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ABSTRACT

p -Adic divergence and gradient operators are constructed giving rise to p -adic vertex Laplacian operators used by Zúñiga in order to study Turing patterns on graphs, as well as their edge Laplacian counterparts. It is shown that the Euler characteristic of a finite graph can be expressed via traces of certain heat kernels associated with these new operators. This result is applied to the extraction of topological information from Mumford curves via heat kernels.

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I. INTRODUCTION

The well-known relationship between the divergence and gradient operators for functions on \mathbb{R}^n and corresponding vector fields reveals them as mutually adjoint linear operators. In equations describing the motion of particles, these two operators are used in order to describe advection and diffusion. The former uses the divergence operator, and the latter the Laplacian which is gradient followed by divergence. In a discrete (finite) setting, these two operators are mimicked by the incidence matrix (divergence) and its transpose (gradient). The mathematical significance of these is that the singular value theorem says that the two Laplacians which can be constructed from these (edge Laplacian and vertex Laplacian) have a common spectrum whose difference lies only in the multiplicity of eigenvalue zero. In the case of finite graphs, this difference turns out to coincide with the Euler characteristic of the graph. Using the heat kernels corresponding to the two Laplacians then leads to the well-known result that the index of the gradient operator equals the difference between the traces of these heat kernels. Already Roth had seen one “half” of this theorem by proving that the trace of the heat kernel associated with the (vertex) Laplacian of a finite graph has an expansion into a term containing the Euler characteristic and decaying exponential functions over the spectrum.²³

On the p -adic side of mathematics and applications, various types of analoga of partial differential equations have been developed: e.g., the diffusion equation,³ Navier-Stokes equation,^{17,22} porous medium equation.¹⁶ What is common to that research is that the equations all live on \mathbb{Q}_p or subdomains thereof, and that a p -adic Laplacian is used without further decomposition. Most of the p -adic analysis needed for these tasks can be found e.g., in Ref. 24 or Ref. 1. The study of the heat equation on other p -adic domains, like compact manifolds, has been initiated in the special case of a so-called *Mumford curve*.⁵ These have a rich structure and are of interest in p -adic string theory, cf. Ref. 11.

In light of the manifold applications in mathematics and physics, which the celebrated Atiyah-Singer Index theorem has,⁴ it is actually a desideratum to obtain results toward a p -adic formulation of an index theorem for p -adic manifolds, possibly using integral operators defined on these. In this context, the present article poses a first contribution for p -adic integral operators defined via finite graphs in a similar manner as Zúñiga-Galindo has done in the study of Turing patterns.²⁵ The difference to that reference is that here, the vertices are represented by compact open subsets of the field \mathbb{Q}_p of p -adic numbers instead merely of discs.

Using a p -adically defined divergence operator d^α and a corresponding gradient operator δ^α for $\alpha > 0$, corresponding p -adic vertex and edge Laplacian operators D^α and Δ^α can be defined which are self-adjoint and positive semi-definite. The vertex Laplacian D^α then is a generalisation of a Zúñiga operator also to be found in Ref. 5, and also slightly generalises the ones in Ref. 9 defined on all of \mathbb{Q}_p . These operators are one of the two types of Laplacian operators which can be constructed from divergence and gradient operators. These operators

are defined on the boundary of the p -adic tree associated with the p -adic numbers, and are thus related to the forward derivative on that tree, defined as in graph theory. In the case of the p -adic integers, it is used in order to construct p -adic versions of Dirac operators,¹⁸ and spectral properties of one of the two possible Laplacians constructed from it are studied.¹⁹ This begs the question of making explicit the relationship between these two kinds of viewpoints: from the tree structure vs from its boundary, and develop a theory of Dirac operators from both viewpoints.

The main result of this article is given as follows:

Theorem. *If every vertex of a finite graph G has a loop-edge, then*

$$\lim_{t \rightarrow \infty} \left(\text{tr}(p_{D^\alpha, t}) - \text{tr}(p_{\Delta_p^\alpha, t}) \right) = \chi(G') = \text{ind}(d^\alpha)$$

for $\alpha > 0$. Otherwise, it holds true that

$$\lim_{\ell \rightarrow \infty} \lim_{t \rightarrow \infty} \left(\text{tr}(p_{D_{\ell}^\alpha, t}) - \text{tr}(p_{\Delta_p^\alpha, t}) \right) = \chi(G') = \text{ind}(d^\alpha)$$

for $\alpha > 0$. The graph G' is obtained from G by removing all loop-edges.

The expression $p_{D^\alpha, t}$ denotes the heat kernel associated with D^α , whereas $p_{D_{\ell}^\alpha, t}$ and $p_{\Delta_p^\alpha}$ are heat kernels associated with finite-dimensional operators derived from D^α and Δ^α where cut-offs are made through selecting eigenfunctions. The reason for the need to take limits is that on the one hand, the kernel of Δ^α is infinite-dimensional, and the heat kernel of D^α does not have a well-defined trace unless all vertices of the graph G have loop-edges attached to them. Furthermore, the index of d^α , in order to be finite, is defined using the kernel of the operator Δ_p^α instead of that of Δ^α , but is using the kernel of D^α .

Finally, the theorem is applied to Mumford curves by constructing three kinds of graphs from a good fundamental domain in the sense of Ref. 14, Definition I.4.1.3. In this way, topological information can be extracted from a Mumford curve by heat kernels. This continues in a sense the hearing of shapes for graphs or p -adic geometric objects via p -adic analysis, as done in Refs. 6 and 7. For applications, this could be of interest, since on the one hand, using hierarchies allows a faster processing of structured data, and on the other hand using integral operators to extract topological information from data is a main topic of applied research.

The goal of the ongoing project *Distributed Simulation of Processes on Buildings and City Models* (project number 469999674) aims to provide a framework for simulating processes like e.g., heat flows on building and city models on distributed systems via p -adic numbers. The building or city model topologies can be represented as a T_0 -topology on the finite set of atomic objects of the model connected by a binary relation which can be interpreted as a boundary relationship.⁸ The Hasse diagram is then a minimal representation of this T_0 -topology. For details on the properties of such topological spaces, cf. Ref. 2.

The choice of p -adic numbers as an underlying structure for the Deutsche Forschungsgemeinschaft (DFG) project seems natural because of their inherent hierarchical structure, and the fact that processing hierarchical information in general is efficient due to their tree-like structuring. Since p -adic analysis is relatively young, the mathematics behind processes on hierarchically structured data still needs to be developed in many cases. The concern of a large part of p -adic analysis is diffusion, whereas other types of processes, e.g., advection, have not yet been studied to the author's knowledge. The gradient and divergence operators developed here can fill a part of this gap by providing a p -adic advection operator:

$$D_+^\alpha = \delta_+^\alpha d_+^\alpha : L^2(\Omega_V) \rightarrow L^2(\Omega_V)$$

for $\alpha > 0$, using the notation of Sec. III. This is in fact a natural generalisation of advection operators built from the outgoing incidence matrix of finite graphs introduced in Ref. 10. The study of p -adic advection equations like

$$\frac{\partial}{\partial t} u(x, t) + D_+^\alpha u(x, t) = 0$$

is left for future work in the ongoing project.

Another significance of the results of this article for the DFG-project is that the task of retrieving topological information from p -adic diffusion processes becomes more feasible. A further significance is that Mumford curves as compact p -adic manifolds are gradually coming into the focus of data analysis, not only because of the results of the previous section, but also because of recent project results showing that a hierarchical dataset can be viewed as being sampled from a Mumford curve together with a regular differential 1-form on it. This leads to the question about the possible shapes such a Mumford curve can take, including the question about its genus.

The following section lists some preliminaries helpful for reading this article. Section III constructs the divergence and gradient operators, shows that they are mutually adjoint, thus giving rise to two kinds of p -adic Laplacians, and shows how to obtain the Betti numbers of a finite graph from these. Section IV constructs the heat kernels, using the result from²⁵ that the p -adic vertex Laplacian defines a strong Markov process, and proves the main theorem. Section V applies the main theorem to Mumford curves in order to extract topological information via heat kernels.

II. PRELIMINARIES

Some familiarity with p -adic numbers is assumed. The field of p -adic numbers is denoted as \mathbb{Q}_p , and the p -adic absolute value with $|\cdot|_p$. As a locally compact Abelian group, \mathbb{Q}_p is endowed with a Haar measure, denoted as dx or dy , depending on the variable of integration used. It is assumed normalised such that

$$\int_{\mathbb{Z}_p} dx = 1$$

where $\mathbb{Z}_p \subset \mathbb{Q}_p$ is the p -adic unit disc, and where integration means that of real- or complex-valued functions on \mathbb{Q}_p against the Haar measure.

From p -adic analysis, variants of the integral operator version of the Vladimirov operator are used. The latter is studied e.g., in Ref. 24. The variants of interest here are first defined in Ref. 25. Of use for determining the spectra of such operators are the Kozyrev wavelets, where such is a function of the form

$$\psi(x) = p^{-\frac{d}{2}} \chi(p^{d-1} jx) \Omega\left(\left|p^d x - n\right|_p\right)$$

for $d \in \mathbb{Z}$, $j \in \{1, \dots, p-1\}$, and $n \in \mathbb{Q}_p$ the fractional part of a representative of an element of $\mathbb{Q}_p/\mathbb{Z}_p$, cf. Ref. 20. By the fractional part of a p -adic number

$$x = \sum_{k=-m} \alpha_k p^k$$

with $m \in \mathbb{Z}$ $\alpha_k \in \{0, \dots, p-1\}$ is meant the expression

$$\{x\}_p = \sum_{k=-m}^{-1} \alpha_k p^k$$

as a rational number. The indicator function $\Omega(|f(x)|_p)$ is defined as

$$\Omega(|f(x)|_p) = \begin{cases} 1, & |f(x)|_p \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

for a given function $f: \mathbb{Q}_p \rightarrow \mathbb{Q}_p$.

Some theory of finite graphs, viewed as 1-dimensional simplicial complexes is also helpful. A lot more than needed for this article about the homology and cohomology of such objects can be found in e.g., Ref. 15.

The last section deals with applications to Mumford curves. These are treated in depth in Refs. 13 and 14. What is needed here, however, is only their relationship to finite simple graphs in order to see that the p -adic version of an index theorem can be applied to these.

III. p -ADIC DIVERGENCE AND GRADIENT OPERATORS

Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set E . The graph G is assumed to have at most one loop-edge attached to a vertex. By a loop-edge is meant an edge connecting a vertex with itself. In particular, it is simple after removing all loop-edges.

The subset of loop-edges is denoted as E_0 , and

$$E' = E \setminus E_0$$

is the set of all simple edges of G . By choosing for each edge from E' a positive orientation, obtain the sets E_+ , E_- of positively and negatively oriented simple edges of G . This leads to one way of viewing simple graphs as 1-dimensional simplicial complexes which can be learned e.g., in Ref. 15, Chap. 2 and elsewhere.

Generalising an idea of Wilson Zúñiga-Galindo, identify the vertices $v \in V$ with pairwise disjoint p -adic compact open sets $U_v \subset \mathbb{Q}_p$ satisfying the condition

$$\forall v, w \in V : v \neq w \Rightarrow \forall x \in U_v \forall y \in U_w : |x - y|_p = \text{dist}(U_v, U_w) \quad (1)$$

where dist is the p -adic distance between sets, defined as

$$\text{dist}(A, B) = \inf \{|x - y|_p : x \in A, y \in B\}$$

for subsets $A, B \subseteq \mathbb{Q}_p$, and define now the compact open sets:

$$\Omega_V = \bigsqcup_{v \in V} U_v \subset \mathbb{Q}_p$$

$$\Omega_{E_0} = \bigsqcup_{e \in E_0} U_{o(e)} \times U_{o(e)} \subset \mathbb{Q}_p^2$$

$$\Omega_{E_+} = \bigsqcup_{e \in E_+} U_{o(e)} \times U_{t(e)} \subset \mathbb{Q}_p^2$$

$$\begin{aligned}\Omega_{E_-} &= \bigsqcup_{e \in E_+} U_{t(e)} \times U_{o(e)} \subset \mathbb{Q}_p^2 \\ \Omega_{E'} &= \Omega_{E_+} \cup \Omega_{E_-} \subset \mathbb{Q}_p^2 \\ \Omega_E &= \Omega_{E_0} \cup \Omega_{E_+} \cup \Omega_{E_-} \subset \mathbb{Q}_p^2\end{aligned}$$

where $o(e) \in V$ denotes the origin vertex of an oriented edge $e \in E_+$, and $t(e)$ its terminal vertex. Note that if $e \in E_0$, then $o(e) = t(e)$.

Let $\alpha > 0$, and define the following p -adic divergence operators:

$$\delta_\bullet^\alpha : L^2(\Omega_{E_\bullet}) \rightarrow L^2(\Omega_V), \quad h \mapsto \delta_\bullet^\alpha h$$

with

$$(\delta_\bullet^\alpha h)(x) = \int_{\Omega_V} h(x, y) |x - y|_p^{-\alpha} dy$$

for $h \in L^2(\Omega_{E_\bullet})$ and $\bullet \in \{+, -, 0\}$. Define also the following p -adic gradient operators:

$$d_0^\alpha : L^2(\Omega_V) \rightarrow L^2(\Omega_{E_0}), \quad f \mapsto d_0^\alpha f$$

with

$$(d_0^\alpha f)(x, y) = \frac{f(x) - f(y)}{|x - y|_p^\alpha}$$

for $(x, y) \in \Omega_{E_0}$,

$$d_+^\alpha : L^2(\Omega_V) \rightarrow L^2(\Omega_{E_+}), \quad f \mapsto d_+^\alpha f$$

with

$$(d_+^\alpha f)(x, y) = \frac{f(x) - f(y)}{2|x - y|_p^\alpha}$$

for $(x, y) \in \Omega_{E_+}$, and

$$d_-^\alpha : L^2(\Omega_V) \rightarrow L^2(\Omega_{E_-})$$

with

$$(d_- f)(x, y) = (d_+ f)(y, x)$$

for $(x, y) \in \Omega_{E_-}$.

The inner products on $L^2(\Omega_{E_\bullet})$, $L^2(\Omega_V)$ will be denoted as

$$\langle \cdot, \cdot \rangle_{E_\bullet}, \quad \langle \cdot, \cdot \rangle_V$$

and are defined in a standard way via integration. Notice that the operators $\delta_0^\alpha, d_0^\alpha$ are each unbounded operators on their respective Hilbert spaces.

Define the spaces

$$\begin{aligned}L^2(\Omega_E)_\sigma &:= \{h \in L^2(\Omega_E) \mid \forall (x, y) \in \Omega_E : h(x, y) = -h(y, x)\} \\ L^2(\Omega_{E_0})_\sigma &:= L^2(\Omega_E)_\sigma \cap L^2(\Omega_{E_0})\end{aligned}$$

with inner product for the larger space denoted as $\langle \cdot, \cdot \rangle_E$. Any function $h \in L^2(\Omega_E)_\sigma$ can be uniquely decomposed as

$$h = h_0 + h_+ - h_-$$

with $h_0 \in L^2(\Omega_{E_0})_\sigma$, $h_+ \in L^2(\Omega_{E_+})$ and $h_- \in L^2(\Omega_{E_-})$. Define now the operator

$$\delta^\alpha : L^2(\Omega_E)_\sigma \rightarrow L^2(\Omega_V)$$

as

$$\delta^\alpha = \delta_0^\alpha + \delta_+^\alpha + \delta_-^\alpha$$

and finally the operator

$$d^\alpha : L^2(\Omega_V) \rightarrow L^2(\Omega_E)_\sigma$$

as

$$d^\alpha = d_0^\alpha + d_+^\alpha + d_-^\alpha$$

for $\alpha > 0$.

Lemma 3.1. It holds true that

$$\langle d^\alpha f, h \rangle_E = \langle f, \delta^\alpha h \rangle_V$$

for $h \in L^2(\Omega_E)_\sigma$, $f \in L^2(\Omega_V)$. In other words, $(d^\alpha, \delta^\alpha)$ is a pair of adjoint operators.

Proof. Let $f \in L^2(\Omega_V)$ and $h \in L^2(\Omega_E)_\sigma$. Observe that

$$h_+(x, y) = h(x, y), \quad h_-(y, x) = -h(x, y)$$

for $(x, y) \in \Omega_{E_+}$. It follows that

$$\begin{aligned} \langle d^\alpha f, h_+ + h_- \rangle_E &= \int_{\Omega_{E_+}} \frac{f(x) - f(y)}{2|x - y|_p^\alpha} \overline{h(x, y)} \, dx \, dy - \int_{\Omega_{E_+}} \frac{f(y) - f(x)}{2|x - y|_p^\alpha} \overline{h(x, y)} \, dx \, dy \\ &= \int_{\Omega_{E_+}} \frac{f(x) - f(y)}{|x - y|_p^\alpha} \overline{h(x, y)} \, dx \, dy \end{aligned}$$

on the one hand. On the other hand,

$$\begin{aligned} \langle f, \delta_+^\alpha h_+ \rangle_V &= \int_{\Omega_V} \int_{\Omega_V} f(x) \overline{h_+(x, y)} |x - y|_p^{-\alpha} \, dx \, dy \\ &= \int_{\Omega_{E_+}} f(x) \overline{h_+(x, y)} |x - y|_p^{-\alpha} \, dx \, dy \end{aligned}$$

and

$$\begin{aligned} \langle f, \delta_-^\alpha h_- \rangle_V &= \int_{\Omega_V} \int_{\Omega_V} f(x) \overline{h_-(x, y)} |x - y|_p^{-\alpha} \, dx \, dy \\ &= \int_{\Omega_{E_-}} f(x) \overline{h_-(x, y)} |x - y|_p^{-\alpha} \, dx \, dy \\ &= - \int_{\Omega_{E_+}} f(x) \overline{h_+(y, x)} |x - y|_p^{-\alpha} \, dy \, dx \\ &= - \int_{\Omega_{E_+}} f(y) \overline{h_+(x, y)} |x - y|_p^{-\alpha} \, dx \, dy \end{aligned}$$

which takes care of the part concerning the simple edges of G . For the loop-edges, assume that $f \in \mathcal{D}(\Omega_V)$, and $h_0 \in \mathcal{D}(\Omega_{E_0})_\sigma$, where $\mathcal{D}(Z)$ means the space of locally constant functions on a space Z with compact support, and the subscript σ again that $h_0(x, y) = -h_0(y, x)$ for $(x, y) \in \Omega_{E_0}$. Then it is readily seen that $d_0^\alpha f \in \mathcal{D}(\Omega_{E_0})_\sigma$ and $\delta_0^\alpha h_0 \in \mathcal{D}(V)$. Furthermore, the supports of h_0 and $d_0^\alpha f$ are away from a neighborhood of the diagonal. Hence, they can be written as a disjoint union of products of two disjoint discs, where both sides each run through a finite set of discs in Ω_V . This allows the decompositions:

$$h_0 = h_{0,+} - h_{0,-}, \quad d_0^\alpha f = (d_0^\alpha f)_+ - (d_0^\alpha f)_-$$

where each part is a function which is positive, locally constant (and compactly supported). Then a similar calculation as in the simple-edge case yields that

$$\langle d_0^\alpha f, h_0 \rangle_{E_0} = \langle f, \delta_0^\alpha h_0 \rangle_V$$

for test functions. By continuity of the inner products, this identity extends to the (restricted) L^2 -spaces. Hence, putting things together, arrive at

$$\langle f, \delta^\alpha h \rangle_V = \langle f, \delta_0^\alpha h_0 \rangle_V + \langle f, \delta_+^\alpha h_+ \rangle_V - \langle f, \delta_-^\alpha h_- \rangle_V = \langle d^\alpha f, h \rangle_E$$

as asserted. \square

Define the following Laplacian operators:

$$\begin{aligned} D^\alpha &= \delta^\alpha d^\alpha : L^2(\Omega_V) \rightarrow L^2(\Omega_V) \quad (p\text{-adic vertex Laplacian}) \\ \Delta^\alpha &= d^\alpha \delta^\alpha : L^2(\Omega_E)_\sigma \rightarrow L^2(\Omega_E)_\sigma \quad (p\text{-adic edge Laplacian}) \end{aligned}$$

for a finite simple graph $G = (V, E)$. Explicitly, D^α is given as follows:

Corollary 3.2. *It holds true that*

$$D^\alpha = \delta_0^\alpha d_0^\alpha + \delta_+^\alpha d_+^\alpha + \delta_-^\alpha d_-^\alpha \quad \text{and} \quad \Delta^\alpha = d_0^\alpha \delta_0^\alpha + d_+^\alpha \delta_+^\alpha + d_-^\alpha \delta_-^\alpha$$

are self-adjoint, positive semi-definite, and

$$D^\alpha f(x) = \int_{\{y \in \Omega_V \mid (x,y) \in \Omega_E\}} \frac{f(x) - f(y)}{|x - y|^{2\alpha}} dy$$

for $f \in L^2(\Omega_V)$ and $\alpha > 0$. In the case that G is a simple graph, i.e., $E_0 = \emptyset$, the operators D^α and Δ^α are bounded, otherwise these operators are unbounded.

Proof. The self-adjointness and positive semi-definiteness statements are an immediate consequence of Lemma 3.1. The decompositions are seen thus:

$$D^\alpha = \delta_0^\alpha d_0^\alpha + \delta_+^\alpha d_+^\alpha + \delta_-^\alpha d_-^\alpha + \delta_0^\alpha d_+^\alpha + \delta_+^\alpha d_-^\alpha + \delta_-^\alpha d_0^\alpha + \delta_+^\alpha d_0^\alpha + \delta_-^\alpha d_+^\alpha + \delta_-^\alpha d_0^\alpha$$

and all the mixed terms vanish, as their non-vanishing loci are complementary. This proves the first decomposition, and the second one follows in a similar manner. The assertion about the integral form of operator D^α is an immediate consequence of the decomposition of D^α . The last assertion is also immediate. \square

Remark 3.3. Notice that according to Corollary 3.2, the vertex Laplacian is the negative of a Vladimirov-type integral Laplacian operator obtained in the case of a complete graph by restricting the domain and target of the well-known Vladimirov operator, cf. e.g., Ref. 24, to functions supported on Ω_V .

The following results are about realising the Betti numbers of a finite graph, viewed as a 1-dimensional simplicial complex. Consult e.g., Ref. 15, Chap. 2.1 for simplicial homology.

Lemma 3.4. *It holds true that*

$$\dim \ker(d^\alpha) = \dim \ker(D^\alpha) = b_0(G)$$

for $\alpha > 0$, where $b_0(G)$ is the zeroth Betti number of G .

Proof. The operator d^α clearly vanishes on all functions $f : \Omega_V \rightarrow \mathbb{C}$ constant on the compact open sets $\Omega_C \subseteq \Omega_V$ defined by the vertices of the connected components C of G . Now assume that $f \in \ker d^\alpha$. First, observe that f is constant on each compact open U_v with $v \in V$. Otherwise, there is an edge (v, w) such that f restricted to U_w misses one of the different values on U_v . Hence, f takes the same value on sets U_w where w is connected to v by an edge in E . It now follows that f is constant on Ω_C , where C is a connected component of G . This proves that

$$\dim \ker d^\alpha = b_0(G)$$

for $\alpha > 0$. The first equality is a consequence of the fact that δ^α and d^α are each other's adjoint operator. \square

The p -adic edge Laplacian Δ^α does not have a finite-dimensional kernel, as can be seen thus:

$$\begin{aligned} (\delta^\alpha h)(x) &= \int_{\Omega_V} h(x, y) |x - y|_p^{-\alpha} dy \\ &= \int_{\{y \in \Omega_V \mid (x,y) \in \Omega_E\}} h(x, y) |x - y|_p^{-\alpha} dy \\ &= \sum_{w \sim v(x)} \int_{U_w} h(x, y) |x - y|_p^{-\alpha} dy \\ &= \sum_{w \sim v(x)} \text{dist}(x, U_w)^{-\alpha} \int_{U_w} h(x, y) dy \end{aligned}$$

where $v \sim w$ means that vertices $v, w \in V$ are connected by an edge, and $v(x)$ means the unique vertex $v \in V$ for which $x \in U_x$. The space of functions $h \in L^2(\Omega_E)_\sigma$ such that each integral in the last sum vanishes, is infinite-dimensional. Define

$$L^2(\Omega_E)_\rho = \{h \in L^2(\Omega_E)_\sigma \mid \forall (v, w) \in E : h|_{U_v \times U_w} = \text{const.}\}$$

which is a finite-dimensional vector space.

Lemma 3.5. It holds true that

$$\dim L^2(\Omega_E)_\rho = |E_+|$$

for any finite graph G being simple after removing the loop-edges.

Proof. In the case that G is itself a simple graph, this is immediate. Otherwise, observe that a function $h \in L^2(\Omega_E)_\rho$ satisfies

$$h|_{U_v \times U_v} = 0$$

for all $v \in V$. This now implies the assertion also in the more general case. \square

The orthogonal complement of $L^2(\Omega_E)_\rho$ in $L^2(\Omega_E)_\sigma$ is

$$L^2(\Omega_E)_a = \left\{ h \in L^2(\Omega_E)_\sigma \mid \forall (v, w) \in E : \int_{U_v \times U_w} h(x, y) \, dx \, dy = 0 \right\}$$

and the following holds true:

Lemma 3.6. It holds true that

$$L^2(\Omega_E)_a \subseteq \ker \delta^\alpha$$

for $\alpha > 0$.

Proof. This is immediate. \square

Define

$$\delta_\rho^\alpha : L^2(\Omega_E)_\rho \rightarrow L^2(\Omega_V)$$

as the restriction of δ^α to $L^2(\Omega_E)_\rho$. Due to the last result, it fits into the commutative diagram:

$$\begin{array}{ccc} L^2(\Omega_E)_\sigma & & \\ \downarrow \kappa & \searrow \delta^\alpha & \\ L^2(\Omega_E)_\rho & \xrightarrow{\delta_\rho^\alpha} & L^2(\Omega_V) \end{array}$$

where κ is the canonical orthogonal projection. There is now the following operator:

$$\Delta_\rho^\alpha = d^\alpha \delta_\rho^\alpha : L^2(\Omega_E)_\rho \rightarrow L^2(\Omega_E)_\rho$$

for $\alpha > 0$, which clearly is well-defined, cf. the Proof of Lemma 3.5.

Define $G' = (V, E')$ to be the simple graph obtained from G by removing all loop-edges.

Lemma 3.7. It holds true that

$$\ker \delta_\rho^\alpha = \ker \Delta_\rho^\alpha = b_1(G')$$

for $\alpha > 0$, where $b_1(G')$ is the first Betti number of G' .

Proof. In the Proof of Lemma 3.4, it was shown that the kernel of d^α coincides with the kernel of the co-boundary map $d_G^\alpha : \mathbb{C}^V \rightarrow \mathbb{C}^E$ of a suitably weighted version of the graph G (using positive powers of p -adic distances between the corresponding vertex discs as weights). The image of the co-boundary map in fact lies in $\mathbb{C}^{E'}$, and δ_ρ^α is clearly the adjoint of that map. This implies the first equality. The second now follows from graph theory. \square

This last Lemma allows to define the index of d^α :

$$\text{ind}(d^\alpha) := \dim \ker d^\alpha - \dim \ker \delta_\rho^\alpha$$

which is a natural number for $\alpha > 0$. Using the Euler characteristic $\chi(G')$, defined as

$$\chi(G') = |V| - |E_+|$$

of the simple graph G' , obtain:

Corollary 3.8. It holds true that

$$\text{ind}(d^\alpha) = \dim \ker D^\alpha - \dim \ker \Delta_\rho^\alpha = \chi(G')$$

for $\alpha > 0$.

Proof. The first equality follows immediately from Lemmas 3.4 and 3.7, using the equality

$$\chi(G') = b_0(G') - b_1(G')$$

from graph theory, cf. e.g., Ref. 15, Theorem 2.44, and the fact that $b_0(G) = b_0(G')$ holds true. \square

IV. HEAT KERNELS

Zúñiga-Galindo showed that his p -adic graph operators are bounded self-adjoint linear Laplacian operators defining a strong Markov process whose transition function is given for each $x \in \Omega_V$ by a probability measure $p_t(x, \cdot)$ on the Borel σ -algebra of Ω_V , and this proves the uniqueness and existence of a solution to the Cauchy problem for the heat equation attached to the operator D^α (Ref. 25, Theorem 4.2). Later on, it will be shown that the probability measure $p_t(x, \cdot)$ has a well-defined distribution function, if every vertex of G has a loop-edge attached to it.

As for the edge Laplacian Δ_ρ^α , which is a finite-dimensional linear operator, the following holds true:

Lemma 4.1. The heat kernel for Δ_ρ^α exists and has the form

$$p_{\Delta_\rho^\alpha, t}(x, y) = \sum_{\text{Spec}(\Delta_\rho^\alpha)} e^{-\lambda t} \phi_\lambda(x) \phi_\lambda(y)$$

for $\alpha > 0$.

Proof. This is well-known from the theory of finite graphs, cf. e.g., Ref. 12, Lemma 10.1. \square

Lemma 4.2. In the case that every vertex of G has a loop-edge, the operator D^α is compact. In all other cases, D^α has an eigenvalue with infinite multiplicity.

Proof. In Ref. 5, Lemma 5, it was observed that in the case of a simple graph, the operator D^α is a Zúñiga operator. These are shown to have Kozyrev wavelets as eigenfunctions (Ref. 25, Theorem 10.1). The eigenvalues corresponding to a Kozyrev wavelet have infinite multiplicity for the following reason: let ψ_B be a Kozyrev eigenfunction supported on $B \subset \Omega_V$. Then

$$D^\alpha \psi_B(x) = \int_{\{y \in \Omega_V | (x, y) \in \Omega_E\}} |x - y|_p^{-2\alpha} dy \psi_B(x)$$

where

$$\lambda_B = \int_{\{y \in \Omega_V | (x, y) \in \Omega_E\}} |x - y|_p^{-2\alpha} dy$$

is a well-defined positive number, and this is an eigenvalue of D^α . If the support C of another Kozyrev wavelet ψ_C is contained in B , then the same computation yields

$$D^\alpha \psi_C(x) = \int_{\{y \in \Omega_V | (x, y) \in \Omega_E\}} |x - y|_p^{-2\alpha} dy \psi_C(x)$$

and $\lambda_C = \lambda_B$ holds true. Since the disc C can be arbitrarily small, the multiplicity of $\lambda = \lambda_B$ is infinite. Notice that this argument makes use only of the fact that $(x, y) \in \Omega_E$ implies that $y \notin U_{v(x)}$.

Now assume that vertex $v(x)$ has a loop-edge attached to it. Observe that the operator D^α is in fact of the form of Laplacian operators studied in Ref. 21. Its kernel function $T(x, y)$ satisfies condition the important condition (6) of that article which says that

$$|x - y|_p = \text{const.} \Rightarrow T(x, y) = \text{const.}$$

and in that case, (Ref. 21, Theorem 3) shows that the Kozyrev wavelets ψ_B supported in $B \subset \Omega_V$ are eigenfunctions, and also provides an explicit expression for the eigenvalues attached to Kozyrev wavelets. In the case of a vertex v having loop-edge attached to it, and $B \subset U_v$, it takes the form:

$$\lambda_B = \lambda_{d,n} = \int_{\{y \in \Omega_V | (x, y) \in \Omega_E\} \setminus B} |p^{-d}n - y|_p^{-2\alpha} dy + p^{d(1+2\alpha)}$$

$$= \int_{\{y \in \Omega_V \mid (x,y) \in \Omega_E\} \setminus B} \text{dist}(B, y)^{-2\alpha} dy + p^{d(1+2\alpha)}$$

where

$$B = \left\{ y \in \mathbb{Q}_p \mid \left| p^d x - n \right|_p \leq 1 \right\}$$

with $d \in \mathbb{Z}$, and $n \in \mathbb{Q}_p$ is the fractional part of a representative of an element of $\mathbb{Q}_p/\mathbb{Z}_p$. This can be compared with the case of a simple graph as follows: it holds true that

$$\lambda_B = \lambda_B^\sigma + p^{d(1+2\alpha)} + \int_{U_v \setminus B} \text{dist}(B, y)^{-2\alpha} dy$$

where

$$\lambda_B^\sigma = \int_{\{y \in \Omega_V \mid (x,y) \in \Omega_E\} \setminus U_v} |x - y|_p^{-2\alpha} dy = \int_{\{y \in \Omega_V \mid (x,y) \in \Omega_E\} \setminus U_v} \text{dist}(B, y)^{-2\alpha} dy$$

is the eigenvalue if v has no loop-edge attached. Since $\lambda_B^\sigma = \lambda_C^\sigma$ if $\text{disc } C \subseteq B$, it now follows that $\lambda_C > \lambda_B$ if C is strictly contained in B . This shows that the multiplicity of λ_B is finite. It now follows that D^α is a compact operator, if all vertices have a loop-edge attached, and in all other cases some eigenvalue has infinite multiplicity, as asserted. \square

Remark 4.3. It has already been observed that Zúñiga operators are in general not compact (Ref. 5, Example 1).

Corollary 4.4. Assume that each vertex of G has a loop-edge attached to it. Then the heat kernel for D^α exists and has the form

$$p_{D^\alpha, t}(x, y) = \sum_{\lambda \in \text{Spec}(D^\alpha)} e^{-\lambda t} f_\lambda(x) \overline{f_\lambda(y)}$$

where $f_\lambda \in L^2(\Omega_V)$ is a normalised eigenfunction of D^α for eigenvalue $\lambda \in \text{Spec}(D^\alpha)$, for $t > 0$ and $\alpha > 0$.

Proof. Taking as initial condition

$$h_0(x) = f_\lambda(x)$$

obtain from the Cauchy problem:

$$h(t, x) = \int_{\Omega_V} h_0(y) p_{D^\alpha, t}(x, dy) = e^{-\lambda t} f_\lambda(x)$$

for $t > 0$ and $\alpha > 0$, where $p_{D^\alpha, t}(x, \cdot)$ is the probability measure giving rise to a strong Markov process according to Ref. 25, Theorem 4.2. Now, the proposed density function integrates the initial condition in this way:

$$\begin{aligned} \int_{\Omega_V} f_\lambda(y) p_{D^\alpha, t}(x, y) dy &= \sum_{\lambda' \in \text{Spec}(D^\alpha)} e^{-\lambda' t} \int_{\Omega_V} f_\lambda(y) f_{\lambda'}(x) \overline{f_{\lambda'}(y)} dy \\ &= e^{\lambda t} f_\lambda(x) \end{aligned}$$

giving rise to the same solution of the Cauchy problem. Hence, this happens for any initial condition in $L^2(\Omega_V)$, and the proposed function is the distribution function of $p_{D^\alpha, t}(x, \cdot)$ for any $x \in \Omega_V$. In other words, the heat kernel for D^α exists and has the asserted form. \square

In order to formulate a p -adic version of an index theorem, two cases need to be considered: either the case where each vertex of G has a loop-edge, or some vertex does not. In the latter case, consider the finite approximation operators D_ℓ^α defined as:

$$D_\ell^\alpha : L^2(\Omega_{V, \ell}) \rightarrow L^2(\Omega_{V, \ell})$$

with

$$D_\ell^\alpha f(x) = \int_{\Omega_{V, \ell}} \frac{f(x) - f(y)}{|x - y|^{2\alpha}} d_\ell y$$

where

$$\Omega_{V, \ell} = \Omega_V \bmod p^\ell \mathbb{Z}_p$$

is obtained by from Ω_V by partitioning into a disjoint union of discs of radius $p^{-\ell}$ for $\ell \in \mathbb{N}$, and where $d_\ell y$ is the counting measure on the finite set $\Omega_{V, \ell}$. Notice that the expression

$$|x - y|_p$$

in the integral is well-defined.

Theorem 4.5. *If every vertex of G has a loop-edge, then*

$$\lim_{t \rightarrow \infty} \left(\text{tr} (p_{D^\alpha, t}) - \text{tr} (p_{\Delta_p^\alpha, t}) \right) = \chi(G') = \text{ind}(d^\alpha)$$

for $\alpha > 0$. Otherwise, it holds true that

$$\lim_{\ell \rightarrow \infty} \lim_{t \rightarrow \infty} \left(\text{tr} (p_{D_{\ell}^\alpha, t}) - \text{tr} (p_{\Delta_p^\alpha, t}) \right) = \chi(G') = \text{ind}(d^\alpha)$$

for $\alpha > 0$. The graph G' is obtained from G by removing all loop-edges.

Proof. In both cases, each last equality is stated in Corollary 3.8.

In the bounded case, i.e., when G is simple, the operator D^α was identified with a Zúñiga operator, and its spectrum and eigenfunctions were identified (Ref. 5, Corollary 5). Namely, the Kozyrev wavelets supported in Ω_V , together with finitely many functions constant on p -adic discs making up the sets U_v , as explained in the proof of Ref. 5, Lemma 4, form an orthogonal eigenbasis of $L^2(\Omega_V)$. In the case that a vertex v does not have a loop-edge attached to it, the Kozyrev wavelets supported in U_v are still eigenfunctions of D^α , cf. the Proof of Lemma 4.2. This does not touch on the part of $L^2(\Omega)$ of functions which are constant on p -adic discs covering the sets U_v . Hence, those finitely many functions identified in the simple graph case remain eigenfunctions also in the non-simple case.

In the general case, the heat kernel for D_ℓ^α exists and has the form:

$$p_{D_\ell^\alpha, t}(x, y) = \sum_{\lambda \in \text{Spec}(D_\ell^\alpha)} e^{-\lambda t} f_\lambda^\ell(x) f_\lambda^\ell(y)$$

where f_λ^ℓ is a normalised eigenfunction for D_ℓ^α . Accordingly, the following holds true:

$$\text{tr} (p_{D_\ell^\alpha, t}) - \text{tr} (p_{\Delta_p^\alpha, t}) = \chi(G') + \sum_{\lambda \in \text{Spec}_{\text{res}}(D_\ell^\alpha)} e^{-\lambda t} \quad (2)$$

where $\text{Spec}_{\text{res}}(D_\ell^\alpha)$ is the part of the spectrum of D_ℓ^α not coming from the (weighted) graph Laplacian of G , i.e., not from Kozyrev wavelets and some of the other eigenfunctions of D^α described above. This does make sense, because a Kozyrev wavelet on \mathbb{Q}_p takes precisely p values, each on a distinct maximal non-trivial subdisc of its support, and by the construction of D_ℓ acting on a finite-dimensional part of $L^2(\Omega_V)$ as the operator D^α .

Hence, it follows in the more general case of a vertex possibly not having a loop-edge that

$$\lim_{\ell \rightarrow \infty} \lim_{t \rightarrow \infty} \left(\text{tr} (p_{D_{\ell}^\alpha, t}) - \text{tr} (p_{\Delta_p^\alpha, t}) \right) = \chi(G')$$

for $\alpha > 0$, as asserted.

In the case that D^α is compact, the identity

$$\lim_{\ell \rightarrow \infty} p_{D_\ell^\alpha, t} = \sum_{\lambda \in \text{Spec}(D^\alpha)} e^{-\lambda t} f_\lambda(x) f_\lambda(y)$$

does make sense where $\{f_\lambda\}$ is an orthonormal eigenbasis of $L^2(\Omega_V)$ for D^α , as the right-hand side converges due to the compactness of the operator. By taking limits in (2), it follows that

$$\text{tr} (p_{D^\alpha, t}) - \text{tr} (p_{\Delta_p^\alpha, t}) = \chi(G') + \sum_{\lambda \in \text{Spec}_{\text{res}}(D^\alpha)} e^{-\lambda t}$$

where $\text{Spec}_{\text{res}}(D^\alpha)$ is defined in an analogous manner as in the more general case. It follows that

$$\lim_{t \rightarrow \infty} \left(\text{tr} (p_{D^\alpha, t}) - \text{tr} (p_{\Delta_p^\alpha, t}) \right) = \chi(G')$$

also in this case, as asserted. This proves the theorem. \square

V. APPLICATION TO p -ADIC MUMFORD CURVES

Ongoing work of the author and collaborators suggests the viewpoint that a locally hierarchical dataset can be viewed as being sampled from a Mumford curve endowed with a regular differential 1-form. This resembles the underpinning concepts of manifold learning in the classical case, as Mumford curves are p -adic analogs of Riemann surfaces. In this way, a connection to p -adic mathematical physics opens up to further investigation. In this viewpoint, topological information, like e.g., the genus or the Euler characteristic χ , of this underlying Mumford curve is of interest. This lends to the issue of describing χ via heat kernels of suitable operators. It turns out that this is possible by applying Theorem 4.5 in different ways.

The fact that the p -adic index theorem (Theorem 4.5) has applications to Mumford curves is because of their property as a compact p -adic manifold with an inner graph structure. The theory of Mumford curves can be found in e.g., Refs. 13 and 14, from which their only property needed is their relationship to finite graphs.

In the following subsections, different kinds of finite graphs are built from a good fundamental domain F in the sense of Ref. 14, Definition I.4.1.3, and then Theorem 4.5 is applied in order to extract information from a Mumford curve via the heat kernels.

A. The genus of a Mumford curve

Let $F \subset \mathbb{Q}_p$ be a good fundamental domain (in the sense of Ref. 14, Definition I.4.1.3) of the action of the Schottky group giving rise to a Mumford curve X of genus $g(X) > 0$. Let $T(F)$ be the subtree of the Bruhat-Tits tree \mathcal{T}_p of \mathbb{Q}_p associated with F , defined in Ref. 5, Sec. 4.1 as the *dendrogram* for the finitely many points, each taken from a distinct inner hole of F . Then a so-called *semi-stable reduction graph* of X is given by attaching $g(X)$ edges to $T(F)$. This is in fact the quotient graph of Ref. 14, Chap. I.3 obtained by the action of the Schottky group of X on an infinite subtree of the Bruhat-Tits tree \mathcal{T}_p of \mathbb{Q}_p . This is a simple graph which becomes a so-called *stable* graph after contracting all edges attached to a vertex of degree 2. Let $G = (V, E(G))$ be the graph obtained from that simple graph by attaching to all vertices a loop-edge. Thus

$$F = \bigcup_{v \in V(T_F)} U_v$$

where $U_v \subset \mathbb{Q}_p$ is a holed disc, as explained in Ref. 5, Sec. 4.1. The corresponding edge space is

$$\Omega_E = \bigsqcup_{e \in E(G)} U_{o(e)} \times U_{t(e)}$$

and the results of the previous section can be applied.

Corollary 5.1. It holds true that

$$\text{ind}(d_G^\alpha) = \lim_{t \rightarrow \infty} \left(\text{tr}(p_{D^\alpha, t}) - \text{tr}(p_{\Delta_p^\alpha}) \right) = 1 - g(X) = \chi(X)$$

where $\chi(X)$ is the Euler characteristic of the Mumford curve X w.r.t. its simplicial cohomology.

Proof. This is an immediate consequence of Theorem 4.5, because the graph G is connected, and $g(X) = b_1(G')$ by the theory of Mumford curves, cf. Ref. 14, Theorem III.2.12.2. \square

B. Counting holes of a good fundamental domain

The graph G is now defined as the complete graph (including all loop-edges) on the set \mathcal{H} of inner holes of a good fundamental domain $F \subset \mathbb{Q}_p$. Assume that the number of holes is N . Then

$$\Omega_V = \bigsqcup_{H \in \mathcal{H}} H \subset \mathbb{Q}_p$$

is the set defined by the vertices of G , whereas

$$\Omega_E = \Omega_V \times \Omega_V$$

is defined by the edges of G .

Corollary 5.2. It holds true that

$$\text{ind}(d_G^\alpha) = \lim_{t \rightarrow \infty} \left(\text{tr}(p_{D^\alpha, t}) - \text{tr}(p_{\Delta_p^\alpha, t}) \right) = -\frac{1}{2}N^2 + \frac{3}{2}N$$

where N is the number of holes of a good fundamental domain $F \subset \mathbb{Q}_p$ for X .

Proof. This is an immediate consequence of Theorem 4.5. \square

Remark 5.3. Observe that the genus $g(X)$ can be extracted from the number of holes:

$$\text{ind}(d_G^\alpha) = -\frac{1}{2}(2g-1)^2 + \frac{3}{2}(2g-1)$$

as $N = 2g - 1$ holds true (Ref. 14, Sec. I.4).

C. The number of vertices of the reduction graph

Take as G now the complete graph on the set of vertices of $T(F)$ for a good fundamental domain $F \subset \mathbb{Q}_p$ for a Mumford curve X . Again, assume that all loop-edges are present.

Corollary 5.4. It holds true that

$$\text{ind}(d_G^\alpha) = \lim_{t \rightarrow \infty} (\text{tr}(p_{D^\alpha, t}) - \text{tr}(p_{\Delta^\alpha, t})) = -\frac{1}{2}|V(T_F)|^2 + \frac{3}{2}|V(T_F)|$$

where $F \subset \mathbb{Q}_p$ is a good fundamental domain of X .

Proof. This follows immediately from Theorem 4.5. □

Remark 5.5. The number of vertices of T_F is a statistic about the geometry of a Mumford curve. E.g., in the case $g = 1$, this number equals the thickness of the annulus F , i.e., the negative p -adic valuation of the parameter $q \in \mathbb{Z}_p$ of the corresponding p -adic Tate curve. Consult e.g., Ref. 13, Chap. 5.1 for more about the Tate elliptic curve.

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Conflict of Interest

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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