

RESEARCH ARTICLE

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On Torelli groups and Dehn twists of smooth 4-manifolds

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Email: krannich@kit.edu**Abstract**

This note has two related but independent parts. Firstly, we prove a generalisation of a recent result of Gay on the smooth mapping class group of S^4 . Secondly, we give an alternative proof of a consequence of work of Saeki, namely that the Dehn twist along the boundary sphere of a simply connected closed smooth 4-manifold X with $\partial X \cong S^3$ is trivial after taking connected sums with enough copies of $S^2 \times S^2$.

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This note serves to record two results on the mapping class group of compact simply connected smooth 4-manifolds, the first on their *Torelli subgroup* and the second on stable *Dehn twists*.

Generating Torelli groups

Recently, Gay [8, Theorem 1] constructed a surjection

$$\operatorname{colim}_{g \rightarrow \infty} \pi_1(\operatorname{Emb}(\sqcup^g S^2, W_{g,1}), \operatorname{inc}) \twoheadrightarrow \pi_0(\operatorname{Diff}^+(S^4)) \quad (1)$$

onto the smooth oriented mapping class group of S^4 , whose domain is a colimit of the fundamental groups of the space of smooth embeddings of $\sqcup^g S^2$ into $W_{g,1} := (S^2 \times S^2)^{\#g} \setminus \operatorname{int}(D^4)$, based at the embedding $\operatorname{inc} : \sqcup^g S^2 \hookrightarrow W_{g,1}$ induced by g copies of the inclusion $S^2 \times \{*\} \subset S^2 \times S^2$. In the first part of this note, we use work of Kreck [10] and Quinn [15] to prove a generalisation of Gay's

result which applies to all simply connected closed oriented smooth 4-manifolds X . For such a general X , the target of (1) needs to be replaced by the *Torelli subgroup* $\text{Tor}(X) \subset \text{Diff}^+(X)$ —the subgroup of those diffeomorphisms that act as the identity on the integral homology of X .

Theorem A. *For a closed simply connected smooth 4-manifold X , there is a surjective morphism*

$$\text{colim}_{g \rightarrow \infty} \pi_1(\text{Emb}(\sqcup^g D^2 \times S^2, X \# W_{g,1}), \text{inc}) \twoheadrightarrow \pi_0(\text{Tor}(X)).$$

Remark. For $X = S^4$, we have $\text{Tor}(S^4) = \text{Diff}^+(S^4)$ and our result recovers Gay's. Indeed, comparing the construction we give in the proof of Theorem A with Gay's description of (1) in [8, p. 8–9], one sees the homomorphism we construct agrees with Gay's, up to precomposition with the surjective homomorphism on fundamental groups induced by the map $\text{Emb}(\sqcup^g D^2 \times S^2, W_{g,1}) \rightarrow \text{Emb}(\sqcup^g S^2, W_{g,1})$ obtained by restriction along $\sqcup^g * \times S^2 \subset \sqcup^g D^2 \times S^2$. The fact that this homomorphism is surjective follows from the long exact sequence in homotopy groups induced by the restriction map, since the homotopy fibre over the inclusion is homotopy equivalent to the path-connected space $\text{Map}(S^2, \text{SO}(2))^g$.

Dehn twists are stably trivial

Related to, but independent of, the proof of Theorem A (see Remark 3), we show in the second part of this note that the Dehn twist $t_X \in \pi_0(\text{Diff}_\partial(X \# D^4))$ along the boundary sphere of $X \# D^4$ for a closed smooth simply connected 4-manifold is stably isotopic to the identity. In addition to Kreck and Quinn's work mentioned above, this relies on Wall's stable classification in [17] and Galatius–Randal-Williams' work on stable moduli spaces of manifolds [7].

Theorem B (Saeki). *For a closed simply connected smooth 4-manifold X , the Dehn twist $t_X \in \pi_0(\text{Diff}_\partial(X \# D^4))$ lies in the kernel of the stabilisation map*

$$\pi_0(\text{Diff}_\partial(X \# D^4)) \rightarrow \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}_\partial(X \# W_{g,1})).$$

Remark.

- (i) As indicated by the attribution, Theorem B can also be deduced (by very different means) from Saeki's computation of the stable mapping class groups of compact simply connected smooth 4-manifolds with nonempty connected boundary [16, Theorem 3.7, Proposition 4.2].
- (ii) In the case where X does not admit a spin structure, there is direct argument that the Dehn twist t_X is isotopic to the identity, even before stabilisation by $S^2 \times S^2$; see [14, Corollary A.5]. In the special case where $X = Y \# \mathbb{C}P^2$ or $X = Y \# \overline{\mathbb{C}P}^2$ for some Y , this was shown in [9, Theorem 2.4].
- (iii) If X is spin, then the Dehn twist t_X is *not* isotopic to the identity *without* stabilisation in general, for example in the case $X = K3$. This was independently shown by Baraglia–Konno [1] and Kronheimer–Mrowka [11, Proposition 1.2] (see also the discussion in [14, Section 1.1]).

We now turn to the proofs of Theorems A and B, in this order.

Proof of Theorem A

Consider the embedding $\sqcup^g D^2 \times S^2 \subset W_{g,1} = (S^2 \times S^2)^{\#g} \setminus \text{int}(D^4)$ obtained by viewing $D^2 \subset S^2$ as the upper hemisphere, performing the connected sums in the complement of $\sqcup^g D^2 \times S^2 \subset \sqcup^g S^2 \times S^2$, and choosing the embedded 4-disc also away from this complement. The complement $Y_{g,1} := W_{g,1} \setminus \sqcup^g \text{int}(D^2 \times S^2)$ has the diffeomorphism type of the connected sum $D^4 \# (D^2 \times S^2)^{\#g}$.

The morphism in Theorem A will arise as the middle row of a commutative diagram of groups

$$\begin{array}{ccccccc}
 & & \pi_0(\text{Tor}_\partial(X \# D^4)) & & & & \\
 & & \downarrow \textcircled{4} & \nearrow & & & \\
 \text{colim}_{g \rightarrow \infty} \pi_1(\text{Emb}(\sqcup^g D^2 \times S^2, X \# W_{g,1})) & \xrightarrow{\textcircled{1}} & \text{colim}_{g \rightarrow \infty} \pi_0(\text{Tor}_\partial(X \# Y_{g,1})) & \xrightarrow{\textcircled{2}} & \pi_0(\text{Tor}_\partial(X \# D^4)) & \xrightarrow{\textcircled{3}} & \pi_0(\text{Tor}(X)) \\
 & & \downarrow \textcircled{5} & & & & \\
 & & \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}_\partial(X \# W_{g,1})) & & & &
 \end{array}$$

which we explain in the following.

① & ⑤. By the parametrised isotopy extension theorem (see, for example, [18, Theorem 6.1.1]) restricting diffeomorphisms of $X \# W_{g,1}$ along the embedding $\sqcup^g D^2 \times S^2 \subset X \# W_{g,1}$ induces a fibre sequence

$$\text{Diff}_\partial(X \# Y_{g,1}) \longrightarrow \text{Diff}_\partial(X \# W_{g,1}) \longrightarrow \text{Emb}(\sqcup^g D^2 \times S^2, X \# W_{g,1}) \quad (2)$$

with fibre taken over the inclusion. Extending diffeomorphisms and embeddings by the identity along the inclusion of pairs $(W_{g-1,1}, \sqcup^{g-1} D^2 \times S^2) \subset (W_{g,1}, \sqcup^g D^2 \times S^2)$ induces a map of fibre sequences between (2) for $g-1$ and g . The long exact sequences of homotopy groups then induce an exact sequence

$$\text{colim}_{g \rightarrow \infty} \pi_1(\text{Emb}(\sqcup^g D^2 \times S^2, X \# W_{g,1}), \text{inc}) \longrightarrow \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}_\partial(X \# Y_{g,1})) \longrightarrow \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}_\partial(X \# W_{g,1})). \quad (3)$$

The right morphism in this sequence gives ⑤ by restriction to the Torelli subgroups in the source and the left morphism yields a homomorphism that qualifies as ① by the following lemma.

Lemma 1. *For a closed simply connected smooth 4-manifold X , the image of the connecting map*

$$\pi_1(\text{Emb}(\sqcup^g D^2 \times S^2, X \# W_{g,1}), \text{inc}) \longrightarrow \pi_0(\text{Diff}_\partial(X \# Y_{g,1}))$$

induced by (2) is contained in the subgroup $\pi_0(\text{Tor}_\partial(X \# Y_{g,1})) \subset \pi_0(\text{Diff}_\partial(X \# Y_{g,1}))$.

Proof. The inclusion $Y_{g,1} \subset W_{g,1}$ is injective on homology, so the same holds for the inclusion $X \# Y_{g,1} \subset X \# W_{g,1}$. Combining this with the observation that the composition $\pi_1(\text{Emb}(\sqcup^g D^2 \times S^2, X \# W_{g,1}), \text{inc}) \rightarrow \pi_0(\text{Diff}_\partial(X \# Y_{g,1})) \rightarrow \pi_0(\text{Diff}_\partial(X \# W_{g,1}))$ is trivial by exactness, the claim follows. \square

② & ④. The effect of gluing $\sqcup^g S^1 \times D^3$ to $Y_{g,1}$ using the canonical identification $\partial(\sqcup^g S^1 \times D^3) = \partial(\sqcup^g D^2 \times S^2) = \partial Y_{g,1}$ agrees with the result of doing surgery along the embeddings constituting $\sqcup^g D^2 \times S^2 \subset W_{g,1}$, so this process yields a manifold diffeomorphic to D^4 . Hence there are embeddings $Y_{g,1} = W_{g,1} \setminus \sqcup^g \text{int}(D^2 \times S^2) \hookrightarrow D^4$, one for each g , which we may choose so that the extension map $\pi_0(\text{Diff}_\partial(X \# Y_{g,1})) \rightarrow \pi_0(\text{Diff}_\partial(X \# D^4))$ is compatible with the extension maps $\pi_0(\text{Diff}_\partial(X \# Y_{g-1,1})) \rightarrow \pi_0(\text{Diff}_\partial(X \# Y_{g,1}))$. Taking colimits and restricting to Torelli subgroups defines ②. The morphism ④ is the canonical map into the colimit (note that $D^4 = Y_{0,1}$). Going through the definition, one sees that the composition ② \circ ④ agrees with the identity, so ② is split surjective and ④ is split injective.

③. This morphism is induced by extension with the identity along $X \# D^4 \subset X \# S^4 \cong X$. We claim that it is surjective and its kernel is generated by the Dehn twist t_X . To see this, we use that the restriction map $\text{Tor}(X) \rightarrow \text{Emb}^+(D^4, X)$ along an embedded disc $D^4 \subset X$ is a fibration by parametrised isotopy extension, whose fibre over the inclusion is $\text{Tor}_\partial(X \setminus \text{int}(D^4)) \cong \text{Tor}_\partial(X \# D^4)$. Taking derivatives induces a homotopy equivalence $\text{Emb}^+(D^4, X) \simeq \text{Fr}^+(X)$ to the oriented frame bundle which in turns fits into a fibre sequence $\text{SO}(4) \rightarrow \text{Fr}^+(X) \rightarrow X$. As X is simply connected, the long exact sequence of homotopy groups yields $\pi_0(\text{Fr}^+(X)) = *$ and $\pi_1(\text{Fr}^+(X)) \cong \mathbf{Z}/2$, so the long exact sequence of homotopy groups for the fibre sequence $\text{Tor}_\partial(X \# D^4) \rightarrow \text{Tor}(X) \rightarrow \text{Emb}^+(D^4, X)$ induces a short exact sequence of the form

$$\mathbf{Z}/2 \xrightarrow{t_X} \pi_0(\text{Diff}_\partial(X \# D^4)) \longrightarrow \pi_0(\text{Diff}^+(X)) \longrightarrow 0, \quad (4)$$

whose first map maps the generator to the Dehn twist t_X . Passing to Torelli subgroups, the claim follows.

In addition to the above diagram, the main ingredient for Theorem A is the following.

Lemma 2. *For a closed simply connected smooth 4-manifold X , the group $\pi_0(\text{Tor}_\partial(X \# D^4))$ is generated by the Dehn twist t_X and the kernel of the stabilisation $(5 \circ 4) : \pi_0(\text{Tor}_\partial(X \# D^4)) \rightarrow \text{colim}_{g \rightarrow \infty} \pi_0(\text{Tor}_\partial(X \# W_{g,1}))$.*

Proof. By a result of Kreck [10, Theorem 1], the subgroup $\pi_0(\text{Tor}(X)) \subset \pi_0(\text{Diff}(X))$ agrees with the subgroup of diffeomorphisms that are pseudo-isotopic to the identity, so it follows that the group $\pi_0(\text{Tor}_\partial(X \# D^4))$ is generated by the Dehn twist $t_X \in \pi_0(\text{Tor}_\partial(X \# D^4))$ and the subgroup of $\pi_0(\text{Tor}_\partial(X \# D^4))$ consisting of diffeomorphisms pseudo-isotopic to the identity. But by a result of Quinn [15, Theorem 1.4] (whose proof was corrected in [5, Theorem 1.1]), any diffeomorphism of $X \# D^4$ that is pseudo-isotopic to the identity lies in the kernel of the map $\pi_0(\text{Diff}_\partial(X \# D^4)) \rightarrow \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}_\partial(X \# W_{g,1}))$. \square

We now conclude the proof of Theorem A by showing that the composition ③ \circ ② \circ ① is surjective. Since ③ is surjective and its kernel is generated by t_X , it suffices to show that $\pi_0(\text{Tor}_\partial(X \# D^4))$ is generated by the Dehn twist and the image of ② \circ ①. By Lemma 2 it thus suffices to show that the kernel of ⑤ \circ ④ is contained in the image of ② \circ ①. But for $x \in \ker(5 \circ 4)$, we have ④(x) $\in \text{im}(1)$ by exactness of (3), so by applying ② and using ② \circ ④ = id, we get $x \in \text{im}(2 \circ 1)$.

Remark 3. Theorem B shows that the Dehn twist generator $t_X \in \pi_0(\text{Tor}_\partial(X \# D^4))$ in Lemma 2 is not actually necessary. This also shows that the composition of ① and ② is already surjective, so one can replace $\pi_0(\text{Tor}(X))$ by $\pi_0(\text{Tor}_\partial(X \# D^4))$ in the statement of Theorem A.

Proof of Theorem B

Recall that the Dehn twist t_X is the image of the nontrivial element under the leftmost map in (4). To study its behaviour under stabilisation, we use that the Dehn twists in $\pi_0(\text{Diff}_\partial(W_{1,1}\#D^4))$ around the two boundary components are isotopic (see [9, Proposition 3.7]), so the stabilisation $\pi_0(\text{Diff}_\partial(X\#W_{g,1})) \rightarrow \pi_0(\text{Diff}_\partial(X\#W_{g+1,1}))$ sends $t_{X\#W_g}$ to $t_{X\#W_{g+1}}$. Combined with (4) and writing $W_g := \#^g S^2 \times S^2$ this shows that there is a stabilisation map $\pi_0(\text{Diff}^+(X\#W_g)) \rightarrow \pi_0(\text{Diff}^+(X\#W_{g+1}))$, a well-defined element $t_{X\#W_\infty} \in \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}_\partial(X\#W_{g,1}))$, an exact sequence

$$\mathbf{Z}/2 \xrightarrow{t_{X\#W_\infty}} \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}_\partial(X\#W_{g,1})) \longrightarrow \text{colim}_{g \rightarrow \infty} \pi_0(\text{Diff}^+(X\#W_g)) \longrightarrow 0, \quad (5)$$

and that Theorem B is equivalent to showing that $t_{X\#W_\infty}$ is trivial. Since every orientation-preserving diffeomorphism can be assumed to fix a codimension 0 disc, this question depends only on the oriented diffeomorphism type of X up to taking connected sums with $S^2 \times S^2$, the *stable diffeomorphism type*. The latter is by [17, Theorem 2 and 3] determined by the *stable isomorphism type* of the intersection form, that is, the isomorphism type of the intersection form up to adding copies of the hyperbolic form $H := (\mathbf{Z}^2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$, the intersection form of $S^2 \times S^2$.

If X is not spin, then the Dehn twist t_X is trivial even before stabilisation (see the remark in the introduction). If X is spin, then its intersection form is even and its signature is divisible by 16 as a consequence of Rokhlin's theorem. From the classification of indefinite unimodular symmetric bilinear forms over \mathbf{Z} [13, Theorem II.5.3], it follows that the stable isomorphism type (that is, isomorphism up to addition of H) of the intersection form of X agrees with that of $(E_8^{\oplus -2} \oplus H^{\oplus 3})^{\oplus k}$ for some $k \in \mathbf{Z}$, which is the intersection form of $\#^k K_3$ (the convention is that $\#^k K_3$ is for negative k the $|k|$ -fold connected sum of K_3 with the opposite orientation). We may thus assume $X = \#^k K_3$ for some $k \in \mathbf{Z}$. To settle this case, we first prove the following result on stable abelianisations.

Lemma 4. *Let $X = \#^k K_3$ for $k \in \mathbf{Z}$, then*

$$\text{colim}_{g \rightarrow \infty} H_1(\pi_0(\text{Diff}_\partial(X\#W_{g,1}))) \cong \begin{cases} \text{colim}_{g \rightarrow \infty} H_1(\pi_0(\text{Diff}_\partial(W_{g,1}))) & \text{if } t_{X\#W_\infty} \text{ is trivial,} \\ \text{colim}_{g \rightarrow \infty} H_1(\pi_0(\text{Diff}_\partial(W_{g,1}))) \oplus \mathbf{Z}/2 & \text{otherwise.} \end{cases}$$

Moreover, we have $\text{colim}_{g \rightarrow \infty} H_1(\pi_0(\text{Diff}_\partial(W_{g,1}))) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

Proof. To ease the notation, given a sequence of groups $G_1 \rightarrow G_2 \rightarrow \dots$ we write G_∞ for its colimit. We first show that the following composition induces a homology isomorphism in all degrees

$$\pi_0(\text{Diff}_\partial(W_{\infty,1})) \longrightarrow \pi_0(\text{Diff}_\partial(X\#W_{\infty,1})) \longrightarrow \pi_0(\text{Diff}^+(X\#W_\infty)), \quad (6)$$

so in particular the second map induces a split surjection on all homology groups. The second map is as in (5) and the first map goes as follows: in terms of the identification of $X\#W_{g,1}$ with $(X\#D^4)\natural W_{g,1}$, the stabilisation map is given by $(-)\natural \text{id}_{W_{1,1}}$, so the map $\text{id}_{X\#D^4}\natural(-)$ induces a map on colimits.

Now by [9, Proposition 3.7], the Dehn twist $t_{W_g} \in \pi_0(\text{Diff}_\partial(W_{g,1}))$ is trivial for all $g \geq 0$, so $\pi_0(\text{Diff}_\partial(W_{g,1})) \rightarrow \pi_0(\text{Diff}^+(W_g))$ is an isomorphism. Similar to the argument in Lemma 2, a

combination of [10, Theorem 1, Remark 2)] and Quinn [15, Theorem 1.4] thus identifies the composition (6) with the induced map on automorphism groups of intersection forms,

$$\mathrm{id}_{E_8^{\oplus k}} \oplus (-) : \mathrm{Aut}(H^{\oplus \infty}) \longrightarrow \mathrm{Aut}(E_8^{\oplus k} \oplus H^{\oplus \infty}). \quad (7)$$

Fixing an isomorphism $E_8 \oplus -E_8 \cong H^{\oplus 8}$, the composition of $E_8^{\oplus k} \oplus (-)$ with $-E_8^{\oplus k} \oplus (-)$ agrees on each finite stage of the colimit with $(-) \oplus H^{\oplus 8k}$ up to an inner automorphism. As inner automorphisms induce the identity on group homology [3, Proposition II.6.2] and taking homology of a group preserves sequential colimits by [3, p. 121], it follows that the map $\mathrm{id}_{-E_8^{\oplus k}} \oplus (-) : \mathrm{Aut}(E_8^{\oplus k} \oplus H^{\oplus \infty}) \rightarrow \mathrm{Aut}(H^{\oplus 8+\infty}) = \mathrm{Aut}(H^{\oplus \infty})$ is inverse to (7) on the level of homology groups, so the composition (6) is indeed a homology isomorphism.

If $t_{X\#W_\infty}$ is trivial, then the second map in (6) is an isomorphism by (5), so the first map induces an isomorphism on all homology groups, in particular on $H_1(-)$ as claimed. If $t_{X\#W_\infty}$ is nontrivial, then (5) is a short exact sequence, so using that $t_{X\#W_\infty}$ is a central element, the 5-term exact sequence (see [3, Corollary VII.6.4]) of this short exact sequence has the form

$$\begin{array}{ccccccc} H_2(\pi_0(\mathrm{Diff}_\partial^+(X\#W_{\infty,1}))) & \xrightarrow{(*)} & H_2(\pi_0(\mathrm{Diff}^+(X\#W_\infty))) & \searrow & & & \\ \swarrow & \mathbb{Z}/2 \longrightarrow & H_1(\pi_0(\mathrm{Diff}_\partial(X\#W_{\infty,1}))) & \xrightarrow{(*)} & H_1(\pi_0(\mathrm{Diff}^+(X\#W_\infty))) & \longrightarrow & 0. \end{array}$$

Since $\pi_0(\mathrm{Diff}_\partial^+(X\#W_{\infty,1})) \rightarrow \pi_0(\mathrm{Diff}^+(X\#W_\infty))$ induces a split surjection on all homology groups by the first part, the two arrows indicated by $(*)$ are split surjective, so by exactness $H_1(\pi_0(\mathrm{Diff}_\partial(X\#W_{\infty,1})))$ is isomorphic to $H_1(\pi_0(\mathrm{Diff}^+(X\#W_\infty))) \oplus \mathbb{Z}/2$, as claimed.

As $\pi_0(\mathrm{Diff}_\partial(X\#W_{\infty,1})) \cong \mathrm{Aut}(H^{\oplus \infty})$ by the first part of the proof, the final part of the claim follows from the known fact that $H_1(\mathrm{Aut}(H^{\oplus \infty})) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (see, for example, [6, Proposition 2.2] for a reference). \square

In view of Lemma 4, to show that $t_{X\#W_\infty}$ is trivial, it suffices to prove that the groups $H_1(\pi_0(\mathrm{Diff}_\partial(W_{\infty,1})))$ and $H_1(\pi_0(\mathrm{Diff}_\partial(X\#W_{\infty,1})))$ are abstractly isomorphic. As $H_1(\pi_0(G)) \cong H_1(BG)$ holds for any topological group G (because $\pi_1(BG) \cong \pi_0(G)$ and $H_1(A)$ agrees with the abelianisation of $\pi_1(A)$ for any connected space A), this is equivalent to showing that the first homology groups of the homotopy colimits

$$\mathrm{hocolim}_{g \rightarrow \infty} \mathrm{BDiff}_\partial(Z\#W_{g,1}) \quad (8)$$

are isomorphic in the two cases $Z = S^4$ and $Z = X = \#^k K_3$. Here we used that taking homology turns sequential homotopy colimits into colimits (for a reference, use the mapping telescope model for homotopy colimits and apply [12, Section 14.6]). We will show the more general statement that the homology groups of the spaces (8) are in any degree independent of the choice of Z up to isomorphism for any closed 1-connected spin manifold Z , in particular for $Z = S^4$ and $Z = X = \#^k K_3$. This independence result follows from work of Galatius–Randal-Williams: an application of [7, Theorem 1.5] to the map $\theta : \mathrm{BSpin}(4) \rightarrow \mathrm{BO}(4)$ shows that there is for any 1-connected closed spin manifold Z a map from (8) to a fixed infinite loop space $\Omega^\infty \mathrm{MT}\theta$, independent of Z , that is a homology equivalence onto the path component of $\Omega^\infty \mathrm{MT}\theta$ that is hit by this map; which path component may depend on Z . Deducing this from the cited result uses that, by an application of obstruction theory, the spaces $\mathrm{BDiff}_\partial(W)$ and $\mathrm{Bun}_{n,\partial}^\theta(TW; \hat{\ell}_P) // \mathrm{Diff}_\partial(W)$

in [7, Definition 1.1] are homotopy equivalent for $n = 2$, $P = S^3$, and $W = \mathbb{Z}\#W_{g,1}$. Since all path components of a loop space are homotopy equivalent to the component of the constant loop (by multiplication with the inverse loop), all path components of $\Omega^\infty \text{MT}\theta$ have the same homology groups, so the claim follows.

Remark 5. This method can be used to resolve the ambiguity in [9, Theorem 7.1]: the kernel is $(\mathbb{Z}/2)^{n-1}$.

Remark 6. One can provide an explicit bound on the number of stabilisations by $S^2 \times S^2$ that are necessary to make t_X in Theorem B *topologically* isotopic to the identity, using the classification of such manifolds up to homeomorphism after a single stabilisation [4, Section 10.1] and the fact t_{K3} is *topologically* trivial (this implies the same for $t_{\overline{K3}}$). The latter uses the following reformulation (see [11, Section 2]): t_{K3} is smoothly (or topologically) nontrivial if and only if every smooth (or topological) bundle $K3 \rightarrow E \rightarrow S^2$ satisfies $w_2(TE) = 0$. The existence of a topological bundle with $w_2(TE) \neq 0$ follows from [2, Section 5]; it is the ‘difference’ of the bundles E and E' over $S^1 \times S^1$ which agree over the 1-skeleton.

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REFERENCES

1. D. Baraglia and H. Konno, *On the Bauer-Furuta and Seiberg-Witten invariants of families of 4-manifolds*, J. Topol. **15** (2022), no. 2, 505–586. MR4441598.
2. D. Baraglia and H. Konno, *A note on the Nielsen realization problem for K3 surfaces*, Proc. Amer. Math. Soc. **151** (2023), no. 9, 4079–4087. MR4607650.
3. K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer, New York, 1994, Corrected reprint of the 1982 original. MR1324339.
4. M. H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990. MR1201584.
5. G. Gabai, D. T. Gay, D. Hartman, V. Krushkal, and M. Powell, *Pseudo-isotopies of simply connected 4-manifolds*, arXiv:2311.11196, 2023.
6. S. Galatius and O. Randal-Williams, *Abelian quotients of mapping class groups of highly connected manifolds*, Math. Ann. **365** (2016), no. 1–2, 857–879. MR3498929.
7. S. Galatius and O. Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. II*, Ann. of Math. (2) **186** (2017), no. 1, 127–204. MR3665002.
8. D. T. Gay, *Diffeomorphisms of the 4-sphere, Cerf theory and Montesinos twins*, arXiv:2102.12890, 2021, to appear in Algebr. Geom. Topol.

9. J. Giansiracusa, *The stable mapping class group of simply connected 4-manifolds*, J. reine angew. Math. **617** (2008), 215–235. MR2400996.
10. M. Kreck, *Isotopy classes of diffeomorphisms of $(k - 1)$ -connected almost-parallelizable $2k$ -manifolds*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Mathematics, vol. 763, Springer, Berlin, 1979, pp. 643–663. MR561244.
11. P. B. Kronheimer and T. S. Mrowka, *The Dehn twist on a sum of two $K3$ surfaces*, Math. Res. Lett. **27** (2020), no. 6, 1767–1784. MR4216604.
12. J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999.
13. J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, Springer, New York, 1973.
14. P. Orson and M. Powell, *Mapping class groups of simply connected 4-manifolds with boundary*, arXiv:2207.05986, 2022, to appear in J. Differential Geom.
15. F. Quinn, *Isotopy of 4-manifolds*, J. Differential Geom. **24** (1986), no. 3, 343–372. MR868975.
16. O. Saeki, *Stable mapping class groups of 4-manifolds with boundary*, Trans. Amer. Math. Soc. **358** (2006), no. 5, 2091–2104. MR2197449.
17. C. T. C. Wall, *On simply-connected 4-manifolds*, J. Lond. Math. Soc. **39** (1964), 141–149. MR163324.
18. C. T. C. Wall, *Differential topology*, Cambridge Studies in Advanced Mathematics, vol. 156, Cambridge University Press, Cambridge, 2016. MR3558600.