Order Σ-invariants, Novikov homology, and nilpotent groups

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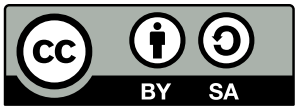


Introduction

It is a curious fact about finitely generated groups that while every quotient of a finitely generated group is again finitely generated, the same cannot be said about subgroups of finitely generated groups. Consider for example the free group F_2 on two generators and any surjective map $\Phi\colon F_2 \twoheadrightarrow \mathbb{Z}$. Then the kernel $\ker\Phi$ is a normal subgroup of F_2 which is generated as a normal subgroup by some $g \in F_2$. But $\ker\Phi$ is not finitely generated as a group. This gives rise to two general questions about finitely generated groups G:

1. If $N \trianglelefteq G$ is a normal subgroup, how can we decide if N is finitely generated?
2. Does there exist a finitely generated normal subgroup $N \trianglelefteq G$ such that $G/N \cong \mathbb{Z}$?

An answer to the first question in case G/N is abelian is given by Bieri, Neumann and Strebel in [BNS87] in form of the Σ-invariant. It is a set containing certain maps $G \to \mathbb{R}$. Bieri, Neumann and Strebel show that N is finitely generated if and only if $\Sigma^1(G)$ contains every map with N contained in its kernel.

An important characterisation of the Σ-invariant is due to Sikorav [Sik87]. He provides an equivalent criterion for a map being in $\Sigma^1(G)$ via vanishing of the homology of G. The coefficient ring Sikorav uses is the *Novikov ring* from [Nov81].

Novikov's original motivation was to generalise Morse theory, hinting that there is a link between Morse theory and Σ-invariants. Indeed, Bestvina and Brady show in [BB97] that seeing a map $\Phi\colon G \twoheadrightarrow \mathbb{Z}$ as a height function on the Cayley graph, finite generatedness of $\ker\Phi$ is equivalent to ascending and descending links being connected.

Let us now elaborate on the second one of the above questions. We call a group *algebraically fibred* if the answer to the second question is positive for that group. Kielak shows in [Kie20] that a group that is virtually RFRS is virtually algebraically fibred if and only if its first ℓ^2-Betti number vanishes. In the proof, the Σ-invariant and its characterisation via Novikov homology play a crucial role.

Another essential tool in Kielak's proof is the Atiyah conjecture. In [Ati76], Atiyah conjectures that all ℓ^2-Betti numbers of torsion-free groups are integers. While the conjecture remains open in general, Kielak shows that it holds for RFRS groups. Linnell provides in [Lin93] a reformulation of Atiyah's conjecture to the question of whether a certain ring $\mathcal{D}(G)$ is a division ring. In this case, figuring out the ℓ^2-Betti numbers of G becomes a question about the dimensions of certain homology groups of G as $\mathcal{D}(G)$-vector spaces. Kielak uses this in his proof to bridge the gap between ℓ^2-Betti numbers and Novikov homology.

Kielak's theorem goes back to a question of Thurston: In [Thu82], he asks if every hyperbolic 3-manifold has a finite-sheeted cover that fibres over the circle. By a theorem

of Stallings' [Sta62], this is equivalent to the corresponding map between fundamental groups having finitely generated kernel.

RFRS groups were introduced by Agol in [Ago08], providing a criterion to answer Thurston's question for certain 3-manifolds: They fibre virtually if their fundamental group is RFRS. Kielak's theorem transfers Agol's result to a purely algebraic setting.

The primary motivation for this thesis is to generalise Kielak's fibration theorem to a larger class of groups. An important observation is that every finitely generated abelian group is RFRS, but non-abelian nilpotent groups are not RFRS. The definition of RFRS requires that certain quotients are abelian. In this work, we instead require those quotients to be nilpotent, thereby enlarging the class of RFRS groups to a class we call *RFN*, which in particular contains all torsion-free nilpotent groups.

In order to approach a generalised fibration theorem for RFN groups, we construct analogues of the Σ-invariant and Novikov homology for nilpotent quotients instead of abelian ones. Thereby, we answer the first of the two questions above in this case. Our approach is to replace maps to $\mathbb{R}$ with partial biorderings on G.

A similar idea to study orders on a group and thereby characterise the Σ-invariant can be found in [Alo+22]. Notably, these authors use total left orderings instead of partial biorderings.

Another approach to get around the restriction that G/N is abelian is due to Heuer and Kielak in [HK22]. There, the authors replace homomorphisms $G \to \mathbb{R}$ with quasi-morphisms. They also provide an analogue of the Σ-invariant in this setting and prove that it can be characterised via vanishing of Novikov homology.

In [Ren88], Renz generalises the Σ-invariant to higher dimensions and shows that it characterises higher finiteness properties of normal subgroups. This generalisation transfers to fibrations: In [Fis22], Fisher shows that there exists a subgroup $N \trianglelefteq G$ of type $\mathcal{FP}_n$ such that $G/N \cong \mathbb{Z}$ if and only if the first n ℓ^2-Betti numbers of G vanish. [HK24] is a very recent release where Hughes and Kielak show that for groups G of type $\mathcal{FP}_n(\mathbb{Q})$, the n'th Σ-invariant is even empty if the n'th ℓ^2-Betti number of G does not vanish.

The higher Σ-invariants are only of minor importance in this thesis, but it seems likely that much of the theory we introduce transfers analogously to a higher dimensional setting.

Another vital topic in Kielak's proof is the Ore localisation introduced in [Ore31]. In order to understand ℓ^2-Betti numbers as the dimension of $\mathcal{D}(G)$-vector spaces, one first needs to understand the *Linnell ring* $\mathcal{D}(G)$. It is the division closure of the group ring $\mathbb{Z}G$ in the Ore localisation of the von Neumann Algebra $\mathcal{L}(G)$. The Ore localisation need not exist, but if it does, it is a particularly nice description of the universal localisation of a ring R.

Let us now give an overview this of thesis' structure and main results. We start in Part I by reviewing the most important foundational material and providing some helpful examples we will use throughout the later parts.

Part II focuses on the Ore localisation. While the description of the Ore localisation is convincing, the actual proof that it is well-defined is notoriously technical. There are published proofs in [Ore31] itself as well as [Pas77] and [Ško04]. However, each of those leaves out at least some of the details. Also, it is nearly impossible to verify those proofs by hand.

The Lean theorem prover [Mou+15] is a relatively recent development that allows for automatic verification of proofs. The accompanying math library [Com20] is an ever-growing collection of objects and proofs that have been formalised in Lean. We contribute to that by adding the definition of the Ore localisation and providing proofs for the following theorems. This project is joint work with Jakob von Raumer. The code has been published in the math library [KR22].

In our formalisation, after defining the Ore localisation in the categories of monoids, semirings and rings, we show that the localisation is a well-defined object in the same category.

Theorem A (Theorem 5.10, Theorem 5.20, Lemma 5.23)**.**

1. *Let R be a monoid and $S \subseteq R$ an Ore subset. Then the multiplication on the Ore localisation of R at S is well defined and turns it into a monoid.*

2. *Let R be a semiring or ring and $S \subseteq R$ an Ore subset. Then the addition on the Ore localisation of R at S is well defined and, together with the multiplication, turns it into a semiring or ring, respectively.*

We also show that the Ore localisation is a description of the universal localisation.

Theorem B (Theorem 5.13, Theorem 5.21)**.** *If R is a monoid or semiring, then the Ore localisation satisfies the universal property of the universal localisation.*

In the commutative case, the localisation has already been formalised in the math library. We provide an isomorphism between the commutative localisation and the Ore localisation for monoids.

Theorem C (Theorem 5.14)**.** *If R is a commutative monoid, then the Ore localisation at S is isomorphic to the group of differences of R and S.*

The main reason why the Ore localisation is interesting to us later on is because it is a canonical way to turn a ring into a division ring.

Theorem D (Theorem 5.24)**.** *If R is a ring without zero divisors, then the Ore localisation of R at the set of non-zero elements is a division ring.*

While these results are not new, the existence of a computer-verified proof is. To the best of my knowledge, they are also the first published proofs that are complete and correct.

There are some immediate consequences of the above theorems.

1. Because the Ore localisation fulfils the universal localisation property, it is isomorphic to the universal localisation.

2. The Ore localisation is also the universal localisation if R is a ring, as every semiring homomorphism between rings is also a ring homomorphism.

3. If R is a commutative domain, then the Ore localisation is the field of fractions.

We will use these corollaries as well as the theorems themselves in Part IV, when we tackle a generalised version of Kielak's fibration theorem. This concludes the results we implemented in Lean.

In Part III, we return our attention to Σ-invariants. As mentioned above, the Σ-invariant provides a criterion for $N \trianglelefteqslant G$ being finitely generated only if G/N is abelian. Our goal in this part is to generalise the Σ-invariant to the case where G/N is nilpotent.

In order to do this, the first important observation is that seeing a map $\Phi\colon G \to \mathbb{R}$ as a height function induces a partial order on G that compares two elements by comparing their heights. We introduce the notion of *full order* to recognise orders that are induced by maps. We show that together with the archimedean property, it provides a way to translate between maps $G \to \mathbb{R}$ and orders on G.

Theorem E (Theorem 7.23)**.** *Let G be a finitely generated abelian group. Every full archimedean order on G is induced by a map $\Phi\colon G \to \mathbb{R}$, where $\mathbb{R}$ carries the standard order.*

We will also see that conversely, every Φ induces a full archimedean order on G, thus obtaining a one-to-one correspondence between maps and orders, up to some equivalence.

This correspondence between maps and orders allows us to translate the classical setting involving finitely generated kernels, Σ-invariants and Novikov homology based on maps $G \to \mathbb{R}$ into a language that is based on orders instead. This enables us to drop the assumption from the classical setting that some groups must be abelian for the above notions to make sense.

Instead, we will assume that the respective groups are nilpotent and proceed to show that many of the classical results transfer to our new setting. We start by giving a complete description of full archimedean orders on nilpotent groups.

Theorem F (Theorem 8.6, Proposition 8.7)**.** *Let G be a finitely generated nilpotent group. For every full archimedean order $\prec$ on G, there exists a normal subgroup $H \trianglelefteqslant G$ such that $\prec$ is induced by some order on the center $Z(G/H)$.*

The correspondence between orders on G and pairs consisting of a subgroup H and an order on $Z(G/H)$ is one-to-one.

Every order on G gives rise to a positive cone. By asking about connectedness of this cone in the Cayley graph, we obtain a definition of Σ-invariant for partial orders.

For full orders, we provide a correspondence between kernels of order-inducing maps and trivially ordered subgroups, so-called *antichains*. We use this link to prove that in analogy to the classical Σ-invariant, our generalisation, the *order Σ-invariant*, characterises finitely generated normal subgroups $N \trianglelefteqslant G$ if G/N is nilpotent.

Theorem G (Theorem 8.11)**.** *Let G be a finitely generated group and $N \trianglelefteq G$ a normal subgroup such that G/N is nilpotent. Then N is finitely generated if and only if the order Σ-invariant of G contains every full archimedean order such that N is an antichain.*

In the proof of this theorem, we use the fact that we can characterise full archimedean orders on nilpotent groups and thus require G/N to be nilpotent. However, it is not clear if the condition G/N nilpotent is necessary in the sense that the theorem becomes wrong if G/N is not nilpotent. I have already released this generalisation of the Σ-invariant up to the previous theorem; For the preprint, see [Kli23].

As we did for Σ-invariants, we introduce a generalisation to the Novikov ring that depends on a full archimedean order instead of a map. We show that many of the structural properties of the classical Novikov ring transfer to the order setting.

The Σ-invariant for orders may be characterised via vanishing of Novikov homology. This shows that two crucial results from the classical setting transfer directly to orders. Namely, finite generatedness of normal subgroups can be seen in the Σ-invariant and via Novikov homology.

Theorem H (Theorem 9.36)**.** *Let G be a finitely generated group and $N \trianglelefteq G$ a normal subgroup such that G/N is nilpotent. Then the following are equivalent.*

1. *N is finitely generated.*
2. *Every full archimedean order on G such that N is an antichain is contained in the order Σ-invariant.*
3. *For every full archimedean order on G such that N is an antichain, the first homology of G with Novikov coefficients vanishes.*

We conclude the thesis by outlining a strategy to extend Kielak's fibration theorem to RFN groups in Part IV. Moreover, we already prove some results that are necessary for the proof of a generalised fibration theorem. This connects all the material we have covered in the previous parts, in particular the Ore localisation and the order Σ-invariant and its characterisation via Novikov homology.

The class of RFN groups is an extension of RFRS groups by nilpotent groups. The following theorem provides a plethora of examples.

Theorem I (Proposition 10.4)**.** *The class of RFN groups contains*

1. *every RFRS group*
2. *and every torsion-free nilpotent group.*

It is closed under the following operations:

3. *taking subgroups,*
4. *taking free products,*

5. taking direct products.

An important observation is that to prove a generalised fibration theorem, we can no longer restrict ourselves to archimedean orders. Hence, we characterise all full orders on nilpotent groups.

Theorem J (Theorem 10.24)**.** *Every full order on a nilpotent group is either archimedean or properly lexicographic.*

This provides an explicit construction of all full orders on nilpotent groups.

We show that a large chunk of the proof of Kielak's fibration theorem translates to the RFN setting. The following theorem depends on some notions, which are not yet clear to me how precisely they transfer to RFN groups. Figuring out the details will be a big part of the work required to prove a fibration theorem for RFN groups. Notably among these notions are those of a *rich set* and a *well representable element*.

Theorem K (Theorem 10.40)**.** *Let G be an RFN group and assume that every element of $\mathcal{D}(G)$ is well representable. Then there exists a finite index subgroup $H \leqslant G$ and a rich set U of orders on H such that the Novikov homology of H vanishes for every order in U.*

Hence, we know that not only Σ-invariants, but also most other tools used in Kielak's proof, apply analogously in the nilpotent setting. This suggests that the fibration theorem itself also holds for RFN groups.

Conjecture L (Conjecture 10.41)**.** *Let G be a finitely generated group that is virtually RFN. Then G is virtually fibred if and only if the first ℓ^2-Betti number of G vanishes.*

Contents

Part I.

Preliminaries

1. Groups and spaces

The premise of geometric group theory is to study a group G by considering actions of G on geometric spaces. An idea of algebraic topology is to take this approach to an algebraic setting and study actions of G on rings or modules. We start by reviewing how to translate between groups, spaces and modules. The main references are [Hat01], [Pas77], [Geo08] and [Bro82].

1.1. Group rings

We start by investigating actions of groups on modules. We will see that the group ring is the natural ring to consider in this situation. To fix our terminology: By a *ring*, we always mean a ring with 1 and not necessarily commutative multiplication. Most rings in this work are some variation of a group ring with coefficients in $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ or another group ring.

A group always acts freely on itself by multiplication. However, there is generally no way to simply define an additional additive structure on G to turn it into a ring. For example, there exists no ring $(R, +, \cdot)$ such that the multiplicative monoid $(R \setminus 0, \cdot)$ is isomorphic to $\mathbb{Z}$. The group ring is a canonical construction of a ring on which G acts freely. For now, since the multiplicative structure of a ring need not be a group, we may loosen our assumption on G and require it to be just a monoid.

Definition 1.1. Let R be a ring and G a monoid. Then the *monoid ring* of G with R-coefficients is the free R-module with basis G. That is, RG is the additive group of finite formal sums of R-multiples of elements of G. The multiplication of two basis elements is the multiplication in G, the multiplication of coefficients is the multiplication in R. and everything else is defined by linear extension. To be explicit,

$$(\sum_{g\in G} r_g g)\ (\sum_{g\in G} t_g g) = \sum_{g\in G}(\sum_{h\in G} r_{gh^{-1}} t_h) g.$$

We also call RG a *group ring* if G is a group.

Note that RG is commutative if and only if G is abelian. Later on, we will want to consider at least nilpotent groups that are not abelian, so we cannot allow ourselves to assume that all rings are commutative.

In the literature, if R is a field, RG is sometimes also called a *group algebra* to emphasise the fact that it is an R-vector space as well as a ring.

Example 1.2. For $G = \mathbb{N}$, the monoid ring RG is the polynomial ring with coefficients in R. Note that if we let X be the generator of $\mathbb{N}$, there is an intentional overlap of

notations for polynomials and for elements of the monoid ring $R\mathbb{N}$. Similarly, if $G = \mathbb{Z}$, the group ring $R\mathbb{Z}$ is the ring of Laurent polynomials.

Note that a sum

$$r = \sum_{g \in G} r_g g$$

is an element of the monoid ring RG if and only if only finitely many r_g are non-zero. The set of those g such that $r_g \neq 0$ is called the *support* of r and is denoted by $\operatorname{supp} r$. In line with Example 1.2, we call a single summand $r_g g$ a *monomial*.

Another way to view r is as a finitely supported map between sets

$$r \colon G \to R, \quad g \mapsto r_g.$$

We denote the set of all those maps, even with infinite support, by R^G.

The embedding $G \to RG$ is a map of monoids and hence defines an action of G on RG. This action is free because G acts freely on a free basis of R. An RG-module is an R-module together with a G-action.

Remark 1.3. To be more precise: A left RG-module is a left R-module with a left G-action. For example, left multiplication by G on RG. The same works for right multiplication and right modules. Often, we get to decide between left multiplication or right multiplication, which will provide left or right module structures, respectively. In these cases, we drop the designation of left or right as long as everything works equally well on either side.

Note that any ring together with its addition is an abelian group and hence a $\mathbb{Z}$-module. Thus, if we only want to construct a ring on which G acts freely, the canonical choice for the coefficient ring R is $\mathbb{Z}$. To make this notion precise, consider the forgetful functor Rings $\to$ Monoids that sends a ring to its multiplicative monoid. The left adjoint of this functor is the functor that sends G to $\mathbb{Z}G$.

In this spirit, the literature sometimes calls $\mathbb{Z}G$-modules just G-modules because they are just abelian groups with a G-action on them. This mimics the definition of an R-module as an abelian group with an R-multiplication.

Remark 1.4. I sometimes make category theoretic remarks like the previous one about adjoint functors. They are meant as an additional explanation for readers familiar with these notions and should not discourage everyone else. They are not essential to what follows. For those interested, a good reference is [Mac78].

1.2. Group homology

The homology of a group may be seen as a way to construct the "correct" modules with G actions on them if one wishes to study a group by its actions. We review the important definitions and will see how to compute the homology via classifying spaces.

Definition 1.5. Let R be a ring. A *chain complex* is a sequence of R-modules M_i for $i \in \mathbb{Z}$ and a map $\partial\colon \oplus_{i\in\mathbb{Z}} M_i \to \oplus_{i\in\mathbb{Z}} M_i$ such that $\partial M_{i+1} \subseteq M_i$ and $\partial^2 = 0$.

∂ is called the *boundary map*.

The restriction $\partial_{|M_i}$ is also denoted ∂_i and the whole chain complex is

$$\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} \ldots.$$

For a finitely long sequence like

$$0 \to M_2 \to M_1 \to M_0 \to 0,$$

all modules outside the shown range are assumed to be trivial.

Definition 1.6. Let (M_i, ∂) be a chain complex. Then its *homology* is

$$H_i(M_*) \coloneqq \ker \partial_i / \operatorname{im} \partial_{i+1}.$$

If $H_i(M) = 0$, then M_* is *exact at* i. M_* is *exact* if it is exact at every position.

Note that $H_i(M_*)$ is again an R-module.

The homology of a group G is the homology of a particular chain complex that arises from G.

Definition 1.7. Let G be a group and M a $\mathbb{Z}G$-module. A *resolution* of M is an exact sequence

$$\ldots M_2 \to M_1 \to M_0 \to M \to 0$$

of $\mathbb{Z}G$-modules M_i. The resolution is *projective* or *free* if all M_i are projective or free, respectively.

Remark 1.8. An R-module P is projective if there exists another R-module Q such that $P \oplus Q$ is a free R-module. In particular, free modules themselves are always projective.

Several other equivalent definitions of projective modules also exist. However, we will not concern ourselves too much with these. For us, free resolutions will usually suffice.

Definition 1.9. Let G be a group and M_* a projective resolution of $\mathbb{Z}$. Note that by taking the trivial G action on $\mathbb{Z}$, it becomes a $\mathbb{Z}G$-module. Let A be another $\mathbb{Z}G$-module.

The *homology* of G with coefficients in A is

$$H_i(G, A) \coloneqq H_i(M_* \otimes_{\mathbb{Z}G} A) = \ker(\partial_i \otimes_{\mathbb{Z}G} A) / \operatorname{im}(\partial_{i+1} \otimes_{\mathbb{Z}G} A).$$

The first important fact about this definition is that it does not depend on the choice of projective resolution. The second fact is that M_* can always be chosen to be a free resolution.

Example 1.10. For any group G with generating set S, consider the sequence

$$(\mathbb{Z}G)^S \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z} \to 0.$$

Here, $(\mathbb{Z}G)^S$ is the free $\mathbb{Z}G$-module with basis S and the basis elements are called e_s. The boundary map ∂_0 sends every $g \in G$ to $1 \in \mathbb{Z}$ and ∂_1 sends a basis element e_s to $1 - s \in \mathbb{Z}G$.

∂_0 is onto, and its kernel is the set of such elements of $\mathbb{Z}G$ whose coefficients add up to 0. Every such element is a $\mathbb{Z}$-linear combination of elements of the form $1 - g$. There exists an $x \in (\mathbb{Z}G)^S$ such that $\partial_1 x = 1 - g$. To construct this x, we write g as a word in S. Take, for example, $g = abc$. Then

$$1 - g = (1 - a) + (a - ab) + (ab - abc) = \partial_1 e_a + \partial_1(a\ e_b) + \partial_1(ab\ e_c).$$

In Definition 1.13, we will make precise how this construction works in general.

We have seen that ∂_0 is onto and $\ker \partial_0 \subseteq \operatorname{im} \partial_1$. As also $\partial_1 \circ \partial_0 = 0$, the sequence defined above is exact at $\mathbb{Z}G$ and at $\mathbb{Z}$.

Now consider the case where $G = F_2 = \langle a, b \rangle$, the free group with generating set $S = \{a, b\}$. Then an element $xe_a + ye_b \in (\mathbb{Z}G)^S$ is in $\ker \partial_1$ if $x(1 - a) + y(1 - b) = 0$. This cannot happen, so the sequence

$$0 \to (\mathbb{Z}G)^S \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

is exact.

To see why there are no $x, y \in \mathbb{Z}G$ such that $x(1-a)+y(1-b) = 0$, one may prove this directly by comparing the supports of the two summands and showing that they cannot be identical. However, the non-existence of such x, y is also precisely the statement that $\mathbb{Z}F_2$ does not satisfy the *Ore condition* which we will see in Chapter 3.

As another example, take $G = \mathbb{Z}^2$, again with generating set $S = \{a, b\}$. Now ∂_1 is no longer injective as for example $(b - 1)(1 - a) + (1 - a)(1 - b) = 0$. Indeed, this is essentially the only case where $x(1 - a) + y(1 - b) = 0$, so the sequence

$$0 \to \mathbb{Z}G \xrightarrow{\partial_2} (\mathbb{Z}G)^S \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

where $\partial_2\ 1 = (b - 1)e_a + (1 - a)e_b$ is exact.

1.3. Classifying spaces

In order to compute the homology of a group, one way to obtain a resolution of $\mathbb{Z}$ is by looking at CW-complexes that come with a sufficiently nice G-action on them. In this way, the CW-complex is a geometric realisation of the group, and the homology of the group is the homology of the space.

Let C be a CW-complex and C_i the set of its i-cells. Let R be a ring. We may associate to C a chain complex by taking M_i to be the free $\mathbb{Z}$-module with basis C_i. The boundary map sends an $i + 1$-cell to the oriented sum of i-cells on its boundary.

If G is a group that acts on C cellularly, that is in such a way that i-cells map to i-cells and cells that are fixed by the action are fixed pointwise, then we call C a *G-CW-complex*. In this case, the action of G also becomes an action on the chain complex associated to C. So the chain complex is actually a sequence of $\mathbb{Z}G$-modules. We call C a classifying space for G if the homology of the associated chain complex is zero in every degree.

If G is a group given by a presentation, we can construct the 2-skeleton of a classifying space in a particularly nice way.

Definition 1.11. Let G be a group with a presentation $G = \langle S \mid R \rangle$. The *presentation complex* associated to this presentation is the CW-complex that consists of one vertex, a loop for every element of S and a 2-cell for every element of R, glued in along a path tracing out the respective relation as a word in the generators. By construction, this space has fundamental group G, so G acts by deck transformations on the universal cover. There, we get a free G-orbit of cells for every cell of the presentation complex. Hence, the chain complex associated to the universal cover is

$$(\mathbb{Z}G)^R \to (\mathbb{Z}G)^S \to \mathbb{Z}G \to 1.$$

We also denote it

$$C_2 \to C_1 \to C_0 \to 1.$$

To define the boundary map, let $\{f_r \mid r \in R\}$ be the basis of C_2 and $\{e_s \mid s \in S\}$ the basis of C_1. Then

$$\partial e_s \coloneqq 1 - s$$

and

$$\partial f_r = \sum_{s \in S} \frac{\partial}{\partial_s}(r) e_s$$

where ∂/∂_s is the *Fox derivative* we define below.

Note that the 1-skeleton of the universal cover of the presentation complex is the *Cayley graph* $\mathrm{Cay}(G, S)$.

Example 1.12. In Figure 1.1, we see the universal cover of the presentation complex of $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$. It has one $\mathbb{Z}^2$-orbit of vertices, two orbits of edges, with the vertical edges corresponding to e_a and the horizontal edges corresponding to e_b, and one orbit of 2-cells.

Next to it is the universal cover of the presentation complex of the free group on two generators $F_2 = \langle a, b \rangle$. Again, there are two orbits of edges, one for each generator. But as F_2 has no relations, the presentation complex is one-dimensional.

Note how these two complexes correspond to the resolutions of the same groups we have seen in Example 1.10.

In order to explicitly compute the boundary map for a given presentation, one may use the *Fox derivative*. They were developed by Fox in a series of papers starting with [Fox53]. A modern reference is an exercise in [Bro82].

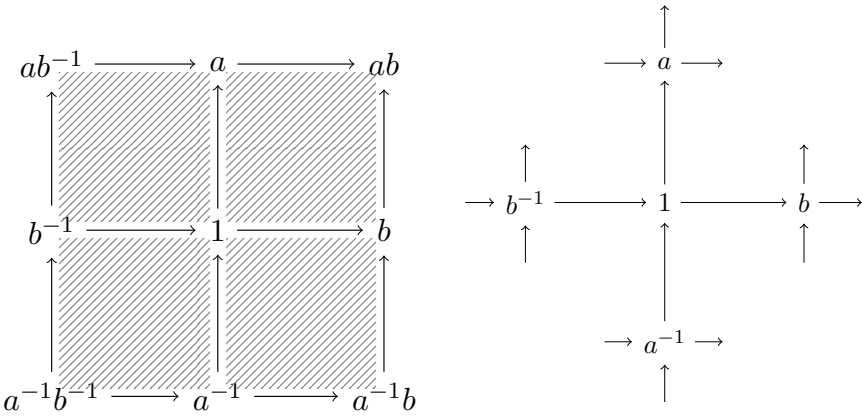


Figure 1.1.: The universal covers of the presentation complexes of $\mathbb{Z}^2$ and F_2.

Definition 1.13. Let F be a free group on the generating set S and $g \in F$ a word in the generators. Let $s \in S$ be a generator. Then the *Fox derivative* ∂/∂_s is a map $F \to \mathbb{Z}F$ defined inductively as follows:

- $\frac{\partial}{\partial_s}(1) = 0$
- $\frac{\partial}{\partial_s}(t) = \begin{cases} 1 & \text{if } t = s \\ -t & \text{if } t = s^{-1} \\ 0 & \text{otherwise} \end{cases}$
- $\frac{\partial}{\partial_s}(wt) = \frac{\partial}{\partial_s}(w) + w\frac{\partial}{\partial_s}(t)$

for any $t \in S \cup S^{-1}$ and $w \in F$.

Note that for any $g \in G$, if w_g is a word in the generators S describing g, the sum

$$\sum_{s \in S} \frac{\partial}{\partial_s}(w_g)e_s \in (\mathbb{Z}G)^S$$

is a path from 1 to g, following edges with labels according to the letters of w_g. In this sense, the Fox derivative of a relation is the algebraic equivalent of the boundary of the respective 2-cell.

Example 1.14. In Example 1.12, we have seen the presentation complex of $\mathbb{Z}^2$ with respect to the presentation $\langle a, b \mid a^{-1}b^{-1}ab \rangle$. In Figure 1.2, we see a smaller portion of the same complex.

If we take $g = 1$ presented by the word $w_g = bab^{-1}a^{-1}$, the sum of fox derivatives described above is

$$\sum_{s \in S} \frac{\partial}{\partial_s}(w_g)e_s = e_b + b\ e_a - a\ e_b - e_a = (b-1)e_a + (1-a)e_b.$$

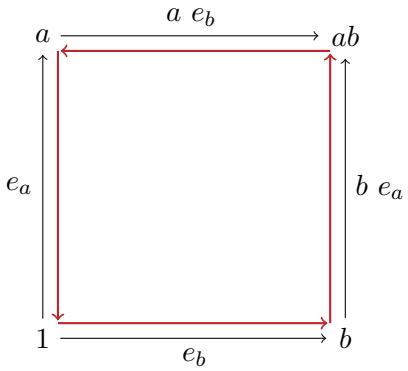


Figure 1.2.: The fox derivative of the relation $bab^{-1}a^{-1} = 1 \in \mathbb{Z}^2$ seen as a path in the Cayley graph.

Each of these summands corresponds to an edge in the presentation complex by interpreting negative summands as backwards edges. Concatenating those edges yields a loop around a 2-cell.

Also, recall Example 1.10. There, we assigned $\partial_1 1 = (b-1)e_a + (1-a)e_b$, which corresponds precisely to sending the single relation to its fox derivatives.

The presentation complex may be used to compute the first homology of G. An *Eilenberg-MacLane space* or $K(G,1)$ is the analogue for higher homology groups. It is a CW-complex C with fundamental group G such that $H_i(C,\mathbb{Z}) = 0$ for every $i > 1$. Such a CW-complex is also called *aspherical*. There always exists a $K(G,1)$ such that the 2-skeleton of its universal cover is the presentation complex. That is, a $K(G,1)$ may be constructed by filling up all non-contractible spheres in the presentation complex, starting in dimension 2 and going up. The universal cover of a $K(G,1)$ is what we called a classifying space above.

Example 1.15. For $G = \mathbb{Z}/2$, we may construct the universal cover of a $K(G,1)$ as follows: Start with two vertices, one for each element of G. Attach an edge connecting the two vertices. Because we want G to act freely on our space, we have to add a second edge. Thus we obtain a 1-sphere. To make it acyclic, we have to fill the sphere up with a 2-cell and again, to satisfy our group action, we have to add a second disc. Interpreting these as hemispheres, we see that we obtain a 2-sphere. The same will happen to any n-sphere we wish to make acyclic in this manner. Thus we obtain the infinite dimensional sphere S^∞: an infinite-dimensional space whose n-skeleton is an n-sphere. Dividing this by the group action, we obtain $\mathbb{R}P^\infty$ as our $K(G,1)$.

1.4. Finiteness properties

The Eilenberg-MacLane spaces we constructed have a 1-cell for every generator of the group. In particular, there exists a $K(G,1)$ with finitely many 1-cells if and only if G is

finitely generated. Similarly, G is finitely presented if and only if there exists a $K(G,1)$ with finite 2-skeleton.

Finitely generated groups will be of particular interest to us later on. However, much of the theory we will see also works in higher dimensions via the following higher finiteness properties.

Definition 1.16. A group G is of type $\mathcal{F}_n$ if there exists a $K(G,1)$ with finitely many cells up to dimension n.

G is of type $\mathcal{F}_\infty$ if it is of type $\mathcal{F}_n$ for every $n \in \mathbb{N}$. That is, if there exists a $K(G,1)$ with finitely many cells in every dimension.

G is of type $\mathcal{F}$ if there is a $K(G,1)$ with finitely many cells.

Every $K(G,1)$ defines a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module. We have seen how this works up to dimension 2 in Definition 1.11. An analogous construction also works in higher dimension: The i'th chain module C_i is the free $\mathbb{Z}G$ module with a basis element for every i-cell in the $K(G,1)$. The boundary map ∂ sends a basis element of C_i to its boundary seen as an element of C_{i-1}. The terminal module $\mathbb{Z}$ may be seen as representing the one connected component of the $K(G,1)$. The boundary map ∂_0 in degree zero sends every group element to the connected component of its associated vertex. That is, to the generator of $\mathbb{Z}$. The G-action on $\mathbb{Z}$ is trivial because all vertices are in the same connected component.

A natural question is if the same thing works in reverse. That is if every projective resolution gives rise to a $K(G,1)$. The answer is negative in general, and therefore, the following definition is meaningful.

Definition 1.17. A group G is of type $\mathcal{FP}_n$ if there exists a projective resolution C_i of $\mathbb{Z}$ such that C_i is finitely generated as a $\mathbb{Z}G$-module for every $1 \leqslant i \leqslant n$.

G is of type $\mathcal{FP}_\infty$ if it is of type $\mathcal{FP}_n$ for every $n \in \mathbb{N}$.

G is of type $\mathcal{FP}$ if there exists a projective resolution C_i of $\mathbb{Z}$ by finitely generated $\mathbb{Z}G$ modules such that all but finitely many C_i are trivial.

Example 1.18. We have seen in Example 1.10 that $\mathbb{Z}^2$ and F_2 are of type $\mathcal{FP}$. In Example 1.12, we saw that they are also of type $\mathcal{F}$.

Finite groups are groups of type $\mathcal{F}_\infty$ and $\mathcal{FP}_\infty$. We have seen this for $G = \mathbb{Z}/2$ in Example 1.15. The same example works similarly for any finite group and indeed, there is no finite dimensional $K(G,1)$ if G is finite so finite groups are not type $\mathcal{FP}$. A torsion-free example of an $\mathcal{FP}_\infty$-group that is not $\mathcal{FP}$ is *Thompson's group F* as was shown in [BG84].

In [BB97], we find for every n an example of a group that is of type $\mathcal{FP}_n$ but not $\mathcal{FP}_{n+1}$.

We have argued that a group of type $\mathcal{F}_n$ is also of type $\mathcal{FP}_n$. However, the converse is not true: There exists a group of type $\mathcal{FP}_n$ but not $\mathcal{F}_n$ for every $n > 1$ and we find an example of that in [BB97]. However, if G is of type $\mathcal{F}_2$, then $\mathcal{FP}_n$ implies $\mathcal{F}_n$. The same applies analogously for types $\mathcal{FP}_\infty$ and $\mathcal{FP}$.

2. Groups

We have established that our prime example of a ring is a group ring. Now, let us see which groups we want to study. We have already seen the free group F_2 and the free-abelian group $\mathbb{Z}^2$ and they will continue to be essential examples. But of course, they are not the only groups we consider. Two important properties that most of our groups have are that the groups are

1. finitely generated
2. and torsion-free.

Finite generation is important because if S and S' are two finite generating sets of a group G, the Cayley graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, S')$ are quasi-isometric. We will discuss this feature in Chapter 6.

If G has torsion, that is if there exists an element $1 \neq g \in G$ such that $g^n = 1$ for some $n \in \mathbb{N}^+$, then

$$(1-g)\sum_{i=0}^{n-1} g^i = \sum_{i=0}^{n-1} g^i - \sum_{i=1}^{n} g^i = 1 - g^n = 0 \in \mathbb{Z}G.$$

Hence, $\mathbb{Z}G$ has zero divisors. We will see how this is an issue in Section 3.1. Sometimes, we can get around this by considering groups which are *virtually* torsion-free. That is, groups that have a torsion-free subgroup of finite index.

2.1. Nilpotent groups

One class of groups that play a vital role in this work is the class of nilpotent groups. They have been studied a lot in different contexts. A comprehensive introduction may be found, for example, in [CMZ17]. But let us recall the facts most important to us.

Definition 2.1. Let G be a group. For $g, h \in G$, denote by $[g, h] \coloneqq g^{-1}h^{-1}gh$ the commutator of g and h. The *lower central series* of G is defined as follows: Set $G^{(0)} = G$ and

$$G^{(i+1)} \coloneqq [G^{(i)}, G] = \langle [g, h] \mid g \in G^{(i)}, h \in G \rangle.$$

G is *nilpotent* if all but finitely many $G^{(i)}$ are trivial. If $G^{(n)}$ is the first trivial group in the lower central series, then n is the *nilpotency class* of G.

Remark 2.2.

1. Setting $[g,h]$ to be $g^{-1}h^{-1}gh$ is just a convention, but it is the convention we use in this work. This way we get $hg[g,h] = gh$.

   The convention $[g,h] = ghg^{-1}h^{-1}$ would be just as valid and is sometimes found in the literature.

2. The subgroup $[G^{(i)}, G]$ is normal in G as for $f, h \in G$ and $g \in G^{(i)}$,

$$f^{-1}[g,h]f = [f^{-1}gf, f^{-1}hf] \in [G^{(i)}, G].$$

While the definition of nilpotent we gave considers smaller and smaller subgroups of G until we reach the trivial group, there is another description, which tries to fill G by starting with the trivial group and then considering a sequence of increasingly large subgroups:

Definition 2.3. Let G be a group. The *center* of G is

$$Z(G) \coloneqq \{g \in G \mid [g,h] = 1 \text{ for every } h \in G\}.$$

Again, this is a normal subgroup of G.

The *upper central series* of G is $Z^{(0)} \coloneqq \{1\}$ and

$$Z^{(i+1)} \coloneqq {\pi_i}^{-1} Z(G/Z^{(i)})$$

where π_i is the projection map $G \twoheadrightarrow G/Z^{(i)}$.

Lemma 2.4. *Let G be a group and $Z^{(i)}$ its upper central series.*

G is nilpotent if and only if all but finitely many $Z^{(i)}$ are equal to G. The nilpotency class of G is the minimal n such that $Z^{(n)} = G$.

Remark 2.5. The trivial group is the only nilpotent group of class 0. G is nilpotent of class $n+1$ if it is not nilpotent of class n but $G/Z(G)$ is. If $G/Z(G)$ is nilpotent of class n, then $[G,G]$ is of class at most n.

In particular, a non-trivial group is abelian if and only if it is nilpotent of class 1.

Let us have a look at some examples of nilpotent groups. These will be our prime examples throughout this work and also explain an intuition behind the terms *upper* and *lower central series*.

Example 2.6. Consider the group

$$H \coloneqq \Big\langle a, b \;\Big|\; 1 = [a,[a,b]] = [b,[a,b]] \Big\rangle.$$

It is called the *Heisenberg group*. Its lower central series is

$$H^{(1)} = [H,H] = \langle [a,b] \rangle \trianglelefteqslant H$$

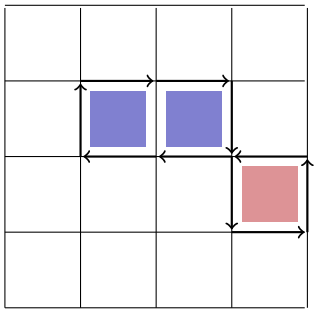


Figure 2.1.: A path in the Cayley graph of $\mathbb{Z}^2$. It goes around two squares in positive direction and one in negative direction.

and every other term is trivial. Its upper central series also is

$$Z^{(1)} = Z(H) = \langle [a,b] \rangle \trianglelefteq H$$

and again, every other term is trivial.

We will often use the Heisenberg group as an example because it is, in a sense, the simplest non-abelian nilpotent group: It has 2 generators and is nilpotent of class 2. Both are the lowest possible numbers for non-abelian groups. It is the only such group that is torsion-free.

The Heisenberg group may also be seen as the group of upper triangular 3×3 matrices with integer entries and 1 on the diagonal. A possible isomorphism is

$$a \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad [a,b] \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This embedding also shows that any element of H may be written uniquely as $a^\alpha b^\beta [a,b]^\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}$, as this element corresponds to the matrix

$$\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}.$$

The Cayley graph of H is not so easy to draw, but consider the following: As $H/[a,b] = \mathbb{Z}^2$, the Cayley graph of H also projects onto the Cayley graph of $\mathbb{Z}^2$. Have a look at Figure 2.1. A path in $\mathrm{Cay}(\mathbb{Z}^2)$ lifts to a unique path of $\mathrm{Cay}(H)$. If the path is a loop, then its endpoint is $[a,b]^n$, where n is the number of squares the path encloses. This is because the number of squares enclosed corresponds to the number of times we need to apply the relation $[a,b] = 1 \in \mathbb{Z}^2$ to show that the path is a loop in $\mathrm{Cay}(\mathbb{Z}^2)$. To be more precise, we count how many times the path goes around each square in clockwise direction minus the number in counter-clockwise direction. Thus, lifting the path in Figure 2.1 to $\mathrm{Cay}(H)$ results in a path from 1 to $[a,b]$. Here, it does not matter which point on the path is the origin.

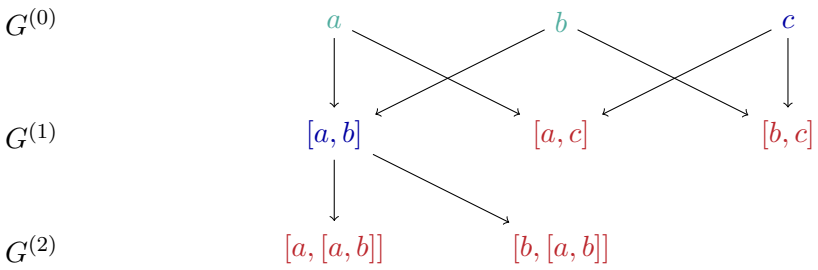


Figure 2.2.: A forest of commutators in the group $H \times \mathbb{Z}$ where H is the Heisenberg group. Each row contains the normal generators of a group in the lower central series. Colours indicate generators of groups in the upper central series.

For a path in $\mathrm{Cay}(H)$ from g to $g[a,b]^n$, one may think about n as the *height* of the path above its projection to $\mathrm{Cay}(\mathbb{Z}^2)$. More precisely, it is the relative height of the path's start and end points. Note that there is no clear notion of the height of a path that is not a loop in $\mathrm{Cay}(\mathbb{Z}^2)$. But two paths to the same point in $\mathbb{Z}^2$ have a relative height, which is the height of their concatenation, reversing one of the paths. This observation will be crucial when we study orders on the Heisenberg group in Example 8.1.

Many features of nilpotent groups can already be seen in the Heisenberg group, so it is often a good example. But sometimes, the Heisenberg group is not sufficient. For example, the non-trivial terms of the upper and lower central series are the same for the Heisenberg group. However, this does not have to be the case.

Example 2.7. Take $G \coloneqq H \times \mathbb{Z}$ where H is the Heisenberg group. That is

$$G = \Big\langle a, b, c \;\Big|\; 1 = [a,[a,b]] = [b,[a,b]] = [a,c] = [b,c] \Big\rangle.$$

It is also nilpotent of class 2. The center is

$$Z^{(1)} = Z(G) = \langle [a,b], c \rangle \trianglelefteq G$$

whereas the commutator subgroup is

$$G^{(1)} = [G,G] = \langle [a,b] \rangle \trianglelefteq G.$$

A depiction of how to find these generators can be seen in Figure 2.2.

If G is nilpotent and $N \trianglelefteq G$ is a normal subgroup, then the terms of the lower central series of G/N are $(G/N)^{(i)} = G^{(i)}/(N \cap G^{(i)})$. Hence N, is also nilpotent and its nilpotency class is no larger than n, the class of G. Similarly, any subgroup $H \leqslant G$ is nilpotent of class at most n. This observation allows for the following definition.

Definition 2.8. The *free-nilpotent group* of class n and rank k is the group

$$F_k / {F_k}^{(n)}$$

where F_k is the free group on k generators.

We have already seen the Heisenberg group, which is the free-nilpotent group of class 2 and rank 2.

A free-nilpotent group is "barely nilpotent". That is, it has only those relations that are necessary to make it nilpotent of class n. Hence, a group is nilpotent of class at most n, if and only if it is a quotient of some class n free-nilpotent group. This is analogous to all groups being quotients of free groups and abelian groups being quotients of free-abelian groups. In contrast to those, not every subgroup of a free-nilpotent group is itself free-nilpotent.

2.2. Residually finite, rationally solvable groups

Another important class of groups besides nilpotent groups are residually finite, rationally solvable or RFRS groups. Their significance for us lies in the following theorem. In fact, they were introduced by Agol in [Ago08], essentially as the class of groups for which a similar theorem holds in a topological setting.

Theorem 2.9 ([Kie20])**.** *Let G be a non-trivial finitely generated group that is virtually RFRS. Then G is virtually fibred if and only if $\beta_1^2(G) = 0$.*

To explain the terminology, a group is said to have some property *virtually* if and only if it admits a finite index subgroup with that property. G is said to be *fibred* if it admits a map onto $\mathbb{Z}$ with finitely generated kernel. Such a map is also called an *algebraic fibration* to distinguish it from the topological analogue. By $\beta_1^2(G)$, we denote the first ℓ^2-Betti number of G, on which we will elaborate in Chapter 4.

For now, it suffices that if we are interested in whether a group fibres, Theorem 2.9 provides an answer if our group is RFRS. One goal in this work is to generalise the theorem to a larger class of groups, which we call RFN groups, and we will define in Chapter 10. It is a class of groups that canonically extends the class of RFRS groups to also include all nilpotent groups. But first, let us have a look at RFRS groups.

Definition 2.10. Let G be a group. If there exist finite index normal subgroups $G_i \trianglelefteqslant G$ for $i \in \mathbb{N}$ such that

1. $G_0 = G$,
2. $G_{i+1} \trianglelefteqslant G_i$ and
3. $\cap_{i\in\mathbb{N}} G_i = \{1\}$,

then we call G *residually finite*.

More generally, a group G is said to have some property P *residually*, if there exist $G_i \trianglelefteqslant G$ not necessarily of finite index but otherwise as above, such that G/G_i has property P for every i. The collection of the G_i is called a *witnessing chain* for the fact that G is residually P.

Definition 2.11. Let G be a residually finite group. Then G is *residually finite, rationally solvable* or *RFRS* if there exists a witnessing chain (G_i) and free-abelian groups (A_i) such that the projection $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through A_i for every $i \in \mathbb{N}$.

Remark 2.12. If G_i is finitely generated, its abelianisation is the direct sum of a free abelian group G_i^{fab} and some finite group. In this case, we can always ask that A_i is the kernel of the projection $G_i \twoheadrightarrow G_i^{\mathrm{fab}}$.

If G_i is not finitely generated, we set G_i^{fab} to be the image of G in $(G/[G,G]) \otimes_{\mathbb{Z}} \mathbb{Q}$ and again we can ask that A_i is the kernel of the projection $G_i \twoheadrightarrow G_i^{\mathrm{fab}}$. It is important to note that in the finitely generated case, the two definitions of G_i^{fab} coincide.

Example 2.13. Every subgroup of a right-angled Artin group is virtually RFRS [Ago08, Chapter 2 and Corollary 2.3]. Notably, this includes free groups, torsion-free abelian groups and the examples of groups that are $\mathcal{FP}_n$ but not $\mathcal{FP}_{n+1}$ from [BB97] that we mentioned in Example 1.18.

On the other hand, non-abelian nilpotent groups are not RFRS. More precisely: A group that is both virtually nilpotent and virtually RFRS is also virtually abelian [Kob10, Theorem 1.6].

In Definition 10.1 we will define RFN groups, which is a class containing all RFRS groups. As we will then re-prove any property of RFRS groups important to us for this larger class of groups, let us defer any further discussion of RFRS until then.

2.3. Baumslag-Solitar groups

When exploring the boundary of one's theory, it is often helpful to find examples where the theory does not apply anymore. A class of groups that is notoriously good at being such a counterexample are the Baumslag-Solitar groups introduced in [BS62]. We will also use them several times in this capacity, so let us review their definition and most important properties.

Definition 2.14. Let $m, n \in \mathbb{N}$. Then the *Baumslag-Solitar group* is the two-generator one-relator group

$$\mathrm{BS}(m,n) \coloneqq \langle a, b \mid b^m a = a b^n \rangle.$$

In Figure 2.3, we see the Cayley graph of $\mathrm{BS}(1,2)$. The single relation $ba = ab^2$ may be seen as a pentagon in the Cayley graph. Arranging several of these pentagons, we obtain a "sheet" as in the left picture as a subset of the Cayley graph.

Note that in the sheet, at every level, we omit the b^{-1}-edge below every other point. Thus, to find such a sheet in the Cayley graph, we have to choose, which half of the edges we omit at every level. Hence, the sheets are arranged as the binary tree in the right picture. Every edge in the tree corresponds to a choice of which half of the edges we omit. This tree may also be thought of as the set of left (or right) b-cosets in $\mathrm{BS}(1,2)$.

Similar to nilpotent groups, Baumslag-Solitar groups can be seen as a group of upper triangular matrices via the embedding

$$a \mapsto A \coloneqq \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto B \coloneqq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

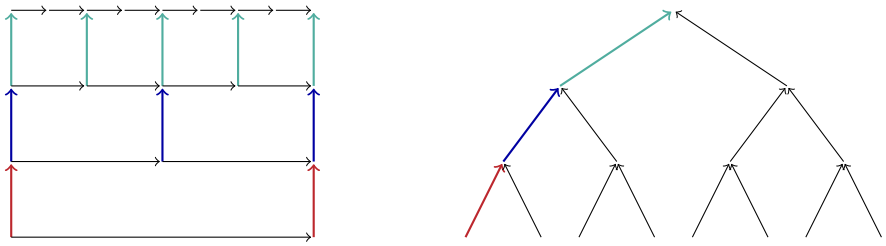


Figure 2.3.: The Cayley graph of $\mathrm{BS}(1,2) = \langle ab \mid ba = ab^2 \rangle$. In the left picture, vertical edges correspond to the generator a and horizontal edges to the generator b. We see only those elements that may be reached from the bottom left corner by using edges a and b in positive orientation. In the right picture, every edge corresponds to the generator a. Colours indicate matching edges under the projection mod b.

To verify this, we compute that $B^m A = AB^n$. So, the above map is a group homomorphism. If $m = 1$ and $n \neq 1$, it is also injective, as can be verified directly. However, this also follows from the following discussion.

We claim that every $g \in \mathrm{BS}(1,n)$ has a presentation as $a^i b^k a^{-j}$ with $i, j, k \in \mathbb{Z}$ such that both i and j are non-negative.

In matrix presentation, any element of $\mathrm{BS}(1,n)$ corresponds to a matrix

$$M(m, m', k_0) \coloneqq \begin{pmatrix} n^{-m} & k_0\, n^{-m'} \\ 0 & 1 \end{pmatrix}$$

for some $m, m', k_0 \in \mathbb{Z}$ as the set of these matrices is closed under multiplication with A and B and it contains $1 = M(0,0,0)$.

The element $a^i b^k a^{-j}$ written as a matrix is

$$a^i b^k a^{-j} \mapsto \begin{pmatrix} \frac{1}{n^i} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n^j & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n^{j-i} & k\, n^{-i} \\ 0 & 1 \end{pmatrix}.$$

This covers every matrix $M(m, m', k_0)$ for some non-negative i and j: Take

$$i = \max(m, m', 0), \quad j = i - m, \quad k = k_0\, n^{i-m'}.$$

As this choice of i, j, k is unique if we ask j to be minimal, the mapping is injective. In particular, every elemen of $\mathrm{BS}(1,n)$ has a unique presentation of the form $a^i b^k a^{-j}$ with minimal j.

3. Localisations

When studying modules over rings, we find that many things become much easier if our base ring is a field. Crucially, every module over a field is free. That is, it is a vector space. Therefore, the question arises if we can somehow pass from modules over a given ring R to vector spaces. For this, the first question is if R embeds into any field and, if so, if there is some universal choice of field into which R embeds.

The ring $\mathbb{Z}$ embeds into the field $\mathbb{Q}$, and any field into which $\mathbb{Z}$ embeds is a field extension of $\mathbb{Q}$. Hence $\mathbb{Q}$ is the canonical choice for a "fieldification" of $\mathbb{Z}$. Similarly, if R is any commutative domain, we may construct its *field of fractions* by taking formal fractions with numerator and denominator in R, and it plays a similar role as $\mathbb{Q}$ does for $\mathbb{Z}$.

For us, R will mostly be a group ring over some non-abelian group, hence not commutative. There is no chance of R embedding into a field if R is non-commutative, but the next best thing would be a *division ring*, which is the non-commutative version of a field. In this chapter, we will see how close we can get to a field of fractions for non-commutative rings. See also [DF03] and [Lam99]. A foundational paper discussing which rings embed into division rings is [Coh61]. A more recent and very thorough treatment of the topic can be found in [Lóp21].

3.1. The universal localisation

A localisation of a ring R is a canonical choice of another ring R_S such that all elements of some subset $S \subseteq R$ are invertible in R_S. Such a ring always exists, and it is called the *universal localisation* of R at S. However, R_S is not always as nice as one might hope. In this section, we review the construction and why it is universal. In Chapter 5, we will see in which cases the universal localisation is particularly well-behaved.

Definition 3.1. Let R be a ring. A *left unit* is an element $r \in R$ such that $rt = 1$ for some $t \in R$. This t is called the *right inverse* of r. A *right unit* is defined analogously. A *unit* is an element that is a unit from both sides such that the inverses are equal.

The set of units in R is denoted by $R^\times$.

Definition 3.2. A *division ring*, also known as a *skew field*, is a ring such that

$$R^\times = R \setminus 0.$$

Remark 3.3. The trivial ring is not a division ring because its one element is a unit, but $R \setminus 0$ is empty. But unfortunately, we have to accept that the trivial ring is a ring for now.

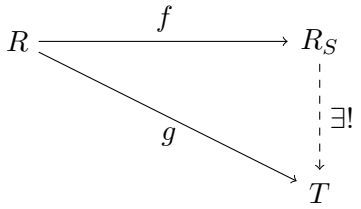


Figure 3.1.: The universal property of the universal localisation of R at the subset S: Every S-inverting map $R \to T$ factors uniquely through R_S.

Like fields, division rings have the property that every module over them is free. This means that algebra over division rings behaves much like linear algebra. Many concepts from linear algebra, such as basis, dimension, and matrix presentations of homomorphisms, apply analogously when studying modules over division rings. Therefore, an important question is if R embeds into a division ring and, if so, if there is a canonical choice of surrounding division ring.

A naïve attempt to simply add inverses to every element of R does not always yield the expected result. But this idea can be made precise in the following way:

Definition 3.4. Let R be a ring and $S \subseteq R$ any subset of R.

1. Let T be another ring and $f\colon R \to T$ a ring homomorphism. Then f is called *S-inverting*, if $f(S) \subseteq T^\times$.

2. The *universal localisation* of R at S is a ring R_S together with an S-inverting map $f\colon R \to R_S$ such that for any ring T, every S-inverting map $g\colon R \to T$ factors uniquely through R_S. That is the diagram in Figure 3.1 commutes. In this case, f is called the *localisation map*.

Example 3.5. The universal localisation of $\mathbb{Z}$ at $\mathbb{Z} \setminus 0$ is $\mathbb{Q}$, with the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ as localisation map. If $g\colon \mathbb{Z} \to T$ is any $(\mathbb{Z} \setminus 0)$-inverting map, then it extends uniquely to a map $h\colon \mathbb{Q} \to T$. This map is defined via

$$h(\frac{p}{q}) = g(p)\ g(q)^{-1}.$$

An important fact is that the universal localisation always exists. However, the localisation map does not need to be injective. This is the case, for example, if S contains any zero-divisors.

Definition 3.6. Let R be a ring. An element $r \in R$ is called a *left zero-divisor* if there exists a $t \in R \setminus 0$ such that $rt = 0$ and analogously for *right zero-divisors*.

A *domain* is a ring without left or right zero-divisors.

If r is a left unit, it cannot also be a right zero-divisor. To see this suppose that $rs = 1$ and $tr = 0$, then

$$0 = 0s = (tr)s = t(rs) = t.$$

Similarly, r cannot be both a right unit and a left zero-divisor.

Hence if $S \subseteq R$ contains a zero-divisor s, let us say $st = 0$ for some $t \neq 0$, then the localisation map $f\colon R \to R_S$ cannot be injective since

$$0 = f(s)^{-1}0 = f(s)^{-1}f(st) = f(t).$$

If S even contains 0, then R_S is the trivial ring because it is the only ring where 0 is a unit.

Example 3.7. If R is a commutative domain, then the universal localisation of R at $R \setminus 0$ is the *field of fractions*. It may be constructed by taking all fractions over entries in R with a non-zero denominator. Addition and multiplication are defined just as for fractions in $\mathbb{Q}$, and expansions and cancellations also work the same way. So this is a generalisation of Example 3.5.

If R is a non-commutative domain, the *field of fractions* does not have to exist. The *Ore condition* is a condition that is weaker than commutativity and guarantees the existence of a *skew field of fractions*. We give this topic extensive treatment in Part II. For now, consider the set

$$R[S]^{-1} \coloneqq \{(r, s) \mid r \in R, s \in S\}/\sim,$$

where we interpret a pair (r, s) as the *right fraction* rs^{-1} and the equivalence $\sim$ allows expansions and cancellations of fractions. In general, it is not clear that this set has any meaningful additive or multiplicative structure. The Ore condition is precisely such that $R[S]^{-1}$ can be turned into a ring. Again, we will see the exact definition and proof in Part II.

We also call $R[S]^{-1}$ the *Ore localisation* of R at S. If it exists, the Ore localisation is isomorphic to the universal localisation R_S. The map

$$R \hookrightarrow R[S]^{-1}, \quad r \mapsto r1^{-1}$$

is an S-inverting embedding of R into the Ore localisation. If $S = R \setminus 0$, then $R[S]^{-1}$ is a division ring.

3.2. Division closures

As we have seen, the universal localisation may be pretty wild, but at least it always exists. On the other hand, the Ore localisation has a nice description but does not always exist. Whenever R embeds into a larger ring U, there is another approach to adding inverses to R. Namely, if we try only localising elements of R that are invertible in U, we may add their inverses to R. Hopefully, the result will again be a ring. If not, we can turn it into a ring by taking the closure under addition and multiplication. Let us make this idea precise.

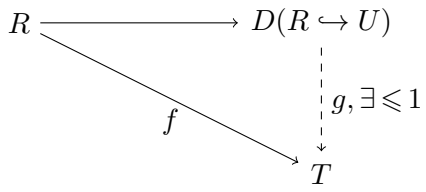


Figure 3.2.: A map $R \to D(R \hookrightarrow U)$ is an epimorphism if for every map $R \to T$ to some ring T there is at most one way of extending this map to $D(R \hookrightarrow U) \to T$.

Definition 3.8. Let U be a ring and $R \leqslant U$ a subring. We call a ring $T \leqslant U$ *division closed* in U if for every $t \in T$ that is a unit in U, t is already a unit in T. That is if

$$T \cap U^\times \subseteq T^\times.$$

The *division closure* of R inside U is the intersection of all division closed rings T with $R \leqslant T \leqslant U$ and denoted by $D(R \hookrightarrow U)$.

Note that the division closure always exists because at least U itself is division closed in U, and the intersection of even infinitely many division closed subrings $T \leqslant U$ is again a division closed subring.

As alluded to above, the division closure may be constructed explicitly by taking all elements of U that are any combination of sums, products and inverses of elements of R.

To be explicit, we add to R all inverses of elements in $S \coloneqq R \cap U^\times$. Let R' be the subring of U generated by $R \cup S^{-1}$. Applying the same construction to R' yields another ring $R'' \leqslant U$. Repeating this process countably infinitely many times or until $R^{(i)} \cap U^\times \subseteq R^{(i)^\times}$ results in the division closure $D(R \hookrightarrow U)$. The details of this construction may be found in [JL20].

Example 3.9. The division closure of $\mathbb{Z}$ inside $\mathbb{R}$ is $D(\mathbb{Z} \hookrightarrow \mathbb{R}) = \mathbb{Q}$. In fact, the division closure of $\mathbb{Z}$ in any field of characteristic 0 is isomorphic to $\mathbb{Q}$.

More generally, if R satisfies the Ore condition with respect to S, and U is a ring containing R such that the embedding $R \hookrightarrow U$ is S-inverting, then $D(R \hookrightarrow U) = R[S]^{-1}$. In particular, this is the case if $S = R \setminus 0$ and U is a division ring. The reason for this is that by the universal property of the localisation, $R[S]^{-1}$ is contained in U. The embedding $R \hookrightarrow R[S]^{-1}$ is S-inverting and adding the inverses to R and closing under multiplication gives the set of fractions rs^{-1} which is precisely $R[S]^{-1}$. The localisation $R[S]^{-1}$ is a ring, and every element of S is invertible in $R[S]^{-1}$, so $R[S]^{-1} = D(R \hookrightarrow U)$.

Let us give another characterisation of the division closure.

Lemma 3.10. *The embedding*

$$R \hookrightarrow D(R \hookrightarrow U)$$

is an epimorphism of rings.

Recall that a map $R \hookrightarrow D(R \hookrightarrow U)$ is an epimorphism if and only if every map $f\colon R \to T$ to some ring T that factors through $D(R \hookrightarrow U)$ does so uniquely. See also Figure 3.2.

Proof. To see that $R \hookrightarrow D(R \hookrightarrow U)$ is epic, note that to extend the map f to a map $g\colon D(R \hookrightarrow U) \to T$ such that $g_{|R} = f$, there is only one choice of where to send the inverse of an element $s \in R \cap U^\times$, namely

$$g(s^{-1}) = g(s)^{-1} = f(s)^{-1}.$$

Similarly, once we know where g sends two elements x and y, there is only one choice of where to send xy and $x + y$. Following our construction of $D(R \hookrightarrow U)$ above, this defines g uniquely on all of $D(R \hookrightarrow U)$, and so $R \hookrightarrow D(R \hookrightarrow U)$ is an epimorphism. □

Even more, a map $f\colon R \to T$ between rings is an epimorphism precisely if

$$D(f(R) \hookrightarrow T) = T.$$

The proof is analogous to the proof from Galois theory of the same fact if T is a field.

3.3. The Mal'cev-Neumann division ring

As we have seen, the division closure of a ring has a nice explicit description of its elements by taking sums, products and inverses of existing elements. If the surrounding ring is a division ring, so is any division closure inside of it. This is because every element of $D(R \hookrightarrow U)$ that is invertible in U is already invertible in the division closure. So, to find a nice division ring into which R embeds, the first question is if it embeds into any division ring whatsoever.

Let us consider the case where R is a group ring $R = kG$ for some field k. The first candidate for a division ring into which kG embeds is its Ore localisation. It is a fact due to Tamari [Tam54] and Kielak [BK16] that the Ore localisation of kG exists if and only if G is amenable and kG admits no zero-divisors.

Even if G is not amenable, we can construct some other division ring containing kG if G is bi-orderable. We treat orders on groups extensively in Chapter 7. For now, it suffices to know that a group is bi-orderable if there exists a total order relation $\prec$ on G such that

$$g \prec g' \iff hg \prec hg' \iff gh \prec g'h$$

for every g, g' and $h \in G$.

Definition 3.11. Let $(G, \prec)$ be a totally ordered group and R a ring. The Mal'cev-Neumann ring is

$$MN_\prec(RG) \coloneqq \{f \in R^G \mid \operatorname{supp} f \text{ is well ordered}\},$$

with addition and multiplication extended from the ones in RG. By *well ordered*, we mean that every subset of $\operatorname{supp} f$ has a unique minimum.

The group ring RG is contained in $MN_\prec(RG)$ because every finite totally ordered set has a minimum.

It may be proven directly that for a field k, the Mal'cev-Neumann ring $MN_\prec(kG)$ is a division ring but in Section 9.1 we will see a particularly nice description of $MN_\prec(kG)$ so we defer the proof until then.

Note that just as in the group ring, we can write maps in R^G and hence also elements of $MN_\prec(RG)$ or of RG as formal sums

$$f\colon G \to R \leftrightarrow \sum_{g\in G} f(g)g.$$

This is just a notation, but it fits nicely with the addition and multiplication one would expect from the notation as a sum. We will discuss this in detail in Section 9.1.

Example 3.12.

1. Let $G = \mathbb{Z}^2 = \langle a, b\rangle_{\text{ab}}$ be the free abelian group on two generators a and b. Take $\prec$ to be the *lexicographic order* on G. That is,

   $$a^i b^j \prec a^k b^l \iff i \leqslant k \text{ and } i = k \Rightarrow j < l.$$

   Then $MN_\prec(\mathbb{Q}G)$ is the set of formal sums $p \in \mathbb{Q}^G$ such that:

   a) There exists a minimal i such that $a^i b^j \in \operatorname{supp} p$ for some j.

   b) For each i such that $a^i b^j \in \operatorname{supp} p$ for some j, the set of these j has a minimum..

   We will see that this is a bi-invariant total order on G in Definition 7.14.

   The support of any element of $MN_\prec(\mathbb{Q}G)$ is well ordered and hence has a unique minimum in the sense of the order. Every monomial in $MN_\prec(\mathbb{Q}G)$ is invertible even in $\mathbb{Q}G$. By multiplying with the inverse of the minimum-degree monomial, we see that every element of $MN_\prec(\mathbb{Q}G)$ is up to multiplication by a unit equal to $1+x$ for some $x \in MN_\prec(\mathbb{Q}G)$ such that $g \in \operatorname{supp} x \Rightarrow 1 \prec g$. A simple computation shows that

   $$(1+x)\ \sum_{i=0}^{\infty}(-x)^i = 1 \in MN_\prec(\mathbb{Q}G)$$

   and that the infinite sum is an element of $MN_\prec(\mathbb{Q}G)$. We will provide the details in Lemma 9.20.

2. Now let $G = F_2 = \langle a, b\rangle$ be the free group in the two generators a and b. Let us pick some order on the generators, say

   $$a^{-1} \prec b^{-1} \prec 1 \prec b \prec a.$$

   The *lexicographic order* in this case is the order that compares two reduced words $w_1, w_2 \in F_2$ by comparing the first letter where they differ. Here by reduced we

mean that w_1, w_2 do not contain any sub words of the form $c^{-1}c$. Unfortunately, this total order on F_2 is not bi-invariant as for example

$$1 \succ b^{-1}a, \text{ whereas } b \prec a = b\, b^{-1}a,$$

so $\prec$ is not invariant under left multiplication by b.

However, a bi-invariant total order on F_2 may be constructed as follows. This construction was introduced in [Mag35] and is hence called the Magnus embedding. A discussion in English may be found in [CMZ17].

Consider the ring of power series in two non-commuting variables

$$\mathbb{Z}\langle\langle a, b\rangle\rangle \coloneqq \mathbb{Z}^{[a,b]}.$$

The polynomials $1 + a$ and $1 + b$ are invertible in $\mathbb{Z}\langle\langle a, b\rangle\rangle$ but have no common right multiple. Hence, the map

$$F_2 \to \mathbb{Z}\langle\langle a, b\rangle\rangle, \quad a \mapsto 1 + a, \quad b \mapsto 1 + b$$

defines an embedding of F_2 into the group of units in the power series. On $\mathbb{Z}\langle\langle a, b\rangle\rangle \setminus 0$, we define a bi-invariant total order by comparing monomials by their degree. If two monomials have the same degree, we order those lexicographically. Then, we compare two sums by comparing the coefficient of the first monomial where these are not equal.

Note that as opposed to $\mathbb{Z}F_2$, in $\mathbb{Z}\langle\langle a, b\rangle\rangle$, we are only dealing with a free monoid, not a free group, so there is no direct contradiction to the lexicographical order not being a bi-invariant total order on F_2.

Restricting the order on $\mathbb{Z}\langle\langle a, b\rangle\rangle \setminus 0$ to the subgroup F_2 via the above embedding yields a bi-invariant total order on F_2. So the group ring $\mathbb{Z}F_2$ embeds into the division ring $MN_{\prec}(\mathbb{Q}F_2)$.

3.4. Rings that do not satisfy the Ore condition

If R satisfies the Ore condition, then the Ore localisation is not only universal in the sense of the universal property but also in that it appears as the division closure of R in any sufficiently large ring. In particular, if R satisfies the Ore condition with respect to $S = R \setminus 0$, then every ring containing R where S is invertible is a free module over the division ring $R[S]^{-1}$.

Let us investigate in which ways R can fail to satisfy the Ore condition - for the remainder of this section, we always mean with respect to $S = R \setminus 0$.

If R admits any non-zero zero-divisor r, it cannot embed into any division ring D. Suppose that $\iota\colon R \hookrightarrow D$ was such an embedding. Then $\iota(r)$ is a zero-divisor in D. But the only zero-divisor in a division ring is 0, so $\iota(r) = 0$. As ι is supposed to be an embedding, this contradicts $r \neq 0$.

The following is an example of a ring without zero-divisors that does not embed into any division ring.

Example 3.13 (Mal'cev, [Mal37])**.** Let M be the monoid

$$M \coloneqq [a, b, c, d, u, v, x, y \mid ax = by, cx = dy, au = bv].$$

M does not embed into any group because in a group, we have

$$cu = cx\ x^{-1}a^{-1}\ au = dy\ y^{-1}b^{-1}\ bv = dv$$

but this relation does not hold in M. Any embedding of $\mathbb{Z}M$ into a division ring D would restrict to an embedding of M into the group of units in D. Hence, such an embedding cannot exist.

To see that $\mathbb{Z}M$ does not admit any zero-divisors, first note that monomials are not zero-divisors because the product of two monomials is again a monomial and the single coefficient is non-zero because $\mathbb{Z}$ is zero-divisor-free. Hence, if

$$(\sum_{m\in M} a_m m)(\sum_{n\in M} b_n n) = 0,$$

if $a_m, b_n \neq 0$ for some m, n, then there must be $m' \neq m$ and n' such that $a_{m'}, b_{n'} \neq 0$ and $mn = m'n'$. Note that every element of M has a unique word length. Restricting the above sums to words of minimal length in their supports gives another zero-divisor pair, so we may assume that all m have the same length and all n have the same length.

Now if $mn = m'n'$, then there must be relations in M transforming mn to $m'n'$. Note that any relation changes letters only within the sets

$$\{a, b\}, \{c, d\}, \{x, y\}, \{u, v\}$$

and a, b, c, d appear only as the first letter and x, y, u, v appear only as the second letter. As $m \neq m'$, there must, in fact, be a single relation transforming mn to $m'n'$ and this relation must involve the last letter of m and the first letter of n.

For brevity, let us assume that m, m', n, n' all have length 1. Then $mn = m'n'$ is precisely one of the three relations defining M. Whichever relation it is, mn' is a word not involved in any of the relations. Hence for mn', there are no m'', n'' such that $mn' = m''n''$ and therefore

$$mn' \in \operatorname{supp}\big((\sum_{m\in M} a_m m)(\sum_{n\in M} b_n n)\big),$$

contradicting the assumption that the product is 0.

This concludes the example of a domain that does not embed into any division ring. The other problem that may arise is that there are two non-isomorphic division rings into which R embeds epically. Both would have to be the Ore localisation, which is impossible because the universal localisation is unique. The following is an adapted version of an example that may be found in [Lam99].

Example 3.14. Let $F = [x, y]$ be the free monoid on two generators. Then the monoid ring $\mathbb{Q}F$ embeds epically into two non-isomorphic division rings.

To construct those division rings, let $\mathrm{BS}(1, n)$ be the Baumslag-Solitar group. That is

$$\mathrm{BS}(1, n) = \langle a, b \mid ba = ab^n \rangle.$$

Recall from Section 2.3 that every element of $\mathrm{BS}(1, n)$ has a unique normal form. We order the normal forms by setting $a^i b^k a^{-j} \prec a^{i'} b^{k'} a^{-j'}$ if and only if one of the following conditions is met.

1. $i - j < i' - j'$,
2. $i - j = i' - j'$ and $i < i'$,
3. $i = i', j = j'$ and $k < k'$.

Otherwise we set $a^i b^k a^{-j} \succcurlyeq a^{i'} b^{k'} a^{-j'}$. This defines a total order on $\mathrm{BS}(1, n)$, and a straightforward computation shows that it is bi-invariant. Therefore, $\mathbb{Q}\,\mathrm{BS}(1, n)$ embeds into the respective Mal'cev-Neumann division ring, which we denote by M_n.

If $n > 1$,

$$\iota \colon F \to \mathrm{BS}(1, n), \quad x \mapsto a, \quad y \mapsto ab$$

is a map of the free monoid into the Baumslag-Solitar group. To see that it is also an embedding, note that a word w in the letters {a,b} of length l maps to $a^l b^{f(w)}$ where $f(w)$ is the following recursive map:

$$\begin{aligned} a &\mapsto 0 \\ b &\mapsto 1 \\ ws &\mapsto n \cdot f(w) + f(s) \end{aligned}$$

Checking that this is true is, again, a straightforward computation.

Put differently, $f(w)$ is the natural number we get by seeing w as an l-digit n-ary number interpreting a as the digit 0 and b as the digit 1. Since every number has at most one n-ary presentation of length l, this shows that ι is injective. The embedding ι extends to an embedding $\iota_n \colon \mathbb{Q}F \hookrightarrow \mathbb{Q}\,\mathrm{BS}(1, n)$ of monoid rings and as the group ring $\mathbb{Q}\,\mathrm{BS}(1, n)$ is contained in the Mal'cev-Neumann division ring, we even get an embedding $\mathbb{Q}F \hookrightarrow M_n$. As M_n is a division ring, so is the division closure of $\mathbb{Q}F$ in M_n. We claim that for $n \neq m$, the division closures $D(\mathbb{Q}F \hookrightarrow M_n)$ and $D(\mathbb{Q}F \hookrightarrow M_m)$ are not isomorphic in a way that respects the embedding of $\mathbb{Q}F$. So let

$$\varphi \colon D(\mathbb{Q}F \hookrightarrow M_n) \to D(\mathbb{Q}F \hookrightarrow M_m)$$

be any ring homomorphism such that $\varphi \circ \iota_n = \iota_m$. That is to say, the diagram in Figure 3.3 commutes. Then

$$\big(\iota_m(y)\iota_m(x)^{-1}\big)^m = (ab\ a^{-1})^m = ab^m a^{-1} = b = \iota_m(x)^{-1}\iota_m(y) \in D(\mathbb{Q}F \hookrightarrow M_m).$$

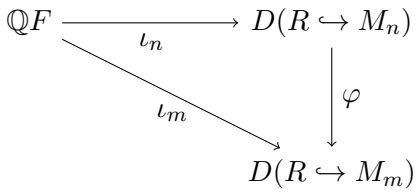


Figure 3.3.: No map between the two division closures can be an isomorphism respecting the embedding of $\mathbb{Q}F$.

But this means

$$\varphi(\iota_n(y)\iota_n(x)^{-1})^m = \varphi(\iota_n(x)^{-1}\iota_n(y)) = \varphi(\iota_n(y)^{-1}\iota_n(x))^n$$

where the second equality is similar to the computation above but in $D(\mathbb{Q}F \hookrightarrow M_n)$. Hence

$$ab^{m-n}a^{-1} - 1 = \varphi(\iota_n(y)\iota_n(x)^{-1})^{m-n} - 1 = 0.$$

As $ab^{m-n}a^{-1} - 1 \neq 0 \in D(\mathbb{Q}F \hookrightarrow M_n)$ this means that φ is not injective and certainly not an isomorphism.

4. ℓ^2-invariants

In Theorem 2.9, we have encountered ℓ^2-Betti numbers as a criterion if a group G fibres. We give a short overview of their definition and most important properties. Most essential for us will be the definition of the *Linnell ring* $\mathcal{D}(G)$. After we have defined that, we will no longer need most of the details of ℓ^2-Betti numbers, so we focus only on the essentials here. A good introduction to the topic may be found in [Kam19], and [Lüc02] is a comprehensive reference.

4.1. ℓ^2-Betti numbers

Recall from Chapter 1 that R^G is the set of maps from G to R and RG is the ring of finitely supported maps in R^G. Consider the case $R = \mathbb{Q}$ or $R = \mathbb{C}$. The case $R = \mathbb{C}$ is the "classical" setting, whereas we will later on be more interested in $R = \mathbb{Q}$. To make sure we remember that we are in a special case, let us take $R = \mathbb{C}$. Nevertheles, everything works the same way for $\mathbb{Q}$.

Unlike a generic ring, $\mathbb{C}$ is not discrete but has a non-trivial metric. Hence, we can make the following definition.

Definition 4.1. Let G be a group. Then $\ell^2(G)$ is defined as

$$\ell^2(G) \coloneqq \{f \in \mathbb{C}^G \mid \sum_{g \in G} |f(g)|^2 < \infty\}.$$

$\ell^2(G)$ is the completion of $\mathbb{C}G$ with respect to the L^2-norm on $\mathbb{C}^G$. Note that the multiplication on $\mathbb{C}G$ extends naturally to a multiplication of $\mathbb{C}G$ on $\mathbb{C}^G$. With respect to this multiplication, the subset $\ell^2(G) \subseteq \mathbb{C}^G$ is a $\mathbb{C}G$-module. We will elaborate on the details in Section 9.1. Generally, $\ell^2(G)$ is not a ring.

Definition 4.2. A map or *operator* $f\colon \ell^2(G) \to \ell^2(G)$ is called *bounded*, if

$$\sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|} < \infty.$$

Here, $\|x\|$ is the L^2-norm on $\mathbb{C}$.

Recall that $\ell^2(G)$ is a $\mathbb{C}G$-module, and therefore every element of $\mathbb{C}G$ can be seen as an operator on ℓ^2 by identifying $x \in \mathbb{C}G$ with the left or right multiplication by x on $\ell^2 G$. Thus the set of bounded operators on $\ell^2(G)$ also forms a $\mathbb{C}G$-module

Definition 4.3. An operator $f\colon \ell^2(G) \to \ell^2(G)$ is *G-equivariant* if $(gf)(x) = f(gx)$ for every $x \in \ell^2(G)$ and $g \in G$.

We denote the *G-equivariant bounded operators* on $\ell^2(G)$ by $\mathcal{L}(G)$. This is also called the *von Neumann algebra*.

$\mathcal{L}(G)$ forms a ring with pointwise addition and composition of functions. $\mathcal{L}(G)$ is also a $\mathbb{C}G$-module and in particular a $\mathbb{Z}G$-module, so homology of G with coefficients in $\mathcal{L}(G)$ is well-defined and $H_*(G, \mathcal{L}(G))$ is an $\mathcal{L}(G)$-module in every dimension.

Definition 4.4. Let G be a group, $f \in \mathcal{L}(G)$ and M an $\mathcal{L}(G)$-module. Then the *von Neumann trace* of f is

$$\operatorname{tr}_{\mathrm{vN}}(f) \coloneqq \langle 1, f(1) \rangle.$$

The *von Neumann dimension* of M is

$$\dim_{\mathrm{vN}} M \coloneqq \operatorname{tr}_{\mathrm{vN}}(\mathrm{id}_M)$$

if M is a projective module. This extends uniquely to a dimension function on all $\mathcal{L}(G)$-modules.

The *n'th ℓ^2-Betti number* of G is

$$\beta_n^2(G) \coloneqq \dim_{\mathrm{vN}} H_n(G, \mathcal{L}(G)).$$

This is analogous to ordinary Betti numbers being the dimension of the n'th homology group with $\mathbb{Z}$-coefficients. It is a fact that $\operatorname{tr}_{\mathrm{vN}}()$ is actually a trace and $\dim_{\mathrm{vN}}$ is a dimension function.

4.2. The Atiyah conjecture

In contrast to ordinary Betti numbers, ℓ^2-Betti numbers may take non-integer values. A question due to Atiyah is which numbers arise as ℓ^2-Betti numbers of a given group G. Any finite group G has $\beta_0^2(G) = 1/|G|$. However, it was open for a long time if all ℓ^2-Betti numbers are rational. While the answer is now known to be negative, as of this writing, all known examples of irrational ℓ^2-Betti numbers come from torsion groups. The following question is still open.

Conjecture 4.5 (Atiyah conjecture)**.** *Let G be a torsion-free group and $n \in \mathbb{N}$. Then $\beta_n^2(G) \in \mathbb{Z}$.*

While the Atiyah conjecture originates from the analytic setting outlined above, there is a purely algebraic formulation of the same statement. It may be stated as asking if a certain ring $\mathcal{D}(G)$ is a division ring.

Definition 4.6. Let $\mathcal{U}(G)$ be the set of potentially unbounded G-equivariant operators on $\ell^2(G)$. Then the *Linnell ring* is

$$\mathcal{D}(G) \coloneqq D(\mathbb{C}G \hookrightarrow \mathcal{U}(G)).$$

An alternative formulation of the Atiyah conjecture is the following one.

Conjecture 4.7 (Atiyah conjecture)**.** *Let G be a torsion-free group. Then $\mathcal{D}(G)$ is a division ring.*

It is a theorem ([Lin93]) that the two formulations of the Atiyah conjecture are equivalent. If G is a group that satisfies the Atiyah conjecture, then the von Neumann dimension and normal $\mathcal{D}(G)$-vector space dimension align, so we may express the ℓ^2-Betti numbers as

$$\beta_n^2(G) = \dim_{\mathcal{D}(G)} H_n(G, \mathcal{D}(G))$$

using the normal vector space dimension over $\mathcal{D}(G)$. Recall that modules over division rings are always free and behave mostly like vector spaces over commutative fields. Hence we also call these modules vector spaces.

An important fact here is that $\mathcal{U}(G)$ is also the Ore localisation of $\mathcal{L}(G)$ at the set of non-zero divisor elements of $\mathcal{L}(G)$. Hence, if $\mathbb{C}G$ has no zero divisors and satisfies the Ore condition, then $\mathcal{D}(G) = \mathbb{C}G[\mathbb{C}G \setminus 0]^{-1}$ is a division ring, meaning that G also satisfies the Atiyah conjecture. On the other hand, there exist groups that satisfy the Atiyah conjecture that do not satisfy the Ore condition. The groups we encounter in this work will usually satisfy the Atiyah conjecture, but we will point this out again when it becomes relevant.

Another long-standing conjecture that has to be mentioned in this context is the zero divisor conjecture.

Conjecture 4.8 (Kaplansky's zero divisor conjecure)**.** *Let G be a torsion-free group. Then $\mathbb{Z}G$ does not admit any zero divisors.*

Note that if G satisfies the Atiyah conjecture, then $\mathbb{Z}G \hookrightarrow \mathbb{C}G \hookrightarrow \mathcal{D}(G)$ is an embedding of $\mathbb{Z}G$ into a division ring, so G also satisfies the zero divisor conjecture.

In Section 3.2, we have seen that if a ring R embeds into a division ring U, then the division closure $D(R \hookrightarrow U)$ is also a division ring. It stands to reason that it would be enough in this situation that instead of asking U to be a division ring, that every element of U is either a unit or a division ring, and the embedding $R \hookrightarrow U$ is $(R \setminus 0)$-inverting. But note that $\mathcal{U}(G)$ is such a ring U. So this would mean that $\mathcal{D}(G) = D(\mathbb{Z}G \hookrightarrow \mathcal{U}(G))$ is a division ring, and hence the zero divisor conjecture would imply the Atiyah conjecture. It is not known if this implication is true.

Part II.

The Ore localisation

5. A formalisation in Lean

We have seen in Chapter 3 that for any ring R and subset $S \subseteq R$, there exists a universal localisation R_S that comes equipped with a map $R \to R_S$ that is S-inverting. We have also seen that for a commutative domain, the role of universal localisation is taken by the field of fractions, which has a much nicer description than the generic universal localisation and also has many nice properties, foremost among them that it is a field.

In this chapter, we explore the middle ground. Vaguely, we want to answer the question: For which non-commutative rings is the universal localisation close enough to the field of fractions? Ore has answered this question in [Ore31]. We provide a computer-verified proof that his construction is verified, using the Lean theorem prover. While the Lean code stands for itself, this part is thought to be a guide through the code. Besides showing how the proofs work, it also translates between Lean and natural language, meaning that the statements Lean accepts to be true correspond with the ones we formulate here.

In this part, instead of proofs, we provide the respective statements in Lean, for which we have also written proofs in Lean. Sometimes, we leave out some parts of the Lean statements if these parts are technicalities that do not help understanding. As a general remark, note that sometimes a statement that is a lemma in natural language need not translate to a `theorem` but may also be an `instance` or `def` in Lean. This is because every definition in Lean has to come with a proof of well-definedness. In these cases, we usually leave out the proof and only state the definition here.

The complete code with all proofs may be found in Appendix B and in the Lean community math library at [KR22]. This chapter, and especially the proofs we provide, largely follow [Ško04]. Writing the code was a joint work with Jakob von Raumer. We originally wrote the code in Lean 3. The version presented here is the Lean 4 code that our code was ported to by the math library.

5.1. Ore sets

We start by defining the *Ore condition*, which we use throughout this part. We are going to show that if $S \subseteq R$ satisfies this condition, then we can describe the universal localisation R_S as the set of fractions with numerator in R and denominator in S together with appropriate addition and multiplication.

Definition 5.1. Let R be a monoid and $S \subseteq R$ a submonoid. Then S is called an *Ore set* if the following conditions hold:

1. For every $r_1, r_2 \in R$ and $s \in S$ such that $sr_1 = sr_2$ there exists an $s' \in S$ such that $r_1 s' = r_2 s'$. (left weak cancellation)

2. For every $r \in R, s \in S$ there exist $r' \in R, s' \in S$ such that $rs' = sr'$. (common right multiple)

In this case, we also say that R *satisfies the Ore condition* with respect to S.

```
class OreSet {R : Type*} [Monoid R] (S : Submonoid R)
  ore_left_cancel :
    ∀ (r₁ r₂ : R) (s : S), ↑s * r₁ = s * r₂ → ∃ s' : S, r₁ * s' = r₂ * s'
  oreNum : R → S → R
  oreDenom : R → S → S
  ore_eq : ∀ (r : R) (s : S), r * oreDenom r s = s * oreNum r s
```

We have only defined the Ore condition if R is a monoid. If R is a ring, we can interpret it as a monoid by considering the set R together with the multiplication. If this monoid satisfies the Ore condition, then we also say that the ring R satisfies the Ore condition. In both cases, S is just a submonoid and not necessarily a subring.

Also, we ask that S is a submonoid, whereas before in Chapter 3, we only asked that S is a subset. This is just for our convenience. One may state the Ore condition for any subset S. But then S is an Ore set in R if and only if the submonoid of R generated by S is an Ore set.

Note that in the code, we phrased the common right multiple condition not as an existence of (r', s'), but we ask that such elements should actually be provided for any given pair (r, s). We call those `oreNum r s` and `oreDenom r s`.

If for some R and S that satisfy the common right multiple condition, the maps `oreNum` and `oreDenom` cannot be explicitly constructed, the two definitions are not equivalent. However, if we wrap our Lean code in a **`noncomputable`** environment, they are equivalent. This environment may be thought of as the Lean equivalent of the axiom of choice and states that if we know something exists, then such an object can actually be provided. We defer the finer points of this discussion to Appendix A.

In any case, we must ensure that whatever we define later on does not depend on the concrete choice of (r', s').

The Ore condition is void if R is commutative. We will see later that, in this case, the universal localisation coincides with well-known constructions from commutative algebra, depending on the structure on R. Namely with the *group of differences* or the *(semi-) field of fractions*.

Lemma 5.2. *If R is a commutative monoid, then any submonoid of R is an Ore set.*

```
instance oreSetComm {R} [CommMonoid R] (S : Submonoid R) : OreSet S
```

To justify the name left weak cancellation property, we show that it is satisfied if R is left cancellative. That is, if for any $a, b, c \in R$ such that $ab = ac$, we also have $b = c$. This allows us to ignore the left weak cancellation condition as long as we work with left cancellative monoids.

Lemma 5.3. *Let R be a monoid satisfying the cancellation property. Then any submonoid of R satisfies the left weak cancellation property.*

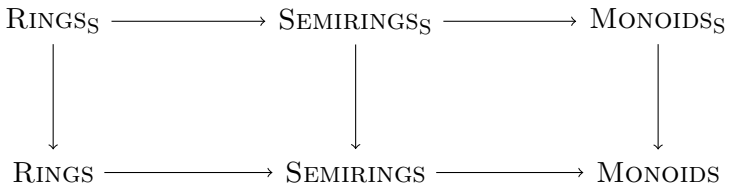


Figure 5.1.: A commutative diagram of functors. Objects in the upper row are pairs (R, S), where S is an Ore set in R. The horizontal lines are forgetful functors, and the vertical lines are the Ore localisations of R at S.

```
def oreSetOfCancelMonoidWithZero {R : Type*} [CancelMonoidWithZero R]
  {S : Submonoid R}
    (oreNum : R → S → R)
    (oreDenom : R → S → S)
    (ore_eq : ∀ (r : R) (s : S), r * oreDenom r s = s * oreNum r s) : OreSet S
```

In Part I, we have discussed zero-divisors in group rings. We have seen that conjecturally, group rings for torsion-free groups admit no zero-divisors. The conjecture is known to be true for many groups. Hence, it is important to know that having no zero-divisors is stronger than being left weak cancellative.

Lemma 5.4. *Let R be a ring without zero-divisors. Then any submonoid of R satisfies the left weak cancellation property.*

```
def oreSetOfNoZeroDivisors {R : Type*} [Ring R] [NoZeroDivisors R]
  {S : Submonoid R}
    (oreNum : R → S → R) (oreDenom : R → S → S)
    (ore_eq : ∀ (r : R) (s : S), r * oreDenom r s = s * oreNum r s) : OreSet S
```

5.2. Monoids

Throughout this section, let R be a monoid and $S \subseteq R$ an Ore set.

We proceed to construct the universal localisation at an Ore subset explicitly. We will see later that the underlying set of R_S is always the same, regardless of whether R is a monoid, a semiring or a ring. Similarly, the multiplicative structure will be the same in each case, and the additive structure will be equal for semirings and rings. This allows us to start with the localisation as monoids and then build on that to proceed to semirings and rings.

To make this precise: The diagram of functors in Figure 5.1 commutes. Note that we do not prove functioriality of the Ore localisation. But in fact, even the universal localisation is functorial, which is a direct consequence of its universal property.

Let us start by constructing the Ore localisation for a monoid R. First, we define the underlying set of the localisation. We denote it by $R[S]^{-1}$ to highlight that we have yet to see that it coincides with R_S. Even after this chapter, we will continue to use $R[S]^{-1}$ to signal that we make use of the explicit description of the Ore localisation.

Definition 5.5. The *Ore localisation* $R[S]^{-1}$ of R at S is the set

$$\{(r, s) \mid r \in R, s \in S\}/\sim$$

where two pairs (r, s) and (r', s') are equivalent if there exist $u \in S, v \in R$ such that

$$(rv, sv) = (r'u, s'u).$$

```
variable (R : Type*) [Monoid R] (S : Submonoid R) [OreSet S]
def oreEqv : Setoid (R × S) where
  r rs rs' := ∃ (u : S) (v : R),
    rs'.1 * u = rs.1 * v ∧ (rs'.2 : R) * u = rs.2 * v
def OreLocalization := Quotient (OreLocalization.oreEqv R S)
```

Note that this also shows that $\sim$ is indeed an equivalence.

One should think of an equivalence class $[(r, s)] \in R[S]^{-1}$ as a fraction rs^{-1} and of the equivalence as right expansions and cancellations of that fraction. In fact, we already denote elements of `R[S⁻¹]` as `r /ₒ s` in the code. We will see soon why this notion is justified and only then adopt that notation in this text as well. This ensures that one does not make assumptions stemming from the interpretation as fractions. For example, one would assume that $r1^{-1} \cdot 1s^{-1} = rs^{-1}$. While this is indeed true in the sense that $(r, 1) \cdot (1, s) = (r, s) \in R[S]^{-1}$, we have not seen the proof yet. In fact, we have not even defined the multiplication on $R[S]^{-1}$ at all, so let us do that next.

Definition 5.6. Let $(r_1, s_1), (r_2, s_2) \in R[S]^{-1}$. Let $u \in R, v \in S$ be such that $r_2 v = s_1 u$, which must exist because S is an Ore set.

Then set

$$(r_1, s_1) \cdot (r_2, s_2) \coloneqq (r_1 v, s_2 u).$$

```
def mul' (r₁ : R) (s₁ : S) (r₂ : R) (s₂ : S) : R[S⁻¹] :=
  r₁ * oreNum r₂ s₁ /ₒ (s₂ * oreDenom r₂ s₁)
```

Recall that `oreNum r₂ s₁` and `oreDenom r₂ s₁` play the roles of u and v here and that they satisfy the condition $r_2 v = s_1 u$ by assumption.

For now, the code only defines a map $(R \times S)^2 \to R[S]^{-1}$, but Definition 5.6 wants to define a multiplication on $R[S]^{-1}$. Hence, we have yet to see that Definition 5.6 is well defined. This involves two things. First, it does not depend on the choice of u, v. And second, we may change the representative within the equivalence class of (r_1, s_1) or (r_2, s_2) without changing the product.

Lemma 5.7. *Definition 5.6 does not depend on the choice of u, v.*

```
theorem mul'_char (r₁ r₂ : R) (s₁ s₂ : S) (u : S) (v : R)
  (huv : r₂ * (u : R) = s₁ * v) :
    OreLocalization.mul' r₁ s₁ r₂ s₂ = r₁ * v /ₒ (s₂ * u)
```

Lemma 5.8. *Let $r_1, r_2 \in R, s_1, s_2 \in S$ and $t_1, t_2 \in R$ such that $s_1 t_1, s_2 t_2 \in S$. Then*

$$(r_1, s_1) \cdot (r_2, s_2) = (r_1 t_1, s_1 t_1) \cdot (r_2 t_2, s_2 t_2)$$

```
def mul : R[S⁻¹] → R[S⁻¹] → R[S⁻¹] :=
  lift₂Expand mul' fun r₂ p s₂ hp r₁ r s₁ hr
```

Note that this Lemma does not quite show that the multiplication is independent of the choice of representative, as not every two representatives of the same equivalence class differ by an expansion. By definition, we might need to perform expansions on both representatives to obtain equality. Hence, it might, a priori, be necessary to do a chain of expansions and cancellations to transform one representative into another. However, the following shows that Lemma 5.8 is indeed sufficient.

Lemma 5.9. *Let $f\colon R \times S \to C$ be some map on $R \times S$ to some set C. In particular, if $C = \{\text{TRUE}, \text{FALSE}\}$, f is a statement about elements of $R \times S$.*

Suppose that $f(r,s) = f(rt,st)$ for every $r,t \in R, s \in S$ such that $st \in S$. Then

$$f^*\colon R[S]^{-1} \to C, \quad (r,s) \mapsto f(r,s)$$

is a well-defined map.

```
def liftExpand {C : Sort*} (P : R → S → C)
    ( hP : ∀ (r t : R) (s : S) (ht : (s : R) * t ∈ S),
      P r s = P (r * t) ⟨s * t, ht⟩) : R[S⁻¹] → C :=
  Quotient.lift (fun p : R × S => P p.1 p.2)
    fun (r₁, s₁) (r₂, s₂) ⟨u, v, hr₂, hs₂⟩
```

The function `lift₂Expand` that we used in the proof of Lemma 5.8 is a version of `liftExpand` for functions on $(R \times S)^2$ rather than on $R \times S$.

Thus the multiplication on $R[S]^{-1}$ is well defined. Now, we show that it gives $R[S]^{-1}$ again the structure of a monoid by checking all the monoid axioms.

Theorem 5.10. *The localisation $R[S]^{-1}$ together with the multiplication defined above is a monoid.*

The neutral element may be represented as any element (s,s).

```
theorem one_def : (1 : R[S⁻¹]) = 1 /₀ 1
theorem one_mul (x : R[S⁻¹]) : 1 * x = x
theorem mul_one (x : R[S⁻¹]) : x * 1 = x
theorem mul_assoc (x y z : R[S⁻¹]) : x * y * z = x * (y * z)
```

We want to show why the interpretation

$$(r,s) = rs^{-1}$$

is justified in $R[S]^{-1}$. The following statements all look obvious once we replace (r,s) by rs^{-1}. We mainly need them as a step in the next proof, but they also serve as reassurance of interpreting (r,s) as a fraction.

Lemma 5.11. *Let $s, s', t \in S$ and $r, r', p \in R$. Then*

1. $(s,s') \cdot (s',s) = 1$,

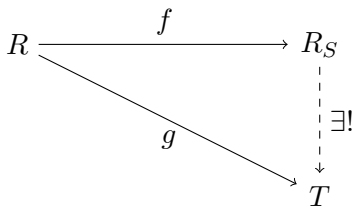


Figure 5.2.: The universal property of the universal localisation of R at the subset S: Every S-inverting map $R \to T$ factors uniquely through R_S.

2. $(r, s)(1, t) = (r, ts)$,
3. $(r, s)(s, t) = (r, t)$,
4. $(r, s) \cdot (sr', t) = (rr', t)$,
5. $(r, 1)(p, s) = (rp, s)$,
6. $(s, 1)^{-1} = (1, s)$. *In particular,* $(s, 1)$ *is a unit in* $R[S]^{-1}$.

```
theorem mul_inv (s s' : S) : ((s : R) /ₒ s') * ((s' : R) /ₒ s) = 1
theorem mul_one_div {r : R} {s t : S} : (r /ₒ s) * (1 /ₒ t) = r /ₒ (t * s)
theorem mul_cancel {r : R} {s t : S} : (r /ₒ s) * ((s : R) /ₒ t) = r /ₒ t
theorem mul_cancel' {r₁ r₂ : R} {s t : S} :
    (r₁ /ₒ s) * ((s * r₂) /ₒ t) = (r₁ * r₂) /ₒ t
theorem div_one_mul {p r : R} {s : S} : (r /ₒ 1) * (p /ₒ s) = (r * p) /ₒ s
def numeratorUnit (s : S) : Units R[S⁻¹] where
  val := (s : R) /ₒ 1
  inv := (1 : R) /ₒ s
```

When we write $(r, 1) \in R[S]^{-1}$ as a fraction, it becomes $r1^{-1} = r$. This suggests that we may think of R as being contained in $R[S]^{-1}$ via the map $r \mapsto (r, 1)$. Indeed, recall that the universal localisation comes with a map $R \to R_S$, and the map we just defined is precisely the localisation map.

Lemma 5.12. *The* localisation map

$$R \to R[S]^{-1}, \quad r \mapsto (r, 1)$$

is an S*-inverting homomorphism of monoids.*

```
def numeratorHom : R →* R[S⁻¹] where toFun r := r /ₒ 1
theorem numerator_isUnit (s : S) : IsUnit (numeratorHom (s : R) : R[S⁻¹])
```

From now on, we will denote elements of $R[S]^{-1}$ as fractions rs^{-1}.

We show that the Ore localisation $R[S]^{-1}$, if it exists, is the universal localisation of monoids. Recall the definition of the universal localisation property from Definition 3.4, particularly the associated commutative diagram. As a reminder, we can see the same diagram again in Figure 5.2.

Theorem 5.13. *The localisation map*

$$R \to R[S]^{-1}$$

satisfies the universal localisation property in the category of monoids.

```
variable (f : R →* T) (fS : S →* Units T)
variable (hf : ∀ s : S, f s = fS s)

def universalMulHom : R[S⁻¹] →* T where
  toFun x := x.liftExpand (fun r s => f r * ((fS s)⁻¹ : Units T)) fun r t s ht
theorem universalMulHom_commutes {r : R} :
    universalMulHom f fS hf (numeratorHom r) = f r := by
theorem universalMulHom_unique (φ : R[S⁻¹] →* T)
    (huniv : ∀ r : R, φ (numeratorHom r) = f r) :
```

The steps of the proof here are as follows: First, we define the proposed unique map $R[S]^{-1} \to T$ as `universalMulHom`. Then, we show that the diagram in Figure 5.2 commutes in `universalMulHom_commutes`. Finally, `universalMulHom_unique` states that every φ such that the diagram commutes is equal to `univeralMulHom`.

We will prove the injectivity of the localisation map in Lemma 5.23.

If R is commutative, then the universal localisation has already been formalised in Lean as `Localization S`. In this case, the localisation is also known as the *group of differences*. We show that the two localisations coincide when R is commutative.

Theorem 5.14. *Let R be a commutative monoid and $S \subseteq R$ an Ore set. Then $R[S]^{-1}$ is isomorphic to the group of differences of R and S.*

```
noncomputable def equivMonoidLocalization : Localization S ≃* R[S⁻¹]
```

Note that the use of **`noncomputable`** is because the existing `Localization` is defined in a **`noncomputable`** environment only. This is probably not a conceptual limitation but a design choice.

5.3. Semirings

From now on, let R be a semiring. $S \subseteq R$ is still an Ore set with respect to the structure of a multiplicative monoid on R.

We already know that $R[S]^{-1}$ has a multiplicative monoid structure. We proceed by defining an addition on the elements of $R[S]^{-1}$ that will turn the localisation into a semiring. Later on, we will also give the localisation a ring structure if R is a ring. But as we do not need the existence of additive inverses in R for anything but the existence of additive inverses in $R[S]^{-1}$, the localisation of semirings comes basically for free.

In the previous chapter, we had to take several steps to show that the multiplication was well-defined. This is even more intricate for the addition, so let us proceed very carefully and only define an addition for pairs in $R \times S$ at first.

Definition 5.15. Let $r_1, r_2 \in R, s_1, s_2 \in S$. Let $r' \in R, s' \in S$ such that $s_1 s' = s_2 r'$. Then set

$$(r_1, s_1) + (r_2, s_2) \coloneqq (r_1 s' + r_2 r', s_1 s').$$

```
def add'' (r₁ : R) (s₁ : S) (r₂ : R) (s₂ : S) : R[S⁻¹] :=
  ( r₁ * oreDenom (s₁ : R) s₂ +
    r₂ * oreNum (s₁ : R) s₂) /ₒ (s₁ * oreDenom (s₁ : R) s₂)
```

Note that we already defined the sum to be an element of $R[S]^{-1}$. This is possible because we have no structural requirements yet. For now, the addition is just a map between sets.

Note that the codomain is $R[S]^{-1}$, and it has to be in order for Definition 5.15 to be well-defined. Otherwise, the sum would definitely depend on the choice of (r', s').

Lemma 5.16. *Definition 5.15 is independent of the choice of* (r', s').

```
theorem add''_char (r₁ : R) (s₁ : S) (r₂ : R) (s₂ : S) (rb : R) (sb : S)
    (hb : (s₁ : R) * sb = (s₂ : R) * rb) :
    add'' r₁ s₁ r₂ s₂ = (r₁ * sb + r₂ * rb) /ₒ (s₁ * sb)
```

Now we want to prove that the addition descends to a map

$$R[S]^{-1} \times R[S]^{-1} \to R[S]^{-1}.$$

That is, it is independent of the choice of representative in either argument. The order in which we do that is we first show independence in the second argument, then commutativity and then independence in the first argument. This is necessary because the proofs of these statements rely on each other in that order. First, we show independence in the second argument.

Lemma 5.17. *For* $r_1, r_2, r_3 \in R, s_1, s_2, s_3 \in S$, *if*

$$r_2 {s_2}^{-1} = r_3 {s_3}^{-1} \in R[S]^{-1},$$

then

$$(r_1, s_1) + (r_2, s_2) = (r_1, s_1) + (r_3, s_3).$$

```
def add' (r₂ : R) (s₂ : S) : R[S⁻¹] → R[S⁻¹] :=
  ( Quotient.lift
    fun r₁s₁ : R × S => add'' r₁s₁.1 r₁s₁.2 r₂ s₂) <|
```

Note that the function `add'` is the same as the function `add''` with the additional knowledge that `add'` does not depend on the choice of representative in the second argument.

Now, we prove the commutativity of the addition. In fact, we show that it is commutative as a map $(R \times S) \times R[S]^{-1} \to R[S]^{-1}$.

Lemma 5.18. *Let* $r_1, r_2 \in R, s_1, s_2 \in S$. *Then*

$$(r_1, s_1) + r_2 {s_2}^{-1} = (r_2, s_2) + r_1 {s_1}^{-1}.$$

```
theorem add'_comm (r₁ r₂ : R) (s₁ s₂ : S) :
    add' r₁ s₁ (r₂ /₀ s₂) = add' r₂ s₂ (r₁ /₀ s₁)
```

Finally, independence in the first argument follows from the previous two statements.

Lemma 5.19. *For $r_1, r_2, r_3 \in R, s_1, s_2, s_3 \in S$, if*

$$r_1 {s_1}^{-1} = r_2 {s_2}^{-1} \in R[S]^{-1},$$

then

$$(r_1, s_1) + (r_3, s_3) = (r_2, s_2) + (r_3, s_3).$$

```
def add : R[S⁻¹] → R[S⁻¹] → R[S⁻¹] := fun x =>
  Quotient.lift (fun rs : R × S => add' rs.1 rs.2 x)
```

Thus, the addition is a well-defined map $R[S]^{-1} \times R[S]^{-1} \to R[S]^{-1}$. It remains to check that this addition turns $R[S]^{-1}$ into a semiring.

Theorem 5.20. *The localisation $R[S]^{-1}$ together with the addition defined above is a semiring.*

The additive unit may be represented as any fraction $0s^{-1}$.

```
theorem add_assoc (x y z : R[S⁻¹]) : x + y + z = x + (y + z)
def zero : R[S⁻¹] := 0 /₀ 1
theorem zero_div_eq_zero (s : S) : 0 /₀ s = 0
theorem zero_add (x : R[S⁻¹]) : 0 + x = x
theorem left_distrib (x y z : R[S⁻¹]) : x * (y + z) = x * y + x * z
theorem right_distrib (x y z : R[S⁻¹]) : (x + y) * z = x * z + y * z
```

Now we show that $R[S]^{-1}$ and the universal localisation R_S coincide in our current setting. Namely when R is a semiring and $S \subseteq R$ is an Ore set. We do this by proving the universal property of the localisation for $R[S]^{-1}$. As we know that the universal localisation is unique, this shows $R[S]^{-1} = R_S$.

Theorem 5.21. *The localisation map $R \to R[S]^{-1}$ as defined in Lemma 5.12 satisfies the universal property from Definition 3.4 in the category of semirings.*

```
variable (f : R →+* T) (fS : S →* Units T)
variable (hf : ∀ s : S, f s = fS s)
def universalHom : R[S⁻¹] →+* T :=
  { universalMulHom f.toMonoidHom fS hf
theorem universalHom_commutes {r : R} :
  universalHom f fS hf (numeratorHom r) = f r
theorem universalHom_unique
  (φ : R[S⁻¹] →+* T) (huniv : ∀ r : R, φ (numeratorHom r) = f r) :
    φ = universalHom f fS hf
```

The proof has essentially the same steps as the proof for the universal property in the category of monoids in Theorem 5.13. Note that the definition of `universalHom` shows that the unique monoid-map $R[S]^{-1} \to T$ is a map of semirings.

Again, injectivity will be shown in Lemma 5.23.

5.4. Rings

From now on, let R be a ring. $S \subseteq R$ is still an Ore set. First, we show that then $R[S]^{-1}$, as defined for semirings, admits additive inverses and is, hence, a ring.

Lemma 5.22. *In $R[S]^{-1}$, $(-r)s^{-1}$ is the additive inverse of rs^{-1}.*
Thus, $R[S]^{-1}$ is a ring.

```
def neg : R[S⁻¹] → R[S⁻¹] :=
  liftExpand (fun (r : R) (s : S) => -r /ₚ s) fun r t s ht
```

Note that this, together with Theorem 5.21, also shows that $R[S]^{-1} = R_S$ if R is a ring.

We have discussed in Chapter 3 the question of whether there exists a universal division ring into which R embeds. Indeed, the Ore localisation is such a division ring if R is a domain and satisfies the Ore condition with respect to the submonoid $R \setminus 0$.

Lemma 5.23. *Suppose that R has no zero divisors and that $0 \notin S$. Then the localisation map $R \to R[S]^{-1}$ is injective.*

```
theorem numeratorHom_inj (hS : S ≤ R⁰) :
  Function.Injective (numeratorHom : R → R[S⁻¹])
```

Note that as it is the same map, this also shows that the localisation map $R \to R[S]^{-1}$ is also injective as a map between semirings or monoids.

Theorem 5.24. *Suppose that R is nontrivial and has no zero divisors. Then $R[R \setminus 0]^{-1}$ is a division ring.*

```
def inv : R[R⁰⁻¹] → R[R⁰⁻¹] :=
  liftExpand
    (fun r s =>
      if hr : r = (0 : R) then (0 : R[R⁰⁻¹])
      else s /ₚ ⟨r, fun _ => eq_zero_of_ne_zero_of_mul_right_eq_zero hr⟩)
theorem mul_inv_cancel (x : R[R⁰⁻¹]) (h : x ≠ 0) : x * x⁻¹ = 1
```

Part III.

$\sum$-invariants

6. Characters

From now on, we will turn our attention to the following question: If G is a finitely generated group and $N \trianglelefteq G$ a normal subgroup, how can we decide if N is also finitely generated? We start by reviewing in this chapter the Σ-invariant, which answers this question in case G/N is abelian. Apart from [BNS87], where Σ-invariants were introduced, [Str13] provides a comprehensive survey of the topic. Everything we do in this chapter may also be found there.

In Chapters 7 and 8, we are going to generalise the material of this chapter to the case where G/N is nilpotent.

6.1. The character sphere

If G is a subgroup, any normal subgroup $N \trianglelefteq G$ is also the kernel of the projection map $G \twoheadrightarrow G/N$. This fact provides a bridge between normal subgroups and maps defined on G. If G/N is finitely generated torsion-free abelian, then there exists an embedding $G/N \hookrightarrow (\mathbb{R}, +)$ and hence a map $\Phi\colon G \to \mathbb{R}$ with $\ker \Phi = N$. Note that scaling $\mathbb{R}$ by some factor does not change the kernel of Φ. So, as long as we are only interested in $\ker \Phi$, we may consider Φ up to a scaling of $\mathbb{R}$.

Definition 6.1. Let G be a group. A *character* of G is an equivalence class of group homomorphisms $\Phi\colon G \to (\mathbb{R}, +)$ where two maps Φ and Ψ are equivalent if they differ only by multiplication by a positive number. That is if there is some positive number $\lambda \in \mathbb{R}^+$ such that $\Phi(g) = \lambda\Psi(g)$ for every $g \in G$.

The *character sphere* $S(G)$ is the set of non-trivial characters of G where by *trivial character* we mean the constant map $g \mapsto 0$.

We will see in the next chapter why we only allow scaling by a *positive* factor.

Remark 6.2. Any group homomorphism $\Phi\colon G \to \mathbb{R}$ is defined entirely by the values Φ takes on a generating set of G. If G has n generators, this defines an embedding of $\mathrm{Hom}(G, \mathbb{R})$ as a linear subspace of $\mathbb{R}^n$. Seeing the n-sphere as $S^n = (\mathbb{R}^{n+1} \setminus 0)/\mathbb{R}^+$, we get an embedding of $S(G)$ into S^{n-1}, hence the name character *sphere*. This also gives $S(G)$ a topological structure as a subspace of S^{n-1}. If G is a free-abelian group, this embedding is actually a homeomorphism.

Given two maps $\Phi, \Psi\colon G \to \mathbb{R}$ that represent the same equivalence class in $S(G)$ we have $\ker \Phi = \ker \Psi$. Hence, the *kernel of a character* is well-defined. This allows us to interchangeably use characters and homomorphisms when we are only interested in their kernels.

Definition 6.3. Let G be a group and $N \trianglelefteqslant G$ a normal subgroup. The *relative character sphere* $S(G, N)$ is the subset of non-trivial characters $\Phi \in S(G)$ such that $N \leqslant \ker \Phi$.

Example 6.4. The character sphere $S(\mathbb{Z})$ has just two elements: The maps that send the generator of $\mathbb{Z}$ to a positive number and those that send the generator to a negative number.

If $N \trianglelefteqslant G$ is a subgroup such that $G/N \cong \mathbb{Z}$, then $S(G, N)$ also has just two elements, namely the projection map concatenated with any of the characters of $\mathbb{Z}$. More generally, $S(G, N) = S(G/N)$ is true for any group G and $N \trianglelefteqslant G$.

Note that as $\mathbb{R}$ is an abelian group, any map $\Phi\colon G \to \mathbb{R}$ necessarily sends every commutator $[g, h]$ to 0. Hence Φ factors through the abelianisation $G_{\mathrm{ab}} \coloneqq G/[G, G]$. Also, Φ sends any torsion element of G_{ab} to 0. If G is finitely generated, then so is G_{ab}, so it may be written as $\mathbb{Z}^n \oplus T$, where T contains the torsion elements. The map Φ factors through the *free abelianisation map* $G \twoheadrightarrow G_{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}^n$ and hence $S(G) = S(G, N)$, where N is the kernel of the free abelianisation map. We call a character *irrational*, if $\ker \Phi = N$.

Example 6.5. To justify the name *irrational*, note that Φ may be written as the free abelianisation map composed with some character $\Psi \in S(G, N) = S(\mathbb{Z}^n)$. The character Φ is irrational if and only if this Ψ is injective. Consider the case $n = 2$. Then any representative of the character Ψ corresponds to some vector $(x, y) \in \mathbb{R}^2$, where x and y are the images of the generators a and b of $\mathbb{Z}^2$. If x or y is 0, then Ψ is not injective. Otherwise, if $x/y \in \mathbb{Q}$, say $x/y = p/q$ for some integers p, q, then

$$\Psi(a^q b^{-p}) = qx - py = 0,$$

so Ψ is again not injective. If, however, x/y is irrational, then the same argument shows that Ψ is injective and hence Φ is irrational.

Similarly, a character of $\mathbb{Z}^n$ is injective if no generator gets mapped to 0, and any pair of generators has images that are at an irrational ratio.

6.2. Connected subsets of groups

A character $\Phi \in S(G)$ divides G into two halves: The preimages of the positive and negative numbers. The Σ-invariant we will see in Section 6.3 asks if those halves are connected. Let us first make precise what connected means in this context.

Definition 6.6. Let Γ be a graph and $K \subseteq V(\Gamma)$ a subset of the vertices. The *full subgraph spanned by* K of Γ is the subgraph whose vertex set is K and whose edge set contains every edge of Γ which has both ends in K.

Definition 6.7. Let G be a group and $K \subseteq G$ any subset. We call K *coarsely connected as a subset* of G if there exists a finite subset $S \subseteq G$ such that the full subgraph of the Cayley graph $\mathrm{Cay}(G, S)$ spanned by K is connected.

If G is finitely generated and the full subgraph of $\mathrm{Cay}(G, S)$ spanned by K is coarsely connected for every finite generating set S, then we say that K is *connected as a subset* of G.

Figure 6.1.: The set of red vertices is coarsely connected as it is connected by a 2-path: A path, where at least every other vertex is red.

Remark 6.8. Note that in the definition of coarsely connected, we do not require that S generates G as a group. Hence, $G \subseteq G$ is coarsely connected if and only if G is finitely generated. In this case, G is also connected.

Remark 6.9.

1. If G itself is finitely generated, then we may always ask that S is a generating set of G as every finite subset of G is contained in a finite generating set of G.

2. Suppose G is finitely generated and S is a finite generating set. Then $K \subseteq G$ is coarsely connected if and only if there exists some constant $n \in \mathbb{N}$ such that for every $g, h \in K$, there exists an n-path in $\mathrm{Cay}(G, S)$ from g to h supported on K. Here, an n-path supported on K is a path in $\mathrm{Cay}(G, S)$ such that any segment of n consecutive vertices on the path contains at least one point in K. See Figure 6.1 for an example.

   To see that this is equivalent to the definition of coarse connection, note the following: If T is another finite generating set of G, then any 1-path in $\mathrm{Cay}(G, S)$ is an n-path in $\mathrm{Cay}(G, T)$ where n is an upper bound to the T-wordlength of elements of S. Conversely, if K is connected by n-paths in $\mathrm{Cay}(G, S)$, then taking T to be the set of all words in S of length at most n, we get that the full subgraph of $\mathrm{Cay}(G, T)$ spanned by K is connected.

3. If G is contained in a finitely generated group G' such that K is coarsely connected as a subset of G', then K is also coarsely connected as a subset of G: Pick a finite generating set S' of G' and any generating set S of G. Let n be such that any two elements of K are connected by an n-path in S'. There are only finitely many words in K of length at most n in S', so their length in S is bounded from above by some m. Also, we use only a finite subset $T \subseteq S$ of the generators to write these words. Hence every n-path in S' is also an m-path in T, so K is coarsely connected as a subset of G.

   Conversely, if K is coarsely connected as a subset of G, then it is also coarsely connected as a subset of G'.

Example 6.10. In the group of integers, the subset of even numbers is coarsely connected as it is connected with respect to the generating set $\{X, X^2\}$. The set of powers of 2 is not coarsely connected because the distance between two adjacent powers of 2 is unbounded.

Similarly, for any surjective map $\Phi\colon F_2 \twoheadrightarrow \mathbb{Z}$, the preimage of the positive numbers is not coarsely connected. We will see an illustration of the latter in Figure 6.2.

In the standard literature, coarse connectedness is usually only defined for finitely generated groups. The extension to non-finitely generated groups is our own addition, and it might not be apparent at first why this is a good definition. But we will see in Section 8.2 how this definition ties in nicely with the finitely generated case.

6.3. Σ-invariants

Now, we have established the background necessary to introduce the Σ-invariant.

Definition 6.11. Let G be a finitely generated group. Then the *(first) Σ-invariant* $\Sigma^1(G) \subseteq S(G)$ is the subset containing all characters Φ such that $\Phi^{-1}([0,\infty))$ is connected as a subset of G.

Remark 6.12. The Σ-invariant is also known as BNS-invariant, where BNS is for Robert Bieri, Walter Neumann, and Ralph Strebel. As I believe that generally, we should prefer names for objects that are not derived from their authors, and the name Σ-invariant is also established, that is the name we use in this work.

Remark 6.13. Note that for a character Φ, connectedness of $\Phi^{-1}([0,\infty)$ as a subgraph of the Cayley graph is independent of the choice of finite generating set. Hence $\Phi^{-1}([0,\infty))$ is connected if and only if it is coarsely connected.

The standard definition of the Σ-invariant is to have it contain all characters $\Phi \in S(G)$ such that the full subgraph of $\mathrm{Cay}(G,S)$ spanned by $\Phi^{-1}([0,\infty))$ is connected for some generating set S. Then, one shows that this does not actually depend on S. With the knowledge that connectedness does not depend on the choice of generating set, the definition we presented here is immediately equivalent to the standard one. Our formulation will be better suited to the generalisation we will see in Section 7.4.

The original definition of $\Sigma^1(G)$ was yet different from the modern definition in the literature and from Definition 6.11.

Lemma 6.14. *Let G be a group with finite generating set S. Then $\Phi \in \Sigma^1(G)$ if and only if the monoid $\Phi^{-1}([0,\infty))$ is of type $\mathcal{FP}_1$.*

Remark 6.15. In Definition 1.17, we have defined the finiteness properties only for groups. But at no point did we use that G is actually a group, and the same definition works for monoids. Beware, however, that in contrast to groups, not all monoids of type $\mathcal{FP}_1$ are finitely generated.

For example, if $\Phi\colon \mathbb{Z}^2 \to \mathbb{R}$ is an irrational - that is, injective - character, then $\Phi^{-1}([0,\infty))$ is $\mathcal{FP}_1$ by the above lemma, and it is also possible to explicitly construct a resolution to prove this fact directly.

But $M \coloneqq \Phi^{-1}([0,\infty))$ is not finitely generated as a monoid: Take for example

$$\Phi\colon \mathbb{Z}^2 = \langle a, b\rangle \to \mathbb{R}, \quad a \mapsto 1, \quad b \mapsto \pi.$$

This is purely for illustration. Any other irrational character would work just as well.

Let $S \subseteq \Phi^{-1}((0,\infty))$ be any finite subset. Note that for now, S does not contain any element that maps to 0 but is otherwise an arbitrary finite subset of our monoid M. Then $S \subseteq \mathbb{Z}^2$ maps via Φ to some finite set of positive real numbers.

Thus, $\Phi(S)$ has a positive minimum, and the submonoid of $\mathbb{R}$ generated by $\Phi(S)$ has a minimal non-zero element. But $\Phi(\mathbb{Z}^2)$ contains a sequence approaching 0 from above: any sequence of rational numbers p/q approaching π from above yields a sequence

$$a^p b^{-q} \mapsto p - q\pi$$

of positive numbers converging to 0. So $\Phi(S)$ is not a generating set of $\Phi(\mathbb{Z}^2) \cap (0,\infty)$ and hence S does not generate $\Phi^{-1}((0,\infty))$.

Since for any generating set $T \subseteq M$, the intersection $T \cap \Phi^{-1}((0,\infty))$ has to be a generating set for $\Phi^{-1}((0,\infty))$, the monoid M is not finitely generated.

While Definition 6.11 is the definition of $\Sigma^1(G)$ we will primarily be using, the characterisation via Lemma 6.14 has a canonical generalisation.

Definition 6.16. Let G be a group with finite generating set S. Then the n'th Σ-invariant of G is the subset $\Sigma^n(G)$ of $S(G)$ containing all characters Φ such that the monoid $\Phi^{-1}([0,\infty))$ is of type $\mathcal{FP}_n$.

We mention this here because much of this theory applies analogously if we replace "finitely generated" by $\mathcal{FP}_n$ and $\Sigma^1(G)$ by $\Sigma^n(G)$. For full details, see [Ren88].

Remark 6.17. It is also possible to generalise $\Sigma^1(G)$ by starting from the geometric characterisation in Definition 6.11. This yields a different notion of $\Sigma^n(G)$. The difference is similar to the difference between the geometric and algebraic finiteness properties $\mathcal{F}_n$ and $\mathcal{FP}_n$.

The relative character sphere and first Σ-invariant are connected to finite generatedness of subgroups via the following theorem.

Theorem 6.18 (Bieri, Neumann, Strebel [BNS87])**.** *Let G be a finitely generated group and $N \trianglelefteq G$ a normal subgroup such that G/N is abelian.*

Then N is finitely generated if and only if $S(G,N) \subseteq \Sigma^1(G)$.

Example 6.19. If $G/N \cong \mathbb{Z}$, then $S(G,N)$ has only two elements, namely the projection map and the projection map concatenated with multiplication by -1. Hence, Theorem 6.18 states that N is finitely generated if and only if the preimages of the positive and the negative numbers under the projection map are connected.

Have a look at Figure 6.2. We see the case where $G = \mathbb{Z}^2$ and N is the subgroup generated by one of the free generators. Here, N is isomorphic to $\mathbb{Z}$. In particular, N is finitely generated, and both the positive and negative half of the Cayley graph are connected.

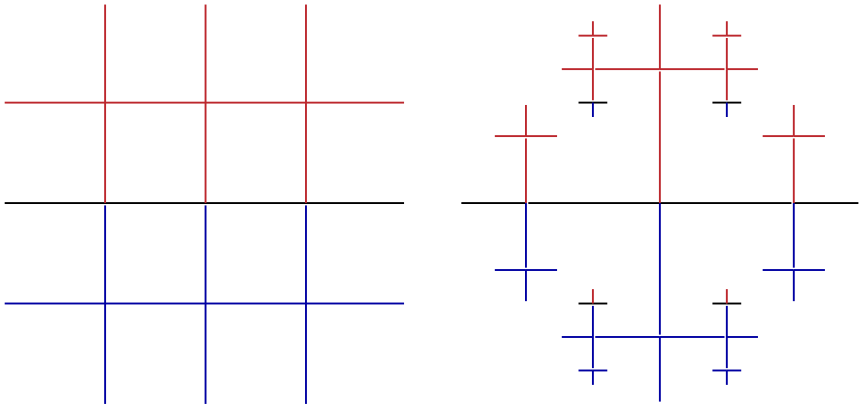


Figure 6.2.: Cayley graphs of $\mathbb{Z}^2$ and F_2 divided into positive and negative elements by the projections onto one generator, distinguished by colour.

On the other hand, if $G = \langle a, b \rangle$ is the free group on two generators, then the kernel of the projection onto one of the generators, say onto $\langle a \rangle$, is

$$\langle\langle b \rangle\rangle = \langle a^{-i} b a^i \ (i \in \mathbb{Z}) \rangle.$$

This kernel is not finitely generated, and the two halves of $\mathrm{Cay}(F_2)$ are not connected.

For the cases $G = \mathbb{Z}^2$ or $G = F_2$, we have just seen that either both characters or neither character in $S(G, N)$ is in $\Sigma^1(G)$. So Theorem 6.18 might as well have stated that $\ker \Phi$ is finitely generated if and only if $\Phi \in \Sigma^1(G)$ in these cases. The Baumslag-Solitar group provides an example where this formulation does not suffice because one character is in $\Sigma^1(G)$ and its negative is not. Recall from Section 2.3 that the Baumslag-Solitar group is $\mathrm{BS}(1,2) = \langle a, b \mid ba = ab^2 \rangle$ and that $\mathrm{Cay}(\mathrm{BS}(1,2), \{a, b\})$ projects onto the infinite binary tree we see in Figure 6.3. It may be seen as the left cosets of $\mathrm{BS}(1,2)$ with respect to the multiplication by b. The character

$$\Phi \colon \mathrm{BS}(1,2) \twoheadrightarrow \mathbb{Z}, \quad a \mapsto 1, \quad b \mapsto 0$$

factors through this projection and the value $\Phi(g)$ is given by the height of g in the tree. The positive half of the tree is connected whereas the negative half is not, so $\Phi \in \Sigma^1(\mathrm{BS}(1,2))$ but $-\Phi \notin \Sigma^1(\mathrm{BS}(1,2))$. By Theorem 6.18, $\ker \Phi$ is hence not finitely generated.

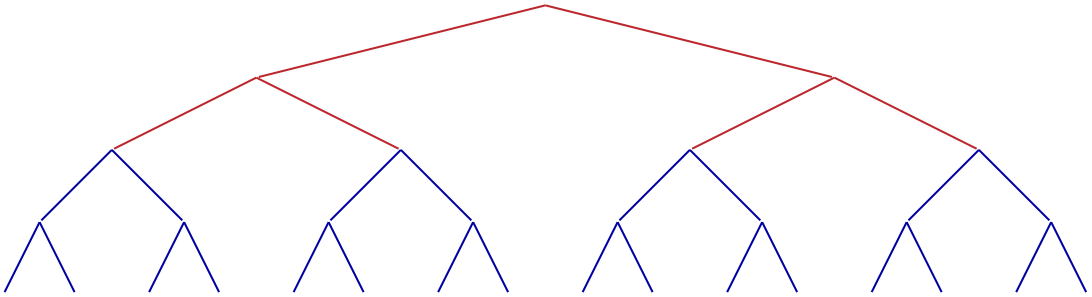


Figure 6.3.: A quotient of the Cayley graph of the Baumslag-Solitar group $BS(1, 2)$. Colours indicate elements that get mapped to positive and negative numbers, respectively, by the projection onto the first generator.

7. From characters to orders

Theorem 6.18 gives us a criterion for the kernel of a map being finitely generated only if the codomain of that map is an abelian group. The criterion then has us consider all maps to $\mathbb{R}$. We will generalise the theorem to maps onto non-abelian groups. For that, we need to find an analogue of "maps to $\mathbb{R}$". The analogue will be partial orders induced by maps onto some group Q. In this chapter, we will establish how we can identify a character $\Phi \in S(G)$ with an order on G. This allows us to see the Σ-invariant as a set of orders on G. We propose an extension of the Σ-invariant for non-abelian groups such that it also includes orders not induced by characters.

7.1. Partially ordered groups

We start by having a look at partially ordered groups in general. The material in this section is treated more thoroughly in [KM96] and [Gla99].

Definition 7.1. Let G be a group. A *partial order* on G is a relation $\prec$ on the set G such that for all $f, g, h \in G$

- $g \not\prec g$ (antireflexive)
- $g \prec h \Rightarrow h \not\prec g$ (antisymmetric)
- $f \prec g$ and $g \prec h \Rightarrow f \prec h$ (transitive)

Additionally, a partial order may have the following properties

- $f \prec g \Rightarrow hf \prec hg$ (left invariant)
- $f \prec g \Rightarrow fh \prec gh$ (right invariant)

and $\prec$ is said to be *bi-invariant* if it satisfies both.

Remark 7.2.

1. The symbol $=$ always refers to honest equality as elements of G.
2. For an order $\prec$, we set

$$g \preccurlyeq h \iff g \prec h \text{ or } g = h.$$

The symbols $\succ$ and $\succcurlyeq$ are defined as the respective opposite orders where

$$g \succ h \iff h \prec g \quad \text{and} \quad g \succcurlyeq h \iff h \preccurlyeq g.$$

3. In the literature, one often finds the definition of $(G, \preccurlyeq)$ instead of $(G, \prec)$ as the definition of a partially ordered group. The symbols $\prec, \succcurlyeq, \succ$ are then derived. However, I find it often makes notation easier to view $\prec$ as "the order" so that is the convention we use here.

4. When we use $<, \leqslant, >$ and $\geqslant$, we always mean them in their well-established, canonical meanings, such as the standard order on the real numbers.

Let us give some important examples. We will review them in more detail in Definition 7.7 and Definition 7.14.

Example 7.3.

1. The *trivial order* $g \not\prec h$ for each pair $g, h \in G$ is a bi-invariant partial order for any group G.

2. For any character $\Phi\colon G \to \mathbb{R}$, we can define a bi-invariant partial order $\prec$ on G by letting
$$g \prec h \iff \Phi(g) < \Phi(h).$$
For $\Phi = 0$, we obtain the trivial order on G.

   Note that the statement $\Phi(g) < \Phi(h)$ does not depend on the choice of representative for the character Φ.

3. For $G = \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$, there is the *lexicographic order*
$$a^i b^j \prec a^k b^l \iff \begin{cases} i < k & \text{or} \\ i = k \text{ and } j < l \end{cases}$$
We have already encountered this order in Example 3.12.

We say $(G, \prec)$ or sometimes just G is an *ordered group* as a shortcut for G being a group and $\prec$ a bi-invariant partial order on G. In this work, any order will be partial and bi-invariant unless noted otherwise.

Definition 7.4. Let G be an ordered group.

1. Two elements $g, h \in G$ are called *comparable* if $g \prec h$ or $h \prec g$ and *incomparable* otherwise.

2. An order is called *total* if any two elements are comparable or equal.

3. $g \in G$ is called *positive* if $1 \prec g$ and *negative* if $g \prec 1$.

4. $G^{\succ} \coloneqq \{g \in G \mid g \succ 1\}$ is called the *positive cone* of $(G, \prec)$.

5. For the inclusion of a subset $\iota\colon S \hookrightarrow G$ the *restriction* is the partial order $\prec_{|S}$ on S such that for any $s, t \in S$
$$s \prec_{|S} t \iff \iota(s) \prec \iota(t).$$

6. A subset $S \subseteq G$ is called an *antichain* if $\prec_{|S}$ is the trivial order.

7. An antichain S is *maximal* if the only antichain containing S is S itself. We say that S is a *maximal antichain subgroup* if it is an antichain and a subgroup, and it is maximal amongst subgroups that are antichains.

8. For a subset $S \subseteq G$ and $*$ any of $\{\prec, \succ, \preccurlyeq, \succcurlyeq\}$, set

$$S_* \coloneqq \{g \in G \mid \exists s \in S \colon g * s\}.$$

Remark 7.5.

1. Being incomparable is, in general, not transitive.

2. A subgroup $H \leqslant G$ is an antichain if and only if $H \cap G^{\succ}$ is the empty set. The same cannot be said if H is just any subset.

3. The set $S_{\succ}$ may also be described as $S_{\succ} = SG^{\succ} = \{sg \mid s \in S, g \in G^{\succ}\}$ and similarly for $S_{\prec}$.

4. For any subset $S \subseteq G$ we have $S_{\prec} \cap S_{\succcurlyeq} = \emptyset$ if and only if S is an antichain. Additionally, $S_{\prec} \cup S_{\succcurlyeq} = G$ if and only if S is a maximal antichain. The latter is not always true if S is a maximal antichain subgroup, as we will see, for instance, in Example 7.20.

5. If g is a torsion element of G, then 1 and g are necessarily incomparable since if g is positive, then
$$1 \prec g \prec g^2 \prec \cdots \prec g^n = 1$$
contradicts antireflexivity. The proof if g is negative is analogous.

   If the set of all torsion elements T is a normal subgroup, then any order on G is induced by an order on G/T - a notion we will make precise in Definition 7.7. Since this will almost always be the case in this work, it suffices for us to think of a generic group G as torsion-free.

It is often helpful and allows for more ergonomic notation to think of the positive cone instead of the order itself. The following lemma tells us that an order is uniquely determined by its positive cone.

Lemma 7.6. *Let G be a group.*

1. *If G is ordered, the positive cone $G^{\succ}$ is closed under multiplication with elements of $G^{\succ}$ and under conjugation with elements of G. If $g \in G^{\succ}$ then $g^{-1} \notin G^{\succ}$.*

2. *For any subset $S \subseteq G$ that is closed under multiplication with S and under conjugation with elements of G and such that $S \cap S^{-1} = \emptyset$, there is a unique order on G such that $G^{\succ} = S$.*

Proof. 1. This is immediate from the definition of a bi-invariant partial order.

2. The postulated order is
$$g \prec h \iff g^{-1}h \in S.$$
Checking that this is indeed an order is, again, essentially just applying the definition. Transitivity follows from the assumption that S is closed under multiplication. For anti-symmetry, we use that $S \cap S^{-1} = \emptyset$. In particular, $1 \notin S$, so $\prec$ is anti-reflexive. Left invariance is straightforward, and for right invariance, we need S closed under conjugation.

As for uniqueness, note that
$$g \prec h \iff 1 \prec g^{-1}h \iff g^{-1}h \in G^{\succ} = S$$
is necessarily true for any bi-invariant relation with $G^{\succ} = S$.

□

7.2. Orders induced by characters

Our goal will later be to translate group homomorphism properties into a language based on partial orders. In this section, we investigate how the two concepts relate. While many of these ideas are used implicitly in, for example, [Kie20], I am not aware of any sources where this is made explicit.

Definition 7.7. Let $(G, \prec)$ be an ordered group.

1. Let Q be another ordered group and $\varphi\colon G \to Q$ a group homomorphism. Then φ is called *order-preserving* if for all $g, h \in G$ we have
$$g \prec h \Rightarrow \varphi(g) \preccurlyeq \varphi(h).$$

2. Let $\prec'$ be another order on G. We say that $\prec$ is a *suborder* of $\prec'$ if
$$G^{\succ} \subseteq G^{\succ'}.$$

3. Let $(Q, \prec_Q)$ be an ordered group and $\Phi\colon G \to Q$ order-preserving. We say that $\prec$ *is induced* by $(\Phi, \prec_Q)$ if the following condition holds: For every order $\prec'$ on G such that $\Phi\colon (G, \prec') \to (Q, \prec_Q)$ is order-preserving, $\prec'$ is a suborder of $\prec$.

   We also say that $\prec$ is induced by Φ or by $\prec_Q$ if the other object is clear from the context.

4. Let $(H, \prec_H)$ be an ordered group and $\iota\colon H \to G$ order-preserving. We say that $\prec$ *is induced* by $(\iota, \prec_H)$ if the following condition holds: For every order $\prec'$ on G such that $\iota\colon (H, \prec_H) \to (G, \prec')$ is order-preserving, $\prec$ is a suborder of $\prec'$.

   Again, we say that $\prec$ is induced by ι or by $\prec_H$ if the other is clear. We denote the induced orders by $\prec_Q^{\Phi^{-1}}$ and $\prec_H^{\iota}$ respectively or just by $\prec$ if there is no chance of confusion.

5. If $(Q, \prec_Q)$ is an ordered group and $\Phi\colon G \to Q$ such that $\prec$ is induced by $(\Phi, \prec_Q)$, then Φ is *order-inducing on the domain*. An *order-inducing map on the codomain* $\iota\colon H \to G$ is defined analogously. We will omit the (co)domain part if it is clear on which side a map is order-inducing.

Remark 7.8. The map ι is not necessarily injective. However, if ι induces an order on its image, we may also describe that order as the order induced by the inclusion $\iota\colon H/\ker\iota \hookrightarrow G$. In this case, the projection $H \twoheadrightarrow H/\ker\iota$ is necessarily order-preserving. Thus, whenever ι is order-inducing, ι can be made injective without losing information about the order on G. We will elaborate on this argument in Lemma 7.9.

The definition of induced order on the domain essentially says that the induced order is the largest order such that the inducing map is order-preserving. Similarly, the induced order on the codomain is the smallest order such that the inducting map is order-preserving. Note the analogy between this definition and the definition of, for example, the division closure. While the definition as phrased above might be most natural, the following description is often easier to handle.

Lemma 7.9. *Let G be a group and Q, H be ordered groups. Let $\Phi\colon G \to Q$ and $\iota\colon H \to G$.*
Then

1. *Φ induces the following order on G:*
$$1 \prec^{\Phi^{-1}} g \iff 1 \prec \Phi(g)$$
2. *Consider the relation*
$$1 \prec' g \iff g \in G \cdot \iota(H^\succ) \smallsetminus 1$$
Here, $G \cdot \iota(H^\succ)$ denotes the image of $\iota(H^\succ)$ under conjugation with elements in G. If ι induces an order on G, then $\prec'$ is an order and it is the order induced by ι.

Proof.

1. This is the order such that $G^\succ = \Phi^{-1}(Q^\succ)$. Checking that $\Phi^{-1}(Q^\succ)$ is closed under multiplication, conjugation with $g \in G$ and does not contain two elements inverse to each other is straightforward since we know that $Q^\succ$ has all these properties. The order is then well-defined by Lemma 7.6.

   As any order on G such that Φ is order-preserving requires $\Phi(G^\succ) \subseteq Q^\succ$, all such orders are suborders of $\prec^{\Phi^{-1}}$.

2. If ι is order-preserving, then every element of $\iota(H^\succ) \smallsetminus 1$ has to be positive. Because the positive cone of the induced order has to be closed under conjugation, this extends to $G \cdot \iota(H^\succ) \smallsetminus 1$. Hence, if the relation $\prec'$ defined in the statement is actually an order, then it is a suborder of every order such that ι is order-preserving.

   Checking that $G \cdot \iota(H^\succ) \smallsetminus 1$ is closed under multiplication and conjugation is straightforward. It may, however, contain two elements inverse to each other. But

in this case, the above argument shows that ι cannot be order-preserving for any order on G and hence ι does not induce any order.

□

Remark 7.10. In particular, any map Φ with an ordered codomain induces an order on the domain. But not every map ι with an ordered domain induces an order on the codomain. However, it is easy to tell if a given order on the codomain is induced by ι. Especially if $\iota(H^{\succ})$ is already closed under conjugation with G, then this is the positive cone of the order induced by ι. Also, if G admits some order such that $G^{\succ} \subseteq \iota(H)$, then that order is induced by ι.

Example 7.11. Let $\Phi\colon G \to \mathbb{R}$ and consider $\mathbb{R}$ to be ordered by the standard order. Then $\prec^{\Phi^{-1}}$ is the order on G with $G^{\succ} = \Phi^{-1}((0,\infty))$. As $(0,\infty)$ is preserved by multiplication with a positive number, two maps $\Phi, \Psi\colon G \to \mathbb{R}$ induce the same order if they represent the same character. Thus, we may say that $\prec$ is induced by the *character* Φ.

We may chain order-inducing maps as expected:

Lemma 7.12. *Let there be three ordered groups $(G, \prec_G)$, $(H, \prec_H)$ and $(K, \prec_K)$. Let $\Phi\colon G \to H$ and $\Psi\colon H \to K$ be maps of groups.*

1. *If $\prec_H$ is induced by Ψ and $\prec_G$ is induced by Φ, then $\prec_G$ is also induced by $\Psi \circ \Phi$.*
2. *If $\prec_H$ is induced by Φ and $\prec_K$ is induced by Ψ, then $\prec_K$ is also induced by $\Psi \circ \Phi$.*

Proof. For the first part, from Lemma 7.9 we know that

$$1 \prec_G g \iff 1 \prec_H \Phi(g) \iff 1 \prec_K (\Psi \circ \Phi)(g)$$

and hence $\prec_G = \prec_K^{(\Psi\circ\Phi)^{-1}}$. The second part may be proven similarly. □

In view of Theorem 6.18, the kernel of a map is particularly interesting to us. Given a map that is order-inducing on the domain, we would like to know if we can recover the kernel of that map just by looking at the induced order. This is not always possible, for example, when both the inducing and the induced order are trivial. But in most other cases, we get some restrictions on what the kernel might have been.

Lemma 7.13. *Let G be a finitely generated group, Q an ordered group, $\Phi\colon G \twoheadrightarrow Q$ onto and let G carry the order induced by Φ. Then:*

1. *If $K \subseteq G$ is an antichain, then so is $\Phi(K)$. If K is maximal then $\Phi(K)$ is also maximal.*
2. *If $P \subseteq Q$ is an antichain, then so is $\Phi^{-1}(P)$. If P is maximal then $\Phi^{-1}(P)$ is also maximal.*
3. *For any maximal antichain $K \subseteq G$ containing 1 we have $\ker\Phi \subseteq K$.*

4. *If Q is totally ordered,* $\ker\Phi$ *is an antichain and every other antichain that contains* 1 *is contained in* $\ker\Phi$*. In particular,* $\ker\Phi$ *is the only maximal antichain subgroup. It is maximal even among all antichains.*

Proof.

1. As Φ induces $\prec$, if $\Phi(k) \prec \Phi(k')$ then $k \prec k'$, so $\Phi(K)$ is an antichain.

   Now suppose there is a $q \in Q \setminus \Phi(Q)$ such that q is incomparable to every $k \in \Phi(K)$. Then any preimage q_0 of q is incomparable to any element of K. This contradicts the maximality of K, so $\Phi(K)$ is also maximal.

2. Suppose there are comparable $g, h \in \Phi^{-1}(P)$. Then $\Phi(g), \Phi(h) \in P$ are also comparable but P is an antichain. Hence, $\Phi^{-1}(P)$ must be an antichain.

   Suppose there is a larger antichain $\Phi^{-1}(P) \subsetneq H \subseteq G$. Then $P \subsetneq \Phi(H)$ also is an antichain by (1). So P cannot be maximal.

3. By (1), $\Phi(K)$ is a maximal antichain in Q that contains 1. (2) tells us that then $\Phi^{-1}(\Phi(K))$ is an antichain containing K. As K is maximal, this means
$$K = \Phi^{-1}(\Phi(K)) \supseteq \Phi^{-1}(1) = \ker\Phi.$$

4. If Q is totally ordered, the only maximal antichain containing 1 is $\{1\}$. By (2), $\ker\Phi$ is a maximal antichain and by (1) there cannot be any other that contains 1. Hence, $\ker\Phi$ contains all other antichains that contain 1.

□

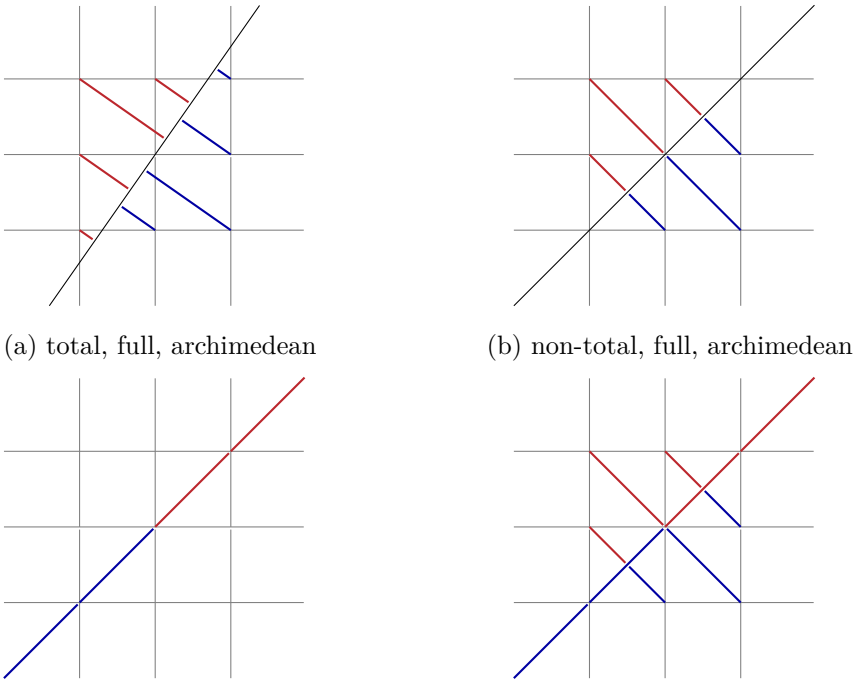
We have already seen the *lexicographic order* several times. Actually, most orders we have seen so far may be seen as lexicographic in some way, so let us make the notion precise.

Definition 7.14. Let
$$0 \to H \overset{\iota}{\hookrightarrow} G \overset{\pi}{\twoheadrightarrow} Q \to 0$$
be an exact sequence of groups and suppose that H and Q are ordered by $\prec_H$ and $\prec_Q$ respectively. Further assume that ι induces an order on G.

Then the *lexicographic order* with respect to this sequence is the order $\prec_G$ such that
$$1 \prec_G g \iff g \in \pi^{-1}(Q^{\succ}) \text{ or } g \in G \cdot \iota(H^{\succ})$$

If both $\prec_H$ and $\prec_Q$ are non-trivial, then $\prec_G$ is *properly lexicographic.*

To see that the lexicographic order is well-defined, we have to check that the positive cone satisfies the conditions of Lemma 7.6. That is, that $\pi^{-1}(Q^{\succ}) \cup G \cdot \iota(H^{\succ})$ is closed under multiplication with itself and under conjugation with elements of G. It is closed under conjugation because both $\pi^{-1}(Q^{\succ})$ and $G \cdot \iota(H^{\succ})$ are. Also, both sets are multiplicatively closed. So we only need to check that for $g \in \pi^{-1}(Q^{\succ})$, $h \in G \cdot \iota(H^{\succ})$, we have $1 \prec_G gh$ and $1 \prec_G hg$. Indeed, as $h \in G \cdot \iota(H^{\succ}) \subseteq \ker\pi$,
$$\pi(gh) = \pi(g) = \pi(hg)$$
and hence $gh, hg \in \pi^{-1}(Q^{\succ})$.



(a) total, full, archimedean

(b) non-total, full, archimedean

(c) non-total, non-full, archimedean

(d) total, full, non-archimedean

Figure 7.1.: Orders on $\mathbb{Z}^2$.

Remark 7.15. For a lexicographic order with respect to some sequence $H \hookrightarrow G \twoheadrightarrow Q$, note that any two given elements of G either have different images in Q or they lie in the same H-coset. Hence, to compare those elements, we first try comparing their images in Q. If they have the same image, we instead compare them using the order induced by H. In case G is a semidirect product $G = H \ltimes Q$, this means comparing the Q factor and then the H factor.

The lexicographic order has both $\prec_Q^{\pi^{-1}}$ and $\prec_H^{\iota}$ as suborders. In fact, it is the smallest such order. So one may think about it as "induced by π and ι together".

7.3. Full archimedean orders

With these tools at hand, we can construct a multitude of orders. Given a character $\Phi \in S(G)$, we can already construct the induced order on G. Now, we study how to reconstruct the inducing character from a given order. Also, we will see how we can decide if such a character even exists.

Let us start with an example. In Figure 7.1 we see some orders on $\mathbb{Z}^2$. For now, we only look at the pictures. The properties *full* and *archimedean* from the captions will be defined in Definition 7.18 and Definition 7.16.

(a) is an order induced by an injective map to $\mathbb{R}$. Such a map corresponds to a line at an irrational slope. Elements are ordered by their oriented distance to that line. This distance is unique for every element, so the order is total.

Similarly for (b). But here, the inducing map has image isomorphic to $\mathbb{Z}$, so the line is at a rational slope. Points on the line are elements of the kernel of the inducing map. Distances are no longer unique. For example, all points on the line have distance 0 from it. Two elements at the same distance are incomparable.

(c) is an order that is induced by the inclusion of $\mathbb{Z}$ into $\mathbb{Z}^2$. All elements comparable to 1 lie in some copy of $\mathbb{Z}$ embedded into $\mathbb{Z}^2$. In the picture, this is the coloured diagonal line. Comparability divides $\mathbb{Z}^2$ into equivalence classes that correspond to parallels of the coloured lines.

Finally, (d) is a lexicographic order corresponding to the sequence $\mathbb{Z} \hookrightarrow \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ where both factors are non-trivially ordered. Note that this picture may be obtained by overlaying (b) with (c). This corresponds to the fact that we may use either copy of $\mathbb{Z}$ in the above sequence to compare elements in the lexicographic order.

In fact, each of (b), (c) and (d) may be seen as lexicographic with respect to the sequence $\mathbb{Z} \hookrightarrow \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$. In (b), the left factor is ordered trivially. In (c), the right factor is ordered trivially. In (d), neither factor is ordered trivially. Thus, (d) are the only properly lexicographic orders.

In Example 7.20, we will see how every non-trivial order on $\mathbb{Z}^2$ falls into exactly one of these four categories.

Recall that our goal is to translate a statement about characters into a statement about orders. Thus, we need some way to recognise orders induced by characters. Note that only (a) and (b) are induced by maps to $\mathbb{R}$. Hence, we must distinguish these orders from (c) and (d).

Definition 7.16. Let G be an ordered group and $g, h \in G$. We say that g is *infinitesimal with respect to* h, if $g^i \prec h$ for all $i \in \mathbb{Z}$. In this case, we write $g \lll h$.

An order is called *archimedean* if it does not admit any positive infinitesimal elements.

Remark 7.17.

1. The archimedean property is well-established for totally ordered groups. However, there are minor differences between the definitions of archimedean for partially ordered groups in different sources. The definition above is the one from [KM96].

2. The neutral element 1 is infinitesimal with respect to any positive element.

3. If $\prec_G$ is a lexicographic order with respect to some orders $\prec_Q$ and $\prec_H$, then $\prec_G$ is archimedean if and only if at least one of $\prec_Q$ and $\prec_H$ is trivial and the other one is archimedean. In particular, an induced order is archimedean if and only if the inducing order is. Moreover, a properly lexicographic order is non-archimedean.

4. It is a fact due to Hölder [Höl01] that every totally ordered archimedean group is a subgroup of $\mathbb{R}$. An account in English may be found in [KM96]. This is not the case for partially ordered groups, as we will see in Example 8.1.

While the archimedean property distinguishes properly lexicographic orders from the others, it is not sufficient to recognise orders induced by characters. Thus, we introduce the notion of *full* orders.

Definition 7.18. Let $(G, \prec)$ be an ordered group.

1. The order $\prec$ is called *primitive*, if for every $g, h \in G$ and every $n \in \mathbb{N}$ we have $g^n \prec h^n \Rightarrow g \prec h$.

2. $\prec$ is called *factorising* if for any antichain normal subgroup $H \trianglelefteqslant G$, $\prec$ is induced by the projection $G \twoheadrightarrow G/H$ for some order on G/H.

3. $\prec$ is called *full* if it has both above properties.

Remark 7.19. Any $1 \prec g \in G$ spans a totally ordered subgroup $\langle g \rangle \leqslant G$. As g cannot be a torsion element, this subgroup is isomorphic to $\mathbb{Z}$. If $\prec$ is primitive, then any other cyclic subgroup of G containing $\langle g \rangle$ is also totally ordered.

To familiarise ourselves with the notion, we have a look at full archimedean orders on abelian groups. For the following example, we don't provide a proof. But it will be a corollary of Theorem 7.23.

Example 7.20.

1. On $\mathbb{Z} = \langle X \rangle$, there are exactly two non-trivial primitive orders: The standard order, where $X \succ 1$ and the opposite order, where we interchange the meanings of "positive" and "negative". That is the unique primitive order where $X \prec 1$. The isomorphism
$$(\mathbb{Z}, \prec) \to (\mathbb{Z}, \succ), \quad X \mapsto X^{-1}$$
is order-inducing in either direction.

   If we do not require our order to be primitive, any submonoid that does not contain any two elements inverse to each other defines an order on $\mathbb{Z}$. This includes for example the monoids $2\mathbb{N} \subseteq \mathbb{Z}$ and $\{1, X^k \mid k \geqslant 2\} \subseteq \mathbb{Z}$.

   For primitive orders, the condition that the order is factorising is void on $\mathbb{Z}$ because any proper quotient $\mathbb{Z}/N$ is finite and hence necessarily trivially ordered. If a non-trivial subgroup $N \trianglelefteqslant \mathbb{Z}$ is an antichain, then all of $\mathbb{Z}$ is trivially ordered. The trivial order is induced by any map to a trivially ordered group.

2. On $\mathbb{Z}^n$, any primitive order is total or lexicographic. Thus, we may construct all non-total primitive orders on a finitely generated abelian group by writing it as an extension $N \hookrightarrow G \twoheadrightarrow Q$. As N and Q are also abelian groups and of lower rank than G, we can construct their orders by inductively using the same construction.

   The resulting order on G is archimedean if among all the steps of the construction, there was at most one factor N or Q that was non-trivially ordered. It is factorising and hence full if at every step of constructing a lexicographic order, the following holds: If N is trivially ordered, then so is Q.

3. To make the previous example more concrete, consider the case $n = 2$ and recall Figure 7.1. When writing $\mathbb{Z}^2$ as a product of abelian groups, we can either use just one factor $\mathbb{Z}^2$ and order that totally. In this case, we obtain an order like (a). Or we decompose it into two factors $A \hookrightarrow \mathbb{Z}^2 \twoheadrightarrow B$, where both A and B are isomorphic to $\mathbb{Z}$. Then we can order A trivially and B totally as in (b), the other way round as in (c), or, if we order both factors totally, we get (d).

   Note that an order on $\mathbb{Z}^2$ is induced by a map to $\mathbb{R}$ if and only if it is full and archimedean. Also note that totality distinguishes (a) and (d) from (b) and (c). Hence, this is not a helpful criterion for finding orders that are induced by characters. This is why we consider partial orders in the first place.

   The fact that every primitive order is full or archimedean is due to $n = 2$ being too small. $\mathbb{Z}^3$ admits an order that is primitive but neither full nor archimedean. Namely the lexicographic order with respect to $\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \twoheadrightarrow \mathbb{Z}^2$, where $\mathbb{Z}$ is trivially ordered and $\mathbb{Z}^2$ carries a total non-archimedean order. The same order can be realised as a lexicographic order with respect to $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3 \twoheadrightarrow \mathbb{Z}$.

Remark 7.21. If $\prec_G$ is induced by $G \twoheadrightarrow (Q, \prec_Q)$ and $\prec_Q$ is full, then so is $\prec_G$. But if $\prec_G$ is induced by $(H, \prec_H) \hookrightarrow G$ and $\prec_H$ is full, then $\prec_G$ need not be full. For example, if $2\mathbb{Z}$ is taken to be ordered by the standard order, the order on $\mathbb{Z}$ induced by $2\mathbb{Z} \hookrightarrow \mathbb{Z}$ is the order with $\mathbb{Z}^{\succ} = 2\mathbb{N}$, which is not full as we have seen in Example 7.20.

While being full is not true for every order induced by a full order, the archimedean property does not have this issue.

Lemma 7.22. *Let $(G, \prec)$ be an ordered group.*

1. *Suppose that $\prec$ is induced by a map $\Phi\colon G \twoheadrightarrow Q$. If Q is archimedean then so is $\prec$.*
2. *Suppose that $\prec$ is induced by a map $H \hookrightarrow G$. If H is archimedean, then so is $\prec$.*

Proof. For the first part let $g, g' \in G$ such that $g^k \prec g'$ for every $k \in \mathbb{Z}$. Then

$$\Phi(g)^k \prec \Phi(g') \in Q.$$

If Q is archimedean, this means that $\Phi(g) \notin Q^{\succ}$. By Lemma 7.9, $G^{\succ} = \Phi^{-1}(Q^{\succ})$ and as $g \in \Phi^{-1}(\Phi(g))$, g cannot be positive in G.

The proof of the second part is analogous. □

To wrap up, we classify full archimedean orders on finitely generated abelian groups. This is also a proof for Example 7.20.

Theorem 7.23. *Let $(G, \prec)$ be a finitely generated abelian group and $\prec$ a full archimedean order. Then*

1. *G contains a unique maximal antichain subgroup H,*
2. *G/H is totally ordered,*

3. *and $\prec$ is induced by a map $\Phi\colon G \to \mathbb{R}$, where $\mathbb{R}$ carries the standard order.*

Proof.

1. We may write $G = F \oplus T$ where F is a free-abelian group and T is the torsion part. As T is an antichain by Remark 7.5, and $\prec$ is full, $\prec$ is induced by the projection $G \twoheadrightarrow F$. Thus, we may assume without loss of generality that $G = \mathbb{Z}^n$ for some $n \in \mathbb{N}_0$.

   We proceed by induction on n. For $n = 0$, the only subgroup is the trivial group, and it is indeed an antichain.

   Otherwise, if G is totally ordered, the trivial group is the only antichain and hence also maximal.

   If G is not totally ordered, any element incomparable to 1 generates an antichain subgroup $C \leqslant G$ with $C \cong \mathbb{Z}$. In this case, $\prec$ is induced by the projection onto $G/C \cong \mathbb{Z}^{n-1} \oplus T'$. But the torsion part T' is again trivially ordered so $\prec$ is actually induced by the projection $G \twoheadrightarrow \mathbb{Z}^{n-1}$.

   By induction, $\mathbb{Z}^{n-1}$ contains a unique maximal antichain subgroup H'. Its preimage under the projection is by Lemma 7.13 the unique maximal antichain subgroup of G.

2. Let $g \in G/H$ be incomparable to 1 and let g_0 be a preimage of g under the projection $G \twoheadrightarrow G/H$. Then g_0 is also incomparable to 1. As H is the only maximal antichain subgroup, $g_0 \in H$. But then, $g = 1$ so G/H is totally ordered.

3. In Remark 7.17, we mentioned that every totally ordered archimedean group is a subgroup of $\mathbb{R}$. In other words, every order on such a group is induced by the inclusion of said group into $\mathbb{R}$.

   We know that $\prec$ is induced by an order on G/H and G/H is a totally ordered abelian group. It is also archimedean as otherwise suppose there are $1 \prec g \ll g' \in G/H$. But then if g_0, g_0' are some preimages of these elements in G, we have $1 \prec g_0 \ll g_0'$. But this contradicts G being archimedean. By Lemma 7.12, $\prec$ is induced by a map $G \twoheadrightarrow G/H \hookrightarrow \mathbb{R}$.

□

7.4. Order Σ-invariants

Now we know that we can identify a character $\Phi \in S(G)$ with the full archimedean order $\prec^{\Phi^{-1}}$ that it induces on G. We have seen that if G is abelian, this relation is one-to-one. That is, every full archimedean order is induced by a unique character on G. Thus, if $N \trianglelefteqslant G$ is a normal subgroup such that G/N is abelian, we get an identification of full archimedean orders on G such that N is an antichain with characters on G.

Using this identification, we get a reformulation of Theorem 6.18 as follows.

Theorem 7.24. *Let G be a finitely generated group and $N \trianglelefteq G$ such that G/N is abelian. Then N is finitely generated if and only if $N_{\succcurlyeq}$ is connected for every full archimedean order $\prec$ on G such that N is an antichain.*

Proof. Suppose that N is finitely generated. If $\prec$ is a full archimedean order on G, then by Theorem 7.23, there exists a unique maximal antichain subgroup $H \trianglelefteq G$. By the same theorem, $\prec$ is induced by some character $\Phi\colon G \to \mathbb{R}$ and since H is an antichain, this character factors through G/H. As H is maximal, it contains N and hence $\Phi \in S(G, N)$. By Theorem 6.18, $\Phi \in \Sigma^1(G)$. Thus $\Phi^{-1}([0, \infty)) = H_{\succcurlyeq}$ is connected. One can use G-invariance of $\prec$ to show that $N_{\succcurlyeq}$ is then also connected. We omit this step here, but it will be covered by the proof of Theorem 8.11.

Conversely, every full archimedean order such that N is an antichain is induced by a character $\Phi \in S(G, N)$. Again, one can show that if $N_{\succcurlyeq}$ is connected then so is $H_{\succcurlyeq}$ and hence $\Phi \in \Sigma^1(G)$. As every character also induces a full archimedean order, this shows that $S(G, N) \subseteq \Sigma^1(G)$ and hence N is finitely generated by Theorem 6.18. □

In this section, we transfer the notions of Σ-invariant and character sphere to orders.

Definition 7.25. Let G be a group. Then the *order Σ-invariant* $\Sigma^1_{\mathrm{ord}}(G)$ is the subset of the set of non-trivial full archimedean orders on G defined as follows.

Let $\prec$ be a non-trivial full archimedean order on G. Then $\prec \in \Sigma^1_{\mathrm{ord}}(G)$ if and only if for every antichain normal subgroup $K \trianglelefteq G$ that is maximal among antichain normal subgroups, $K_{\succcurlyeq}$ is coarsely connected.

Remark 7.26. As opposed to the definition of $\Sigma^1(G)$ in Definition 6.11, this definition of $\Sigma^1_{\mathrm{ord}}(G)$ is not independent of the choice of generating set, as we will see in Example 9.13. Thus, we must ask that $K_{\succcurlyeq}$ is coarsely connected instead of just connected.

On the other hand, we can state this definition even if G is not finitely generated.

To understand how this definition generalises Definition 6.11, we note that they align if we restrict to orders induced by characters and then identify a character with the order it induces on G.

Lemma 7.27. *Let G be a finitely generated group and $\Phi\colon G \to \mathbb{R}$ a character. Then*

$$\Phi \in \Sigma^1(G) \iff \prec^{\Phi^{-1}} \in \Sigma^1_{\mathrm{ord}}(G),$$

where $\prec$ is the standard order on $\mathbb{R}$.

Proof. Take G to be ordered by $\prec^{\Phi^{-1}}$. Then $\ker \Phi$ is the only maximal antichain subgroup of G by Lemma 7.13. Hence $\prec^{\Phi^{-1}} \in \Sigma^1_{\mathrm{ord}}(G)$ if and only if $(\ker \Phi)_{\succcurlyeq}$ is coarsely connected. The latter is equivalent to $(\ker \Phi)_{\succcurlyeq}$ being connected for some finite generating set, which is the definition of $\Phi \in \Sigma^1(G)$. □

The other missing ingredient of Theorem 6.18 is the relative character sphere.

Definition 7.28. Let G be a group, N a normal subgroup and $\pi\colon G \twoheadrightarrow G/N$ the projection map. Then define the *relative order sphere* as

$$S_{\mathrm{ord}}(G,N) \coloneqq \{\prec^{\pi^{-1}} \mid \prec \text{ a non-trivial full archimedean order on } G/N\}.$$

Recall that in case G/N is abelian, Theorem 7.23 tells us that every full archimedean order on G/N is induced by a map to $\mathbb{R}$. $S_{\mathrm{ord}}(G,N)$ contains all orders induced by maps to G/N. As G/N is abelian, every order on G/N is induced by a map to $\mathbb{R}$. That is, $S_{\mathrm{ord}}(G,N)$ contains precisely those orders induced by maps $G \to G/N \to \mathbb{R}$, which are by definition exactly the maps in $S(G,N)$. So this is a generalisation of $S(G,N)$ for G/N abelian in the same sense as $\Sigma^1_{\mathrm{ord}}(G)$ is for $\Sigma^1(G)$. If G/N is abelian, then the identification of a character with its induced order is an isomorphism

$$S(G,N) \to S_{\mathrm{ord}}(G,N).$$

Note that if G itself is non-abelian, we do not get an identification of the character sphere $S(G)$ with the set of orders on G. This is because there might be orders on G that are not induced by a map onto any abelian group, but every character does factor through some abelian group.

The following is an often useful description of the relative order sphere.

Lemma 7.29. *For a group G and $N \trianglelefteq G$ a normal subgroup, $S_{\mathrm{ord}}(G,N)$ is the set*

$$\{\prec \mid \prec \textit{a non-trivial full archimedean order on } G \textit{ such that } N \textit{ is an antichain}\}.$$

Proof. By definition of $\prec^{\pi^{-1}}$, for every order in $S_{\mathrm{ord}}(G,N)$, N is an antichain.

Now let $\prec$ be a full order such that N is an antichain. Then it is induced by the projection $G \twoheadrightarrow G/N$ and hence $\prec \in S_{\mathrm{ord}}(G,N)$. □

In Lemma 7.27, we have seen that $\Sigma^1_{\mathrm{ord}}(G)$ contains every order induced by a character in $\Sigma^1(G)$. But there may still be full archimedean orders on G that are not induced by any character. However, if G/N is abelian, the statements $S(G,N) \subseteq \Sigma^1(G)$ and $S_{\mathrm{ord}}(G,N) \subseteq \Sigma^1_{\mathrm{ord}}(G)$ are equivalent because in this case $S_{\mathrm{ord}}(G,N)$ contains precisely those orders induced by characters that factor through N and Lemma 7.27 provides that those orders are contained in $\Sigma^1_{\mathrm{ord}}(G)$.

We may use the new terminology to obtain a more concise formulation of Theorem 7.24.

Theorem 7.30. *Let G be a finitely generated group and $N \trianglelefteq G$ a normal subgroup such that G/N is abelian. Then N is finitely generated if and only if $S_{\mathrm{ord}}(G,N) \subseteq \Sigma^1_{\mathrm{ord}}(G)$.*

Proof. For a full archimedean order $\prec$ on G, the normal subgroup N is an antichain if and only if $\prec \in S_{\mathrm{ord}}(G,N)$. Furthermore, $N_{\succcurlyeq}$ is coarsely connected if and only if $\prec \in \Sigma^1_{\mathrm{ord}}(G)$. In case G/N is abelian, $N_{\succcurlyeq}$ is coarsely connected if and only if it is connected as follows from Remark 6.13.

Applying these equivalences to the claim yields Theorem 7.24. □

8. Order Σ-invariants for nilpotent quotients

Let G be a finitely generated group and $N \trianglelefteqslant G$ a normal subgroup. In Theorem 7.30, we have established a connection between finite generatedness of N, the order Σ-invariant $\Sigma^1_{\mathrm{ord}}(G)$ and the relative order sphere $S_{\mathrm{ord}}(G, N)$. However, we only know that the theorem holds if G/N is abelian. It is not clear that it becomes false if we drop that assumption. In this chapter, we prove Theorem 8.11, which is an analogue of Theorem 7.30 in the case where G/N is nilpotent. The primary tool we use is the transfer between characters and full archimedean orders we have established in Chapter 7. We restrict to nilpotent groups because for them, we can classify the full archimedean orders.

8.1. Classification of orders

Theorem 7.30 asks if $S_{\mathrm{ord}}(G, N) \subseteq \Sigma^1_{\mathrm{ord}}(G)$. To have any chance of proving it in a more general setting, we need to know what $S_{\mathrm{ord}}(G, N)$ looks like. By definition, $S_{\mathrm{ord}}(G, N)$ is in 1-to-1 correspondence with the non-trivial full archimedean orders on G/N. As we will focus on the case where G/N is nilpotent, our actual objective is to understand orders on nilpotent groups. Furthermore, the torsion-elements of a nilpotent group form a normal subgroup, so an order on G/N is induced by an order on $G/N/T$ where T is the torsion part of G/N.

As Theorem 7.30 requires G to be finitely generated and any order the theorem asks about is full and archimedean, we allow ourselves to make these assumptions whenever convenient. At the end of this section, we will prove Theorem 8.6. It provides a way to construct all full archimedean orders on a nilpotent group G. Let us start by looking at some examples.

Example 8.1.

1. Recall Example 7.20 constructing all full orders on finitely generated free-abelian groups.

2. We construct all full orders on the Heisenberg group. Recall from Example 2.6 that the Heisenberg group has the presentation

$$H = \left\langle a, b \;\middle|\; 1 = [a, [a, b]] = [b, [a, b]] \right\rangle$$

and that any element $g \in H$ can be written uniquely as $g = a^\alpha b^\beta [a, b]^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$.

Any order on

$$\mathbb{Z}^2 = H_{\mathrm{ab}} = H/\langle [a,b] \rangle$$

induces an order on H via the projection map. Even more, for any full order on H, the projection modulo $[a,b]$ is order-preserving. Thus, any full order is lexicographic with respect to the exact sequence

$$\langle [a,b] \rangle \hookrightarrow H \twoheadrightarrow H_{\mathrm{ab}}.$$

To obtain an archimedean order, at least one of the two factors has to be ordered trivially by Remark 7.17.

So any full archimedean order on H is either induced by the projection $H \twoheadrightarrow H_{\mathrm{ab}}$ or by the inclusion $\langle [a,b] \rangle \hookrightarrow H$. Orders induced by the projection are in one-to-one correspondence with orders on $\mathbb{Z}^2$. As $\langle [a,b] \rangle \cong \mathbb{Z}$, there are two non-trivial full orders on that subgroup. They induce the orders

$$a^\alpha b^\beta [a,b]^\gamma \succ 1 \iff \alpha = \beta = 0 \text{ and } \gamma > 0$$

and

$$a^\alpha b^\beta [a,b]^\gamma \succ 1 \iff \alpha = \beta = 0 \text{ and } \gamma < 0$$

on H.

Recall Example 2.6. There, we had a notion of relative height for paths in $\mathrm{Cay}(\mathbb{Z}^2)$, as depicted in Figure 2.1.

The orders induced by the inclusion $\langle [a,b] \rangle \hookrightarrow H$ make precise what we mean by the relative height. Two paths have a relative height precisely if they have the same endpoint in $\mathbb{Z}^2$. Analogously, with the two "special" orders above, two elements of H are comparable if they project onto the same element of $\mathbb{Z}^2 = H/\langle [a,b] \rangle$. That is if they differ by some element of $\langle [a,b] \rangle$. The relative height of two such paths is the difference in the powers of $[a,b]$ in the normal forms of the corresponding elements of H. In particular, which path is higher than the other is entirely determined by whether that difference is positive or negative.

3. Now let G be the free-nilpotent group of class 2 and rank 3. That is

$$G = \left\langle a,b,c \;\middle|\; [x,[y,z]] = 1 \; \forall x,y,z \in \{a,b,c\} \right\rangle$$

It contains the Heisenberg group H as the subgroup generated by $\{a,b\}$. Any order on G therefore restricts to an order on H. If $\prec_{|H}$ is induced by an order on H_{ab}, then $\prec$ is induced by an order on G_{ab}. If $\prec_{|H}$ is one of the two archimedean orders such that $[a,b]$ and 1 are comparable, then $\prec$ is induced by one of the inclusions

$$\mathbb{Z}^2 = \langle [a,b], c \rangle_{\mathrm{ab}} \hookrightarrow G$$

or

$$\mathbb{Z}^3 \cong \langle [a,b], [a,c], [b,c] \rangle_{\mathrm{ab}} = G^{(1)} \hookrightarrow G.$$

Recall that $G^{(1)}$ denotes the first term of the lower central series.

Up to the choice of embedding of H and hence isomorphism of G, these are all archimedean orders.

For now, we omit the proof. We will review the example at the end of this section after we give the characterisation of full archimedean orders in Theorem 8.6. That theorem is the main tool to verify that the above examples are indeed correct.

The examples show that the orders are largely determined by orders on G_{ab}. For instance, in the third example, if there is more than a single generator comparable to 1, all of $G^{(1)}$ is necessarily trivially ordered. This leads to an intuition saying the more elements of $G \setminus G^{(1)}$ are comparable to 1, the fewer possibilities there are to extend an order on $G \setminus G^{(1)}$ to an order on G. With these ideas in mind, we aim to make precise what orders on nilpotent groups look like. The case of nilpotent groups of class 1, that is, abelian groups, has already been dealt with in Theorem 7.23.

Remark 8.2. Let G be a finitely generated free-abelian group and $\Phi\colon G \to \mathbb{R}$. Then $\ker\Phi$ is also finitely generated free-abelian. And so is $G/\ker\Phi$, as Φ maps any torsion element of the quotient to some torsion element of $\mathbb{R}$, but $\mathbb{R}$ is torsion-free.

Hence, $\ker\Phi \oplus G/\ker\Phi$ is also finitely generated free-abelian. By counting dimensions, we see that it is even isomorphic to G.

Take $\mathbb{R}$ to be ordered by the standard order. As Φ factors through $G/\ker\Phi$, Φ induces a total order on $G/\ker\Phi$ and the projection map then induces an order on G. This is the same order that Φ induces on G.

Any isomorphism $G \cong \ker\Phi \oplus G/\ker\Phi$ produces a free-abelian generating set of G from such sets for $\ker\Phi$ and $G/\ker\Phi$. That is, for any full archimedean order on G, there are free generators such that

1. every generator is either positive or incomparable to 1,
2. and the positive cone is a subset of the subgroup spanned by the positive generators.

This is not true for arbitrary nilpotent groups. For example, in the Heisenberg group, none of the generators may be positive, but their commutator is, as we have seen in Example 8.1.

Let us now consider the case where G is a non-abelian nilpotent group. Any order on the abelianisation G_{ab} induces an order on G, and these orders we already understand by Theorem 7.23. They are the ones such that $G^{(1)}$ is an antichain. For all other orders, the following lemma restricts the order on $G^{(1)}$.

Lemma 8.3. *Let G be a finitely generated nilpotent group carrying a full order and $g \in G^{\succ}, h \in G$.*

Then $[g,h] \lll g$.

Proof. Let $N = \langle\!\langle g \rangle\!\rangle \trianglelefteq G$ be the normal subgroup generated by g and n the nilpotency class of N. Note that $[g,h] = g^{-1}(h^{-1}gh) \in N$.

We show that for $1 \leqslant i$, every element of $N^{(i)}$ is infinitesimal with respect to g. We know that this is true for $i \geqslant n$, where n is the nilpotency class of N, because then $N^{(i)}$ is trivial and $1 \lessdot g$.

So let $1 \leqslant i < n$ and suppose that every element of $N^{(i+1)}$ is infinitesimal with respect to g. $N^{(i)}$ is generated by elements of the form $[f, g]$. It is enough to show that these generators are infinitesimal with respect to g, because then

$$([f_1, g][f_2, g])^k = [f_1, g]^k [f_2, g]^k c$$

for some c which is a product of commutators of powers of $[f_1, g]$ and $[f_2, g]$. In particular, $c \in N^{(i+1)}$. So

$$([f_1, g][f_2, g])^k = [f_1, g]^k [f_2, g]^k c \prec g^3$$

and hence $[f_1, g][f_2, g] \lessdot g^3$. As $\prec$ is full, this also means $[f_1, g][f_2, g] \lessdot g$.

It remains to show that $[f, g] \lessdot g$. We compute

$$g[f, g]^{-k} = f^{-k} g f^k c \succ f^{-k} f^k c = c,$$

again for some c consisting of commutators involving $[f, g]$ and thus $c \in N^{(i+1)}$ and in particular $c^{-1} \prec g$. Thus

$$[f, g]^k \prec c^{-1} g \prec g^2$$

and hence $[f, g] \lessdot g$. □

Remark 8.4. In the following proof, we will do an induction on all nilpotent groups. For a partially ordered set X admitting no infinite descending sequences, in order to prove some statement about every element $x \in X$, it is enough to prove that statement under the assumption that it is true for every $y \in X$ that is smaller than x in the sense of the order.

Definition 8.5. Let A, B be finitely generated nilpotent groups. For any $i \in \mathbb{N}$, $A^{(i)}/A^{(i+1)}$ is a finitely generated abelian group. Denote the rank of its free part by $\operatorname{rk} A^{(i)}/A^{(i+1)}$ and set $\operatorname{rk}^{(i)} A \coloneqq \operatorname{rk} A^{(i)}/A^{(i+1)}$.

We say that A *comes before* B if and only if $\operatorname{rk}^{(i)} A < \operatorname{rk}^{(i)} B$ for the largest i such that these ranks are not equal. Unless all the ranks are equal, such a largest i must exist since A being nilpotent means that $\operatorname{rk}^{(i)} A = 0$ for all but finitely many i, and the same is true for B. So, in particular, the ranks differ for only finitely many i. This defines a partial well-ordering on the class of finitely generated nilpotent groups.

We use the terminology *comes before* instead of *is smaller* here to avoid confusing the order on a given nilpotent group with the order on the class of nilpotent groups we defined above.

Theorem 8.6. *Let G be a finitely generated partially ordered nilpotent group with a full archimedean order $\prec$. Then there exists a normal subgroup $H \trianglelefteqslant G$ such that G/H is torsion free and $\prec$ is induced by an order on G/H, which itself is induced by a total order on the center $Z(G/H)$.*

That is, we may obtain any full archimedean order on G from the standard order on $\mathbb{R}$ via the following chain of maps.

$$\mathbb{R} \hookleftarrow \mathbb{Z}^n \cong Z(G/H) \hookrightarrow G/H \twoheadleftarrow G$$

Proof. As the torsion elements of G form a normal antichain subgroup T, and any order on G is induced by the projection $G \twoheadrightarrow G/T$, we may assume without loss of generality that G is torsion free.

We do an induction as outlined in Remark 8.4 by either proving the theorem directly, or reducing it to the same theorem on another group that comes before G in the sense of Definition 8.5.

Suppose first that $Z(G)$ is totally ordered. If $1 \prec g$ for some $g \notin Z(G)$, we may choose some $h \in G$ such that $[g,h] \in Z(G) \smallsetminus 1$. By Lemma 8.3, we get $[g,h] \ll g$. Since $1 \prec [g,h]$ or $1 \prec [g,h]^{-1}$, this contradicts $\prec$ being archimedean. Thus $G^{\succ} \subseteq Z(G)$. That is, $\prec$ is induced by the inclusion $Z(G) \hookrightarrow G$. So, by letting $H = \{1\}$, we see that the claim is true.

If $Z(G)$ is not totally ordered, pick a maximal cyclic subgroup $C \leqslant Z(G)$ that is an antichain. As a subgroup of the center, C is automatically normal in G. Since $\prec$ is full, it is induced by the projection $G \twoheadrightarrow G/C$.

Let k be the largest number such that $C \leqslant G^{(k)}$. Then we have

$$\mathrm{rk}^{(i)} G/C = \begin{cases} \mathrm{rk}^{(i)} G & \text{if } i \neq k \\ \mathrm{rk}^{(i)} G - 1 & \text{if } i = k. \end{cases}$$

Note that $\mathrm{rk}^{(i)} G/C \leqslant \mathrm{rk}^{(i)} G$ for any i and $\mathrm{rk}^{(k)} G/C < \mathrm{rk}^{(k)} G$. That is, G/C comes before G in the sense of Definition 8.5. Hence by induction, we may assume that there is some $H' \trianglelefteqslant G/C$ such that $\prec$ is induced by the projection $G \twoheadrightarrow G/C \twoheadrightarrow (G/C)/H'$ and the order on $(G/C)/H'$ is induced by a total order on its center.

By Lemma 7.12, $\prec$ is then induced by the projection $G \twoheadrightarrow (G/C)/H'$ and setting H to be the kernel of this projection finishes the proof. □

Conversely, every choice of $H \trianglelefteqslant G$ such that G/H is torsion-free and $\iota \colon Z(G/H) \hookrightarrow \mathbb{R}$ induces a unique order on G. Two choices H, ι and H', ι' yield the same order if and only if $H = H'$ and $\iota = \lambda \iota'$ for some $\lambda \in \mathbb{R}^+$:

Proposition 8.7. *Let G be a nilpotent group, $H, H' \trianglelefteqslant G$ normal subgroups and*

$$\iota \colon Z(G/H) \hookrightarrow \mathbb{R}, \quad \iota' \colon Z(G/H') \hookrightarrow \mathbb{R}.$$

The maps ι, ι' induce orders on the quotients G/H and G/H' which in turn induce orders on G via the projection maps. Call these orders $\prec$ and $\prec'$. Then $\prec = \prec'$ if and only if $H = H'$ and ι and ι' are representatives of the same character in $S(Z(G/H))$.

Proof. It is immediate that the induced orders are equal if the inducing maps are.

Now suppose that $\prec = \prec'$. Suppose that $H = H'$ but ι and ι' represent distinct characters. Then there exists some $g \in Z(G/H)$ such that $\iota(g) > 0$ and $\iota'(g) < 0$.

Hence, any preimage of g under the projection $G \twoheadrightarrow G/H$ is positive with respect to $\prec$ but negative with respect to $\prec'$. Thus, the two orders are not equal.

It remains to show that $H = H'$. Suppose they are not equal. Furthermore, suppose that H is not a subset of H', not losing generality if we allow swapping H and H'. Let $h \in H \smallsetminus H'$. Then the normal subgroup $\langle\!\langle h \rangle\!\rangle \trianglelefteqslant G$ generated by h is a subgroup of H and hence an antichain with respect to $\prec$. However, $\langle\!\langle h \rangle\!\rangle$ projects onto a non-trivial normal subgroup of G/H'. As G/H' is nilpotent, its center has non-trivial intersection with $\langle\!\langle h \rangle\!\rangle$. As $Z(G/H')$ is totally ordered with respect to $\prec'$, $\langle\!\langle h \rangle\!\rangle$ therefore contains some positive element and, in particular, is not an antichain with respect to $\prec'$, showing that $\prec$ and $\prec'$ are not the same order. □

Theorem 8.6 and Proposition 8.7 together provide a full characterisation of all full archimedean orders on finitely generated nilpotent groups.

Remark 8.8. Note that if G is abelian, $G/H = Z(G/H)$, so we get that $\prec$ is induced by

$$\mathbb{R} \hookleftarrow \mathbb{Z}^n \cong Z(G/H) \cong G/H \twoheadleftarrow G.$$

Hence, we recover that every full archimedean order on G is induced by a map to $\mathbb{R}$.

To conclude the section now is a good time to revisit Example 8.1. The center of the Heisenberg group is

$$Z(H) = \langle [a, b] \rangle \cong \mathbb{Z}.$$

Let $\prec$ be an order on H. By Theorem 8.6, we know that there is $P \trianglelefteqslant H$ such that $\prec$ is induced by a total order on $Z(H/P)$. If P is trivial, then $Z(H)$ is totally ordered by one of the two total orders on $\mathbb{Z}$. Spelling this out, we obtain one of the two orders of the form

$$a^\alpha b^\beta [a, b]^\gamma \succ 1 \iff \alpha = \beta = 0 \text{ and } \gamma \succ 0.$$

If P is non-trivial it contains $Z(H)$. For any full order $\prec$ on H that is induced by the projection $H \twoheadrightarrow H/P$, $Z(H) \trianglelefteqslant P$ is an antichain. Hence $\prec$ is induced by an order on $H/Z(H) = H_{\mathrm{ab}} \cong \mathbb{Z}^2$.

The case of the free-nilpotent group of class 2 and rank 3 may be handled similarly by looking at all possible intersections of P and $Z(G)$.

8.2. Finitely generated kernels

In this section, we prove Theorem 8.11, which is the generalisation of Theorem 7.30 to the case where G/N is nilpotent. As we know from Theorem 8.6 that all orders on nilpotent groups are induced by the inclusion of the center into some quotient, let us investigate how $S_{\mathrm{ord}}(G, N)$ and $\Sigma^1_{\mathrm{ord}}(G)$ behave when passing to subgroups.

Lemma 8.9. *Let G be a group, $H \leqslant G$ a subgroup and $\Phi\colon G \twoheadrightarrow Q$ a map onto some group Q such that* $\ker \Phi \trianglelefteqslant H$.

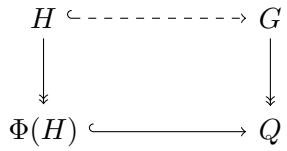


Figure 8.1.: $H \hookrightarrow G$ is order-inducing if all the other maps are.

Let $\prec_G \in S_{\mathrm{ord}}(G, \ker\Phi)$. It is induced by some order $\prec_Q$ on Q. Suppose that $\prec_Q$ is induced by the inclusion $\Phi(H) \hookrightarrow Q$. The restriction $\prec_{Q_{|\Phi(H)}}$ induces via Φ^{-1} an order $\prec_H$ on H and $\prec_H \in S_{\mathrm{ord}}(H, \ker\Phi)$.

Then $\prec_G$ is induced by $\prec_H$ via the inclusion $H \hookrightarrow G$. This statement is also visualized in Figure 8.1.

Proof. Recall Lemma 7.6 stating that an order is characterised entirely by its positive cone.

Let $1 \prec_G g \in G$. Then $1 \prec_Q \Phi(g) \in Q$. So $\Phi(g) \in \Phi(H)$. Hence there is some $h \in H$ such that $\Phi(h) = \Phi(g)$. That is, $gh^{-1} \in \ker\Phi \trianglelefteq H$, so $g = (gh^{-1})h \in H$. □

Lemma 8.10. *Let G be a group and Q a finitely generated nilpotent group. Let $\Phi\colon G \twoheadrightarrow Q$ be onto.*

Take Q to be ordered by $\prec_Q$ and G ordered by $\prec_Q^{\Phi^{-1}}$. Let P be the subgroup of Q such that $\prec_Q$ is induced by a total order on $Z(Q/P)$ as provided by Theorem 8.6. The projection $Q \twoheadrightarrow Q/P$ is called π and the order on Q/P that induces $\prec_Q$ is $\prec$. Set $\Psi \coloneqq \pi \circ \Phi$ and $H \coloneqq \Psi^{-1}(Z(Q/P))$. This situation is summed up in Figure 8.2.

Then

$$\prec^{\Psi^{-1}} \in \Sigma^1_{\mathrm{ord}}(G) \iff \prec_{|Z(Q/P)}{}^{\Psi^{-1}} \in \Sigma^1_{\mathrm{ord}}(H).$$

Proof. By Lemma 7.13, the only maximal antichain subgroup of H is $\ker\Psi$. Also, any maximal antichain normal subgroup of G gets mapped by Ψ to a maximal antichain normal subgroup of Q/P. Since Q/P is nilpotent, every nontrivial normal subgroup of Q/P has nontrivial intersection with $Z(Q/P)$. Thus, the only normal antichain subgroup of Q/P is the trivial group. Hence, the only maximal normal antichain subgroup of G is $\ker\Psi$.

The set $\ker\Psi_{\succcurlyeq} \subseteq G$ is the same set as $\ker\Psi_{\succcurlyeq_{|H}} \subseteq H$. Thus, if the full subgraph spanned by $\ker\Psi_{\succcurlyeq_{|H}} \subseteq \mathrm{Cay}(H, S)$ is connected for some $S \subseteq H$, then the full subgraph of $\mathrm{Cay}(G, S)$ spanned by $\ker\Psi_{\succcurlyeq}$ is also connected. The other direction uses the same argument after we note that we can always ask that

$$S \subseteq \ker\Psi_{\succcurlyeq} = \ker\Psi_{\succcurlyeq_{|H}} \subseteq H.$$

by Remark 6.9. □

Finally, we are ready to prove the result we were aiming for.

Theorem 8.11. *Let G be a finitely generated group, Q finitely generated nilpotent and $\Phi\colon G \twoheadrightarrow Q$ onto. The following are equivalent.*

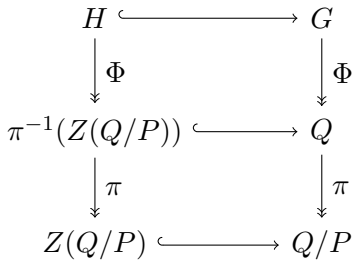


Figure 8.2.: If we start with an ordered nilpotent group Q and a map Φ, we get induced orders on all the groups in this diagram.

1. $\ker\Phi$ *is finitely generated*
2. $S_{\mathrm{ord}}(G, \ker\Phi) \subseteq \Sigma^1_{\mathrm{ord}}(G)$
3. $S_{\mathrm{ord}}(\bar{Z}, \ker\Phi) \subseteq \Sigma^1_{\mathrm{ord}}(\bar{Z})$

where $\bar{Z} \coloneqq \Phi^{-1}(Z(Q))$.

Proof. (1) $\Rightarrow$ (2): Let $\prec \in S_{\mathrm{ord}}(G, \ker\Phi)$. By definition, $\prec$ is induced by a unique order on Q. Theorem 8.6 tells us that there is a normal subgroup $P \trianglelefteqslant Q$ such that the order on Q is induced by a total order on $Z(Q/P)$.

Let π be the projection map $Q \twoheadrightarrow Q/P$ and set

$$H \coloneqq (\pi \circ \Phi)^{-1}(Z(Q/P)) \subseteq G.$$

We know that

$$g \succ 1 \iff \Phi(g)P \succ 1 \in Z(Q/P),$$

so $\prec_{|H} \in \Sigma^1_{\mathrm{ord}}(H)$ implies $\prec \in \Sigma^1_{\mathrm{ord}}(G)$ by Lemma 8.10. Hence, it is enough to show that $\prec_{|H} \in \Sigma^1_{\mathrm{ord}}(H)$.

$(\pi \circ \Phi)_{|H} \colon H \twoheadrightarrow Z(Q/P)$ is a surjective map onto an abelian group. So by Theorem 6.18, if its kernel is finitely generated, then $\prec_{|H} \in \Sigma^1_{\mathrm{ord}}(H)$. By construction, $\ker(\pi \circ \Phi) \trianglelefteqslant H$, so $\ker(\pi \circ \Phi) = \ker(\pi \circ \Phi)_{|H}$.

Note that we may write $\ker(\pi \circ \Phi)$ as an extension

$$\ker\Phi \hookrightarrow \ker(\pi \circ \Phi) \twoheadrightarrow \ker\pi.$$

Using different names for the same groups, we obtain the extension

$$\ker\Phi \hookrightarrow \ker(\pi \circ \Phi)_{|H} \twoheadrightarrow P.$$

By assumption, $\ker\Phi$ was finitely generated. As P is a subgroup of a finitely generated nilpotent group and hence itself finitely generated, $\ker(\pi \circ \Phi)_{|H}$ is also finitely generated. Hence $\prec_{|H} \in \Sigma^1_{\mathrm{ord}}(H)$ and $\prec \in \Sigma^1_{\mathrm{ord}}(G)$.

(2) $\Rightarrow$ (3): Let $\prec \in S_{\mathrm{ord}}(\bar{Z}, \ker\Phi)$. Then $\prec$ is induced by an order on $\bar{Z}/\ker\Phi \cong Z(Q)$. This order induces an order on Q, and that order induces an order $\prec'$ on G.

$\prec'$ is hence induced by the inclusion $\bar{Z} \hookrightarrow G$ by Lemma 8.9 and

$$\prec'_{|\bar{Z}} = \prec.$$

Thus if $\prec' \in \Sigma^1_{\mathrm{ord}}(G)$, then $\prec \in \Sigma^1_{\mathrm{ord}}(\bar{Z})$ by Lemma 8.10. As $\ker\Phi$ is an antichain with respect to $\prec$ and $\ker\Phi \trianglelefteq \bar{Z}$, the kernel is also an antichain with respect to $\prec'$. Hence

$$\prec' \in S_{\mathrm{ord}}(G, \ker\Phi) \subseteq \Sigma^1_{\mathrm{ord}}(G)$$

and therefore $\prec \in \Sigma^1_{\mathrm{ord}}(\bar{Z})$.

(3) $\Rightarrow$ (1): Let H be a finitely generated subgroup of $\bar{Z}$ containing $\ker\Phi$. If no such H exists, then $(\ker\Phi)_{\succcurlyeq}$ is not coarsely connected for any order on $\bar{Z}$. That is no order in $S_{\mathrm{ord}}(\bar{Z}, \ker\Phi)$ is contained in $\Sigma^1_{\mathrm{ord}}(\bar{Z})$. As the former contains one element for every order on $\bar{Z}/\ker\Phi = Z(Q)$ and $Z(Q)$ is a nontrivial torsion-free abelian group, the relative order sphere is in, particular, non-empty. Hence if (3) is true, H must exist.

We have $S_{\mathrm{ord}}(H, \ker\Phi) \subseteq S_{\mathrm{ord}}(\bar{Z}, \ker\Phi) \subseteq \Sigma^1_{\mathrm{ord}}(\bar{Z})$. By Lemma 8.10, this also means $S_{\mathrm{ord}}(H, \ker\Phi) \subseteq \Sigma^1_{\mathrm{ord}}(H)$. As $H/\ker\Phi$ is a subgroup of $Z(Q)$ and hence abelian, Theorem 6.18 shows that $\ker\Phi$ is finitely generated, finishing the proof.

□

To obtain the same formulation we used before, and as this is the essential part of the result, let us restate just the equivalence of the first two points.

Corollary 8.12. *Let G be a finitely generated group and $N \trianglelefteq G$ a normal subgroup such that G/N is nilpotent. Then N is finitely generated if and only if $S_{\mathrm{ord}}(G, N) \subseteq \Sigma^1_{\mathrm{ord}}(G)$.*

9. Novikov homology for orders

For a character Φ on a group G, one may define the *Novikov ring* $\widehat{RG}^{\Phi}$. It is an extension of the group ring that contains not only maps in R^G with finite support but also admits some infinite supports.

Definition 9.1. Let R be a ring, G a finitely generated group and $\Phi\colon G \to \mathbb{R}$ a character of G. Then the *Novikov ring* is

$$\widehat{RG}^{\Phi} \coloneqq \big\{ f \in R^G \;\big|\; |\operatorname{supp} f \cap \Phi^{-1}((-\infty, k])| < \infty \text{ for every } k \in \mathbb{R} \big\}.$$

This gives rise to an important characterisation of $\Sigma^1(G)$.

Theorem 9.2 (Sikorav [Sik87], [Sik17])**.** *Let G be a finitely generated group and Φ a character on G. Then $\Phi \in \Sigma^1(G)$ if and only if $H_1(G, \widehat{\mathbb{Z}G}^{\Phi}) = 0$.*

In this chapter, we extend the established definition of the Novikov ring to be with respect to a full archimedean order instead of only characters. The main result of this chapter is that the above theorem transfers to the order-setting: We show in Theorem 9.35 that for a full archimedean order $\prec$ on G, $\prec \in \Sigma^1_{\text{ord}}(G)$ if and only if the Novikov homology $H_1(G, \widehat{\mathbb{Z}G}^{\prec})$ vanishes.

9.1. Novikov rings

We start by reviewing the Novikov ring, which was introduced in [Nov81] and has since been studied in various settings. A comprehensive treatment may be found in [Sik17]. We will focus on how the multiplication in the Novikov ring works. In particular, we will see how the multiplicative structure restricts possible extensions of the group ring.

Recall from Section 1.1 the definition of a group ring. As a set, the group ring RG contains the finitely supported maps in R^G. The restriction to finite supports is important because, for example, if $G = \mathbb{Z} = \langle X \rangle$, then the product

$$\Big(\sum_{i=-\infty}^{0} X^i\Big)\Big(\sum_{i=0}^{\infty} X^i\Big)$$

is not well-defined. The coefficient of X^0 in the product would have to be the sum of infinitely many summands, each of which equals 1. So R^G is not a ring.

However, the Novikov ring is a ring that admits infinite supports. First note that G acts on R^G by multiplication and hence R^G is an RG-module. Recall that this works

with multiplication on either side and for brevity, we do not write left and right module everywhere. R^G being an RG-module is a different way of saying that the product of two maps is already defined if at least one of them is finitely supported. To make this idea precise, we introduce a topology on R^G.

Take R to be endowed with the discrete topology and R^G with the product topology. Then, a sequence in R converges if and only if it eventually becomes stationary, and a sequence in R^G converges if and only if every coefficient converges individually. More concretely, a series $\sum_{i=0}^{\infty} f_i \in R^G$ converges if for every $g \in G$ there are only finitely many i such that $f_i(g) \neq 0$. In particular, the sum $\sum_{g\in G} f_g g$ interpreted as a sequence of finite partial sums converges to the map $g \mapsto f_g$. That is, if G is countable and we fix an enumeration g_i of all elements of G, then

$$\sum_{i=0}^{n} f_{g_i} g_i \overset{(n\to\infty)}{\longrightarrow} \sum_{i=0}^{\infty} f_{g_i} g_i = \sum_{g\in G} f_g g.$$

Thus, the notational overlap between limits of series $\sum_{g\in G} f_g g$ and elements of RG is well-justified. As every partial sum $\sum_{i=0}^{n} f_{g_i} g_i$ is an element of RG and every element of R^G is a limit of such partial sums, the group ring RG is a dense subset of R^G.

Hence, we may naturally continue the multiplication on RG to a partial multiplication on R^G as follows. Every element of R^G is a limit of some sequence in RG. For two such sequences set

$$(\lim a_n)\ (\lim b_n) \coloneqq \lim(a_n b_n)$$

if the sequence of products $(a_n b_n)$ converges. Otherwise, we leave the left-hand side undefined. To be specific, we get

$$(\sum_{g\in G} a_g g)\ (\sum_{g\in G} b_g g) = \sum_{g\in G} (\sum_{h\in G} a_{gh^{-1}} b_h) g$$

if for each $g \in G$ there are at most finitely many $h \in G$ such that $a_{gh^{-1}} b_h \neq 0$ and an undefined product otherwise.

In this sense, if $a \in R^G$ is a finite sum and hence an element of RG, we may multiply a with any $b \in R^G$. This turns R^G into an RG-module. Note that for any $g \in G$, the multiplication of g on R^G is continuous. Hence, the multiplication by any element of RG on R^G is also continuous.

More generally, we say that a subset $T \subseteq R^G$ is an RG-module if for every $a \in RG$ and $b \in T$, the product ab is defined and $ab \in T$. Similarly, if for all $c, d \in T$ the product cd is defined and in T, then T is a ring.

The Novikov ring we defined in Definition 9.1 is a subset of R^G that is a ring in this sense. We will prove that it is a ring in Lemma 9.14, after we have extended the notion of Novikov ring.

The following is a notation that we will use often in the context of Novikov rings.

Definition 9.3. For $a = \sum_{g\in G} a_g g \in R^G$ and a subset $S \subseteq G$ we may *restrict* a to S by zeroing all values outside of S. That is

$$a_{|S} \coloneqq \sum_{g\in S} a_g g \in R^G.$$

Note that the sum ranges over S instead of G.

Note that $f_{|S} \in RG$ if and only if $S \cap \operatorname{supp} f$ is finite. Thus a map $f \in R^G$ is in the Novikov ring $\widehat{RG}^\Phi$ if and only if $f_{|\varphi^{-1}((-\infty,k])} \in RG$ for every $k \in \mathbb{R}$. Also note that while a character is an equivalence class of maps $G \to \mathbb{R}$, the above condition does not depend on the choice of representative, and hence $\widehat{RG}^\Phi$ is well-defined for a character Φ.

9.2. Novikov rings for orders

Our goal in this chapter is to characterise $\Sigma^1_{\text{ord}}(G)$ via vanishing of some Novikov homology. We first introduce a new viewpoint that allows us to generalise the Novikov ring such that it depends on an order instead of a character. If the order is induced by a character, the two rings will be identical. We will see this generalisation in Definition 9.11.

The first thing to note is that whether a map $f \in R^G$ is an element of $\widehat{RG}^\Phi$ is entirely determined by $\operatorname{supp} f$. Hence, there exists a set $\mathcal{A}$ of subsets of G such that $f \in \widehat{RG}^\Phi$ if and only if $\operatorname{supp} f \in \mathcal{A}$. To be explicit, by Definition 9.1, the set $\mathcal{A}$ contains all subsets $A \subseteq G$ such that $A \cap \Phi^{-1}((-\infty, k])$ is finite for every $k \in \mathbb{R}$. It is this property, that we can decide membership in the Novikov ring by only looking at the support of f, that is the defining property for our generalised Novikov ring.

Thus let us drop the explicit description of $\mathcal{A}$ and just consider any subset $T \subseteq R^G$ with the following property: There exists some $\mathcal{A} \subseteq 2^G$ such that $f \in T$ if and only if $\operatorname{supp} f \in \mathcal{A}$.

If we allow just any $\mathcal{A}$, we have little control over the elements of T. To retain some structure, we ask that T is at least an RG-module. Note that T being closed under multiplication by R is not an issue, as multiplication by R does not change the support of $f \in R^G$. But for $x, y \in R^G$ we have $\operatorname{supp}(x + y) \subseteq \operatorname{supp} x \cup \operatorname{supp} y$. Similarly, if $x \in R^G$ and $g \in G$, then $\operatorname{supp}(gx) = g \operatorname{supp} x$. This motivates the following definition.

Definition 9.4. Let X be a set and 2^X the set of its subsets. We call $I \subseteq 2^X$ an *ideal*, if for all $Y, Z \in 2^X$ the following hold:

1. $\emptyset \in I$
2. If $Y \in I$ and $Z \subseteq Y$, then $Z \in I$.
3. If $Y \in I$ and $Z \in I$, then $Y \cup Z \in I$.

I is *left G-invariant* if $gY \in I$ for every $Y \in I$. Right invariance and bi-invariance are defined analogously.

Remark 9.5. This is a well-established definition that appears in many contexts, many of which have little to do with Novikov rings. See for example [BS12].

Some readers may be more familiar with *filters*. Filters and ideals are dual notions in the following sense: A set I is an ideal if and only if its complement $2^X \setminus I$ is a filter.

Remark 9.6. For any set $S \subseteq 2^X$, there is a unique smallest ideal containing S. Starting with S and adding the empty set and all subsets of elements of S and then adding finite unions of such sets yields said ideal.

Similarly, if X is a group, then there exists a smallest X-invariant ideal containing S, namely the smallest ideal containing XS for left invariance, SX for right invariance or XSX for bi-invariance.

Finally, this allows us to extend the definition of the Novikov ring.

Definition 9.7. Let G be a group, $\mathcal{A} \subseteq 2^G$ a left invariant ideal and R a ring. Then define the *Novikov module* with respect to $\mathcal{A}$ to be

$$\widehat{RG}^{\mathcal{A}} := \{f \in R^G \mid \operatorname{supp} f \in \mathcal{A}\}.$$

Lemma 9.8. *Let G be a group, R a ring and $\mathcal{A} \subseteq 2^G$ a left-invariant ideal. Then $\widehat{RG}^{\mathcal{A}}$ is a left RG-module.*

Proof. Let $x, y \in \widehat{RG}^{\mathcal{A}}$ and write $x = \sum_{g\in G} x_g g$ and similar for y. Further let $r \in R$ and $g \in G$. Then

$$g \in \operatorname{supp}(x+y) \iff x_g + y_g \neq 0 \Rightarrow x_g \neq 0 \text{ or } y_g \neq 0 \Rightarrow g \in \operatorname{supp} x \cup \operatorname{supp} y.$$

Hence

$$\operatorname{supp} x + y \subseteq \operatorname{supp} x \cup \operatorname{supp} y \in \mathcal{A}$$

and so $x + y \in \widehat{RG}^{\mathcal{A}}$.

Also, $\operatorname{supp}(rgx) = g \operatorname{supp} x$ is a g-multiple of an element of $\mathcal{A}$ and hence itself in $\mathcal{A}$ by left invariance. □

Remark 9.9. If $\mathcal{A}$ is a right G-invariant ideal, then we can still define $\widehat{RG}^{\mathcal{A}}$ exactly as we did, and it will be a right RG-module. In general, $\widehat{RG}^{\mathcal{A}}$ is not a ring.

Example 9.10.

1. For the trivial ideal $\mathcal{A} = \{\emptyset\}$, $\widehat{RG}^{\mathcal{A}}$ is the trivial RG-module.
2. If $\mathcal{A} = 2^G$, then $\widehat{RG}^{\mathcal{A}} = R^G$. In this case, $\widehat{RG}^{\mathcal{A}}$ is not a ring.
3. If $\mathcal{A}$ is the set of finite subsets of G, then $\widehat{RG}^{\mathcal{A}} = RG$.
4. We have seen in Definition 9.1 that for every character Φ there exists some set $\mathcal{A} \subseteq 2^G$ such that $\widehat{RG}^{\mathcal{A}} = \widehat{RG}^{\Phi}$.
5. If G is a totally ordered group and $\mathcal{A}$ is the set of well-ordered subsets of G, then $\widehat{RG}^{\mathcal{A}}$ is the Mal'cev-Neumann ring $MN_{\prec}(RG)$ we have seen in Section 3.3.

With this setup, we can define the Novikov ring with respect to an order.

Definition 9.11. Let G be a group carrying a full archimedean order $\prec$ and let R be a ring. Then the *Novikov ring* with respect to $\prec$ is

$$\widehat{RG}^{\prec} \coloneqq \widehat{RG}^{\mathcal{A}^{(\prec)}}$$

where

$$\mathcal{A}^{(\prec)} \coloneqq \big\{A \subseteq G \;\big|\; |S_{\preccurlyeq} \cap A| < \infty \text{ for every antichain } S \subseteq A\big\}.$$

Recall that we mainly define the Novikov ring to obtain a description of $\Sigma^1_{\mathrm{ord}}(G)$. As $\Sigma^1_{\mathrm{ord}}(G)$ contains only full archimedean orders, it suffices for now to define the Novikov ring for these orders. In Definition 10.27, we will see an even more general definition of the Novikov ring that also allows non-archimedean full orders.

We check that the definitions of $\widehat{RG}^{\Phi}$ and $\widehat{RG}^{\prec}$ align if we identify characters with the orders they induce.

Lemma 9.12. *Let G be a group, R a ring, $\Phi\colon G \to \mathbb{R}$ a character and $\prec$ the order on G induced by Φ. Then $\widehat{RG}^{\prec} = \widehat{RG}^{\Phi}$.*

Proof. Let $f \in \widehat{RG}^{\Phi}$ and $S \subseteq G$ an antichain. As $\mathbb{R}$ is totally ordered and $\prec$ is indued by Φ, the kernel $\ker\Phi$ is the unique maximal antichain subset of G that contains 1 by Lemma 7.13. So every antichain $S \subseteq G$ may be written as $S \subseteq g\ker\Phi$. Again because Φ induces $\prec$,

$$S_{\preccurlyeq} \subseteq (g\ker\Phi)_{\preccurlyeq} = \Phi^{-1}(\Phi(g\ker\Phi_{\preccurlyeq})) = \Phi^{-1}((-\infty, \Phi(g)]).$$

By definition of $\widehat{RG}^{\Phi}$, $f_{|\Phi((-\infty,\Phi(g)]^{-1}} \in RG$. Hence $\operatorname{supp} f \cap S_{\preccurlyeq}$ is finite and $f \in \widehat{RG}^{\prec}$.

Conversely, if $f \in \widehat{RG}^{\prec}$, let $k \in \mathbb{R}$. We may pick $g \in G$ such that $\Phi(g) \geqslant k$. For this g,

$$\operatorname{supp} f \cap \Phi^{-1}((-\infty, k]) \subseteq \operatorname{supp} f \cap \Phi^{-1}((-\infty, \Phi(g)]) = \operatorname{supp} f \cap g\ker\Phi_{\preccurlyeq}$$

is finite and hence $f_{|\Phi^{-1}((-\infty,k])} \in RG$. □

Let us look at an example of a Novikov ring that does not come from a character. As we will want to look at nilpotent groups specifically and the distinction between characters and orders does not become apparent for abelian groups, we examine the simplest nilpotent group that is not abelian.

Example 9.13. Let $H = \left\langle a, b \;\middle|\; 1 = [a,[a,b]] = [b,[a,b]] \right\rangle$ be the Heisenberg group. We have seen in Example 8.1 that there exists a unique full archimedean order on H such that $1 \prec [a,b]$.

Recall that any element of H can be uniquely written as $a^\alpha b^\beta [a,b]^\gamma$ and that such an element is positive with respect to $\prec$ if and only if $\alpha = \beta = 0$ and $\gamma > 0$. Also recall that we use the convention $[a,b] = a^{-1}b^{-1}ab$.

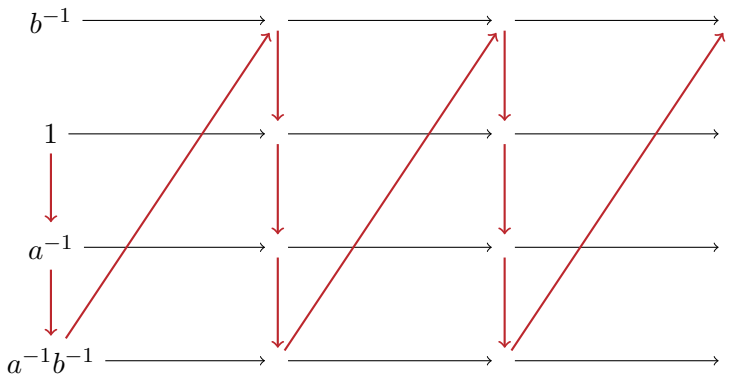


Figure 9.1.: A path in the Cayley graph of the Heisenberg group with free-nilpotent generators a, b. Arrows to the right correspond to the commutator $[a, b]$. Moving to the right along a horizontal arrow increases the order, and every relation in the order is described by the transitive closure of these arrows. The support of the red path is an element of the Novikov ring with respect to an archimedean order induced by the inclusion of the center.

Consider the infinitely long word $w = (a^{-1}b^{-1}ab)^\infty$ in the generators a, b. We can understand it as an infinite ray in the Cayley graph of H, which we see in Figure 9.1. Take $g_i \in H$ to be the prefix of w of length i. The first few g_i are

$$\begin{aligned}
g_0 &= 1 \\
g_1 &= a^{-1} \\
g_2 &= a^{-1}b^{-1} \\
g_3 &= a^{-1}b^{-1}a && = b^{-1}[a,b] \\
g_4 &= a^{-1}b^{-1}ab && = [a,b] \\
g_5 &= a^{-1}b^{-1}aba^{-1} && = a^{-1}[a,b] \\
g_6 &= a^{-1}b^{-1}aba^{-1}b^{-1} && = a^{-1}b^{-1}[a,b] \\
g_7 &= a^{-1}b^{-1}aba^{-1}b^{-1}a && = b^{-1}[a,b]^2 \\
g_8 &= a^{-1}b^{-1}aba^{-1}b^{-1}ab && = [a,b]^2
\end{aligned}$$

Then

$$x \coloneqq \sum_{i=0}^{\infty} g_i$$

is an element of $\widehat{\mathbb{Z}H}^{\prec}$. To see why this is true, let us reorder the summands of x:

$$x = (1 + a^{-1} + a^{-1}b^{-1} + b^{-1}[a,b]) \sum_{i=0}^{\infty} [a,b]^i.$$

That is, the support of x is the union of the four disjoint sets $\{1\}_{\succcurlyeq}, \{a^{-1}\}_{\succcurlyeq}, \{a^{-1}b^{-1}\}_{\succcurlyeq}$ and $\{b^{-1}[a,b]\}_{\succcurlyeq}$. Note that each set is isomorphic to $\mathbb{N}$ as an ordered set. Hence, an antichain $S \subseteq \operatorname{supp} x$ is a choice of four points, one from each of these sets. $S_{\preccurlyeq} \cap \operatorname{supp} x$ is the union of four intervals of the form $[0,k]$ in sets that are each isomorphic to $\mathbb{N}$, so $S_{\preccurlyeq} \cap \operatorname{supp} x$ is finite. In Figure 9.1, this corresponds to cutting each horizontal line at some point and retaining only the part of the line to the left of the cutting point.

Also note that the support of this particular sum is the union of four sets that each are disconnected in the Cayley graph of H with respect to the generating set $\{a,b\}$. But in each set, any two neighbouring points of the path are at distance at most 4, so it is coarsely connected. Also, each of the four sets is connected in $\operatorname{Cay}(H, \{a, b, [a,b]\})$.

On the other hand, the sequence $a^i[a,b]^i$ might look at first sight as though it was increasing because the exponent of $[a,b]$ increases. But it is an infinite antichain as every point of the sequence lies in its own translate of $H^{\succ}$, and hence the sum

$$\sum_{i \in \mathbb{N}} a^i [a,b]^i$$

is not an element of $\widehat{\mathbb{Z}H}^{\prec}$.

While we have been calling it the Novikov *ring* so far, an important fact we have yet to check is that it is actually a ring.

Lemma 9.14. *Let G be an ordered group and R a ring. Then $\widehat{RG}^{\prec}$ is a ring.*

Proof. Let $a, b \in \widehat{RG}^{\prec}$. To show that ab is defined, we need to show that for any $g \in G$, $a_{gh^{-1}} b_h = 0$ for all but finitely many $h \in G$.

Let $S \subseteq G$ be a maximal antichain subset. Then $b_{|S_{\preccurlyeq}}$ is finitely supported. So it is enough to show that $a_{gh^{-1}} = 0$ for all but finitely many $h \in G \smallsetminus S_{\preccurlyeq} = S_{\succ}$.

As $\prec$ is bi-invariant, gS^{-1} is also an antichain. If $h \in S_{\succ}$, then $gh^{-1} \in (gS^{-1})_{\prec}$. As $\operatorname{supp} a \cap (gS^{-1})_{\prec}$ is finite, this means there are only finitely many $h \in S_{\succ}$ such that $a_{gh^{-1}} \neq 0$, so ab is defined.

To see that $ab \in \widehat{RG}^{\prec}$, let $T \subseteq G$ be an antichain subset. We need to show that $(ab)_{|T_{\preccurlyeq}}$ has finite support. We may assume without loss of generality that T is a maximal antichain as this makes $T_{\preccurlyeq}$ only larger.

For any $g \in \operatorname{supp} ab$ there exists at least some h such that $a_{gh^{-1}} b_h \neq 0$. Just as above, in each case, both $a_{gh^{-1}}$ and b_h need to be non-zero, but this happens only finitely many times. □

Remark 9.15. We have seen before that if an order $\prec$ is induced by a character Φ, then $\widehat{RG}^{\prec} = \widehat{RG}^{\Phi}$. Hence, $\widehat{RG}^{\prec}$ being a ring shows that the classical Novikov ring $\widehat{RG}^{\Phi}$ is also a ring.

We conclude the section by giving two alternative descriptions of the Novikov ring.

A fact that is often important in the classical character setting is that $\Phi(\operatorname{supp} f)$ has a minimum for any $f \in \widehat{RG}^{\Phi}$. For example, it implies that if $\prec$ is a total order on G such

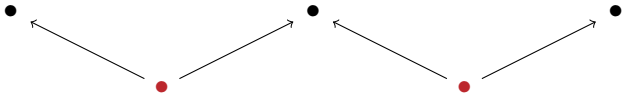


Figure 9.2.: A partially ordered set. Arrows point from smaller elements to larger elements. The elements on the bottom row form the minimum of this set.

that Φ is order-preserving, then the Mal'cev-Neumann ring $MN_{\prec}(RG)$ contains $\widehat{RG}^{\Phi}$. We want to make a similar statement for Novikov rings that depend on orders. Note that there are partially ordered sets X such that some element $x \in X$ admits no smaller element, but still, not every $y \in X$ is larger than x. See Figure 9.2 for an example.

Definition 9.16. Let G be an ordered group and $S \subseteq G$. Then the *minimum* of S is

$$\min S \coloneqq \{s \in S \mid s \not\succ t \ \forall t \in S\}.$$

Remark 9.17.

1. For any $S \subseteq G$, $\min S$ is an antichain.
2. In general, $s \in S$ does not imply that s is larger than some element of $\min S$. For example, if $S = G$, and the order on G is non-trivial, then $\min S$ is empty.
3. For $S, T \subseteq G$ and $s \in (\min S)_{\succcurlyeq}, t \in (\min T)_{\succcurlyeq}$,

   $$st \in (\min S \min T)_{\succcurlyeq}$$

   This is a direct consequence of the fact that this remains true if we replace $\min S$ and $\min T$ with arbitrary subsets of G.

This characterises the Novikov ring as follows.

Lemma 9.18. *Let G be an ordered group, R a ring and $f \in \widehat{RG}^{\prec}$. Then*

1. $\operatorname{supp} f \subseteq (\min \operatorname{supp} f)_{\succcurlyeq}$.
2. $|\min \operatorname{supp} f| < \infty$.

Proof.

1. Let $g \in \operatorname{supp} f$. As $\{g\}$ is an antichain, there are only finitely many $g_i \in \operatorname{supp} f$ such that $g_i \preccurlyeq g$. There is at least one g' among the g_i such that $g_i \not\succ g'$ for every i. This g' is also minimal in $\operatorname{supp} f$, as any element $h \prec g'$ in particular satisfies $h \prec g$, so h is one of the g_i. But g' was minimal among those.

   In total, $g \succcurlyeq g' \in \min \operatorname{supp} f$, proving the claim.

2. As $\min \operatorname{supp} f$ is an antichain by Remark 9.17 and the minimum is contained in $\operatorname{supp} f$,

$$|\min \operatorname{supp} f| = |\min \operatorname{supp} f \cap \operatorname{supp} f| \leqslant |(\min \operatorname{supp} f)_{\preccurlyeq} \cap \operatorname{supp} f| < \infty$$

by definition of $\widehat{RG}^{\prec}$.

□

By definition, the set $\mathcal{A}^{(\prec)}$ of admissible supports for elements of the Novikov ring is a G-invariant ideal. We may use this fact to obtain a compact description of $\mathcal{A}^{(\prec)}$: It is the G-invariant ideal generated by the elements of $\mathcal{A}$ that contain only positive elements.

Lemma 9.19. *Let (G) be an ordered group and R a ring. Let $\mathcal{A}^{(\prec)}$ as in Definition 9.11. Then $\mathcal{A}^{(\prec)}$ is the smallest G-invariant ideal in 2^G containing $\mathcal{A}^{(\prec)} \cap 2^{G^{\succ}}$.*

Proof. Checking that $\mathcal{A}^{(\prec)}$ is a G-invariant ideal is straightforward. So it is enough to show that every $A \in \mathcal{A}^{(\prec)}$ may be written as a finite union

$$A = \bigcup_{g \in M} gA_g$$

where $M \subseteq G$ is a finite set and $A_g \in \mathcal{A}^{(\prec)} \cap 2^{G^{\succ}}$.

For this, consider the finite antichain $\min A$. Then

$$A \subseteq \bigcup_{g \in \min A} gG^{\succ}$$

by Lemma 9.18. And

$$A \cap gG^{\succ} \subseteq A \in \mathcal{A}^{(\prec)}$$

so $g^{-1}(A \cap gG^{\succ}) \in \mathcal{A}^{(\prec)}$ by G-invariance. Taking this set as A_g and $M = \min A$ finishes the proof. □

9.3. Units in the Novikov ring

While $\widehat{RG}^{\prec}$ is a ring, it is usually not a division ring. But we can at least explicitly describe the units in $\widehat{RG}^{\prec}$. Knowing the units will be necessary when we compute the Novikov homology in Section 9.5. While this is well-established for the classical Novikov ring, we adapt the proof to our generalisation.

Lemma 9.20. *Let R be a ring, G an archimedean group.*

1. *For any $x \in \widehat{RG}^{\prec}$ with $\operatorname{supp} x \subseteq G^{\succ}$, $1 - x$ is invertible in $\widehat{RG}^{\prec}$ and*

$$(1 - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

2. *For any square matrix $A \in (\widehat{RG}^{\prec})^{n\times n}$ such that every entry of A is supported on $G^{\succ}$, $1-A$ is invertible in $(\widehat{RG}^{\prec})^{n\times n}$ and*

$$(1-A)^{-1} = \sum_{i=0}^{\infty} A^i$$

Proof.

1. We first show that $\sum_{i=0}^{\infty} x^i$ converges and is hence a well-defined element in R^G. Suppose that there is a $g \in G$ with $g \in \operatorname{supp} x^i$ for infinitely many i. For those i, Remark 9.17 tells us that $g \succcurlyeq (\min \operatorname{supp} x)^i$. That is, for arbitrarily large numbers i, there are $g_1, \dots, g_i \in \min \operatorname{supp} x$ such that g is larger than or equal to the product $g_1 \dots g_i$.

   By Lemma 9.18, $\min \operatorname{supp} x$ is finite. If $i > n \cdot |\min \operatorname{supp} f|$, at least one h from $\min \operatorname{supp} x$ has to appear at least n times among the g_j. Thus

$$g \succcurlyeq g_1 \dots g_i \succcurlyeq h^n.$$

   By picking for every i the element $h \in \min \operatorname{supp} x$ that appears the most times among the g_j, we obtain an element $h \in \min \operatorname{supp} x$ such that $g \succcurlyeq h^n$ for arbitrarily large n. As $g, h \in G^{\succ}$, this contradicts G being archimedean. So g appears in only finitely many summands, and hence the sum converges.

   We proceed by showing that $y := \sum_{i=0}^{\infty} x^i$ is actually an element of $\widehat{RG}^{\prec}$. Suppose there was an antichain $S \subseteq \operatorname{supp} y$ such that $|S_{\preccurlyeq} \cap \operatorname{supp} y| = \infty$. Without loss of generality, we extend S to a maximal antichain in G. As $\operatorname{supp} y \subseteq \bigcup_{i\in\mathbb{N}} \operatorname{supp} x^i$ and $|S_{\preccurlyeq} \cap \operatorname{supp} x^i| < \infty$, there are infinitely many $j \in \mathbb{N}$ such that $S_{\preccurlyeq} \cap \operatorname{supp} x^j$ is non-empty, so contains some g_j. Let us pick $h_j \in \min \operatorname{supp} x^j$ such that $g_j \succcurlyeq h_j$. Just as in the first part of the proof, there is some $h \in \min \operatorname{supp} x$ such that $h_j \succ h^j$ for infinitely many j. In summary, we found some $1 \prec h$ such that for each $i \in \mathbb{N}$, there is a $g_j \in \operatorname{supp} y$ such that $h^i \prec g_j$.

   As the antichain S is maximal, it must contain at least one element in $G^{\succ} \cup \{1\} \cup G^{\prec}$. In particular, there exists $1 \prec s \in S_{\succ}$, unless the order is trivial, in which case there is nothing to show. Because $\prec$ is archimedean, $h^i \succ s$ for some i and hence $g_j \succ h^i \in S_{\succ}$ for some, and indeed, almost all j. This contradicts $g_j \in S_{\preccurlyeq}$ and hence the existence of S. Thus $y \in \widehat{RG}^{\prec}$.

   To see that y is the inverse of $1-x$, we compute

$$(1-x)\sum_{i=0}^{\infty} x^i = \sum_{i=0}^{\infty} x^i - \sum_{i=0}^{\infty} x^{i+1} = \sum_{i=0}^{\infty} x^i - \sum_{i=1}^{\infty} x^i = x^0 = 1.$$

2. The proof goes essentially the same as for (1). We note that as A has only finitely many entries, the minimum of the union of their supports is still a finite set. Every

entry of A^i is a finite sum of products of i factors that are entries of A. So, we may use the same reasoning as above to show that any $g \in G$ appears in the supports of only finitely many entries among all A^i. Hence, the sum converges and, again by the same argument as above, is an element of $\widehat{RG}^{\prec}$.

The exact same computation as in (1) now shows that

$$(1 - A) \sum_{i=0}^{\infty} A^i = 1.$$

□

Remark 9.21. Note that for $x, y \in \widehat{RG}^{\prec}$,

$$(xy)_{|\min \operatorname{supp} xy} = x_{|\min \operatorname{supp} x} \cdot y_{|\min \operatorname{supp} y} .$$

These restrictions are finitely supported and thus elements of RG. If x is a unit in $\widehat{RG}^{\prec}$, then $x_{|\min \operatorname{supp} x}$ is therefore a unit in RG. Let y be the inverse of $x_{|\min \operatorname{supp} x}$. Then xy is of the form $1 - x'$ for some positively supported x'. Hence, Lemma 9.20 describes all units in $\widehat{RG}^{\prec}$ up to multiplication by units in RG.

If $x \in \widehat{RG}^{\prec}$ is positively supported, then $1 - x$ is invertible in $\widehat{RG}^{\prec}$. As the multiplication on $\widehat{RG}^{\prec}$ is a restriction of the partially defined multiplication on R^G, $1 - x$ is invertible in every ring $T \leqslant R^G$ such that $\sum_{i=0}^{\infty} x^i \in T$.

The converse is also true: If $1 - x$ is a unit in $T \leqslant R^G$, then $(1 - x)^{-1} = \sum_{i=0}^{\infty} x^i$ and the infinite sum is an element of T. To see this, convince yourself that for no other $y \in R^G$, the product $(1 - x)y$ is defined and $(1 - x)y = 1$.

Thus, to recognise units in the Novikov ring, $f_{|\min \operatorname{supp} f}$ plays an important role. We may use this idea to understand elements in $\widehat{RG}^{\prec}$. An element $f \in \widehat{RG}^{\prec}$ decomposes into the part that is supported on $\min \operatorname{supp} f$ and the rest $R_0 = f - f_{|\min \operatorname{supp} f}$. As $\min \operatorname{supp} f$ is an antichain, $f_{|\min \operatorname{supp} f}$ is an element of RG.

We may continue this idea as follows: R_0 is again an element of $\widehat{RG}^{\prec}$ and thus has a part supported on its minimum and a smaller rest R_1. In total, we get a decomposition of f into *levels*, where each level set is the minimum of the support of some R_i or of f itself. The restriction of f to any level set yields a finitely supported sum and, hence, an element of RG. If the Novikov ring is with respect to a character Φ, that Φ is constant on each level. Interpreting the value of a given level under Φ as the *height* of that level provides an order on the levels such that the order embeds into the standard order on $\mathbb{N}$.

When passing to orders, we can generally no longer see the height of a level as the image under a map. To recover a similar notion, we fix a positive element $s \in G$ and a maximal antichain normal subgroup $N \trianglelefteq G$. Think of N as "cutting through G", dividing it into elements above N, below N and incomparable to N. By translating this cut with our element s in either direction, we cut up G into slices of thickness s. This idea is made precise in the following lemma and visualised in Figure 9.3.

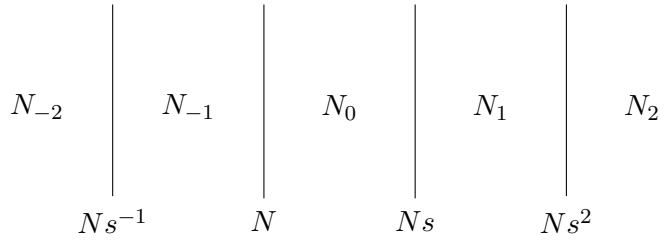


Figure 9.3.: A maximal normal antichain subgroup N divides $N_\succ \cup N_\preccurlyeq$ into slices N_i.

Lemma 9.22. *Let G be an ordered group with $\prec$ a full order, $s \in G^\succ$ non-infinitesimal and $N \trianglelefteqslant G$ a maximal antichain normal subgroup. Suppose that G/N is totally ordered, that is, that N is maximal antichains.*

For $i \in \mathbb{Z}$ set

$$N_i \coloneqq (Ns^i)^c_\preccurlyeq \cap (Ns^{i+1})_\preccurlyeq$$

where $(Ns^i)^c_\preccurlyeq$ denotes the complement. Then G is the disjoint union of all the N_i for $i \in \mathbb{Z}$.

Proof. As $\prec$ is full, it is induced by the projection map $\pi\colon G \twoheadrightarrow G/N$. Note that $\pi(Ns^i) = \{\pi(s)^i\}$, so $g \in (Ns^i)_\preccurlyeq$ is equivalent to $\pi(g) \preccurlyeq \pi(s)^i$.

To see that the N_i are pairwise disjoint, let $i < j \in \mathbb{Z}$. Then

$$g \in N_i \Rightarrow g \in (Ns^{i+1})_\preccurlyeq \Rightarrow \pi(g) \preccurlyeq \pi(s)^{i+1} \preccurlyeq \pi(s)^j \Rightarrow g \notin N_j$$

and vice-versa.

It remains to show that every $g \in G$ is contained in some N_i. Note that as G/N is totally ordered, $G = N_\preccurlyeq \cup N_\succ$.

We show that there is a minimal i such that $g \in (Ns^i)_\preccurlyeq$. Then for that i we have $g \in N_{i-1}$. So let I be the set of $i \in \mathbb{Z}$ such that $g \in (Ns^i)_\preccurlyeq$. We first show that I is non-empty. If $g \in N_\preccurlyeq$, then $0 \in I$. Otherwise

$$g \in N_\succ \Rightarrow \pi(g) \succ 1 \Rightarrow g \succ 1.$$

So $gs \succ s \succ 1$. As $\prec$ is archimedean, this means that there is some i such that $s^i \succ gs$ and hence $g \prec s^{i-1} \in N(s^{i-1})$. That is, $i - 1 \in I$.

Now, if I contains only finitely many negative numbers, then it has a minimum. Otherwise, $\pi(g) \prec \pi(s)^i$ for infinitely many negative i and hence even for all $i \in \mathbb{Z}$. But then $1 \prec \pi(s) \ll (\pi(g))^{-1}$ contradicts s being non-infinitesimal. So I does indeed have a minimum and $g \in N_{\min I - 1}$. □

Remark 9.23. We ask that G/N be totally ordered to make sure that every element of G is above or below N. If this is not the case, Lemma 9.22 still provides a slicing of every subgroup H for which H/N is totally ordered.

It is plausible that Lemma 9.22 with slightly different assumptions will give a similar result. However, one has to be very careful to make sure that every element of G actually

ends up in some slice, with the issue being that if $f \prec g \succ h$, this tells us nothing at all about the relation between f and h. They may be comparable or incomparable; in the latter case, they may or may not be in the same N-coset.

The version of Lemma 9.22 stated here is enough for the proofs we present in the following section.

9.4. Novikov homology

We have seen in Chapter 7 that if G is a group and $N \trianglelefteq G$ a normal subgroup such that G/N is abelian, then a character $\Phi \in S(G,N) = S(G)$ can be identified with an order $\prec \in S_{\mathrm{ord}}(G,N)$. Theorem 9.2 states that in this case $\Phi \in \Sigma^1(G)$ if and only if $H_1(G, \widehat{\mathbb{Z}G}^{\Phi}) = 0$. As we know from Lemma 9.12 that the Novikov rings for Φ and $\prec$ coincide, we get the following result:

Corollary 9.24. *Let G be a finitely generated group, $N \trianglelefteq G$ a normal subgroup such that G/N is abelian and $\prec \in S_{\mathrm{ord}}(G,N)$. Then $\prec \in \Sigma^1_{\mathrm{ord}}(G)$ if and only if $H_1(G, \widehat{\mathbb{Z}G}^{\prec}) = 0$.*

This is a direct translation of Theorem 9.2 to the language of orders.

By Theorem 6.18, a normal subgroup $N \trianglelefteq G$ is finitely generated if and only if $S(G,N) \subseteq \Sigma^1(G)$ as long as G/N is abelian. In Chapter 8, we replaced the assumption that G/N is abelian with G/N nilpotent. Now, we do the same thing for Theorem 9.2. For this, we first investigate how to compute the first homology of G with coefficients in $\widehat{\mathbb{Z}G}^{\prec}$.

Let G be any finitely generated group. Recall the resolution C_* provided by the presentation complex from Definition 1.11. We will use this concrete resolution to compute $H_1(G, \widehat{\mathbb{Z}G}^{\prec})$. In general, for any $\mathbb{Z}G$-module M, we may see the kernel $\ker(M \otimes_{\mathbb{Z}G} \partial_1)$ as the space of paths without boundary points in the presentation complex, with M providing an interpretation of what constitutes a path. In the case where M is the Novikov ring, this means that we look at paths that may be potentially infinite, but if they are, they have to be increasing in the order. Recall Example 9.13, where we have seen an infinite path supported on Novikov coefficients. The path from the example has a single boundary point in 1. If we concatenate another infinitely long path that ends in 1, this yields some element of $\ker \widehat{\mathbb{Z}G}^{\prec} \otimes_{\mathbb{Z}G} \partial_1$ that is usually not already in $\ker \partial_1$. The following lemma generalises this idea to provide a generating set of $\ker(\widehat{\mathbb{Z}G}^{\prec} \otimes_{\mathbb{Z}G} \partial_1)$.

Lemma 9.25. *Let G be an archimedean group with some generating set S and $1 \prec s \in S$. Denote by e_t for $t \in S$ the set of free generators of $(\widehat{\mathbb{Z}G}^{\prec})^S$ and consider the map*

$$\partial\colon (\mathbb{Z}G)^S \to \mathbb{Z}G, \quad e_t \mapsto 1 - t,$$

just as in Definition 1.11. We write

$$\widehat{\partial} \coloneqq \widehat{\mathbb{Z}G}^{\prec} \otimes_{\mathbb{Z}G} \partial.$$

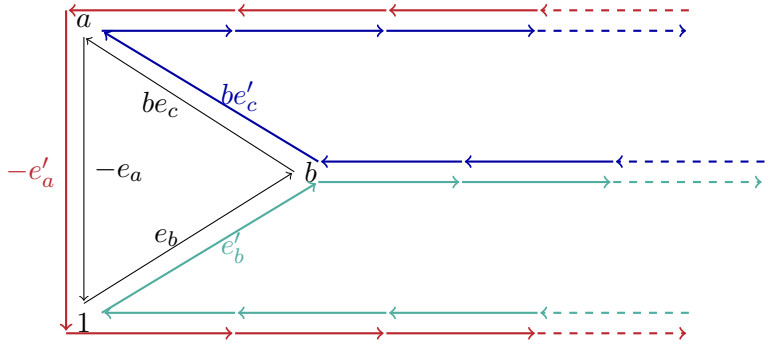


Figure 9.4.: A presentation of the relation $bca^{-1} = 1$ as a sum of generators of the kernel $\ker(\widehat{\mathbb{Z}G}^{\prec} \otimes \partial_1)$ for the group $G = \langle a, b, c, s \mid bca^{-1} \rangle$. All horizontal edges have labels ge_s for varying g.

Let

$$e'_t := e_t - (1-t)(1-s)^{-1}e_s.$$

Then

1. $\ker \widehat{\partial}$ *is generated by* $\{e'_t \mid t \in S \smallsetminus s\}$.
2. *For any cycle*

$$\sum_{t \in S} \lambda_t e_t \in \ker \partial$$

with coefficients $\lambda_t \in \mathbb{Z}G$*, only finitely many of which are non-zero, we have*

$$\sum_{t \in S} \lambda_t e_t = \sum_{t \in S} \lambda_t e'_t \in \ker \widehat{\partial}.$$

We can see an illustration of the second statement in Figure 9.4 for the cycle

$$1 \to b \to a \to 1$$

in some group where the corresponding edges in the presentation complex exist.

Proof. We start with the second statement. Note that

$$e'_s = e_s - (1-s)(1-s)^{-1}e_s = 0.$$

Further note that

$$\widehat{\partial} e_t = (\widehat{\mathbb{Z}G}^{\prec} \otimes_{\mathbb{Z}G} \partial)\ (1 \otimes e_t) = 1 \otimes \partial e_t = \partial e_t,$$

using the identification of C_* with the subspace $\mathbb{Z}G \otimes_{\mathbb{Z}G} C_* \leqslant \widehat{\mathbb{Z}G}^{\prec} \otimes_{\mathbb{Z}G} C_*$ for the last equality. For a cycle $\sum_{t\in S} \lambda_t e_t \in \ker \widehat{\partial}$ we compute

$$\begin{aligned}\sum_{t\in S\smallsetminus s} \lambda_t e'_t &= \sum_{t\in S} \lambda_t e'_t \\ &= \sum_{t\in S} \lambda_t e_t - \sum_{t\in S} \lambda_t (1-t)(1-s)^{-1} e_s \\ &= \sum_{t\in S} \lambda_t e_t - \partial(\sum_{t\in S} \lambda_t e_t)\ (1-s)^{-1} e_s \\ &= \sum_{t\in S} \lambda_t e_t\end{aligned}$$

Since any element of $(\widehat{\mathbb{Z}G}^{\prec})^S = \widehat{\mathbb{Z}G}^{\prec} \otimes_{\mathbb{Z}G} (\mathbb{Z}G)^S$ can be written as $\sum_{t\in S} \lambda_t e_t$ for some $\lambda_t \in \widehat{\mathbb{Z}G}^{\prec}$, this shows that in particular every cycle can be written as a $\widehat{\mathbb{Z}G}^{\prec}$-linear combination of the e'_t.

It remains to show that $e'_t \in \ker \widehat{\partial}$ for every $t \in S \smallsetminus s$:

$$\widehat{\partial} e'_t = \widehat{\partial} e_t - (1-t)(1-s)^{-1} \widehat{\partial} e_s = (1-t) - (1-t)(1-s)^{-1}(1-s) = 0.$$

□

It will be essential for us later on to see how the homology behaves when passing to subgroups. More precisely: If $H \leqslant G$ is a subgroup, what is the relation between $H_1(G, \widehat{\mathbb{Z}G}^{\prec})$ and $H_1(H, \widehat{\mathbb{Z}H}^{\prec_{|H}})$? In general, not much can be said, with the issue being that we change both the group and the coefficient ring. In Remark 9.30, we will see an example where one of the homologies vanishes and the other does not. But at least there is an embedding $\widehat{\mathbb{Z}H}^{\prec_{|H}} \hookrightarrow \widehat{\mathbb{Z}G}^{\prec}$ compatible with the embedding of the group rings.

Lemma 9.26. *Let R be a ring, G an ordered group, $H \leqslant G$ and $x \in \widehat{RG}^{\prec}$ with $\operatorname{supp} x \subseteq H$. Then $x \in \widehat{RH}^{\prec_{|H}}$.*

Proof. As $x \in \widehat{RG}^{\prec}$, Lemma 9.18 provides a finite set M such that

$$x = \sum_{g\in M} x_{|gG^{\succ}}\ .$$

Since the sum is finite, it is enough to show that $x_{|gG^{\succ}} \in \widehat{RH}^{\prec_{|H}}$. As $\operatorname{supp} x \subseteq H$, each summand is supported on

$$gG^{\succ} \cap H = \min(gG^{\succ} \cap H) H^{\succ},$$

again using Lemma 9.18 for the equality, and to see that the minimum is finite. Hence each support is a finite union of translates of $H^{\succ}$. The claim follows from the fact that for each such translate $hH^{\succ}$, the restriction $x_{|hH^{\succ}}$ is an element of $\widehat{RH}^{\prec}$. □

If G is ordered by an order induced by the inclusion of H, this means that all the structure of $(G, \prec)$ is already contained in the ordered group H. This transfers to the Novikov ring $\widehat{\mathbb{Z}G}^{\prec}$, which is a $\mathbb{Z}G$-module generated by $\widehat{\mathbb{Z}H}^{\prec}$. So all of the additional structure of the Novikov ring $\widehat{\mathbb{Z}G}^{\prec}$ in comparison to the group ring $\mathbb{Z}G$ can already be seen in $\widehat{\mathbb{Z}H}^{\prec}$. Let us make this notion precise.

Lemma 9.27. *Let G be an ordered group and $H \trianglelefteqslant G$ a normal subgroup such that the order on G is induced by the inclusion of H. Suppose that $H^{\succ}$ is closed under conjugation with G. Let R be a ring. Then*

$$\widehat{RG}^{\prec} = RG \otimes_{RH} \widehat{RH}^{\prec}$$

as left RG-modules.

Proof. Let

$$s\colon G/H \to G, \quad gH \mapsto \tilde{g}$$

be a set-theoretic section of the projection map $G \twoheadrightarrow G/H$. That is $\tilde{g}H = gH \subseteq G$ for every $g \in G$.

We may write any $g \in G$ as $g = \tilde{g}h$ for $h = \tilde{g}^{-1}g \in H$. Thus any $x \in \widehat{RG}^{\prec}$ may be written as

$$x = \sum_{g \in G} a_g g = \sum_{gH \in G/H} \sum_{h \in H} a_{\tilde{g}h} \tilde{g} h = \sum_{gH \in G/H} \tilde{g} \Big(\sum_{h \in H} a_{\tilde{g}h} h \Big).$$

By Lemma 9.18, we know that $\operatorname{supp} x$ is a union of finitely many sets $\tilde{g}G^{\succ}$. Because $\prec$ is induced by the inclusion $H \hookrightarrow G$, the positive cone $G^{\succ}$ is the closure of $H^{\succ}$ under conjugation with G. As $H^{\succ}$ is already closed in this sense by assumption, this means $G^{\succ} = H^{\succ}$. Thus, for all but finitely many gH, the sum $\sum_{h \in H} a_{\tilde{g}h}$ must be zero. The isomorphism we are looking for is

$$\widehat{RG}^{\prec} \to RG \otimes_{RH} \widehat{RH}^{\prec}, \quad x = \sum_{gH \in G/H} \tilde{g} \Big(\sum_{h \in H} a_{\tilde{g}h} h \Big) \mapsto \sum_{gH \in G/H} \Big(\tilde{g} \otimes \sum_{h \in H} a_{\tilde{g}h} h \Big).$$

Note that the element on the right-hand side is well-defined since the outer sum is finite as we have seen above and $\operatorname{supp}(\sum_{h \in H} a_{\tilde{g}h} h) \subseteq H$. Hence, the sum is an element of $\widehat{RH}^{\prec}$ by Lemma 9.26.

The inverse of that isomorphism is

$$RG \otimes_{RH} \widehat{RH}^{\prec} \to \widehat{RG}^{\prec}, \quad g \otimes x \mapsto gx.$$

Checking these are inverse maps of left RG-modules is straightforward. □

Let us check what this means on the level of homology.

Corollary 9.28. *Let G be an ordered group and $H \leqslant G$ a subgroup such that the order on G is induced by the inclusion of H. Suppose that $H^{\succ}$ is closed under conjugation with G. Then $H_*(G, \widehat{\mathbb{Z}G}^{\prec}) = H_*(H, \widehat{\mathbb{Z}H}^{\prec})$.*

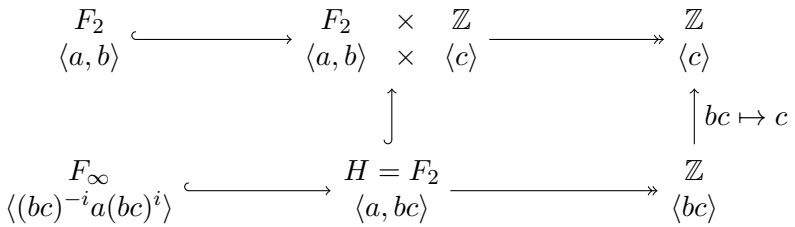


Figure 9.5.: A commuting diagram of order-preserving maps. Each copy of $\mathbb{Z}$ is ordered either by the standard order or its opposite. The groups to the left are ordered trivially, and the groups in the middle are ordered via their projections onto $\mathbb{Z}$. The horizontal lines are exact.

Proof. Let F_* be a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. Note that $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. Hence the modules F_* are also free as $\mathbb{Z}H$-modules and F_* is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}H$-modules: The cartesian product of a free basis of F_* as $\mathbb{Z}G$-module times a free basis of $\mathbb{Z}G$ as $\mathbb{Z}H$-module is a free $\mathbb{Z}H$-basis of F_*. We use Lemma 9.27 to compute

$$\begin{aligned} H_*(G, \widehat{\mathbb{Z}G}^{\prec}) &= H_*(F_* \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}^{\prec}) \\ &= H_*(F_* \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}H} \widehat{\mathbb{Z}H}^{\prec}) \\ &= H_*(F_* \otimes_{\mathbb{Z}H} \widehat{\mathbb{Z}H}^{\prec}) \\ &= H_*(H, \widehat{\mathbb{Z}H}^{\prec}). \end{aligned}$$

□

Remark 9.29. If G is nilpotent and the order is full archimedean, the assumption that $H^\succ$ is closed under conjugation with G is automatic: By Theorem 8.6, the order on G is induced by some projection $G \twoheadrightarrow G/N$ and the order on G/N is induced by a total order on the center. Hence H contains the preimage of $Z(G/N)$ under the projection $G \twoheadrightarrow G/N$. As N is an antichain, the order on H is induced by the projection onto $H/(H \cap N)$. By Lemma 7.9, $(G/N)^\succ$ is the closure of $(H/(H \cap N))^\succ$ under conjugation with elements of G/N. But $(G/N)^\succ$ is central in G/N and hence the two positive cones are already equal. Thus for their preimages under the projection map, we also get $H^\succ = G^\succ$.

The same argument also holds if G itself is not necessarily nilpotent, but the order is induced by the projection onto a nilpotent quotient.

Remark 9.30. Note that in Corollary 9.28, it is not enough that H is ordered by the restriction of the order on G. The order on G really has to be induced by the inclusion of H. Figure 9.5 shows an example where the inclusion of the subgroup is not order-inducing: The direct product $F_2 \times \mathbb{Z}$ contains a normal subgroup that is isomorphic to F_2 such that the projection onto the $\mathbb{Z}$-component restricts to a non-trivial map on the

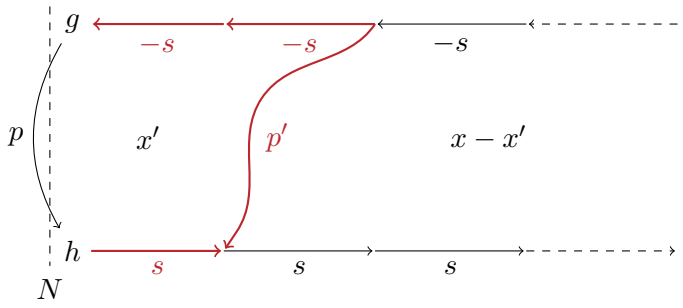


Figure 9.6.: A path connecting g to h supported on $N_{\succcurlyeq}$.

subgroup. We call this subgroup H to distinguish it from the left factor of $F_2 \times \mathbb{Z}$. It also plays the role of the subgroup H in Corollary 9.28.

As the projections $F_2 \times \mathbb{Z} \twoheadrightarrow \mathbb{Z}$ and $H \twoheadrightarrow \mathbb{Z}$ are order-inducing, the inclusion $H \hookrightarrow F_2 \times \mathbb{Z}$ is order-preserving. That is, the order on $F_2 \times \mathbb{Z}$ restricts to the order on H. Note, however, that $\prec$ is not induced by the inclusion of H as not every positive element is a conjugate of an element of H. The kernel of the projection $F_2 \times \mathbb{Z} \twoheadrightarrow \mathbb{Z}$ is F_2 and hence finitely generated. However, the kernel of that projection's restriction to H is isomorphic to F_∞ and, in particular, not finitely generated. We will see in Theorem 9.36 that this obstructs equal homologies.

9.5. Sikorav's theorem for orders

Recall Theorem 9.2 stating that a character is in the Σ-invariant if and only if the homology with coefficients in the Novikov ring vanishes. While this is a statement only about characters, we now have notions of the Σ-invariant and the Novikov ring for orders on G. So, it is natural to ask if Sikorav's theorem still holds for orders. In this section, more precisely in Theorem 9.35, we prove it for orders that are induced by maps onto nilpotent groups.

We start by assuming vanishing homology and proving that $\prec \in \Sigma^1_{\mathrm{ord}}(G)$. Let us first review the case where the order is induced by a character. The following proof is an adapted version of the proof for Theorem 9.2 as communicated via [Kie20, Theorem 3.11], which is the analogous statement in the character-setting.

Theorem 9.31. *Let $(G, \prec)$ be a finitely generated partially ordered group such that $\prec$ is full, archimedean and non-trivial and such that there is some normal antichain subgroup N such that G/N is abelian.*

If $H_1(G, \widehat{\mathbb{Z}G}^{\prec}) = 0$, then $\prec \in \Sigma^1_{\mathrm{ord}}(G)$.

Proof. All tensoring is over $\mathbb{Z}G$.

The abelian group G/N contains a maximal normal antichain subgroup $K \trianglelefteq G/N$. As $\prec$ is a full order, it is induced by the projection $G \twoheadrightarrow G/N/K$. The kernel of that

projection is a maximal antichain normal subgroup of G. As $G/N/K$ is also abelian, by replacing N with said kernel, we may assume without loss of generality that $N \trianglelefteq G$ is already a maximal normal antichain subgroup. In particular, G/N admits no normal antichain subgroups and is hence totally ordered by Theorem 8.6. Thus N is maximal among all antichains and $N_\prec \cup N_\succcurlyeq = G$.

In this proof, we need to show that $N_\succ$ is coarsely connected. There is some $1 \prec s \in G$ and a finite generating set $S \subseteq G$ with $s \in S$. It suffices to show that the full subgraph of $\mathrm{Cay}(G, S)$ spanned by $N_\succcurlyeq$ is connected. That is, for every $g, h \in N_\succcurlyeq$, there is a path in $\mathrm{Cay}(G, S)$ connecting g to h supported on $N_\succcurlyeq$.

Denote by (C_*, ∂) the presentation complex of some presentation using the generating set S as in Definition 1.11. As $\mathrm{Cay}(G, S)$ is connected, there is at least some path p connecting g to h. However, it need not be supported on $N_\succcurlyeq$. Have a look at Figure 9.6. We see the path p, the ray starting at h taking infinitely many edges labeled s, and the ray ending in g, coming from edges labeled s. These three paths may be interpreted as elements of $\widehat{\mathbb{Z}G}^\prec \otimes C_1$, and their sum ξ is a cycle. As $\mathrm{H}_1(G, \widehat{\mathbb{Z}G}^\prec) = 0$, there is an element $x \in \widehat{\mathbb{Z}G}^\prec \otimes C_2$ with $(\widehat{\mathbb{Z}G}^\prec \otimes \partial)x = \xi$.

As $\widehat{\mathbb{Z}G}^\prec \otimes C_2$ is a free $\widehat{\mathbb{Z}G}^\prec$-module generated by any free basis of C_2, we can write this x as $x = \sum_{i=1}^n x_i r_i$ where $x_i \in \widehat{\mathbb{Z}G}^\prec$ and r_i are finitely many of the basis elements of C_2.

To avoid cluttering notation too much, let us fix one i and set $y \coloneqq x_i$. Also, we write just ∂ instead of $\widehat{\mathbb{Z}G}^\prec \otimes \partial$.

We use slicing as in Lemma 9.22 to write $y = \sum_{j=0}^\infty y_{|N_j}$. Then as $N_j \subseteq (Ns^{j+1})_\prec$ and Ns^{j+1} is an antichain, $y_{|N_j}$ has finite support. Hence $\partial\, y_{|N_j}$ also has finite support. Recall from Section 9.1 that multiplication by an element of $\mathbb{Z}G$ is continuous on $\widehat{\mathbb{Z}G}^\prec$ and so in particular ∂ is continuous. That is

$$\partial y = \sum_{j=0}^\infty \partial(y_{|N_j}).$$

Thus, the right-hand side is a well-defined sum, and therefore, any $g \in G$ can only be contained in the supports of finitely many $\partial(y_{|N_j})$. As $\operatorname{supp} \partial y \cap N_\prec$ is finite by definition of the Novikov ring, this means that there is some $n_0 \in \mathbb{N}$ such that

$$\operatorname{supp}\Big(\sum_{j=n_0+1}^\infty \partial(y_{|N_j})\Big)$$

is disjoint from $N_\prec$. Let

$$x_i' = y' \coloneqq \sum_{j=0}^{n_0} y_{|N_j} \in \mathbb{Z}G$$

and

$$x' \coloneqq \sum_{i=1}^n x_i' r_i \in C_2 \subseteq \widehat{\mathbb{Z}G}^\prec \otimes C_2.$$

Now by construction,

$$\partial(x - x') = \partial x - \partial x' = \xi - \partial x'$$

and we have

$$\operatorname{supp}(\xi - \partial x') \cap N_{\prec} = \emptyset.$$

Also, $\xi - \partial x'$ is a cycle, as

$$\partial(\xi - \partial x') = \partial\partial(x - x') = 0.$$

As $x' \in C_2$, we have that $\partial x' \in C_1$. That is, $\partial x'$ is finitely supported. The support of ξ contains all but finitely many elements of the form gs^i and hs^i for positive i, so $\operatorname{supp}(\xi - \partial x')$ still contains all but finitely many.

Take i, j to be the largest numbers such that $gs^{i-1}, hs^{j-1} \notin \operatorname{supp}(\xi - \partial x')$. As $\xi - \partial x'$ is a cycle, it contains a path p' from gs^i to hs^j. Since p' lies on the boundary of $x - x'$, we get in particular $\operatorname{supp} p' \subseteq N_{\succcurlyeq}$ by construction of x'.

The path from g to h on $N_{\succcurlyeq}$ is then constructed as follows: Start at g, move up to gs^i along edges labeled s, follow the path p' in $\partial(x - x')$ to hs^j and go down to h along edges labeled s. □

Now we use the characterisation of orders on nilpotent groups from Theorem 8.6 to generalise Theorem 9.31 to the case where G/N is any nilpotent group.

Lemma 9.32. *Let G be group, H a subgroup and $\prec_H$ a full archimedean order on H such that the following implication holds:*

$$H_1(H, \widehat{\mathbb{Z}H}^{\prec_H}) = 0 \Rightarrow \prec_H \in \Sigma^1_{\mathrm{ord}}(H).$$

Suppose that G is ordered by some order $\prec_G$ that is induced by the inclusion of H and that admits a normal antichain subgroup $N \trianglelefteq G$ such that G/N is nilpotent. Further suppose that $H_1(G, \widehat{\mathbb{Z}G}^{\prec_G}) = 0$. Then $\prec_G \in \Sigma^1_{\mathrm{ord}}(G)$.

Proof. As G/N is nilpotent, there exists an antichain normal subgroup $K \trianglelefteq G/N$ such that the order on G/N is induced by the projection to $(G/N)/K$. Let N' be the kernel of the projection $G \twoheadrightarrow (G/N)/K$. Then G/N' is also nilpotent and its order is induced by the inclusion of H/N'. We know by assumption that $H_1(G, \widehat{\mathbb{Z}G}^{\prec_G}) = 0$. By Remark 9.29, we may apply Corollary 9.28, which tells us that $H_1(H, \widehat{\mathbb{Z}H}^{\prec_H}) = 0$ as well. Thus $\prec_H \in \Sigma^1_{\mathrm{ord}}(H)$ by assumption and hence $\prec_G \in \Sigma^1_{\mathrm{ord}}(G)$ by Lemma 8.10. □

Corollary 9.33. *Let $(G, \prec)$ be a finitely generated partially ordered group such that $\prec$ is full, archimedean and non-trivial and such that there is some normal antichain subgroup N such that G/N is nilpotent.*

If $H_1(G, \widehat{\mathbb{Z}G}^{\prec}) = 0$, then $\prec \in \Sigma^1_{\mathrm{ord}}(G)$.

Proof. As $\prec$ is a full archimedean order, it is induced by the projection $G \twoheadrightarrow G/N$ and some full archimedean order on the finitely generated nilpotent group G/N. By Theorem 8.6, there exists a normal subgroup $K \trianglelefteq G/N$ such that the order on G/N

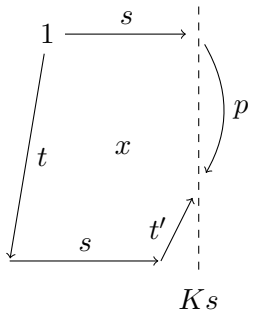


Figure 9.7.: While we do not know where the generators t and t' are, s is known to be positive and tst' ends up in Ks. There is a path p connecting s to tst' supported on $(Ks)_{\succcurlyeq}$.

is induced by the inclusion of the center $Z(G/N/K) \hookrightarrow G/N/K$. Let $H \trianglelefteqslant G$ be the preimage of the center under the projections. Then the order on G is induced by the inclusion of H.

As $Z(G/N/K)$ is an abelian group, Theorem 9.2 tells us that $H_1(H, \widehat{\mathbb{Z}H}^{\prec}) = 0$ implies $\prec \in \Sigma^1_{\mathrm{ord}}(H)$. The claim follows from Lemma 9.32. □

This concludes the proof of the first direction of a generalised version of Theorem 9.2. Now for the other direction.

Theorem 9.34. *Let G be a finitely generated group, $\prec \in \Sigma^1_{\mathrm{ord}}(G)$ and $1 \prec s \in G$.*

Suppose that there is a normal antichain subgroup $K \leqslant G$ and a finite generating set $S \subseteq G$ with $s \in S$ such that for every $t \in S$ there is a $t' \in S^{-1}$ such that $tst' \in Ks$.

Then $H_1(G, \widehat{\mathbb{Z}G}^{\prec}) = 0$.

Proof. Fix a presentation $G = \langle S \mid R \rangle$ using the generating set S and construct C_* as in Definition 1.11.

Let e_t be the generators of C_1. Then according to Lemma 9.25, $\ker \partial_1 \otimes \widehat{\mathbb{Z}G}^{\prec}$ is generated by $e'_t \coloneqq e_t - (1-t)(1-s)^{-1}e_s$ for $t \in S \smallsetminus s$.

Let us, for now, fix some $t \in S$. We have $s \in Ks$, and by assumption, there is some $t' \in S^{-1}$ such that $tst' \in Ks$. Thus $\prec \in \Sigma^1(G)$ provides a path p connecting s to tst' with $\operatorname{supp} p \subseteq (Ks)_{\succcurlyeq}$. We can see this setup in Figure 9.7.

Extend p by one edge labeled ${t'}^{-1}$ to obtain a path p' from s to ts and write that path as

$$p' = \sum_{u \in S} \lambda_u e_u = \sum_{u \in S} e_u \otimes \lambda_u \in C_1 \otimes \widehat{\mathbb{Z}G}^{\prec}.$$

As p' has finite length we even get $\lambda_u \in \mathbb{Z}G$ and by construction

$$\operatorname{supp} \lambda_u \subseteq \operatorname{supp} p \cup \{tst'\} \subseteq (Ks)_{\succcurlyeq}.$$

Also, $x \coloneqq e_t - p' - (1-t)e_s$ is a cycle as

$$\begin{aligned}\partial x &= \partial e_t \quad - \partial p' \quad - (1-t)\partial e_s \\ &= (1-t) - (s-ts) - (1-t)(1-s) \\ &= 0.\end{aligned}$$

So there is $c_t \in C_2$ with $\partial c_t = x$. At the same time, by Lemma 9.25, we have

$$x = e'_t - \sum_{u \in S \smallsetminus s} \lambda_u e'_u \in \ker(\partial \otimes \widehat{\mathbb{Z}G}^{\prec}).$$

The map $e'_t \mapsto (\partial \otimes \widehat{\mathbb{Z}G}^{\prec})(c_t) = e'_t - \sum_{u \in S \smallsetminus s} \lambda_u e'_u$ is a map from the one-cycles to the one-cycles. Writing this map as a matrix M, we see that M is the unit matrix plus some matrix with entries λ_u, which are all supported on $(Ks)_{\succcurlyeq}$.

As $\prec \in \Sigma^1_{\mathrm{ord}}(G)$ means that $\prec$ is in particular full, it is induced by the projection $G \twoheadrightarrow G/K$. As this projection maps Ks to a single point $[s]$, we get $Ks \subseteq G^{\succ}$ and hence $\operatorname{supp} \lambda_u \subseteq (Ks)_{\succcurlyeq} \subseteq G^{\succ}$.

By Lemma 9.20, M is then invertible. In particular, it is onto, meaning that $\partial \otimes \widehat{\mathbb{Z}G}^{\prec}$ is also onto the cycles. This finishes the proof. □

Merging this with Corollary 9.33 yields the following equivalence.

Theorem 9.35. *Let G be a finitely generated group carrying a full archimedean order $\prec$ that is induced by some map $\Phi\colon G \twoheadrightarrow Q$ for some nilpotent group Q. Then the following are equivalent.*

1. $\prec \in \Sigma^1_{\mathrm{ord}}(G)$.
2. $H_1(G, \widehat{\mathbb{Z}G}^{\prec}) = 0$.

Proof. The implication (2) ⇒ (1) is Corollary 9.33. For the other direction we use Theorem 8.6 to obtain some $P \trianglelefteqslant Q$ such that $\prec$ is induced by the inclusion of the center $Z(Q/P) \hookrightarrow Q/P$ and some total order on $Z(Q/P)$. Every maximal antichain subgroup of Q contains $K' \coloneqq \pi^{-1}(P)$, where π is the projection $Q \twoheadrightarrow Q/P$. Hence every maximal antichain subgroup of G contains $K \coloneqq \Phi^{-1}(K')$.

Let S be any finite generating set of G such that $S = S^{-1}$ and such that there is at least some positive $s \in S$. Then $(\pi \circ \Phi)(s) \in Z(Q/P)$ and hence for any $t \in S$ we have $(\pi \circ \Phi)(tst^{-1}s^{-1}) = 1$. That is $tst^{-1}s^{-1} \in K$ or equivalently, $tst^{-1} \in Ks$.

Applying Theorem 9.34 to this setup proves the claim. □

Note that if Q is abelian, and we identify a character with the order it induces on G, we recover Theorem 9.2.

We can compactly state the main results of the previous chapters as the following equivalence.

Theorem 9.36. *Let G be a finitely generated group and $N \trianglelefteqslant G$ a normal subgroup such that G/N is nilpotent. Then the following are equivalent.*

1. *N is finitely generated.*

2. *$S_{\mathrm{ord}}(G,N) \subseteq \Sigma^1_{\mathrm{ord}}(G)$.*

3. *$H_1(G, \widehat{\mathbb{Z}G}^{\prec}) = 0$ for every $\prec \in S_{\mathrm{ord}}(G,N)$.*

Proof. The equivalence (1) $\iff$ (2) is Corollary 8.12. The equivalence (2) $\iff$ (3) is Theorem 9.35 applied to every order in $S_{\mathrm{ord}}(G,N)$. □

Part IV.

Algebraic fibrations

10. Fibring of RFN groups

Recall Theorem 2.9 stating that a finitely generated group that is virtually RFRS admits a subgroup mapping onto $\mathbb{Z}$ with finitely generated kernel if and only if the first ℓ^2-Betti number of G vanishes. Further, recall that a group that is both RFRS and nilpotent is abelian. On the other hand, Theorem 2.9 also holds for virtually nilpotent groups as we will see in Lemma 10.9.

In this chapter, we follow the proof of Theorem 2.9 in [Kie20] to see how it translates to the RFN setting. One crucial ingredient in that proof is the equivalence of finite generatedness and vanishing Novikov homology in Theorem 9.36 in the case where G/N is abelian. We now have generalised this equivalence to G/N nilpotent. A RFRS group is, in particular, a residually finite group such that G/G_i is abelian for every i. In this chapter, we replace these abelian quotients in the definition of RFRS with nilpotent quotients to obtain a larger class of groups which we call RFN. It seems reasonable that Theorem 2.9 will hold for RFN groups, as we know that at least in Theorem 9.36 we may indeed replace the condition that some quotients are abelian with the weaker condition that they are nilpotent.

Following Kielak's proof yields a strategy for proving that fibring and vanishing of first ℓ^2-Betti number are also equivalent for RFN groups. However, there are some loose ends where it is unclear how the transition from RFRS to RFN works. We will point these out when they come up. It will be the subject of future research to fill these gaps and try to turn the strategy outlined in this chapter into a full proof.

10.1. RFRS and nilpotency

We start by defining RFN groups. They are the groups that we will study in this chapter. So far, we have been extending concepts that exist for abelian groups to nilpotent groups. In the same spirit, we now replace RFRS groups with RFN groups.

Definition 10.1. A group G is *residually finite with nilpotent quotients* or *RFN*, if there exists a sequence of finite index normal subgroups $G_i \trianglelefteq G$ such that

$$G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \dots$$

and

$$\bigcap_{i \in \mathbb{N}} G_i = \{1\}$$

and for every i there exists some torsion-free nilpotent group A_i such that the projection $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through A_i.

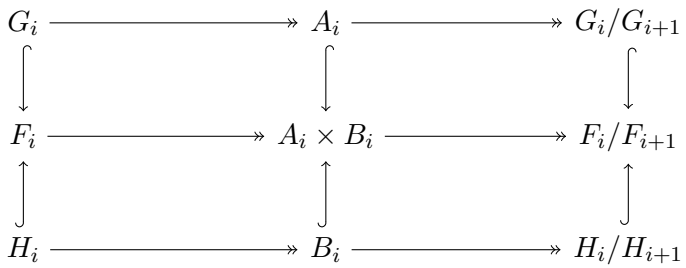


Figure 10.1.: A commutative diagram of RFN groups. F_i is a residual chain for the free product $G * H$. The projection $F_i \twoheadrightarrow F_i/F_{i+1}$ factors through the torsion-free nilpotent group $A_i \times B_i$.

Remark 10.2. Let $i \in \mathbb{N}$. Let n be the nipotency class of A_i and recall that we denote the lower central series of G_i by ${G_i}^{(n)}$.

The group $Q \coloneqq G_i/{G_i}^{(n)}$ is nilpotent of class at most n and any quotient of G_i that is nilpotent of class at most n factors through Q The set of torsion elements $T \subseteq Q$ is a normal subgroup. Hence, we may always ask that A_i is the kernel of the projection $G_i \twoheadrightarrow Q/T$.

This is analogous to RFRS groups where we did the same construction with $Q = G_i/{G_i}^{(1)}$.

Remark 10.3. Every RFN group is torsion-free: Since $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through a torsion-free group, every torsion element of G_i is contained in G_{i+1}. Inductively, G_{i+1} contains every torsion element of $G_0 = G$. Hence, if G is not torsion-free, the intersection of the G_i cannot be trivial.

The following provides a large number of RFN groups. This is analogous to [Ago08, Corollary 2.3].

Proposition 10.4. *The class of RFN groups contains*

1. *every RFRS group*
2. *and every torsion-free nilpotent group.*

It is closed under the following operations:

3. *taking subgroups,*
4. *taking free products,*
5. *taking direct products.*

Proof.

1. If G_i is a witnessing chain for the fact that G is RFRS, then every projection $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through the torsion-free abelian group G_i^{fab}, which is also torsion-free nilpotent. Thus, G_i is also a witnessing chain for G RFN.

2. Every nilpotent group G is residually finite, and every quotient of a nilpotent group is again nilpotent. Let G_i be a residual chain for G. For every i, G_i is torsion-free and if n is the nilpotency class of G_i/G_{i+1} then the projection $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through $G_i/{G_i}^{(n)}$. So any residual chain of G is automatically a witnessing chain for G RFN. As always, ${G_i}^{(n)}$ denotes the lower central series.

3. Let $H \leqslant G$ where G is RFN with witnessing chain G_i. Then $H_i \coloneqq H \cap G_i$ is a witnessing chain for H being RFN. To see this, first note that for $h \in H_i, f \in H$, the conjugate $f^{-1}hf$ is in H because all factors are, and is in G_i because G_i is normal. So H_i is normal in H.

   Secondly, $\cap_{i\in\mathbb{N}} H_i \leqslant \cap_{i\in\mathbb{N}} G_i = \{1\}$.

   And the embedding $H_i \hookrightarrow G_i$ descends to an embedding

   $$H_i/H_{i+1} = H_i/(G_{i+1} \cap H_i) \hookrightarrow G_i/G_{i+1}.$$

   If the projection $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through the torsion-free nilpotent group A_i, say via the map π_i, then $H_i \twoheadrightarrow H_i/H_{i+1}$ factors through $\pi_i(H_i)$, which is a subgroup of A_i and hence also torsion-free nilpotent.

4. If G and H are RFN with witnessing chains G_i and H_i, set

   $$F_i \coloneqq \langle G_i, H_i, [G_{i-1}, H_{i-1}]\rangle \trianglelefteqslant G * H.$$

   Then F_i is a witnessing chain for $G * H$ RFN, as

   $$\bigcap_{i\in\mathbb{N}} (G_i * H_i) = (\bigcap_{i\in\mathbb{N}} G_i) * (\bigcap_{i\in\mathbb{N}} H_i) = \{1\}$$

   and so

   $$\bigcap_{i\in\mathbb{N}} [G_{i-1}, H_{i-1}] = \{1\}$$

   as well, so the intersection of the F_i is trivial.

   Also if $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through A_i and $H_i \twoheadrightarrow H_i/H_{i+1}$ factors through B_i, then $F_i \twoheadrightarrow F_i/F_{i+1}$ factors through $A_i \times B_i$. See also Figure 10.1. If A_i and B_i are torsion-free nilpotent then so is $A_i \times B_i$.

5. The proof is the same as for free products. Note that $[G_{i-1}, H_{i-1}]$ is trivial in $G \times H$, so the witnessing chain for $G \times H$ is just $G_i \times H_i$.

□

Every RFN group is residually finite and torsion-free. Let us provide a counterexample to see that the converse is not true. Positive examples we have already seen in Proposition 10.4.

Example 10.5. The Baumslag-Solitar group $G = \mathrm{BS}(1,2) = \langle a, b \mid ba = ab^2 \rangle$ is not RFN. To see this, a simple computation shows that every group $G^{(k)}$ in the lower central series contains the generator b. So if we suppose that G_i is a witness for G being RFN, then G_1 necessarily contains b. We claim that by induction, every G_i must contain b. So suppose that $b \in G_i$. As G_i is finite index in G, there is some $n \in \mathbb{N}$ such that $a^n \in G_i$. Again, a simple computation shows $b^n \in {G_i}^{(k)}$ for every k. As $G_i \twoheadrightarrow G_i/G_{i+1}$ factors through some nilpotent group N_i, this means $b^n \in G_{i+1}$. Because N_i is also torsion-free, we even get $b \in G_{i+1}$. But then the intersection of the G_i is non-trivial, contradicting the assumption that a witnessing chain G_i for G being RFN exists.

The details about the lower central series of Baumslag Solitar groups can be found for example in [BN20].

An important fact about RFRS groups is that they satisfy the Atiyah conjecture ([Kie20, Proposition 4.2]), so let us prove the same for RFN groups.

Lemma 10.6. *Every RFN group G satisfies the Atiyah conjecture.*

Proof. Corollary 2.7 from [Sch02] states that every residually torsion-free elementary amenable group satisfies the Atiyah conjecture. As every virtually nilpotent group is elementary amenable, it suffices to show that G is residually {torsion-free and virtually nilpotent}. That is, there exists a chain K_i of normal subgroups of G such that $\cap K_i = \{1\}$ and G/K_i is torsion-free and virtually nilpotent.

Take G_i to be a witnessing chain of G being RFN. It comes with subgroups $K_i \leqslant G_{i+1}$ such that G/K_i is torsion free nilpotent. We claim that these K_i witness that G is residually {torsion free and virtually nilpotent}.

The intersection of all K_i is trivial because every K_i is a subgroup of G_i and the G_i have trivial intersection. Similarly, every K_i is a normal subgroup of G.

K_i is a chain because

$$K_{i+1} = G_{i+1}^{(m_{i+1})} \leqslant G_{i+1}^{(m_i)} \leqslant G_i^{(m_i)} = K_i.$$

G/K_i is virtually nilpotent because it fits into the extension

$$G_i/K_i \hookrightarrow G/K_i \twoheadrightarrow G/G_i.$$

G_i/K_i is nilpotent and G/G_i is finite.

It remains to show that G/K_i is torsion free. First note that $G/K_0 = G_0/K_0$ is torsion free by construction. For any i, we have the extension

$$K_i/K_{i+1} \hookrightarrow G/K_{i+1} \twoheadrightarrow G/K_i.$$

K_i/K_{i+1} is a subgroup of the torsion free group G_{i+1}/K_{i+1} and hence itself torsion free. G/K_i is torsion free by induction. So G/K_{i+1} is torsion free as an extension of torsion free groups.

□

If RFN groups share so many properties with RFRS groups, the following question seems natural.

Question 10.7. *Let G be a finitely generated group that is virtually RFN. Is it true that G is virtually fibred if and only if $\beta_1^2(G) = 0$?*

One direction is immediately true by the first part of the proof of [Kie20, Theorem 5.3], which we repeat here for convenience.

Lemma 10.8. *If G is a group that is virtually RFN and virtually fibred, then $\beta_1^2(G) = 0$.*

Proof. Let $H \leqslant G$ be a finite index subgroup of G that is fibred. Let $\Phi\colon H \twoheadrightarrow \mathbb{Z}$ be a map with finitely generated kernel. Then we may write H as an extension

$$\ker \Phi \hookrightarrow H \twoheadrightarrow \mathbb{Z}.$$

As $\ker \Phi$ is finitely generated and infinite, [Lüc02, Theorem 7.2(5)] tells us that $\beta_1^2(H)$ vanishes. Thus $\beta_1^2(G)$ also vanishes by [Lüc02, Theorem 1.35(9)]. □

The other direction of Question 10.7 is true for RFRS groups by Theorem 2.9. It is also true for nilpotent groups, direct products and free products, as we will see in Lemma 10.9 and Lemma 10.10.

This covers many of our known examples from Proposition 10.4, notably omitting subgroups, for which such a general statement cannot be made: They may or may not fibre and may or may not have vanishing first ℓ^2-Betti number. Nevertheless, this gives hope that the answer to Question 10.7 is positive for all RFN groups.

Lemma 10.9. *Let G be a torsion-free nilpotent group. Then $\beta_1^2(G) = 0$ and G fibres.*

Proof. The commutator subgroup $[G, G]$ is a subgroup of G and hence finitely generated. Also, $G_{\mathrm{ab}} = G/[G, G]$ is a finitely generated free-abelian group. Hence any projection

$$G \twoheadrightarrow G_{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}$$

has finitely generated kernel and thus G fibres. As G is in particular RFN, Lemma 10.8 shows that $\beta_1^2(G)$ vanishes. □

Lemma 10.10. *Let G, H be finitely generated groups admitting maps onto $\mathbb{Z}$, not necessarily with finitely generated kernel. Then*

1. *$\beta_1^2(G \times H) = 0$ and $G \times H$ fibres, whereas*
2. *$\beta_1^2(G * H) > 0$ and $G * H$ does not fibre.*

Proof. The statements about vanishing or non-vanishing of ℓ^2-Betti numbers are Theorem 4.15 from [Kam19]. Note that as G and H project onto $\mathbb{Z}$, they are infinite and therefore have vanishing zeroth ℓ^2-Betti numbers.

In this proof, we see $\mathbb{Z}$ as an additive group. This corresponds with the notation for characters as maps to the additive group $\mathbb{R}$.

To see that $G \times H$ fibres let $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ and $\psi\colon H \twoheadrightarrow \mathbb{Z}$. These induce a map

$$f\colon G \times H \twoheadrightarrow \mathbb{Z}, \quad (g,h) \mapsto \varphi(g) + \psi(h).$$

Then $(g,h) \in \ker f \iff \varphi(g) = -\psi(h)$.

Let g_0, h_0 such that $\varphi(g_0) = \psi(h_0) = -1$. Further let $A \subseteq G$ and $B \subseteq H$ be finite generating sets such that $g_0 \in A$ and $h_0 \in B$. We claim that $\ker f$ is generated by

$$S = \{(a, h_0^{\varphi(a)}) \mid a \in A\} \cup \{(g_0^{\psi(b)}, b) \mid b \in B\}.$$

It is a straightforward computation that these generators are elements of $\ker f$. And if $(g,h) \in \ker f$, we may write $g = a_1 a_2 \dots a_n$ and $h = b_1 b_2 \dots b_m$ for some $a_i \in A, b_j \in B$. Now

$$(a_1, h_0^{\varphi(a_1)}) \dots (a_n, h_0^{\varphi(a_n)}) = (a_1 \dots a_n, h_0^{\varphi(a_1 \dots a_n)}) = (g, h_0^{\varphi(g)}) \in \langle S \rangle$$

and simlarly, $(g_0^{\psi(h)}, h) \in \langle S \rangle$. But then

$$(g, h_0^{\varphi(g)})(g_0^{\psi(h)}, h) = (g,h)(g_0^{\psi(h)}, h_0^{\varphi(g)}) \in \langle S \rangle.$$

As $(g,h) \in \ker f$ and so $\psi(h) = -\varphi(g)$,

$$(g_0^{\psi(h)}, h_0^{\varphi(g)}) = (g_0, {h_0}^{-1})^{\psi}(g) \in \langle (g_0, h_0^{-1}) \rangle \subseteq \langle S \rangle$$

and so $(g,h) \in \langle S \rangle$, proving the claim.

It remains to show that the free product $G * H$ does not fibre. Let $f\colon G * H \twoheadrightarrow \mathbb{Z}$ be any surjective map. Then as any commutator in $G * H$ is in $\ker f$, f factors through $G_{\mathrm{fab}} * H$ where G_{fab} is the torsion-free part of the abelianisation G_{ab}. G_{fab} is a finitely generated free-abelian group. Restricting f to G_{fab} yields a map

$$G_{\mathrm{fab}} \cong \mathbb{Z}^n \to \mathbb{Z}.$$

We may choose free generators of G_{fab} such that at most one of these generators is not in the kernel of f. This is by linear algebra, seeing $\mathbb{Z}^n$ as a free $\mathbb{Z}$-module. Thus f factors through $\mathbb{Z} * H$.

Using the same argument on the right factor, we get that f even factors through $\mathbb{Z} * \mathbb{Z} \cong F_2$. Any finite generating set of $\ker f \leqslant G * H$ projects onto a finite generating set of $\ker f \leqslant F_2$. But F_2 does not fibre by Theorem 2.9 using the fact from [Kam19, Theorem 4.15] that $\beta_1^2(F_2) > 0$. Hence, a finite generating set of $\ker f$ does not exist, finishing the proof. □

10.2. Twisted group rings

We continue investigating Question 10.7. The only part we still have to answer is if vanishing of $\beta_1^2(G)$ implies that G is virtually fibred for every RFN group G. We approach this by translating the proof of the same statement for RFRS groups into the language of orders we have established in the previous chapters.

Recall that we have seen that orders on nilpotent groups are always induced by the inclusion of a subgroup, and the order on that subgroup is induced by the projection onto some quotient G/N. This defines some subgroup $H = Z(G/N) \leqslant G$ of the group G that contains all the information necessary to reconstruct the given order.

The group ring RH is contained in RG, leading to the equality of RG-modules

$$RG = (RH)G/H.$$

However, this is not an equality of rings. In Lemma 9.27, we have seen that this transfers to the Novikov rings, so $\widehat{RG}^{\prec} = \widehat{RH}^{\prec} G/H$ as RG-modules but not as rings. Similarly, $\widehat{RH}^{\prec} = (\widehat{RN)H/N}^{\prec}$ as RH-modules but not as rings. In this section, we tweak the multiplication on the left-hand side of these equations such that they become ring equalities. We do this by replacing group rings with *twisted group rings* and show that this replacement extends to Novikov rings. This follows the setup in [Kie20], where we find twisted Novikov rings for characters.

Definition 10.11. Let R be any ring. Let $N \hookrightarrow G \twoheadrightarrow Q$ be an extension of groups with a set-theoretic section $s\colon Q \to G$.

For every $a, b \in RN$ and $p, q \in Q$ we set

$$ap \cdot bq \coloneqq \big(as(p)bs(q)s(pq)^{-1}\big)pq.$$

Extending this RN-linearly to $(RN)Q$ turns the latter into a ring, which we call a *twisted group ring*.

Note that here we multiply $a, s(p), b, s(q)$ and $s(pq)^{-1}$ as elements of RG, and their product ends up in the subset RN.

Remark 10.12. Note that the twisted group ring $(RN)Q$ and the group ring for the group Q with coefficients in RN are identical as abelian groups and even as RN-modules. But they carry different ring multiplications. In fact, every section s defines a different multiplication on $(RN)Q$. We will have to make sure that it is clear which multiplication $(RN)Q$ carries in every instance. If s is a group homomorphism, then the resulting twisted group ring is $(RN)Q$ with the usual group ring multiplication.

Remark 10.13. A much more general notion of twisted group rings exists, but the one presented here covers all cases we will see.

Lemma 10.14. *Let RG be a group ring and $(RN)Q$ a twisted group ring induced by some section $s\colon Q \to G$. Then*

1. *RG and $(RN)Q$ are isomorphic as RN-modules and*
2. *the RN-module map*
$$\varphi\colon (RN)Q \to RG, \quad q \mapsto s(q)$$
is an isomorphism of rings.

Proof. To see that φ is an isomorphism of RN-modules, it suffices to note that Q is a free basis of the RN-module $(RN)Q$ and $s(Q)$ is a free basis of the RN-module RG.

Note that the inverse of φ is the map

$$\varphi^{-1}\colon RG \to (RN)Q, \quad g \mapsto \left(gs(\pi(g))^{-1}\right)\, \pi(g)$$

where π is the projection map $G \twoheadrightarrow Q$. To parse the notation, $g\pi(g)^{-1} \in N \subseteq RN$ is the coefficient of $\pi(g) \in Q$, making $(gs(\pi(g)^{-1}))\ \pi(g)$ an element of $(RN)Q$.

Now, we show that φ^{-1} is a ring isomorphism. For $r, r' \in R, g, h \in G$,

$$\begin{aligned}
&\varphi^{-1}(rg)\ \cdot\ \varphi^{-1}(r'h)\\
&= rgs(\pi(g))^{-1}\pi(g)\ \cdot\ r'hs(\pi(h))^{-1}\pi(h)\\
&= \left(rgs(\pi(g))^{-1}s(\pi(g))r'hs(\pi(h))^{-1}s(\pi(h))s(\pi(g)\pi(h))^{-1}\right)\ \pi(g)\pi(h)\\
&= \varphi^{-1}(rg\ r'h)\\
&= \varphi^{-1}(rr'\ gh).
\end{aligned}$$

□

Remark 10.15. If T is any ring that is also an RN-module, then the multiplication on $(RN)Q$ extends to componentwise multiplication on $T \otimes_{RN} (RN)Q = TQ$, turning TQ into another ring, which we will also call a twisted group ring.

In Chapter 9, we have studied Novikov modules. And in Section 4.2 we have seen the Linnell ring $\mathcal{D}(G)$. Both are extensions of the group ring. Let us see what happens if, in their definitions, we allow the group ring to be twisted, starting with the Linnell ring.

The following is analogous to [Kie20, Proposition 2.23] and [Kie20, Lemma 2.22].

Lemma 10.16. *Let $N \hookrightarrow G \twoheadrightarrow Q$ be an extension of groups such that G satisfies the Atiyah conjecture. Take $(\mathbb{Q}N)Q$ to be a twisted group ring isomorphic to $\mathbb{Q}G$ and consider the twisted group ring $\mathcal{D}(N)Q = \mathcal{D}(N) \otimes_{\mathbb{Q}N} (\mathbb{Q}N)Q$.*

If $\mathcal{D}(N)Q$ satisfies the Ore condition with respect to $\mathcal{D}(N)Q \setminus 0$, then

$$\mathcal{D}(G) = \mathcal{D}(N)Q[\mathcal{D}(N)Q \setminus 0]^{-1}.$$

If instead Q is finite, then

$$\mathcal{D}(G) = \mathcal{D}(N)Q.$$

Proof. Recall that $\mathcal{D}(G)$ is the division closure of $\mathbb{Q}G$ inside $\mathcal{U}(G) \coloneqq \mathcal{L}(G)[S]^{-1}$ where S is the set of non-zero-divisors in $\mathcal{L}(G)$. As $\mathcal{L}(N) \leqslant \mathcal{L}(G)$, we have

$$\mathcal{D}(N) = D(\mathbb{Q}N \hookrightarrow \mathcal{U}(N)) \leqslant D(\mathbb{Q}N \hookrightarrow \mathcal{U}(G)) \leqslant D(\mathbb{Q}G \hookrightarrow \mathcal{U}(G)) = \mathcal{D}(G).$$

Thus we get inclusions

$$\mathbb{Q}G = (\mathbb{Q}N)Q \hookrightarrow \mathcal{D}(N)Q \hookrightarrow \mathcal{D}(G)$$

and so $\mathcal{D}(G) = D(\mathcal{D}(N)Q \hookrightarrow \mathcal{D}(G))$. Recall from Definition 3.4 the universal property of the Ore localisation stating that the $(\mathcal{D}(N)Q \setminus 0)$-inverting inclusion $\mathcal{D}(N)Q \hookrightarrow \mathcal{D}(G)$ extends uniquely to a map $f\colon \mathcal{D}(N)Q[\mathcal{D}(N)Q \setminus 0]^{-1} \to \mathcal{D}(G)$. This f is a map between division rings, so its kernel is either $\{0\}$ or the whole domain. But the latter cannot be the case as we already know that f is injective on $\mathcal{D}(N)Q$. Hence, f is an inclusion and $\mathcal{D}(N)Q[\mathcal{D}(N)Q \setminus 0]^{-1} \leqslant \mathcal{D}(G)$.

As $\mathcal{D}(N)Q[\mathcal{D}(N)Q \setminus 0]^{-1}$ is a division ring containing $\mathbb{Q}G$,

$$D(\mathbb{Q}G \hookrightarrow \mathcal{D}(G)) \leqslant \mathcal{D}(N)Q[\mathcal{D}(N)Q \setminus 0]^{-1}.$$

But $D(\mathbb{Q}G \hookrightarrow \mathcal{D}(G)) = \mathcal{D}(G)$, finishing the proof.

The second statement is [Kie20, Lemma 2.22]. The proof is analogous to [Lüc02, Lemma 10.59]. □

Remark 10.17. Note that by [Tam54], in Lemma 10.16 for $\mathcal{D}(N)Q$ to satisfy the Ore condition with respect to $\mathcal{D}(N)Q \setminus 0$, it is sufficient that Q is amenable and $\mathcal{D}(N)Q$ is a domain. This is particularly the case if N satisfies the Atiyah conjecture and additionally Q is solvable, or, even more specifically but also more relevant for us, nilpotent.

Now, let us investigate the Novikov ring for twisted group rings. If RG is a twisted group ring with respect to two different multiplications $\cdot$ and $*$, then

$$\operatorname{supp}(a \cdot b) = \operatorname{supp}(a * b)$$

for every $a, b \in R^G$ such that one and then both products are defined. The proof is a straightforward computation.

This is in particular true if $\cdot$ is the normal group ring multiplication. Thus for an ideal $\mathcal{A} \subseteq 2^G$ and $a, b \in \widehat{RG}^{\mathcal{A}}$ we get

$$\operatorname{supp}(a * b) = \operatorname{supp}(a \cdot b) \in \mathcal{A},$$

allowing for the following definition.

Definition 10.18. Let RG be a twisted group ring with multiplication $*$ and $\mathcal{A} \subseteq 2^G$ an ideal. Then the RG-module $\widehat{RG}^{\mathcal{A}}$ together with the multiplication

$$(\sum_{g \in G} a_g g) \ * \ (\sum_{g \in G} b_g g) \coloneqq \sum_{g \in G} (\sum_{h \in G} a_g g \ * \ b_h h)$$

is the *twisted Novikov ring*.

Remark 10.19. In particular, if G is an ordered group with an archimedean order and $N \trianglelefteqslant G$ is an antichain normal subgroup, the Novikov ring $\widehat{RG}^{\prec}$ and the twisted Novikov ring $(\widehat{RN)G/N}^{\prec}$ are isomorphic as rings.

10.3. Orders on nilpotent groups revisited

In the preceding chapters, our assumption has always been that all "interesting" orders on a group G are full and archimedean. We have also seen the notion of irrational characters, which for abelian groups is the same as injective characters. In the proof of Theorem 2.9, irrational characters play a vital role. They allow seeing elements of $\mathcal{D}(G)$ as elements of the Novikov ring. An important property used in that proof is that irrational characters induce total orders on the abelianisation. But by Theorem 8.6, no full archimedean total orders exist on non-abelian nilpotent groups. Thus, we must drop the assumption that all orders are archimedean. In this section, we classify all full orders on nilpotent groups.

Definition 10.20. Let G be a group. We denote the *set of full orders* on G by $T_{\mathrm{ord}}(G)$.

The subset containing all orders such that $N \trianglelefteqslant G$ is an antichain normal subgroup is the *set of full orders relative to* N, and we denote it by $T_{\mathrm{ord}}(G, N)$.

Note that because the orders are full, we get $T_{\mathrm{ord}}(G, N) = T_{\mathrm{ord}}(G/N)$ just as for archimedean orders. The proof is the same as for Lemma 7.29.

In this section, we extend the classification of $S_{\mathrm{ord}}(G)$ from Theorem 8.6 to $T_{\mathrm{ord}}(G)$. Recall Example 7.20 where we have seen that every full order $\prec$ on $\mathbb{Z}^n$ is lexicographic with respect to some extension $N \hookrightarrow \mathbb{Z}^n \twoheadrightarrow Q$. If $\prec$ is *properly lexicographic* in the sense that both Q and N carry a non-trivial order and, in particular, are non-trivial groups, then $\prec$ is non-archimedean: Every element of N is infinitesimal with respect to every positive element of Q.

Conversely, if one of Q and N is trivially ordered, then the other has to be archimedean if $\prec$ is archimedean. Thus, every full order on $\mathbb{Z}^n$ is either properly lexicographic or archimedean.

In Example 8.1, we saw that every order on the Heisenberg group H is lexicographic with respect to the exact sequence $Z(H) \hookrightarrow H \twoheadrightarrow H_{\mathrm{ab}}$. Again, $\prec$ is archimedean if one of $Z(H)$ and H_{ab} is trivially ordered and the other is archimedean.

In Theorem 10.24, we show that every order on a nilpotent group is either properly lexicographic or archimedean. Even more, the theorem shows that we can construct every ordered nilpotent group by starting with some archimedean nilpotent groups and constructing lexicographic orders on extensions of these groups.

We start by showing that being properly lexicographic and being archimedean are mutually exclusive.

Lemma 10.21. *Let G be a nilpotent group and $N \trianglelefteqslant G$ a normal subgroup. Let $\prec \in T_{\mathrm{ord}}(G)$ be lexicographic with respect to the sequence $N \hookrightarrow G \twoheadrightarrow G/N$. Then $h \ll g$ for every $h \in N$ and $g \in G^{\succ} \smallsetminus N$.*

Proof. Let $h \in N$ and $g \in G^{\succ} \smallsetminus N$. Let $\pi\colon G \twoheadrightarrow G/N$ be the projection map. Then $\pi(h) = 1$ and $\pi(g) \in (G/N)^{\succ}$ by definition of the lexicographic order. Therefore $\pi(h^{-1}g) \in (G/N)^{\succ}$ and hence

$$h^{-1}g \in \pi^{-1}((G/N)^{\succ}) \subseteq G^{\succ}$$

or equivalently, $h \prec g$.

As h^k is also an element of N for every k, the same argument shows that $h^k \prec g$ and so $h \ll g$. □

Lemma 10.22. *Let G be a group, $\prec$ a full order on G and $N \trianglelefteq G$ a normal subgroup such that $h \ll g$ for every $h \in N$ and $g \in G^{\succ} \setminus N$. Then there exists some order on G/N such that $\prec$ is lexicographic with respect to the sequence $N \hookrightarrow G \twoheadrightarrow G/N$.*

Proof. Let $g \in G \setminus N$ and $h \in N$. If $g \in G^{\succ}$, by assumption $h^{-1} \prec g$ and so $hg \in G^{\succ}$. Similarly, $g \notin G^{\succ}$ implies $gh \notin G^{\succ}$ as $gh \notin N$ and if $gh \in G^{\succ}$ then $h \ll gh$ and hence $1 \prec g$.

This allows us to define an order on G/N by setting

$$(G/N)^{\succ} := \{\pi(g) \mid g \in G^{\succ} \setminus N\}$$

where $\pi\colon G \twoheadrightarrow G/N$ is the projection map. The above argument shows that an element of G/N is positive if and only if *all* of its preimages in G are positive, so $(G/N)^{\succ}$ satisfies all the properties required for a positive cone by Lemma 7.6. In particular, $(G/N)^{\succ}$ is closed under multiplication.

The order on G/N defined by $(G/N)^{\succ}$ induces an order $\prec'$ on G that is a suborder of $\prec$. More precisely, if $g, h \in G$ such that $h^{-1}g \notin N$, then $g \prec h \iff g \prec' h$. That is,

$$G^{\succ} = \pi^{-1}((G/N)^{\succ}) \cup N^{\succ},$$

which is the definition of the lexicographic order. □

Hence, if G is non-archimedean, there must be some positive infinitesimals. We show that we may always locate one of those infinitesimals in the center of G.

Lemma 10.23. *Let G be a finitely generated nilpotent group and $\prec \in T_{\mathrm{ord}}(G)$ non-archimedean. Then there exists some $\{1\} \neq N \trianglelefteq Z(G)$ such that $h \ll g$ for every $h \in N$ and $g \in G^{\succ} \setminus N$.*

Proof. If $\prec_{|Z(G)}$ is not a total order, then there exists some $h \in Z(G)$ that is incomparable to 1. Because $\prec$ is full, it is induced by the projection $G \twoheadrightarrow G/\langle h \rangle$. Equivalently, $\prec$ is lexicographic with respect to the sequence $\langle h \rangle \hookrightarrow G \twoheadrightarrow G/\langle h \rangle$ where $\langle h \rangle$ is ordered trivially. By Lemma 10.21, h is infinitesimal with respect to every $g \in G^{\succ} \setminus \langle h \rangle$.

Now take $\prec$ to be total on $Z(G)$. We claim that there exists some $h \in Z(G)^{\succ}$ such that no $g \in Z(G) \setminus 1$ is infinitesimal with respect to h. Note that then the set of such h together with their inverses forms a non-trivial subgroup $N \leqslant G$: If $h, h' \in N^{\succ}$ and $g \in Z(G)$ with $g \ll hh'$, then without loss of generality, $h' \prec h$ as $\prec$ is total and so $g \ll hh' \prec h^2$ and hence $g \ll h$, contradicting $h \in N$. The argument if one or both of h, h' are negative is similar.

Now to prove that some such h exists. Suppose to the contrary that h does not exist. Then for every $g \in Z(G)$ we find some $g' \in Z(G)$ such that $g' \ll g$. That is, there exist sets of elements

$$g_0 \ll g_1 \ll \cdots \ll g_k$$

of arbitrary length k and $g_i \neq 1$ for every i.

If $g \lll g'$, then g and g' cannot lie in the same cyclic subgroup of G as every full order on $\mathbb{Z}$ is archimedean. So $\langle g, g' \rangle$ is a free-abelian subgroup of $Z(G)$ of rank 2.

Similarly, if $g \lll g' \lll g''$, then every element of $\langle g, g' \rangle$ is infinitesimal with respect to g'', so $g'' \notin \langle g, g' \rangle$. Inductively, we can construct a free-abelian subgroup $\langle g_0, \dots g_k \rangle$ of G of rank $k+1$ for arbitrary $k \in \mathbb{N}$. Taking k to be the rank of $Z(G)$ leads to a contradiction.

Let $N \leqslant Z(G)$ be the subgroup from above and take $g \in G^{\succ} \smallsetminus N$ and $h \in N$. As every subgroup of a nilpotent group has non-trivial intersection with the center, either $g \in Z(G)$, or there exists some $g' \in G$ such that $1 \neq [g, g'] \in Z(G)$. If $g \in Z(G)$, then we already know that $h \lll g'$.

Otherwise, by Lemma 8.3, $[g, g'] \lll g$. As $[g, g']$ is not infinitesimal with respect to h, there exists some $k \in \mathbb{Z}$ such that $h \prec [g, g']^k$. Since $[g, g']^k$ is infinitesimal with respect to g, so is h. □

We are now prepared to prove the following classification of full orders on nilpotent groups.

Theorem 10.24. *Let G be a nilpotent group and $\prec \in T_{\mathrm{ord}}(G)$.*

Then there exists a normal subgroup $N \trianglelefteqslant G$ with $N \neq G$ and an archimedean order $\prec' \in T_{\mathrm{ord}}(G/N)$ such that $\prec$ is lexicographic with respect to the sequence

$$(N, \prec_{|N}) \hookrightarrow G \twoheadrightarrow (G/N, \prec').$$

Proof. If $\prec$ is archimedean, we take $N = \{1\}$ and there is nothing to prove. Otherwise, Lemma 10.23 provides an $N_0 \trianglelefteqslant Z(G)$ such that every element of N_0 is infinitesimal with respect to every positive element of $G \smallsetminus N_0$. By Lemma 10.22, $\prec$ is lexicographic with respect to the sequence $N_0 \hookrightarrow G \twoheadrightarrow G/N_0$.

Let $Q = G/N_0$. Using induction as in Definition 8.5, we may assume that the order on Q is lexicographic with respect to some chain $N_1 \hookrightarrow Q \twoheadrightarrow Q/N_1$ and some archimedean order on Q/N_1.

Every element of N_1 is infinitesimal with respect to every $g \in Q^{\succ} \smallsetminus N_1$ by Lemma 10.21 and so is every element of N_0. Hence $\prec$ is lexicographic with respect to

$$N \hookrightarrow G \twoheadrightarrow Q/N_1$$

by Lemma 10.22, where N is the kernel of the projection $G \twoheadrightarrow Q/N_1$, finishing the proof. □

In the above theorem, as N is itself nilpotent, we may apply the same theorem to N. For a given group, we can do this only finitely many times before one of the factors becomes trivial. Thus, repeated application of Theorem 10.24 decomposes G into several archimedean factors such that we can reassemble G by doing stepwise extensions of these factors and ordering the products lexicographically.

Example 10.25. On $\mathbb{Z}^2$, for every non-archimedean full order $\prec \in T_{\mathrm{ord}}(G)$, there exists some embedding of a subgroup that is isomorphic to $\mathbb{Z}$ such that $\prec$ is lexicographic with respect to the extension $\mathbb{Z} \hookrightarrow \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ and both factors are non-trivially ordered.

On the Heisenberg group H, every non-trivial full order falls into exactly one of these categories:

1. Orders induced by the projection onto the abelianisation $H_{\mathrm{ab}} \cong \mathbb{Z}^2$,
2. orders induced by the inclusion of the center $Z(H) \cong \mathbb{Z}$,
3. and orders that are properly lexicographic with respect to the extension

$$Z(H) \hookrightarrow H \twoheadrightarrow H_{\mathrm{ab}}.$$

10.4. Novikov rings for non-archimedean orders

The crucial property of irrational characters used in the proof of Theorem 2.9 is the following: Let G be a finitely generated torsion-free group with abelianisation G_{ab}. Then for any field k, the Novikov ring $\widehat{kG_{\mathrm{ab}}}^{\Phi}$ is a division ring for every irrational character Φ on G.

In Lemma 9.20, we have seen that an element x of a Novikov ring $\widehat{RG}^{\prec}$ is a unit if and only if $x_{|\min \operatorname{supp} x}$ is a unit in RG. If R is a division ring and $\prec$ is a total archimedean order, this condition is void, and thus every $x \neq 0$ is a unit in $\widehat{RG}^{\prec}$. In particular, this is the case for orders induced by irrational characters on abelian groups. This provides the following connection between the Linnell ring, the Ore localisation and the Novikov ring:

Remark 10.26. If $N \hookrightarrow G \twoheadrightarrow Q$ is an extension of groups and Q is abelian, then

$$\mathcal{D}(G) = \mathcal{D}(N)Q[\mathcal{D}(N)Q \setminus 0]^{-1} = \widehat{\mathcal{D}(N)Q}^{\prec}$$

for every total archimedean order $\prec$ on Q.

For the first equality recall Lemma 10.16. The second equality is a consequence of [Kie20, Section 3.3]. In fact, we do not even explicitly require for this argument that Q is abelian, but other groups do not admit total archimedean orders.

Now suppose that Q is nilpotent and non-abelian. Then $\widehat{\mathcal{D}(N)Q}^{\prec}$ will usually not be a division ring as if $\prec$ is total, it is properly lexicographic by Theorem 10.24 and hence cannot be archimedean by Remark 7.17. But in Section 3.3, we have defined the Mal'cev-Neumann division ring, another extension of the group ring that depends on an order on G and is indeed a division ring. We have also seen in Section 9.2 that it shares many properties with the Novikov ring for orders. In particular, if $\prec$ is a total archimedean order on G, then the Novikov ring and Mal'cev-Neumann ring are equal, as we will see in Lemma 10.30. We use this idea to define a Novikov ring between the two: For an order that is potentially non-archimedean but also not total.

Definition 10.27. Let G be a group carrying a full order $\prec$ and R a ring. Fix some possibly twisted multiplication on RG. Then the $\prec$-*Novikov ring* with R-coefficients is the twisted Novikov ring

$$\widehat{RG}^{\prec} \coloneqq \widehat{RG}^{\mathcal{A}}$$

where

$$\mathcal{A} = \{A \subseteq G \mid A \text{ is well-ordered with respect to } \prec\}.$$

Recall that for totally ordered sets, well-ordered means that every subset has a minimum. In our setting of partial orders, we say that a set A is *well-ordered* if it is well-founded and admits no infinite antichain subsets. A set is *well-founded* if every totally ordered subset is well-ordered in the sense of total orders.

If $N \trianglelefteqslant G$ is an antichain normal subgroup, we denote the *twisted Novikov ring with* $\mathcal{D}(N)$*-coefficients* by

$$F_{\prec}(G, N) \coloneqq \mathcal{D}(\widehat{N)G/N}^{\prec}.$$

Remark 10.28. A set S is well-founded if and only if every subset $T \subseteq S$ admits some $t \in T$ such that $t' \not\prec t$ for every $t' \in T$.

Thus, a set S is well-ordered as a partially ordered set if and only if every sequence of pairwise distinct elements in S admits an ascending subsequence. This is equivalent to no sequence of pairwise distinct elements in S being descending or an antichain.

By ascending sequences we always mean $g_i \in S$ for $i \in \mathbb{N}$ such that $g_i \prec g_{i+1}$ and similar for descending.

If $\prec$ is a full archimedean order, then we have two definitions for $\widehat{RG}^{\prec}$, namely Definition 10.27 and Definition 9.11. The following lemma shows that the two definitions are not in conflict.

Lemma 10.29. *Let G be a group with a full archimedean order $\prec$ and R a ring Let $A \subseteq G$ be any subset.*

Then A is well-ordered if and only if $|S_{\preccurlyeq} \cap A| < \infty$ for every antichain $S \subseteq A$.

Proof. Assume that $|S_{\preccurlyeq} \cap A| < \infty$ for every antichain $S \subseteq A$. Then, in particular, $|S \cap A| < \infty$ and hence A admits no infinite antichains. So, we only have to show that A is well-founded. So let $B \subseteq A$ be totally ordered and $b \in B$. Then $\{b\} \subseteq A$ is an antichain and hence

$$|\{b\}_{\preccurlyeq} \cap B| \leqslant |\{b\}_{\preccurlyeq} \cap A| < \infty.$$

Hence the totally ordered finite set $\{b\}_{\preccurlyeq} \cap B$ has a unique minimum b_0. Let $b' \in B$. Then either $b' > b \geqslant b_0$. Or $b' \leqslant b$ in which case $b' \in \{b\}_{\preccurlyeq} \cap B$ and hence $b_0 \leqslant b'$. Hence b_0 is also the minimum of B and thus A is well-founded.

Now assume that A is well-ordered and let $S \subseteq A$ be an antichain. Suppose $S_{\preccurlyeq} \cap A$ was infinite. By Remark 10.28 there is an ascending sequence $(a_i)_{i \in \mathbb{N}}$ supported on $S_{\preccurlyeq} \cap A$. Assume without loss of generality that the set $\{a_{i+1}{a_i}^{-1} \mid i \in \mathbb{N}\}$ is totally ordered. If not, we may replace the sequence (a_i) with a subsequence with this property. To do this, note that the set $\{a_{i+1}{a_i}^{-1}\}$ has a finite minimum. If for each minimal element m

we keep only those a_i where ${a_i a_{i-1}}^{-1} \succ m$, we obtain a sequence where ${a_{j+1} a_j}^{-1} \succ m$ for every j which is an index of an element we kept. There are only finitely many m, so at least one of those sequences must be infinite. Inductively, we can assume that the remaining elements are well-ordered.

Let $b \coloneqq \min\{{a_{i+1} a_i}^{-1}\}$, which exists because A is well-ordered and is unique because the set is totally ordered. Then by construction, $a_i \leqslant a_0 b^i$ for every $i \in \mathbb{N}$. Also note that b is positive since $a_{i+1} \succ a_i$. As $a_i \in S_{\preccurlyeq}$ and the set of a_i is totally ordered, there exists some $s \in S$ such that $a_i < s$ for every i and hence

$$b^i \leqslant a_i {a_0}^{-1} < s {a_0}^{-1}$$

contradicting $\prec$ being archimedean. □

We show that for total orders, Definition 10.27 is a redefinition of the Mal'cev-Neumann ring $MN_{\prec}(RG)$.

Lemma 10.30. *Let G be a totally ordered group and R a ring. Then $MN_{\prec}(RG) = \widehat{RG}^{\prec}$.*

Proof. First note that a subset $A \subseteq G$ is well-ordered as a totally ordered set if and only if it is well-ordered in the sense of Definition 10.27. Recall that a map $f \colon G \to R$ is an element of $MN_{\prec}(RG)$ if and only if $\operatorname{supp} f$ is well-ordered. This is precisely the definition of $\widehat{RG}^{\prec}$. □

Thus, the Novikov ring we defined in Definition 10.27 generalises both the Novikov ring for archimedean orders and the Mal'cev-Neumann ring.

An essential and well-known fact is that if $\mathcal{D}(N)$ is a division ring and $\prec \in T_{\mathrm{ord}}(G/N)$ is a total order, then $F_{\prec}(G, N)$ is a division ring. See for example [Kie20, Definition 2.29]. This is why we consider the Mal'cev-Neumann ring $F_{\prec}(G, N)$ in the first place.

I am unaware of any sources where this fact is proven, so we include proof here. As we have done for Novikov rings for archimedean orders in Lemma 9.20, we investigate what units in the expanded Novikov rings look like. The proof and statement are essentially analogous to the archimedean case, but things get a bit more complicated. We allow ourselves to assume that our group is nilpotent as that will be the case we are interested in. First, a lemma that we will need for the main proof.

Lemma 10.31. *Let G be a nilpotent group, $\prec \in T_{\mathrm{ord}}(G)$, R a ring and $x \in \widehat{RG}^{\prec}$ such that $\operatorname{supp} x \subseteq G^{\succ}$. Then*

$$\bigcup_{i \in \mathbb{N}} \operatorname{supp} x^i$$

admits no descending sequence.

Proof. By Theorem 10.24, there exists a filtration of G by normal subgroups $H_i \trianglelefteqslant G$ such that $\prec_{|H_i}$ is lexicographic with respect to $H_{i-1} \hookrightarrow H_i \twoheadrightarrow H_i/H_{i-1}$ and $H_0 = \{1\}$.

Because the orders are lexicographic, every element of H_{i-1} is infinitesimal with repsect to any positive element of $H_i \smallsetminus H_{i-1}$ by Lemma 10.21.

Suppose that (g_k) is a descending sequence in $\bigcup_{i\in\mathbb{N}} \operatorname{supp} x^i$. We may write g_0 as a product of finitely many elements of $\operatorname{supp} x$. Hence, there is some minimal n such that one of the factors of g_0 is in H_n. As $g_k \prec g_0$ for $k > 0$, in particular, every g_k can be written as a product of finitely many factors in $H_n \cap \operatorname{supp} x$: g_k can be written as a product of factors in $\operatorname{supp} x$. As these factors are all positive, if one factor g' is in H_N for some $N > n$, then $g_k \succ g' \gg g_0$.

By dropping some prefix of the sequence (g_k), we can assume that every g_k contains at least one factor in $H_n \smallsetminus H_{n-1}$.

Recall that $\prec$ is lexicographic with respect to $H_{n-1} \hookrightarrow H_n \twoheadrightarrow H_n/H_{n-1}$. Let $[g_k]$ be the image of g_k under the projection $H_n \twoheadrightarrow H_n/H_{n-1}$. Then $[g_{k+1}] \preccurlyeq [g_k]$ for every k as $g_{k+1} \prec g_k$ means either $[g_{k+1}] \prec [g_k]$ or ${g_{k+1}}^{-1} g_k \in {H_{n-1}}^{\succ} \subseteq H_{n-1}$ by definition of the lexicographic order. The group H_n/H_{n-1} is archimedean, as if $1 \prec g \ll h$ for some $h \in H_n$, then $g \in H_{n-1}$. We have already seen in the proof of Lemma 9.20 that then $\bigcup_{i\in\mathbb{N}} H_n^i$ admits no descending sequence. Hence, there is an infinite subset $J \subseteq \mathbb{N}$ such that all $[g_j]$ with $j \in J$ are pairwise equal.

The sequence $g_j {g_{\min J}}^{-1}$ is a sequence in H_{n-1}, and it is descending because it is a translate of a subsequence of (g_k). Inductively, every H_i for $i \leqslant n$ admits a descending sequence. This is, in particular, true for H_0. But $H_0 = \{1\}$. This contradicts the existence of the sequence (g_k). □

Proposition 10.32. *Let G be a nilpotent group, $\prec \in T_{\mathrm{ord}}(G)$ and R a ring. Let $x \in \widehat{RG}^{\prec}$ be such that $\operatorname{supp} x \subseteq G^{\succ}$. Then $1 - x$ is invertible in $\widehat{RG}^{\prec}$ and $(1-x)^{-1} = \sum_{i=0}^{\infty} x^i$.*

Moreover, an element $y \in \widehat{RG}^{\prec} \smallsetminus 0$ is invertible if and only if $y_{|\min \operatorname{supp} y}$ is a unit in RG.

Proof. We only need to show that $\sum_{i=0}^{\infty} x^i$ is a well-defined element of $MN_{\prec}(RG)$ as then

$$(1-x)\sum_{i=0}^{\infty} x^i = \sum_{i=0}^{\infty} x^i - \sum_{i=0}^{\infty} x^{i+1} = 1.$$

So let $g \in G$. To see that the sum is well-defined, We need to show that $g \in \operatorname{supp} x^i$ for at most finitely many i. To the contrary, let us assume that $g \in \operatorname{supp} x^i$ for infinitely many i. Without loss of generality, no element of any $\operatorname{supp} x^j$ is smaller than g and also contained in infinitely many $\operatorname{supp} x^i$. If this is not the case, we replace g with some smaller element. If after this, there is still some element smaller than g and the process of replacing g does not terminate after finitely many steps, then we have constructed a descending sequence in $\operatorname{supp} x$. This contradicts the fact that since $x \in \widehat{RG}^{\prec}$, $\operatorname{supp} x$ is well-ordered and thus admits no descending sequences by Remark 10.28.

For brevity, suppose that $g \in \operatorname{supp} x$.

Then there are $g_i \in \operatorname{supp} x$ and $h_i \in \operatorname{supp} x^i$ such that $g_i h_i = g$ for infinitely many i.

By Lemma 10.31, there is no descending sequence in the set of h_i. Hence, there is no ascending sequence in the set of g_i. There is also no descending sequence in g_i and no infinite antichain in g_i. Thus, the set of g_i is finite.

But then, the set of $h_i = {g_i}^{-1}g$ is also finite and hence some $h \in \{h_i\}$ is contained in infinitely many $\operatorname{supp} x^i$. As $h = h_i \prec g_i h_i = g$, this contradicts the assumption that no element smaller than g is contained in infinitely many $\operatorname{supp} x^i$.

It remains to show that $\sum_{i=0}^{\infty} x^i \in MN_{\prec}(RG)$. That is, that the support is well-ordered. Suppose to the contrary that the support contained a descending sequence (g_i). As no $\operatorname{supp} x^i$ contains a descending sequence and hence at most finitely many g_i, there exists a subsequence of (g_i) and an ascending sequence of indices j such that $g_i \in \operatorname{supp} x^j$. But analogous to the proof of Lemma 9.20, this cannot happen. □

So in, particular, let $N \hookrightarrow G \twoheadrightarrow Q$ be an extension of groups, $\mathcal{D}(N)$ a division ring, Q nilpotent and $\prec$ a total order on Q. Then every $x \in F_{\prec}(G, N)$ may be written as $x = x_0 g_0 (1 - x_+)$ where $\{g_0\} = \min \operatorname{supp} x$ and $x_0 = {x_{|\min \operatorname{supp} x}\ g_0}^{-1} \in \mathcal{D}(G)$ and $\operatorname{supp} x_+ \subseteq G^{\succ}$. Hence $F_{\prec}(G, N)$ is a division ring.

10.5. Non-archimedean Novikov homology

Now, we have transferred most objects that appear in [Kie20] to the setting of RFN groups. We have omitted two things so far: the topology on the character sphere and the definition of irrational characters. This is because it is not clear to me yet what precisely their analogues should be. In this section, we will make a leap of faith and assume that such analogues exist.

Assumption 10.33. *There is a topology on the space of full orders on G and a notion of* irrational order *that are in some sense sufficiently nice.*

Then, we outline the proof of Theorem 2.9 in [Kie20] to see how it might generalise to RFN groups.

Throughout this section, we fix the following objects:

1. G is a RFN group with witnessing chain G_k.
2. For $k \in \mathbb{N}$ let $G'_k \trianglelefteq G_{k+1}$ such that G_k/G'_k is torsion-free nilpotent. We also write $G' = G'_0$.

   Recall Remark 10.2. We will assume that $G'_k = {G_k}^{(n_k)}$ where n_k is the nilpotency class of G_k/G'_k.

   Note that the projection $G_k \twoheadrightarrow G_k/G_{k+1}$ factors not only through ${G_k}^{(n_k)}$ but also through every subgroup of ${G_k}^{(n_k)}$. In particular, it factors through ${G_k}^{(N)}$ for $N \geqslant n_k$. This allows us to additionally assume that the sequence $(n_k)_{k\in\mathbb{N}}$ is non-decreasing. Hence
   $$G'_k = {G_k}^{(n_k)} \geqslant {G_{k+1}}^{(n_{k+1})} = G'_{k+1}.$$
3. For every k, Lemma 10.16 provides a twisted group ring structure on $\mathcal{D}(G'_k)G_k/G'_k$ such that it is isomorphic to $\mathcal{D}(G)$. We identify the two rings by fixing one such isomorphism for every k. Whenever we speak about $\mathcal{D}(G'_k)G_k/G'_k$, we interpret it as this twisted group ring.

We start by elaborating on the meaning of "sufficiently nice" in Assumption 10.33. In Remark 6.2, we have seen that the character sphere $S(G)$ is a topological space. Note that every character factors through the abelianisation of G. Hence $S(G) = S(G, N)$ if N is the kernel of the projection $G \twoheadrightarrow G_{\mathrm{ab}}$. The analogue of $S(G)$ in our setting is $T_{\mathrm{ord}}(G, G')$, so we will assume that $T_{\mathrm{ord}}(G, G')$ is a topological space as well.

There are several ways of defining such a topology. Whatever the topology, knowing the structure of $T_{\mathrm{ord}}(G, G')$ as a set by using the characterisation of orders on nilpotent groups from Theorem 10.24 and Theorem 8.6 will be crucial. Similarly, we assume a topology on all the $T_{\mathrm{ord}}(G_k, {G_k}')$.

To define such a topology, a good start is to aks that it restricts to the topology on the character sphere $S(G)$. Further, the projection that sends an order that is lexicographic with respect to the extension $N \hookrightarrow G \twoheadrightarrow Q$ to the order induced by Q should be continuous. There is a unique minimal topology satisfying these two assumptions on $T_{\mathrm{ord}}(\mathbb{Z}^2)$, namely

$$T_{\mathrm{ord}}(\mathbb{Z}^2) = (S(\mathbb{Z}^2) \cup \{1\}) \times (S(\mathbb{Z}) \cup \{1\}) / \sim$$

where $(\varphi, \psi) \sim (\varphi', \psi')$ if $\varphi = \varphi'$ and φ is irrational. To see this, recall that any full order on $\mathbb{Z}^2$ is either induced by an irrational character or it is lexicographic with respect to $\mathbb{Z} \hookrightarrow \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$. Every full order on $\mathbb{Z}$ is induced by a character. Further, recall that $S(\mathbb{Z}^n) \cup \{1\}$ is an $(n-1)$-sphere with an added isolated point for the trivial character. Hence $T_{\mathrm{ord}}(()\mathbb{Z}^2)$ is three copies of S^1 glued at all irrational points. It has yet to be determined if this construction is sufficient in general.

Remark 10.34. Note that we may restrict any order on G_k to an order on the subgroup G_{k+1}. As $G_k' \geqslant G_{k+1}'$, an order in $T_{\mathrm{ord}}(G_k, {G_k}')$ restricts to an order in $T_{\mathrm{ord}}(G_{k+1}, {G_{k+1}}')$. Thus, the restriction of orders induces a map $T_{\mathrm{ord}}(G_k, {G_k}') \to T_{\mathrm{ord}}(G_{k+1}, {G_{k+1}}')$.

Suppose that $\prec \in T_{\mathrm{ord}}(G_{k+1}, {G_{k+1}}')$ is a restriction of some order $\prec' \in T_{\mathrm{ord}}(G_k, {G_k}')$. Then this $\prec'$ is unique, as for any $g \in G_k$ there exists some n such that $g^n \in G_{k+1}$. Since $\prec'$ is full we get

$$g \in {G_k}^{\succ'} \iff g^n \in {G_k}^{\succ'} \iff g^n \in {G_{k+1}}^{\succ}.$$

Hence we get an embedding $T_{\mathrm{ord}}(G_k, {G_k}') \hookrightarrow T_{\mathrm{ord}}(G_{k+1}, {G_{k+1}}')$. We implicitly use this embedding to see $T_{\mathrm{ord}}(G_k, {G_k}')$ as a subspace of $T_{\mathrm{ord}}(G_{k+1}, {G_{k+1}}')$.

As an additional requirement for our topology, let us assume that $T_{\mathrm{ord}}(G_k, {G_k}')$ carries the subspace topology with respect to $T_{\mathrm{ord}}(G_{k+1}, {G_{k+1}}')$. Thus, we get a sequence of topological spaces

$$T_{\mathrm{ord}}(G_0, {G_0}') \hookrightarrow T_{\mathrm{ord}}(G_1, {G_1}') \hookrightarrow \dots$$

where each map is a homeomorphism onto its image.

The other important definition will be that of an irrational order. Recall that a character is irrational if it is injective on the abelianisation. Equivalently, a character is irrational if the order it induces on the abelianisation is total. The important property is that $F_{\prec}(G, N)$ is a division ring if N is the kernel of the free abelianisation map and $\prec$ is induced by an irrational character. Thus, we will require at least that irrational

orders are induced by total orders on G/G'. But maybe not all such orders should be considered irrational. Whatever they are, let us denote the space of irrational orders on G by $T^{\mathrm{irr}}_{\mathrm{ord}}(G)$.

We will require that all orders in $T_{\mathrm{ord}}(G, G')$ are suborders of some irrational order. This is justified by the observation that every order on G/G' can be extended to a total order:

Let $\prec \in T_{\mathrm{ord}}(G, G') = T_{\mathrm{ord}}(G/G')$. Recall that by Theorem 10.24 together with Theorem 8.6, we may write G/G' as a sequence of extensions. Each factor carries a total archimedean or trivial order, and every extension carries the lexicographic order. By replacing every trivial order with some total order, we can construct a total order $\prec'$ on G/G' such that $\prec$ is a suborder of $\prec'$.

As we did for the topology on $T_{\mathrm{ord}}(G, G')$, we may analogously ask for a notion of irrational orders on the G_k.

The following definitions will depend on the topology on $T_{\mathrm{ord}}(G, G')$ and the definition of $T^{\mathrm{irr}}_{\mathrm{ord}}(G)$. They are generalisations for the same notions in the RFRS setting where $T_{\mathrm{ord}}(G) = S(G)$ and $T^{\mathrm{irr}}_{\mathrm{ord}}(G)$ is the space of orders induced by irrational characters.

Definition 10.35.

1. A subset $U \subseteq T_{\mathrm{ord}}(G, G')$ is *rich* if it is open and $T^{\mathrm{irr}}_{\mathrm{ord}}(G) \subseteq U$.
2. A subset $U \subseteq T_{\mathrm{ord}}(G_k, {G_k}')$ for $k > 0$ is *rich*, if $\overline{U}^{\mathrm{o}} \cap T_{\mathrm{ord}}(G_{k-1}, {G_{k-1}}')$ is rich as a subset of $T_{\mathrm{ord}}(G_{k-1}, {G_{k-1}}')$.

   By $\overline{U}^{\mathrm{o}}$ we denote the inner part of the closure of U in $T_{\mathrm{ord}}(G_k, {G_k}')$.

Recall Lemma 10.16 stating that $\mathcal{D}(G) = \mathcal{D}(G')G/G'[S]^{-1}$ where $S = \mathcal{D}(G')G/G' \setminus 0$. Let $\prec \in T_{\mathrm{ord}}(G, G')$ and $\prec' \in T^{\mathrm{irr}}_{\mathrm{ord}}(G)$ such that $\prec$ is a suborder of $\prec'$. Then $F_{\prec'}(G, G')$ is a division ring containing $\mathcal{D}(G')G/G'$. Thus $F_{\prec'}(G, G')$ even contains $\mathcal{D}(G)$ by the universal localisation property.

The ring $F_{\prec'}(G, G') = \mathcal{D}(\widehat{G'G}/G')^{\prec'}$ also contains $\widehat{\mathbb{Q}G}^{\prec} = (\mathbb{Q}\widehat{G'G}/G')^{\prec}$ via the embedding of coefficients $\mathbb{Q}G' \hookrightarrow \mathcal{D}(G')$. We also have to use the fact that $\prec$ is a suborder of $\prec'$, and hence every set that is well-ordered with respect to $\prec$ is also well-ordered with respect to $\prec'$. If $x \in \mathcal{D}(G)$, we may hence ask if $x \in \mathcal{D}(G) \cap \widehat{\mathbb{Q}G}^{\prec} \subseteq F_{\prec'}(G, G')$. In this case, we simply write $x \in \widehat{\mathbb{Q}G}^{\prec}$. Note that this statement is relative to some irrational order $\prec'$. This observation leads to the following definition.

Definition 10.36. Let $\prec \in T_{\mathrm{ord}}(G, G')_k$ and $x \in \mathcal{D}(G_k)$. Then x is *representable* with respect to $\prec$ if $x \in \widehat{\mathbb{Q}G_k}^{\prec}$ for every $\prec' \in T^{\mathrm{irr}}_{\mathrm{ord}}(G_k)$ such that $\prec$ is a suborder of $\prec'$.

For $U \subseteq T_{\mathrm{ord}}(G, G')$, we denote by $\mathcal{D}(G, U)$ the set of elements of $\mathcal{D}(G)$ that are representable for every $\prec \in U$.

Remark 10.37. Note that the embedding $\mathcal{D}(G, \{\prec\}) \hookrightarrow \mathcal{D}(G)$ is independent of the choice of $\prec'$.

Thus

$$\mathcal{D}(G, \{\prec\}) = \mathcal{D}(G) \cap \widehat{\mathbb{Q}G}^{\prec} \leqslant F_{\prec'}(G, G')$$

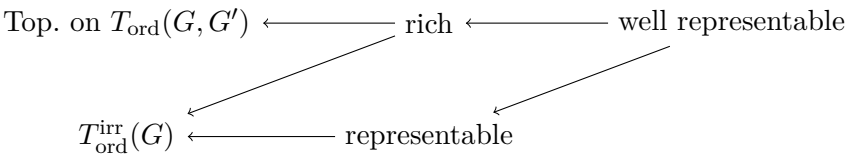


Figure 10.2.: Dependencies between the definitions in this section.

is an intersection of subrings of $F_{\prec'}(G, G')$ and therefore itself a ring. Similarly,

$$\mathcal{D}(G, U) = \bigcap_{\prec \in U} \mathcal{D}(G, \{\prec\})$$

is a ring.

Definition 10.38. Let $k \in \mathbb{N}$. We say that an element $x \in \mathcal{D}(G)$ is *well representable* if there exists an $n \geqslant 0$ and rich sets $U_i \subseteq T_{\mathrm{ord}}(G_i, G_i{}')$ for every $i \geqslant n$ such that $x \in \mathcal{D}(G_i, U_i)G/G_i$.

In Figure 10.2, we see a dependency graph for the definitions we just made. Note that, in particular, the definition of well representable depends on the topology on $T_{\mathrm{ord}}(G, G')$ and the definition of $T^{\mathrm{irr}}_{\mathrm{ord}}(G)$.

Many statements and proofs in [Kie20] translate directly to the generalised definitions, as long as our topology satisfies the right assumptions. This makes me reasonably confident that the following statement is true. If G is RFRS, this is a simplified version of [Kie20, Theorem 4.13].

Conjecture 10.39. *There exist topologies on $T_{\mathrm{ord}}(G_i, G_i{}')$ and notions of irrational orders $T^{\mathrm{irr}}_{\mathrm{ord}}(G_i)$ such that every element of $\mathcal{D}(G)$ is well representable.*

For G RFRS, the following theorem is very close to [Kie20, Theorem 5.2]. The proof presented here is an adapted version of Kielak's proof. We include this here to illustrate that much is independent of whether G is RFRS or RFN and also not on the concrete topology on $T_{\mathrm{ord}}(G)$ or notion of irrationality.

Theorem 10.40. *Assume that Conjecture 10.39 is true.*

Let G be a finitely generated infinite RFN group such that $\beta_1^2(G) = 0$. Let G_k be a witnessing chain and C_* a free resolution of $\mathbb{Q}$ by free $\mathbb{Q}G$-modules such that C_i is finite-dimensional for $i < 2$.

Then there exists a $k \in \mathbb{N}$ and a rich set $U \subseteq T_{\mathrm{ord}}(G_k, G_k{}')$ such that

$$H_1(\widehat{\mathbb{Q}G_k}^{\prec} \otimes_{\mathbb{Q}G} C_*) = 0$$

for every $\prec \in U$.

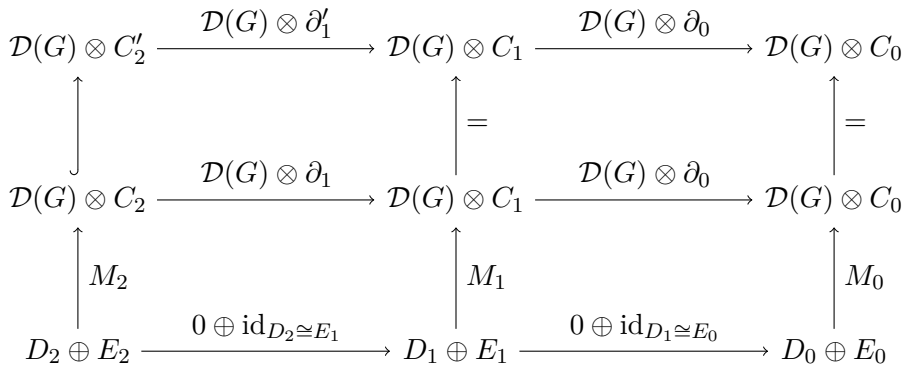


Figure 10.3.: A commutative diagram of $\mathcal{D}(G)$-vector spaces. Every row is exact. All maps except for ∂_1' may be represented as finite matrices with entries in $\mathcal{D}(G)$. Except for the M_i, these matrices even have entries in $\mathbb{Q}G$.

Proof. Figure 10.3 is an illustration of this proof. All tensoring is over $\mathbb{Q}G$.

First, we may replace C_2 with some finite-dimensional submodule such that the sequence $C_2 \to C_1 \to C_0 \to \mathbb{Q} \to 0$ is still exact. For notational convenience, let us denote the original module by C_2' and the finite-dimensional one by C_2. Similarly, ∂_1 is actually the restriction of the boundary map ∂_1' of the original sequence.

As $\beta_1^2(G) = 0$, the sequence $\mathcal{D}(G) \otimes C_*$ is an exact sequence of $\mathcal{D}(G)$-vector spaces. For $i \in \{-1, 0, 1\}$, let $D_i = \mathcal{D}(G) \otimes \ker \partial_i$ and E_i a $\mathcal{D}(G)$-vector space such that

$$D_i \oplus E_i = \mathcal{D}(G) \otimes_{\mathbb{Q}G} C_i.$$

Then

$$(\mathcal{D}(G) \otimes \partial_i) D_i = 0$$

and $(\mathcal{D}(G) \otimes \partial_i)_{|E_i}$ is an isomorphism onto D_{i-1}.

That is, up to change of basis, $\mathcal{D}(G) \otimes \partial_i = 0 \oplus \mathrm{id}$. Let $M_i \in \mathrm{GL}(\mathcal{D}(G))$ be the matrix realising this change of basis. Or, put differently,

$$\mathcal{D}(G) \otimes \partial_i = M_i \circ (0 \oplus \mathrm{id}) \circ {M_{i+1}}^{-1}.$$

The entries of the M_i and their inverses are elements of $\mathcal{D}(G)$. By Conjecture 10.39 they are well representable. Thus for each such entry x, there exists some n and rich sets U_i for $i \geqslant n$ such that $x \in \mathcal{D}(G_i, U_i)$. As the matrices M_i and their inverses have in total only finitely many entries, we may take k to be the maximum of the n and U the intersection of the U_k. If we know that the intersection of finitely many rich sets is again rich, then this U is a rich set. For RFRS groups, this is [Kie20, Lemma 4.4], and the proof for RFN groups is exactly the same.

Now the M_i and their inverses are matrices with entries in $\mathcal{D}(G_k, U)$. Let $\prec$ be an order in U. The entries of the M_i are then also elements of $\widehat{\mathbb{Q}G}^{\prec}$. Hence, if we replace every $\mathcal{D}(G) \otimes C_i$ in Figure 10.3 by $\widehat{\mathbb{Q}G}^{\prec} \otimes C_i$ and D_i, E_i by free $\widehat{\mathbb{Q}G}^{\prec}$-modules of the

same dimension, we can still make sense of all the maps in the diagram, and the diagram still commutes. The bottom row is still exact, and the M_i are still isomorphisms. Thus, the middle row is also exact, proving

$$H_1(\widehat{\mathbb{Q}G}^{\prec} \otimes C_*) = 0.$$

□

This concludes a major step in proving that G is virtually fibred. To see what obstacles remain, we outline how Kielak's argument in case G is RFRS continues from here:

Take k and U as in Theorem 10.40. Recall that in this setting, U will be a set of characters on G_k. Show that every rich set - and hence in particular U - contains a rich subset U' such that $\Phi \in U' \iff -\Phi \in U'$. Then use richness to show that U' contains some character Φ with image isomorphic to $\mathbb{Z}$. By construction of U, $H_1(G_k, \widehat{\mathbb{Q}G_k}^{\Phi}) = 0$ and thus $H_1(G_k, \widehat{\mathbb{Q}G_k}^{-\Phi}) = 0$ as well. Theorem 9.2 states that then $\ker \Phi$ is finitely generated. Hence G_k fibres. As G_k is a finite index subgroup of G the latter fibres virtually. In fact, the version of Theorem 9.2 we stated here is for $\widehat{\mathbb{Z}G}^{\Phi}$, but the theorem holds analogously for $\widehat{\mathbb{Q}G}^{\Phi}$.

It is plausible that these arguments apply analogously to RFN groups. The critical steps are Conjecture 10.39 and Theorem 9.2. We have shown in Theorem 9.36 that Theorem 9.2 holds analogously for full archimedean orders on G.

A crucial point will be to find the correct notion of rich set as this dictates what assumptions we may make about the elements of U. For the above proof to work for RFN groups, we must either ensure that U contains enough archimedean orders. Or, we will need an even more general version of Theorem 9.36 that also works with Novikov rings for non-archimedean orders. This would provide a positive answer to Question 10.7 for all RFN groups:

Conjecture 10.41. *Let G be a finitely generated group that is virtually RFN, and suppose that $\beta_1^2(G) = 0$. Then G is virtually fibred.*

Part V.

Appendix

A. Noncomputability and the axiom of choice

In Chapter 5, we have discussed the use of the `noncomputable` environment in Lean. In our code, we tried to avoid it if possible. This makes all our definitions constructive, allowing us to actually inspect the objects we defined. On the other hand, using `noncomputable` is warranted whenever it simplifies things. In this chapter, we spend some thought on what `noncomputable` means for the natural language interpretation of the Lean code.

Consider the following two definitions.

Definition A.1. A *nonempty set* is a set S such that there exists an $s \in S$.

Definition A.2. A *nonempty set* is a pair (S, s) of a set S and an element s such that $s \in S$. Two pairs (S, s) and (T, t) are *equivalent* if $S = T$.

At first glance, the difference between Definition A.1 and Definition A.2 up to equivalence seems moot. However, in a language as precise as Lean, we have to choose between these two options when making a definition. So, it is worth investigating what precisely the differences are.

Those definitions in Lean might look something like

```
class nonempty_set (S : Type*) [Set S] := (nonempty : ∃(s : S), s ∈ S)
```

and

```
class nonempty_set' (S : Type*) [Set S] := (s : S; hs: s ∈ S)
```

leaving out the equivalence in the second definition. The fact that neither definition would really be useful most of the time is beside the point here. In Chapter 5, we have seen that the same principle applies to more general and more sensible settings.

Let us denote the category of objects that match Definition A.1 by Set and the category of objects that match Definition A.2 by Set_*. In the second case, we really mean the objects, not their equivalence classes.

In either category, the morphisms are normal maps between sets. Note that even though sets in Set_* have a distinguished point, we do not require that maps send the distinguished point of the domain to the distinguished point in the codomain. Every map of sets is a map in Set_*. To make our question what the difference between the two definitions is precise, we ask if Set and Set_* are equivalent categories.

Proposition A.3. *The equivalence of the categories* Set *and* Set_* *is equivalent to the axiom of choice.*

The formulation of the axiom of choice that we use here is the following one:

Axiom A.4. *Let $\mathcal{A}$ be a set of nonempty sets. Here, we mean nonempty in the sense that they are not equal to the empty set. Then there exists a map*

$$f\colon \mathcal{A} \to \bigcup_{A\in\mathcal{A}} A$$

such that $f(A) \in A$ for every $A \in \mathcal{A}$.

An equivalence of categories is a pair of adjoint functors

$$F\colon \text{Set} \to \text{Set}_*, \quad G\colon \text{Set}_* \to \text{Set}$$

that are both full and faithful. That is for every $A, A' \in \text{Set}$ and $B \in \text{Set}_*$,

$$\text{Hom}_{\text{Set}}(A, G(B)) = \text{Hom}_{\text{Set}_*}(F(A), B),$$

and the map

$$\text{Hom}_{\text{Set}}(A, A') \to \text{Hom}_{\text{Set}_*}(F(A), F(A'))$$

induced by F is a natural bijection and analogously for G.

Let us now prove Proposition A.3. The condition that the hom-sets are isomorphic means that on the level of objects, A and $F(A)$ have to be isomorphic as sets. Thus the functor F maps every set A to a pair (A, a_A). When $\mathcal{A}$ is a set of sets, we may apply F to the subcategory of Set containing all sets in $\mathcal{A}$. This produces a map

$$f\colon \mathcal{A} \to \bigcup_{A\in\mathcal{A}} A, \quad A \mapsto a_A$$

as required by the axiom of choice.

Conversely, any such f may be turned into a functor F by setting $F(A) = (A, f(A))$. This functor is adjoint to the forgetful functor

$$\text{Set}_* \to \text{Set}, \quad (S, s) \mapsto S.$$

Hence, the axiom of choice provides an equivalence of categories.

We interpret the equivalence of the categories Set and Set_* as Definition A.1 and Definition A.2 defining essentially the same thing. In Lean, the `noncomputable` environment may be seen as assuming the axiom of choice. Indeed, the two Lean definitions above can be proven equivalent in Lean if we assume noncomputability. But Definition A.2 has the added benefit that we do not need noncomputability in case it is easy to provide an actual element of each set. This covers every case where existence is proved constructively. Only if no constructive proof exists or none can be provided, we need to transfer to a noncomputable environment to obtain Definition A.1 from Definition A.2.

It is a nice bonus to see that most of our theory is independent of the axiom of choice.

B. Lean code for the Ore localisation

```
/- File: OreSet.lean -/

import Mathlib.Algebra.Ring.Regular
import Mathlib.GroupTheory.Submonoid.Basic

namespace OreLocalization

section Monoid

class OreSet {R : Type*} [Monoid R] (S : Submonoid R) where
  ore_left_cancel
    : ∀ (r₁ r₂ : R) (s : S), ↑s * r₁ = s * r₂ → ∃ s' : S, r₁ * s' = r₂ * s'
  oreNum : R → S → R
  oreDenom : R → S → S
  ore_eq : ∀ (r : R) (s : S), r * oreDenom r s = s * oreNum r s

variable {R : Type*} [Monoid R] {S : Submonoid R} [OreSet S]

theorem ore_left_cancel (r₁ r₂ : R) (s : S) (h : ↑s * r₁ = s * r₂) :
  ∃ s' : S, r₁ * s' = r₂ * s' := OreSet.ore_left_cancel r₁ r₂ s h

def oreNum (r : R) (s : S) : R :=
  OreSet.oreNum r s

def oreDenom (r : R) (s : S) : S :=
  OreSet.oreDenom r s

theorem ore_eq (r : R) (s : S) : r * oreDenom r s = s * oreNum r s :=
  OreSet.ore_eq r s

def oreCondition (r : R) (s : S) : Σ'r' : R, Σ's' : S, r * s' = s * r' :=
  ⟨oreNum r s, oreDenom r s, ore_eq r s⟩

instance oreSetBot : OreSet (⊥ : Submonoid R)
    where
  ore_left_cancel _ _ s h :=
    ⟨s, by
      rcases s with ⟨s, hs⟩
      rw [Submonoid.mem_bot] at hs
      subst hs
      rw [one_mul, one_mul] at h
      subst h
      rfl⟩
  oreNum r _ := r
  oreDenom _ s := s
  ore_eq _ s := by
```

```
    rcases s with ⟨s, hs⟩
    rw [Submonoid.mem_bot] at hs
    simp [hs]

instance (priority := 100) oreSetComm {R} [CommMonoid R] (S : Submonoid R) :
    OreSet S where
  ore_left_cancel m n s h := ⟨s, by rw [mul_comm n s, mul_comm m s, h]⟩
  oreNum r _ := r
  oreDenom _ s := s
  ore_eq r s := by rw [mul_comm]

end Monoid

def oreSetOfCancelMonoidWithZero {R : Type*} [CancelMonoidWithZero R]
    {S : Submonoid R}
    (oreNum : R → S → R) (oreDenom : R → S → S)
    (ore_eq : ∀ (r : R) (s : S), r * oreDenom r s = s * oreNum r s) :
    OreSet S :=
  { ore_left_cancel := fun _ _ s h =>
      ⟨s, mul_eq_mul_right_iff.mpr (mul_eq_mul_left_iff.mp h)⟩
    oreNum
    oreDenom
    ore_eq }

def oreSetOfNoZeroDivisors {R : Type*} [Ring R] [NoZeroDivisors R]
    {S : Submonoid R}
    (oreNum : R → S → R) (oreDenom : R → S → S)
    (ore_eq : ∀ (r : R) (s : S), r * oreDenom r s = s * oreNum r s) :
    OreSet S :=
  letI : CancelMonoidWithZero R := NoZeroDivisors.toCancelMonoidWithZero
  oreSetOfCancelMonoidWithZero oreNum oreDenom ore_eq

end OreLocalization
```

```
/- File: Basic.lean -/

import Mathlib.GroupTheory.MonoidLocalization
import Mathlib.Algebra.GroupWithZero.NonZeroDivisors
import Mathlib.RingTheory.OreLocalization.OreSet
import Mathlib.Tactic.NoncommRing


universe u

open OreLocalization

namespace OreLocalization

variable (R : Type*) [Monoid R] (S : Submonoid R) [OreSet S]

def oreEqv : Setoid (R × S) where
  r rs rs' := ∃ (u : S) (v : R), rs'.1 * u = rs.1 * v ∧
    (rs'.2 : R) * u = rs.2 * v
  iseqv := by
    refine ⟨fun _ => ⟨1, 1, by simp⟩, ?_, ?_⟩
    · rintro ⟨r, s⟩ ⟨r', s'⟩ ⟨u, v, hru, hsu⟩; dsimp only at *
      rcases oreCondition (s : R) s' with ⟨r₂, s₂, h₁⟩
      rcases oreCondition r₂ u with ⟨r₃, s₃, h₂⟩
      have : (s : R) * ((v : R) * r₃) = (s : R) * (s₂ * s₃) := by
        rw [← mul_assoc _ (s₂ : R), h₁, mul_assoc, h₂,
            ← mul_assoc, ← hsu, mul_assoc]
      rcases ore_left_cancel (v * r₃) (s₂ * s₃) s this with ⟨w, hw⟩
      refine ⟨s₂ * s₃ * w, u * r₃ * w, ?_, ?_⟩ <;>
        simp only [Submonoid.coe_mul, ← hw]
      · simp only [← mul_assoc, hru]
      · simp only [← mul_assoc, hsu]
    · rintro ⟨r₁, s₁⟩ ⟨r₂, s₂⟩ ⟨r₃, s₃⟩ ⟨u, v, hur₁, hs₁u⟩
        ⟨u', v', hur₂, hs₂u⟩
      rcases oreCondition v' u with ⟨r', s', h⟩; dsimp only at *
      refine ⟨u' * s', v * r', ?_, ?_⟩ <;>
        simp only [Submonoid.coe_mul, ← mul_assoc]
      · rw [hur₂, mul_assoc, h, ← mul_assoc, hur₁]
      · rw [hs₂u, mul_assoc, h, ← mul_assoc, hs₁u]

end OreLocalization

def OreLocalization (R : Type*) [Monoid R] (S : Submonoid R)
    [OreSet S] :=
  Quotient (OreLocalization.oreEqv R S)

namespace OreLocalization

section Monoid

variable {R : Type*} [Monoid R] {S : Submonoid R}

variable (R S) [OreSet S]
```

```
@[inherit_doc OreLocalization]
scoped syntax:1075 term noWs atomic("[" term "⁻¹" noWs "]") : term
macro_rules | `($R[$S⁻¹]) => ``(OreLocalization $R $S)

attribute [local instance] oreEqv

variable {R S}

def oreDiv (r : R) (s : S) : R[S⁻¹] :=
  Quotient.mk' (r, s)

@[inherit_doc]
infixl:70 " /ₒ " => oreDiv

@[elab_as_elim]
protected theorem ind {β : R[S⁻¹] → Prop} (c : ∀ (r : R) (s : S),
    β (r /ₒ s)) : ∀ q, β q := by
  apply Quotient.ind
  rintro ⟨r, s⟩
  exact c r s

theorem oreDiv_eq_iff {r₁ r₂ : R} {s₁ s₂ : S} :
    r₁ /ₒ s₁ = r₂ /ₒ s₂ ↔ ∃ (u : S) (v : R),
      r₂ * u = r₁ * v ∧ (s₂ : R) * u = s₁ * v :=
  Quotient.eq''

protected theorem expand (r : R) (s : S) (t : R) (hst : (s : R) * t ∈ S) :
    r /ₒ s = r * t /ₒ ⟨s * t, hst⟩ := by
  apply Quotient.sound
  refine' ⟨s, t * s, _, _⟩ <;> dsimp <;> rw [mul_assoc]

protected theorem expand' (r : R)
    (s s' : S) : r /ₒ s = r * s' /ₒ (s * s') :=
  OreLocalization.expand r s s' (by norm_cast; apply SetLike.coe_mem)

protected theorem eq_of_num_factor_eq {r r' r₁ r₂ : R} {s t : S}
    (h : r * t = r' * t) : r₁ * r * r₂ /ₒ s = r₁ * r' * r₂ /ₒ s := by
  rcases oreCondition r₂ t with ⟨r₂', t', hr₂⟩
  calc
    r₁ * r * r₂ /ₒ s = r₁ * r * r₂ * t' /ₒ (s * t') :=
      OreLocalization.expand (r₁ * r * r₂) s t' _
    _ = r₁ * r * (r₂ * t') /ₒ (s * t') := by simp [← mul_assoc]
    _ = r₁ * r * (t * r₂') /ₒ (s * t') := by rw [hr₂]
    _ = r₁ * (r * t) * r₂' /ₒ (s * t') := by simp [← mul_assoc]
    _ = r₁ * (r' * t) * r₂' /ₒ (s * t') := by rw [h]
    _ = r₁ * r' * (t * r₂') /ₒ (s * t') := by simp [← mul_assoc]
    _ = r₁ * r' * (r₂ * t') /ₒ (s * t') := by rw [hr₂]
    _ = r₁ * r' * r₂ * t' /ₒ (s * t') := by simp [← mul_assoc]
    _ = r₁ * r' * r₂ /ₒ s := (OreLocalization.expand _ _ _ _).symm

def liftExpand {C : Sort*} (P : R → S → C)
    ( hP : ∀ (r t : R) (s : S) (ht : (s : R) * t ∈ S),
      P r s = P (r * t) ⟨s * t, ht⟩) : R[S⁻¹] → C :=
  Quotient.lift (fun p : R × S => P p.1 p.2)
```

```
      fun (r₁, s₁) (r₂, s₂) ⟨u, v, hr₂, hs₂⟩ => by
    dsimp at *
    have s₁vS : (s₁ : R) * v ∈ S := by
      rw [← hs₂, ← S.coe_mul]
      exact SetLike.coe_mem (s₂ * u)
    replace hs₂ : s₂ * u = ⟨(s₁ : R) * v, s₁vS⟩
    · ext; simp [hs₂]
    rw [hP r₁ v s₁ s₁vS, hP r₂ u s₂ (by norm_cast; rwa [hs₂]), hr₂]
    simp only [← hs₂]; rfl

@[simp]
theorem liftExpand_of {C : Sort*} {P : R → S → C}
    { hP : ∀ (r t : R) (s : S) (ht : (s : R) * t ∈ S),
      P r s = P (r * t) ⟨s * t, ht⟩} (r : R)
    (s : S) : liftExpand P hP (r /ₒ s) = P r s :=
  rfl

def lift₂Expand {C : Sort*} (P : R → S → R → S → C)
    (hP :
      ∀ (r₁ t₁ : R) (s₁ : S) (ht₁ : (s₁ : R) * t₁ ∈ S)
        (r₂ t₂ : R) (s₂ : S) (ht₂ : (s₂ : R) * t₂ ∈ S),
        P r₁ s₁ r₂ s₂ =
        P (r₁ * t₁) ⟨s₁ * t₁, ht₁⟩ (r₂ * t₂) ⟨s₂ * t₂, ht₂⟩) :
    R[S⁻¹] → R[S⁻¹] → C :=
  liftExpand
    (fun r₁ s₁ => liftExpand (P r₁ s₁) fun r₂ t₂ s₂ ht₂ => by
      have := hP r₁ 1 s₁ (by simp) r₂ t₂ s₂ ht₂
      simp [this])
    fun r₁ t₁ s₁ ht₁ => by
    ext x; induction' x using OreLocalization.ind with r₂ s₂
    dsimp only
    rw [liftExpand_of, liftExpand_of, hP r₁ t₁ s₁ ht₁ r₂ 1 s₂ (by simp)];
    simp

@[simp]
theorem lift₂Expand_of {C : Sort*} {P : R → S → R → S → C}
    {hP :
      ∀ (r₁ t₁ : R) (s₁ : S) (ht₁ : (s₁ : R) * t₁ ∈ S) (r₂ t₂ : R) (s₂ : S)
        (ht₂ : (s₂ : R) * t₂ ∈ S),
        P r₁ s₁ r₂ s₂ = P (r₁ * t₁) ⟨s₁ * t₁, ht₁⟩ (r₂ * t₂) ⟨s₂ * t₂, ht₂⟩}
    (r₁ : R) (s₁ : S) (r₂ : R) (s₂ : S) :
      lift₂Expand P hP (r₁ /ₒ s₁) (r₂ /ₒ s₂) = P r₁ s₁ r₂ s₂ :=
  rfl

private def mul' (r₁ : R) (s₁ : S) (r₂ : R) (s₂ : S) : R[S⁻¹] :=
  r₁ * oreNum r₂ s₁ /ₒ (s₂ * oreDenom r₂ s₁)

private theorem mul'_char (r₁ r₂ : R) (s₁ s₂ : S) (u : S) (v : R)
    (huv : r₂ * (u : R) = s₁ * v) :
    OreLocalization.mul' r₁ s₁ r₂ s₂ = r₁ * v /ₒ (s₂ * u) := by
  simp only [mul']
  have h₀ := ore_eq r₂ s₁; set v₀ := oreNum r₂ s₁; set u₀ := oreDenom r₂ s₁
  rcases oreCondition (u₀ : R) u with ⟨r₃, s₃, h₃⟩
  have :=
```

```
    calc
      (s₁ : R) * (v * r₃) = r₂ * u * r₃ := by rw [← mul_assoc, ← huv]
      _ = r₂ * u₀ * s₃ := by rw [mul_assoc, mul_assoc, h₃]
      _ = s₁ * (v₀ * s₃) := by rw [← mul_assoc, h₀]
  rcases ore_left_cancel _ _ _ this with ⟨s₄, hs₄⟩
  symm; rw [oreDiv_eq_iff]
  use s₃ * s₄; use r₃ * s₄; simp only [Submonoid.coe_mul]; constructor
  · rw [mul_assoc (b := v₀), ← mul_assoc (a := v₀), ← hs₄]
    simp only [mul_assoc]
  · rw [mul_assoc (b := (u₀ : R)), ← mul_assoc (a := (u₀ : R)), h₃]
    simp only [mul_assoc]

protected def mul : R[S⁻¹] → R[S⁻¹] → R[S⁻¹] :=
  lift₂Expand mul' fun r₂ p s₂ hp r₁ r s₁ hr => by
    have h₁ := ore_eq r₁ s₂
    set r₁' := oreNum r₁ s₂
    set s₂' := oreDenom r₁ s₂
    rcases oreCondition (↑s₂ * r₁') ⟨s₂ * p, hp⟩ with ⟨p', s_star, h₂⟩
    dsimp at h₂
    rcases oreCondition r (s₂' * s_star) with ⟨p_flat, s_flat, h₃⟩
    simp only [S.coe_mul] at h₃
    have : r₁ * r * s_flat = s₂ * p * (p' * p_flat) := by
      rw [← mul_assoc, ← h₂, ← h₁, mul_assoc, h₃]
      simp only [mul_assoc]
    rw [mul'_char (r₂ * p) (r₁ * r) ⟨↑s₂ * p, hp⟩ ⟨↑s₁ * r, hr⟩ _ _ this]
    clear this
    have hsssp : ↑s₁ * ↑s₂' * ↑s_star * p_flat ∈ S := by
      rw [mul_assoc, mul_assoc, ← mul_assoc (s₂' : R), ← h₃, ← mul_assoc]
      exact S.mul_mem hr (SetLike.coe_mem s_flat)
    have : (⟨↑s₁ * r, hr⟩ : S) * s_flat =
        ⟨s₁ * s₂' * s_star * p_flat, hsssp⟩ := by
      ext
      simp only [Submonoid.coe_mul]
      rw [mul_assoc, h₃, ← mul_assoc, ← mul_assoc]
    rw [this]
    clear this
    rcases ore_left_cancel (p * p') (r₁' * (s_star : R)) s₂
        (by simp [← mul_assoc, h₂])
      with ⟨s₂'', h₂''⟩
    rw [← mul_assoc, mul_assoc r₂, OreLocalization.eq_of_num_factor_eq h₂'']
    norm_cast at hsssp ⊢
    rw [← OreLocalization.expand _ _ _ hsssp, ← mul_assoc]
    apply OreLocalization.expand

instance instMulOreLocalization : Mul R[S⁻¹] :=
  ⟨OreLocalization.mul⟩

theorem oreDiv_mul_oreDiv {r₁ r₂ : R} {s₁ s₂ : S} :
    r₁ /ₒ s₁ * (r₂ /ₒ s₂) = r₁ * oreNum r₂ s₁ /ₒ (s₂ * oreDenom r₂ s₁) :=
  rfl

theorem oreDiv_mul_char (r₁ r₂ : R) (s₁ s₂ : S) (r' : R) (s' : S)
    (huv : r₂ * (s' : R) = s₁ * r') :
    r₁ /ₒ s₁ * (r₂ /ₒ s₂) = r₁ * r' /ₒ (s₂ * s') :=
```

```
  mul'_char r₁ r₂ s₁ s₂ s' r' huv

def oreDivMulChar' (r₁ r₂ : R) (s₁ s₂ : S) :
    Σ'r' : R, Σ's' : S, r₂ * (s' : R) = s₁ * r' ∧
    r₁ /ₒ s₁ * (r₂ /ₒ s₂) = r₁ * r' /ₒ (s₂ * s') :=
  ⟨oreNum r₂ s₁, oreDenom r₂ s₁, ore_eq r₂ s₁, oreDiv_mul_oreDiv⟩

instance instOneOreLocalization : One R[S⁻¹] :=
  ⟨1 /ₒ 1⟩

protected theorem one_def : (1 : R[S⁻¹]) = 1 /ₒ 1 :=
  rfl

instance : Inhabited R[S⁻¹] :=
  ⟨1⟩

@[simp]
protected theorem div_eq_one' {r : R} (hr : r ∈ S) : r /ₒ ⟨r, hr⟩ = 1 := by
  rw [OreLocalization.one_def, oreDiv_eq_iff]
  exact ⟨⟨r, hr⟩, 1, by simp, by simp⟩

@[simp]
protected theorem div_eq_one {s : S} : (s : R) /ₒ s = 1 :=
  OreLocalization.div_eq_one' _

protected theorem one_mul (x : R[S⁻¹]) : 1 * x = x := by
  induction' x using OreLocalization.ind with r s
  simp [OreLocalization.one_def,
        oreDiv_mul_char (1 : R) r (1 : S) s r 1 (by simp)]

protected theorem mul_one (x : R[S⁻¹]) : x * 1 = x := by
  induction' x using OreLocalization.ind with r s
  simp [OreLocalization.one_def, oreDiv_mul_char r 1 s 1 1 s (by simp)]

protected theorem mul_assoc (x y z : R[S⁻¹]) : x * y * z = x * (y * z) := by
  induction' x using OreLocalization.ind with r₁ s₁
  induction' y using OreLocalization.ind with r₂ s₂
  induction' z using OreLocalization.ind with r₃ s₃
  rcases oreDivMulChar' r₁ r₂ s₁ s₂ with
    ⟨ra, sa, ha, ha'⟩; rw [ha']; clear ha'
  rcases oreDivMulChar' r₂ r₃ s₂ s₃ with
    ⟨rb, sb, hb, hb'⟩; rw [hb']; clear hb'
  rcases oreCondition rb sa with ⟨rc, sc, hc⟩
  rw [oreDiv_mul_char (r₁ * ra) r₃ (s₂ * sa) s₃ rc (sb * sc)
      (by
        simp only [Submonoid.coe_mul]
        rw [← mul_assoc, hb, mul_assoc, hc, ← mul_assoc])]
  rw [mul_assoc, ← mul_assoc s₃]
  symm; apply oreDiv_mul_char
  rw [mul_assoc, hc, ← mul_assoc (b := ra), ← ha, mul_assoc]

instance instMonoidOreLocalization : Monoid R[S⁻¹] :=
  { OreLocalization.instMulOreLocalization,
    OreLocalization.instOneOreLocalization with
```

```
    one_mul := OreLocalization.one_mul
    mul_one := OreLocalization.mul_one
    mul_assoc := OreLocalization.mul_assoc }

protected theorem mul_inv (s s' : S) :
    ((s : R) /ₒ s') * ((s' : R) /ₒ s) = 1 := by
  simp [oreDiv_mul_char (s : R) s' s' s 1 1 (by simp)]

@[simp]
protected theorem mul_one_div {r : R} {s t : S} :
    (r /ₒ s) * (1 /ₒ t) = r /ₒ (t * s) := by
  simp [oreDiv_mul_char r 1 s t 1 s (by simp)]

@[simp]
protected theorem mul_cancel {r : R} {s t : S} :
    (r /ₒ s) * ((s : R) /ₒ t) = r /ₒ t := by
  simp [oreDiv_mul_char r s s t 1 1 (by simp)]

@[simp]
protected theorem mul_cancel' {r₁ r₂ : R} {s t : S} :
    (r₁ /ₒ s) * ((s * r₂) /ₒ t) = (r₁ * r₂) /ₒ t := by
  simp [oreDiv_mul_char r₁ (s * r₂) s t r₂ 1 (by simp)]

@[simp]
theorem div_one_mul {p r : R} {s : S} :
    (r /ₒ 1) * (p /ₒ s) = (r * p) /ₒ s := by
  simp [oreDiv_mul_char r p 1 s p 1 (by simp)]

def numeratorUnit (s : S) : Units R[S⁻¹] where
  val := (s : R) /ₒ 1
  inv := (1 : R) /ₒ s
  val_inv := OreLocalization.mul_inv s 1
  inv_val := OreLocalization.mul_inv 1 s

def numeratorHom : R →* R[S⁻¹] where
  toFun r := r /ₒ 1
  map_one' := rfl
  map_mul' _ _ := div_one_mul.symm

theorem numeratorHom_apply {r : R} : numeratorHom r = r /ₒ (1 : S) :=
  rfl

theorem numerator_isUnit (s : S) : IsUnit (numeratorHom (s : R) : R[S⁻¹]) :=
  ⟨numeratorUnit s, rfl⟩

section UMP

variable {T : Type*} [Monoid T]

variable (f : R →* T) (fS : S →* Units T)

variable (hf : ∀ s : S, f s = fS s)

def universalMulHom : R[S⁻¹] →* T
```

```
    where
  toFun x :=
    x.liftExpand (fun r s => f r * ((fS s)⁻¹ : Units T)) fun r t s ht => by
      simp only []
      have : (fS ⟨s * t, ht⟩ : T) = fS s * f t := by
        simp only [← hf, MonoidHom.map_mul]
      conv_rhs =>
        rw [MonoidHom.map_mul, ← mul_one (f r),
          ← Units.val_one, ← mul_left_inv (fS s)]
        rw [Units.val_mul, ← mul_assoc, mul_assoc _ (fS s : T),
          ← this, mul_assoc]
      simp only [mul_one, Units.mul_inv]
  map_one' := by
    simp only []; rw [OreLocalization.one_def, liftExpand_of]; simp
  map_mul' x y := by
    simp only []
    induction' x using OreLocalization.ind with r₁ s₁
    induction' y using OreLocalization.ind with r₂ s₂
    rcases oreDivMulChar' r₁ r₂ s₁ s₂ with ⟨ra, sa, ha, ha'⟩;
      rw [ha']; clear ha'
    rw [liftExpand_of, liftExpand_of, liftExpand_of]
    conv_rhs =>
      congr
      · skip
      congr
      rw [← mul_one (f r₂), ← (fS sa).mul_inv, ← mul_assoc,
          ← hf, ← f.map_mul, ha, f.map_mul]
    rw [mul_assoc, mul_assoc, mul_assoc, ← mul_assoc _ (f s₁),
        hf s₁, (fS s₁).inv_mul, one_mul,
      f.map_mul, mul_assoc, fS.map_mul, ← Units.val_mul]
    rfl

theorem universalMulHom_apply {r : R} {s : S} :
    universalMulHom f fS hf (r /ₒ s) = f r * ((fS s)⁻¹ : Units T) :=
  rfl

theorem universalMulHom_commutes {r : R} :
    universalMulHom f fS hf (numeratorHom r) = f r := by
  simp [numeratorHom_apply, universalMulHom_apply]

theorem universalMulHom_unique (φ : R[S⁻¹] →* T)
    (huniv : ∀ r : R, φ (numeratorHom r) = f r) :
    φ = universalMulHom f fS hf := by
  ext x; induction' x using OreLocalization.ind with r s
  rw [universalMulHom_apply, ← huniv r, numeratorHom_apply,
    ← mul_one (φ (r /ₒ s)), ← Units.val_one, ← mul_right_inv (fS s),
    Units.val_mul, ← mul_assoc, ← hf, ← huniv, ← φ.map_mul,
    numeratorHom_apply, OreLocalization.mul_cancel]

end UMP

end Monoid

section CommMonoid
```

```
variable {R : Type*} [CommMonoid R] {S : Submonoid R} [OreSet S]

theorem oreDiv_mul_oreDiv_comm {r₁ r₂ : R} {s₁ s₂ : S} :
    r₁ /ₒ s₁ * (r₂ /ₒ s₂) = r₁ * r₂ /ₒ (s₁ * s₂) := by
  rw [oreDiv_mul_char r₁ r₂ s₁ s₂ r₂ s₁ (by simp [mul_comm]), mul_comm s₂]

instance : CommMonoid R[S⁻¹] :=
  { OreLocalization.instMonoidOreLocalization with
    mul_comm := fun x y => by
      induction' x using OreLocalization.ind with r₁ s₁
      induction' y using OreLocalization.ind with r₂ s₂
      rw [oreDiv_mul_oreDiv_comm, oreDiv_mul_oreDiv_comm,
        mul_comm r₁, mul_comm s₁] }

variable (R S)

protected def localizationMap : S.LocalizationMap R[S⁻¹]
    where
  toFun := numeratorHom
  map_one' := rfl
  map_mul' r₁ r₂ := by simp
  map_units' := numerator_isUnit
  surj' z := by
    induction' z using OreLocalization.ind with r s
    use (r, s); dsimp
    rw [numeratorHom_apply, numeratorHom_apply]; simp
  exists_of_eq r₁ r₂ := by
    dsimp
    intro h
    rw [numeratorHom_apply, numeratorHom_apply, oreDiv_eq_iff] at h
    rcases h with ⟨u, v, h₁, h₂⟩
    dsimp at h₂
    rw [one_mul, one_mul] at h₂
    subst h₂
    use u
    simpa only [mul_comm] using h₁.symm

protected noncomputable def equivMonoidLocalization :
    Localization S ≃* R[S⁻¹] :=
  Localization.mulEquivOfQuotient (OreLocalization.localizationMap R S)

end CommMonoid

section Semiring

variable {R : Type*} [Semiring R] {S : Submonoid R} [OreSet S]

private def add'' (r₁ : R) (s₁ : S) (r₂ : R) (s₂ : S) : R[S⁻¹] :=
  ( r₁ * oreDenom (s₁ : R) s₂ +
    r₂ * oreNum (s₁ : R) s₂) /ₒ (s₁ * oreDenom (s₁ : R) s₂)

private theorem add''_char (r₁ : R) (s₁ : S) (r₂ : R) (s₂ : S) (rb : R)
    (sb : S) (hb : (s₁ : R) * sb = (s₂ : R) * rb) :
```

```
    add'' r₁ s₁ r₂ s₂ = (r₁ * sb + r₂ * rb) /ₒ (s₁ * sb) := by
  simp only [add'']
  have ha := ore_eq (s₁ : R) s₂
  set! ra := oreNum (s₁ : R) s₂ with h
  rw [← h] at *
  clear h
  -- r tilde
  set! sa := oreDenom (s₁ : R) s₂ with h
  rw [← h] at *
  clear h
  -- s tilde
  rcases oreCondition (sa : R) sb with ⟨rc, sc, hc⟩
  -- s*, r*
  have : (s₂ : R) * (rb * rc) = s₂ * (ra * sc) := by
    rw [← mul_assoc, ← hb, mul_assoc, ← hc, ← mul_assoc, ← mul_assoc, ha]
  rcases ore_left_cancel _ _ s₂ this with ⟨sd, hd⟩
  -- s#
  symm
  rw [oreDiv_eq_iff]
  use sc * sd
  use rc * sd
  constructor <;> simp only [Submonoid.coe_mul]
  · noncomm_ring
    rw [← mul_assoc (a := rb), hd, ← mul_assoc (a := (sa : R)), hc]
    noncomm_ring
  · rw [mul_assoc (a := (s₁ : R)), ← mul_assoc (a := (sa : R)), hc]
    noncomm_ring

attribute [local instance] OreLocalization.oreEqv

private def add' (r₂ : R) (s₂ : S) : R[S⁻¹] → R[S⁻¹] :=
  (--plus tilde
      Quotient.lift
      fun r₁s₁ : R × S => add'' r₁s₁.1 r₁s₁.2 r₂ s₂) <| by
    rintro ⟨r₁', s₁'⟩ ⟨r₁, s₁⟩ ⟨sb, rb, hb, hb'⟩
    -- s*, r*
    rcases oreCondition (s₁' : R) s₂ with ⟨rc, sc, hc⟩
    --s~~, r~~
    rcases oreCondition rb sc with ⟨rd, sd, hd⟩
    -- s#, r#
    dsimp at *
    rw [add''_char _ _ _ _ rc sc hc]
    have : ↑s₁ * ↑(sb * sd) = ↑s₂ * (rc * rd) := by
      simp only [Submonoid.coe_mul]
      rw [← mul_assoc, hb', mul_assoc, hd, ← mul_assoc, hc, mul_assoc]
    rw [add''_char _ _ _ _ (rc * rd : R) (sb * sd : S) this]
    simp only [Submonoid.coe_mul]
    rw [← mul_assoc (a := r₁) (b := (sb : R)), hb,
      mul_assoc (a := r₁') (b := (rb : R)), hd,
      ← mul_assoc, ← mul_assoc, ← add_mul, oreDiv_eq_iff]
    use 1
    use rd
    constructor
    · simp
```

```
    · simp only [mul_one, Submonoid.coe_one, Submonoid.coe_mul] at this ⊢
      rw [hc, this, mul_assoc]

private theorem add'_comm (r₁ r₂ : R) (s₁ s₂ : S) :
    add' r₁ s₁ (r₂ /ₒ s₂) = add' r₂ s₂ (r₁ /ₒ s₁) := by
  simp only [add', oreDiv, add'', Quotient.mk', Quotient.lift_mk]
  rw [Quotient.eq]
  have hb := ore_eq (↑s₂) s₁
  set rb := oreNum (↑s₂) s₁
  set sb := oreDenom (↑s₂) s₁
  have ha := ore_eq (↑s₁) s₂
  set ra := oreNum (↑s₁) s₂
  set sa := oreDenom (↑s₁) s₂
  rcases oreCondition ra sb with ⟨rc, sc, hc⟩
  -- r#, s#
  have : (s₁ : R) * (rb * rc) = s₁ * (sa * sc) := by
    rw [← mul_assoc, ← hb, mul_assoc, ← hc, ← mul_assoc, ← ha, mul_assoc]
  rcases ore_left_cancel _ _ s₁ this with ⟨sd, hd⟩
  -- s+
  use sc * sd
  use rc * sd
  dsimp
  constructor
  · rw [add_mul, add_mul, add_comm, mul_assoc (a := r₁) (b := (sa : R)),
      ← mul_assoc (a := (sa : R)), ← hd, mul_assoc (a := r₂) (b := ra),
      ← mul_assoc (a := ra) (b := (sc : R)), hc]
    simp only [mul_assoc]
  · rw [mul_assoc, ← mul_assoc (sa : R), ← hd, hb]
    simp only [mul_assoc]

private def add : R[S⁻¹] → R[S⁻¹] → R[S⁻¹] := fun x =>
  Quotient.lift (fun rs : R × S => add' rs.1 rs.2 x)
    (by
      rintro ⟨r₁, s₁⟩ ⟨r₂, s₂⟩ hyz
      induction' x using OreLocalization.ind with r₃ s₃
      dsimp; rw [add'_comm, add'_comm r₂]
      simp [(· /ₒ ·), Quotient.mk', Quotient.sound hyz])

instance instAddOreLocalization : Add R[S⁻¹] :=
  ⟨add⟩

theorem oreDiv_add_oreDiv {r r' : R} {s s' : S} :
    r /ₒ s + r' /ₒ s' =
      ( r * oreDenom (s : R) s' +
        r' * oreNum (s : R) s') /ₒ (s * oreDenom (s : R) s') :=
  rfl

theorem oreDiv_add_char {r r' : R} (s s' : S) (rb : R)
    (sb : S) (h : (s : R) * sb = s' * rb) :
    r /ₒ s + r' /ₒ s' = (r * sb + r' * rb) /ₒ (s * sb) :=
  add''_char r s r' s' rb sb h

def oreDivAddChar' (r r' : R) (s s' : S) :
    Σ'r'' : R, Σ's'' : S, (s : R) * s'' = s' * r'' ∧
```

```
      r /ₒ s + r' /ₒ s' = (r * s'' + r' * r'') /ₒ (s * s'') :=
  ⟨ oreNum (s : R) s',
    oreDenom (s : R) s',
    ore_eq (s : R) s',
    oreDiv_add_oreDiv⟩

@[simp]
theorem add_oreDiv {r r' : R} {s : S} : r /ₒ s + r' /ₒ s = (r + r') /ₒ s :=
  by simp [oreDiv_add_char s s 1 1 (by simp)]

protected theorem add_assoc (x y z : R[S⁻¹]) : x + y + z = x + (y + z) := by
  induction' x using OreLocalization.ind with r₁ s₁
  induction' y using OreLocalization.ind with r₂ s₂
  induction' z using OreLocalization.ind with r₃ s₃
  rcases oreDivAddChar' r₁ r₂ s₁ s₂ with
    ⟨ra, sa, ha, ha'⟩; rw [ha']; clear ha'
  rcases oreDivAddChar' r₂ r₃ s₂ s₃ with
    ⟨rb, sb, hb, hb'⟩; rw [hb']; clear hb'
  rcases oreDivAddChar' (r₁ * sa + r₂ * ra) r₃ (s₁ * sa) s₃ with
    ⟨rc, sc, hc, q⟩; rw [q]; clear q
  rcases oreDivAddChar' r₁ (r₂ * sb + r₃ * rb) s₁ (s₂ * sb) with
    ⟨rd, sd, hd, q⟩; rw [q]; clear q
  simp only [right_distrib, mul_assoc, add_assoc]
  simp only [← add_oreDiv]
  congr 1
  · rw [← OreLocalization.expand', ← mul_assoc, ← mul_assoc,
    ← OreLocalization.expand', ← OreLocalization.expand']
  congr 1
  · simp_rw [← Submonoid.coe_mul] at ha hd
    rw [Subtype.coe_eq_of_eq_mk hd, ← mul_assoc, ← mul_assoc,
      ← mul_assoc, ← OreLocalization.expand, ← OreLocalization.expand',
      Subtype.coe_eq_of_eq_mk ha, ← OreLocalization.expand]
    apply OreLocalization.expand'
  · rcases oreCondition (sd : R) (sa * sc) with ⟨re, _, _⟩
    · simp_rw [← Submonoid.coe_mul] at hb hc hd
      rw [← mul_assoc, Subtype.coe_eq_of_eq_mk hc]
      rw [← OreLocalization.expand, Subtype.coe_eq_of_eq_mk hd, ← mul_assoc,
        ← OreLocalization.expand, Subtype.coe_eq_of_eq_mk hb]
      apply OreLocalization.expand

private def zero : R[S⁻¹] :=
  0 /ₒ 1

instance : Zero R[S⁻¹] :=
  ⟨zero⟩

protected theorem zero_def : (0 : R[S⁻¹]) = 0 /ₒ 1 :=
  rfl

@[simp]
theorem zero_div_eq_zero (s : S) : 0 /ₒ s = 0 := by
  rw [OreLocalization.zero_def, oreDiv_eq_iff]
  exact ⟨s, 1, by simp⟩
```

```
protected theorem zero_add (x : R[S⁻¹]) : 0 + x = x := by
  induction x using OreLocalization.ind
  rw [← zero_div_eq_zero, add_oreDiv]; simp

protected theorem add_comm (x y : R[S⁻¹]) : x + y = y + x := by
  induction x using OreLocalization.ind
  induction y using OreLocalization.ind
  change add' _ _ (_ /ₒ _) = _; apply add'_comm

instance instAddCommMonoidOreLocalization : AddCommMonoid R[S⁻¹] :=
  { OreLocalization.instAddOreLocalization with
    add_comm := OreLocalization.add_comm
    add_assoc := OreLocalization.add_assoc
    zero := zero
    zero_add := OreLocalization.zero_add
    add_zero := fun x => by
      rw [OreLocalization.add_comm, OreLocalization.zero_add] }

protected theorem zero_mul (x : R[S⁻¹]) : 0 * x = 0 := by
  induction' x using OreLocalization.ind with r s
  rw [OreLocalization.zero_def, oreDiv_mul_char 0 r 1 s r 1 (by simp)]; simp

protected theorem mul_zero (x : R[S⁻¹]) : x * 0 = 0 := by
  induction' x using OreLocalization.ind with r s
  rw [OreLocalization.zero_def, oreDiv_mul_char r 0 s 1 0 1 (by simp)]; simp

protected theorem left_distrib (x y z : R[S⁻¹]) :
    x * (y + z) = x * y + x * z := by
  induction' x using OreLocalization.ind with r₁ s₁
  induction' y using OreLocalization.ind with r₂ s₂
  induction' z using OreLocalization.ind with r₃ s₃
  rcases oreDivAddChar' r₂ r₃ s₂ s₃ with ⟨ra, sa, ha, q⟩
  rw [q]
  clear q
  rw [OreLocalization.expand' r₂ s₂ sa]
  rcases oreDivMulChar' r₁ (r₂ * sa) s₁ (s₂ * sa) with ⟨rb, sb, hb, q⟩
  rw [q]
  clear q
  have hs₃rasb : ↑s₃ * (ra * sb) ∈ S := by
    rw [← mul_assoc, ← ha]
    norm_cast
    apply SetLike.coe_mem
  rw [OreLocalization.expand _ _ _ hs₃rasb]
  have ha' : ↑(s₂ * sa * sb) = ↑s₃ * (ra * sb) := by simp [ha, ← mul_assoc]
  rw [← Subtype.coe_eq_of_eq_mk ha']
  rcases oreDivMulChar' r₁ (r₃ * (ra * sb)) s₁ (s₂ * sa * sb) with
    ⟨rc, sc, hc, hc'⟩
  rw [hc']
  rw [oreDiv_add_char (s₂ * sa * sb) (s₂ * sa * sb * sc) 1 sc (by simp)]
  rw [OreLocalization.expand' (r₂ * ↑sa + r₃ * ra) (s₂ * sa) (sb * sc)]
  conv_lhs =>
    congr
    · skip
    congr
```

```
    rw [add_mul, S.coe_mul, ← mul_assoc, hb, ← mul_assoc, mul_assoc r₃,
      hc, mul_assoc, ← mul_add]
  rw [OreLocalization.mul_cancel']
  simp only [mul_one, Submonoid.coe_mul, mul_add, ← mul_assoc]

theorem right_distrib (x y z : R[S⁻¹]) : (x + y) * z = x * z + y * z := by
  induction' x using OreLocalization.ind with r₁ s₁
  induction' y using OreLocalization.ind with r₂ s₂
  induction' z using OreLocalization.ind with r₃ s₃
  rcases oreDivAddChar' r₁ r₂ s₁ s₂ with
    ⟨ra, sa, ha, ha'⟩; rw [ha']; clear ha'; norm_cast at ha
  rw [OreLocalization.expand' r₁ s₁ sa]
  rw [OreLocalization.expand r₂ s₂ ra (by rw [← ha]; apply SetLike.coe_mem)]
  rw [← Subtype.coe_eq_of_eq_mk ha]
  repeat rw [oreDiv_mul_oreDiv]
  simp only [add_mul, add_oreDiv]

instance instSemiringOreLocalization : Semiring R[S⁻¹] :=
  { OreLocalization.instAddCommMonoidOreLocalization,
    OreLocalization.instMonoidOreLocalization with
    zero_mul := OreLocalization.zero_mul
    mul_zero := OreLocalization.mul_zero
    left_distrib := OreLocalization.left_distrib
    right_distrib := right_distrib }

section UMP

variable {T : Type*} [Semiring T]

variable (f : R →+* T) (fS : S →* Units T)

variable (hf : ∀ s : S, f s = fS s)

def universalHom : R[S⁻¹] →+* T :=
  {
    universalMulHom f.toMonoidHom fS
      hf with
    map_zero' := by
      change (universalMulHom f.toMonoidHom fS hf : R[S⁻¹] → T) 0 = 0
      rw [OreLocalization.zero_def, universalMulHom_apply]
      simp
    map_add' := fun x y => by
      change (universalMulHom f.toMonoidHom fS hf : R[S⁻¹] → T) (x + y)
        = (universalMulHom f.toMonoidHom fS hf : R[S⁻¹] → T) x
        + (universalMulHom f.toMonoidHom fS hf : R[S⁻¹] → T) y
      induction' x using OreLocalization.ind with r₁ s₁
      induction' y using OreLocalization.ind with r₂ s₂
      rcases oreDivAddChar' r₁ r₂ s₁ s₂ with ⟨r₃, s₃, h₃, h₃'⟩
      rw [h₃']
      clear h₃'
      simp only [universalMulHom_apply, RingHom.toMonoidHom_eq_coe,
        MonoidHom.coe_coe]
      simp only [mul_inv_rev, MonoidHom.map_mul, RingHom.map_add,
        RingHom.map_mul, Units.val_mul]
```

```
      rw [add_mul, ← mul_assoc, mul_assoc (f r₁), hf, ← Units.val_mul]
      simp only [mul_one, mul_right_inv, Units.val_one]
      congr 1
      rw [mul_assoc]
      congr 1
      norm_cast at h₃
      have h₃' := Subtype.coe_eq_of_eq_mk h₃
      rw [← Units.val_mul, ← mul_inv_rev, ← fS.map_mul, h₃']
      have hs₂r₃ : ↑s₂ * r₃ ∈ S := by
        rw [← h₃]
        exact SetLike.coe_mem (s₁ * s₃)
      apply (Units.inv_mul_cancel_left (fS s₂) _).symm.trans
      conv_lhs =>
        congr
        · skip
        rw [← Units.mul_inv_cancel_left (fS ⟨s₂ * r₃, hs₂r₃⟩) (fS s₂),
          mul_assoc, mul_assoc]
        congr
        · skip
        rw [← hf, ← mul_assoc (f s₂), ← f.map_mul]
        conv =>
          congr
          · skip
          congr
          rw [← h₃]
        rw [hf, ← mul_assoc, ← h₃', Units.inv_mul]
      rw [one_mul, ← h₃', Units.mul_inv, mul_one] }

theorem universalHom_apply {r : R} {s : S} :
    universalHom f fS hf (r /ₒ s) = f r * ((fS s)⁻¹ : Units T) :=
  rfl

theorem universalHom_commutes {r : R} :
    universalHom f fS hf (numeratorHom r) = f r := by
  simp [numeratorHom_apply, universalHom_apply]

theorem universalHom_unique (φ : R[S⁻¹] →+* T) (huniv : ∀ r : R,
    φ (numeratorHom r) = f r) : φ = universalHom f fS hf :=
  RingHom.coe_monoidHom_injective <| universalMulHom_unique
    (RingHom.toMonoidHom f) fS hf (↑φ) huniv

end UMP

end Semiring

section Ring

variable {R : Type*} [Ring R] {S : Submonoid R} [OreSet S]

protected def neg : R[S⁻¹] → R[S⁻¹] :=
  liftExpand (fun (r : R) (s : S) => -r /ₒ s) fun r t s ht => by
    simp only []
    rw [neg_mul_eq_neg_mul, ← OreLocalization.expand]
```

```
instance instNegOreLocalization : Neg R[S⁻¹] :=
  ⟨OreLocalization.neg⟩

@[simp]
protected theorem neg_def (r : R) (s : S) : -(r /ₒ s) = -r /ₒ s :=
  rfl

protected theorem add_left_neg (x : R[S⁻¹]) : -x + x = 0 := by
  induction' x using OreLocalization.ind with r s; simp

instance ring : Ring R[S⁻¹] :=
  { OreLocalization.instSemiringOreLocalization,
    OreLocalization.instNegOreLocalization with
    add_left_neg := OreLocalization.add_left_neg }

open nonZeroDivisors

theorem numeratorHom_inj (hS : S ≤ R⁰) :
    Function.Injective (numeratorHom : R → R[S⁻¹]) :=
  fun r₁ r₂ h => by
  rw [numeratorHom_apply, numeratorHom_apply, oreDiv_eq_iff] at h
  rcases h with ⟨u, v, h₁, h₂⟩
  simp only [S.coe_one, one_mul] at h₂
  rwa [← h₂, mul_cancel_right_mem_nonZeroDivisors
    (hS (SetLike.coe_mem u)), eq_comm] at h₁

theorem nontrivial_of_nonZeroDivisors [Nontrivial R] (hS : S ≤ R⁰) :
    Nontrivial R[S⁻¹] :=
  ⟨⟨0, 1, fun h => by
      rw [OreLocalization.one_def, OreLocalization.zero_def] at h
      apply nonZeroDivisors.coe_ne_zero 1 (numeratorHom_inj hS h).symm⟩⟩

end Ring

noncomputable section DivisionRing

open nonZeroDivisors

open Classical

variable {R : Type*} [Ring R] [Nontrivial R] [OreSet R⁰]

instance nontrivial : Nontrivial R[R⁰⁻¹] :=
  nontrivial_of_nonZeroDivisors (refl R⁰)

variable [NoZeroDivisors R]

protected def inv : R[R⁰⁻¹] → R[R⁰⁻¹] :=
  liftExpand
    (fun r s =>
      if hr : r = (0 : R) then (0 : R[R⁰⁻¹])
      else s /ₒ ⟨r, fun _ => eq_zero_of_ne_zero_of_mul_right_eq_zero hr⟩)
    (by
      intro r t s hst
```

```
      by_cases hr : r = 0
      · simp [hr]
      · by_cases ht : t = 0
        · exfalso
          apply nonZeroDivisors.coe_ne_zero ⟨_, hst⟩
          simp [ht, mul_zero]
        · simp only [hr, ht, dif_neg, not_false_iff,
            or_self_iff, mul_eq_zero]
          apply OreLocalization.expand)

instance inv' : Inv R[R⁰⁻¹] :=
  ⟨OreLocalization.inv⟩

protected theorem inv_def {r : R} {s : R⁰} :
    (r /ₒ s)⁻¹ =
      if hr : r = (0 : R) then (0 : R[R⁰⁻¹])
      else s /ₒ ⟨r, fun _ => eq_zero_of_ne_zero_of_mul_right_eq_zero hr⟩ :=
  rfl

protected theorem mul_inv_cancel (x : R[R⁰⁻¹]) (h : x ≠ 0) :
    x * x⁻¹ = 1 := by
  induction' x using OreLocalization.ind with r s
  rw [OreLocalization.inv_def, OreLocalization.one_def]
  by_cases hr : r = 0
  · exfalso
    apply h
    simp [hr]
  · simp [hr]
    apply OreLocalization.div_eq_one'

protected theorem inv_zero : (0 : R[R⁰⁻¹])⁻¹ = 0 := by
  rw [OreLocalization.zero_def, OreLocalization.inv_def]
  simp

instance divisionRing : DivisionRing R[R⁰⁻¹] :=
  { OreLocalization.nontrivial,
    OreLocalization.inv',
    OreLocalization.ring with
    mul_inv_cancel := OreLocalization.mul_inv_cancel
    inv_zero := OreLocalization.inv_zero }

end DivisionRing

end OreLocalization
```

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