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Schottky-Invariant p-Adic Diffusion Operators

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Abstract

A parametrised diffusion operator on the regular domain Ω of a p-adic Schottky group is constructed. It is defined as an integral operator on the complex-valued functions on Ω which are invariant under the Schottky group Γ, where integration is against the measure defined by an invariant regular differential 1-form ω. It is proven that the space of Schottky invariant L^2-functions on Ω outside the zeros of ω has an orthonormal basis consisting of Γ-invariant extensions of Kozyrev wavelets which are eigenfunctions of the operator. The eigenvalues are calculated, and it is shown that the heat equation for this operator provides a unique solution for its Cauchy problem with Schottky-invariant continuous initial conditions supported outside the zero set of ω, and gives rise to a strong Markov process on the corresponding orbit space for the Schottky group whose paths are càdlàg.

Keywords Schottky group · Mumford curves · p-adic numbers · Heat equation · Diffusion

1 Introduction

The first diffusion operators on p-adic domains are Vladimirov–Taibleson operators [15, 18]. These are convolution operators on non-archimedean local fields, and hence diagonalisable by the Fourier transform. From this as a starting point, they were extended to the adèles, and their connection to integration on path spaces via Feynman-Kac formulas was explored, including proofs that such types of diffusion are scaling limits, cf. e.g. [17, 20, 21]. As a p-adic ball is itself a compact abelian group, the Fourier transform method can be adapted to that case in the study of the heat equation [9].

Of importance is also their representation as a Laplacian integral operator. This allows the extension to compact p-adic subdomains which are not necessarily endowed

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with a group structure, and where Turing patterns can be observed [24, 25]. Also, certain compact p-adic manifolds known as Mumford curves became amenable to their own diffusion operators in integral form [2].

The spectrum of such Laplacian operators can be studied via Kozyrev wavelets, introduced in order to find an orthonormal basis of the Hilbert space $L^2(\mathbb{Q}_p)$ consisting of eigenfunctions of the Vladimirov operator [10, 11]. These turned out to be extendable to Mumford curves, [2]. And in recent work, efforts were made in order to rid the constructions on Mumford curves from their dependence on a fundamental domain. Whereas in [2], the construction is exclusively on a compact fundamental domain, in the case of genus one, theta functions are used in order to construct an invariant meromorphic function [3], and this method was also extended to higher genus [4], allowing to hear the genus of a Mumford curve from the spectrum of a diffusion operator.

The main goal of this article is to not require the removal of an essential part of a given Mumford curve by resorting to a meromorphic function as previously done in [4], where it was necessary to exclude the pre-image of the limit set of the Schottky group under taking differences $x - y$ of two variables x, y in the regular domain of the Schottky group action. And this is in general a set of positive measure. In the new approach here, only the zeros of a regular differential 1-form need to be removed, and these form a zero set, thus obtaining a diffusion on a given Mumford curve almost everywhere through a Schottky invariant diffusion operator almost everywhere on the domain of regularity of the Schottky group. This is obtained by simply taking the kernel function locally as a positive power of the p-adic absolute difference $|\beta x - \gamma y|^\alpha$, with β, γ taken from the Schottky group, appropriately weighted by a function of the length $\ell(\beta^{-1}\gamma)$ of the word $\beta^{-1}\gamma$ in a given set of g generators of the Schottky group, and their inverses as a reference alphabet. The only set now which needs to be excluded are the zeros of the invariant regular differential 1-form ω giving rise to the measure $|\omega|$ on the regular domain $\Omega(K)$ of the Schottky group Γ. The main results of this article can be stated as follows:

Theorem 1 *The space of Γ-invariant L^2-functions on Ω outside the zeros of ω has an orthonormal basis consisting of Γ-invariant extensions of Kozyrev wavelets supported on discs outside the zero set of ω. These are eigenfunctions of the self-adjoint diffusion operator $-\Delta_\alpha^{\frac{1}{2}}$ on that Hilbert space. The eigenvalue corresponding to such a wavelet $\psi_{B,j}$, where B is a disc, and j an element of the residue field of the non-Archimedean local field K, is*

$$\lambda_B = \mu^\Gamma(F)^{-1} \sum_{\gamma \in \Gamma} |\pi|^{\alpha_g \ell(\gamma)} \left(\int_{F \setminus B} |x - \gamma y|^{-\alpha} \, |\omega(y)| + \mu^\Gamma(B)^{1-\alpha} \right)$$

with $\alpha > 0$, and depending on B and a good fundamental domain F, and for any $\gamma \in \Gamma$, and where

$$\mu^\Gamma(C) = \int_C |\omega|$$

for any $|\omega|$-measurable set C. The eigenvalues have finite multiplicities, and are invariant under shifting from a given fundamental domain F to γF with $\gamma \in \Gamma$. Here, $x \in B$, and λ_B does not depend on $x \in B$.

The technical notion of "good fundamental domain" is introduced in [6, I.4.1.3], whose existence is guaranteed for any Mumford curve. Also, a straightforward transition formula for the eigenvalues under the replacement $F \to \gamma F$ with $\gamma \in \Gamma$ is given in Lemma 4.11 below, because there could be a possible effect of the actual arrangement the "holes" cut out of a disc in order to form a good fundamental domain.

The next theorem deals with the Cauchy problem for the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta_\alpha^{\frac{1}{2}}\right) h(t,x) = 0$$

having initial condition $h(0,x) = h_0(x)$ which is a continuous Γ-invariant function defined on $\Omega(K) \setminus V(\omega)$, where $V(\omega)$ is the zero set of the differential form ω, and the solution space is assumed to be

$$C^1((0,\infty), \Omega(K) \setminus V(\omega))^\Gamma$$

where the superscript $^\Gamma$ denotes that the functions are assumed invariant under the action of Γ.

Theorem 2 *The heat equation for operator $-\Delta_\alpha^{\frac{1}{2}}$ provides a unique solution for its Cauchy problem with Γ-invariant continuous initial condition $h_0(x)$ supported outside the zero set $V(\omega)$ of ω, and is given as*

$$h(t,x) = \int_{\Omega(K)\setminus V(\omega)} h_0(y) p_t(x, |\omega(y)|)$$

given by a probability measure $p_t(x,\cdot)$ on the Borel σ-algebra on $\Omega(K) \setminus V(\omega)$, which is also the transition function of a strong Markov process on the orbit space $(\Omega(K) \setminus V(\omega))/\Gamma$ whose paths are càdlàg.

2 Notation and Concepts Used

Assume that K is a p-adic number field, i.e. a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. Denote the Haar measure on K as μ_K, or as $|dx|$ if the dependence on a variable x is to be emphasised. It is normalised such that $\mu_K(O_K) = 1$, where O_K is the ring of integers of K. The absolute value on K is denoted as $|\cdot|$, and is chosen such that

$$|\pi| = p^{-f}$$

where π is a uniformiser of O_K, and f is the degree of the residue field $O_K/\pi O_K$ as an extension of the finite field $\mathbb{F}_p$ with p elements. Indicator functions will often be written as

$$\Omega(x \in B) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases}$$

where B is a Borel measurable subset of K.

Any n-dimensional K-analytic manifold X with a regular differential n-form ω has a measure $|\omega|$ on X outside the vanishing set $V(\omega)$ in X, which locally on $U \subset X$ has the form

$$|\omega|\,|_U = |f|\,|\mu_K|$$

with $f \colon U \to K$ an analytic function, cf. [19, Ch. II.2.2], or [7, Ch. 7.4]. Unlike in [4], the measure $|\omega|$ will not be extended to $V(\omega)$, here. This exceptional set is a zero set according to [22, Lem. 3.1]. More about K-analytic manifolds can be learned in [14] or [13], if the reader wishes so.

A Mumford curve can be viewed as a 1-dimensional compact K-analytic manifold X having an atlas consisting of pieces bi-analytically isomorphic to holed discs in K. They are explained in depth e.g. in [6]. What is needed for this article is that they have a universal covering space which is open in the projective line $\mathbb{P}^1(K)$, and the topological fundamental group Γ of X is a free group generated by g hyperbolic Möbius transformations in $\mathrm{PGL}_2(K)$, where g is the genus of X. The group Γ is also known as a so-called Schottky group. A Mumford curve is also a projective algebraic curve defined over K, and posseses regular differential 1-forms which are in fact algebraic. Namely, according to [6, Prop. VI.4.2], the space of regular differential 1-forms on a Mumford curve of genus g has dimension g over the ground field K. A regular algebraic differential 1-form on the K-rational points $X(K)$ of X is given by a Γ-invariant holomorphic differential 1-form on $\Omega(K)$, where $\Omega \subset \mathbb{P}^1_K$ is the universal covering space of X which exists as an open analytic domain, cf. [6, Ch. IV.3].

Assumption 1 It is assumed that the differential 1-form $\omega \in \Omega^1_{X/K}$ has all its zeros in $X(K)$, the set of K-rational points of the Mumford curve X.

This assumption can be fulfilled for a given algebraic differential 1-form after a finite extension of the field K, if necessary. The Γ-invariant differential 1-form corresponding to ω of Assumption 1 is also denoted as ω. This should not be a cause for confusion, as the points of the Mumford curve X themselves are Γ-orbits.

Let $L^2(\Omega(K), |\omega|)$ be the Hilbert space of L^2-functions on $\Omega(K)$, on which the inner product

$$\langle f, g \rangle_\omega = \int_{\Omega(K)} f(x)\overline{g(x)}\ |\omega(x)|$$

is used. The space of continuous functions on $\Omega(K)$ is denoted as $C(\Omega(K), \|\cdot\|_\infty)$, and is a Banach space w.r.t. the supremum norm $\|\cdot\|_\infty$.

Let $\mathscr{F}(\Omega(K))$ be a space of functions $\Omega(K) \to \mathbb{C}$ and define

$$\mathscr{F}(\Omega(K))^\Gamma = \{u \in \mathscr{F} \mid \forall\gamma \in \Gamma\ \forall x \in \Omega(K)\colon u(\gamma x) = u(x)\}$$

as the corresponding subspace of Γ-invariant functions.

Similarly, a corresponding notation will be used for function spaces on $\Omega(K) \setminus V(\omega)$, where $V(\omega) \setminus \Omega(K)$ denotes the vanishing set of ω. Since the differential 1-form is algebraic as a differential form on X, this vanishing set is countable. An example is the space $L^2(\Omega(K) \setminus V(\omega))^\Gamma$.

3 Kernel Function for Γ-Invariant Functions

Let $\Gamma = \langle\gamma_1, \ldots, \gamma_g\rangle \subset \mathrm{PGL}_2(K)$ be a Schottky group on $g \geq 1$ generators with K a non-archimedean local field. As an abstract group, Γ is isomorphic to the free group F_g with g generators. Each element of F_g can be uniquely represented as a reduced word over the alphabet $\left\{\gamma_1^{\pm 1}, \ldots, \gamma_g^{\pm 1}\right\}$, i.e. by deleting all expressions of the form

$$\gamma_i\gamma_i^{-1} = 1 \quad \text{or} \quad \gamma_i^{-1}\gamma_i = 1$$

for $i = 1, \ldots, g$. The length of a reduced word w over a finite alphabet $\mathcal{A}$ is defined as the sum of the occurrence counts of each letter from $\mathcal{A}$ in w, and is denoted as $\ell(w)$.

The following result is well-known:

Lemma 3.1 *Fix $\beta \in \Gamma$. The number of elements $\gamma \in \Gamma$ such that $\beta^{-1}\gamma$ has length ℓ is at most*

$$2g(2g-1)^{\ell(\beta)+\ell}$$

for any natural number $\ell > 0$.

Proof Assume first that $\beta = 1$. Then any of the $2g$ letters in $\gamma_1^{\pm 1}, \ldots, \gamma_g^{\pm 1}$ can be appended by any letter from this alphabet, except the inverse of that letter. So, initially, there are $2g$ choices, after which there are $2g-1$ choices in each further step in constructing a reduced word in Γ.

For any $\beta \in \Gamma$, observe that

$$\ell(\beta^{-1}\gamma) \leq \ell(\beta) + \ell(\gamma)$$

which yields the desired upper bound by using the previous case and taking care of possible cancelling with suffixes of β^{-1}. □

Gerritzen and van der Put in [6, I.4.1.3] introduce the notion of *good fundamental domain* for a p-adic Schottky group Γ, which is needed below. This is the complement in the projective line $\mathbb{P}^1(K)$ of $2g$ open discs $B_1, C_1, \ldots, B_g, C_g$ whose "closures" B_i^+, C_i^+ (i.e. where in the defining inequalities "<" is replaced with "≤", and radii are assumed to be in the valuation group of K) are mutually disjoint, and there are g generators $\gamma_1, \ldots, \gamma_g$ such that

$$\gamma_i(\mathbb{P}^1(K) \setminus B_i) = C_i^+, \quad \gamma_i(\mathbb{P}^1(K) \setminus B_i^+) = C_i$$

for $i = 1, \ldots, g$.

Let $\Omega(K) \subset \mathbb{P}^1(K)$ be defined as the complement of the set $\mathscr{L} \subset \mathbb{P}^1(K)$ of limit points of the action of Γ, assuming that $\infty \in \mathscr{L}$. Let $F = F(K) \subset \Omega(K)$ be a good fundamental domain for Γ. Now, let $\alpha_g > 0$ such that

$$p^{f\alpha_g} > 2g \tag{1}$$

and define

$$H_\alpha(\beta x, \gamma y) = \mu^\Gamma(F)^{-1} \, |\pi|^{\alpha_g \ell(\beta^{-1}\gamma)} \, |\beta x - \gamma y|^{-\alpha} \tag{2}$$

for $x, y \in F$, $\beta, \gamma \in \Gamma$, and $\alpha > 0$, and where the Γ-invariant Borel measure on $\Omega(K) \setminus V(\omega)$ evaluated on sets is as

$$\mu^\Gamma(B) = \int_B |\omega|$$

for any $|\omega|$-measurable set $B \subset \Omega(K) \setminus V(\omega)$.

Now, define the operator

$$\mathcal{H}_\alpha u(\beta x) = \sum_{\gamma \in \Gamma} \int_F H_\alpha(\beta x, \gamma y)(u(y) - u(x)) \, |\omega(y)|$$

where $\omega \in \Omega^1(\Omega(K))^\Gamma$ is a Γ-invariant holomorphic differential 1-form on $\Omega(K)$, and $u \in C\,(\Omega(K), |\cdot|_\infty)^\Gamma$ or $u \in L^2(\Omega(K), |\omega|)$, and $x \in F$. Observe that $\mathcal{H}_\alpha$ is an operator of the following:

$$\begin{aligned}
&\mathcal{H}_\alpha \colon L^2(\Omega(K), |\omega|)^\Gamma \to L^2(\Omega(K), |\omega|)\\
&\mathcal{H}_\alpha \colon C(\Omega(K), \|\cdot\|_\infty)^\Gamma \to C(\Omega(K), \|\cdot\|_\infty)
\end{aligned}$$

for $\alpha > 0$. Further, there is a bilinear Dirichlet form

$$\begin{aligned}
\mathscr{E}_\alpha(u, v) &= \langle \mathcal{H}_\alpha u, \mathcal{H}_\alpha v \rangle_\omega\\
&= \sum_{\beta, \gamma \in \Gamma} \int_F \int_F H_\alpha(\beta x, \gamma y)(u(y) - u(x)) \left(\overline{v(y)} - \overline{v(x)}\right) |\omega(y)| \, |\omega(x)|
\end{aligned}$$

and a quadratic Dirichlet form

$$\mathscr{E}_\alpha(u) = \langle \mathcal{H}^\alpha u, \mathcal{H}^\alpha u \rangle_\omega$$

for $u, v \in L^2(\Omega(K), |\omega|)^\Gamma$.

Lemma 3.2 *The operator $\mathcal{H}_\alpha$ is densely defined for $\alpha > 0$.*

Denote the space of Γ-invariant locally constant functions on $\Omega(K)$ as $\mathcal{D}(\Omega(K))^\Gamma$.

Proof Let $u \in \mathcal{D}(\Omega(K))^\Gamma$. Then

$$\begin{aligned}\mathcal{H}_\gamma^\alpha u(\beta x) &:= \int_F H_\alpha(\beta x, \gamma y)(u(y) - u(x)) \, |\omega(y)| \\ &= \mu^\Gamma(F)^{-1} \, |\pi|^{\alpha_g \ell(\beta^{-1}\gamma)} \int_F |\beta x - \gamma y|^{-\alpha} \, (u(y) - u(x)) \, |\omega(y)|\end{aligned}$$

for $x \in F$. Now, the distance between x and γy can be arbitrarily large for fixed $x, y \in F$. It takes as values natural powers of $|\pi|^\alpha$. Since u is locally constant, it now follows that the integral term in $\mathcal{H}_\gamma^\alpha u(\beta x)$ converges for all $\gamma \in \Gamma$ to a value bounded from above by a positive constant times a power of $|\pi|$. By assumption (1), the number of $\gamma \in \Gamma \setminus \beta$ for which the values $\mathcal{H}_\gamma^\alpha u(x)$ are fixed, is bounded from above by $(2g)^{\ell(\beta)+\ell}$ with $\ell = \ell(\gamma) > 0$ fixed, cf. Lemma 3.1, meaning that the infinite sum

$$\mathcal{H}_\alpha u(\beta x) = \sum_{\gamma \in \Gamma} \mathcal{H}_\gamma^\alpha u(\beta x)$$

is bounded from above by a constant times a geometric series in a power of

$$2g \, |\pi|^{\alpha_g} \, |\pi|^\alpha < 1$$

and hence converges for any $\beta \in \Gamma$, $x \in F$, and $\alpha > 0$. □

Lemma 3.3 *The quadratic Dirichlet form $u \mapsto \mathcal{E}_\alpha(u)$ is densely defined.*

Proof Let $u \in \mathcal{D}(\Omega(K))^\Gamma$. The value of $\mathcal{E}_\alpha(u)$ is

$$\begin{aligned}\mathcal{E}_\alpha(u) &= \langle \mathcal{H}_\alpha u, \mathcal{H}_\alpha u \rangle_\omega \\ &= \mu^\Gamma(F)^{-2} \sum_{\beta, \gamma \in \Gamma} |\pi|^{2\alpha_g \ell(\beta^{-1}\gamma)} \\ &\quad \times \iint_{F^2} |\beta x - \gamma y|^{-\alpha} \, |u(y) - u(x)|^2 \, |\omega(y)| \, |\omega(x)|\end{aligned}$$

whose convergence is shown similarly as in the proof of Lemma 3.2. □

Let $A = L^2(\Omega(K), |\omega|)$ and $\mathcal{H}_\alpha^* \colon A \to A^\Gamma$ the adjoint operator of $\mathcal{H}_\alpha \colon A^\Gamma \to A$. Now, define

$$\Delta_\alpha := \mathcal{H}_\alpha^* \circ \mathcal{H}_\alpha \colon A^\Gamma \to A^\Gamma$$

as an operator on Γ-invariant functions on $\Omega(K)$, or on functions on the Mumford curve $X(K)$, which is the same thing. There is also an operator

$$\Delta_\alpha^\dagger := \mathcal{H}_\alpha \circ \mathcal{H}_\alpha^* \colon A \to A$$

for $\alpha > 0$.

Lemma 3.4 *The operators $\mathcal{H}_\alpha$, $\mathcal{H}_\alpha^*$ are closed, the operators Δ_α and $\Delta_\alpha^\dagger$ are self-adjoint, and the operators $I + \Delta_\alpha$, $I + \Delta_\alpha^\dagger$ have bounded inverses for $\alpha > 0$.*

Proof In order to see that $\mathcal{H}_\alpha$ is closed, assume $u_n \in \operatorname{dom}(\mathcal{H}_\alpha)$ such that $u_n \to u \in L^2(\Omega(K), |\omega|)^\Gamma$, and $\mathcal{H}_\alpha u_n \to v \in L^2(\Omega(K), |\omega|)$. Then

$$\|\mathcal{H}_\alpha u - v\|_\omega \le \|\mathcal{H}_\alpha u - \mathcal{H}_\alpha u_n\|_\omega + \|\mathcal{H}_\alpha u_n - v\|_\omega$$

and the second summand tends to zero by assumption. The square of the first summand is

$$\|\mathcal{H}_\alpha u - \mathcal{H}_\alpha u_n\|_\omega^2 = \mathcal{E}_\alpha(u - u_n) \to 0$$

because

$$\int_F |\beta x - \gamma y|^{-\alpha} \left(u_n(y) - u(y) + u_n(x) - u(x)\right) |\omega(y)|$$

tends to zero for $n \to \infty$ for all $\beta, \gamma \in \Gamma$, as $u - u_n$ tends to the constant zero function. It follows that $\mathcal{H}_\alpha u = v \in L^2(\Omega(K), |\omega|)$, i.e. $u \in \operatorname{dom}(\mathcal{H}_\alpha)$ for $\alpha > 0$. The closedness of the adjoint is now a standard fact, and the remaining assertions follow from von Neumann's Theorem on the adjoint [23, p. 200]. □

A consequence is that it is also possible to write

$$\mathcal{E}_\alpha(u) = \langle \mathcal{H}_\alpha u, \mathcal{H}_\alpha u \rangle_\omega = \langle \Delta_\alpha u, u \rangle_\omega$$

for $u \in \operatorname{dom}(\mathcal{E}_\alpha)$ using the self-adjoint operator Δ_α on $L^2(\Omega(K), |\omega|)^\Gamma$.

4 Spectrum

A Kozrev wavelet is a function

$$\psi_{B,j}(x) = \mu_K(B)^{-\frac{1}{2}} \chi(\pi^{d-1}\tau(j)\,x)\Omega(x \in B)$$

where $B \subset K$ is a disc of radius $|\pi|^{-d}$, $d \in \mathbb{Z}$, $j \in (O_K/\pi O_K)^\times$, and $\tau \colon O_K/\pi O_K \to K$ a lift. They were introduced by S. Kozyrev as an eigenbasis in $L^2(\mathbb{Q}_p, |dx|)$ for the p-adic Vladimirov operator [10].

Lemma 4.1 *It holds true that*

$$\int_{|x|=|\pi|^k} \chi(ax)\,|x|^m\,|dx| = \begin{cases} |\pi|^{k(m+1)}\,(1-|\pi|), & |a| \le |\pi|^{-k} \\ -\,|\pi|^{k(m+1)+1}\,, & |a| = |\pi|^{-k-1} \\ 0, & \textit{otherwise} \end{cases}$$

for $k, m \in \mathbb{Z}$.

Proof It holds true that

$$\int_{|x|=|\pi|^k} \chi(ax)\,|x|^m\,|dx| = |\pi|^{km} \int_{|x|=|\pi|^k} \chi(ax)\,|dx|$$

which shows how the assertion follows from the case $m = 0$. That case is shown e.g. in [12, Lem. 3.6] in the case $K = \mathbb{Q}_p$. His proof carries over to general K in a straightforward manner. □

Lemma 4.2 *Let* $a \in K$ *with* $|a| = |\pi|^{d-1}$ *for* $d \in \mathbb{Z}$, *and let* $m \in \mathbb{N}$. *Then it holds true that*

$$\int_{|x|\le|\pi|^\ell} \chi(ax)\,|x|^m\,|dx| = \begin{cases} C(m)\,|\pi|^{\ell(m+1)}\,, & \ell \ge 1-d \\ C(m)\,|\pi|^{(1-d)(m+1)} - |\pi|^{1-d(m+1)}\,, & \ell = -d \\ 0 & \textit{otherwise} \end{cases}$$

with

$$C(m) = \frac{1-|\pi|}{1-|\pi|^{m+1}}$$

In particular, the integral vanishes, if and only if $m = 0$ *and* $\ell \le -d$.

Proof It holds true that

$$\int_{|x|=|\pi|^\ell} \chi(ax)\,|x|^m\,|dx| = \sum_{k=\ell}^{\infty} \int_{|x|=|\pi|^k} \chi(ax)\,|x|^m\,|dx|$$

and according to Lemma 4.1, the right hand side vanishes, if and only if $d < -\ell$, as asserted. If $d > -\ell$, then the right hand side equals

$$\sum_{k=\ell}^{\infty} |\pi|^{k(m+1)}\,(1-|\pi|) = C(m)\,|\pi|^{\ell(m+1)}$$

as asserted. In the remaining case that $d = -\ell$, it holds true that the right hand side equals

$$-\,|\pi|^{1-d(m+1)} + \sum_{k=1-d} |\pi|^{k(m+1)} = C(m)\,|\pi|^{(1-d)(m+1)} - |\pi|^{1-d(m+1)}$$

again asserted. □

Lemma 4.3 *Let $B = B_\ell(a) \subset K$ be a disc not containing the r points $a_1, \dots, a_r \in K$. then the polynomial*

$$h(x) = \prod_{i=1}^{r}(x - a_i)$$

restricted to B has the constant absolute value

$$|h|_B(x)| = \prod_{i=1}^{r}|a - a_i|$$

for $x \in B$.

Proof Since $a_1, \dots, a_r$ are not in B, it follows that

$$|a - a_i| > |\pi|^\ell$$

for all $i = 1, \dots, r$. Hence, for all $x \in B$, it holds true that

$$|x - a_i| = |x - a + a - a_i| = |a - a_i|$$

for $i = 1, \dots, r$. This proves the assertion. □

Corollary 4.4 *Let $\psi_{B,j}$ be a Kozyrev wavelet on $\Omega(K)$. Then*

$$\int_B \psi_{B,j}(x)\,|\omega(x)|$$

vanishes if B does not contain any zero of ω.

Proof This follows immediately from Lemma 4.3 and the well-known result by Kozyrev, cf. [8, Thm. 3.29] or [1, Thm. 9.4.2]. □

Let ω be a Γ-invariant regular 1-form on $\Omega(K)$. Then, according to Lemma 4.3,

$$|\omega(x)| = C_B\,|dx| \tag{3}$$

where

$$C_B = C \cdot \prod_{i=1}^{r}|x - a_i|$$

for some $C > 0$ and $a_1, \dots, a_r \in F$ are the zeros of ω in F.

Corollary 4.5 *It holds true that*

$$C_B = C_{\beta B} \quad \text{and} \quad \left|\beta'(x)\right| = 1$$

for all $\beta \in \Gamma$ and $x \in \Omega(K) \setminus V(\omega)$, where $B \subset \Omega(K) \setminus V(\omega)$ is a disc.

Proof Assume w.l.o.g. that $B \subset F$. The first statement now follows immediately from

$$C_B\,|dx| = |\omega(x)| = |\omega(\beta x)| = C_{\beta B}\,|dx|$$

for $\beta \in \Gamma$, because, since ω is Γ-invariant, βB also does not contain any zeros of ω, and a similar reasoning as in the proof of Lemma 4.3 can be used. This also explains why the constant factor $C_{\beta B}$ exists in the first place.

Now, β' does not have any zeros or poles in B, because the zeros and poles of β' are zeros or poles of ω. But B is away from the zeros of ω, and ω is a regular differential form, i.e. has no poles. Hence, $\left|\beta'\right|$ is readily seen to be locally constant on B. From the Γ-invariance of ω, it follows that this is actually constant equalling to one, because

$$C_{\tilde{B}}\,\mu_K = |\omega| = \left|\omega \circ \beta^{-1}\right| = C_{\tilde{B}}\left|\beta'\right|^{-1}\mu_K$$

as measures on B. This implies that $\left|\beta'(z)\right| = 1$ for $z \in B$. But since $\Omega(K) \setminus V(\Omega)$ can be covered by discs, it follows that $\left|\beta'(x)\right| = 1$ for all $x \in \Omega(K) \setminus V(\omega)$. □

Remark 4.6 If B contains a zero of ω, then it does happen that the $|\omega|$-mean of a Kozyrev wavelet supported in B does not vanish, as can be seen in the case of $0 \in V(\omega)$ and B a small disc containing 0, by using Lemma 4.2. However, it is not clear to the author whether this holds true in all cases, i.e. the converse implication in Corollary 4.4 might possibly not hold true.

Lemma 4.7 *Let $\gamma \in \Gamma$. Then*

$$\gamma(x) - \gamma(y) = \gamma'(x)^{\frac{1}{2}}\gamma'(y)^{\frac{1}{2}}(x - y)$$

for a suitable choice of square root in K.

Proof Assume that $\gamma \in \Gamma$ is represented by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K)$$

Then

$$\gamma'(z) = \frac{1}{(cz + d)^2}$$

and

$$\gamma(x)-\gamma(y)=\frac{ax+b}{cx+d}-\frac{ay+b}{cy+d}=\frac{(ax+b)(cy+d)-(ay+b)(cx+d)}{(cx+d)(cy+d)}$$
$$=\frac{(ad-bc)(x-y)}{(cx+d)(cy+d)}=\gamma'(x)^{\frac{1}{2}}\gamma'(y)^{\frac{1}{2}}(x-y)$$

for a suitable choice of square roots in K, as asserted. □

Lemma 4.8 *Let $x, y \in \Omega(K)\setminus V(\omega)$. Then*

$$|\beta x-\gamma y|=\left|x-\beta^{-1}\gamma y\right|$$

for $\beta, \gamma \in \Gamma$.

Proof Let $x, y \in \Omega(K)\setminus V(\omega)$. Then

$$\left|x-\beta^{-1}\gamma y\right|=\left|\beta^{-1}\beta x-\beta^{-1}\gamma y\right|=|\beta x-\gamma y|\left|\beta'(\beta x)\right|^{-\frac{1}{2}}\left|\beta'(\gamma y)\right|^{-\frac{1}{2}}$$

for $\beta, \gamma \in \Gamma$. Hence, the assertion follows from Corollary 4.5. □

Lemma 4.9 *It holds true that*

$$\int_F |x-y|^{-\alpha}\left(\psi_{B,j}(y)-\psi_{B,j}(x)\right)|\omega(y)|$$
$$=-\left(\int_{F\setminus B}|x-y|^{-\alpha}|\omega(y)|+\mu^{\Gamma}(B)^{1-\alpha}\right)\psi_{B,j}(x)$$

for $x \in K$, $B \subset \Omega(K)\setminus V(\omega)$ a disc, and $j \in O_K/\pi O_K$.

Proof This follows from [11, Thm. 3], as the conditions for that theorem to be valid are satisfied. □

A Kozyrev wavelet $\psi_{B,j}(x)$ supported on a disc $B \subset F$ can be extended to a Γ-invariant function

$$\psi^{\Gamma}_{B,j}(\gamma x):=\psi_{B,j}(x)$$

for all $\gamma \in \Gamma$. Call this function a Γ-*invariant Kozyrev wavelet*.

Define the number

$$N_F(B):=\left|\left\{\text{discs } \tilde{B}\subset F \mid \mu^{\Gamma}(\tilde{B})=\mu^{\Gamma}(B)\ \wedge\ \forall\gamma\in\Gamma\colon\ I_F(\gamma\tilde{B})=I_F(\gamma B)\right\}\right|$$

for a given disc $B \subset F$ and

$$I_F(\gamma B):=\int_{F\setminus B}|x-\gamma y|^{-\alpha}|\omega(y)|$$

for $\gamma \in \Gamma, \alpha > 0$.

Theorem 4.10 *The space $L^2(\Omega(K) \setminus V(\omega), |\omega|)^\Gamma$ of Γ-invariant L^2-functions on $\Omega(K) \setminus V(\omega)$ has an orthonormal basis consisting of the Γ-periodic wavelets $\psi^\Gamma_{B,j}$ supported in $\Omega(K) \setminus V(\omega)$, and these are eigenfunctions of $\Delta_\alpha^{\frac{1}{2}}$ for $\alpha > 0$. The eigenvalue corresponding to $\psi^\Gamma_{B,j}$ is*

$$\lambda_B = \mu^\Gamma(F)^{-1} \sum_{\gamma\in\Gamma} |\pi|^{\alpha_g \ell(\gamma)} \left(\int_{F\setminus B} |x - \gamma y|^{-\alpha} \, |\omega(y)| + \mu^\Gamma(B)^{1-\alpha} \right)$$

for $j \in O_K/\pi O_K \setminus \{0\}$, $B \subset F \setminus V(\omega)$ a disc whose Γ-translates form the support of $\psi_{B,j}$, and F a good fundamental domain for the action of Γ. Here, $x \in B$, and λ_B does not depend on $x \in B$. The multiplicity of eigenvalue λ_B is $N_F(B) \cdot (p^f - 1)$. Both, λ_B and its multiplicity, are invariant under replacing F with γF for any $\gamma \in \Gamma$. The restriction of $\Delta_\alpha^{\frac{1}{2}}$ to $L^2(\Omega(K) \setminus V(\omega), |\omega|)^\Gamma$ coincides with $-\mathcal{H}_\alpha$ for $\alpha > 0$.

This is Theorem 1.

Proof The Γ-invariant function $\psi^\Gamma_{B,j}(x)$ is an element of $L^2(\Omega(K) \setminus V(\omega), |\omega|)^\Gamma$, because

$$\int_{\Omega(K)} \psi^\Gamma_{B,j}(x) \; |\omega(x)| = \sum_{\gamma\in\Gamma} \int_B \psi^\Gamma_{B,j}(\gamma x) \, |\omega(x)| = \sum_{\gamma\in\Gamma} \int_B \psi_{B,j}(x) \, |\omega(x)| = 0$$

where the last equation holds true by Corollary 4.4. Since any Γ-periodic L^2-function on $\Omega(K) \setminus V(\omega)$ has to have mean zero, it now follows that the space of Γ-periodic L^2-functions on that space is spanned by the Γ-invariant Kozyrev wavelets supported in $\Omega(K)\setminus V(\omega)$, as these are in $1-1$-correspondence with the Kozyrev wavelets which are an orthonormal basis of $L^2(F \setminus V(\omega), \mu_K)$. Notice that the measure $|\omega|$ differs from μ_K on the support of any Kozyrev wavelet only by a constant factor according to (3). Therefore, the different choices of measures for those Hilbert spaces are not an issue.

Now, let $\beta, \gamma \in \Gamma$. Then

$$\begin{aligned}
&\mu^\Gamma(F) \, |\pi|^{-\alpha_g \ell(\beta^{-1}\gamma)} \; \mathcal{H}^\alpha_\gamma \psi^\Gamma_{B,j}(\beta x) \\
&\overset{\text{Lem. 4.8}}{=} \int_F \left| x - \beta^{-1}\gamma y \right|^{-\alpha} \left(\psi^\Gamma_{B,j}(y) - \psi^\Gamma_{B,j}(x) \right) |\omega(y)| \\
&= -\left(\int_{F\setminus B} \left| x - \beta^{-1}\gamma y \right|^{-\alpha} |\omega(y)| + \mu^\Gamma(B)^{1-\alpha} \right) \psi^\Gamma_{B,j}(x)
\end{aligned}$$

for $x \in F$, where the last equality uses [11, Thm. 3.1] in a similar manner as Lemma 4.9.

What has been established so far, is that $\psi^\Gamma_{B,j} \in L^2(\Omega(K) \setminus V(\omega), |\omega|)^\Gamma$ is an eigenfunction of $\mathcal{H}^\alpha_\gamma$ for any $\gamma \in \Gamma$ and $\alpha > 0$. This means that $\mathcal{H}_\alpha$ takes the closed subspace $L^2(\Omega(K)\setminus V(\omega), |\omega|)^\Gamma$ to itself. Hence, Δ_α equals the square of the restriction of $\mathcal{H}_\alpha$ to that space, since the Γ-invariant Kozyrev eigenvalues of $\mathcal{H}_\alpha$ are real numbers.

Now, it follows that

$$\begin{aligned}
&\mathcal{H}_\alpha \psi_{B,j}^\Gamma(\beta x)\\
&\quad= -\mu^\Gamma(F)^{-1}\sum_{\gamma\in\Gamma}|\pi|^{\alpha_g \ell(\beta^{-1}\gamma)}\left(\int_{F\setminus B}\left|x-\beta^{-1}\gamma y\right|^{-\alpha}|\omega(y)| + \mu^\Gamma(B)^{1-\alpha}\right)\psi_{B,j}^\Gamma(x)\\
&\quad= -\mu^\Gamma(F)^{-1}\sum_{\gamma\in\Gamma}|\pi|^{\alpha_g \ell(\gamma)}\left(\int_{F\setminus B}|x-\gamma y|^{-\alpha}|\omega(y)| + \mu^\Gamma(B)^{1-\alpha}\right)\psi_{B,j}^\Gamma(x)
\end{aligned}$$

where the last equality follows from the fact that summation over $\gamma\in\Gamma$ is the same as summation over $\beta^{-1}\gamma\in\Gamma$. Hence, the expression does not depend on the choice of $\beta\in\Gamma$. Hence, the $\psi_{B,j}^\Gamma$ is an eigenfunction of $\Delta_\alpha^{\frac{1}{2}}$ for $\alpha>0$ with eigenvalue λ_B as stated. Indeed, it can be checked that the infinite sum does converge, because γy never falls into B, implying that $|x-\Gamma y|$ does not become arbitrarily small. This proves the value of eigenvalue $-\lambda_B$ of $\mathcal{H}_\alpha$, or, equivalently, that of eigenvalue λ_B of $\Delta_\alpha^{\frac{1}{2}}$, and that λ_B does not depend on $x\in B$. Hence,

$$\Delta_\alpha^{\frac{1}{2}} = -\mathcal{H}_\alpha$$

for $\alpha>0$, as asserted.

As to the multiplicities, clearly, λ_B does not depend on the choice of $j\in O_K/\pi O_K$. This accounts for the factor (p^f-1) in its multiplicity. The other factor is obtained by observing that λ_B only depends on the Γ-invariant volume of disc $B\subset F$ and a summation of $I_F(\gamma B)$ terms, which is invariant by Lemma 4.8. This again yields a finite contribution to the multiplicity, as F is compact, and it also follows that both, λ_B and its multiplicity, are invariant under replacing F with γF for $\gamma\in\Gamma$. This proves the theorem. □

The eigenvalue of $\psi_{B,j}$ is invariant under the action of Γ, but there is a dependence on the choice of a good fundamental domain modulo Γ-equivalence, which likely leads to different spectra for different such choices. Anyway, if $\phi\colon F\to\tilde{F}$ is a bianalytic map between two fundamental domains, then

$$H_\alpha(\beta\phi(x),\gamma\phi(y)) = \mu^\Gamma(\tilde{F})^{-1}|\pi|^{\alpha_g\ell(\beta^{-1}\gamma)}|\beta\phi(x)-\gamma\phi(y)|^{-\alpha}$$

and

$$|\omega(\phi(y))| = |f(\phi(y))|\left|\phi'(y)\right||dy|$$

where ω on $\tilde{F}$ takes the form:

$$\omega(z) = f(z)\,dz$$

for some holomorphic function $f\colon F \to K$. This leads to

$$\mathcal{H}_\alpha u(\beta\phi(x)) = \mu^\Gamma(\tilde{F})^{-1} \sum_{\gamma\in\Gamma} |\pi|^{\alpha_g \ell(\beta^{-1}\gamma)} \int_F |\beta\phi(x) - \gamma\phi(y)|^{-\alpha} \, |f(\phi(y))| \left|\phi'(y)\right| |dy| \quad (4)$$

for functions $u\colon \tilde{F} \to \mathbb{C}$, and $\beta \in \Gamma$.

Lemma 4.11 *The quantity λ_B corresponding to $\psi_{B,j}(x)$ transforms under ϕ to $\lambda_{\phi(B)}$ with*

$$\lambda_{\phi(B)} = \mu^\Gamma(\phi(F))^{-1} \times \sum_{\gamma\in\Gamma} |\pi|^{\alpha_g \ell(\gamma)} \left(\int_{F\setminus B} |x - \gamma\phi(z)|^{-\alpha} \left|\phi'(z)\right| |\omega(z)| - \mu^\Gamma(\phi(B)) \right)$$

where $B \subset \Omega(K) \setminus V(\omega)$ is a disc.

Proof Since the bi-analytic pre-image of a p-adic disc is a p-adic disc, the expression $\lambda_{\phi(B)}$ is a well-defined eigenvalue of a well-defined Γ-periodic wavelet. The expression for $\lambda_{\phi(B)}$ follows in a straightforward manner. □

Remark 4.12 Both, the genus and the geometry of a Mumford curve are encoded in the spectrum of $-\Delta_\alpha^{\frac{1}{2}}$, as can be seen in Theorem 4.10. Firstly, via the number of elements of Γ of a given length, leading to a given coefficient in the infinite sum making up λ_B. This coefficient thus depends on the number g of free generators of Γ, i.e. the genus of X. Secondly, via the integral $\int_{F\setminus B}$ which is determined by the geometry of a Mumford curve via the holes in a good fundamental domain.

5 Feller Property

Lemma 5.1 *The linear operator $\mathcal{H}_\alpha = -\Delta_\alpha^{\frac{1}{2}}$ generates a Feller semigroup $\exp\left(-t\Delta_\alpha^{\frac{1}{2}}\right)$ with $t \geq 0$ on $C(\Omega(K), \|\cdot\|_\infty)^\Gamma$ for $\alpha > 0$.*

Proof The criteria given by the Hille–Yosida–Ray Theorem are verified, cf. [5, Ch. 4, Lem. 2.1].

1. The domain of $-\Delta_\alpha^{\frac{1}{2}}$ is dense in $C(\Omega(K), \mathbb{R})^\Gamma$. This follows from Lemma 3.2.
2. $-\Delta_\alpha^{\frac{1}{2}}$ satisfies the positive maximum principle. For this, let $h \in \mathcal{D}(\Omega(K))^\Gamma$, and $x_0 \in \Omega(K)$ such that $h(x_0) = \sup\limits_{x\in\Omega(K)} h(x)$. This exists, because h is Γ-periodic, and

the fundamental domain F is compact. Then

$$-\Delta_{\alpha}^{\frac{1}{2}} h(x_0) \leq \int_{\Omega(K)} H_\alpha(x_0, y)(h(x_0) - h(x_0)) \, |\omega(y)| \leq 0$$

which implies the positive maximum principle.

3. $\mathrm{Ran}(\eta I + \Delta_\alpha)$ is dense in $C(\Omega(K), \mathbb{R})^\Gamma$ for some $\eta > 0$. Since $-\Delta_{\alpha}^{\frac{1}{2}}$ is unbounded, an approach different the proof of [24, Lem. 4.1] is required. Let $h \in C(\Omega(K), \mathbb{R})$, $\eta > 0$. The task is to find a solution of the equation

$$\left(\eta I + \Delta_{\alpha}^{\frac{1}{2}}\right) u = h \tag{5}$$

for some $\eta > 0$ and h in some dense subspace of $C(\Omega(K), \mathbb{R})^\Gamma$. The equation formally can be rewritten as

$$u(z) - \frac{\int H_\alpha(z, y) u(y) \, |\omega(y)|}{\eta + \deg(z)} = \frac{h(z)}{\eta + \deg(z)} \tag{6}$$

with

$$\deg(z) = \int_{\Omega(K)} H_\alpha(z, y) \, |\omega(y)|$$

which does not converge, as the operator $\Delta_{\alpha}^{\frac{1}{2}}$ is unbounded. That is why the operator

$$T_k u(z) = \frac{\int_{\Omega_{z,k}} H_\alpha(z, y) u(y) \, |\omega(y)|}{\eta + \deg_k(z)}$$

with

$$\Omega_{z,k} = \bigsqcup_{\gamma \in \Gamma} \gamma F_{z,k}$$

and

$$F_{z,k} = F \setminus B_k(z)$$

for $k \gg 0$ is now being studied. Let

$$\deg_k(z) = \int_{\Omega_{z,k}} H_\alpha(z, y) \, |\omega(y)|$$

which is finite for $k \gg 0$, and

$$|T_k u(z)| \leq \frac{\deg_k(z)}{\eta + \deg_k(z)} \, \|u\|_\infty$$

where the supremum norm of u is finite, as u is Γ-invariant and F is compact. Hence,

$$\|T_k\| \leq \frac{1}{\eta/\deg_k(z)+1} < 1$$

for any $\eta > 0$, and $k \gg 0$, and in this case it follows that $I + T_k$ has a bounded inverse as an operator on $C(\Omega(K), \mathbb{R})^\Gamma$. This proves the denseness of its range for $k \gg 0$.

Now, let $h \in \mathcal{D}(\Omega(K))^\Gamma$, and $u_k, u_\ell \in C(\Omega(K), \mathbb{R})^\Gamma$ be solutions of

$$(I+T_k)u_k = \frac{h}{\eta+\deg_k}, \quad (I+T_\ell)u_\ell = \frac{h}{\eta+\deg_\ell}$$

for $k, \ell \gg 0$. Then

$$u_k - u_\ell = \frac{(I+T_\ell)(\eta+\deg_\ell)-(I+T_k)(\eta+\deg_k)}{(I+T_k)(I+T_\ell)(\eta+\deg_k)(\eta+\deg_\ell)}h \tag{7}$$

shows that u_k is a Cauchy sequence w.r.t. $\|\cdot\|_\infty$. The reason is that, firstly,

$$\|T_k\| = \sup_{z\in F}\frac{\deg_k(z)}{\eta+\deg_k(z)}$$

clearly holds true, and this is a (strictly increasing) sequence convergent to 1, and this implies the convergence of the sequence of operators T_k to a bounded linear operator T on $C(\Omega(K), \mathbb{R})^\Gamma$. Secondly, the numerator of the right hand side of (7) is

$$\eta(T_\ell - T_k) + (\deg_\ell - \deg_k) + (T_\ell \deg \ell - T_k \deg_k)$$

whose first and second terms become arbitrarily small in norm as $\ell \geq k \to \infty$. The third term is

$$T_\ell \deg_\ell - T_k \deg_k = (T_\ell \deg_\ell - T_k \deg_\ell) + (T_k \deg_\ell - T_k \deg_k)$$

both of whose terms in norm become arbitrarily small as $\ell \geq k \to \infty$. Hence, u_k converges to some $u \in C(\Omega(K), \mathbb{R})^\Gamma$ which is seen to be a solution of (5) by using the limit operator T as follows: Namely,

$$(\eta+\deg_k)T_k \to (\eta+\deg)T \quad (k\to\infty)$$

where the limit operator coincides with the unbounded integral operator

$$u \mapsto Au = \int_{\Omega(K)} H_\alpha(\cdot, y)u(y)\,|\omega(y)|$$

which shows that the operator

$$\frac{A}{\eta + \deg} = T$$

appearing in (6) is bounded. Now, u_k is a solution of

$$(\eta I - H_k)u_k = h$$

with

$$H_k = (\eta + \deg_k)T_k - \deg_k$$

which for $k \to \infty$ converges to $-\Delta_\alpha^{\frac{1}{2}}$. As $u_k \to u$, it follows that

$$(\eta I + \Delta_\alpha^{\frac{1}{2}})u = (\eta I - H_k)u + (H_k + \Delta_\alpha^{\frac{1}{2}})u$$

where

$$(\eta I - H_k)u = (\eta I - H_k)u_k + H_k(u_k - u) = h + H_k(u_k - u) \to h$$

and

$$(H_k + \Delta_\alpha^{\frac{1}{2}})u \to 0$$

for $k \to \infty$. Hence, u is a solution of (5). This proves that the range of $\eta I + \Delta_\alpha^{\frac{1}{2}}$ contains the real-valued functions in $\mathcal{D}(\Omega(K))^\Gamma$ which is dense in $C(\Omega, \mathbb{R})^\Gamma$.

Now, by Hille–Yosida–Ray, the assertion follows. □

Theorem 5.2 *There exists a probability measure $p_t(x, \cdot)$ with $t \geq 0$, $x \in \Omega(K) \setminus V(\omega)$ on the Borel σ-algebra of $\Omega(K) \setminus V(\omega)$ such that the Cauchy problem for the heat equation*

$$\left(\frac{\partial}{\partial t} + \Delta_\alpha^{\frac{1}{2}}\right) h(t, x) = 0$$

having initial condition $h(0, x) = h_0(x) \in C(\Omega(K) \setminus V(\omega), \|\cdot\|_\infty)^\Gamma$ has a unique solution in $C^1((0, \infty), \Omega(K) \setminus V(\omega))^\Gamma$ of the form

$$h(t, x) = \int_{\Omega(K)\setminus V(\omega)} h_0(y) p_t(x, |\omega(y)|$$

Additionally, $p_t(x, \cdot)$ is the transition function of a strong Markov process on $(\Omega(K) \setminus V(\omega))/\Gamma$ whose paths are càdlàg.

This is Theorem 2. The notation $C^1\left((0,\infty),\Omega(K)\setminus V(\omega)\right)^\Gamma$ indicates that for each $t>0$, any such function $h(t,x)$ is Γ-invariant.

Proof According to Lemma 5.1, $-\Delta_\alpha$ generates a Feller semigroup on the Banach space $C(\Omega(K),\|\cdot\|_\infty)^\Gamma$. Using [2, Prop. 15] allows to restrict to the the closed subspace $C(\Omega(K)\setminus V(\omega),\|\cdot\|_\infty)^\Gamma$ invariant under $-\Delta_\alpha^{\frac{1}{2}}$. Hence, $-\Delta_\alpha^{\frac{1}{2}}$ generates a Feller semigroup also on that space. Now, it can be argued as in the proof of [24, Thm. 4.2], namely that there exists a uniformly stochastically continuous C_0-transition function $p_t(x,|\omega(y)|)$ satisfying condition (L) of [16, Thm. 2.10] such that

$$e^{-t\Delta_\alpha^{\frac{1}{2}}}h_0(x)=\int_{\Omega(K)\setminus V(\omega)}h_0(y)p_t(x,|\omega(y)|)$$

cf. [16, Thm. 2.15]. From the correspondence between transition functions and Markov processes, there now exists a strong Markov process on the quotient space $(\Omega(K)\setminus V(\omega))/\Gamma$, which consists of $X(K)$ minus finitely many points, and whose paths are càdlàg, cf. [16, Thm. 2.12]. □

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