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A note on the multicolour version of the Erdős-Hajnal conjecture

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ABSTRACT

Informally, the multicolour version of the Erdős-Hajnal conjecture (shortly EH-conjecture) asserts that if a sufficiently large host clique on n vertices is edge-coloured avoiding a copy of some fixed edge-coloured clique, then there is a large homogeneous set of size n^β for some positive β, where a set of vertices is homogeneous if it does not induce all the colours. This conjecture, if true, claims that imposing local conditions on edge-partitions of cliques results in a global structural consequence such as a large homogeneous set, a set avoiding all edges of some part. While this conjecture attracted a lot of attention, it is still open even for two colours.

In this note, we reduce the multicolour version of the EH-conjecture to the case when the number of colours used in a host clique is either the same as in the forbidden pattern or one more. We exhibit a non-monotonicity behaviour of homogeneous sets in coloured cliques with forbidden patterns by showing that allowing an extra colour in the host graph could actually decrease the size of a largest homogeneous set.

1. Introduction

We shall be considering edge-coloured cliques. Here, a *clique* of size n is a complete graph on n vertices denoted K_n. A *co-clique* of size n is a graph on n vertices with no edges. For a graph H, we denote by $\alpha(H)$ and $\omega(H)$ the sizes of a largest induced co-clique and clique in H, respectively. An *s-edge-colouring* c of the clique K_n on vertex set $[n]$ is a map $c : \binom{[n]}{2} \to [s]$. We denote by $|c|$ the number of colours from $[s]$ for which c^{-1} is not empty. Note that an s-edge-colouring c of K_n can be seen as an edge-partition of K_n into s colour classes, i.e. $K_n = G_1 \cup \cdots \cup G_s$, where G_i corresponds to a maximal subgraph of K_n whose edges are assigned colour i under c. Here G_i can be an empty graph if $|c| < s$. For an s-edge-colouring c of K_n on vertex set V and an s'-edge-colouring f of K_k on vertex set $[k]$, we define a *copy* of f in c to be a clique on a set $U \subseteq V$ of size k, such that c restricted to this clique is isomorphic to f, i.e., there is a vertex bijection $\phi : U \to [k]$ so that for any two vertices $x, y \in U$, $c(xy) = f(\phi(x)\phi(y))$. We say that c is f-free if there is no copy of f in c. Typically we assume that k is fixed and n is large, i.e. the f-free property is a local condition on the colouring. One can think of the colouring f as a forbidden colour pattern with a fixed set of colours. One of the key questions considered is how the local restrictions impact global properties, in particular how large the homogeneous number must be.

A *homogeneous set* in an s-edge-colouring c of K_n is a set $X \subseteq [n]$ that has a colour "missing", i.e. $|\{c(xy) : x, y \in X\}| < s$. The size of a largest homogeneous set in c is denoted by $h_s(c)$, or if the number of colours is clear from the context,

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simply $h(c)$. Note that any homogeneous set is an independent set in some colour class G_i, $i \in [s]$. Thus, we have $h(c) = \max\{\alpha(G_i) : i \in [s]\}$. Note that for $s = 2$ one colour of c corresponds to the edges of some n-vertex graph G and the other colour corresponds to the edges of the complement of G. Then in particular, we have $h(c) = h(G) = \max\{\alpha(G), \omega(G)\}$, which coincides with the definition of a homogeneous set in graphs. For $s' \leq s$, an s'-edge-colouring f of K_k, $k \leq n$, we define

$$h(n, f) = h_s(n, f) = \min\{h_s(c) \mid c \text{ is an } f\text{-free } s\text{-edge-colouring of } K_n\}.$$

Definition 1. Let f be an s'-edge-colouring of K_k and let $s \geq s'$. If there are positive constants $\epsilon = \epsilon(f, s)$ and C, such that $h_s(n, f) \geq Cn^{\epsilon}$, we say that f has the EH-property for s colours.

For example, when f is an edge-colouring of K_3, i.e. a triangle, with two edges of colour 1 and one edge of colour 2, one can show that in any f-free edge-colouring of a clique on n vertices using colours 1, 2, and 3 there is a set of vertices of size $n^{1/2}$ inducing edges only of two of these three colours, see for example Axenovich, Snyder, Weber [3]. This shows that f has the EH-property for 3 colours.

Conjecture 1 *(Multicolour version of the Erdős-Hajnal conjecture). Let k, s' be integers with $k, s' \geq 2$. Then for any $s \geq s'$, any edge-colouring f of K_k with $|f| = s'$ has the EH-property for s colours.*

We shall call this conjecture the "multicolour EH-conjecture".

Even in the case of two colours, i.e., $s' = s = 2$, the conjecture remains open, see for example a survey by Chudnovsky [7], as well as [1,5,12,19], to name a few. Note that the Erdős-Hajnal conjecture [10] was first stated for the case of only two colours. When F is a fixed graph and G is any F-free n-vertex graph, Erdős and Hajnal proved that $h(G) \geq 2^{d\sqrt{\log n}}$, for a positive constant d. It was mentioned in [10] and again by Hajnal [16] that one could extend the arguments from two colours to s colours to show that in the above setting $h_s(n, f) = \Omega(2^{\sqrt{\log n}})$. In particular, for any s and any forbidden colouring f, we have for any y and any sufficiently large n, that

$$h_s(n, f) \geq \log^y n. \tag{1}$$

The bound for two colours was recently improved to $h(G) \geq 2^{d\sqrt{\log n \log\log n}}$, for a positive constant d, by Bucić, Nguyen, Scott, and Seymour [6].

Here, we consider Conjecture 1 in general. Note that f might not use all colours in $[s]$ and it is not immediately obvious whether a larger number of colours in the edge-colouring of the host clique forces larger homogeneous sets. We show that we can reduce the problem to the case when the number of colours in the edge-colouring of a large clique is the same or one more than the number of colours in the forbidden pattern f.

Theorem 1. *Let f be an s'-edge-colouring of a clique such that $|f| = s'$ and let s be an integer with $s > s'$. For any positive n, $h_{s+1}(n, f) \geq h_s(n, f)$ and for any positive ξ, for sufficiently large n, $h_{s+1}(n, f) \leq h_s^{1+\xi}(n, f)$. In particular, f has the EH-property for s colours if and only if f has the EH-property for $s + 1$ colours.*

Corollary 1. *Let f be an s'-edge-colouring of a clique. Then the multicolour EH-conjecture holds for f if and only if f has the EH-property for s' and $s' + 1$ colours.*

Note that Theorem 1 does not hold if $s = s'$. Indeed, as we shall show in the next proposition, there is a colouring f of a clique on 4 vertices in 2 colours, such that $h_3(n, f) = o(h_2(n, f))$. Moreover, there is a colouring f' of a clique on three vertices in three colours, such that $h_4(n, f') \neq \omega(h_3(n, f'))$. Here, an edge colouring of a graph is *rainbow* if it assigns distinct colours to distinct edges.

Proposition 1. *Let f be a rainbow colouring of K_3 with colours 1, 2, and 3. Let f' be an edge-colouring of K_4 with colours 1 and 2 in which each class forms a path on three edges. Then*

$$h_3(n, f) = \Theta\left(n^{1/3}\log^2 n\right) \text{ and } h_4(n, f) = O(n^{1/3}\log^2 n),$$

$$h_2(n, f') = n^{1/2} \text{ and } h_3(n, f') = O(n^{1/3}\log^{7/3} n).$$

This paper is structured as follows. We prove Theorem 1, Corollary 1, and Proposition 1 in Section 2. We state some concluding remarks and open problems in Section 3. We omit floors and ceilings when clear from the context.

2. Proofs of the main results

Proof of Theorem 1. Let f be an edge-colouring of a clique, s be an integer with $s > |f|$, and n be a sufficiently large integer.

Let c be an f-free $(s+1)$-edge-colouring of K_n with $h(c) = h_{s+1}(n, f)$. Since $s > |f|$, there exists a colour $a \in [s]$ which is not used in f. The colour $s+1$ is also not used in f. Now recolour all edges of colour $s+1$ in c with colour a and call the resulting colouring c'. Then c' is an s-edge-colouring of K_n which is f-free, since the edges having colours from f are the same in c and c'. Thus, there is a homogeneous set X in K_n under c' of size at least $h_s(n, f)$. In particular, X avoids some colour $a' \in [s]$ under c'. If $a' \neq a$, then X also avoids a' under c. If $a = a'$, then X avoids a and $s+1$ under c. Thus, in any case, X is a homogeneous set under c, so $h_{s+1}(n, f) \geq |X| \geq h_s(n, f)$.

Let $0 < \xi < 1$ and n be sufficiently large. Assume that $h_{s+1}(n, f) = h^{1+\xi}$, for some h. Note that from (1) we have $h > \log^{2/\xi} n$. We shall show that $h_s(n, f) \geq h$.

Let c be an f-free s-colouring of K_n on vertex set V. We want to show that $h(c) \geq h$. We shall construct an $(s+1)$-edge-colouring c' of K_n starting with c as follows: Recolour each edge with the colour $s+1$ with probability $\frac{1}{2}$, and leave the colour from c with probability $\frac{1}{2}$. Since the colour $s+1$ is not used in f, the new colouring c' is f-free. Assume that $h(c) < h$. Then under c every subset of V of size h contains an edge of each colour in $[s]$. Using the properties of a random graph $G \in \mathcal{G}(n, \frac{1}{2})$, we see that each vertex set of size $2\log n$ and thus, each vertex set of size h, induces an edge of colour $s+1$ under c' with probability close to 1. On the other hand, since $h(c) < h$, we know that in c each subset of V of size h induces an edge of colour i for each $i \in [s]$. Thus, using Turán's theorem [20], a given subset V of size $h^{1+\xi}$ induces at least $x = \Omega\left(\binom{h^{1+\xi}}{2}\frac{1}{h}\right) = \Omega(h^{1+2\xi})$ edges of colour i, for any $i \in [s]$, under the colouring c. The probability that all these edges of colour i are recoloured with colour $s+1$ is at most $(1/2)^x$. Thus, the probability that some subset of $h^{1+\xi}$ vertices misses some colour from $[s]$ under c' is at most

$$\binom{n}{h^{1+\xi}} s \left(\frac{1}{2}\right)^x \leq 2^{h^{1+\xi}\log n - \Omega(h^{1+2\xi})},$$

i.e., close to zero for large n. Therefore, with positive probability, all subsets of $h^{1+\xi}$ vertices induce edges of all colours under c' and so $h(c') < h^{1+\xi}$, a contradiction to our assumption.

As a consequence of the two inequalities on $h_s(n, f)$ and $h_{s+1}(n, f)$, we have that f has the EH-property for s colours if and only if f has the EH-property for $s+1$ colours. □

Proof of Proposition 1. We shall first forbid a rainbow triangle. Let f be an edge-colouring of K_3 in which the edges have colours 1, 2, and 3. The structure of an f-free 3-colouring of a clique is known and is called a *Gallai colouring* [13,15]. It is known that f has the EH-property for 3 colours, see for example [11], where it is shown that $h_3(n, f) = \Theta\left(n^{1/3}\log^2 n\right)$. Next we shall show that $h_4(n, f) = O(n^{1/3}\log^2 n)$.

We shall consider lexicographic products of colourings. If c^* and c^{**} are colourings, where c^* colours a clique on vertex set $\{v_1, \ldots, v_k\}$ and c^{**} colours a clique on y vertices, then the *lexicographic product* of c^* and c^{**}, denoted $c^* \times c^{**}$ is a colouring of a clique on vertex set $V_1 \cup \cdots \cup V_k$, where $|V_i| = y$, $i = 1, \ldots, k$, the edges induced by V_i are coloured according to c^{**}, $i = 1, \ldots, k$, and all edges between V_i and V_j are coloured with $c^*(v_i v_j)$, for $1 \leq i < j \leq k$. Here, we refer to the V_i's as *blobs*. Let for a colouring c^* of a clique and a subset of colours I, $S_I^{c^*}$ be the size of a largest clique using colours only from I. Note that if c^* and c^{**} are colourings, then $S_I^{c^* \times c^{**}} = S_I^{c^*} S_I^{c^{**}}$.

Let c_i, $i \in [3]$ be a 3-edge-colouring of $K_{n^{1/3}}$ using colours from $[4] - \{i\}$ and satisfying $h_3(c_i) = O(\log n)$, i.e. for some positive C any clique on $C\log n$ vertices induces all three colours from $[4] - \{i\}$. Note that such colourings exist and could be chosen by randomly assigning one of the three colours to each edge uniformly. Also note that c_i is f-free for $i \in [3]$.
Let $c = c_1 \times c_2 \times c_3$ be the lexicographic product of c_1, c_2 and c_3, it is a 4-edge-colouring of K_n. This is a construction very similar to one used in [11].

To verify that c is f-free, consider first $c_2 \times c_3$. Since c_2 and c_3 are f-free, we only need to check each triangle with two vertices in one blob and one vertex in a different blob of the blow-up of $c_2 \times c_3$. Since the edges between two different blobs all have the same colour, the triangle is not rainbow. Thus, $c_2 \times c_3$ is f-free. Similarly, we conclude that $c = c_1 \times (c_2 \times c_3)$ is f-free.

Next, we shall bound the size of a largest homogeneous set in c. For a subset $\{i, j, k\}$ of $[4]$, we shall simply write ijk. We have that $S_I^{c_j} = n^{1/3}$ if $I = [4] - \{j\}$ and $S_I^{c_j} = S_{I-\{j\}}^{c_j} = O(\log n)$, if $j \in I$. Then

$$\begin{aligned}
S_{123}^c &= S_{123}^{c_1} \cdot S_{123}^{c_2} \cdot S_{123}^{c_3} = O(\log n)O(\log n)O(\log n),\\
S_{124}^c &= S_{124}^{c_1} \cdot S_{124}^{c_2} \cdot S_{124}^{c_3} = O(\log n)O(\log n)n^{1/3},\\
S_{134}^c &= S_{134}^{c_1} \cdot S_{134}^{c_2} \cdot S_{134}^{c_3} = O(\log n)n^{1/3}O(\log n), \quad \text{and}\\
S_{234}^c &= S_{234}^{c_1} \cdot S_{234}^{c_2} \cdot S_{234}^{c_3} = n^{1/3}O(\log n)O(\log n).
\end{aligned}$$

Since $h_4(c) = \max\{S^c_{ijk} : \{i, j, k\} \subseteq [4], |\{i, j, k\}| = 3\}$, we have that $h_4(n, f) \le h_4(c) = O(n^{1/3} \log^2 n)$.

Now, we shall forbid induced P_4 in two colours. Let f' be a 2-edge-colouring of K_4 in which colour class 1 and colour class 2 form a P_4. Note that f' having the EH-property for 2 colours is equivalent to P_4 having the EH-property. Any P_4-free graph G is a co-graph (see for example [4,9] for properties of co-graphs), which is in particular a perfect graph, i.e., $\omega(G) = \chi(G)$, where $\chi(G)$ is the chromatic number. As observed by Erdős and Hajnal [10], if G has n vertices, $n \le \alpha(G)\chi(G) = \alpha(G)\omega(G)$. Therefore $h(G) = \max\{\alpha(G), \omega(G)\} \ge n^{1/2}$. In particular, we have $h_2(n, f') \ge n^{1/2}$. It is also not difficult to show and is proven in [10], that $h_2(n, f') \le n^{1/2}(1 + o(1))$.

Next we shall construct an f'-free colouring c' of K_n on a vertex set V using colours 1, 2, and 3, for sufficiently large n. By the lower bound on the Ramsey number $R(4, t) = \Omega(t^3/\log^4 t)$ by Mattheus and Verstraete [17], there exists a graph H on the vertex set V such that $\omega(H) < 4$ and $\alpha(H) < Cn^{1/3} \log^{4/3} n$, for some positive constant C. To define c', let the edges not in H be coloured 3, and each edge of H be coloured 1 with probability 1/2 and 2 with probability 1/2. Note that in this colouring each K_4 has an edge of colour 3, and therefore there is no copy of f'. We shall argue that with positive probability $h_3(c') = O(n^{1/3} \log^{7/3} n)$. Letting $q(n) = 8\alpha(H) \log(n)$, we shall show that any set of $q(n)$ vertices induces edges of all three colours under c'. Let X be a fixed set of $q(n)$ vertices. By Turán's theorem [20], the number of edges induced by X in H is at least

$$e_X = \frac{1}{\alpha(H)} \binom{q(n)}{2} \ge \frac{q^2(n)}{4\alpha(H)}.$$

If p_X is the probability that X induces only edges of colours 2 and 3 in c' or that X induces only edges of colours 1 and 3 in c', then

$$p_X \le 2 \cdot 2^{-e_X} \le 2 \cdot 2^{-q^2(n)/4\alpha(H)}.$$

Using the union bound over all $q(n)$-element subsets of V, we have that the probability that c' contains a $q(n)$-vertex set inducing edges of only two colours is at most

$$\binom{n}{q(n)} p_X \le n^{q(n)} 2^{1-q^2(n)/4\alpha(H)} = 2^{8\alpha(H)\log^2(n)+1-16\alpha(H)\log^2(n)} < 1.$$

Thus, with positive probability there exists a desired colouring. □

We remark that we did not attempt to optimise any of the constants involved.

3. Concluding remarks

The multicolour version of the Erdős-Hajnal conjecture is concerned with the existence of large homogeneous sets in edge-coloured cliques that do not contain a copy of a given colouring of a small clique. It could be that the number of colours used in a large clique is strictly larger than the number of colours used in a forbidden clique-colouring.

We showed that the multicolour EH-conjecture could be reduced to the situation when the large clique uses the same set of colours as the forbidden colouring or maybe one more. This brings us to the following special cases, in a sense smallest, for which the EH-conjecture is known to be true for the number of colours used in the forbidden colouring, but not any more once additional colours are allowed:

Question 1. Does the 2-edge-colouring of K_4 in which each colour class is isomorphic to P_4 have the EH-property for 3 colours?

Question 2. Does the rainbow triangle have the EH-property for 4 colours?

Note that Question 2 was formulated by Conlon, Fox, and Rödl [8] in a connection with a hypergraph Ramsey number $R_3(H; 3)$ for a specific 3-uniform hypergraph H. Here $R_3(H; 3)$ is the smallest n such that any colouring of triples from an n-element set using three colours results in a monochromatic copy of H.

Let H_t be a 3-uniform hypergraph on a vertex set $[t] \cup \binom{[t]}{2}$ and edge set $\{\{i, j, \{i, j\}\} : i, j \in [t]\}$. It was shown in [8], that

$$F(t) \le R_3(H_t; 3) = O(t^4 F(t^3)^2),$$

where $F(t)$ is a dual function for $h_4(n, f)$, so that f is an edge colouring of a triangle using three colours 1, 2, and 3, i.e., $F(t)$ is the smallest n such that any edge colouring of a clique on n vertices using colours 1, 2, 3, and 4 contains either f or a clique of size t inducing at most 3 colours. So, if Question 2 has a positive answer, it would imply a polynomial behaviour of $R_3(H_t; 3)$, contrasting the exponential behaviour of the 4-colour Ramsey number $R_3(H_t; 4)$.

While the multicolour version of the EH-conjecture is still wide open, we conjecture that its weaker size-version holds. Here, instead of forbidding a specific pattern, we forbid a palette $T = (t_1, \dots, t_{s'})$, where t_i's add up to the number of edges in a k-vertex clique. We say that a colouring c *avoids the palette* T if there exists no clique of size k so that the number of edges of colour i in that clique is exactly t_i, $i = 1, \dots, s'$.

Conjecture 2. *Let k, s', and s be integers, $s \geq s'$. Let $T = (t_1, \dots, t_{s'})$ be a tuple of nonnegative integers adding up to $\binom{k}{2}$. Then there is a positive constant ϵ such that for any colouring c of K_n in colours from $[s]$ avoiding T, we have $h_s(c) \geq n^{\epsilon}$.*

When $s = s' = 2$, a quantitative version of the above conjecture is proven in [2]. Moreover, one can show that the conjecture holds if $s' = 2$ and $s > s'$ using a generalisation of a result by Alon, Pach, and Solymosi [1] to arbitrary number of colours, see Weber [21]. In general, the conjecture still might be challenging as it coincides with the EH-conjecture when $(t_1, \dots, t_{s'}) = (1, \dots, 1)$, i.e., in the case of the rainbow pattern.

The EH-conjecture fails for r-graphs, $r \geq 3$, already when F is a clique of size $r+1$. Indeed, well-known results on off-diagonal hypergraph Ramsey numbers show that there are n-vertex r-graphs that do not have a clique on $r+1$ vertices and do not have cocliques on $g_r(n)$ vertices, where g_r is an iterated logarithmic function (see [18] for the best known results). Moreover, a result (Claim 1.3. in Gishboliner and Tomon [14]) tells us that for any $r \geq 3$, if F is an r-graph on at least $r+1$ vertices, $F \neq D_2$, then there is an F-free r-graph H on n vertices such that $h(H) = (\log n)^{O(1)}$. Here D_2 is a unique 3-graph on four vertices and two edges.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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