub-process

In this section we explain how the $\mathcal{O}(\lambda)$ power correction to the top quark decay sub-process is computed. Following the discussion in section 2, the top quark is polarised and its polarisation vector s_P is determined by the kinematics of the production sub-process. We note that corrections to the decay of an unpolarised top quark considered in isolation can be found in appendix B in ref. [1].

Our starting point is eq. (2.16). The momenta assignments differ from the ones in appendix B of ref. [1]; for this reason we emphasise that we consider the decay process

$$t(p_t) \to \nu(p_1) + e^+(p_2) + b(p_f). \tag{4.1}$$

The calculation of the real-emission contributions proceeds similarly to the case of the single top production and along the lines of appendix B of ref. [1]. We use the following momenta assignment to describe the real-emission process

$$t(p_t) \to \nu(p_1) + e^+(q_2) + b(q_f) + g(k), \tag{4.2}$$

with $k^2 = \lambda^2$. We again use the Low-Burnett-Kroll theorem [20–22] shown in eq. (3.13) where for the process in eq. (4.2) we have

$$J^\mu = J_t^\mu + J_f^\mu, \quad J_t^\mu = \frac{2p_t^\mu - k^\mu}{d_t}, \quad J_f^\mu = \frac{2q_f^\mu + k^\mu}{d_f}, \tag{4.3}$$

with $d_t = (p_t - k)^2 - m_t^2$ and $d_f = (q_f + k)^2$.

In order to factorise the phase space of the gluon momentum from the rest of the decay phase space, we employ a momentum mapping. In variance with the case of the production sub-process, this mapping does not need to involve the top quark momentum and, hence, the production process remains unaffected. Following ref. [1], we write

$$q_f = p_f - k + \frac{(p_f k)}{(p_2 p_f)} \, p_2, \quad q_2 = \left(1 - \frac{(p_f k)}{(p_2 p_f)}\right) p_2. \tag{4.4}$$

Using this transformation, the phase space changes as follows [1]

$$\mathrm{d}\Phi_D(p_t; p_1, q_2, q_f, k) = \mathrm{d}\Phi_D(p_t; p_1, p_2, p_f) \, [\mathrm{d}k]_\lambda \left(1 + \frac{(p_2 k)}{(p_f p_2)} - \frac{(p_f k)}{(p_f p_2)} + \mathcal{O}(\lambda^2)\right). \tag{4.5}$$

Since these momenta transformations do not impact p_t and, therefore, the "spin" of the top quark as defined by the production process, the only addition to the unpolarised case for the width arises because of the analog of the S_t^μ term in eq. (3.13) which is already $\mathcal{O}(\lambda)$ and, hence, can be easily integrated over k. We therefore find

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD}^{R,\mathrm{dec}}\right] = 2 \frac{\mathcal{T}_\lambda \left[\mathrm{d}\Gamma_t^R(s_P)\right]}{\Gamma_t} \, \mathrm{d}\sigma_t + \mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\;\mathrm{new}}^{R,\mathrm{dec}}\right], \tag{4.6}$$

where in the first term formulas from unpolarised case can be employed except that the leading order matrix element squared should be replaced with the polarised one. The second term is new; after integration over the momentum of the soft gluon it evaluates to

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\mathrm{new}}^{R,\mathrm{dec}}\right] = -\frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \, 2 \, \frac{\mathrm{d}\sigma_t}{\Gamma_t} \, s_{P,\mu} \, \omega_{tf}^{\mu\nu} \frac{\partial}{\partial s_P^\nu} \, \mathrm{d}\Gamma_t(s_P). \tag{4.7}$$

We also need to compute virtual corrections and perform mass and wave function renormalisation for the decay sub-process. The virtual corrections are computed in the same way as what was described for the production sub-process and in appendix B of ref. [1]. We find

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD}^{V,\mathrm{dec}}\right] = 2 \frac{\mathcal{T}_\lambda \left[\mathrm{d}\Gamma_t^V(s_P)\right]}{\Gamma_t} \, \mathrm{d}\sigma_t + \mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\mathrm{new}}^{V,\mathrm{dec}}\right], \tag{4.8}$$

where

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD,\mathrm{new}}^{V,\mathrm{dec}}\right] = \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \, 2 \, \frac{\mathrm{d}\sigma_t}{\Gamma_t} \, s_{P,\mu} \, \omega_{tf}^{\mu\nu} \frac{\partial}{\partial s_P^\nu} \, \mathrm{d}\Gamma_t(s_P). \tag{4.9}$$

The renormalisation contributions are not affected by the polarisation of the top quark and, therefore, can be directly borrowed from the results in appendix B in ref. [1] except that the differential decay width has to be computed for the polarised top quark. Hence, we write

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_{PD}^{\mathrm{Ren,dec}}\right] = 2 \frac{\mathcal{T}_\lambda \left[\mathrm{d}\Gamma_t^{\mathrm{Ren}}(s_P)\right]}{\Gamma_t} \, \mathrm{d}\sigma_t. \tag{4.10}$$

5 Redefinition of the top quark mass parameter

In cases when the top quark production and the top quark decay are considered separately, it is known [1] that the cancellation of $\mathcal{O}(\lambda)$ contributions is only possible if the production cross section and the decay rate are expressed through a short-distance top quark mass and *not through the pole mass*. For reasons of technical convenience, we performed the renormalisation in the on-shell scheme, similar to what was done in ref. [1].[6] To derive the final results, we need to switch to a short-distance mass parameter. We explain below how to do this in case when top quark production and decay are considered simultaneously.

As explained in ref. [1], we can switch to a short-distance mass parameter by redefining momenta of final-state particles. Our goal will be to do this in such a way that, when spin correlations are neglected, we obtain formulas which are identical to the ones in ref. [1], where production and decay are considered separately.

We begin with the momenta transformations for particles that originate from top decay and write

$$q_f^\mu = \tilde{p}_f^\mu - \kappa q_t + \kappa \frac{(\tilde{p}_f q_t)}{(\tilde{p}_f \tilde{p}_2)}\, \tilde{p}_2^\mu, \quad q_2^\mu = \tilde{p}_2^\mu \left(1 - \kappa \frac{(\tilde{p}_f q_t)}{(\tilde{p}_f \tilde{p}_2)}\right). \tag{5.1}$$

This momentum transformation leads to the following change in the decay phase space [1]

$$\mathrm{d}\Phi_D(q_t; q_f, q_2, q_1) = \mathrm{d}\Phi_D((1+\kappa)q_t; \tilde{p}_f, \tilde{p}_2, q_1)\left(1 + \kappa\frac{(\tilde{p}_2 q_t)}{(\tilde{p}_f \tilde{p}_2)} - \kappa\frac{(\tilde{p}_f q_t)}{(\tilde{p}_f \tilde{p}_2)} + \mathcal{O}(\lambda^2)\right). \tag{5.2}$$

It follows from the above equation that the mass of the decaying top quark becomes

$$\tilde{m}_t = \sqrt{(1+\kappa)^2 q_t^2} = (1+\kappa)m_t. \tag{5.3}$$

Hence,

$$\tilde{m}_t - m_t = \kappa m_t, \tag{5.4}$$

which implies that κm_t is the shift in the mass parameter and where κ is defined as in ref. [1],

$$\kappa = \frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}. \tag{5.5}$$

To proceed further, it is convenient to define the top quark momentum that appears in the decay phase space

$$\tilde{p}_t = (1+\kappa)q_t. \tag{5.6}$$

When the production and decay processes are considered together, the top quark momentum appears in the phase space of the production sub-process; hence, the above redefinition will modify the production phase space and the matrix element. We begin with the analysis of the production phase space and write[7]

$$\mathrm{d}\Phi_P(\ldots; q_t, q_d) = \frac{\mathrm{d}^4 q_t}{(2\pi)^3}\delta(q_t^2 - m_t^2)\,[\mathrm{d}q_d]\,(2\pi)^4\delta(p_u + p_i - q_t - q_d)$$
$$= (1 - 2\kappa)\frac{\mathrm{d}^4\tilde{p}_t}{(2\pi)^4}\delta(\tilde{p}_t^2 - \tilde{m}_t^2)[\mathrm{d}q_d](2\pi)^4\delta(p_u + p_i - \tilde{p}_t + \kappa\tilde{p}_t - q_d). \tag{5.7}$$

---

[6]A calculation that directly uses the mass parameter defined in a short-distance scheme, see the appendix.

[7]To derive this formula, one needs to account for the fact that $q_t^2 \neq m_t^2$ a priori. Hence, $\delta(q_t^2 - m_t^2) = \delta((1 - 2\kappa)(\tilde{p}_t^2 - \tilde{m}_t^2)) = (1 + 2\kappa)\delta(\tilde{p}_t^2 - \tilde{m}_t^2)$.

We then perform one more momentum redefinition but this time we change the momenta in such a way that the top quark remains on a (new) mass shell. We write

$$\tilde{p}_t = p_t + \kappa p_t - \frac{\kappa m_t^2}{(p_t p_d)} p_d, \qquad q_d = \left(1 + \frac{\kappa m_t^2}{(p_t p_d)}\right) p_d. \tag{5.8}$$

This gives

$$\mathrm{d}\Phi_P(p_u, p_i; q_t, q_d) = \mathrm{d}\Phi_P(p_u, p_i; p_t, p_d)\left(1 + \frac{\kappa m_t^2}{(p_t p_d)} + \mathcal{O}(\lambda^2)\right), \tag{5.9}$$

where $p_t^2 = \tilde{m}_t^2$.

We note that the change $\tilde{p}_t \to p_t$ impacts the decay phase space *again*. However, it is easy to solve this problem because this momentum change can be written as a Lorentz transformation

$$\tilde{p}_t^\mu = \Lambda_m^{\mu\nu} p_{t,\nu}, \tag{5.10}$$

where

$$\Lambda_m^{\mu\nu} = g^{\mu\nu} + \kappa \, \omega_{td}^{\mu\nu}. \tag{5.11}$$

It follows that the decay phase transforms as follows

$$\mathrm{d}\Phi_D\left(\tilde{p}_t; \{\tilde{p}_{\mathrm{dec}}\}\right) = \mathrm{d}\Phi_D(\Lambda_m p_t, \{\Lambda_m p_{\mathrm{dec}}\}) = \mathrm{d}\Phi_D(p_t, \{p_{\mathrm{dec}}\}). \tag{5.12}$$

We have worked out the momenta transformations required to modify the mass of a top quark in a process where it is produced and then decays. We now need to combine the several transformations and write down formulas that elucidate phase-space and matrix-element transformations of the full process. We use the representation shown in eq. (2.14) where unpolarised decay width and polarised production cross section are combined. We use the invariance of the decay matrix element squared F_d under Lorentz transformations and find

$$\mathrm{d}\Phi_P(p_u, p_i; q_t, q_d) F_p(p_u, p_i; q_t, q_d, s_D(q_t, q_2)) \, \mathrm{d}\Phi_D(q_t; q_f, q_2, q_1) F_d(q_t; q_f, q_2, q_1)$$

$$= \mathrm{d}\Phi_P \, j_p \, F_p\Big(\ldots; \, p_t - \kappa\xi_d \, p_d, \, (1 + \kappa\xi_d) \, p_d, \, \Lambda_m s_D\big((1 - \kappa) \, p_t, (1 - \kappa\xi_2) \, p_2\big)\Big) \tag{5.13}$$

$$\times \, \mathrm{d}\Phi_D \, j_d \, F_d\Big((1 - \kappa) \, p_t; \, p_f + \kappa\delta p_f, \, (1 - \kappa\xi_2) \, p_2, \, p_1\Big) + \mathcal{O}(\kappa^2),$$

where the phase spaces $\mathrm{d}\Phi_{p,d}$ depend on the transformed momenta $\{p\}$, Λ_m is the boost defined in eq. (5.11) and

$$j_p = 1 + \frac{\kappa m_t^2}{(p_t p_d)}, \qquad j_d = 1 + \kappa \, \frac{(p_2 p_t)}{(p_f \tilde{p}_2)} - \kappa \, \frac{(p_f p_t)}{(p_f p_2)},$$

$$\xi_d = \frac{m_t^2}{(p_t p_d)}, \qquad \xi_2 = \frac{(p_f p_t)}{(p_f p_2)}, \qquad \delta p_f^\mu = -p_t^\mu + \frac{(p_f p_t)}{(p_f p_2)} \, p_2^\mu. \tag{5.14}$$

To obtain $\mathcal{O}(\lambda)$ correction to the cross section of the process in eq. (2.1) related to mass redefinition, we expand eq. (5.13) in κ and keep linear terms. These terms can be combined into three groups:

1. the term that originates from the expansion of F_p caused by the $\mathcal{O}(\kappa)$ contribution to the matrix Λ_m acting on s_D;

2. all $\mathcal{O}(\kappa)$ terms that appear from the expansion of the second line in eq. (5.13) but without a term discussed in the previous item *and* without a correction to the argument of the spin vector s_D;

3. terms that originate from the expansion of the third line in eq. (5.13) in κ *and* terms that originate from the expansion of the argument of spin vector s_D in function F_p.

We now discuss these three groups of terms separately. The term in the first item is new. Terms in the second item provide the required contribution to cancel all "old" $\mathcal{O}(\lambda)$ corrections to the production sub-process discussed in section 3. Note that this cancellation occurs for the polarised matrix element squared $F_p(\ldots, s_D)$ since this is what appears in eq. (5.13).

The contribution of the third group of terms should, in principle, cancel all "old" $\mathcal{O}(\lambda)$ terms to the decay sub-process, described in section 4. However, it follows from eq. (5.13) that this contribution lacks the polarisation vector s_P, which is present in the similar contributions in section 4. Hence, to claim this cancellation, we need to put it back into the decay matrix element squared. This is possible because the following identity holds

$$F_p\Big(\ldots; p_t, p_d, s_D\big((1-\kappa)\,p_t,\ (1-\kappa\xi_2)\,p_2\big)\Big)F_d\Big((1-\kappa)\,p_t;\ p_f+\kappa\delta p_f,\ (1-\kappa\xi_2)p_2,\ p_1\Big)$$

$$= F_p\Big(\ldots; p_t, p_d\Big)F_d\Big((1-\kappa)\,p_t;\ p_f+\kappa\delta p_f,\ (1-\kappa\xi_2)p_2,\ p_1,\ s_P(p_t,p_d)\Big), \tag{5.15}$$

thanks to the relation between polarised production and decay cross sections shown in eq. (2.20).

We conclude that the only new term that we need to consider is the term in the first item that arises from the boost of the spin vector. It evaluates to

$$\mathcal{T}_\lambda\,[\mathrm{d}\sigma_{\mathrm{mass}}^{\mathrm{new}}] = -\frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}\,2\,\frac{\mathrm{d}\Gamma_t}{\Gamma_t}s_{D,\mu}\,\omega_{td}^{\mu\nu}\,\frac{\partial}{\partial s_D^\nu}\,\mathrm{d}\sigma_t(s_D). \tag{5.16}$$

Other terms that arise from the mass redefinition combine with "old" contributions to the production and decay sub-processes and cancel in the same way as discussed in ref. [1].

6 Results and corrections to observables

The final result is obtained by combining the $\mathcal{O}(\lambda)$ contributions to single top production and decay process derived in sections 3, 4 and 5. As we argued extensively during the calculation many $\mathcal{O}(\lambda)$ contributions cancel in the sum; the only ones that survive involve polarisation effects which is an important new feature of a process with a long-lived particle that is first produced and then decays. They are obtained by adding eqs. (3.28), (3.40), (4.7), (4.9), (5.16). We find

$$\mathcal{T}_\lambda[\mathrm{d}\sigma_{PD}] = \frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}\,2\,\frac{\mathrm{d}\Gamma_t}{\Gamma_t}\,s_{D,\mu}\left(2\omega_{ti}^{\mu\nu}+2\omega_{dt}^{\mu\nu}+\frac{2m_t^2(p_ip_d)}{(p_tp_i)(p_tp_d)}\omega_{id}^{\mu\nu}\right)\frac{\partial}{\partial s_D^\nu}\,\mathrm{d}\sigma_t(s_D). \tag{6.1}$$

We can use eq. (2.20) as well as the relations $s_P \cdot p_t = s_D \cdot p_t = 0$, to find

$$\mathcal{T}_\lambda[\mathrm{d}\sigma_{PD}] = -\frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}\,\frac{\mathrm{d}\Gamma_t\mathrm{d}\sigma_t}{\Gamma_t}\,\frac{2m_t^2(p_ip_d)}{(p_tp_i)(p_tp_d)}\,s_{D,\mu}\omega_{id}^{\mu\nu}s_{P,\nu}. \tag{6.2}$$

The expression in eq. (6.2) assumes a particularly simple form in the *top quark rest frame*. Indeed, in this case

$$s_D^\mu = (0, \vec{n}_2), \quad s_P^\mu = (0, \vec{n}_d), \tag{6.3}$$

where $\vec{n}_2$ and $\vec{n}_d$ are unit vectors aligned with directions of the positron and d-quark in this frame, respectively. This implies that

$$\frac{2m_t^2(p_i p_d)}{(p_t p_i)(p_t p_d)} s_{D,\mu} \omega_{id}^{\mu\nu} s_{P,\nu} = 2\left((\vec{n}_2 \cdot \vec{n}_i) - (\vec{n}_2 \cdot \vec{n}_d)(\vec{n}_i \cdot \vec{n}_d)\right) = 2\left[\vec{n}_2 \times \vec{n}_d\right] \cdot \left[\vec{n}_i \times \vec{n}_d\right], \tag{6.4}$$

where $\vec{n}_i$ is the direction of the incoming b quark in the top quark rest frame. We conclude that, in the top quark rest frame, eq. (6.2) takes a remarkably simple form

$$\mathcal{T}_\lambda[\mathrm{d}\sigma_{PD}] = -\frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \frac{\mathrm{d}\Gamma_t \mathrm{d}\sigma_t}{\Gamma_t} 2 \left[\vec{n}_2 \times \vec{n}_d\right] \cdot \left[\vec{n}_i \times \vec{n}_d\right]. \tag{6.5}$$

For the case, when the observable does not depend on decay momenta, we recover our previous result presented in ref. [1]. Indeed, integrating over the total phase space of the decay products and using the fact that

$$\int \mathrm{d}\Gamma_t \, \vec{n}_2 = 0, \tag{6.6}$$

in the top quark rest frame, we recover the stable-top-quark result [1]

$$\mathcal{T}_\lambda[\sigma_{PD}] = 0. \tag{6.7}$$

Nevertheless, eq. (6.5) shows that, in general, there *is* an $\mathcal{O}(\lambda)$ contribution to the differential cross section related to polarisation effects. However, because we have used momenta redefinitions to derive this result, it is important to account for them also in the *observables* since they are defined using the original momenta.

To this end, we consider an observable X and study the following integral

$$O_X = \int \mathrm{d}\sigma_{PD} \, X. \tag{6.8}$$

In principle, the observable X is generic; however, we would like to focus upon observables that are used in practice to study polarisation effects in single top production [14–16]. For this reason, we assume that the observable X depends on the top quark momentum, the d-quark momentum, the incoming b-quark momentum (the collision axis) and the positron momentum

$$X = X(q_t, q_d, q_i, q_2). \tag{6.9}$$

When different contributions to eq. (6.8) are studied and different mappings are performed, there will be shifts in the arguments of the function X that are proportional to the gluon momentum k or to the mass-redefinition parameter κ. We are interested in terms that originate in the expansion of the function X in these small parameters.

There are three contributions that affect the arguments of X: real radiation in production, real radiation in decay and mass redefinition. As the first step, we summarise the momenta redefinitions for each of these contributions. Since none of these momenta redefinitions changes the collision axis, we will not show q_i among the arguments of X in what follows. We find:

- Radiation in the production subprocess:

$$X \to X \left(\Lambda p_t, \left(1 - \frac{(p_t k)}{(p_t p_d)} \right) p_d, \Lambda p_2 \right), \tag{6.10}$$

where

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \frac{p_d^\mu k^\nu - k^\mu p_d^\nu}{(p_t p_d)}. \tag{6.11}$$

- Radiation in the decay subprocess:

$$X \to X \left(p_t, p_d, \left(1 - \frac{(p_f k)}{(p_f p_2)} \right) p_2 \right), \tag{6.12}$$

- Mass redefinition:

$$X \to X \left((1-\kappa) \Lambda_m p_t, \left(1 + \kappa \frac{m_t^2}{(p_t p_d)} \right) p_d, \left(1 - \kappa \frac{(p_f p_t)}{(p_f p_2)} \right) \Lambda_m p_2 \right), \tag{6.13}$$

where

$$\Lambda_m^{\mu\nu} = g^{\mu\nu} + \kappa \frac{p_t^\mu p_d^\nu - p_d^\mu p_t^\nu}{(p_t p_d)}. \tag{6.14}$$

We then expand X in series for each of the three contributions and integrate over the gluon momentum k where appropriate. This is straightforward, and the matrix element squared is only needed in the eikonal approximation. The result reads

$$\delta X = \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \left[\left(p_t^\mu - \frac{2m_t^2}{(p_t p_i)} p_i^\mu \right) \frac{\partial X}{\partial p_t^\mu} - 2p_{2,\nu} \left(\omega_{td}^{\nu\mu} + \frac{m_t^2 (p_d p_i)}{(p_t p_d)(p_t p_i)} \omega_{di}^{\nu\mu} \right) \frac{\partial X}{\partial p_2^\mu} \right]. \tag{6.15}$$

We note that the above result assumes that the mass parameter does not appear in the definition of the observable; if this is not the case, the mass parameter needs to be replaced with $\sqrt{p_t^2}$.

Eq. (6.15) is applicable to any observable; the only constraint is that it can only depend on the momenta of final-state particles shown in eq. (6.9). Hence, we conclude that the complete linear correction to the expectation value of such an observable reads

$$\mathcal{T}_\lambda[\mathcal{O}_X] = \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \int \frac{d\Gamma_t d\sigma_t}{\Gamma_t} \left[- \frac{2m_t^2 (p_i p_d)}{(p_t p_i)(p_t p_d)} s_{D,\mu} \omega_{id}^{\mu\nu} s_{P,\nu} X \right.$$
$$\left. + (1 - s_D \cdot s_P) \left[\left(p_t^\mu - \frac{2m_t^2}{(p_t p_i)} p_i^\mu \right) \frac{\partial X}{\partial p_t^\mu} - 2p_{2,\nu} \left(\omega_{td}^{\nu\mu} + \frac{m_t^2 (p_d p_i)}{(p_t p_d)(p_t p_i)} \omega_{di}^{\nu\mu} \right) \frac{\partial X}{\partial p_2^\mu} \right] \right]. \tag{6.16}$$

We will now analyse this general formula. First we note that one can consider observables that depend on the top quark momentum, but are inclusive with respect to the momenta of its decay products. Then X is a function of p_t only. For such observables, we can integrate over the momenta of the top quark decay products. Then, considering the integrand in the top quark rest frame and using eq. (6.6), we conclude that the first term on the right hand side in eq. (6.16) vanishes. The second term then coincides with the correction to observable discussed in ref. [1] and the last term vanishes if X is a function of p_t only.

There are also observables that are designed to study polarisation effects in single top production. Perhaps the simplest observable that belongs to this class is the one used by the CMS collaboration where the asymmetry between the direction of the outgoing light jet in single top production (d-jet in our case) and the direction of positron in top decay is studied in the top quark rest frame [15]. We can construct such an observable by simply multiplying the production and decay spin polarisation vectors, s_P and s_D. Since in the top rest frame

$$s_D \cdot s_P = -\vec{n}_2 \cdot \vec{n}_d = -\cos\theta_{d2}, \tag{6.17}$$

any function of this variable will provide a probe of polarisation effects; the observable used by the CMS collaboration corresponds to

$$X_{\text{CMS}} = \theta(-s_D s_P) - \theta(s_D s_P). \tag{6.18}$$

Using eq. (6.15), it is easy to show that

$$\delta X_{\text{CMS}} = 0, \tag{6.19}$$

which then implies that the only relevant term in eq. (6.16) that contributes for such observables is the first term in the integrand of eq. (6.16).

It is interesting to note that one can arrive at the same result without any computation. In fact, there is a simple argument that can be used to argue that for any observable X that depends upon the *directions* of p_t, p_d and p_2 *only*[8] there cannot be any change in X after the remapping described in this paper. To illustrate this argument, consider radiation in the production. According to eq. (6.10) the momenta redefinitions lead to

$$X(p_t, p_d, p_2) \to X\left(\Lambda p_t, \left(1 - \frac{(p_t k)}{(p_t p_d)}\right) p_d, \Lambda p_2\right) = X\left(\Lambda p_t, \left(1 + \frac{(p_d k)}{(p_t p_d)}\right) p_d, \Lambda p_2\right), \tag{6.20}$$

where in the last step we used the fact that the observable X depends on the direction of p_d. This implies that the exact form of rescaling is irrelevant, and we can change it at will. Since

$$\left(1 + \frac{(p_d k)}{(p_t p_d)}\right) p_d = \Lambda p_d, \tag{6.21}$$

we find

$$X\left(\Lambda p_t, \left(1 + \frac{(p_d k)}{(p_t p_d)}\right) p_d, \Lambda p_2\right) = X\left(\Lambda p_t, \Lambda p_d, \Lambda p_2\right) = X\left(p_t, p_d, p_2\right), \tag{6.22}$$

where in the last step Lorentz invariance of the observable was used. Hence, we conclude that the momenta redefinitions employed in the description of the real emission in production do not change an observable which depends on directions of final-state particles. The same reason also applies to the momenta transformations employed to describe radiation in decay and the mass redefinition.

To complete the analysis of the CMS asymmetry, we need to understand the fate of the first term in eq. (6.16). Considering this term in the top rest frame, we find that it involves the following integral

$$\int d\Gamma_t \left[\vec{n}_2 \times \vec{n}_d\right] X(\vec{n}_2 \cdot \vec{n}_d). \tag{6.23}$$

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[8]We note that s_D and s_P belong to this category.

Since for any function X

$$\int \mathrm{d}\Gamma_t \, \vec{n}_2 \, X(\vec{n}_2 \cdot \vec{n}_d) \sim \vec{n}_d, \tag{6.24}$$

the integral in eq. (6.23) vanishes. We conclude that the asymmetries in single top production studied by the CMS collaboration [14, 15] are not affected by the non-perturbative effects that can be modelled with renormalons.

A more complex polarisation observable was studied by the ATLAS collaboration [16]. To define it, a reference system in the top rest frame is introduced, where the three axes are[9]

$$\vec{e}_z = \vec{n}_d, \quad \vec{e}_y = \frac{\vec{n}_i \times \vec{n}_d}{|\vec{n}_i \times \vec{n}_d|}, \quad \vec{e}_x = \vec{e}_y \times \vec{e}_z = \frac{\vec{n}_i \times \vec{n}_d}{|\vec{n}_i \times \vec{n}_d|} \times \vec{n}_d. \tag{6.25}$$

The observable Q is defined as follows

$$Q(\vec{n}_2, \{\vec{e}\}) = 4\theta(\vec{n}_2 \cdot \vec{e}_z) + 2\theta(\vec{n}_2 \cdot \vec{e}_x) + \theta(\vec{n}_2 \cdot \vec{e}_y). \tag{6.26}$$

We now determine the expectation value of Q at leading order and the non-perturbative correction to it. First, writing the leading order cross section using the reference frame described above, we obtain

$$\mathrm{d}\sigma_{PD} = \mathrm{d}\sigma_t \frac{\mathrm{d}\Gamma_t}{\Gamma_t} \left(1 + \vec{e}_z \cdot \vec{n}_2\right). \tag{6.27}$$

If we integrate over the top quark decay products without imposing any cuts on final-state particles, the following equations hold

$$\int \mathrm{d}\Gamma_t \, \theta(\vec{n}_2 \cdot \vec{a}) = \frac{1}{2}\Gamma_t, \quad \int \mathrm{d}\Gamma_t \, \vec{n}_2 \, \theta(\vec{n}_2 \cdot \vec{a}) = \frac{1}{4}\Gamma_t \, \vec{a}, \tag{6.28}$$

where $\vec{a}$ is an arbitrary unit vector. We use eq. (6.28) together with the leading order cross section in eq. (6.27) to find

$$\langle Q(\vec{n}_2, \{\vec{e}\}) \rangle = \frac{\int \mathrm{d}\sigma_{PD} \, Q(\vec{n}_2, \{\vec{e}\})}{\int \mathrm{d}\sigma_{PD}} = \frac{9}{2}. \tag{6.29}$$

To compute the power corrections to this result, we need to combine the corrections to the cross section and to the observable. We begin with the latter. The correction to the observable is computed using eq. (6.15). To apply this equation to the observable Q, we should write it in a Lorentz-covariant form. To this end, we write

$$Q(\vec{n}_2, \{\vec{e}\}) = 4\theta(\hat{Q}_z) + 2\theta(\hat{Q}_x) + \theta(\hat{Q}_y), \tag{6.30}$$

with

$$\begin{aligned}
\hat{Q}_x &= \vec{n}_2 \cdot \vec{e}_x = \frac{1}{|\vec{n}_i \times \vec{n}_d|} \left[(\vec{n}_2 \cdot \vec{n}_d)(\vec{n}_i \cdot \vec{n}_d) - (\vec{n}_i \cdot \vec{n}_2) \right], \\
\hat{Q}_y &= \vec{n}_2 \cdot \vec{e}_y = \frac{1}{|\vec{n}_i \times \vec{n}_d|} \frac{p_t^2}{(p_t p_2)(p_t p_i)(p_t p_d)} \, \epsilon_{\mu\nu\rho\sigma} \, p_t^\mu p_2^\nu p_i^\rho p_d^\sigma, \\
\hat{Q}_z &= \vec{n}_2 \cdot \vec{e}_z = \vec{n}_2 \cdot \vec{n}_d,
\end{aligned} \tag{6.31}$$

---

[9]We note that in our calculation $\vec{n}_i$ denotes the direction of the incoming b-quark in the top quark rest frame, whereas in the ATLAS paper [16] the direction of the incoming light quark is chosen to define the reference system. These vectors are not back-to-back in the top quark rest frame but, thanks to momentum conservation, in this reference frame their vector products with $\vec{n}_d$ are the same up to a sign.

and note that the covariant generalisation of the scalar product of two vectors in the top rest frame is given by

$$\vec{n}_i \cdot \vec{n}_j = 1 - \frac{p_t^2 (p_i p_j)}{(p_t p_i)(p_t p_j)}. \tag{6.32}$$

After that, the calculation becomes straightforward. We obtain

$$\delta \hat{Q}_x = 0, \qquad\qquad \delta \hat{Q}_y = 0, \qquad\qquad \delta \hat{Q}_z = 0, \tag{6.33}$$

so that also in this case there is no change in the observable

$$\delta Q = 0. \tag{6.34}$$

Written in the reference system defined in eq. (6.25), the non-perturbative shift in the cross section shown in eq. (6.5) becomes

$$\mathcal{T}_\lambda[\mathrm{d}\sigma_{PD}] = \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \frac{\mathrm{d}\Gamma_t \mathrm{d}\sigma_t}{\Gamma_t} \, 2 \, |\vec{n}_i \times \vec{n}_d| \, (\vec{e}_x \cdot \vec{n}_2). \tag{6.35}$$

We then integrate the product of this quantity with the observable Q over the top quark decay products, and find

$$
\begin{aligned}
\int \mathcal{T}_\lambda[\mathrm{d}\sigma_{PD}] \, Q(\vec{n}_2, \{\vec{e}\}) &= \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \, \mathrm{d}\sigma_t \frac{2|\vec{n}_i \times \vec{n}_d|}{\Gamma_t} \, \vec{e}_x \cdot \int \mathrm{d}\Gamma_t \, Q(\vec{n}_2, \{\vec{e}\}) \, \vec{n}_2 \\
&= \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \, \mathrm{d}\sigma_t \frac{4|\vec{n}_i \times \vec{n}_d|}{\Gamma_t} \, \vec{e}_x \cdot \int \mathrm{d}\Gamma_t \, \theta(\vec{n}_2 \cdot \vec{e}_x) \, \vec{n}_2 \\
&= \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \, \mathrm{d}\sigma_t \, |\vec{n}_i \times \vec{n}_d|.
\end{aligned} \tag{6.36}
$$

We use this equation to determine the non-perturbative correction to the expectation value of the observable Q

$$\langle Q \rangle = \frac{1}{\sigma_t} \int \mathrm{d}\sigma_t \left(\frac{9}{2} + \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} |\vec{n}_i \times \vec{n}_d| \right), \tag{6.37}$$

at fixed center-of-mass collision energy $\sqrt{s}$. We note that in the center-of-mass frame of partonic collision, the absolute value of the vector product of $\vec{n}_i$ and $\vec{n}_d$ reads

$$|\vec{n}_i \times \vec{n}_d| = \sqrt{\frac{4m_t^2 \, s \, t \, u}{(s - m_t^2)^2 (m_t^2 - t)^2}} = \frac{2m_t \, s \, p_{d\perp}}{(m_t^2 - t)(s - m_t^2)}, \tag{6.38}$$

where $p_{d\perp}$ is the transverse momentum of the d-jet relative to the collision axis. We note that in the last step we used the fact that $t \, u = s \, p_\perp^2$.

Integrating over the scattering angle, we find

$$\frac{1}{\sigma_t} \int \mathrm{d}\sigma_t \, |\vec{n}_i \times \vec{n}_d| = f_Q(s, m_t, m_W), \tag{6.39}$$

where the function f_Q reads

$$f_Q(s, m_t, m_W) = \frac{\pi m_t m_W \sqrt{s}\sqrt{\bar{s}} \left(-m_t^4 + m_t^2 (m_W^2 + s) - 2m_t m_W \sqrt{s}\sqrt{\bar{s}} + m_W^2 s \right)}{(m_t^2 - m_W^2)^2 (s - m_t^2)^2}, \tag{6.40}$$

and we have defined the quantity

$$\bar{s} = s - m_t^2 + m_W^2.$$ (6.41)

The non-perturbative correction to the expectation value of the variable Q in proton collisions is obtained by convoluting the above result with parton distribution functions. In principle, since the function $f_Q(s, m_t, m_W)$ depends on the center-of-mass energy, parton distribution functions do not decouple. However, in practice, $f(s, m_t, m_W)$ is a slowly changing function of s. Indeed, it changes from the value

$$\lim_{s \to m_t^2} f(s, m_t, m_W) = \frac{\pi}{4} \approx 0.785,$$ (6.42)

at the threshold, to

$$\lim_{s \to \infty} f(s, m_t, m_W) = \frac{\pi\, m_t\, m_W}{(m_t + m_W)^2} \approx 0.68,$$ (6.43)

at $s = \infty$ for physical values of m_t and m_W. Hence, we find the following estimate for the ATLAS variable Q in proton-proton collisions

$$\langle Q \rangle \approx \frac{9}{2} + \frac{\alpha_s C_F}{2\pi}\, \frac{\pi\lambda}{m_t}\, \frac{\pi}{4}.$$ (6.44)

The above result does not account for realistic event selection criteria which in many ways introduce additional directions into the integration over top quark decay products. However, it does illustrate the point that non-perturbative effects that we discuss in this paper have a small but direct impact on the measured values of the top quark polarisation observables at hadron colliders.

An outstanding problem in collider physics is the measurement of the top quark mass with an ultrahigh precision in a credible way [23]. The tricky issue is the control (or lack of it) of non-perturbative corrections, which is very hard to do for exclusive observables that are used currently for the highest-precision measurements. In this regard, suggestions were made to study lepton observables from top quark decay because they are considered to be less prone to contaminations by non-perturbative effects. Interestingly, our analysis allows us to make exact statements to this effect, albeit in the narrow width approximation.

To this end, consider the following quantity

$$L_\perp = |\vec{p}_2 \cdot \vec{e}_y|,$$ (6.45)

where $\vec{p}_2$ is the positron three-momentum in the top quark rest frame. The vector $\vec{e}_y$ is defined in eq. (6.25); it is orthogonal to the collision plane of single top production process. Hence, $L_\perp$ measures the component of the lepton momentum that points *outside* of the collision plane. We are interested in computing the average value of $L_\perp$. Since

$$\int d\Gamma_t\, \vec{n}_2\, \theta(\vec{p}_2 \cdot \vec{e}_y) \sim \vec{e}_y,$$ (6.46)

it follows that neither the term $s_P \cdot s_D$ in the leading order cross section, nor the power correction $\mathcal{T}_\lambda[d\sigma_{PD}]$ receive contribution from the above integral. Since it is also easy to check that $L_\perp$ does not receive any corrections from momenta redefinitions, $\delta L_\perp = 0$, it follows that

$$\langle L_\perp \rangle = \frac{1}{\Gamma_t} \int d\Gamma_t\, L_\perp + \mathcal{O}(\lambda^2) = \frac{1}{2\Gamma_t} \int d\Gamma_t\, \frac{(p_2 p_t)}{m_t} = \frac{m_t^2 + m_W^2}{8 m_t} + \mathcal{O}(\lambda^2),$$ (6.47)

where in the last step we employed the narrow width approximation for the W-boson to integrate over the positron momentum.

In summary, the point of the above calculation is to demonstrate that there are no linear power corrections to $L_\perp$, so that the (short-distance) top quark mass can be determined from this observable with very high precision. An obvious reservation is that the above calculation is valid in the case when no fiducial cuts are imposed on final-state particles but it is also obvious that to perform such measurements in practice, cutting-edge simulations are required that account for perturbative and parton shower effects.

7 Conclusions

In this paper we have studied linear power corrections to the process of single top production followed by the top quark decay. Our primary interest is the impact of top quark instability on these corrections. Working in the narrow width approximation, we have found that linear power corrections do affect the top quark production cross section if the top quark is allowed to decay, at variance with the case of a stable top quark that was studied earlier in ref. [1].

The non-perturbative corrections that we have found in this article do affect measurements of the top quark polarisation in such processes, and also influence the kinematic distributions of leptons in top quark decays that were suggested as "clean" observables for measuring the top quark mass. However, the particular form of power corrections, that we derived in this paper, allows us to show that "out of the collision plane" component of the positron momentum from top quark decays does not receive linear non-perturbative corrections. Since the average value of this observable depends on m_t, it is an interesting candidate for measuring the top quark mass.

Finally, the results discussed in this paper are obtained in the narrow width approximation for the top quark which corresponds to an unphysical limit $\Gamma_t \ll \Lambda_{\rm QCD}$. The next important step is to extend these result to the *physical* case $\Gamma_t \gg \Lambda_{\rm QCD}$. Then, the analysis becomes significantly more complicated because top quark production process and top quark decay do not factorise any more. Nevertheless, we hope that our understanding of non-perturbative power corrections to top quark production processes achieved in this paper as well as in refs. [1, 2] will allow us to successfully analyse this challenging problem.

A Alternative derivation of power corrections

The goal of this appendix is to discuss an alternative derivation of power corrections to the single top production and decay process. Here we deal directly with the amplitudes as opposed to amplitudes squared, as was done in earlier papers [1, 2]. Below, we first discuss the Born amplitude and cross section and then continue with the real emission and virtual corrections to the top quark production and decay process. We use directly a short-distance

mass scheme for the top quark mass in the calculation and we explain how to do this by considering the self-energy insertion in the (nearly) on-shell top quark line.

The Born cross section

The Born diagram for single top production and decay is shown in figure 1. The colour structure of this process is quite simple. We will ignore it for now and reconstruct it at the end. In the narrow width approximation, we write the Born amplitude as

$$\mathcal{B}_{PD} = \frac{i}{p_t^2 - m_t^2 + i m_t \Gamma_t} \bar{u}(p_f) N_D \left[\sum_{\lambda_s = \pm 1} u(p_t, s, \lambda_s) \bar{u}(p_t, s, \lambda_s) \right] N_P u(p_i), \qquad \text{(A.1)}$$

where the subscripts P and D denote production and decay, and we have displayed explicitly the $b_i - t - b_f$ fermion line. The functions N_P (N_D) contain all remaining structures pertinent to the production and decay processes. We assume that a quantisation axis s, satisfying the conditions $s^2 = -1$ and $p_t \cdot s = 0$, has been chosen for the top quark spin. We denote the signs of the top quark spin along the quantisation axis s with λ_s. We then define

$$\begin{aligned} \mathcal{B}_P(s, \lambda_s) &= \bar{u}(p_t, s, \lambda_s) N_P u(p_i), \\ \mathcal{B}_D(s, \lambda_s) &= \bar{u}(p_f) N_D u(p_t, s, \lambda_s), \end{aligned} \qquad \text{(A.2)}$$

and write the Born cross section for single top production and decay as follows

$$\begin{aligned} |A|^2 &= \frac{1}{2 m_t \Gamma_t} 2\pi \delta(p_t^2 - m_t^2) |\mathcal{B}_{PD}|^2, \\ |\mathcal{B}_{PD}|^2 &= \sum_{\lambda_s, \lambda_s'} [\mathcal{B}_P(s, \lambda_s) \mathcal{B}_P^*(s, \lambda_s')][\mathcal{B}_D(s, \lambda_s) \mathcal{B}_D^*(s, \lambda_s')], \end{aligned} \qquad \text{(A.3)}$$

where we have used the narrow width approximation, see eq. (2.7). The form of the amplitude in eq. (A.3) is the expected product of spin correlation matrices. In single top production process, a further simplification occurs since there are choices of the top quark spin quantisation axes such as $\mathcal{B}_D(s, -1) = 0$ or $\mathcal{B}_P(s, -1) = 0$. The fact that such quantisation axes must exist is a consequence of the fact that the helicities of all massless particles in single top production and decay are fixed by the charged-current interactions, so that also the top quark must be in a pure spin state.[10] If we call such a quantisation axis for the production process s_P, we can write

$$\begin{aligned} \mathcal{B}_P(s, \lambda_s) &= \bar{u}(p_t, s, \lambda_s) B = \bar{u}(p_t, s, \lambda_s) \left[\frac{1 + \gamma_5 \slashed{s}_P}{2} + \frac{1 - \gamma_5 \slashed{s}_P}{2} \right] B \\ &= \bar{u}(p_t, s, \lambda_s) \frac{1 + \gamma_5 \slashed{s}_P}{2} B. \end{aligned} \qquad \text{(A.4)}$$

In eq. (A.4) B denotes whatever is left of $\mathcal{B}_P(s, \lambda_s)$ when the $\bar{u}$ spinor is removed, and the last step follows from the fact that

$$\bar{u}(p_t, s, \lambda_s) \left[\frac{1 - \gamma_5 \slashed{s}_P}{2} \right], \qquad \text{(A.5)}$$

is an eigenstate of the projection of the top quark spin operator on the axis s_P with an eigenstate $-1/2$, which by assumption does not contribute to the production process.

---

[10]This suggests that this property should also be valid in a class of single top production processes with the addition of colour-neutral particles with definite spin.

Upon squaring the amplitude, we find

$$|\mathcal{B}_P(s, \lambda_s)|^2 = \bar{B}\frac{1 + \gamma_5 \slashed{s}_P}{2} u(p_t, s, \lambda_s) \bar{u}(p_t, s, \lambda_s) \frac{1 + \gamma_5 \slashed{s}_P}{2} B$$
$$= \bar{B}\frac{1 + \gamma_5 \slashed{s}_P}{2} \left[(\slashed{p}_t + m_t) \frac{1 + \lambda_s \gamma_5 \slashed{s}}{2} \right] \frac{1 + \gamma_5 \slashed{s}_P}{2} B. \tag{A.6}$$

Working out the simple Dirac algebra we find

$$\frac{1 + \gamma_5 \slashed{s}_P}{2} \left[(\slashed{p}_t + m_t) \frac{1 + \lambda_s \gamma_5 \slashed{s}}{2} \right] \frac{1 + \gamma_5 \slashed{s}_P}{2} = \frac{1 - \lambda_s s \cdot s_P}{2} (\slashed{p}_t + m_t) \frac{1 + \gamma_5 \slashed{s}_P}{2}. \tag{A.7}$$

We then insert this result into eq. (A.6) and obtain

$$|\mathcal{B}_P(s, \lambda_s)|^2 = \frac{1 - \lambda_s s \cdot s_P}{2} \sum_{\pm \lambda'_s} \mathcal{B}_P^*(s_P, \lambda'_s) \mathcal{B}_P(s_P, \lambda'_s) = \frac{1 - \lambda_s s \cdot s_P}{2} |\mathcal{B}_P|^2, \tag{A.8}$$

where we have introduced the notation $|\mathcal{B}_P|^2 = \sum_{\lambda_s} |\mathcal{B}_P(s, \lambda_s)|^2$.

A similar formula can be derived for the decay amplitude,

$$|\mathcal{B}_D(s, \lambda_s)|^2 = \frac{1 - \lambda_s s \cdot s_D}{2} |\mathcal{B}_D|^2, \tag{A.9}$$

where the proper quantisation axis s_D differs from the one in the production. Combining the results for the production and decay amplitudes, we obtain

$$|\mathcal{B}_{PD}|^2 = \frac{1 - s_P \cdot s_D}{2} |\mathcal{B}_P|^2 |\mathcal{B}_D|^2. \tag{A.10}$$

This result can be derived from eq. (A.3) by choosing the quantisation axis to be either s_P or s_D in which case only a single term $\lambda_s = 1$ contributes to sums over spin projections, and using either eq. (A.8) or (A.9). We also note that the differential decay width takes the form

$$d\Gamma_t = \frac{1}{4m_t} d\Phi_D |\mathcal{B}_D|^2, \tag{A.11}$$

where we had to divide by two for the spin average. On the other hand our expression for the differential cross section (ignoring spin and colour averages for the initial fermions) is given by

$$d\sigma_{PD} = \frac{1}{2m_t \Gamma_t} d\Phi_P d\Phi_D |\mathcal{B}_D|^2 |\mathcal{B}_P(s_D)|^2 = 2\frac{d\Gamma_t}{\Gamma_t} d\sigma(s_D), \tag{A.12}$$

which agrees with eq. (2.14). For the case of single top production depicted in figure 1, we can easily identify the quantisation axes s_P and s_D. We begin by computing s_D. The decay amplitude is proportional to

$$\mathcal{B}_D \sim \bar{u}_f \gamma^\mu (1 - \gamma_5) u_t \, \bar{u}_1 \gamma_\mu (1 - \gamma_5) v_2 = -\bar{u}_f \gamma^\mu (1 - \gamma_5) u_t \, \bar{u}_{2R} \gamma_\mu (1 + \gamma_5) v_{1R}$$
$$= -\left[\bar{u}_{2R}(1 - \gamma_5) u_t \right] \left[\bar{u}_f (1 + \gamma_5) v_{1R} \right], \tag{A.13}$$

where we have introduced the charge conjugate spinors u_{2R} and v_{1R} for the positron and the neutrino, and the last step uses a Fierz identity. The subscript R on the conjugate spinors

is to remind that they are right-handed, i.e. $\bar{u}_{2,R}\gamma_5 = -\bar{u}_{2,R}$. We write

$$
\begin{aligned}
\bar{u}_{2,R}(p_2)u(p_t,s) &= \frac{1}{2}\bar{u}_{2,R}(p_2)\left(2 + \gamma_5\frac{m_t}{(p_2 p_t)}\left(\not{p}_2 - \frac{(p_2 p_t)}{m_t^2}(\not{p}_t - m_t)\right)\right)u(p_t,s) \\
&= \frac{1}{2}\bar{u}_{2,R}(p_2)\left(1 + \gamma_5\frac{m_t}{(p_2 p_t)}\left(\not{p}_2 - \frac{(p_2 p_t)}{m_t^2}\not{p}_t\right)\right)u(p_t,s) \\
&= \bar{u}_{2,R}(p_2)\frac{1 + \gamma_5\not{s}_D}{2}u(p_t,s),
\end{aligned}
\tag{A.14}
$$

where s_D reads

$$
s_D^\mu = \frac{m_t}{(p_2 p_t)}p_2^\mu - \frac{1}{m_t}p_t^\mu,
\tag{A.15}
$$

and satisfies the conditions $s_D^2 = -1$ and $s_D \cdot p_t = 0$. We note that in deriving eq. (A.14), we used the fact that the spinors $\bar{u}_{2,R}$ and $u(p_t,s)$ satisfy the respective Dirac equations, and that $\bar{u}_{2,R}\gamma_5 = -\bar{u}_{2,R}$ as follows from its definition. It follows from eq. (A.14) that the top quark in the decay is polarised along the axis s_D, and we will refer to this quantity as the top quark spin vector in the decay.

Repeating the same calculation for the production amplitude, we easily find that top quarks are produced polarised along the quantisation axis which is given by the following equation

$$
s_P^\mu = \frac{m_t}{(p_d p_t)}p_d^\mu - \frac{1}{m_t}p_t^\mu.
\tag{A.16}
$$

Again, we will refer to this vector as the top quark spin vector in the production. Furthermore, in the following we will use a simplified notation, where omitting the λ_s argument implies that it is taken equal to one.

Real corrections in production

We will use the letter q rather than p to indicate momenta of particles that are affected by recoil when a soft gluon is emitted. We will also denote the top spin vector as s_q, since it must be orthogonal to q_t. Momentum conservation is given by

$$
p_i + p_u = q_t + q_d + k.
\tag{A.17}
$$

The difference between q's and p's (and between s and s_q) are of order k, so we can change q's into p's and s_q into s when dealing with subleading terms.

It was mentioned several times that to compute linear power corrections, we only need to consider gluon radiation off the heavy quark line. We split this contribution into two diagrams, one that describes radiation off the final-state top quark, and the other one that describes gluon radiation off the b quark in the initial state.

We begin by computing the contribution to the amplitude of the gluon emission from the final state top quark. It is given by

$$
\begin{aligned}
\mathcal{R}_{P,\mathrm{f}}^\mu &= \bar{u}(q_t,s_q)\gamma^\mu\frac{\not{q}_t + \not{k} + m_t}{(q_t + k)^2 - m_t^2}N_P(q_t + k, q_i)u(q_i) \\
&= \bar{u}(q_t,s_q)\frac{2q_t^\mu + k^\mu + \sigma^{\mu\nu}k_\nu}{(q_t + k)^2 - m_t^2}N_P(q_t + k, q_i)u(q_i),
\end{aligned}
\tag{A.18}
$$

where the gluon polarisation vector has been omitted. For simplicity, we have omitted the arguments of $\mathcal{R}_{P,\mathrm{f}}$. We should remind the reader, however, that it depends upon all the q and p momenta, and upon the spin vector s_q. The arguments in N_P show that this function depends on the q momenta. N_P is similar to the Born diagram case, except that the incoming top quark momentum is off-shell. Nevertheless, it is a well-defined function of the external momenta. We are interested in the leading $\mathcal{O}(k^{-1})$ and next-to-leading $\mathcal{O}(k^0)$ terms in the limit of small gluon momentum k. When performing the manipulations below, we will always discard terms that vanish in the $k \to 0$ limit.

We focus on the term $\sigma^{\mu\nu}k_\nu$ in eq. (A.18) acting on the $\bar{u}$ spinor. Consider the following equation

$$\bar{u}(q_t, s_q)(1 + a_\mu \sigma^{\mu\nu} k_\nu) = \bar{u}_a, \tag{A.19}$$

where u_a is defined as

$$u_a \equiv (1 + k_\nu \sigma^{\nu\mu} a_\mu) u(q_t, s_q), \tag{A.20}$$

and a^μ is an arbitrary four-vector. We note that, up to an irrelevant phase, a general Lorentz transformation of a spinor is given by the following expression

$$\hat{S}(\Lambda) u(p, s) = u(\Lambda p, \Lambda s), \tag{A.21}$$

where for an infinitesimal transformation

$$\Lambda_{\alpha\beta} = g_{\alpha\beta} + \omega_{\alpha\beta}, \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha}, \tag{A.22}$$

the spinor transformation matrix $\hat{S}(\Lambda)$ reads

$$\hat{S}(\Lambda) = e^{\frac{1}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}} \approx 1 + \frac{1}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}. \tag{A.23}$$

If we choose

$$\omega_{\mu\nu} = 2(a_\mu k_\nu - a_\nu k_\mu), \tag{A.24}$$

we find the following equation for the spinor u_a

$$u_a = u(\Lambda q_t, \Lambda s_q). \tag{A.25}$$

Then, writing

$$\sigma^{\nu\mu} k_\nu a_\mu u(q_t, s_q) = u(\Lambda q_t, \Lambda s_q) - u(q_t, s_q), \tag{A.26}$$

and expanding the right-hand side in powers of k through linear terms, we find

$$
\begin{aligned}
k_\nu \sigma^{\nu\mu} a_\mu u &= \frac{\partial u}{\partial p_t^\sigma} \Lambda^{\sigma\rho} p_{t,\rho} + \frac{\partial u}{\partial s^\sigma} \Lambda^{\sigma\rho} s_\rho, \\
\frac{\partial u}{\partial p_t^\sigma} \Lambda^{\sigma\rho} p_{t,\rho} &= 2a_\mu \left(p_t^\mu k^\sigma \frac{\partial}{\partial p_t^\sigma} - (p_t k) \frac{\partial}{\partial p_{t,\mu}} \right) u = d_t\, a_\mu L_t^\mu\, u, \\
\frac{\partial u}{\partial s^\sigma} \Lambda^{\sigma\rho} s_\rho &= 2a_\mu \left(s^\mu k^\sigma \frac{\partial}{\partial s^\sigma} - (sk) \frac{\partial}{\partial s_\mu} \right) u = d_t\, a_\mu S_t^\mu\, u,
\end{aligned}
\tag{A.27}
$$

where $d_t = 2(p_t k)$ is the denominator of the top propagator without the subleading $k^2 = \lambda^2$ term. Eqs. (A.27) implicitly define the operators L_t and S_t as given by

$$L_t^\mu = \frac{2}{d_t}\left(p_t^\mu\, k^\nu \frac{\partial}{\partial p_t^\nu} - (p_t k)\,\frac{\partial}{\partial p_{t,\mu}}\right), \qquad S_t^\mu = \frac{2}{d_t}\left(s^\mu\, k^\nu \frac{\partial}{\partial s^\nu} - (sk)\,\frac{\partial}{\partial s_\mu}\right). \qquad \text{(A.28)}$$

Since the vector a is arbitrary, from eq. (A.27) we infer the following result

$$k_\nu \sigma^{\nu\mu} u = d_t (L_t^\mu + S_t^\mu) u. \qquad \text{(A.29)}$$

Next, using the definition of the current J_t in eq. (3.5), and discarding terms of order k, we write the amplitude as

$$\mathcal{R}_{P,\mathrm{f}}^\mu = J_t^\mu \bar{u}(q_t, s_q) N(q_t + k, q_i) u(q_i) + [(L_t^\mu + S_t^\mu)\bar{u}(p_t, s)] N u(p_i). \qquad \text{(A.30)}$$

We note that if the arguments of the function N are not written explicitly, it is to be understood as $N(p_t, p_i)$. To simplify the leading term, we write

$$J_t^\mu \bar{u}(q_t, s_q) N(q_t + k, q_i) u(q_i) = J_t^\mu \bar{u}(q_t, s_q) N(q_t, q_i) u(q_i) + \frac{2p_t^\mu}{d_t} \bar{u}(p_t, s)\, k^\alpha \frac{\partial N}{\partial p_t^\alpha} u(p_i). \qquad \text{(A.31)}$$

As stated earlier, the function $N(q_t + k, q_i)$ is a well-defined function of its arguments and can be constructed from Feynman graphs. It is not uniquely defined, however, if momentum conservation is violated, and this is exactly what happens in eq. (A.31) both in the leading term $N(q_t, q_i)$ and when derivative with respect to p_t in the last term is taken. To interpret this equation, we need to assume that N is extended in some way to account for the momentum non-conservation. The ambiguity introduced by such arbitrariness must cancel in the end, since it was not present in the initial formula. We will see later that this, in fact, is the case. We finally write

$$\mathcal{R}_{P,\mathrm{f}}^\mu = J_t^\mu \mathcal{B}_P(s_q, q) + \bar{u}(p_t, s)\left[\left(L_t^\mu + \frac{\partial}{\partial p_{t,\mu}}\right) N\right] u(p_i) + [(L_t^\mu + S_t^\mu)\bar{u}(p_t, s)] N u(p_i), \qquad \text{(A.32)}$$

where

$$\mathcal{B}_P(s_q, q) = \bar{u}(q_t, s_q) N(q_t, q_i) u(q_i). \qquad \text{(A.33)}$$

A similar calculation can be performed for the radiation off the b-quark in the initial state. We obtain

$$\mathcal{R}_{P,\mathrm{i}}^\mu = J_i^\mu \mathcal{B}_P(s_q, q) + \bar{u}(p_t, s)\left[\left(-L_i^\mu + \frac{\partial}{\partial p_{i,\mu}}\right) N\right] u(p_i) - \bar{u}(p_t, s)[L_i^\mu u(p_i)], \qquad \text{(A.34)}$$

where J_i and L_i are defined in eqs. (3.5) and (3.8). Unlike the case of radiation off the top quark, no term analogous to the S_t operator arises here, since the $u(p_i)$ spinor is a helicity eigenstate, and helicity is Lorentz invariant. We thus find

$$\begin{aligned}
\mathcal{R}_{P,\mathrm{f}} + \mathcal{R}_{P,\mathrm{i}} &= J\, \mathcal{B}_P(s_q, q) + (L_t + S_t - L_i)\,[\bar{u}(p_t, s) N u(p_i)] \\
&\quad + \bar{u}(p_t, s)\left[\left(\frac{\partial}{\partial p_t} + \frac{\partial}{\partial p_i}\right) N\right] u(p_i).
\end{aligned} \qquad \text{(A.35)}$$

where, as usual, $J = J_t + J_i$.

We note that all terms that appear in the first line in eq. (A.35) vanish if we multiply the equation by k^μ, while the term on the second line does not.[11] On the other hand, since this term is non-singular in the soft limit, its lack of transversality must be compensated by non-singular contributions caused by the radiation from internal lines, that must have the form

$$\mathcal{R}^\mu_{P,\text{int}} = \bar{u}(p_t, s) N^\mu_{\text{reg}} u(p_i). \tag{A.36}$$

Current conservation implies

$$[\mathcal{R}_{P,\text{f}} + \mathcal{R}_{P,\text{i}} + \mathcal{R}_{P,\text{int}}] \cdot k = \bar{u}(p_t, s)\, k \cdot \left[\frac{\partial N}{\partial p_t} + \frac{\partial N}{\partial p_i} + N_{\text{reg}}\right] u(p_i) = 0. \tag{A.37}$$

In order for this equation to hold for any value of k we therefore must have

$$N_{\text{reg}} = -\left(\frac{\partial N}{\partial p_t} + \frac{\partial N}{\partial p_i}\right). \tag{A.38}$$

Thus, the full result for gluon emission in production reads

$$\mathcal{R}_P = \mathcal{R}_{P,\text{f}} + \mathcal{R}_{P,\text{i}} + \mathcal{R}_{P,\text{int}} = J\,\mathcal{B}_P(s_q, q) + (L_t + S_t - L_i)\,\mathcal{B}_P(s). \tag{A.39}$$

We now introduce the mapping from q- to p-momenta. We employ the mapping already used in ref. [1], and discussed at length near eq. (3.16). Since the mass of the top is not changed by the mapping, it must be possible to write it as a Lorentz transformation Λ which is given in eq. (3.21). In the present context, we should remember that we also need a transformation for s_q, that can be conveniently chosen to be given by the same Lorentz transformation, so that the identities $s_q^2 = s^2$ and $q_t \cdot s_q = p_t \cdot s$ hold. For convenience we report here the complete mapping transformation:

$$q_t = \Lambda p_t = p_t - k + \frac{(p_t k)}{(p_t p_d)} p_d, \quad q_d = p_d - \frac{(p_t k)}{(p_t p_d)} p_d,$$
$$s_q = \Lambda s = s + \frac{(sk)}{(p_t p_d)} p_d - \frac{(sp_d)}{(p_t p_d)} k. \tag{A.40}$$

We recall that also the decay momenta must change, since the top quark momentum has changed. However, since this change is the Lorentz transformation Λ, the decay amplitude does not change. Our final result is then

$$\mathcal{R}^\mu_P = J^\mu \mathcal{B}_P(s, p) + D^\mu_{P,r} \mathcal{B}_P(s, p), \tag{A.41}$$

where

$$D_{P,r} = J D_{\text{rec}} + (L_t + S_t - L_i),$$
$$D_{\text{rec}} = \left(-k + \frac{(p_t k)}{(p_t p_d)} p_d\right) \cdot \frac{\partial}{\partial p_t} - \frac{(p_t k)}{(p_t p_d)} p_d \cdot \frac{\partial}{\partial p_d} + \left(\frac{(ks)}{(p_t p_d)} p_d - \frac{(p_d s)}{(p_t p_d)} k\right) \cdot \frac{\partial}{\partial s}. \tag{A.42}$$

D_{rec} is the differential operator associated with the momenta and spin mappings, and it can be immediately read out of eq. (A.40). It is straightforward to verify that $D_{P,r}$ preserves

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[11]In fact it does vanish in the single top production case. It does not necessarily vanish if we consider some associated production process, and we prefer to keep the discussion general.

physical conditions, such as the momentum conservation, the on-shell conditions and the spin transversality condition,

$$D^\mu_{P,r}\,(p_i + p_u - p_t - p_d)^\nu = 0, \qquad D^\mu_{P,r}\,p_i^2 = 0, \qquad D^\mu_{P,r}\,p_d^2 = 0,$$
$$D^\mu_{P,r}\,p_t^2 = 0, \qquad D^\mu_{P,r}\,s \cdot p_t = 0, \qquad D^\mu_{P,r}\,s^2 = 0. \tag{A.43}$$

Thus, eq. (A.41) depends upon $\mathcal{B}_P(s, p)$ evaluated with momenta and spin satisfying the physical conditions, since the derivative acts in a direction tangent to the manifold where the Born amplitude is unambiguously defined.

To obtain the full amplitude for the top production and decay process, we should multiply eq. (A.41) with the decay amplitude. Since, as discussed earlier, we can assume that the momentum mapping satisfies the equation

$$\mathcal{B}_D(s_q, q) = \mathcal{B}_D(s, p), \tag{A.44}$$

the full amplitude for the production and decay reads

$$\mathcal{R}^\mu_{PD} = \sum_{\lambda_s = \pm 1} \left[J^\mu \mathcal{B}_P(s, \lambda_s) \mathcal{B}_D(s, \lambda_s) + \mathcal{B}_D(s, \lambda_s) D^\mu_{P,r} \mathcal{B}_P(s, \lambda_s) \right]. \tag{A.45}$$

As we explained earlier, if we choose $s = s_D$ only contribution with $\lambda_s = +1$ survives in the sum. Thus, upon squaring the above formula we arrive at

$$-g_{\mu\nu} \mathcal{R}^\mu_{PD} \mathcal{R}^{\nu,+}_{PD} = |\mathcal{R}^\mu_{PD}|^2 = |\mathcal{B}_D|^2 \left\{ -J^2 |\mathcal{B}_P(s)|^2 - J \cdot D_{P,r} |\mathcal{B}_P(s)|^2 \right\}_{s=s_D}. \tag{A.46}$$

The k-dependence is exposed in the above formula and, after momenta redefinitions, integration over gluon momentum factorises from the rest of phase space. The needed integrals in k can be found in ref. [1]. The result for the linear term in λ arising from the integration is given by

$$\mathcal{T}_\lambda \left[|R^\mu_{PD}|^2 \right] = -|\mathcal{B}_{PD}|^2 \mathcal{T}_\lambda \left[\int \frac{\mathrm{d}^3 k}{2k^0 (2\pi)^3}\, J^2 \right] + |\mathcal{B}_D|^2 \left[\tilde{D}_{P,r} |\mathcal{B}_P(s)|^2 \right]_{s=s_D}, \tag{A.47}$$

where

$$\tilde{D}_{P,r} = -\mathcal{T}_\lambda \left[\int \frac{\mathrm{d}^3 k}{2k^0 (2\pi)^3}\, J \cdot D_{P,r} \right]. \tag{A.48}$$

We will not show the result of the integration of the first term in the above equation because, as we will see later, it can be combined with other contributions and argued to cancel in a way similar to what was found in ref. [1]. Computing the required integral explicitly for the second term in eq. (A.47), we find

$$\tilde{D}_{P,r} = -\beta \left[\frac{(p_i s)(2m_t^2 p_d - (p_t p_d) p_t) - (p_d s)(2m_t^2 p_i - (p_t p_i) p_t)}{(p_t p_d)(p_t p_i)} \cdot \frac{\partial}{\partial s} \right.$$
$$\left. + \left(\frac{m_t^2}{(p_t p_d)} p_d - \frac{m_t^2}{(p_t p_i)} p_i \right) \cdot \frac{\partial}{\partial p_t} - \frac{m_t^2}{(p_t p_i)} p_i \cdot \frac{\partial}{\partial p_i} - \frac{m_t^2}{(p_t p_d)} p_d \cdot \frac{\partial}{\partial p_d} \right], \tag{A.49}$$

where

$$\beta = \frac{1}{2(2\pi)^2} \frac{\lambda \pi}{m_t}. \tag{A.50}$$

Virtual corrections in production

We continue with the discussion of the virtual correction to the heavy line in the production subprocess. We write it as follows

$$\mathcal{V}_P = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \, F_{VP}(k, \dots), \qquad (A.51)$$

where

$$
\begin{aligned}
F_{VP} = \Bigg[& \bar{u}(p_t, s)\gamma^\mu \frac{\slashed{p}_t + \slashed{k} + m_t}{(p_t + k)^2 - m_t^2 + i\epsilon} N(p_t + k, p_i + k) \frac{\slashed{p}_i + \slashed{k}}{(p_i + k)^2 + i\epsilon} \gamma_\mu u(p_i) \\
& + \bar{u}(p_t, s)\gamma^\mu \frac{\slashed{p}_t + \slashed{k} + m_t}{(p_t + k)^2 - m_t^2 + i\epsilon} N_\mu u(p_i) + \bar{u}(p_t, s) N_\mu \frac{\slashed{p}_i + \slashed{k}}{(p_i + k)^2 + i\epsilon} \gamma^\mu u(p_i) \Bigg].
\end{aligned}
\qquad (A.52)
$$

The first line provides a contribution where a virtual gluon is emitted by an incoming bottom and absorbed by the outgoing top quark, and the terms in the second line describe contributions where virtual gluons are emitted by either bottom or top quarks and are absorbed by the internal lines of the diagrams. Potential contributions where gluons are emitted and absorbed by internal lines are not shown as they cannot produce $\mathcal{O}(\lambda)$ corrections [1]. Using the Dirac equations, and neglecting contributions that cannot produce $\mathcal{O}(\lambda)$ corrections, we rewrite the above expression as follows

$$
\begin{aligned}
F_{VP} = & \bar{u}(p_t, s) \frac{2p_t^\mu + k^\mu + \sigma^{\mu\nu} k_\nu}{(p_t + k)^2 - m_t^2 + i\epsilon} N(p_t + k, p_i + k) \frac{2p_{t,\mu} + k_\mu - \sigma_{\mu\rho} k^\rho}{(p_i + k)^2 + i\epsilon} u(p_i) \\
& + \bar{u}(p_t, s) \frac{2p_t^\mu}{(p_t + k)^2 - m_t^2 + i\epsilon} N_\mu u(p_i) + \bar{u}(p_t, s) N_\mu \frac{2p_i^\mu}{(p_i + k)^2 + i\epsilon} u(p_i).
\end{aligned}
\qquad (A.53)
$$

Using again eq. (A.38) and eq. (A.29) we arrive at

$$
\begin{aligned}
\mathcal{V}_P = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \Bigg\{ & J_t \cdot J_i \, \mathcal{B}_P(s) + [(J_i \cdot (L_t + S_t) + J_t \cdot L_i)\mathcal{B}_P(s)] \\
& - \bar{u}(p_t, s) \left(J_t \cdot \frac{\partial N}{\partial p_t} + J_i \cdot \frac{\partial N}{\partial p_i} \right) u(p_i) \Bigg\},
\end{aligned}
\qquad (A.54)
$$

where the currents J_t and J_i have now changed appropriately for eq. (A.52), and the definitions of L_t, S_t and L_i are given in eqs. (A.28) and (3.8) except that the newly defined denominators d_t, d_i should be used there.

The second term on the right hand side of eq. (A.54) is not a total derivative. To remedy this, we assume that spinors can be written as functions of their momentum alone.[12] To do this, in the expression for $u(p_t, s, \lambda_s)$ we systematically replace the mass m_t with $\sqrt{p_t^2}$. This implies the following modification in the density matrix

$$\sum_{\lambda_s = \pm 1} u(p_t, s, \lambda_s) \bar{u}(p_t, s, \lambda_s) = \left(\slashed{p}_t + \sqrt{p_t^2} \right). \qquad (A.55)$$

---

[12]This choice does not affect the $L_t + S_t$ and the L_i derivatives, since they act as Lorentz transformations on the argument of the spinor, leaving the mass and the constraints on the spin parameter s unchanged.

The replacement $m_t \to \sqrt{p_t^2}$ is important for simplifying eq. (A.54), because it implies

$$p_t \cdot \frac{\partial}{\partial p_t} u(p_t.s) = \frac{1}{2} u(p_t.s). \tag{A.56}$$

This result easily follows from the fact that the mass dimension of a spinor is $1/2$ and that once the mass is eliminated in favour of $\sqrt{p_t^2}$, p_t becomes the only mass scale that appears in the formula for the spinor. Hence, we find

$$\begin{aligned} &\bar{u}(p_t, s) \left(J_t \cdot \frac{\partial N}{\partial p_t} + J_i \cdot \frac{\partial N}{\partial p_i} \right) u(p_i) \\ &= \left(J_i \cdot \frac{\partial}{\partial p_i} + J_t \cdot \frac{\partial}{\partial p_t} \right) [\bar{u}(p_t, s) N u(p_i)] - \left(\frac{1}{d_i} + \frac{1}{d_t} \right) \bar{u}(p_t, s) N u(p_i). \end{aligned} \tag{A.57}$$

Defining

$$D_{P,v} = J_i \cdot (L_t + S_t) + J_t \cdot L_i - J_i \cdot \frac{\partial}{\partial p_i} - J_t \cdot \frac{\partial}{\partial p_t}, \tag{A.58}$$

we write eq. (A.54) as

$$\mathcal{V}_P = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \left[\left(J_t \cdot J_i + \frac{1}{d_i} + \frac{1}{d_t} \right) \mathcal{B}_P(s) + D_{P,v} \mathcal{B}_P(s) \right]. \tag{A.59}$$

We note that the derivative $D_{P,v}$ violates the physicality constraint related to momentum conservation; we will see that it is restored once the mass renormalisation is accounted for.

Since the dependence on the gluon momentum k is exposed in eq. (A.59), we can integrate over it. Similar to the discussion of the real-emission contribution, we leave terms proportional to $\mathcal{B}_P(s)$ as they are, since we will argue later that their cancellation is already demonstrated in ref. [1]. We obtain

$$\begin{aligned} \mathcal{T}_\lambda[\mathcal{V}_P] &= \mathcal{T}_\lambda \left[\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \left(J_t \cdot J_i + \frac{1}{d_i} + \frac{1}{d_t} \right) \right] \mathcal{B}_P(s) + \tilde{D}_{P,v} \mathcal{B}_P(s), \\ \tilde{D}_{P,v} &= \frac{\beta}{(p_t p_i)} \left(-(p_i s) \, p_t \cdot \frac{\partial}{\partial s} + m_t^2 \, p_i \cdot \frac{\partial}{\partial p_i} + m_t^2 \, p_i \cdot \frac{\partial}{\partial p_t} \right). \end{aligned} \tag{A.60}$$

We then multiply eq. (A.60) by the decay amplitude, set $s = s_D$ to get rid of the spin summation, compute the interference with the Born amplitude, and finally obtain

$$\begin{aligned} 2|\mathcal{B}_{PD}|^2 \mathcal{T}_\lambda &\left[\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \left(J_t \cdot J_i + \frac{1}{d_i} + \frac{1}{d_t} \right) \right] \\ &+ |\mathcal{B}_D|^2 \left[\tilde{D}_{P,v} |\mathcal{B}_P(s)|^2 \right]_{s=s_D}. \end{aligned} \tag{A.61}$$

Real corrections in decay

We can describe radiation in the top quark decay following the approach used in the discussion of the production process. The only essential difference is in the momentum mapping, that we choose to coincide with the one discussed in ref. [1]. Details are given in eq. (4.4). Both the initial top quark momentum and its spin vector are not affected by the mapping. The decay amplitude expanded through linear terms in the gluon momentum reads

$$\mathcal{R}_D^\mu(s) = J^\mu \mathcal{B}_D(s) + D_{D,r}^\mu \mathcal{B}_D(s), \qquad D_{D,r}^\mu = J^\mu D_{\mathrm{rec}} - (L_t^\mu + S_t^\mu - L_f^\mu), \tag{A.62}$$

where the currents are given in eq. (4.3). The differential operator associated with the mapping can be immediately read out of eq. (4.4). It reads

$$D_{\rm rec} = \left(-k + \frac{(p_f k)}{(p_2 p_f)} p_2 \right) \cdot \frac{\partial}{\partial p_f} - \frac{(p_f k)}{(p_2 p_f)} p_2 \cdot \frac{\partial}{\partial p_2}. \tag{A.63}$$

The operators L_t and S_t are defined in eq. (A.27) but now $d_t = -2(p_t k)$ has to be used there. The definition of L_f is the same as the one for L_i after the replacement $i \to f$ is performed. The operator $D_{D,r}$ is easily seen to preserve the physicality conditions for the decay. Proceeding as for eq. (A.46), we get

$$|\mathcal{R}_{PD}^\mu|^2 = |\mathcal{B}_P|^2 \left\{ -J^2 |\mathcal{B}_D(s)|^2 - J \cdot D_{D,r} |\mathcal{B}_D(s)|^2 \right\}_{s=s_P}. \tag{A.64}$$

After the momentum mapping, the integration over k factorises and can be performed. Defining

$$
\begin{aligned}
\tilde{D}_{D,r} &= \mathcal{T}_\lambda \left[\int \frac{\mathrm{d}^3 k}{2k^0 (2\pi)^4} (-J \cdot D_{D,r}) \right] \\
&= \beta \left[\frac{(p_f s)}{(p_t p_f)} p_t \cdot \frac{\partial}{\partial s} + \frac{(p_t p_f)}{(p_2 p_f)} p_2 \cdot \left(\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_f} \right) + \left(p_t - \frac{m_t^2}{(p_t p_f)} p_f \right) \cdot \left(\frac{\partial}{\partial p_f} + \frac{\partial}{\partial p_t} \right) \right],
\end{aligned} \tag{A.65}
$$

we find for the total amplitude squared

$$\mathcal{T}_\lambda \left[|\mathcal{R}_{PD}|^2 \right] = |\mathcal{B}_{PD}|^2 \mathcal{T}_\lambda \left[\int \frac{\mathrm{d}^3 k}{2k^0 (2\pi)^4} (-J^2) \right] + |\mathcal{B}_P|^2 \left[\tilde{D}_{D,r} |\mathcal{B}_D(s)|^2 \right]_{s=s_P}. \tag{A.66}$$

Virtual corrections in decay

The calculation of the virtual corrections in the decay process closely follows the production case, and the result can be obtained from eq. (A.60) after the substitution $i \to f$. We obtain

$$
\begin{aligned}
\mathcal{T}_\lambda[\mathcal{V}_D] &= \mathcal{T}_\lambda \left[\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \left(J_t \cdot J_f + \frac{1}{d_f} + \frac{1}{d_t} \right) \right] \mathcal{B}_D(s) + \tilde{D}_{D,v} \mathcal{B}_D(s), \\
\tilde{D}_{D,v} &= \frac{\beta}{(p_t p_f)} \left(-(p_f s) \, p_t \cdot \frac{\partial}{\partial s} + m_t^2 \, p_f \cdot \frac{\partial}{\partial p_f} + m_t^2 \, p_f \cdot \frac{\partial}{\partial p_t} \right).
\end{aligned} \tag{A.67}
$$

The final contribution is

$$
\begin{aligned}
& 2|\mathcal{B}_{PD}|^2 \mathcal{T}_\lambda \left[\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \left(J_t \cdot J_f + \frac{1}{d_f} + \frac{1}{d_t} \right) \right] \\
& + |\mathcal{B}_P|^2 \left[\tilde{D}_{D,v} |\mathcal{B}_D(s)|^2 \right]_{s=s_P}.
\end{aligned} \tag{A.68}
$$

Top quark self-energy contribution

Finally, we need to account for the top quark self-energy correction shown in figure 2. Since we perform the calculation in a short-distance scheme (e.g. the $\overline{\rm MS}$ scheme), no $\mathcal{O}(\lambda)$ correction can arise from the mass counter-terms, and we do not need to account for them. Following
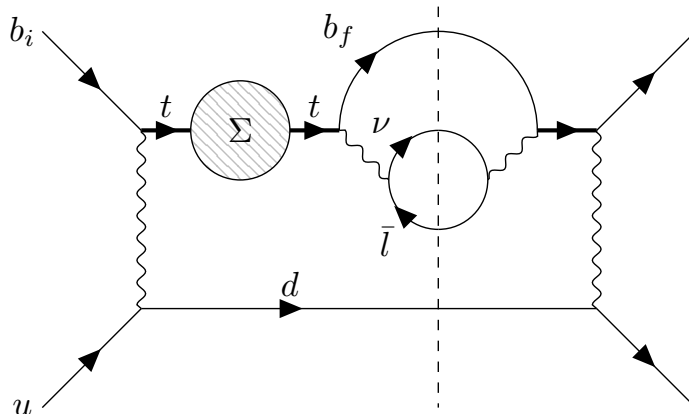
Figure 2. The graph with the top quark self-energy insertion. The complex conjugate diagram should also be added.

the discussion in section 7 in ref. [1], we find that the term of order λ from the self-energy insertion to the left of the cut line in figure 2 reads

$$\Sigma = \kappa \left[-\frac{1}{4m_t}(p_t^2 - m_t^2) - (\not{p}_t - m_t) + m_t \right],$$

$$(A.69)$$

where

$$\kappa = 4\pi\alpha_s C_F \beta = \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t}.$$

$$(A.70)$$

The product of the denominators of the top propagators in figure 2 yields

$$\frac{1}{(p_t^2 - m_t^2)^2 + (m_t\Gamma_t)^2} \frac{1}{(p_t^2 - m_t^2) + im_t\Gamma_t}.$$

$$(A.71)$$

When this expression is combined with the complex conjugate of figure 2, it becomes

$$\frac{2(p_t^2 - m_t^2)}{((p_t^2 - m_t^2)^2 + (m_t\Gamma_t)^2)^2} = -\frac{\partial}{\partial p_t^2} \frac{1}{(p_t^2 - m_t^2)^2 + (m_t\Gamma_t)^2} \approx -\frac{\pi}{m_t\Gamma_t}\delta'(p_t^2 - m_t^2),$$

$$(A.72)$$

where $\delta'(p_t^2 - m_t^2)$ is the derivative of a delta function with respect to p_t^2. Thus, the net effect of the self-energy correction amounts to the following replacement in the Born cross section

$$i(\not{p}_t + m_t)\delta(p_t^2 - m_t^2) \rightarrow -i(\not{p}_t + m_t)i\Sigma i(\not{p}_t + m_t)\delta'(p_t^2 - m_t^2)$$
$$= i\kappa \left[\frac{3}{2}(\not{p}_t + m_t) - m_t \right]\delta(p_t^2 - m_t^2) + i\kappa(\not{p}_t + m_t)2m_t^2\delta'(p_t^2 - m_t^2),$$

$$(A.73)$$

where the relation $\delta'(x)x = -\delta(x)$ was used.

In the self-energy computation, one needs to consider slightly off-shell momenta of the top quark. This means that we cannot set $p_t^2 = m_t^2$ in the factor involving the derivative of the delta function. We thus rewrite this term as follows

$$\not{p}_t + m_t = \left(\not{p}_t + \sqrt{p_t^2} \right) + \frac{m_t^2 - p_t^2}{m_t + \sqrt{p_t^2}}.$$

$$(A.74)$$

Inserting this identity in eq. (A.73) we obtain

$$(\not{p}_t + m_t)\delta(p_t^2 - m_t^2) \to \kappa \frac{3}{2}(\not{p}_t + m_t)\delta(p_t^2 - m_t^2) + \kappa\left(\not{p}_t + \sqrt{p_t^2}\right)2m_t^2\delta'(p_t^2 - m_t^2). \tag{A.75}$$

The first term in the above equation leads to a term proportional to $|\mathcal{B}_{PD}|^2$, and it can be set aside with all other terms of this form. Using for consistency eq. (A.55), we obtain for the second term

$$|\mathcal{B}_{PD}|^2(2m_t^2\kappa\delta'(p_t^2 - m_t^2)) = |\mathcal{B}_{PD}|^2\left[\delta\left(p_t^2 - m_t^2(1 - 2\kappa)\right) - \delta(p_t^2 - m_t^2)\right]. \tag{A.76}$$

In order to compute the result we need to perform a change of variables in the first term. We first relabel all the p's into q's and s into s_q. In particular

$$\delta\left(p_t^2 - m_t^2(1 - 2\kappa)\right) \to \delta\left(q_t^2 - m_t^2(1 - 2\kappa)\right). \tag{A.77}$$

Then we recall that in the unpolarised case [1] two different mappings were used for single-top production process and for top decay. The momenta and spin transformations for the decay are given in eq. (5.1), and we report them here for convenience

$$q_t = p_t(1-\kappa), \quad q_f = p_f - \kappa p_t + \kappa\frac{(p_f p)}{(p_f p_2)}p_2, \quad q_2 = p_2\left(1 - \kappa\frac{(p_f p_t)}{(p_f p_2)}\right), \quad s_q = s. \tag{A.78}$$

Notice that the spin is unchanged, since the top momentum is only rescaled. For the production we use

$$q_t = p_t - \kappa\frac{p_t^2}{(p_d p_t)}p_d, \quad q_d = p_d\left(1 + \kappa\frac{p_t^2}{(p_d p_t)}\right), \quad s_q = \Lambda_m s, \tag{A.79}$$

where $\Lambda_m = 1 + \kappa\omega_{td}$ (see eq. (5.11)) and ω_{td} is given in eq. (3.30). The transformation for s_q can be seen to satisfy the condition $s_q \cdot q_t = s \cdot p_t$ and $s_q^2 = s^2$, as will become clear in the following. We should modify these transformations in such a way that the top quark momentum transforms in the same way in both production *and* decay processes. This is achieved by using the fact that Λ_m is a Lorentz transformation, and that the top momentum mapping can be written as the product of a rescaling times Λ_m. It is sufficient then to apply Λ_m also to all decay products. In summary, we write

$$q_t = (1-\kappa)\Lambda_m p_t, \quad q_d = p_d\left(1 + \kappa\frac{p_t^2}{(p_d p_t)}\right), \quad s_q = \Lambda_m s,$$

$$q_f = \Lambda_m\left(p_f - \kappa p_t + \kappa\frac{(p_f p_t)}{(p_f p_2)}p_2\right), \quad q_2 = \left(1 - \kappa\frac{(p_f p_t)}{(p_f p_2)}\right)\Lambda_m p_2, \quad q_1 = \Lambda_m p_1, \tag{A.80}$$

where only $\mathcal{O}(\kappa)$ terms need to be retained on the right hand sides of the above equations. Notice that the first line is just a rewriting of eq. (A.79), while the second line is the Lorentz transformation applied to the eq. (A.78). Momenta modifications induce changes in the squared amplitudes. Depending on whether momenta in the production or decay undergo these transformations, we group such terms into modification of production and decay amplitudes and write

$$D^m\left|\sum_{\pm 1}\mathcal{B}_D(\lambda_s)\mathcal{B}_P(\lambda_s)\right|^2 = |\mathcal{B}_P|^2[D^m|\mathcal{B}_D(s)|^2]_{s=s_P} + |\mathcal{B}_D|^2[D^m|\mathcal{B}_P(s)|^2]_{s=s_D}. \tag{A.81}$$

The differential operator D^m is associated with the transformations shown in eq. (A.80). It can be written as

$$D^m|\mathcal{B}_D(s)|^2 = \kappa\left[-p_t\cdot\frac{\partial}{\partial p_t} + \left(-p_t + \frac{(p_f p_t)}{(p_f p_2)}p_2\right)\cdot\frac{\partial}{\partial p_f} - \frac{(p_f p_t)}{(p_f p_2)}p_2\cdot\frac{\partial}{\partial p_2}\right]|\mathcal{B}_D(s)|^2, \qquad (A.82)$$

$$D^m|\mathcal{B}_P(s)|^2 = \kappa\left[(-p_t + \omega_{td}p_t)\cdot\frac{\partial}{\partial p_t} + \frac{p_t^2}{(p_d p_t)}p_d\cdot\frac{\partial}{\partial p_d} + (\omega_{td}s)\cdot\frac{\partial}{\partial s}\right]|\mathcal{B}_P(s)|^2, \qquad (A.83)$$

where we have simply dropped from D^m the derivatives with respect to variables not contained in the corresponding amplitude and, in the case of the derivative of the decay amplitude, we have removed the Lorentz transformation, since it affects all the decay momenta and the spin vector s, and the decay amplitude is Lorentz invariant.

Assembling everything

We begin by considering the non-derivative terms that appear in eqs. (A.47), (A.61), (A.66), (A.68), (A.82), (A.83). They arise from the dominant terms in the cross sections, from Jacobians due to momenta transformations and in the calculation of the virtual contributions in production and decay. Their calculation is straightforward, and can be carried out along the lines of ref. [1], where it has been shown that they cancel.

We continue with terms that contain derivatives with respect to momenta and spins of external particles. Combining such contributions in the virtual corrections for production, eq. (A.61), the real emission contributions in production, eq. (A.47), and the corresponding part of the self-energy correction, eq. (A.83), we find

$$|\mathcal{B}_D|^2\left[\left(D^m + \frac{\kappa}{\beta}[\tilde{D}_{P,v} + \tilde{D}_{P,r}]\right)|\mathcal{B}_P(s)|^2\right]_{s=s_D}. \qquad (A.84)$$

It is a matter of simple algebra to verify that the derivative operator $D^m + \kappa/\beta\tilde{D}_{P,v}$, with D^m restricted to the production variables, and, independently, $\tilde{D}_{P,r}$ preserve all physicality conditions, cf. eq. (A.43). The result shown in eq. (A.84) becomes

$$|\mathcal{B}_D|^2\kappa\left[\left\{s\cdot\left(2\omega_{ti} + 2\omega_{dt} + \frac{2m_t^2(p_i p_d)}{(p_t p_i)(p_t p_d)}\omega_{id}\right)\cdot\frac{\partial}{\partial s}\right\}|\mathcal{B}_P(s)|^2\right]_{s=s_D}. \qquad (A.85)$$

For the decay contribution, assembling eqs. (A.66), (A.68) and (A.82) we get

$$|\mathcal{B}_P|^2\left[\left(D^m + \frac{\kappa}{\beta}[\tilde{D}_{D,v} + \tilde{D}_{D,r}]\right)|\mathcal{B}_D(s)|^2\right]_{s=s_P} = 0. \qquad (A.86)$$

The sum of eqs. (A.85) and (A.86) agrees with eq. (6.1).

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