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A stable splitting of factorisation homology of generalised surfaces

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Abstract

For a manifold W and an E_d-algebra A, the factorisation homology $\int_W A$ can be seen as a generalisation of the classical configuration space of labelled particles in W. It carries an action by the diffeomorphism group $\mathrm{Diff}_\partial(W)$, and for the generalised surfaces $W_{g,1} := (\#^g S^n \times S^n) \setminus \mathring{D}^{2n}$, we have stabilisation maps among the quotients $\int_{W_{g,1}} A \,/\!\!/\, \mathrm{Diff}_\partial(W_{g,1})$ which increase the genus g. In the case where a highly-connected tangential structure θ is taken into account, this article describes the stable homology of these quotients in terms of the iterated bar construction $\mathrm{B}^{2n}A$ and a tangential Thom spectrum $\mathrm{MT}\theta$, and addresses the question of homological stability.

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1 | MOTIVATION AND OVERVIEW

1.1 | Stable factorisation homology of oriented surfaces

Let S be a smooth oriented surface, possibly with boundary. For each $r \geqslant 0$, the topological group $\mathrm{Diff}^+_\partial(S)$ of orientation-preserving diffeomorphisms fixing the boundary acts on the configuration space $C_r(\mathring{S})$ of r unordered particles in the interior $\mathring{S}$, and the homotopy quotient $C_r(\mathring{S}) \,/\!\!/\, \mathrm{Diff}^+_\partial(S)$ is a moduli space for surfaces that are diffeomorphic to S and carry r permutable punctures in their interior.

If $S_{g,1}$ is an oriented surface of genus $g \geqslant 0$ with one boundary curve, then we obtain maps $C_r(\mathring{S}_{g,1}) /\!/ \mathrm{Diff}^+_\partial(S_{g,1}) \to C_r(\mathring{S}_{g+1,1}) /\!/ \mathrm{Diff}^+_\partial(S_{g+1,1})$ by forming the boundary-connected sum with $S_{1,1}$ that carries no particles. It follows essentially from Harer's stability theorem [18] that the sequence of these maps is homologically stable. Moreover, the stable homology has been described by [6] in terms of an 'undecorated' part $\mathrm{hocolim}_{g\to\infty}\mathrm{BDiff}^+_\partial(S_{g,1})$ and $\mathrm{B}(\mathfrak{S}_r \wr \mathrm{SO}(2))$.

A more recent work [7] studies generalisations of these configuration spaces, which capture more of the local structure of the surface: We fix a framed E_2-algebra A, that is, a space on which the operad $E_2 \rtimes \mathrm{SO}(2)$ acts, and consider, for each oriented surface S as above, the space $\int_S A$ of oriented embeddings $\coprod^r \mathring{D}^2 \hookrightarrow S$, for arbitrary $r \geqslant 0$, where each disc carries a label in A and where the framed E_2-action on A is balanced with compositions of embeddings of discs as in [32]. This is an instance of factorisation homology developed in [26, section 5.5]. Again, we have an action of the diffeomorphism group $\mathrm{Diff}^+_\partial(S)$ on $\int_S A$, now by postcomposing embeddings of discs, and stabilisation maps $\int_{S_{g,1}} A /\!/ \mathrm{Diff}^+_\partial(S_{g,1}) \to \int_{S_{g+1,1}} A /\!/ \mathrm{Diff}^+_\partial(S_{g+1,1})$.

As before, it turns out that this sequence is homologically stable [7, Theorem E] and that the stable homology splits as follows: Bonatto constructs a geometrically flavoured semi-simplicial space $D_\bullet(A)$ out of A, whose geometric realisation is an E_∞-algebra, and identifies the group-completion of the A_∞-algebra $\coprod_{g\geqslant 0} \int_{S_{g,1}} A /\!/ \mathrm{Diff}^+_\partial(S_{g,1})$ with the infinite loop space $\Omega^\infty \mathrm{MTSO}(2) \times \Omega\mathrm{B}|D_\bullet(A)|$, see [7, Theorem G]. Here $\mathrm{MTSO}(2)$ is the oriented tangential Thom spectrum as in [27]. Via the group-completion theorem from [29], see also [12, Apx. Q], this implies that the colimit of the above stabilisations splits homologically into $\mathrm{hocolim}_{g\to\infty}\mathrm{BDiff}^+_\partial(S_{g,1})$ and $\Omega\mathrm{B}|D_\bullet(A)|$.

1.2 | Aim and setting of this work

The main point of this work is to establish a description of Bonatto's second factor in homotopy-theoretic terms. Our methods to pursue this aim allow for a more general input: Recall that for each dimension d and each d-dimensional tangential structure $\theta : L \to \mathrm{BO}(d)$ (in the sense of Definition 2.16), there is an operad E^θ_d of θ-framed embeddings of d-dimensional discs, and for each θ-framed manifold (W, ℓ_W) and each E^θ_d-algebra A, we can consider the factorisation homology $\int^\theta_W A$, which geometrically is defined exactly as in the case of surfaces. An explicit model as a two-sided bar construction is given in [24]; we recall it in Section 2.3.

As before, one can consider the *moduli space* of such manifolds W with decorations in A, essentially by letting the θ-framing vary and quotienting out the action of the group $\mathrm{Diff}_\partial(W)$, see Construction 2.26 for details. The result is called $W^\theta[A]$ and plays a central role in this paper. We are interested in these spaces for the following reasons:

- They generalise at the same time of the aforementioned generalised configuration spaces of surfaces from [7] and the classifying spaces for punctured diffeomorphism groups for manifolds of arbitrary dimension from [6, 8].
- They appear in a homotopy fibre sequence $\int^\theta_W A \to W^\theta[A] \to \mathcal{M}^\theta_\partial(W, \ell_W)$, where $\mathcal{M}^\theta_\partial(W, \ell_W)$ is the classical moduli space of θ-framed manifolds of type (W, ℓ_W) studied in [16, 17]; see Proposition 2.31 for details.
- If $\Omega^d_\theta X$ is the θ-framed loop space for some retractive space X over L, see Example 6.15 for a definition, then $W^\theta[\Omega^d_\theta X]$ is the moduli space $\mathcal{M}^\theta_\partial(W, \ell_W)\langle X\rangle$ of θ-framed manifolds of type (W, ℓ_W), together with a map to X over L.

In the case of generalised surfaces $W_{g,1} := (\#^g S^n \times S^n) \setminus \mathring{D}^{2n}$, we obtain stabilisation maps $W^\theta_{g,1}[A] \to W^\theta_{g+1,1}[A]$ by taking boundary-connected sums with an 'empty' $W_{1,1}$. Here we assume our tangential structure to be spherical in the sense of [17], that is, we require that S^{2n} admits a θ-framing, in order to ensure the existence of well-behaved θ-framings on all $W_{g,1}$. We study the above stabilisation maps and their colimit — or, in other words, the group-completion of the A_∞-algebra $W^\theta_{*,1}[A] := \coprod_{g\geqslant 0} W^\theta_{g,1}[A]$.

1.3 | Results

Let $\theta : L \to \mathrm{BO}(d)$ be a tangential structure with connected L. For any E^θ_d-algebra A, we denote by $\mathrm{B}^d UA$ the iterated bar construction of its underlying E_d-algebra, the latter depending on a choice of basepoint $b_0 \in L$. Our main result is the following:

Theorem 6.6. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L, and let A be an E^θ_{2n}-algebra. Then there is an A_∞-action of ΩL on the spectrum $\Sigma^{\infty-2n}\mathrm{B}^{2n}UA$ and we have a weak equivalence of loop spaces*

$$\Omega\mathrm{B}W^\theta_{*,1}[A] \simeq \mathbb{Z} \times \Omega^\infty_0 \mathrm{MT}\theta \times \Omega^\infty\big(\big(\Sigma^{\infty-2n}\mathrm{B}^{2n}UA\big)_{\mathrm{h}\Omega L}\big).$$

Here $\Omega^\infty_0 \subseteq \Omega^\infty$ denotes the path-component of the basepoint and $(-)_{\mathrm{h}\Omega L}$ denotes the homotopy quotient in the category of spectra. If θ is of the form $\mathrm{B}G \to \mathrm{BO}(2n)$ for some group homomorphism $G \to \mathrm{O}(2n)$, then we actually take homotopy orbits of a given G-action on $\Sigma^{\infty-2n}\mathrm{B}^{2n}UA$. We point out that our assumptions cover the aforementioned case of $2n = 2$ and $\theta : \mathrm{BSO}(2) \to \mathrm{BO}(2)$.

Moreover, this result can be used to calculate the homology of $W^\theta_{g,1}[A]$ in a range, as the stability results for moduli spaces of surfaces [18, 31] and high-dimensional manifolds [17] rather directly imply the following statement:

Theorem 3.19. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with π_1-injective θ, and let A be an E^θ_{2n}-algebra. If $2n \geqslant 6$ (or $2n = 2$ and θ is admissible to [31, Theorem 7.1]), then the maps $W^\theta_{g,1}[A] \to W^\theta_{g+1,1}[A]$ induce isomorphisms in $H_i(-;\mathbb{Z})$ for $i \leqslant \frac{1}{2}g - \frac{3}{2}$ (for $2n = 2$, the slope is different, but coincides with the one from [31, Theorem 7.1]).*

If L is n-connected, as in Theorem 5.10, then θ is in particular π_1-injective. The case of $2n = 2$ and θ being orientations is covered by [31, Theorem 7.1]; here the slope is $i \leqslant \frac{2}{3}g - \frac{2}{3}$ and we recover [7, Theorem E]. Combining both theorems, we obtain the following:

Corollary 6.7. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L. If $2n \geqslant 6$ (or $2n = 2$ and θ is admissible to [31, Theorem 7.1]), we have, for each path-connected E^θ_{2n}-algebra A and each $g \geqslant 0$, isomorphisms*

$$H_i(W^\theta_{g,1}[A]) \cong H_i\big(\Omega^\infty_0\mathrm{MT}\theta \times \Omega^\infty\big(\big(\Sigma^{\infty-2n}\mathrm{B}^{2n}UA\big)_{\mathrm{h}\Omega L}\big)\big)$$

for every i small enough compared to g to satisfy the conditions of Theorem 3.19.

We also discuss several special cases: If the E_{2n}^{θ}-action on A is only partially defined, then one defines the factorisation homology $\int_W^{\theta} A$ by first completing A to an honest E_{2n}^{θ}-algebra, see Construction 5.8, and then applying the above definition. For example, if $M \subseteq E_{2n}^{\theta}(1)$ is a submonoid in the monoid of unary operations, then each based M-space X can be regarded as a partial E_{2n}^{θ}-algebra. In this case, we get the following:

Theorem 5.10. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L. If $M \subseteq E_{2n}^{\theta}(1)$ is a well-pointed submonoid and X is a based M-space, regarded as a partial E_{2n}^{θ}-algebra, then we have a weak equivalence of loop spaces*

$$\Omega \mathrm{B} W_{*,1}^{\theta}[X] \simeq \Omega^{\infty}\mathrm{MT}\theta \times \Omega^{\infty}\Sigma^{\infty}(X_{\mathrm{h}M}).$$

Here $X_{\mathrm{h}M}$ is the based homotopy quotient. If θ is of the form $\mathrm{B}G \to \mathrm{BO}(2n)$ for some $(n-1)$-connected subgroup $G \subset \mathrm{O}(2n)$, then $E_{2n}^{\theta}(1)$ contains a submonoid equivalent to G, see Example 5.5. In the case of surfaces, θ being orientations and M being equivalent to $\mathrm{SO}(2)$, Theorem 5.10 hence recovers [7, Corollary D]. In the case of arbitrary dimensions and X being S^0 with the trivial $E_{2n}^{\theta}(1)$-action, Theorem 5.10 becomes a (puncture-stabilised) variation of [8, Theorem A], see Example 5.13.

Finally, we deduce a description for a stabilised version of the aforementioned moduli spaces $\mathcal{M}_{\partial}^{\theta}(W, \ell_W)\langle X\rangle$ of θ-framed moduli spaces together with a map to a retractive space X over L, whose fibre over $b_0 \in L$ we denote by X_{b_0}:

Corollary 6.16. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L. Moreover, let X be a $2n$-connective retractive space over L. Then we have a weak equivalence*

$$\operatorname{hocolim}_{g\to\infty}\left(\mathcal{M}_{\partial}^{\theta}(W_{g,1}, \ell_{g,1})\langle X\rangle\right)^{+} \simeq \Omega_0^{\infty}\mathrm{MT}\theta \times \Omega^{\infty}\left((\Sigma^{\infty-2n}X_{b_0})_{\mathrm{h}\Omega L}\right).$$

1.4 | Strategy

For the most part of this paper, we work in the model category of (compactly generated) spaces, and consider algebras over topological operads. Only the last section is written in the language of ∞-categories, after translating the leftover question into this setting.

First, we recall from [2, 36] the generalised surface operad $\mathcal{W}^{\theta}$, whose operation spaces are moduli spaces of θ-framed manifolds of type $W_{g,1}$ for varying g, together with several embedded discs that serve as inputs. We then establish a zig-zag $E_{2n}^{\theta} \xleftarrow{\kappa} F_{2n}^{\theta} \to \mathcal{W}^{\theta}$ of operad maps, where κ is an equivalence. Very generally, for each operad map $\rho : \mathcal{P} \to \mathcal{O}$, the forgetful functor ρ^* from $\mathcal{O}$-algebras to $\mathcal{P}$-algebras admits a (derived) left-adjoint $\mathcal{O} \otimes_{\mathcal{P}}^{\mathbb{L}} (-)$, called *pushforward*. For any F_{2n}^{θ}-algebra B, the pushforward $\mathcal{W}^{\theta} \otimes_{F_{2n}^{\theta}}^{\mathbb{L}} B$ has a canonical grading by genus, and we show the following:

Proposition 3.1. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure. Then we have, for each E_{2n}^{θ}-algebra A, a graded equivalence $W_{*,1}^{\theta}[A] \simeq \mathcal{W}^{\theta} \otimes_{F_{2n}^{\theta}}^{\mathbb{L}} \kappa^* A$.*

We point out that, instead of constructing the stabilisation maps and the A_{∞}-algebra structure on $W_{*,1}^{\theta}[A]$ 'by hand', we use this equivalence and the fact that the right side canonically carries this structure.

We thus are left to understand the group-completion of pushforwards to $\mathcal{W}^\theta$-algebras. Here we use that if L is n-connected, then $\mathcal{W}^\theta$ is an operad with homological stability (OHS) in the sense of [2]. For such operads $\mathcal{O}$, a description of the group-completion $\Omega\mathrm{B}(\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A)$ has been established in [2, 4] in the case where $\mathcal{P} = E_0$ (the operad of based spaces) and where $\mathcal{P}$ is the operad of based G-spaces for some topological group G mapping to $\mathcal{O}(1)$. Both results are based on the observation that in the derived setting, E_∞ is the terminal operad, and hence each operad $\mathcal{O}$ has an essentially unique operad map to E_∞. We show the following generalisation:

Proposition 4.8. *Let $\mathcal{O}$ be an OHS, let $\mathcal{P}$ be a proper operad with $\mathcal{P}(0) \simeq *$, and let $\mathcal{P} \to \mathcal{O}$ be a map of operads under E_0. Then the map of $\mathcal{O}$-algebras*

$$\left(\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A\right) \to \left(\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} *\right) \times \left(E_\infty \otimes^{\mathbb{L}}_{\mathcal{P}} A\right)$$

that is comprised of $A \to *$ and $\mathcal{O} \to E_\infty$ induces an equivalence on group-completions.

Here, properness is a mild point-set topological assumption, see Definition 2.5 for details. In the case of $\mathcal{P} = F^\theta_{2n}$ and $\mathcal{O} = \mathcal{W}^\theta$, the group-completion of the first factor is equivalent to the infinite loop space associated with the tangential Thom spectrum $\mathrm{MT}\theta$ [15, 27]. Then Propositions 3.1 and 4.8, together with the fact that the counit $E^\theta_{2n} \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A \to A$ is an equivalence, show the following:

Corollary 4.12. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L, and let A be an E^θ_{2n}-algebra. Then we have a weak equivalence of loop spaces*

$$\Omega\mathrm{B}W^\theta_{*,1}[A] \simeq \Omega^\infty_0 \mathrm{MT}\theta \times \Omega\mathrm{B}\left(E_\infty \otimes^{\mathbb{L}}_{E^\theta_{2n}} A\right).$$

This reduces the original question to understanding the group-completion of the E_∞-algebra $E_\infty \otimes^{\mathbb{L}}_{E^\theta_d} A$ for a given E^θ_d-algebra A. We start by discussing several special cases in Section 5. A general answer is established in Section 6:

Proposition 6.3. *Let $\theta : L \to \mathrm{BO}(d)$ be a tangential structure with connected L and let A be an E^θ_d-algebra. Then the shifted suspension spectrum $\Sigma^{\infty-d}\mathrm{B}^d UA$ carries an E_1-action by the loop space ΩL and we have an equivalence of connective spectra*

$$\mathrm{B}^\infty\left(E_\infty \otimes^{\mathbb{L}}_{E^\theta_d} A\right) \simeq \left(\Sigma^{\infty-d}\mathrm{B}^d UA\right)_{\mathrm{h}\Omega L}.$$

The main result, Theorem 6.6, is obtained by combining these propositions.

2 | BASIC NOTIONS

2.1 | Monads and operads

Definition 2.1. By a *space*, we mean a compactly generated topological space. Limits are taken in the category Top of compactly generated topological spaces.

Definition 2.2. Let $\mathcal{C}$ be a category and let T be a monad in $\mathcal{C}$. We denote the forgetful functor from T-algebras back to $\mathcal{C}$ by U^T. If $S : \mathcal{C} \to \mathcal{C}'$ is a right T-functor, then each T-algebra A gives rise to a simplicial object in $\mathcal{C}'$, which we call the *two-sided bar construction*, given by $\mathrm{B}_\bullet(S, T, A) := \big([p] \mapsto ST^pU^TA\big)_{p\in\mathbf{\Delta}^{\mathrm{op}}}$.

Example 2.3. For each topological group G, the assignment $\mathbb{G}(X) = G \times X$ is a monad $\mathbb{G}$ in Top, and $\mathbb{G}$-algebras are the same as G-spaces. Moreover, the identity functor $\mathbb{1}$ is a right $\mathbb{G}$-functor by projecting $\mathbb{G}(X)$ to the second factor. If G is well-pointed and X is a G-space, then $|\mathrm{B}_\bullet(\mathbb{1}, \mathbb{G}, X)|$ is our preferred model for the homotopy quotient $X /\!\!/ G$.

Lemma 2.4. *Let T be a monad in $\mathcal{C}$, A a T-algebra, G a well-pointed topological group and $S : \mathcal{C} \to \mathrm{Top}^G$ a right T-functor. Then $S /\!\!/ G : \mathcal{C} \to \mathrm{Top}$ is a right T-functor and*

$$|\mathrm{B}_\bullet(S /\!\!/ G, T, A)| \cong |\mathrm{B}_\bullet(S, T, A) /\!\!/ G|.$$

Proof. This follows from the fact that the two spaces in question are the two possibilities of realising the bisimplicial space $\mathrm{B}_\bullet(\mathrm{B}_\circ(\mathbb{1}, \mathbb{G}, S), T, A) = \mathrm{B}_\circ(\mathbb{1}, \mathbb{G}, \mathrm{B}_\bullet(S, T, A))$. □

Definition 2.5. By an *operad* $\mathcal{O}$, we mean a symmetric, monochromatic operad in spaces, see [28, Definition 1.1], and we additionally require that all operation spaces $\mathcal{O}(r)$ are Hausdorff. The identity operation is denoted by $\mathbf{1} = \mathbf{1}_{\mathcal{O}}$. An *equivalence* $\mathcal{P} \to \mathcal{O}$ is an operad map such that all $\mathcal{P}(r) \to \mathcal{O}(r)$ are weak equivalences of spaces.

We say that an operad $\mathcal{O}$ is *$\mathfrak{S}$-free* if for each $r \geqslant 0$, the $\mathfrak{S}_r$-action on $\mathcal{O}(r)$ is free, and we call $\mathcal{O}$ *well-pointed* if the inclusion $\{\mathbf{1}_{\mathcal{O}}\} \hookrightarrow \mathcal{O}(1)$ is a Hurewicz cofibration. Finally, we call $\mathcal{O}$ *proper* if it is $\mathfrak{S}$-free and well-pointed.

Example 2.6. We consider the *d-discs operad E_d* where $E_d(r)$ is the space of embeddings $\underline{r} \times D^d \hookrightarrow D^d$ (with $\underline{r} := \{1, \ldots, r\}$), which are on each disc of the form $z \mapsto \dot{z} + \rho_i \cdot z$ for some fixed $\dot{z} \in D^d$ and $\rho > 0$. We have an inclusion $E_d \hookrightarrow E_{d+1}$ by extending the above description along $D^d = D^d \times \{0\} \hookrightarrow D^{d+1}$, and we call the colimit E_∞.

For each $0 \leqslant d \leqslant \infty$, we have $E_d(0) = *$ and the operad E_d is proper. Moreover, the operation spaces $E_\infty(r)$ are contractible for each $r \geqslant 0$.

Definition 2.7. An *$\mathcal{O}$-algebra* is a space A, together with maps $\mathcal{O}(r) \times_{\mathfrak{S}_r} A^r \to A$ that are associative and unital [28, Definition 1.2]. A map of $\mathcal{O}$-algebras is called *equivalence* if it is a weak equivalence on underlying spaces.

Example 2.8. For each operad $\mathcal{O}$, the space $\mathcal{O}(0)$ of arity-0 operations is itself an $\mathcal{O}$-algebra; it is actually the *initial* $\mathcal{O}$-algebra.

Definition 2.9. We denote by $U^{\mathcal{O}}$ the forgetful functor from $\mathcal{O}$-algebras to spaces. Its left-adjoint $F^{\mathcal{O}}$ is given by taking X to $\coprod_r \mathcal{O}(r) \times_{\mathfrak{S}_r} X^r$, the $\mathcal{O}$-action induced by the composition inside $\mathcal{O}$. We denote the monad of this adjunction by $\mathbb{O}$; and in general, the monad associated to an operad gets the same letter in blackboard bold. Note that algebras over the monad $\mathbb{O}$ are the same as algebras over the operad $\mathcal{O}$.

Definition 2.10. If $\rho : \mathcal{P} \to \mathcal{O}$ is a map of operads, then the forgetful functor ρ^* from $\mathcal{O}$-algebras to $\mathcal{P}$-algebras has a left-adjoint, given by taking the free $\mathcal{O}$-algebra and quotienting out all relations from the existing $\mathcal{P}$-action. We call this left-adjoint *pushforward* and denote it by $\mathcal{O} \otimes_{\mathcal{P}} (-)$. Note that $\mathcal{O} \otimes_{\mathcal{P}} F^{\mathcal{P}} \cong F^{\mathcal{O}}$.

Under mild point-set topological assumptions, there is a homotopy-invariant replacement for the $\mathcal{O} \otimes_{\mathcal{P}} -$ that has a convenient simplicial description:

Remark 2.11. If $\mathcal{P}$ and $\mathcal{O}$ are $\mathfrak{S}$-free, then their categories of algebras carry a model structure [3] and $\mathcal{O} \otimes_{\mathcal{P}} (-)$ is a left Quillen functor, whose left-derivation we denote by $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} (-)$. If $\mathcal{P}$ is well-pointed, then we have the following explicit model: The augmented simplicial $\mathcal{P}$-algebra $\mathrm{B}_\bullet(F^{\mathcal{P}}, \mathbb{P}, A) \to A$ is proper and has an extra degeneracy, and hence is a $\mathbb{P}$-free simplicial resolution of A in the sense of [13, Definition 8.18]. Using $U^{\mathcal{O}}(\mathcal{O} \otimes_{\mathcal{P}} F^{\mathcal{P}}) = \mathbb{O}$, a model for $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A$ is given by

$$|U^{\mathcal{O}}\mathrm{B}_\bullet(\mathcal{O} \otimes_{\mathcal{P}} F^{\mathcal{P}}, \mathbb{P}, A)| = |\mathrm{B}_\bullet(\mathbb{O}, \mathbb{P}, A)|,$$

together with the $\mathbb{O}$-action $\mathbb{O}|\mathrm{B}_\bullet(\mathbb{O}, \mathbb{P}, A)| \cong |\mathrm{B}_\bullet(\mathbb{O}^2, \mathbb{P}, A)| \to |\mathrm{B}_\bullet(\mathbb{O}, \mathbb{P}, A)|$, using that $\mathbb{O}$ commutes with geometric realisations [28, Lemma 9.7]. This description is functorial in A, and will be our preferred model throughout the paper.

If $\mathcal{Q}$ is a third proper operad, together with a map $\mathcal{Q} \to \mathcal{P}$, then we have a natural weak equivalence $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} (\mathcal{P} \otimes^{\mathbb{L}}_{\mathcal{Q}} (-)) \to \mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{Q}} (-)$. An elementary simplicial argument for this fact has been spelled out in the proof of [4, Lemma 5.12].

For the following lemma, recall that for an operad $\mathcal{P}$, an operad *under* $\mathcal{P}$ is an operad $\mathcal{O}$ that comes with a preferred operad map $\mathcal{P} \to \mathcal{O}$. A map $\mathcal{O} \to \mathcal{O}'$ between operads under $\mathcal{P}$ is required to make the obvious triangle commute.

Lemma 2.12. *Let $\mathcal{P}$ be a proper operad.*

1. *If $\mathcal{O}$ is $\mathfrak{S}$-free operad under $\mathcal{P}$ and $A \to A'$ is a map of $\mathcal{P}$-algebras, then the induced map $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A \to \mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A'$ is an equivalence of $\mathcal{O}$-algebras.*
2. *If A is a $\mathcal{P}$-algebra and $\rho : \mathcal{O} \to \mathcal{O}'$ is an equivalence of $\mathfrak{S}$-free operads under $\mathcal{P}$, then $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A \to \rho^*(\mathcal{O}' \otimes^{\mathbb{L}}_{\mathcal{P}} A)$ is an equivalence of $\mathcal{O}$-algebras.*

By abuse of notation, we will occasionally skip the symbol ρ^* when it is clear from the context that only the underlying $\mathcal{O}$-algebra structure is taken into account.

Proof. As each $\mathcal{O}(r)$ is Hausdorff and $\mathfrak{S}_r$ acts freely on $\mathcal{O}(r)$, the map $\mathcal{O}(r) \to \mathcal{O}(r)/\mathfrak{S}_r$ is a covering. We hence get, for each X, a fibre sequence $X^r \to \mathcal{O}(r) \times_{\mathfrak{S}_r} X^r \to \mathcal{O}(r)/\mathfrak{S}_r$ which is natural in X. By the five lemmas, this shows that the monad $\mathbb{O}$ preserves weak equivalences among arbitrary spaces. The same argument applies to $\mathbb{P}$, and therefore, the simplicial map $\mathrm{B}_\bullet(\mathbb{O}, \mathbb{P}, A \to A')$ is a levelwise equivalence. Similarly, the map $\mathbb{O}X \to \mathbb{O}'X$ is a weak equivalence for each space X, see [24, Lemma 11] for details, whence also $\mathrm{B}_\bullet(\mathbb{O} \to \mathbb{O}', \mathbb{P}, A)$ is a levelwise equivalence.

Finally, since $\mathcal{O}$ and $\mathcal{O}'$ are $\mathfrak{S}$-free and $\mathcal{P}$ is proper, all involved simplicial spaces are proper. This shows that the maps induced on realisations are weak equivalences. □

Construction 2.13. Let $\mathcal{P}$ be a proper operad. Then the operad $\mathcal{P} \times E_\infty$ is again proper, the projection to the first factor $\pi_1 : \mathcal{P} \times E_\infty \to \mathcal{P}$ is an equivalence of proper operads and the projection to the second factor is an operad map $\pi_2 : \mathcal{P} \times E_\infty \to E_\infty$.

Let A be a $\mathcal{P}$-algebra. If $\mathcal{P} \overset{\nu_1}{\leftarrow} \mathcal{Q} \overset{\nu_2}{\rightarrow} E_\infty$ is any other zig-zag of proper operads such that ν is an equivalence, then we have an equivalence $E_\infty \otimes^{\mathbb{L}}_{\mathcal{P}\times E_\infty} \pi_1^* A \simeq E_\infty \otimes^{\mathbb{L}}_{\mathcal{Q}} \nu_1^* A$ of E_∞-algebras. We will denote this homotopy type simply by $E_\infty \otimes^{\mathbb{L}}_{\mathcal{P}} A$, noting that in the case where $\mathcal{P}$ already comes with a map ρ to E_∞, the case of $\nu_1 = \mathrm{id}_{\mathcal{P}}$ and $\nu_2 = \rho$ shows that the two competing definitions agree up to equivalence.

Construction 2.14. Let Ass denote the associative operad; its algebras are topological monoids. An A_∞-*operad* is a proper operad $\mathcal{A}$ with an equivalence $\mathcal{A} \to \mathrm{Ass}$. If A is an A_∞-*algebra*, that is, an algebra over some A_∞-operad $\mathcal{A}$, then it admits a *bar construction*$\mathrm{B}A := \mathrm{B}(\mathrm{Ass} \otimes^{\mathbb{L}}_{\mathcal{A}} A)$, and hence a *group-completion* $\Omega\mathrm{B}A$.

Via the map $\pi_2 : \mathrm{Ass} \times E_\infty \to E_\infty$, each E_∞-algebra A is an algebra over the A_∞-operad $\mathrm{Ass} \times E_\infty$. If $\rho : \mathcal{A} \to E_\infty$ is another map from an A_∞-operad to E_∞, then the monoids $\mathrm{Ass} \otimes^{\mathbb{L}}_{\mathrm{Ass}\times E_\infty} \pi_2^* A$ and $\mathrm{Ass} \otimes^{\mathbb{L}}_{\mathcal{A}} \rho^* A$ are equivalent, and so are their bar constructions.

2.2 | Tangential structures and framed little discs

Definition 2.15. For two smooth manifolds M and N, possibly with boundary, we denote by $\mathrm{Emb}(M, N)$ the space of smooth embeddings $M \hookrightarrow N$. Note that so far, we do not impose any boundary condition on the embeddings.

Definition 2.16. A *tangential structure* is a fibration $\theta : L \to \mathrm{BO}(d)$ such that the total space L is connected. If V_d denotes the universal vector bundle over $\mathrm{BO}(d)$, then a θ-*framing* on a smooth d-dimensional manifold W is a bundle map $\ell_W : TW \to \theta^* V_d$. The space $\mathrm{Fr}^\theta(W)$ of all θ-framings on W is by definition $\mathrm{Bun}(TW, \theta^* V_d)$.

Construction 2.17. Given θ-framed manifolds (W, ℓ_W) and $(W', \ell_{W'})$, the space of θ-*framed embeddings*$(W', \ell_{W'}) \hookrightarrow (W, \ell_W)$ should model the homotopy fibre of the map $\mathrm{Emb}(W', W) \to \mathrm{Fr}^\theta(W')$ with $\alpha \mapsto \ell_W \circ T\alpha$ at $\ell_{W'}$. In [24, Definition 17], the authors give a 'Moore path' description as

$$\mathrm{Emb}^\theta(W', W) := \left\{ \begin{array}{l} (\alpha, t^\bullet, \gamma) \in \mathrm{Emb}(W', W) \times [0, \infty)^{\pi_0(W')} \times \mathrm{Fr}^\theta(W')^{[0,\infty)} \\ \gamma(0) = \ell_{W'} \text{ and } \gamma|_{[t^i,\infty)} \equiv \ell_W \circ T\alpha \text{ on each } i \in \pi_0(W'). \end{array} \right\}.$$

It admits a strict composition $\mathrm{Emb}^\theta(W', W) \times \mathrm{Emb}^\theta(W'', W') \to \mathrm{Emb}^\theta(W'', W)$ by setting $(\alpha, t, \gamma) \circ (\alpha', t', \gamma') = (\alpha \circ \alpha', t' + t, \bar{\gamma})$ on each path-component, with

$$\bar{\gamma}(s) := \begin{cases} \gamma'(s) & \text{for } 0 \leqslant s \leqslant t', \\ \gamma(s - t') \circ T\alpha' & \text{for } t' \leqslant s \leqslant t' + t, \\ \ell_W \circ T\alpha \circ T\alpha' & \text{for } t' + t \leqslant s, \end{cases}$$

and since we treated different components of W' separately, we can also take disjoint unions of θ-framed embeddings, resulting in a family of maps

$$\mathrm{Emb}^{\theta}(W_1', W_1) \times \mathrm{Emb}^{\theta}(W_2', W_2) \to \mathrm{Emb}^{\theta}(W_1' \sqcup W_2', W_1 \sqcup W_2).$$

This constitutes a topologically enriched symmetric monoidal category with objects being θ-framed manifolds, and morphisms being θ-framed embeddings [24, Definition 20].

We have maps $\mathrm{Emb}^{\theta}(W', W) \to \mathrm{Emb}(W', W)$ compatible with compositions and disjoint unions. In the case where θ is the identity on $\mathrm{BO}(d)$, these maps are equivalences.

Notation 2.18. We often suppress the length of the Moore path in the tuple and just write (α, γ), meaning that for each $i \in \pi_0(W')$, with $W_i' \subseteq W'$ being the corresponding path-component, we have a path $\gamma^i : [0, t^i] \to \mathrm{Fr}^{\theta}(W_i')$.

Definition 2.19. We fix, once and for all, a θ-framing $\ell_{\mathbb{R}^d}$ of $\mathbb{R}^d$ which under the tautological trivialisation $\mathbb{R}^d \times \mathbb{R}^d \cong T\mathbb{R}^d$ only depends on the fibre factor (this is the same as choosing a basepoint $b_0 \in L$ and parameterising the fibre $\theta^* V_d|_{b_0}$).

Let $\ell_{0,1}$ be the restriction of $\ell_{\mathbb{R}^d}$ to D^d (the index stands for 'genus 0 and one boundary component'). Then we define the *θ-framed d-discs operad* with $E_d^{\theta}(r) := \mathrm{Emb}^{\theta}(\underline{r} \times D^d, D^d)$, with $\underline{r} = \{1, \dots, r\}$ as before, where the operadic composition is given by composition of disjoint unions of embeddings.

Example 2.20. We have an operad map $\iota : E_d \to E_d^{\theta}$ by endowing each $\alpha : D^d \hookrightarrow D^d$ with $\alpha(z) = \dot{z} + r \cdot z$ with the path $[0, \log(\frac{1}{r})] \to \mathrm{Fr}^{\theta}(D^d)$ taking s to $e^{-s} \cdot \ell_{0,1}$. If θ is the universal bundle $\mathrm{EO}(d) \to \mathrm{BO}(d)$, this map is an equivalence of operads, leading to the usual defect in nomenclature that the classical ('unframed') operad E_d is equivalent to the E_d-operad for the tangential structure of framings.

Lemma 2.21. *The space $E_d^{\theta}(1)$ of unary operations is equivalent to ΩL.*

Proof. The map $\mathrm{Bun}(TD^d, \theta^* V_d) \to L$ that only remembers the value of $0 \in D^d$ is a fibration, whose fibre is homeomorphic to $\mathrm{Bun}_0(TD^d, TD^d)$, the space of bundle maps fixing $0 \in D^d$. Moreover, we have a zig-zag of maps

$$\begin{array}{ccccc}
\mathrm{Emb}(D^d, D^d) & \xleftarrow{\ \omega|_{D^d} \leftarrow\!\shortmid\, \omega\ } & \mathrm{O}(d) & \xrightarrow{\ \omega \mapsto T\omega|_{D^d}\ } & \mathrm{Bun}_0(TD^d, TD^d) \\
{\scriptstyle \alpha \mapsto \ell_{0,1} \circ T\alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle \ell_{0,1} \circ (-)} \\
\mathrm{Bun}(TD^d, \theta^* V_d) & = & \mathrm{Bun}(TD^d, \theta^* V_d) & = & \mathrm{Bun}(TD^d, \theta^* V_d),
\end{array}$$

where the horizontal maps are equivalences. This induces a zig-zag of equivalences among homotopy fibres. They, in turn, are $\mathrm{Emb}^{\theta}(D^d, D^d) = E_d^{\theta}(1)$ and ΩL. □

2.3 | Factorisation homology and decorated moduli spaces

We start by repeating the model from [24, Definition 34+43]:

Reminder 2.22. Let (W, ℓ_W) be a θ-framed manifold and let A be a E_d^θ-algebra. We define the *factorisation homology* $\int_W^\theta A := |\mathrm{B}_\bullet(\mathbb{E}_W^\theta, \mathbb{E}_d^\theta, A)|$, where $\mathbb{E}_W^\theta$ is the right $\mathbb{E}_d^\theta$-functor that takes a space X to $\coprod_{r\geqslant 0} \mathrm{Emb}^\theta(\underline{r} \times D^d, W) \times_{\mathfrak{S}_r} X^r$.

Informally, $\int_W^\theta A$ is the space of configurations of discs inside W, each disc carrying a label in A and the E_d^θ-action on A is balanced with precomposing embeddings of discs; we call such a datum a 'decoration' of W. We want to consider the *moduli space* of such decorated manifolds. As done in [16, 17] for moduli spaces of manifolds without decorations, this can be implemented by extending $\int_W^\theta A$ so that the θ-framing of W is allowed to vary, and then quotient out the action of the usual topological group $\mathrm{Diff}_\partial(W)$. We make precise what we mean by this:

Construction 2.23. Let (W, ℓ_W) be a θ-framed manifold, possibly with boundary. We fix a collar of ∂W and let $\mathrm{Fr}_\partial^\theta(W) \subseteq \mathrm{Fr}^\theta(W)$ be the space of θ-framings ℓ whose restriction to that collar agrees with the restriction of ℓ_W. Moreover, let $\mathrm{Fr}_\partial^\theta(W, \ell_W) \subseteq \mathrm{Fr}_\partial^\theta(W)$ be the subspace containing those path-components intersecting the $\mathrm{Diff}_\partial(W)$-orbit of ℓ_W.

If $(W', \ell_{W'})$ is another θ-framed manifold, then we want the space $\underline{\mathrm{Emb}}^\theta(W', W)$ to model the homotopy fibre of

$$\mathrm{Emb}(W', W) \times \mathrm{Fr}_\partial^\theta(W, \ell_W) \to \mathrm{Fr}^\theta(W'), \quad (\alpha, \ell) \mapsto \ell \circ T\alpha$$

at $\ell_{W'}$. Similar to Construction 2.17, this is achieved by defining $\underline{\mathrm{Emb}}^\theta(W', W)$ as the subspace of $\mathrm{Emb}(W', W) \times [0, \infty)^{\pi_0(W')} \times \mathrm{Fr}^\theta(W')^{[0,\infty)} \times \mathrm{Fr}_\partial^\theta(W, \ell_W)$ containing all tuples $(\alpha, t, \gamma, \ell)$ such that (α, t, γ) is a θ-framed embedding $(W', \ell_{W'}) \hookrightarrow (W, \ell)$. We then have compositions $\underline{\mathrm{Emb}}^\theta(W', W) \times \mathrm{Emb}^\theta(W'', W') \to \underline{\mathrm{Emb}}^\theta(W'', W)$ that are given by

$$(\alpha, t, \gamma, \ell) \circ (\alpha', t', \gamma') = ((\alpha, t, \gamma) \circ (\alpha', t', \gamma'), \ell).$$

Construction 2.24. The topological group $\mathrm{Diff}_\partial(W)$ of diffeomorphisms of W that preserve the collar of W acts on $\underline{\mathrm{Emb}}^\theta(W', W)$ via $\phi \cdot (\alpha, t, \gamma, \ell) = (\phi \circ \alpha, t, \gamma, \ell \circ T\phi^{-1})$ and we denote the homotopy quotient by

$$\mathcal{M}_\partial^\theta(W, \ell_W)^{(W', \ell_{W'})} := \underline{\mathrm{Emb}}^\theta(W', W) /\!\!/ \mathrm{Diff}_\partial(W).$$

Example 2.25. The embedding space $\underline{\mathrm{Emb}}^\theta(\varnothing, W)$ is the same as $\mathrm{Fr}_\partial^\theta(W, \ell_W)$, and so $\mathcal{M}_\partial^\theta(W, \ell_W)^\varnothing = \mathrm{Fr}_\partial^\theta(W, \ell_W) /\!\!/ \mathrm{Diff}_\partial(W)$, which agrees with the classical description of the moduli space of (W, ℓ_W) as in [17]; hence the notation.

We point out that in general, $\mathcal{M}_\partial^\theta(W, \ell_W)^{(W', \ell_{W'})}$ need not be path-connected, but in the special case of $(W', \ell_{W'}) = \varnothing$, it is path-connected by construction.

Construction 2.26. Let (W, ℓ_W) be a θ-framed manifold. We have a right $\mathbb{E}_d^\theta$-functor from spaces to $\mathrm{Diff}_\partial(W)$-spaces by

$$\underline{\mathbb{E}}_W^\theta(X) := \coprod_{r\geqslant 0} \underline{\mathrm{Emb}}^\theta(\underline{r} \times D^d, W) \times_{\mathfrak{S}_r} X^r,$$

where the transformation $\underline{\mathbb{E}}_W^\theta \circ \mathbb{E}_d^\theta \to \underline{\mathbb{E}}_W^\theta$ is induced by the above composition, using that precomposition is $\mathrm{Diff}_\partial(W)$-equivariant. When passing to homotopy quotients, we get a right $\mathbb{E}_d^\theta$-functor from spaces to spaces by $\mathbb{D}^\theta_W := \underline{\mathbb{E}}^\theta_W /\!\!/ \mathrm{Diff}_\partial(W)$, and we define the *moduli space of manifolds of type W with decorations in A* as

$$W^\theta[A] := |\mathrm{B}_\bullet(\mathbb{D}^\theta_W, \mathbb{E}^\theta_d, A)| \cong |\mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{E}^\theta_d, A)| /\!\!/ \mathrm{Diff}_\partial(W).$$

Example 2.27. The one-point space $*$, together with its unique E^θ_d-algebra structure, is the free E^θ_d-algebra over the empty space $\emptyset$. We hence have an augmentation

$$\mathrm{B}_\bullet(\mathbb{D}^\theta_W, \mathbb{E}^\theta_d, *) = \mathrm{B}_\bullet(\mathbb{D}^\theta_W, \mathbb{E}^\theta_d, \mathbb{E}^\theta_d(\emptyset)) \to \mathbb{D}^\theta_W(\emptyset) = \mathcal{M}^\theta_\partial(W, \ell_W),$$

induced by the transformation $\mathbb{D}^\theta_W \circ \mathbb{E}^\theta_d \to \mathbb{D}^\theta_W$, and this augmentation admits an extra degeneracy induced by the unit of the monad $\mathbb{E}^\theta_d$ in the last argument. This shows that after geometric realisation, we obtain an equivalence

$$W^\theta[*] = |\mathrm{B}_\bullet(\mathbb{D}^\theta_W, \mathbb{E}^\theta_d, *)| \xrightarrow{\simeq} \mathcal{M}^\theta_\partial(W, \ell_W).$$

Remark 2.28. The assignment $A \mapsto W^\theta[A]$ is functorial in E^θ_d-algebras and $*$ is both initial and terminal in E^θ_d-algebras. It follows that $W^\theta[*]$ is a retract of $W^\theta[A]$ for each E^θ_d-algebra A. In combination with Example 2.27, we can conclude that each $W^\theta[A]$ contains $\mathcal{M}^\theta_\partial(W, \ell_W)$ as a homotopy retract.

Lemma 2.29. *If A is path-connected, then $W^\theta[A]$ is path-connected as well.*

Proof. We show that the aforementioned retract $W^\theta[*] \hookrightarrow W^\theta[A]$ is 0-connected (i.e. surjective on π_0); then the statement follows from the fact that $W^\theta[*] \simeq \mathcal{M}^\theta_\partial(W, \ell_W)$ is path-connected. Since the map in question is the geometric realisation of the simplicial map $\iota_\bullet : \mathrm{B}_\bullet(\mathbb{D}^\theta_W, \mathbb{E}^\theta_d, *) \to \mathrm{B}_\bullet(\mathbb{D}^\theta_W, \mathbb{E}^\theta_d, A)$, it suffices to check that the map ι_0 among 0-simplices is 0-connected. The map ι_0, however, is explicitly given by the union

$$\coprod_{r \geqslant 0} \mathrm{id}_{\underline{\mathrm{Emb}}^\theta(\underline{r} \times D^d, W)} \times_{\mathfrak{S}_r} (* \to A)^r,$$

which clearly is 0-connected if A is path-connected. ☐

Example 2.30. In the case where $\theta : \mathrm{BSO}(2) \to \mathrm{BO}(2)$ is orientation of surfaces, A is an E^θ_2-algebra and W is an oriented surface, $W^\theta[A]$ is a 'disc model' for the generalised configuration space that has been described in [7, section 4].

We will see further special cases in Section 5, for example, moduli spaces $\mathcal{M}^{\theta,r}_\partial(W, \ell_W)$ of manifolds with r permutable punctures studied in [6, 8], see Example 5.13.

Next, we establish a fibre sequence showing that $W^\theta[A]$ relates factorisation homology $\int^\theta_W A$ and the classical moduli space $\mathcal{M}^\theta_\partial(W, \ell_W)$. This generalises the well-known fibre sequence $\coprod_r C_r(\mathring{W}) \to \coprod_r \mathcal{M}^{\theta,r}_\partial(W, \ell_W) \to \mathcal{M}^\theta_\partial(W, \ell_W)$, where $C_r(\mathring{W})$ is the space of unordered configurations of r particles inside $\mathring{W}$:

Proposition 2.31. *For each E_d^θ-algebra A, we have a homotopy fibre sequence with a section*

$$\int_W^\theta A \longrightarrow W^\theta[A] \longrightarrow \mathcal{M}_\partial^\theta(W, \ell_W).$$

Proof. Consider the augmented proper simplicial space

$$\mathrm{B}_\bullet(\underline{\mathbb{E}}_W^\theta, \mathbb{E}_d^\theta, A) \to \mathrm{B}_\bullet(\underline{\mathbb{E}}_W^\theta, \mathbb{E}_d^\theta, *) \to \underline{\mathbb{E}}_W^\theta(\varnothing) = \mathrm{Fr}_\partial^\theta(W, \ell_W).$$

The augmentations $B_p(\underline{\mathbb{E}}_W^\theta, \mathbb{E}_d^\theta, A) \to \mathrm{Fr}_\partial^\theta(W, \ell_W)$ are of the form $\underline{\mathbb{E}}_W^\theta(Y) \to \mathrm{Fr}_\partial^\theta(W, \ell_W)$ taking a tuple $[\alpha, t, \gamma, \ell; y_1, \dots, y_r] \in \underline{\mathrm{Emb}}^\theta(\underline{r} \times D^d, W) \times_{\mathfrak{S}_r} Y^r$ to ℓ. These maps can easily be checked to be fibrations, in particular quasifibrations. The actual simplicial fibre of the augmentation is the proper simplicial space $\mathrm{B}_\bullet(\mathbb{E}_W^\theta, \mathbb{E}_d^\theta, A)$. By [11, Lemma 2.14] (and the fact that all involved simplicial spaces are proper, enabling us to switch between think and thin realisations), it follows that the homotopy fibre of the realisation $|\mathrm{B}_\bullet(\underline{\mathbb{E}}_W^\theta, \mathbb{E}_d^\theta, A)| \to \mathrm{Fr}_\partial^\theta(W, \ell_W)$ is equivalent to $|\mathrm{B}_\bullet(\mathbb{E}_W^\theta, \mathbb{E}_d^\theta, A)| = \int_A^\theta$, and so we obtain a homotopy fibre sequence

$$\int_W^\theta A \to |\mathrm{B}_\bullet(\underline{\mathbb{E}}_W^\theta, \mathbb{E}_d^\theta, A)| \to \mathrm{Fr}_\partial^\theta(W, \ell_W).$$

Now we observe that the second map in this sequence is $\mathrm{Diff}_\partial(W)$-equivariant, and a diagram chase shows that the induced map among homotopy quotients has the same homotopy fibre. This gives rise to the desired homotopy fibre sequence. Finally, the map $W^\theta[A] \to \mathcal{M}_\partial^\theta(W, \ell_W)$ has a section as it is given by $W^\theta[A] \to W^\theta[*]$, followed by the equivalence $W^\theta[*] \to \mathcal{M}_\partial^\theta(W, \ell_W)$ from Example 2.27, and the first map as a section by the functoriality of $W^\theta[-]$, as mentioned before. □

2.4 | Generalised surfaces

In order to be able to 'stabilise' the spaces $W^\theta[A]$, we restrict our attention to a certain class of even-dimensional manifolds $W_{g,1} = \#^g(S^n \times S^n) \setminus \mathring{D}^{2n}$, which we introduce in this subsection. We will use a slightly different model for $W_{g,1}$ in order to have more control over their θ-framings, especially close to their boundary.

Definition 2.32. Let $n \geqslant 1$. Then we abbreviate $C := D^{2n} \setminus \frac{1}{2}D^{2n}$. We consider the manifold $W_{0,1} := D^{2n}$, together with the collar $\jmath_0 : C \hookrightarrow D^{2n}$ and the θ-framing $\ell_{0,1}$.

Moreover, we consider the manifold $W_{1,1} := (S^n \times S^n) \setminus \mathring{D}^{2n}$, where $\mathring{D}^{2n}$ is the interior of a disc inside $S^n \times S^n$, and fix, once and for all, a collar $\jmath_1 : C \hookrightarrow W_{1,1}$.

In order to prescribe a rather strict θ-framing near the boundary of $W_{1,1}$, we need the notion of spherical tangential structures from [16]:

Definition 2.33. Let $D^d \hookrightarrow S^d$ the inclusion of a hemisphere. We call a tangential structure $\theta : L \to \mathrm{BO}(d)$ *spherical* if each θ-framing on D^d can be extended on S^d. As L is assumed to be path-connected, this is equivalent to requiring that S^d admits a θ-structure.

Lemma 2.34. *Let $\theta : L \to \mathrm{BO}(2n)$ be spherical. Then there is a θ-framing $\ell_{1,1}$ on $W_{1,1}$ which is admissible in the sense of [17, Definition 1.3] and satisfies $\ell_{1,1} \circ T\jmath_1 = \ell_{0,1}|_C$.*

Proof. As θ is spherical, we can extend $\ell_{0,1}$ to a θ-framing ℓ of S^{2n}. Now we fix an embedding of the open disc $S^{2n} \setminus \frac{1}{2}D^{2n}$ into the interior of $W_{1,1}$, and by using that $\ell|_{\mathring{C}}$ factors through the trivial $\mathbb{R}^{2n}$-bundle over a point, we find an admissible θ-framing of $W_{1,1}$ that restricts to ℓ on $S^{2n} \setminus \frac{1}{2}D^{2n}$. By [17, Lemma 7.9], we can extend this θ-framing further onto the closed manifold $S^n \times S^n$. Finally, we remove $S^{2n} \setminus D^{2n}$ from $S^n \times S^n$ to obtain $W_{1,1}$ with the desired θ-framing. □

Construction 2.35. For $g \geqslant 2$, we fix a rectilinear embedding $\alpha_g : \underline{g} \times D^{2n} \hookrightarrow D^{2n}$ that satisfies $\alpha_g(\underline{g} \times \frac{1}{2}D^{2n}) \subseteq \frac{1}{2}D^{2n}$. If we abbreviate $D^{2n}_g := D^{2n} \setminus \alpha_g(\underline{g} \times \frac{1}{2}D^{2n})$, then α_g (co-)restricts to a map $\underline{g} \times C \hookrightarrow D^{2n}_g$, and we define the smooth manifold

$$W_{g,1} := (\underline{g} \times W_{1,1}) \cup_{\underline{g} \times C} D^{2n}_g.$$

We define $\ell_{g,1} : TW_{g,1} \to \theta^* V_{2n}$ to be $r_i^{-1} \cdot \ell_{1,1}$ on each $\{i\} \times TW_{1,1}$, where $r_i > 0$ is the radius of the ith disc, and to be $\ell_{0,1}$ on D^{2n}_g, noting that these maps agree on the intersection of their domains. Then $\jmath_g : C \hookrightarrow D^{2n}_g \hookrightarrow W_{g,1}$ satisfies $\ell_{g,1} \circ T\jmath_g = \ell_{0,1}|_C$.

We thus have defined, for any $g \geqslant 0$, a θ-framed manifold $(W_{g,1}, \ell_{g,1})$ with a collar $\jmath_g : C \to W_{g,1}$ of the boundary. Accordingly, we require each $\phi \in \mathrm{Diff}_\partial(W_{g,1})$ to satisfy $\phi \circ \jmath_g = \jmath_g$, and each θ-framing $\ell \in \mathrm{Fr}^\theta_\partial(W_{g,1}, \ell_{g,1})$ to satisfy $\ell \circ T\jmath_g = \ell_{g,1} \circ T\jmath_g$.

Remark 2.36. By picking θ-framed embeddings $(W_{g,1}, \ell_{g,1}) \hookrightarrow (W_{g+1,1}, \ell_{g+1,1})$, we can construct stabilisation maps $W^\theta_{g,1}[A] \to W^\theta_{g+1,1}[A]$. As it will turn out, these stabilisation maps are actually part of an A_∞-structure on the disjoint union $W^\theta_{*,1}[A] := \coprod_{g \geqslant 0} W_{g,1}[A]$.

Instead of making that formal, we will establish in Proposition 3.1 a graded equivalence between $W^\theta_{*,1}[A]$ and a space that canonically carries an A_∞-structure.

3 | FACTORISATION HOMOLOGY AND THE GENERALISED SURFACE OPERAD

We fix a dimension $d = 2n$ and a spherical tangential structure $\theta : L \to \mathrm{BO}(2n)$. The goal of this section is to prove the following statement, where $\mathcal{W}^\theta$ is the θ-framed generalised surface operad (also called 'manifold operad') from [2]:

Proposition 3.1. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure. Then we have, for each E^θ_{2n}-algebra A, a graded equivalence $W^\theta_{*,1}[A] \simeq \mathcal{W}^\theta \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A$.*

Here F^θ_{2n} is an operad that comes with an operad map $F^\theta_{2n} \to \mathcal{W}^\theta$ and an equivalence $\kappa : F^\theta_{2n} \to E^\theta_{2n}$ of operads, and the pushforward $\mathcal{W}^\theta \otimes^{\mathbb{L}}_{F^\theta_{2n}} (-)$ is graded by genus. Each of these objects needs to be constructed first, and this is the content of the next subsection.

3.1 Geometric models for operads

Informally, the generalised surface operad $\mathcal{W}^\theta$ is associated to the products and permutations category (PROP) given by the subcategory of the θ-framed bordism category, containing as objects disjoint unions of S^{2n-1} and as morphisms bordisms diffeomorphic to unions of $W_{g,r+1} = W_{1,1} \setminus (\underline{r} \times \mathring{D}^{2n})$, considered to have r incoming and one outgoing boundary components. This description can be found in an ∞-operadic setting in [23, section 6.2]. In particular, the operation space $\mathcal{W}^\theta(r)$ is a model for the moduli space $\mathcal{M}^\theta_\partial(W_{g,r+1}, \ell_{g,r+1})$. When working with actual topological operads, one uses a different model [2, 36], due to the fact that the topological bordism category [14] is not strictly monoidal.

We study yet a third model that uses spaces of submanifolds of $\mathbb{R}^\infty$, and I thank the referee for showing a way to significantly simplify my original technical construction. The main merit of this model is the fact that we can make the aforementioned zig-zag $E^\theta_{2n} \leftarrow F^\theta_{2n} \rightarrow \mathcal{W}^\theta$ explicit. The existence of such operad maps has been mentioned as a 'folk theorem' in [21, Remark 6.15], but I am not aware of any reference for it.

Construction 3.2. For each $d \geqslant 0$, we let $D^{d,\infty} := D^d \times D^\infty$, and we identify D^d with the subspace $D^d \times \{0\}^\infty \subset D^{d,\infty}$.

As a slight variation of the operad E_∞ from Example 2.6, let $E_{d,\infty}(r)$ be the space of all embeddings $\beta : \underline{r} \times D^{d,\infty} \hookrightarrow D^{d,\infty}$ which are on each disc of the form $z \mapsto \dot{z} + \rho \cdot z$ for some $\dot{z} \in D^{d,\infty}$ and $\rho > 0$, such that $\beta(\underline{r} \times \frac{1}{2}D^d \times D^\infty) \subseteq \frac{1}{2}D^d \times D^\infty$ holds. These spaces assemble into a proper operad $E_{d,\infty}$ with contractible operation spaces.[†]

Construction 3.3. Recall the annulus $C := D^{2n} \setminus \frac{1}{2}D^{2n}$, the manifolds $W_{g,1}$ and the collars $\jmath_g : C \hookrightarrow W_{g,1}$ from Section 2.4. We define $\mathrm{Emb}_\partial(W_{g,1}, D^{2n,\infty})$ as the space of smooth embeddings $\eta : W_{g,1} \hookrightarrow D^{2n,\infty}$ such that $\eta \circ \jmath_g$ agrees with the canonical embedding $C \hookrightarrow D^{2n,\infty}$ and the remainder $\eta(W_{g,1} \setminus \jmath_g(C))$ lies inside $\frac{1}{2}D^{2n} \times \mathring{D}^{2n,\infty}$.

Construction 3.4. Let $\tilde{\mathcal{W}}^\theta_g(r) \subset \mathrm{Emb}_\partial(W_{g,1}, D^{2n,\infty}) \times \mathrm{Fr}^\theta_\partial(W_{g,1}, \ell_{g,1}) \times E_{2n,\infty}(r)$ the subspace of triples (η, ℓ, β) such that $\eta(W_{g,1}) \cap \beta(\underline{r} \times D^{2n,\infty}) = \beta(\underline{r} \times D^{2n})$ holds and for the (co-)restriction $\beta|_{\underline{r}\times D^{2n}} : \underline{r} \times D^{2n} \rightarrow W$, we have a (strict) equality

$$\ell \circ T\eta^{-1} \circ T\beta|_{\underline{r}\times D^{2n}} = \underline{r} \times \ell_{0,1}.$$

We have an action of $\mathrm{Diff}_\partial(W_{g,1})$ on $\tilde{\mathcal{W}}^\theta_g(r)$ by precomposing η with ϕ and ℓ with $T\phi$, and we call its (actual) quotient $\mathcal{W}^\theta_g(r)$. Then elements in $\mathcal{W}^\theta_g(r)$ are given by triples (W, ℓ_W, β) where $W \subset D^{2n,\infty}$ is a smooth submanifold of type $W_{g,1}$ satisfying the above conditions near the boundary, ℓ_W is a θ-framing of W such that for some identification of W with $W_{g,1}$, the pullback of ℓ_W lies in $\mathrm{Fr}^\theta_\partial(W_{g,1}, \ell_{g,1})$, and $\beta \in E_{2n,\infty}(r)$ satisfying $W \cap \beta(\underline{r} \times D^{2n,\infty}) = \beta(\underline{r} \times D^{2n})$ and $\ell_W \circ T\beta|_{\underline{r}\times D^{2n}} = \underline{r} \times \ell_{0,1}$.

Construction 3.5. For any $r, r_1, \dots, r_r, g, g_1, \dots, g_r \geqslant 0$, we have a map

$$\mathcal{W}^\theta_g(r) \times \prod_{i=1}^{r} \mathcal{W}^\theta_{g_i}(r_i) \rightarrow \mathcal{W}^\theta_{g+\sum_i g_i}(\textstyle\sum_i r_i)$$

[†] The operad $E_{d,\infty}$ is actually equivalent to E_∞, but we will not use this fact.

as follows: Let $(W, \ell_W, \beta) \in \mathcal{W}_g^\theta(r)$ and $(W_i, \ell_{W_i}, \beta_i) \in \mathcal{W}_{h_i}^\theta(r_i)$, then the result is given by $(\hat{W}, \ell_{\hat{W}}, \beta \circ (\beta_1 \sqcup \cdots \sqcup \beta_r))$, where

$$\hat{W} := \left(W \setminus \beta(\underline{r} \times \tfrac{1}{2}D^{2n})\right) \cup_{\beta(\underline{r}\times C)} \bigcup_{i=1}^{r} \beta(\{i\} \times W_i),$$

and $\ell_{\hat{W}}$ is defined to be ℓ_W on $W \setminus \beta(\underline{r} \times \frac{1}{2}D^{2n})$ and $\ell_{W_i} \circ T\beta^{-1}$ on each $\beta(\{i\} \times W_i)$, noting that both terms agree on the intersection of their domains. Using a diffeomorphism of W that moves the discs $\beta(\underline{r} \times D^{2n})$ near the collar, we obtain an identification of $\hat{W}$ with $W_{g+h_1\cdots+h_r,1}$ that pulls back $\ell_{\hat{W}}$ to a θ-framing in the Diff_∂-orbit of $\ell_{g+h_1+\cdots+h_r,1}$.

This turns the collection of $\mathcal{W}^\theta(r) := \coprod_{g\geqslant 0} \mathcal{W}_g^\theta(r)$ into a proper operad, called the *θ-framed generalised surface operad*, with identity $(D^{2n}, \ell_{0,1}, \mathrm{id}_{D^{2n,\infty}}) \in \mathcal{W}_0^\theta(1)$.

Construction 3.6. Recall that a θ-framed embedding $(\alpha, \gamma) \in E_{2n}^\theta(r)$ contains Moore paths $\gamma \in \mathrm{Fr}_\partial^\theta(\underline{r} \times D^{2n})^{[0,\infty)}$ which are on each $\{i\} \times D^{2n}$ of a given length $t^i \geqslant 0$. Let $F_{2n}^\theta(r) \subset E_{2n}^\theta(r) \times \mathrm{Emb}_\partial(D^{2n}, D^{2n,\infty}) \times [0,\infty) \times \mathrm{Fr}_\partial^\theta(D^{2n})^{[0,\infty)}$ be the subspace containing all $(\alpha, \gamma, \eta, u, \zeta)$ that satisfy the following:

- There is a $\beta \in E_{2n,\infty}(r)$ that satisfies $\eta(D^{2n}) \cap \beta(\underline{r} \times D^{2n,\infty}) = \beta(\underline{r} \times D^{2n})$ and $\eta \circ \alpha = \beta|_{\underline{r}\times D^{2n}}$. Note that if such a β exists, it is uniquely determined.
- ζ is constant at $[u, \infty)$, $\zeta(0) = \ell_{0,1}$ and $\zeta(s) \circ T\alpha = \gamma(t^i - s)$ for all $s \geqslant 0$ (where we put $\gamma(s) := \gamma(0)$ for $s < 0$) on $\{i\} \times D^{2n}$; in particular, $\zeta(u) \circ T\alpha = \underline{r} \times \ell_{0,1}$.

As done before, we skip the entry u from the tuple and regard ζ as a path that is defined on $[0, u]$. Then we have an operadic composition $F_{2n}^\theta(r) \times \prod_{i=1}^r F_{2n}^\theta(r_i) \to F_{2n}^\theta(\sum_i r_i)$ by taking $(\alpha, \gamma, \eta, \zeta)$ and $(\alpha_i, \gamma_i, \eta_i, \zeta_i)$ to $((\alpha, \gamma) \circ \bigsqcup_i (\alpha_i, \gamma_i), \hat{\eta}, \hat{\zeta})$, where $\hat{\eta}$ is defined to be $\beta \circ \eta_i \circ \alpha^{-1}$ inside $\alpha(\{i\} \times D^{2n})$, and η everywhere else, and where

$$\hat{\zeta}(s) := \begin{cases} \zeta_i(s - t^i) \circ (T\alpha)^{-1} & \text{inside } \alpha(\{i\} \times D^{2n}) \text{ if } s \geqslant t^i, \\ \zeta(s) & \text{else.} \end{cases}$$

We have a map of proper operads $F_{2n}^\theta \to E_{2n}^\theta$ that remembers (α, γ) from each tuple. Moreover, we have maps $F_{2n}^\theta \to \tilde{\mathcal{W}}_0^\theta(r)$ by taking $(\alpha, \gamma, \eta, \zeta)$ to $(\eta, \zeta(u), \beta)$, where β is uniquely determined as described above. Quotienting out by the action of $\mathrm{Diff}_\partial(W_{g,1})$, the above assignment constitutes a map of proper operads $F_{2n}^\theta \to \mathcal{W}^\theta$.

Construction 3.7. Recall the operad map $\iota : E_{2n} \to E_{2n}^\theta$ from Example 2.20. For any $1 \leqslant k \leqslant 2n$, we let $F_k(r)$ be the subspace of $E_k(r) \times \mathrm{Fr}_\partial^\theta(D^{2n})^{[0,\infty)}$ containing all (α, ζ) such that if $\alpha' : \underline{r} \times D^{2n} \hookrightarrow D^{2n}$ denotes the canonical extension, then $(\iota(\alpha'), \eta_0, \zeta)$ lies in $F_{2n}^\theta(r)$, where $\eta_0 : D^{2n} \hookrightarrow D^{2n,\infty}$ is the standard inclusion. Then the above composition law for paths ζ turns F_k into a proper operad.

We have operad inclusions $F_1 \to F_2 \to \cdots \to F_{2n}$ and an operad map $F_{2n} \to F_{2n}^\theta$ by taking (α, ζ) to $(\iota(\alpha'), \eta_0, \zeta)$. Moreover, we have maps $F_k \to E_k$ by taking (α, ζ) to α.

Construction 3.8. We fix, once and for all, a nullary operation $\mathbf{0} \in F_1(0)$, for example, the path in $\mathrm{Fr}_\partial^\theta(D^{2n})$ that is constantly $\ell_{0,1}$. Along the operad map $F_1 \to \mathcal{W}^\theta(r)$, this becomes a genus-0 nullary operation in $\mathcal{W}^\theta(r)$ and we obtain *capping maps* cap : $\mathcal{W}^\theta_g(r) \to \mathcal{W}^\theta_g(0)$ by precomposing each operation with r copies of $\mathbf{0}$.

Similarly, we by fixing a genus-1 unary operation $\sigma \in \mathcal{W}^\theta_1(1)$, we obtain *stabilisation maps* stab : $\mathcal{W}^\theta_g(r) \to \mathcal{W}^\theta_{g+1}(r)$ by postcomposing with s. Since stabilisation commutes with capping, we have a stable capping map $\operatorname{hocolim}_{g\to\infty}(\mathrm{cap} : \mathcal{W}^\theta_g(r) \to \mathcal{W}^\theta_g(0))$.

3.2 | Properties of the new operads

The goal of this subsection is to compare the spaces $\mathcal{W}^\theta_g(r)$ to moduli spaces of θ-framed manifolds with multiple boundary components, and to show that the operad maps $F^\theta_{2n} \to E^\theta_{2n}$ and $F_k \to E_k$ are equivalences.

Definition 3.9. For $g, r \geqslant 0$, we fix an embedding $\alpha : \underline{r} \times D^{2n} \hookrightarrow \mathring{W}_{g,1}$ and consider the θ-framed manifold $(W_{g,r+1}, \ell_{g,r+1})$ arising from $(W_{g,1}, \ell_{g,1})$ by removing $\alpha(\underline{r} \times \mathring{D}^{2n})$.

We have inclusions $\mathrm{Diff}_\partial(W_{g,r+1}) \hookrightarrow \mathrm{Diff}_\partial(W_{g,1})$ by extending diffeomorphisms trivially on $\alpha(\underline{r} \times \mathring{D}^{2n})$, and similarly $\mathrm{Fr}^\theta_\partial(W_{g,r+1}) \hookrightarrow \mathrm{Fr}^\theta_\partial(W_{g,1})$; their images are given by the subspace of diffeomorphisms ϕ with $\phi \circ \alpha = \alpha$ and framings ℓ with $\ell \circ T\alpha = \ell_{g,1} \circ T\alpha$. We suggestively denote these maps by $\phi \mapsto \phi \cup_\partial (\underline{r} \times D^{2n})$ and $\ell \mapsto \ell \cup_\partial (\underline{r} \times D^{2n})$.

Lemma 3.10. *Let $\theta : L \to \mathrm{BO}(2n)$ be π_1-injective and let $\ell \in \mathrm{Fr}^\theta_\partial(W_{g,r+1})$. Then ℓ lies in $\mathrm{Fr}^\theta_\partial(W_{g,r+1}, \ell_{g,r+1})$ if and only if $\ell \cup_\partial (\underline{r} \times D^{2n})$ lies in $\mathrm{Fr}^\theta_\partial(W_{g,1}, \ell_{g,1})$.*

Proof. If $\ell \in \mathrm{Fr}^\theta_\partial(W_{g,r+1}, \ell_{g,r+1})$, then we find $\psi \in \mathrm{Diff}_\partial(W_{g,r+1})$ and a path γ from ℓ to $\ell_{g,r+1} \circ T\psi$, so $s \mapsto \gamma(s) \cup_\partial (\underline{r} \times D^{2n})$ is a path from $\ell \cup_\partial (\underline{r} \times D^{2n})$ to $\ell_{g,1} \circ T(\psi \cup_\partial \underline{r} \times D^{2n})$.

For the converse, we can assume that α extends to an embedding $\tilde{\alpha} : \underline{r} \times 2D^n \hookrightarrow W_{g,1}$ and $\ell_{g,1} \circ T\tilde{\alpha}$ factors over the trivial $\mathbb{R}^{2n}$-bundle over a point on each $\tilde{\alpha}(\{i\} \times \mathring{2}D^{2n})$. Now we note that $\mathrm{Fr}^\theta_\partial(W_{g,r+1}) \to \mathrm{Fr}^\theta_\partial(W_{g,1})$ is the (homotopy) fibre of $(-) \circ T\alpha$, which lands in $\mathrm{Fr}^\theta(\underline{r} \times D^{2n}) = \mathrm{Fr}^\theta(D^{2n})^r$. As in the proof of Lemma 2.21, this space fits into a fibre sequence $\mathrm{O}(2n)^r \to \mathrm{Fr}^\theta(D^{2n})^r \to L^r$, which can be continued by $\theta^r : L^r \to \mathrm{BO}(2n)^r$. Since θ is assumed to be π_1-injective, the map $\pi_1\mathrm{O}(2n)^r \to \pi_1\mathrm{Fr}^\theta(D^{2n})^r$ is surjective.

Postcomposing with the connecting morphism $\pi_1\mathrm{Fr}^\theta(D^{2n})^r \to \pi_0\mathrm{Fr}^\theta_\partial(W_{g,r+1})$ of the first fibre sequence, we obtain a map $\Phi : \pi_1\mathrm{O}(2n)^r \to \pi_0\mathrm{Fr}^\theta_\partial(W_{g,r+1})$ and by exactness, this shows that if a component of $\mathrm{Fr}^\theta_\partial(W_{g,r+1})$ gets identified with the component of $\ell_{g,r+1}$ when applying $(-) \cup_\partial (\underline{r} \times D^{2n})$, then it lies in the image of Φ. This map Φ, in turn, sends each loop $\gamma^\bullet : [0,1] \to \mathrm{O}(2n)^r$ to the component of ℓ^γ, which is given as follows: Let $\hat{\alpha} : \underline{r} \times S^{2n-1} \times [0,1] \cong \underline{r} \times (2D^{2n} \setminus \mathring{D}^{2n}) \hookrightarrow W_{g,r+1}$ be the restriction of $\tilde{\alpha}$, then a trivialisation of $\tilde{\alpha}^* TW_{g,1}$ restricts to $\hat{\alpha}^* TW_{g,r+1} \cong (\underline{r} \times S^{2n-1} \times [0,1]) \times \mathbb{R}^{2n}$. Under these identifications, ℓ^γ is given by perturbing $\ell_{g,r+1}$ by the vector bundle automorphism $(i, z, t, v) \mapsto (i, z, t, \gamma^i(t) \cdot v)$ inside $\hat{\alpha}(\underline{r} \times S^{2n-1} \times [0,1])$. Now we note that the morphism $\pi_1\mathrm{SO}(2)^r \to \pi_1\mathrm{O}(2n)^r$ is surjective, so we may assume that each γ^i is a mere rotation. Then we find a diffeomorphism $\psi \in \mathrm{Diff}_\partial(W_{g,r+1})$, comprised of r Dehn twists near the r boundary spheres, such that $\ell^\gamma_{g,r+1}$ lies in the same component as $\ell_{g,r+1} \circ T\psi$.

Coming back to the statement, let $\phi \in \mathrm{Diff}_\partial(W_{g,1})$ such that $\ell \cup_\partial (\underline{r} \times D^{2n})$ lies in the same component as $\ell_{g,1} \circ T\phi$. Then ϕ can be isotoped to a diffeomorphism preserving α, and since this new diffeomorphism still satisfies the above condition, we can assume that ϕ itself is of the form $\psi \cup_\partial (\underline{r} \times D^{2n})$ for some $\psi \in \mathrm{Diff}_\partial(W_{g,r+1})$. Then the framing $(\ell \circ T\psi^{-1}) \cup_\partial (\underline{r} \times D^{2n})$ lies in the same component as $\ell_{g,1}$. By the previous discussion, we find a diffeomorphism $\psi' \in \mathrm{Diff}_\partial(W_{g,r+1})$ such that $\ell \circ T(\psi^{-1} \circ \psi')$ lies in the same component as $\ell_{g,r+1}$. □

Lemma 3.11. *If L is π_1-injective, then we have equivalences $\mathcal{W}_g^\theta(r) \simeq \mathcal{M}_\partial^\theta(W_{g,r+1}, \ell_{g,r+1})$.*

Proof. Let $E' \subset E_{2n,\infty}(r)$ be the subspace of all β with $\beta(\underline{r} \times 2D^{2n} \times D^\infty) \subseteq \frac{1}{2}D^{2n} \times D^\infty$, where we extend β on $\underline{r} \times 2D^{2n} \times D^\infty$ with the same term; and we let $\mathcal{W}' \subset \mathcal{W}_g^\theta(r)$ be the subspace of tuples (W, ℓ_W, β) with $\beta \in E'$. Then $\mathcal{W}_g^\theta(r)$ deformation retracts onto $\mathcal{W}'$ by shrinking the radii of the given embeddings and rescaling ℓ_W close to the discs. One can now easily construct, around any $\beta \in E'$, an open neighbourhood $U \subset E'$ and a map $\tau : U \to \mathrm{Diff}(D^{2n,\infty})$ such that, for each $\hat\beta \in U$, the diffeomorphism $\tau(\hat\beta)$ restricts to the identity on $C \times D^\infty$ and satisfies $\tau(\hat\beta) \circ \hat\beta = \beta$. These maps can then be employed to show $\mathcal{W}' \to E'$ is locally trivial and hence a fibration. As the target space E' is still contractible, it hence suffices to study the actual fibre of a fixed element $\beta \in E'$.

This fibre is given by the (strict) $\mathrm{Diff}_\partial(W_{g,1})$-quotient of the subspace S of the product $\mathrm{Emb}_\partial(W_{g,1}, D^{2n,\infty}) \times \mathrm{Fr}_\partial^\theta(W_{g,1}, \ell_{g,1})$ of all (η, ℓ) with $\eta(W_{g,1}) \cap \beta(\underline{r} \times D^{2n,\infty}) = \beta(\underline{r} \times D^{2n})$ and $\ell \circ T\eta^{-1} \circ T\beta|_{\underline{r} \times D^{2n}} = \underline{r} \times \ell_{0,1}$.

We fix an embedding $\alpha : \underline{r} \times D^{2n} \hookrightarrow \mathring{W}_{g,1}$. We note that the subgroup of $\mathrm{Diff}_\partial(W_{g,1})$ containing all ϕ that preserve α agrees with the diffeomorphism group $\mathrm{Diff}_\partial(W_{g,r+1})$, and it acts on the subspace $S' \subseteq S$ of all (η, ℓ) where the first of the above conditions is replaced by the stronger assumption $\eta \circ \alpha = \beta|_{\underline{r} \times D^{2n}}$; then the second assumption reads $\ell \circ T\alpha = \underline{r} \times \ell_{0,1}$. We moreover note that the map $S' \times \mathrm{Diff}_\partial(W_{g,1}) \to S$ taking (η, ℓ, ϕ) to $(\eta \circ \phi, \ell \circ T\phi)$ is a $\mathrm{Diff}_\partial(W_{g,1})$-equivariant principal $\mathrm{Diff}_\partial(W_{g,r+1})$-bundle (where $\mathrm{Diff}_\partial(W_{g,r+1})$ acts by $\psi \cdot (\eta, \ell, \phi) = (\eta \circ \psi, \ell \circ T\psi, \psi^{-1} \circ \phi)$ and where $\mathrm{Diff}_\partial(W_{g,1})$ right acts on $\mathrm{Diff}_\partial(W_{g,1})$ and S by precomposition), so it induces a homeomorphism

$$S'/\mathrm{Diff}_\partial(W_{g,r+1}) = (S' \times_{\mathrm{Diff}_\partial(W_{g,r+1})} \mathrm{Diff}_\partial(W_{g,1}))/\mathrm{Diff}_\partial(W_{g,1}) \to S/\mathrm{Diff}_\partial(W_{g,1}).$$

Finally, we note that S' is the product two spaces: the space of embeddings of $W_{g,r+1}$ into $D^{2n,\infty}$ with a fixed boundary behaviour and the subspace of $\mathrm{Fr}_\partial^\theta(W_{g,r+1})$ of all θ-framings ℓ such that $\ell \cup_\partial (\underline{r} \times D^{2n}) \in \mathrm{Fr}_\partial^\theta(W_{g,1}, \ell_{g,1})$. The first factor is contractible by an application of Whitney's embedding theorem and the second factor is identified with $\mathrm{Fr}_\partial^\theta(W_{g,r+1}, \ell_{g,r+1})$ by Lemma 3.10. As the action of $\mathrm{Diff}_\partial(W_{g,r+1})$ on the first factor is free and proper, we hence get the desired equivalence

$$\begin{aligned} S'/\mathrm{Diff}_\partial(W_{g,r+1}) &\simeq S' /\!\!/ \mathrm{Diff}_\partial(W_{g,r+1}) \\ &\simeq \mathrm{Fr}_\partial^\theta(W_{g,r+1}, \ell_{g,r+1}) /\!\!/ \mathrm{Diff}_\partial(W_{g,r+1}) \\ &= \mathcal{M}_\partial^\theta(W_{g,r+1}, \ell_{g,r+1}). \end{aligned}$$ □

Corollary 3.12. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with π_1-injective θ.*

1. *If $2n \geqslant 6$ (or $2n = 2$ and θ is admissible to [31, Theorem 7.1]), then the stabilisation maps* $\mathrm{stab}: \mathcal{W}_g^\theta(r) \to \mathcal{W}_{g+1}^\theta(r)$ *induce isomorphisms in $H_i(-;\mathbb{Z})$ for $i \leqslant \frac{1}{2}g - \frac{3}{2}$ (for $2n = 2$, the slope has to be replaced by the one from [31, Theorem 7.1]) for any r.*
2. *If L is n-connected, then the stable capping map* $\mathrm{hocolim}_{g\to\infty}(\mathrm{cap}: \mathcal{W}_g^\theta(r) \to \mathcal{W}_g^\theta(0))$ *is a homology equivalence for each $r \geqslant 0$.*

Proof. We note that under the equivalence of Lemma 3.11, the stabilisation and capping maps are identified up to homotopy with the usual maps appearing in the classical stability theorems. Then statement 1 is Harer's stability theorem [18] in the version of [31, Theorem 7.1] for $2n = 2$ and [17, Theorem 1.4] for $2n \geqslant 6$, using that θ is spherical to get the improved slope and that $\ell_{1,1}$ is admissible in the sense of [17, Definition 1.3].

Statement 2 is implied by [16, Theorem 1.3], noting that $\mathcal{M}_\partial^\theta(W_{g,r+1}, \ell_{g,r+1})$ is a single path component of what they would call $\mathcal{N}_n^\theta(\underline{r+1} \times S^{2n-1}, \underline{r+1} \times \ell_{0,1}|_{S^{2n-1}})$. Here n-connectivity of L is needed as the bundle maps $\ell: TW_{g,r+1} \to \theta^* V_{2n}$ constituting $\mathcal{N}_n^\theta(W_{g,r+1}, \ell_{g,r+1})$ are required to cover an n-connected map $W_{g,r+1} \to L$ among base spaces. Since $W_{g,r+1}$ is $(n-1)$-connected, this condition is automatically satisfied if L is n-connected, see also [2, section 7.2] for a similar argument. □

Lemma 3.13. *The operad maps $F_{2n}^\theta \to E_{2n}^\theta$ and $F_k \to E_k$ are equivalences of proper operads.*

Proof. We shall only prove that the first map is an equivalence, as the argument for $F_k \to E_k$ is nearly identical. If $E' \subset E_{2n}^\theta(r)$ contains all (α, γ) with $\alpha(\underline{r} \times \frac{1}{2}D^{2n}) \subseteq \frac{1}{2}D^{2n}$, then E' is a deformation retract of $E_{2n}^\theta(r)$ and the map $F_{2n}^\theta(r) \to E_{2n}^\theta(r)$ corestricts to a fibration over E'. For any $(\alpha,\gamma) \in E'$, the actual fibre $F_{2n}^\theta(r)_{\alpha,\gamma}$ is given by the space of tuples $(\alpha,\gamma,\eta,\zeta) \in F_{2n}^\theta$ satisfying the two conditions from Construction 3.6, and it is our aim to show that this space is contractible.

To do so, we note that the map $F_{2n}^\theta(r)_{\alpha,\gamma} \to E_{2n,\infty}(r)$ assigning to any such tuple the (unique) β with $\eta \circ \alpha = \beta|_{\underline{r}\times D^{2n}}$ is again locally trivial, and therefore a fibration. Since $E_{2n,\infty}(r)$ is contractible, it hence suffices to study its fibre. Similar as in the previous proof, we see that it is the product of two spaces:

- The space of embeddings of $W_{g,r+1}$ into $D^{2n,\infty}$ with a fixed boundary behaviour. This factor is contractible by Whitney's embedding theorem.
- The space S of Moore paths $\zeta: [0,u] \to \mathrm{Fr}_\partial^\theta(D^{2n})$ for some $u \geqslant 0$, satisfying $\zeta(0) = \ell_{0,1}$ and $\zeta(s) \circ T\alpha = \gamma(t_i - s)$.

It hence suffices to show that the second factor S is contractible. Being only interested in its homotopy type, we may replace Moore paths by paths of length 1. If $P_x X$ denotes the space of paths starting at $x \in X$ for any space X, then we see that S is the fibre of

$$(-) \circ T\alpha : P_{\ell_{0,1}} \mathrm{Fr}_\partial^\theta(D^{2n}) \to P_{\ell_{0,1} \circ T\alpha} \mathrm{Fr}_\partial^\theta(\underline{r} \times D^{2n}),$$

the fibre taken at the reversed paths γ. We conclude the argument by noting that both source and target of this map are contractible, and since α is a cofibration, the restriction $(-)\circ T\alpha$ is a fibration, so the fibre S coincides with the contractible homotopy fibre. □

Corollary 3.14. *F_1 is an A_∞-operad and the operad map $F_1 \to F_{2n}^\theta$ sends both path components of $F_1(2)$ to the same component. Moreover, $F_{2n}^\theta(0) \simeq E_{2n}^\theta(0) = *$ is contractible.*

Through the operad map $F_1 \to \mathcal{W}^\theta$, each $\mathcal{W}^\theta$-algebra A is in particular an A_∞-algebra. In the case of $A = \mathcal{W}^\theta(0)$, we get the following:

Corollary 3.15. *If L is n-connected, then the A_∞-algebra $\mathcal{W}^\theta(0)$ group-completes to the infinite loop space $\mathbb{Z} \times \Omega_0^\infty \mathrm{MT}\theta$, where $\mathrm{MT}\theta$ is the tangential Thom spectrum.*

Proof. Recall that Lemma 3.11 provides an equivalence $\mathcal{W}^\theta(0) \simeq \coprod_{g\geqslant 0} \mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1})$ (this does not even use that θ is π_1-injective — which is satisfied anyway, as L is n-connected). By the group-completion theorem, it follows that $\Omega B\mathcal{W}^\theta(0)$ is equivalent to $\mathbb{Z} \times \mathrm{hocolim}_{g\to\infty} \mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1})^+$. Again, using that any map $W_{g,1} \to L$ is n-connected, $\mathcal{W}^\theta(0)$ is a union of several path-components of $\mathcal{N}^\theta_n(S^{2n-1}, \ell_{0,1}|_{S^{2n-1}})$ from the setting of [16], and the stabilisations from [16] restrict on path-components to stabilisations among $\mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1})$. We therefore can apply [16, Theorem 1.5] and conclude that the Pontrjagin–Thom map $\mathrm{hocolim}_{g\to\infty} \mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1}) \to \Omega_0^\infty \mathrm{MT}\theta$ is acyclic. By applying the Quillen plus-construction to the left side, we get an equivalence. □

In the case of $2n = 2$ and $\theta : \mathrm{BSO}(2) \to \mathrm{BO}(2)$, Corollary 3.15 is an instance of the Madsen–Weiss theorem [27] telling us $\Omega B \coprod_{g\geqslant 0} \mathrm{BDiff}^+_\partial(S_{g,1}) \simeq \Omega^\infty \mathrm{MTSO}(2)$.

3.3 | Decorated moduli spaces as a pushforward

Remark 3.16. As the operad map $F^\theta_{2n} \to \mathcal{W}^\theta$ lands in the genus-0 components of $\mathcal{W}^\theta$, the right $\mathbb{F}^\theta_{2n}$-functor $\mathbb{W}^\theta$ splits as a disjoint union $\coprod_{g\geqslant 0} \mathbb{W}^\theta_g$. Thus, the pushforward $\mathcal{W}^\theta \otimes^{\mathbb{L}}_{F^\theta_{2n}} (-)$ splits as a disjoint union of bar constructions $|B_\bullet(\mathbb{W}^\theta_g, \mathbb{F}^\theta_{2n}, -)|$. This is the 'grading by genus' that we promised at the beginning of this section.

The aim of this subsection is to give the proof of Proposition 3.1, that is, to provide, for each E^θ_{2n}-algebra A and each $g \geqslant 0$, equivalences $W^\theta_{g,1}[A] \simeq |B_\bullet(\mathbb{W}^\theta_g, \mathbb{F}^\theta_{2n}, \kappa^* A)|$.

Construction 3.17. Recall the spaces $\tilde{\mathcal{W}}^\theta_g(r)$ from Construction 3.4. For any $g \geqslant 0$, the family $(\tilde{\mathcal{W}}^\theta_g(r))_{r\geqslant 0}$ is a right F^θ_{2n}-module: We have maps $\tilde{\mathcal{W}}^\theta_g(r) \times \prod_i F^\theta_{2n}(r_i) \to \tilde{\mathcal{W}}^\theta_g(\sum_i r_i)$ taking (η, ℓ, β) and $(\alpha_i, \gamma_i, \eta_i, \zeta_i)$ to $(\hat{\eta}, \hat{\ell}, \beta \circ (\beta_1 \sqcup \cdots \sqcup \beta_r))$, where β_i are uniquely determined as before, and where $\hat{\eta}$ is $\beta \circ \eta_i \circ \beta^{-1} \circ \eta$ inside $\eta^{-1}(\beta(\{i\} \times D^{2n})$ and η elsewhere, and $\hat{\ell}$ is $\zeta_i(u_i) \circ T\beta|^{-1}_{r\times D^{2n}} \circ T\eta$ inside each $\eta^{-1}(\beta(\{i\} \times D^{2n})$ and ℓ else.

These structure maps are equivariant with respect to the action of $\mathrm{Diff}_\partial(W_{g,1})$ on each $\tilde{\mathcal{W}}^\theta_g(r)$, and the induced map $\mathcal{W}^\theta_g(r) \times \prod_i F^\theta_{2n}(r_i) \to \mathcal{W}^\theta_g(\sum_i r_i)$ is part of the operadic composition of $\mathcal{W}^\theta$, precomposed with the operad map $F^\theta_{2n} \to \mathcal{W}^\theta$.

Proof of Proposition 3.1. As $F^\theta_{2n} \to E^\theta_{2n}$ is an equivalence of proper operads, the induced map on bar constructions $|B_\bullet(\mathbb{D}^\theta_{W_{g,1}}, \mathbb{F}^\theta_{2n}, \kappa^* A)| \to |B_\bullet(\mathbb{D}^\theta_{W_{g,1}}, \mathbb{E}^\theta_{2n}, A)|$ is an equivalence. It is hence our goal to establish an equivalence of $\mathbb{F}^\theta_{2n}$-functors between $\mathbb{W}^\theta_g$ and $\mathbb{D}^\theta_{W_{g,1}}$.

To this end, we note that the $\mathrm{Diff}_\partial(W_{g,1})$-equivariant right F^θ_{2n}-module structure on $(\tilde{\mathcal{W}}^\theta_g(r))_{r\geqslant 0}$ gives rise to a right $\mathbb{F}^\theta_{2n}$-functor $\tilde{\mathbb{W}}^\theta$ with values in $\mathrm{Diff}_\partial(W_{g,1})$-spaces, and since the action of $\mathrm{Diff}_\partial(W_{g,1})$ on each $\tilde{\mathcal{W}}^\theta(r)$ is free and proper, the canonical augmentation of $\mathbb{F}^\theta_{2n}$-functors $\tilde{\mathbb{W}}^\theta_g /\!\!/ \mathrm{Diff}_\partial(W_{g,1}) \to \tilde{\mathbb{W}}^\theta_g / \mathrm{Diff}_\partial(W_{g,1}) = \mathbb{W}^\theta_g$ is an equivalence.

Next, we generalise Construction 3.6 in two ways: We consider embeddings into $W_{g,1}$ instead of only into $D^{2n} = W_{0,1}$, and we let the framing of $W_{g,1}$ vary. More precisely, let $\mathcal{V}_g(r) \subset \underline{\mathrm{Emb}}^\theta(\underline{r} \times D^{2n}, W_{g,1}) \times \mathrm{Emb}_\partial(W_{g,1}, D^{2n,\infty}) \times [0,\infty) \times \mathrm{Fr}^\theta_\partial(W_{g,1}, \ell_{g,1})^{[0,\infty)}$ contain all tuples $(\alpha, \gamma, \ell, \eta, u, \zeta)$ satisfying the very same conditions as in Construction 3.6, we only replace $\zeta(0) = \ell_{0,1}$ by $\zeta(0) = \ell$. We have a map $\mathcal{V}_g(r) \to \tilde{\mathcal{W}}^\theta_g(r)$ taking such a tuple to $(\eta, \zeta(u), \beta)$, and we have a map $\mathcal{V}_g(r) \to \underline{\mathrm{Emb}}^\theta(\underline{r} \times D^{2n}, W_{g,1})$ that only remembers (α, γ, ℓ). Both maps are equivalences: For the second map, we can run the same proof as Lemma 3.13, and for the first one, we note that the fibre of any (η, ℓ, β) is parameterised by the contractible space of real numbers $t^1, \dots, t^r \geqslant 0$ and paths ζ inside $\mathrm{Fr}^\theta_\partial(W_{g,1}, \ell_{g,1})$ *ending* in ℓ, such that $\zeta(s) \circ T\eta^{-1} \circ T\beta|_{\underline{r} \times D^{2n}} = \ell_{0,1}$ holds inside $\{i\} \times D^{2n}$ for all $s \geqslant t_i$.

Both maps are morphisms of right F^θ_{2n}-modules, and hence induce morphisms of $\mathbb{F}^\theta_{2n}$-functors $\underline{\mathbb{E}}^\theta_{W_{g,1}} \leftarrow \mathbb{V}_g \to \tilde{\mathbb{W}}^\theta_g$. As all three $\mathbb{F}^\theta_{2n}$-functors attain values in $\mathrm{Diff}_\partial(W_{g,1})$-spaces and the maps are equivariant, we obtain the desired zig-zag of F^θ_{2n}-functors

$$\mathbb{D}^\theta_{W_{g,1}} \longleftarrow \mathbb{V}_g \,/\!\!/\, \mathrm{Diff}_\partial(W_{g,1}) \longrightarrow \tilde{\mathbb{W}}^\theta_g \,/\!\!/\, \mathrm{Diff}_\partial(W_{g,1}) \longrightarrow \mathbb{W}^\theta_g.$$

By applying $\mathrm{B}_\bullet(-, \mathbb{F}^\theta_{2n}, \kappa^* A)$, we get a zig-zag of levelwise equivalences of simplicial spaces, all of which are proper as all components of the above right F^θ_{2n}-modules are Hausdorff and $\mathfrak{S}$-free. This gives the desired equivalence on geometric realisations. □

Construction 3.18. Using the equivalence from Proposition 3.1, the unary operation $\sigma \in \mathcal{W}^\theta_1(1)$ induces stabilisation maps $W^\theta_{g,1}[A] \to W^\theta_{g+1,1}[A]$. The equivalence from Proposition 3.1 also tells us that the homotopy type of $W^\theta_{*,1}[A]$ carries an A_∞-structure, and by abuse of notation, we may write $\Omega \mathrm{B} W^\theta_{*,1} := \Omega \mathrm{B}(\mathcal{W}^\theta \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A)$.

If A is path-connected, then $W^\theta_{g,1}[A]$ is path-connected by Lemma 2.29, so by the group-completion theorem, we obtain an equivalence

$$\mathrm{hocolim}_{g\to\infty} W^\theta_{g,1}[A]^+ \simeq \Omega_0 \mathrm{B}(\mathcal{W}^\theta \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A) =: \Omega_0 \mathrm{B} W^\theta_{*,1}[A].$$

We close this section by noting that we can use Proposition 3.1 to deduce homological stability of $W^\theta_{\bullet,1}[A]$, generalising [7, Theorem E], from the stability results for moduli spaces of surfaces [18, 31] and of high-dimensional manifolds [17]:

Theorem 3.19. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with π_1-injective θ, and let A be an E^θ_{2n}-algebra. If $2n \geqslant 6$ (or $2n = 2$ and θ is admissible to [31, Theorem 7.1]), then the maps $W^\theta_{g,1}[A] \to W^\theta_{g+1,1}[A]$ induce isomorphisms in $H_i(-;\mathbb{Z})$ for $i \leqslant \frac{1}{2}g - \frac{3}{2}$ (for $2n = 2$, the slope is different, but coincides with the one from [31, Theorem 7.1]).*

Proof. Let us write $i_g := \frac{1}{2}g - \frac{3}{2}$ (or the slope from [31, Theorem 7.1] for $2n = 2$). The map in question is equivalent to the map $|\mathrm{B}_\bullet(\mathbb{W}^\theta_g, \mathbb{F}^\theta_{2n}, \kappa^* A)| \to |\mathrm{B}_\bullet(\mathbb{W}^\theta_{g+1}, \mathbb{F}^\theta_{2n}, \kappa^* A)|$, induced by the morphism $\mathbb{W}^\theta_g \to \mathbb{W}^\theta_{g+1}$ of right $\mathbb{E}^\theta_{2n}$-functors that, on each space X, is a union of maps $\mathrm{stab} \times_{\mathfrak{S}_r} \mathrm{id} : \mathcal{W}^\theta_g(r) \times_{\mathfrak{S}_r} X^r \to \mathcal{W}^\theta_{g+1}(r) \times_{\mathfrak{S}_r} X^r$. Since stab induces isomorphisms in homology in degree at most i_g, and $\mathfrak{S}_r$ acts freely on both sides, and $\mathcal{W}^\theta_g(r)$ and $\mathcal{W}^\theta_{g+1}(r)$ are Hausdorff, a spectral sequence argument shows that the same holds for each of the maps $\mathrm{stab} \times_{\mathfrak{S}_r} \mathrm{id}$.

This shows that the map $\mathrm{B}_\bullet(\mathbb{W}^\theta_g, \mathbb{F}^\theta_{2n}, \kappa^* A) \to \mathrm{B}_\bullet(\mathbb{W}^\theta_{g+1}, \mathbb{F}^\theta_{2n}, \kappa^* A)$ induces levelwise isomorphisms in homology in degrees at most i_g. Since both source and target are proper simplicial spaces, [35, Proposition A1:iv] enables us to consider the map among 'thick' geometric realisations instead. Here we can invoke the spectral sequence associated with the skeletal filtration, and a comparison argument for spectral sequences shows that the geometric realisation is a homology equivalence in that range as well. □

4 | SPLITTING PUSHFORWARDS TO OPERADS WITH HOMOLOGICAL STABILITY

Proposition 3.1 motivates us to study the group-completion of $\mathcal{W}^\theta \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A$ for a spherical tangential structure θ and an E^θ_{2n}-algebra A. Here we use that if $\theta : L \to \mathrm{BO}(2n)$ is n-connected, then $\mathcal{W}^\theta$ is an OHS as in [2], and we prove a splitting result similar to [2, section 5] and [4, section 5] for pushforwards. Before carrying out this approach, we first have to switch to a based framework.

Construction 4.1. The operad E_0 has exactly two operations, namely the identity **1** and a single nullary operation **0**. Thus, E_0-algebras are the same as based spaces. An operad under E_0 is the same as an operad $\mathcal{O}$ together with a preferred nullary operation $\mathbf{0}_{\mathcal{O}}$. For any operad $\mathcal{O}$ under E_0, we have *capping maps*$\mathrm{cap} : \mathcal{O}(r) \to \mathcal{O}(0)$ given by precomposing an r-ary operation with r copies of $\mathbf{0}_{\mathcal{O}}$.

Let $\mathcal{O}$ be an operad under E_0. The forgetful functor from $\mathcal{O}$-algebras to based spaces has a strict left-adjoint, namely $\tilde{F}^{\mathcal{O}} := \mathcal{O} \otimes_{E_0} (-)$. We denote the monad associated to this adjunction by $\tilde{\mathbb{O}}$; it should be seen as a *reduced* version of $\mathbb{O}$. More explicitly, $\tilde{\mathbb{O}}X$ is the (strict) coequaliser of the two canonical maps $\mathbb{O}\mathbb{E}_0 X \rightrightarrows \mathbb{O}X$.

Under mild point-set topological assumptions, we can also use these reduced monads to describe the derived pushforward, similar to Remark 2.11:

Remark 4.2. If $\mathcal{P}$ is proper and A is a $\mathcal{P}$-algebra whose underlying based space is well-pointed (in short, a 'well-pointed $\mathcal{P}$-algebra'), then the augmented simplicial $\mathcal{P}$-algebra $\mathrm{B}_\bullet(\tilde{F}^{\mathcal{P}}, \tilde{\mathbb{P}}, A) \to A$ is a proper simplicial space (see, e.g. [37, Proposition 1.4.42] for details), has an extra degeneracy, and hence is a $\tilde{\mathbb{P}}$-free simplicial resolution of A in the sense of [13, Definition 8.18]. If $\mathcal{O}$ is $\mathfrak{S}$-free as well and $\mathcal{P} \to \mathcal{O}$ is a map of operads, then, since $\tilde{\mathbb{O}}$ commutes with geometric realisations, the left-derived pushforward $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A$ (whose preferred point-set model for us remains $|\mathrm{B}_\bullet(\mathbb{O}, \mathbb{P}, A)|$ as in Remark 2.11) is equivalent, as an $\mathcal{O}$-algebra, to the 'reduced' two-sided bar construction $|\mathrm{B}_\bullet(\tilde{\mathbb{O}}, \tilde{\mathbb{P}}, A)|$.

Lemma 4.3. *Let $\mathcal{P}$ a proper operad, let $\mathcal{O}$ be a $\mathfrak{S}$-free operad under E_0 such that $\{\mathbf{0}_{\mathcal{O}}\} \hookrightarrow \mathcal{O}(0)$ is a cofibration and let $\mathcal{P} \to \mathcal{O}$ be an operad map. Moreover, let A be a $\mathcal{P}$-algebra. Then the $\mathcal{O}$-algebra $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A$ is well-pointed.*

Proof. If $*_\bullet$ denotes the simplicial singleton, then the basepoint inclusion of $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A$ is the geometric realisation of the simplicial map $f_\bullet : *_\bullet \to \mathrm{B}_\bullet(\mathbb{O}, \mathbb{P}, A)$ that freely extends

$$\{\mathbf{0}_{\mathcal{O}}\} \hookrightarrow \mathcal{O}(0) \subset \coprod_{r \geqslant 0} \mathcal{O}(r) \times_{\mathfrak{S}_r} A^r = \mathbb{P}A = \mathrm{B}_0(\mathbb{O}, \mathbb{P}, A).$$

As this map and all degeneracies of $\mathrm{B}_\bullet(\mathbb{O}, \mathbb{P}, A)$ are cofibrations, $f_\bullet$ is a Reedy cofibration. By [20, Theorem 18.6.7:1], $|f_\bullet|$ is a cofibration as well. □

The following definition is a special case of [2, Definition 4.5]:

Definition 4.4. An OHS is given by:

1. a proper operad $\mathcal{O}$ with a grading $\mathcal{O}(r) = \coprod_{g\geqslant 0} \mathcal{O}_g(r)$ such that the $\mathfrak{S}_r$-actions restrict to $\mathcal{O}_g(r)$ and the composition is graded,
2. an A_∞-operad $\mathcal{A}$, a preferred nullary $\mathbf{0}_{\mathcal{A}} \in \mathcal{A}(1)$ and a map of operads $\mu : \mathcal{A} \to \mathcal{O}$ (turning $\mathcal{O}$ into an operad under E_0) landing inside $\mathcal{O}_0$, such that $\mu(\mathcal{A}(2)) \subseteq \mathcal{O}_0(2)$ lies in a single path-component,

and we require the following stability condition: There is a $\sigma \in \mathcal{O}_1(1)$, inducing maps $\mathrm{stab} : \mathcal{O}_g(r) \to \mathcal{O}_{g+1}(r)$ by postcomposition, such that for each r, the map of towers

$$\begin{array}{ccccccc}
\mathcal{O}_0(r) & \xrightarrow{\mathrm{stab}} & \mathcal{O}_1(r) & \xrightarrow{\mathrm{stab}} & \mathcal{O}_2(r) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{cap}} & & \downarrow{\scriptstyle \mathrm{cap}} & & \downarrow{\scriptstyle \mathrm{cap}} & & \\
\mathcal{O}_0(0) & \xrightarrow[\mathrm{stab}]{} & \mathcal{O}_1(0) & \xrightarrow[\mathrm{stab}]{} & \mathcal{O}_2(0) & \longrightarrow & \cdots
\end{array}$$

induces a homology equivalence on homotopy colimits. Then $\mu(\mathcal{A}) \subseteq \mathcal{O}_0$, and each $\mathcal{O}$-algebra A is in particular a homotopy-commutative $\mathcal{A}$-algebra.

Example 4.5. For each spherical tangential structure $\theta : L \to \mathrm{BO}(2n)$, the θ-framed generalised surface operad $\mathcal{W}^\theta$ has a natural grading $\mathcal{W}^\theta(r) = \coprod_{g\geqslant 0} \mathcal{W}^\theta_g(r)$, and it receives a map $\mu : F_1 \to F^\theta_{2n} \to \mathcal{W}^\theta$ from an A_∞-operad. Since $2n \geqslant 2$, this map factors through F_2, and hence $\mu(F_1(2)) \subseteq \mathcal{W}^\theta_0(2)$ lies in a single path-component.

If L is n-connected, then Corollary 3.12:2 tells us that the stability condition is satisfied as well, using the operation $\sigma \in \mathcal{W}^\theta_1(1)$ from Construction 3.8 as a propagator. Therefore, $\mathcal{W}^\theta$ is an OHS, compare [2, Theorem 7.4]. Its initial algebra $\mathcal{W}^\theta(0)$ group-completes to $\mathbb{Z} \times \Omega^\infty_0 \mathrm{MT}\theta$ by Corollary 3.15.

Remark 4.6. Let $\mathcal{O}$ be an OHS and let $\mathcal{P}$ be another operad under E_0 with $\mathcal{P}(0) \simeq *$, which comes with a map $\rho : \mathcal{P} \to \mathcal{O}$ of operads under E_0. Then we automatically have $\rho(\mathcal{P}(r)) \subseteq \mathcal{O}_0(r)$: Because $\mathcal{P}(0)$ is connected and ρ is a map under E_0, implying $\rho(\mathbf{0}_{\mathcal{P}}) = \mu(\mathbf{0}_{\mathcal{A}}) \in \mathcal{O}_0(0)$, we have $\rho(\mathcal{P}(0)) \subseteq \mathcal{O}_0(0)$, and therefore

$$\begin{aligned}
\mathrm{cap}(\rho(\mathcal{P}(r)) \cap \mathcal{O}_g(r)) &\subseteq \rho(\mathrm{cap}(\mathcal{P}(r))) \cap \mathrm{cap}(\mathcal{O}_g(r)) \\
&\subseteq \rho(\mathcal{P}(0)) \cap \mathcal{O}_g(0) \\
&\subseteq \mathcal{O}_0(0) \cap \mathcal{O}_g(0).
\end{aligned}$$

Construction 4.7. For what follows, we assume that $(\mathcal{O}, \mu : \mathcal{A} \to \mathcal{O})$ is an OHS and $\mathcal{P}$ is another proper operad with $\mathcal{P}(0) \simeq *$ and a map $\rho : \mathcal{P} \to \mathcal{O}$ under E_0.

We furthermore assume the existence of an operad map $\mathcal{O} \to E_\infty$ under E_0. One way to achieve this map is to replace the above structure maps μ and ρ by the equivalent maps $\mu \times \mathrm{id}_{E_\infty}$ and $\rho \times \mathrm{id}_{E_\infty}$, respectively, with E_0 being diagonally included into these product operads, and then consider the projection $\mathcal{O} \times E_\infty \to E_\infty$, compare [2, Corollary 4.10] and [4, Lemma 5.11]. As in their proof, one checks that this replacement changes neither a condition nor a consequence of Proposition 4.8.

Proposition 4.8. *Let $\mathcal{O}$ be an OHS, let $\mathcal{P}$ be a proper operad with $\mathcal{P}(0) \simeq *$ and let $\mathcal{P} \to \mathcal{O}$ be a map of operads under E_0. Then the map of $\mathcal{O}$-algebras*

$$(\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A) \to (\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} *) \times (E_\infty \otimes^{\mathbb{L}}_{\mathcal{P}} A)$$

that is induced by $A \to *$ and $\mathcal{O} \to E_\infty$ induces an equivalence on group-completions.

Remark 4.9. Since $\mathcal{P}(0)$ is contractible, the basepoint inclusion into each $\tilde{\mathbb{P}}^p(*)$ is an equivalence, and hence the canonical map

$$\mathcal{O}(0) = \tilde{\mathbb{O}}(*) = \mathrm{B}_0(\tilde{\mathbb{O}}, \tilde{\mathbb{E}}_0, *) \hookrightarrow |\mathrm{B}_\bullet(\tilde{\mathbb{O}}, \tilde{\mathbb{E}}_0, *)| \to |\mathrm{B}_\bullet(\tilde{\mathbb{O}}, \tilde{\mathbb{P}}, *)| = \mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} *$$

is an equivalence of $\mathcal{O}$-algebras, so Proposition 4.8 in particular tells us

$$\Omega\mathrm{B}\big(\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A\big) \simeq \Omega\mathrm{B}\mathcal{O}(0) \times \Omega\mathrm{B}\big(E_\infty \otimes^{\mathbb{L}}_{\mathcal{P}} A\big).$$

We point out that Proposition 4.8 is a variation of [2, Theorem 5.4] and [4, Theorem 5.9], and consequently, the proof is very similar:

Construction 4.10. We introduce a 'stable' version of $\tilde{\mathbb{O}}$: For each based space X, the space $\tilde{\mathbb{O}}(X)$ splits as $\coprod_g \tilde{\mathbb{O}}_g(X)$, and we consider the mapping telescope

$$\tilde{\mathbb{O}}_\infty(X) := \mathrm{hocolim}\Big(\; \mathbb{O}_0(X) \xrightarrow{\text{stab}} \tilde{\mathbb{O}}_1(X) \xrightarrow{\text{stab}} \tilde{\mathbb{O}}_2(X) \longrightarrow \cdots \Big).$$

Then the terminal map $X \to *$ of based spaces gives rise to maps $\tilde{\mathbb{O}}_g(X) \to \tilde{\mathbb{O}}_g(*)$ for each g, which are compatible with the stabilisations, and hence constitute a map among homotopy colimits $\tilde{\mathbb{O}}_\infty(X) \to \tilde{\mathbb{O}}_\infty(*)$. On the other hand, by fixing a path from the image of $\hat{s}$ inside $E_\infty(1)$ to $\mathbf{1}$, we obtain a homotopy from $\tilde{\mathbb{O}}_g(X) \to \tilde{\mathbb{E}}_\infty(X)$ to $\tilde{\mathbb{O}}_g(X) \to \tilde{\mathbb{O}}_{g+1}(X)$, followed by the map to $\tilde{\mathbb{E}}_\infty(X)$. We hence obtain a map out of the mapping telescope $\tilde{\mathbb{O}}_\infty(X) \to \tilde{\mathbb{E}}_\infty(X)$. The product map is called

$$\Psi_X : \tilde{\mathbb{O}}_\infty(X) \to \tilde{\mathbb{O}}_\infty(*) \times \tilde{\mathbb{E}}_\infty(X).$$

The following statement is shown in the proof of [2, Theorem 5.4] (it does not use the second operad $\mathcal{P}$ at all):

Lemma 4.11. *For each based space X, the map Ψ_X is a homology equivalence.*

We can now show Proposition 4.8 by adapting the proof of [4, Theorem 5.9]:

Proof of Proposition 4.8. By replacing the algebra A by $\mathcal{P} \otimes^{\mathbb{L}}_{\mathcal{P}} A$ and applying Lemma 4.3 if necessary, we may assume that the algebra A to start with is well-pointed, enabling us to employ the reduced monads to describe the pushforward.

As $\mathcal{P}(r) \to \mathcal{O}(r)$ lands in $\mathcal{O}_0(r)$ for each $r \geqslant 0$, each $\tilde{\mathbb{O}}_g$ is a itself right $\tilde{\mathbb{P}}$-functor and $\mathrm{B}_\bullet(\tilde{\mathbb{O}}, \tilde{\mathbb{P}}, A)$ decomposes into $\mathrm{B}_\bullet(\tilde{\mathbb{O}}_g, \tilde{\mathbb{P}}, A)$. Then the map Φ from the statement is a union of realisations of simplicial maps

$$\Phi_g \colon \mathrm{B}_\bullet(\tilde{\mathbb{O}}_g, \tilde{\mathbb{P}}, A) \to \mathrm{B}_\bullet(\tilde{\mathbb{O}}_g, \tilde{\mathbb{P}}, *) \times \mathrm{B}_\bullet(\tilde{\mathbb{E}}_\infty, \tilde{\mathbb{P}}, A).$$

Adding the propagator gives rise to maps $\tilde{\mathbb{O}}_g \to \tilde{\mathbb{O}}_{g+1}$ of $\tilde{\mathbb{P}}$-functors, its homotopy colimit being the $\tilde{\mathbb{P}}$-functor $\tilde{\mathbb{O}}_\infty$. Since the maps Φ_g commute with these stabilisations, we obtain a colimit map Φ_∞. It suffices to show that Φ_∞ is a homology equivalence, as this implies, by a classical telescope argument, that Φ induces an isomorphism among localised Pontrjagin rings $H_*\big(\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} A\big)[\pi_0^{-1}] \to H_*\big((\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} *) \times (E_\infty \otimes^{\mathbb{L}}_{\mathcal{P}} A)\big)[\pi_0^{-1}]$ and by the group-completion theorem, $\Omega\mathrm{B}\Phi$ is a homology equivalence of loop spaces, and hence a weak equivalence as desired.

To this end, we note that Φ_∞ is levelwise given by $\tilde{\mathbb{O}}_\infty(\tilde{\mathbb{P}}^p A) \to \tilde{\mathbb{O}}_\infty(\tilde{\mathbb{P}}^p *) \times \tilde{\mathbb{E}}_\infty(\tilde{\mathbb{P}}^p A)$. Postcomposing with $\tilde{\mathbb{O}}_\infty(\tilde{\mathbb{P}}^p * \to *) \times \mathrm{id}$, the resulting map agrees with $\Psi_{\tilde{\mathbb{P}}^p A}$, which is a homology equivalence by Lemma 4.11. However, since $\mathcal{P}(0)$ is contractible, $\tilde{\mathbb{P}}^p * \to *$ is an equivalence of cofibrant spaces, and since $\tilde{\mathbb{O}}_\infty$ preserves such equivalences, the map to postcompose with is an equivalence. This shows that Φ_∞ is levelwise a homology equivalence. Finally, as both sides are proper, we can invoke the spectral sequence for 'thick' geometric realisations (as in the proof of Theorem 3.19) to conclude that Φ_∞ itself is a homology equivalence. □

Corollary 4.12. *Let $\theta \colon L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L, and let A be an E^θ_{2n}-algebra. Then we have a weak equivalence of loop spaces*

$$\Omega\mathrm{B}W^\theta_{*,1}[A] \simeq \mathbb{Z} \times \Omega^\infty_0 \mathrm{MT}\theta \times \Omega\mathrm{B}\big(E_\infty \otimes^{\mathbb{L}}_{E^\theta_{2n}} A\big).$$

Proof. As explained in Construction 3.18, the left side silently stands for the group completion of the pushforward $\mathcal{W}^\theta \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A$. By Proposition 4.8, this splits into the group-completion of $\mathcal{W}^\theta(0)$, which is $\mathbb{Z} \times \Omega^\infty_0 \mathrm{MT}\theta$ by Corollary 3.15, and the group completion of $E_\infty \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A$. However, since $E^\theta_{2n} \xleftarrow{\kappa} F^\theta_{2n} \xleftarrow{\pi_1} F^\theta_{2n} \times E_\infty \to E_\infty$ is a zig-zag of operads as in Construction 2.13, it follows that $E_\infty \otimes^{\mathbb{L}}_{F^\theta_{2n}} \kappa^* A$ is a model for the homotopy type of the E_∞-algebra $E_\infty \otimes^{\mathbb{L}}_{E^\theta_{2n}} A$. □

5 | CALCULATIONS AND EXAMPLES

The considerations of Section 4 reduce the original problem to understanding, for each tangential structure $\theta \colon L \to \mathrm{BO}(d)$ and each E^θ_d-algebra A, the spectrum $\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E^\theta_d} A)$. We will give a general, yet rather abstract answer to this question in Section 6, but we first discuss several special cases, which rediscover results by [7, 8].

5.1 | Suboperads and submonoids

Remark 5.1. Let $\mathcal{H}$ be any proper operad mapping to E^θ_{2n}. By Remark 2.11, we obtain, for each $\mathcal{H}$-algebra X, an equivalence $E_\infty \otimes^{\mathbb{L}}_{E^\theta_{2n}} (E^\theta_{2n} \otimes^{\mathbb{L}}_{\mathcal{H}} X) \simeq E_\infty \otimes^{\mathbb{L}}_{\mathcal{H}} X$ of E_∞-algebras. This shows that if $\theta \colon L \to \mathrm{BO}(2n)$ is a spherical tangential structure with n-connected L, then we have a weak equivalence of loop spaces

$$\Omega\mathrm{B}W^\theta_{*,1}[E^\theta_{2n} \otimes^{\mathbb{L}}_{\mathcal{H}} X] \simeq \mathbb{Z} \times \Omega^\infty_0 \mathrm{MT}\theta \times \Omega\mathrm{B}\big(E_\infty \otimes^{\mathbb{L}}_{\mathcal{H}} X\big).$$

Particularly easy examples of $\mathcal{H}$ are operads that arise from topological monoids:

Definition 5.2. Each topological monoid M constitutes an operad M_+ with M as its monoid of unary operations, with a single nullary operation, and with no operations of higher arity. Then M_+ is a proper operad if and only if M is well-pointed. Moreover, M_+-algebras are the same as based M-spaces. For a based M-space X, we define its *based homotopy quotient* as $X_{\mathrm{h}M} := E_0 \otimes^{\mathbb{L}}_{M_+} X$. In the case where the M-action on X is trivial, we have an equivalence $X_{\mathrm{h}M} \simeq (\mathrm{B}M)_+ \wedge X$

If M maps to the monoid E^{θ}_{2n}, then we obtain an operad map $M_+ \to E^{\theta}_{2n}$ by taking the unique nullary of M_+ to the unique nullary of E^{θ}_{2n}.

Proposition 5.3. *Let M be a well-pointed topological monoid mapping to $E^{\theta}_{2n}(1)$. Then we have, for each based M-space X, a weak equivalence of loop spaces*

$$\Omega \mathrm{B} W^{\theta}_{*,1}[E^{\theta}_{2n} \otimes^{\mathbb{L}}_{M_+} X] \simeq \mathbb{Z} \times \Omega^{\infty}_0 \mathrm{MT}\theta \times \Omega^{\infty}\Sigma^{\infty}(X_{\mathrm{h}M}).$$

Proof. By Remark 5.1, it suffices to show $\Omega\mathrm{B}(E_{\infty} \otimes^{\mathbb{L}}_{M_+} X) \simeq \Omega^{\infty}\Sigma^{\infty}(X_{\mathrm{h}M})$. To this end, we note that by Construction 2.13, each operad map $M_+ \to E_{\infty}$ can be used to calculate the homotopy type of the left-hand side. One operad map is given by the composition of $M_+ \to E_0$, taking each unary operation in M to the identity, and the inclusion $E_0 \to E_{\infty}$. By Remark 2.11, we get an equivalence of E_{∞}-algebras

$$E_{\infty} \otimes^{\mathbb{L}}_{M_+} X \simeq E_{\infty} \otimes^{\mathbb{L}}_{E_0} \left(E_0 \otimes^{\mathbb{L}}_{M_+} X\right) \simeq E_{\infty} \otimes^{\mathbb{L}}_{E_0} X_{\mathrm{h}M}.$$

We then conclude the proof by invoking the equivalence $\Omega\mathrm{B}(E_{\infty} \otimes^{\mathbb{L}}_{E_0} (-)) \simeq \Omega^{\infty}\Sigma^{\infty}$ on well-pointed spaces from [1, 34] and noting that $X_{\mathrm{h}M} = E_0 \otimes^{\mathbb{L}}_{M_+} X$ is well-pointed by Lemma 4.3. □

Our main example for a submonoid of E^{θ}_d comes from the following observation:

Lemma 5.4. *If θ is of the form $\mathrm{B}G \to \mathrm{BO}(d)$ for a well-pointed subgroup $G \subseteq \mathrm{O}(d)$, then there is a zig-zag of equivalences of well-pointed monoids $G \leftarrow \tilde{G} \hookrightarrow E^{\theta}_d(1)$.*

We point out that there are models for E^{θ}_d, for which G is even a strict subgroup of the monoid of unaries. For the model we use, this is not the case.

Proof. We start by noting that $\theta^* V_d$ is given by the Borel construction $\mathrm{E}G \times_G \mathbb{R}^d$. In this description, there is an element $e_0 \in \mathrm{E}G$ such that under the standard identification $D^d \times \mathbb{R}^d \cong TD^d$, we have $\ell_{0,1}(z, v) = [e_0, v]$ up to a fixed linear automorphism of $\mathbb{R}^d$.

Now we let $\tilde{G} \subseteq E^{\theta}_d(1) = \mathrm{Emb}^{\theta}(D^d, D^d)$ be the submonoid consisting of all (α, γ, t) where $\alpha : D^d \hookrightarrow D^d$ is of the form $\omega|_{D^d}$ for some $\omega \in G \subset \mathrm{O}(d)$ and where γ is of the form $s \mapsto ((z, v) \mapsto [\beta(s), v])$ for some path $\beta : [0, t] \to \mathrm{E}G$ from e_0 to $e_0 \cdot \omega$. Then the map $f : \tilde{G} \to G$ taking $(\omega|_{D^d}, \gamma, t)$ to ω is a homomorphism.

In slightly different words, $\tilde{G}$ is a Moore model for the homotopy fibre of the map $G \to \mathrm{E}G$ taking ω to $e_0 \cdot \omega$ and f is the canonical map from the homotopy fibre to G. Since $\mathrm{E}G$ is contractible, this shows that f is an equivalence. Moreover, we employ the equivalence $\mathrm{Emb}(D^d, D^d) \simeq \mathrm{Bun}_0(TD^d, TD^d)$ from the proof of Lemma 2.21 to see that the inclusion $\tilde{G} \hookrightarrow E^{\theta}_d(1)$ fits into the following morphism of Puppe sequences

$$\begin{array}{ccccccc}
\tilde{G} & \longrightarrow & G & \longrightarrow & \mathrm{E}G & \longrightarrow & \mathrm{B}G \\
\big\downarrow & & {\scriptstyle \omega\mapsto\omega|_{D^d}}\big\downarrow{\scriptstyle \omega\mapsto T\omega|_{D^d}} & & \big\downarrow{\scriptstyle e\mapsto((z,v)\mapsto[e,v])} & & \big\| \\
E_d^\theta(1) & \longrightarrow & \mathrm{Emb}(D^d,D^d)\simeq\mathrm{Bun}_0(TD^d,TD^d) & \longrightarrow & \mathrm{Bun}(TD^d,\mathrm{E}G\times_G\mathbb{R}^d) & \longrightarrow & \mathrm{B}G,
\end{array}$$

so since the right vertical map is an equivalence, first one is an equivalence as well. □

Example 5.5. Let $G \subset \mathrm{O}(2n)$ be an $(n-1)$-connected well-based subgroup such that the tangential structure $\theta : \mathrm{B}G \to \mathrm{BO}(2n)$ is spherical. If X is a based G-space, then it is in particular a based $\tilde{G}$-space and the based homotopy quotients $X_{\mathrm{h}G}$ and $X_{\mathrm{h}\tilde{G}}$ are equivalent. Therefore, Proposition 5.3 provides us with an equivalence

$$\Omega\mathrm{B}W_{*,1}^\theta\left[E_{2n}^\theta \otimes_{\tilde{G}_+}^{\mathbb{L}} X\right] \simeq \mathbb{Z}\times\Omega_0^\infty\mathrm{MT}\theta\times\Omega^\infty\Sigma^\infty(X_{\mathrm{h}G}).$$

In the case of $2n = 2$ and θ being $\mathrm{BSO}(2)\to\mathrm{BO}(2)$, this recovers [7, Corollary D].

5.2 | Partial algebras

A slightly more rigid viewpoint, which appears in both [7] and [24], starts with a *partially defined* E_d^θ-algebra X, and instead of taking the relatively free E_d^θ-algebra along an operad map to E_d^θ, we *complete* the E_d^θ-algebra:

Definition 5.6. Let $\mathcal{P}$ be an operad. A *partial $\mathcal{P}$-algebra* is a space X, together with $\mathrm{Comp}(X) \subseteq \mathbb{P}X$ and a map $\lambda : \mathrm{Comp}(X) \to X$ satisfying the axioms of [24, Example 41].

Clearly, a partial $\mathcal{P}$-algebra X with $\mathrm{Comp}(X) = \mathbb{P}X$ is the same as a $\mathcal{P}$-algebra in the usual sense, hereafter called *honest* $\mathcal{P}$-algebra in order to avoid confusion.

Construction 5.7. Let X be a partial $\mathcal{P}$-algebra. We define an inverse system

$$\left(\cdots \xrightarrow{c_2} \mathrm{Comp}_1(X) \xrightarrow{c_1} \mathrm{Comp}_0(X)\right),$$

with $c_1 = \lambda$ and $\mathrm{Comp}_p(X) \subseteq \mathbb{P}\mathrm{Comp}_{p-1}(X)$ being the subspace of all x satisfying $(\mathbb{P}c_{p-1})(x) \in \mathrm{Comp}_{p-1}(X)$; then c_p is the restriction of $\mathbb{P}c_{p-1}$ to that subspace.

If S is a right $\mathbb{P}$-functor with values in spaces, then we obtain a simplicial space $S\mathrm{Comp}_\bullet(X)$, where the degeneracy maps $s_i : S\mathrm{Comp}_p(X) \to S\mathrm{Comp}_{p+1}(X)$ are induced by the unit of $\mathbb{P}$, the face maps $d_i : S\mathrm{Comp}_p(X) \to S\mathrm{Comp}_{p-1}(X)$ for $0 \leqslant i \leqslant p-1$ are induced by the natural transformations $\mathbb{P}^2 \to \mathbb{P}$ and $S\mathbb{P} \to S$, and the last face map d_p is given by Sc_p, compare [24, Lemma 4.2]. If X is an honest $\mathcal{P}$-algebra, then the simplicial space $S\mathrm{Comp}_\bullet(X)$ agrees with the usual two-sided bar construction $\mathrm{B}_\bullet(S,\mathbb{P},X)$.

Construction 5.8. For a partial $\mathcal{P}$-algebra X, we define its *completion* $\hat{X} := |\mathbb{P}\mathrm{Comp}_\bullet(X)|$, which is a $\mathcal{P}$-algebra by $\mathbb{P}|\mathbb{P}\mathrm{Comp}_\bullet(X)| \cong |\mathbb{P}^2\mathrm{Comp}_\bullet(X)| \to |\mathbb{P}\mathrm{Comp}_\bullet(X)|$. If X is an honest $\mathcal{P}$-algebra, then $\mathbb{P}\mathrm{Comp}_\bullet(X)$ is the $\mathbb{P}$-free simplicial resolution of X from Remark 2.11, so its realisation $\hat{X} = \mathcal{P} \otimes^{\mathbb{L}}_{\mathcal{P}} X$ is equivalent to X.

If S is any right $\mathbb{P}$-functor, then there is an equivalence among geometric realisations $|\mathrm{B}_\bullet(S, \mathbb{P}, \hat{X})| \simeq |S\mathrm{Comp}_\bullet(X)|$, which is established as in the proof of [24, Lemma 44] by looking at the bisimplicial space $\mathrm{B}_\bullet(S, \mathbb{P}, \mathbb{P}\mathrm{Comp}_\bullet(X))$.

Construction 5.9. Let W be a θ-framed manifold and let X be a partial E_d^θ-algebra. Then we put $\int_W^\theta X := \int_W^\theta \hat{X}$ and $W^\theta[X] := W^\theta[\hat{X}]$, that is, we first complete and then apply the old definitions. We have an equivalence

$$\int_W^\theta X = |\mathrm{B}_\bullet(\mathbb{E}_W^\theta, \mathbb{E}_d^\theta, \hat{X})| \simeq |\mathbb{E}_W^\theta(\mathrm{Comp}_\bullet X)|,$$

and similarly for $W^\theta[X]$, showing that in the case where X is already an honest E_d^θ-algebra, the two competing definitions are equivalent.

Let $d = 2n$. If $\mathcal{H} \subseteq E_{2n}^\theta$ is a proper suboperad, and X is an $\mathcal{H}$-algebra, regarded as a partial E_{2n}^θ-algebra, then $\hat{X} \simeq E_{2n}^\theta \otimes^{\mathbb{L}}_{\mathcal{H}} X$, so if $\theta : L \to \mathrm{BO}(2n)$ is spherical with n-connected L, we are in the situation of Remark 5.1. This leads us to the desired reformulation of Proposition 5.3 in terms of partial algebras:

Theorem 5.10. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L. If $M \subseteq E_{2n}^\theta(1)$ is a well-pointed submonoid and X is a based M-space, regarded as a partial E_{2n}^θ-algebra, then we have a weak equivalence of loop spaces*

$$\Omega \mathrm{B}W_{*,1}^\theta[X] \simeq \mathbb{Z} \times \Omega_0^\infty \mathrm{MT}\theta \times \Omega^\infty \Sigma^\infty (X_{\mathrm{h}M}).$$

5.3 | Configuration spaces and punctured diffeomorphism groups

Two further calculations are based on a reinterpretation of some cases of factorisation homology $\int_W^\theta A$ as labelled configuration spaces $C(\mathring{W}; X)$ studied in [5, 34]. This reinterpretation appears to be well known, but as I could not find a reference, I spelled out a proof. Let us first repeat the definition from [5, section 1] in our language:

Definition 5.11. Let Q be a space. For each $r \geqslant 0$, let $\tilde{C}_r(Q) \subset Q^r$ be the space of ordered configurations of r particles in Q. Then we have a right $\mathbb{E}_0$-functor $\mathbb{C}_Q$ that takes a space Y to $\coprod_r \tilde{C}_r(Q) \times_{\mathfrak{S}_r} Y^r$. For every based space X, we define the *labelled configuration space* $C(Q; X)$ as the (strict) coequaliser of the two maps $\mathbb{C}_Q \mathbb{E}_0(X) \rightrightarrows \mathbb{C}_Q(X)$. If X is well pointed, this coequaliser is equivalent to the bar construction $|\mathrm{B}_\bullet(\mathbb{C}_Q, \mathbb{E}_0, X)|$.

Proposition 5.12. *Let X be a well-pointed space, considered as a based $E_d^\theta(1)$-space with trivial action, and hence as a partial E_d^θ-algebra. Then we have weak equivalences $\int_W^\theta X \simeq C(\mathring{W}; X)$ and $W^\theta[X] \simeq (C(\mathring{W}; X) \times \mathrm{Fr}_\partial^\theta(W, \ell_W)) /\!\!/ \mathrm{Diff}_\partial(W)$ that are compatible with embeddings.*

Proof. We show the second equivalence, as the first one is proven analogously. Throughout the proof let M be the monoid $E^\theta_d(1)$. We then have an equivalence

$$W^\theta[X] = |\mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{E}^\theta_d, E^\theta_d, \otimes^{\mathbb{L}}_{\tau E^\theta_d} X)| /\!/ \mathrm{Diff}_\partial(W) \simeq |\mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{M}_+, X)| /\!/ \mathrm{Diff}_\partial(W).$$

We have fibrations $\underline{\mathrm{Emb}}^\theta(\underline{r} \times \mathring{D}^d, W) \to \tilde{C}_r(\mathring{W}) \times \mathrm{Fr}^\theta_\partial(W, \ell_W)$ with $(\ell, \alpha, \gamma) \mapsto (\alpha|_{\underline{r}\times\{0\}}, \ell)$, inducing a transformation $\underline{\mathbb{E}}^\theta_W \to \mathbb{C}_W \times \mathrm{Fr}^\theta_\partial(W, \ell_W)$. We want to show that the $\mathrm{Diff}_\partial(W)$-equivariant augmentation $|\mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{M}_+, \mathbb{E}_0)| \to \mathbb{C}_W \times \mathrm{Fr}^\theta_\partial(W, \ell_W)$ is an equivalence of right $\mathbb{E}_0$-functors; then the claim follows from the chain of equivalence (all of which are compatible with postcomposing with θ-framed embeddings)

$$\begin{aligned}\left(C(\mathring{W}; X) \times \mathrm{Fr}^\theta_\partial(W, \ell_W)\right) /\!/ \mathrm{Diff}_\partial(W) &\simeq |\mathrm{B}_\bullet(\mathbb{C}_W \times \mathrm{Fr}^\theta_\partial(W, \ell_W), \mathbb{E}_0, X)| /\!/ \mathrm{Diff}_\partial(W)\\ &\simeq |\mathrm{B}_\bullet(\mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{M}_+, \mathbb{E}_0), \mathbb{E}_0, X)| /\!/ \mathrm{Diff}_\partial(W)\\ &\simeq |\mathrm{B}_\bullet(\mathbb{E}^\theta_W, \mathbb{M}_+, X)| /\!/ \mathrm{Diff}_\partial(W)\\ &\simeq W^\theta[X].\end{aligned}$$

In order to reach this goal, we first note that for any trivial (unbased) M-space Y, the canonical map $\mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{M}, Y) \to \mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{M}_+, Y_+)$ induces an equivalence on geometric realisations, so by 2-out-of-3, we can likewise show that for any such Y, the augmentation $f : \mathrm{B}_\bullet(\underline{\mathbb{E}}^\theta_W, \mathbb{M}, Y) \to \mathbb{C}_W(Y) \times \mathrm{Fr}^\theta_\partial(W, \ell_W)$ induces an equivalence on geometric realisations. Since all augmentation maps $\mathrm{B}_p(\underline{\mathbb{E}}^\theta_W, \mathbb{M}, Y) \to \mathbb{C}_W \times \mathrm{Fr}^\theta_\partial(W, \ell_W)$ are fibrations and all involved simplicial spaces are proper, [11, Lemma 2.14] tells us that the homotopy fibre of $|f|$ is the geometric realisation of the actual simplicial fibre.

To calculate this fibre, let $([w_1, \dots, w_r; y_1, \dots, y_r], \ell) \in (\tilde{C}_r(\mathring{W}) \times_{\mathfrak{S}_r} Y^r) \times \mathrm{Fr}^\theta_\partial(W, \ell_W)$ be any point. Since Y is a trivial M-space, the actual fibre $F_\bullet$ is of the form

$$F_p = \mathrm{Emb}^\theta_{w_1,\dots,w_r}(\underline{r} \times \mathring{D}^d, W) \times_{\mathfrak{S}_r} (M^p \times \{y_i\})^r_{i=1},$$

where the θ-framing ℓ of W is taken into account and where $\mathrm{Emb}^\theta_{w_1,\dots,w_r} \subseteq \mathrm{Emb}^\theta$ is the subspace of those θ-framed embeddings (α, γ) with $\alpha(i, 0) = w_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_r$. We fix a θ-framed embedding $(\bar{\alpha}, \bar{\gamma}) : \underline{r} \times \mathring{D}^d \hookrightarrow W$ with $\bar{\alpha}(i, 0) = w_i$; then the map

$$\mathrm{Emb}^\theta_0(\mathring{D}^d, \mathring{D}^d)^r \times \mathfrak{S}_r \to \mathrm{Emb}^\theta_{w_1,\dots,w_r}(\underline{r} \times \mathring{D}^d, W),$$

$$(\alpha_1, \gamma_1, \dots, \alpha_r, \gamma_r; \sigma) \mapsto (\bar{\alpha}, \bar{\gamma}) \circ ((\alpha_1, \gamma_r) \sqcup \cdots \sqcup (\alpha_r, \gamma_r)) \circ (\sigma \times \mathrm{id}_{\mathring{D}^d}),$$

is a deformation retract (the deformation given by shrinking the radii of the discs). The left-hand side, in turn, is equivalent to $M^r \times \mathfrak{S}_r$ itself. Altogether, we have established a simplicial equivalence between the actual fibre $F_\bullet$ and the simplicial space $\mathrm{B}_\bullet(\mathbb{M}, \mathbb{M}, *)^r$, which is contractible as desired. □

Example 5.13. Let X be the based space S^0, considered as a trivial $E^\theta_{2n}(1)$-space, and hence as a partial $E^\theta_{2n}(1)$-algebra. By Proposition 5.12, we have an equivalence

$$W^\theta[X] \simeq \coprod_{r\geqslant 0} (C_r(\mathring{W}) \times \mathrm{Fr}^\theta_\partial(W, \ell_W)) /\!/ \mathrm{Diff}_\partial(W),$$

where $C_r(\mathring{W})$ is the *unordered* configuration space of r particles inside $\mathring{W}$. Moreover, we have an action of $\mathrm{Diff}_\partial(W)$ on each $C_r(\mathring{W})$ and the evaluation $\mathrm{Diff}_\partial(W) \to C_r(\mathring{W})$ at a given configuration is a fibration, with the stabiliser being the subgroup $\mathrm{Diff}^r_\partial(W)$ of diffeomorphisms that fix a subset of cardinality r, allowing permutations. Therefore, the canonical map $* /\!\!/ \mathrm{Diff}^r_\partial(W) \to C_r(\mathring{W}) /\!\!/ \mathrm{Diff}_\partial(W)$ is a weak equivalence, so by a five lemma argument, we obtain an equivalence

$$W^\theta[X] \simeq \coprod_{r\geqslant 0} \mathrm{Fr}^\theta_\partial(W, \ell_W) /\!\!/ \mathrm{Diff}^r_\partial(W).$$

These are the punctured moduli spaces $\mathcal{M}^{\theta,r}_\partial(W, \ell_W)$ considered in [8]. Using that $*_{\mathrm{h}E^\theta_{2n}(1)} \simeq (\mathrm{B}E^\theta_{2n}(1))_+ \simeq L_+$, Theorem 5.10 provides us with an equivalence

$$\Omega\mathrm{B} \coprod_{g,r\geqslant 0} \mathcal{M}^{\theta,r}_\partial(W_{g,1}, \ell_{g,1}) \simeq \mathbb{Z} \times \Omega^\infty_0 \mathrm{MT}\theta \times \Omega^\infty\Sigma^\infty_+ L.$$

By noting that $\Omega^\infty\Sigma^\infty_+ L$ is the group-completion of the labelled configuration space $C(\mathbb{R}^\infty; L_+) \simeq \coprod_{r\geqslant 0} L^r /\!\!/ \mathfrak{S}_r$, we obtain an equivalence among Quillen plus-constructions

$$\mathrm{colim}_{g,r\to\infty}(\mathcal{M}^{\theta,r}_\partial(W_{g,1}, \ell_{g,1}))^+ \simeq \left(\mathrm{colim}_{g\to\infty}\mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1})\right)^+ \times (\mathrm{colim}_{r\to\infty} L^r /\!\!/ \mathfrak{S}_r)^+.$$

This is an instance of [8, Theorem A]. We point out, however, that Bonatto's result is stronger: In her case, one gets, very much as in [6], a decoupling result for a fixed, finite number of punctures.

Example 5.14. Recall the fibre sequence $\int^\theta_W A \to W^\theta[A] \to \mathcal{M}^\theta_\partial(W, \ell_W)$ from Proposition 2.31. In the case where $W = W_{g,1}$, we have stabilisation maps for all three terms in this sequence, and as they are compatible with each other, we reach a fibre sequence

$$\mathrm{colim}_{g\to\infty} \int^\theta_{W_{g,1}} A \to \mathrm{colim}_{g\to\infty} W^\theta_{g,1}[A] \to \mathrm{colim}_{g\to\infty}\mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1}).$$

In the case where A is path-connected, the group-completion theorem and the main result of this paper provides a square

$$\begin{array}{ccc} \mathrm{colim}_{g\to\infty} W^\theta_{g,1}[A] & \longrightarrow & \mathrm{colim}_{g\to\infty}\mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1}) \\ \downarrow & & \downarrow \\ \Omega^\infty_0\mathrm{MT}\theta \times \Omega\mathrm{B}(E_\infty \otimes^{\mathbb{L}}_{E^\theta_{2n}} A) & \xrightarrow[\mathrm{pr}]{} & \Omega^\infty_0\mathrm{MT}\theta, \end{array}$$

where the vertical maps are acyclic. One might be tempted to hope that the induced map on homotopy fibres

$$\mathrm{colim}_{g\to\infty} \int^\theta_{W_{g,1}} A \to \Omega\mathrm{B}(E_\infty \otimes^{\mathbb{L}}_{E^\theta_{2n}} A) \tag{1}$$

is an equivalence as well. Here is an example showing that this is not the case: Let $2n = 2$ and consider the tangential structure $\theta\colon \mathrm{BSO}(2) \to \mathrm{BO}(2)$ of orientations. Moreover, consider the based space $X = S^3$, together with the trivial SO(2)-action. Under the identifications from

Proposition 5.12 and Proposition 5.3, the map from Equation 1 is of the form

$$\operatorname{colim}_{g\to\infty} C(\mathring{W}_{g,1}; S^3) \longrightarrow \Omega^\infty\Sigma^\infty(S^3 \wedge \mathrm{BSO}(2)_+).$$

We want to show that this map does not induce an isomorphism in $H_5(-;\mathbb{F}_2)$, and hence cannot be acyclic: On the one hand, we see that the right-hand side is the free E_∞-algebra over the based space $S^3 \wedge \mathrm{BSO}(2)_+ \simeq \Sigma^3\mathbb{C}\mathrm{P}^\infty_+$, so by [10, Theorem I.4.1], its rational homology is the free Dyer-Lashof algebra over based graded $\mathbb{F}_2$-vector space $H_*(\Sigma^3\mathbb{C}\mathrm{P}^\infty_+;\mathbb{F}_2)$, which, in turn, is non-trivial in every odd degree. This shows that $H_5(\Omega^\infty\Sigma^\infty(S^3 \wedge \mathrm{BSO}(2)_+);\mathbb{F}_2)$ is non-trivial. Since homology commutes with filtered colimits, it suffices to show that $H_5(C(\mathring{W}_{g,1}; S^3);\mathbb{F}_2)$ is trivial for each $g \geqslant 0$. To this end, we note that the stable splitting from [5, Proposition 3] and the Thom isomorphism provides us with an isomorphism of graded $\mathbb{F}_2$-vector spaces

$$\tilde{H}_*(C(\mathring{W}_{g,1}; S^3);\mathbb{F}_2) \cong \bigoplus_{r\geqslant 0} H_{*-3r}(C_r(\mathring{W}_{g,1});\mathbb{F}_2).$$

Since $C_r(\mathring{W}_{g,1})$ is a non-compact $2r$-dimensional manifold, the shifted homology groups $H_{*-3r}(C_r(\mathring{W}_{g,1});\mathbb{F}_2)$ are concentrated in degrees $3r \leqslant * < 5r$. Hence the claim follows from the fact that there is no integer $r \geqslant 0$ with $3r \leqslant 5 < 5r$.

Remark 5.15. In higher dimensions, the question whether the map from Equation 1 is acyclic or not is equivalent to the question whether the maximal perfect subgroup of $\operatorname{colim}_{g\to\infty}\pi_1(\mathcal{M}^\theta_\partial(W_{g,1}, \ell_{g,1}))$ acts trivially on the fibre $\operatorname{colim}_{g\to\infty}\int^\theta_{W_{g.1}} A$ or not. However, I am not aware of any work that has dealt with this question.

6 | RELATIVELY FREE E_∞-ALGEBRAS

In this section, we develop a general approach to the remaining question of the homotopy type of $\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E^\theta_d} A)$ for a given E^θ_d-algebra A. By doing so, we assume familiarity with the language of ∞-categories and ∞-operads as in [25, 26]. Note that the question we want to address is only about the operads E^θ_d and E_∞ and has nothing to do with diffeomorphisms of manifolds any more.

Construction 6.1. Recall from Example 2.20 the operad map $E_d \to E^\theta_d$. If A is an E^θ_d-algebra, then we denote by UA its underlying E_d-algebra. For any E_d-algebra $\mathfrak{A}$, we consider the classical *iterated bar construction* $\mathrm{B}^d\mathfrak{A} := |\mathrm{B}_\bullet(\Sigma^d, \tilde{\mathbb{E}}_d, \mathfrak{A})|$ as in [28], which is a $(d-1)$-connected based space.

Remark 6.2. Let C be an ∞-category, let L be a connected space and let $X : L \to C$ be a diagram. Since L is connected, the homotopy type (i.e. the equivalence class in C) of any $X(b_0)$ is independent of $b_0 \in L$, and we say that X *takes value* $X(b_0)$.

For each choice of basepoint $b_0 \in L$, the diagram X may be re-expressed in a slightly more classical language by the object $X(b_0)$ in C, together with an E_1-action by the loop space $\Omega L = \Omega_{b_0} L$ on $X(b_0)$, and the colimit $\operatorname{colim}_L(X)$ agrees with the (homotopy) quotient $X(b_0)_{\mathrm{h}\Omega L}$ if either side (hence both sides) exist(s) in C.

We point out that in the case where L is part of a tangential structure $\theta : L \to \mathrm{BO}(d)$, then a canonical choice for a basepoint of L would be the one coming from the bundle map $\ell_{0,1} : T\mathring{D}^d \to \theta^* V_d$, whose map among base spaces factors through a single point. The aim of this section is the following:

Proposition 6.3. *Let $\theta : L \to \mathrm{BO}(d)$ be a tangential structure with connected L and let A be an E_d^θ-algebra. Then the shifted suspension spectrum $\Sigma^{\infty-d}\mathrm{B}^d UA$ carries an E_1-action by the loop space ΩL and we have an equivalence of connective spectra*

$$\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E_d^\theta} A) \simeq \left(\Sigma^{\infty-d}\mathrm{B}^d UA\right)_{\mathrm{h}\Omega L}.$$

Example 6.4. If L is contractible (i.e. $E_d^\theta \simeq E_d$), then $\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E_d} A) \simeq \Sigma^{\infty-d}\mathrm{B}^d A$.

Example 6.5. If θ is of the form $\mathrm{B}G \to \mathrm{BO}(d)$ for some homomorphism $G \to \mathrm{O}(d)$, then Proposition 6.3 gives rise to a naïve G-action on $\Sigma^{\infty-d}\mathrm{B}^d UA$, and we have an equivalence $\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E_d^\theta} A) \simeq \left(\Sigma^{\infty-d}\mathrm{B}^d UA\right)_{\mathrm{h}G}$.

Altogether, Proposition 6.3 then implies our main theorem:

Theorem 6.6. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L, and let A be an E_{2n}^θ-algebra. Then there is an A_∞-action of ΩL on the spectrum $\Sigma^{\infty-2n}\mathrm{B}^{2n}UA$ and we have a weak equivalence of loop spaces*

$$\Omega \mathrm{B}W^\theta_{*,1}[A] \simeq \mathbb{Z} \times \Omega_0^\infty \mathrm{MT}\theta \times \Omega^\infty\left(\left(\Sigma^{\infty-2n}\mathrm{B}^{2n}UA\right)_{\mathrm{h}\Omega L}\right).$$

In combination with Theorem 3.19, we finally have achieved the following homological computation:

Corollary 6.7. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L. If $2n \geqslant 6$ (or $2n = 2$ and θ is admissible to [31, Theorem 7.1]), we have, for each path-connected E_{2n}^θ-algebra A and each $g \geqslant 0$, isomorphisms*

$$H_i(W^\theta_{g,1}[A]) \cong H_i\left(\Omega_0^\infty \mathrm{MT}\theta \times \Omega^\infty\left(\left(\Sigma^{\infty-2n}\mathrm{B}^{2n}UA\right)_{\mathrm{h}\Omega L}\right)\right)$$

for every i small enough compared to g to satisfy the conditions of Theorem 3.19.

In order to prove Proposition 6.3, we have to reformulate parts of our operadic framework in ∞-categorical terms:

Notation 6.8. For each ∞-category $\mathcal{C}$ and objects X, Y, we denote the corresponding mapping space by $\mathrm{Map}_{\mathcal{C}}(X, Y)$. The (large) ∞-category of ∞-categories is denoted by Cat_∞. For ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we denote the ∞-category of functors $\mathcal{C} \to \mathcal{D}$ by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We call the ∞-category of spaces $\mathcal{S}$, the ∞-category of d-connective (i.e. $(d-1)$-connected) based spaces $\mathcal{S}_*^{\geqslant d}$ and the ∞-category of connective spectra $\mathrm{Sp}^{\mathrm{cn}}$.

The ∞-category of (coloured) ∞-operads is denoted by Op_∞. Recall that ∞-operads are modelled in [26, § 2.1] as functors $\mathcal{O} \to \mathrm{Fin}_*$ to the skeletal category of finite pointed sets satisfying certain conditions [26, 2.1.1.10]. A map of ∞-operads $\mathcal{P} \to \mathcal{O}$ is a functor of categories over Fin_*

satisfying a further condition [26, 2.1.2.7]. Let $\mathrm{Op}_\infty(\mathcal{P},\mathcal{O})$ be the full subcategory of $\mathrm{Fun}_{\mathrm{Fin}_*}(\mathcal{P},\mathcal{O})$ consisting of ∞-operad maps. We define the ∞-category $\mathrm{Alg}_\mathcal{O} = \mathrm{Alg}_\mathcal{O}(\mathcal{S})$ of $\mathcal{O}$-algebras (in spaces) to be $\mathrm{Op}_\infty(\mathcal{O},\mathcal{S})$.

Remark 6.9. Each topological operad is implicitly regarded as an ∞-operad through its operadic nerve [26, 2.1.1.23]. For a $\mathfrak{S}$-free topological operad $\mathcal{O}$, the ∞-category $\mathrm{Alg}_\mathcal{O}$ of algebras over the operadic nerve of $\mathcal{O}$ is equivalent to the coherent nerve of the simplicial model category of $\mathcal{O}$-algebras in the sense of [3], see [19, Theorem 4.2.2], which is an instance of [30, Theorem 7.11].

It then follows from [25, 5.2.4.6+7] that if $\mathcal{P} \to \mathcal{O}$ is a map of $\mathfrak{S}$-free operads and $\mathcal{P}$ is proper, then $\mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{P}} (-)$ models the left-adjoint to the forgetful functor $\mathrm{Alg}_\mathcal{O} \to \mathrm{Alg}_\mathcal{P}$, which, in turn, is given by precomposing an operad map $\mathcal{O} \to \mathcal{S}$ with $\mathcal{P} \to \mathcal{O}$.

Example 6.10. The operad E_∞, called the 'commutative ∞-operad' in [26, 2.1.1.18], is by definition the identity $\mathrm{Fin}_* \to \mathrm{Fin}_*$ and hence the terminal object in Op_∞. It agrees with the operadic nerve of the topological operad E_∞, see [26, 5.1.1.5].

Remark 6.11. May's recognition principle [28] can be formulated homotopy-coherently as in [26, 5.2.6.10]: Taking d-fold loop spaces becomes an equivalence of ∞-categories $\Omega^d : \mathcal{S}_*^{\geqslant d} \to \mathrm{Alg}_{E_d}$, and the d-fold bar construction B^d in the sense of Construction 6.1 is a model for its essential inverse.

Remark 6.12. In [26, section 2.4.3], the author constructs, out of an ∞-category $\mathcal{C}$, an operad $\mathcal{C}^\sqcup$, called the *co-cartesian operad*. Intuitively, its colours are the objects of $\mathcal{C}$, and its space of operations $(X_1, \dots, X_r) \to Y$ is given by $\prod_i \mathrm{Map}_\mathcal{C}(X_i, Y)$. If $\mathcal{O}$ is a *unital* operad (i.e. for each object X, the space of nullary operations $() \to X$ is contractible), then we have a natural equivalence $\mathrm{Op}_\infty(\mathcal{O}, \mathcal{C}^\sqcup) \to \mathrm{Fun}(\mathcal{O}\langle 1\rangle, \mathcal{C})$ of ∞-categories, where $(-)\langle 1\rangle$ denotes the underlying ∞-category of unaries, see [26, 2.4.3.16]. This implies that $(-)^\sqcup$ is a right-adjoint functor from Cat_∞ to the full subcategory of Op_∞ spanned by unital ∞-operads, its left-adjoint given by $(-)\langle 1\rangle$.

As right-adjoints preserve terminal objects, the unital operad E_∞ is of the form $*^\sqcup$, where $*$ is a point. For two ∞-operads $\mathcal{O}$ and $\mathcal{O}'$ and an ∞-category $\mathcal{C}$, [26, 2.4.3.18] establishes an equivalence $\mathrm{Op}_\infty(\mathcal{C}^\sqcup \times_{\mathrm{Fin}_*} \mathcal{O}, \mathcal{O}') \to \mathrm{Fun}(\mathcal{C}, \mathrm{Op}_\infty(\mathcal{O}, \mathcal{O}'))$, which is natural in all arguments. For $\mathcal{O} = E_\infty$ and $\mathcal{O}' = \mathcal{S}$, we get an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{C}^\sqcup} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Alg}_{E_\infty})$, which is natural in $\mathcal{C}$.

Notation 6.13. For any ∞-category $\mathcal{C}$, precomposing functors $* \to \mathcal{C}$ with the terminal map $L \to *$ gives rise to the *constant diagram functor*$\Delta_L : \mathcal{C} \to \mathrm{Fun}(L, \mathcal{C})$. It is right-adjoint to $\mathrm{colim}_L : \mathrm{Fun}(L, \mathcal{C}) \to \mathcal{C}$ if the latter exists.

Lemma 6.14. *Let Q be an $L^\sqcup$-algebra, regarded as a functor $L \to \mathrm{Alg}_{E_\infty}$ as above. Then we have an equivalence of connective spectra*

$$\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{L^\sqcup} Q) \simeq \mathrm{colim}_L(\mathrm{B}^\infty Q).$$

Proof. By [26, 5.2.6.26], the classical equivalence between the category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra and group-like E_∞-algebras has been made precise in the context of ∞-categories, and in [9, section 2.1], it has been shown that group-completion is left-adjoint to the inclusion of group-like E_∞-algebras into the category of all E_∞-algebras. This shows $\mathrm{B}^\infty : \mathrm{Alg}_{E_\infty} \to \mathrm{Sp}^{\mathrm{cn}}$ is a left-adjoint functor between ∞-categories and hence preserves colimits. Second, under the equivalence $\mathrm{Alg}_{L^{\sqcup}} \simeq \mathrm{Fun}(L, \mathrm{Alg}_{E_\infty})$, the forgetful functor from E_∞-algebras to $L^{\sqcup}$-algebras is identified with precomposing functors $* \to \mathrm{Alg}_{E_\infty}$ with the terminal map $L \to *$, that is, with the constant diagram functor Δ_L. Its left-adjoint is given by taking colimits in the ∞-category of E_∞-algebras. □

Proof of Proposition 6.3. Using [22, Proposition 2.2], there is a diagram $\Theta : L \to \mathrm{Op}_\infty$, which takes value E_d, whose colimit is equivalent to the operadic nerve of E_d^θ, and for $b_0 \in L$ coming from the θ-framing $\ell_{0,1}$ of D^d, the inclusion $E_d \simeq \Theta(b_0) \to \mathrm{colim}(\Theta) \simeq E_d^\theta$ is equivalent to the operad map from Example 2.20.

Moreover, we also have an equivalence of ∞-operads $L^{\sqcup} \simeq \mathrm{colim}_L(\Delta_L E_\infty)$, as one sees by applying the ∞-Yoneda lemma to the groupoid cores of the natural equivalence

$$\begin{aligned}\mathrm{Op}_\infty(L^{\sqcup}, -) &\simeq \mathrm{Fun}(L, \mathrm{Op}_\infty(E_\infty, -)) \\ &\simeq \lim_L(\mathrm{Op}_\infty(\Delta_L E_\infty, -)) \\ &\simeq \mathrm{Op}_\infty(\mathrm{colim}_L(\Delta_L E_\infty), -).\end{aligned}$$

As the ∞-category of ∞-operads has L-indexed colimits, the functor Δ_L is right-adjoint and hence preserves terminal objects. This shows that $\Delta_L E_\infty$ is a terminal diagram, so we have an (essentially unique) map $\Theta \to \Delta_L E_\infty$. Its colimit is an operad map $E_d^\theta \to L^{\sqcup}$, and since E_∞ is terminal, the map $E_d^\theta \to E_\infty$ is homotopic to the composition $E_d^\theta \to L^{\sqcup} \to E_\infty$. This shows that we have a commutative diagram of right-adjoints

$$\begin{array}{ccccc}
\mathrm{Fun}(L, \mathrm{Sp}^{\mathrm{cn}}) & & \xrightarrow{\quad\simeq\quad} & & \lim_L(\Delta_L \mathrm{Sp}^{\mathrm{cn}}) \\
{\scriptstyle \mathrm{Fun}(L,\Omega^\infty)}\big\downarrow & & & & \big\downarrow{\scriptstyle \lim_L(\Delta_L \Omega^\infty)} \\
\mathrm{Fun}(L, \mathrm{Alg}_{E_\infty}) & \xrightarrow{\simeq} & \mathrm{Alg}_{L^{\sqcup}} & \xrightarrow{\simeq} & \lim_L(\mathrm{Alg}_{\Delta_L E_\infty}) \\
 & & {\scriptstyle (E_d^\theta \to L^{\sqcup})^*}\big\downarrow & & \big\downarrow{\scriptstyle \lim_L(\Theta \to \Delta_L E_\infty)^*} \\
 & & \mathrm{Alg}_{E_d^\theta} & \xrightarrow[\simeq]{} & \lim_L(\mathrm{Alg}_\Theta),
\end{array}$$

whence the corresponding diagram of left-adjoints commutes as well. The basepoint $b_0 \in L$ gives rise to projection functors $\mathrm{pr}_{b_0} : \lim_L(\mathrm{Alg}_\Theta) \to \mathrm{Alg}_{E_d}$ and so forth out of the limit categories, resulting in a diagram (where $\Delta_L E_\infty \otimes_\Theta^{\mathbb{L}} (-)$ is the system of left-adjoints induced by the map $\Theta \to \Delta_L E_\infty$ of diagrams of ∞-operads)

$$\begin{array}{ccccc}
\mathrm{Alg}_{E_d^\theta} & \xrightarrow{\quad\simeq\quad} & \lim_L(\mathrm{Alg}_\Theta) & \xrightarrow{\mathrm{pr}_{b_0}} & \mathrm{Alg}_{E_d} \\
{\scriptstyle L^{\sqcup} \otimes^{\mathbb{L}}_{E_d^\theta}(-)}\big\downarrow & & \big\downarrow{\scriptstyle \lim_L(\Delta_L E_\infty \otimes^{\mathbb{L}}_\Theta(-))} & & \big\downarrow{\scriptstyle E_\infty \otimes^{\mathbb{L}}_{E_d}(-)} \\
\mathrm{Fun}(L, \mathrm{Alg}_{E_\infty}) \simeq \mathrm{Alg}_{L^{\sqcup}} & \xrightarrow{\simeq} & \lim_L(\mathrm{Alg}_{\Delta_L E_\infty}) & \xrightarrow{\mathrm{pr}_{b_0}} & \mathrm{Alg}_{E_\infty} \\
{\scriptstyle \mathrm{Fun}(L, \mathrm{B}^\infty)}\big\downarrow & & \big\downarrow{\scriptstyle \lim_L(\Delta_L \mathrm{B}^\infty)} & & \big\downarrow{\scriptstyle \mathrm{B}^\infty} \\
\mathrm{Fun}(L, \mathrm{Sp}^{\mathrm{cn}}) & \xrightarrow{\simeq} & \lim_L(\Delta_L \mathrm{Sp}^{\mathrm{cn}}) & \xrightarrow{\mathrm{pr}_{b_0}} & \mathrm{Sp}^{\mathrm{cn}}.
\end{array}$$

Here the commutativity of the left squares is justified by the aforementioned diagram of right-adjoints, while the commutativity of the right squares is immediate as the central vertical maps are induced by morphisms of diagrams. We moreover point out the top horizontal composition agrees up to equivalence with $\mathrm{Op}_\infty(\Theta(b_0) \to \mathrm{colim}(\Theta), \mathcal{S})$ and hence with the 'underlying E_d-algebra' functor U from Construction 6.1.

Thus, for each E_d^θ-algebra A, the left vertical composition $(\mathrm{Fun}(L, \mathrm{B}^\infty))(L^\sqcup \otimes^{\mathbb{L}}_{E_d^\theta} A)$, which is an L-indexed diagram in connective spectra, takes value $\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E_d} UA)$. We hence are left to determine, for a general E_d-algebra $\mathfrak{A}$, the spectrum $\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E_d} \mathfrak{A})$. Here we see that for a given connective spectrum Y, we have a chain of natural equivalences of mapping spaces

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathrm{B}^\infty(E_\infty \otimes^{\mathbb{L}}_{E_d} \mathfrak{A}), Y) &\simeq \mathrm{Map}_{\mathrm{Alg}_{E_d}}(\mathfrak{A}, \Omega^d \Omega^{\infty-d} Y)\\ &\simeq \mathrm{Map}_{\mathcal{S}_*^{\geqslant d}}(\mathrm{B}^d \mathfrak{A}, \Omega^{\infty-d} Y)\\ &\simeq \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(\Sigma^{\infty-d}\mathrm{B}^d \mathfrak{A}, Y).\end{aligned}$$

This proves the statement by virtue of the ∞-Yoneda lemma. □

Example 6.15. A *d-connective retractive space* is a fibration $\xi : X \to L$ with a section σ such that the fibre X_b over each $b \in L$ is $(d-1)$-connected. For a compact θ-framed manifold (W, ℓ) with boundary, we let $\mathrm{Map}^\theta_\partial(W, X)$ be the space of maps $s : W \to X$ with $\xi \circ s = \bar{\ell}$ and $s|_{\partial W} = \sigma \circ \bar{\ell}$, where $\bar{\ell} : W \to L$ is the map induced by ℓ among base spaces. Put differently, it is the space of sections of $\bar{\ell}^* X$ that coincide with $\bar{\ell}^*\sigma$ at ∂W.

For the θ-framed manifold $(D^d, \ell_{0,1})$, the space $\Omega^d_\theta X := \mathrm{Map}^\theta_\partial(D^d, X)$ carries the structure of an E_d^θ-algebra, generalising the usual E_d-structure on $\Omega^d X$ (here we use that $\bar{\ell}_{0,1}$ is constant with value b_0). Its underlying E_d-algebra $U\Omega^d_\theta X$ is equivalent to $\Omega^d X_{b_0}$. Generalising the scanning map from [5, Proposition 2], *non-abelian Poincaré duality* [26, Theorem 5.5.6.6] establishes an equivalence between $\int^\theta_W \Omega^d_\theta X$ and $\mathrm{Map}^\theta_\partial(W, X)$.

As before, we would like to consider the *moduli space* of such objects, and we do so by letting the θ-framing vary: Let $\mathrm{Map}_\partial(W, L)$ be the space of maps $f : W \to L$ with $f|_{\partial W} = \bar{\ell}|_{\partial W}$, and let $\mathrm{Map}_\partial(W, X)$ be the space of maps $g : W \to L$ with $g|_{\partial W} = \sigma \circ \bar{\ell}|_{\partial W}$. Then we consider the homotopy pullback

$$\begin{array}{ccc}\underline{\mathrm{Map}}^\theta_\partial(W, X) & \longrightarrow & \mathrm{Map}_\partial(W, X)\\ \downarrow & \lrcorner & \downarrow\\ \mathrm{Fr}^\theta_\partial(W, \ell) & \longrightarrow & \mathrm{Map}_\partial(W, L)\end{array}$$

The space $\underline{\mathrm{Map}}^\theta_\partial(W, X)$ carries a $\mathrm{Diff}_\partial(W)$-action, and the above equivalence can be enhanced to a $\mathrm{Diff}_\partial(W)$-equivariant equivalence $|\mathrm{B}_\bullet(\mathbb{E}^\theta_W, \mathbb{E}^\theta_d, \Omega^d_\theta X)| \simeq \underline{\mathrm{Map}}^\theta_\partial(W, X)$. We hence obtain an equivalence

$$W^\theta[\Omega^d_\theta X] \simeq \underline{\mathrm{Map}}^\theta_\partial(W, X) /\!\!/ \mathrm{Diff}_\partial(W).$$

The right side can be interpreted as the moduli space of θ-framed manifolds of type (W, ℓ_W), together with a map to the retractive space X, abbreviated by $\mathcal{M}^\theta_\partial(W, \ell_W)\langle X\rangle$.

Coming back to our situation of $d = 2n$ and $(W, \ell_W) = (W_{g,1}, \ell_{g,1})$, we use the equivalence $B^{2n} U\Omega^{2n}_\theta X = B^{2n}\Omega^{2n} X_{b_0} \simeq X_{b_0}$ to conclude via Theorem 6.6 the following:

Corollary 6.16. *Let $\theta : L \to \mathrm{BO}(2n)$ be a spherical tangential structure with n-connected L. Moreover, let X be a $2n$-connective retractive space. Then we have a weak equivalence*

$$\operatorname{colim}_{g\to\infty}\left(\mathcal{M}^{\theta}_{\partial}(W_{g,1},\ell_{g,1})\langle X\rangle\right)^{+} \simeq \Omega^{\infty}_{0}\mathrm{MT}\theta \times \Omega^{\infty}\left((\Sigma^{\infty-2n}X_{b_0})_{\mathrm{h}\Omega L}\right).$$

Outlook 6.17. It would be interesting to have, for an E^{θ}_{d}-algebra A, a description for the action of ΩL on $\Sigma^{\infty-d}\mathrm{B}^{d}UA$. I expect the following to hold: For each euclidean d-dimensional vector space V, there is an operad E_V equivalent to E_d, defined exactly as in Example 2.6, and a d-fold bar construction $\mathrm{B}^{V} : \mathrm{Alg}_{E_V} \to \mathcal{S}^{\geqslant d}_{*}$. If $\theta : L \to \mathrm{BO}(d)$ is a tangential structure, then we obtain the diagram $\Theta : L \to \mathrm{Op}_{\infty}$ from [22, Proposition 2.2] by applying the construction $V \mapsto E_V$ fibrewise to the euclidean vector bundle $\theta^{*}V_d$. By applying $V \mapsto \mathrm{B}^{V}$ fibrewise, we then should obtain a morphism $\mathrm{Alg}_{\Theta} \to \Delta_L \mathcal{S}^{\geqslant d}_{*}$ of L-indexed diagrams in Cat_{∞}. Passing to the limit, this gives rise to a functor

$$\mathrm{B}^{d}_{\theta} : \mathrm{Alg}_{E^{\theta}_{d}} \to \mathrm{Fun}(L, \mathcal{S}^{\geqslant d}_{*}),$$

which we shall call the *θ-framed bar construction*. If θ comes from a subgroup $G \subseteq \mathrm{O}(d)$, then I expect this construction to coincide with the (1-categorical) G-equivariant bar construction from [33]. For each d-connective retractive space X over L, I expect $\mathrm{B}^{d}_{\theta}\Omega^{d}_{\theta}X$ to be the functor corresponding to the fibration $X \to L$.

The fibrewise one-point compactification $\theta^{*}V_d$ gives rise to a diagram $\theta^{*}V^{\infty}_{d} : L \to \mathcal{S}^{\geqslant d}_{*}$, and I expect that a variation of the above Yoneda argument shows that $\mathrm{B}^{\infty}(L^{\sqcup} \otimes^{\mathbb{L}}_{E^{\theta}_{d}} A)$ is equivalent to the L-parameterised mapping spectrum $\mathrm{map}(\Sigma^{\infty}\theta^{*}V^{\infty}_{d}, \Sigma^{\infty}\mathrm{B}^{d}_{\theta}A)$.

It seems to me that a formal argument for such a description would require a deeper analysis of the ∞-categorical bar construction and its naturality, and hence would go beyond the scope of this paper.

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