International Mathematics Research Notices, 2025, **2025**(5), 1–15
https://doi.org/10.1093/imrn/rnaf028
Article

Segal K-theory of vector spaces with an automorphism

Andrea Bianchi[1,*] and Florian Kranhold[2]

[1]Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
[2]Karlsruhe Institute of Technology, Englerstraße 2, 76131 Karlsruhe, Germany
[*]Correspondence to be sent to: e-mail: bianchi@mpim-bonn.mpg.de
Communicated by Prof. Akhil Mathew

We describe the Segal K-theory of the symmetric monoidal category of finite-dimensional vector spaces over a field $\mathbb{F}$ together with an automorphism, or, equivalently, the group-completion of the E_∞-algebra of maps from S^1 to the disjoint union of classifying spaces $\mathrm{BGL}_d(\mathbb{F})$, in terms of the K-theory of finite field extensions of $\mathbb{F}$. A key ingredient for this is a computation of the Segal K-theory of the category of finite-dimensional vector spaces with a nilpotent endomorphism. We also discuss the topological cases of $\mathbb{F} = \mathbb{C}, \mathbb{R}$.

1 Introduction and overview

1.1 Setting and question

If $(\mathbf{C}, \otimes)$ is a symmetric monoidal category, the core groupoid $\mathbf{C}^{\simeq}$ is canonically an E_∞-algebra in spaces, whose group-completion is the infinite loop space of a connective spectrum $K^{\mathrm{S}}(\mathbf{C})$, known as the *Segal K-theory of* $\mathbf{C}$ [28, 36, 40].

We are interested in the relation between the Segal K-theories of $\mathbf{C}$ and of the functor category $\mathbf{C}^{\mathrm{aut}} = \mathrm{Map}(B\mathbb{Z}, \mathbf{C})$ of objects in $\mathbf{C}$ with automorphism, together with the pointwise symmetric monoidal structure. Note that $\mathbf{C}$ is a symmetric monoidal retract of $\mathbf{C}^{\mathrm{aut}}$, comprising objects of $\mathbf{C}$ together with the identity morphism. As a consequence, the connective spectrum $K^{\mathrm{S}}(\mathbf{C}^{\mathrm{aut}})$ contains $K^{\mathrm{S}}(\mathbf{C})$ as a direct summand, and we are interested in determining the remaining summand. Moreover, the E_∞-algebra $\mathbf{C}^{\mathrm{aut},\simeq}$ is equivalent to the free loop space $\mathrm{Map}(S^1, \mathbf{C}^{\simeq})$, see lemma 2.8, so our question can also be phrased in terms of the group-completion of the latter.

When $\mathbf{C}$ is an abelian category with the direct sum monoidal structure, $K^{\mathrm{S}}(\mathbf{C})$ agrees with the connective cover of the spectrum $K^{\oplus}(\mathbf{C})$, which we shall refer to as *additive K-theory* of $\mathbf{C}$, and which is defined as the (not necessarily connective) Quillen K-theory of $\mathbf{C}$ endowed with the split exact structure. We mention that, more generally, an additive K-theory spectrum $K^{\oplus}(\mathbf{C})$ can be defined for any small additive ∞-category $\mathbf{C}$ by first taking the stabilisation of $\mathbf{C}$, which is a small stable ∞-category, and by then taking K-theory of the latter in the sense of [6], see [11, Appendix A], [24, Subsection 3.3]. The connective cover of the latter is equivalent to $K^{\mathrm{S}}(\mathbf{C})$ [18, Corollary 8.1.3].

On the other hand, each abelian category $\mathbf{C}$ can be endowed with its *maximal* exact structure; we denote its associated Quillen's K-theory by $K(\mathbf{C})$ and simply call it *Quillen's K-theory* of $\mathbf{C}$ in the rest of the article. By the theorem of the heart [4], this agrees with the K-theory of the bounded derived category of $\mathbf{C}$, which is a small, stable ∞-category.

Received: July 10, 2024. **Revised:** October 16, 2024. **Accepted:** October 17, 2024

If $\mathbf{C}$ is abelian, then the aforementioned category $\mathbf{C}^{\mathrm{aut}}$ is again abelian. In this work, we study $K^{\mathrm{S}}(\mathbf{C}^{\mathrm{aut}})$ and $K^{\oplus}(\mathbf{C}^{\mathrm{aut}})$ in the case where $\mathbf{C} = \mathbf{Mod}_{\mathbb{F}}$ is the abelian category of finite-dimensional vector spaces over a field $\mathbb{F}$.

1.2 Results

We will see in corollary 4.7 that $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})$ is connective, and hence agrees with Segal's K-theory $K^{\mathrm{S}}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})$. Hence we do not have to distinguish between K^{S} and $K^{\oplus}$ when stating our results in the introduction.

As mentioned above, the E_∞-algebra $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut},\simeq}$ is equivalent to the E_∞-algebra of maps from S^1 to $\mathbf{Mod}_{\mathbb{F}}^{\simeq} \simeq \coprod_{d\geqslant 0} \mathrm{BGL}_d(\mathbb{F})$. Using this description, the main result of this article can be phrased as follows in terms of Segal K-theory: For each field $\mathbb{F}$, there is an equivalence of spectra

$$\mathrm{B}^\infty \mathrm{Map}\left(S^1, \coprod_{d\geqslant 0} \mathrm{BGL}_d(\mathbb{F})\right) \simeq \bigoplus_{\substack{(t)\neq \mathfrak{m}\subset \mathbb{F}[t] \\ \text{maximal ideal}}} \bigoplus_{i=1}^{\infty} K(\mathbb{F}[t]/\mathfrak{m}). \tag{1}$$

In other words, the group completion of the E_∞-algebra $\mathrm{Map}\left(S^1, \coprod_{d\geqslant 0} \mathrm{BGL}_d(\mathbb{F})\right)$ can be identified with the infinite loop space of the right hand side.

Here $\mathbb{F}[t]$ is a polynomial algebra in one variable, (t) is the ideal generated by the monomial t, and $\mathbb{F}[t]/\mathfrak{m}$ is the residue field of $\mathfrak{m}$. Moreover, $K(\mathbb{F})$ is the usual algebraic K-theory spectrum of $\mathbb{F}$, which agrees with the formely introduced spectra $K(\mathbf{Mod}_{\mathbb{F}}) \simeq K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}) \simeq K^{\mathrm{S}}(\mathbf{Mod}_{\mathbb{F}})$. Its homotopy type has been determined in many cases, e.g. [9, 32], see also [42,§VI] for an overview.

The equivalence in equation (1) is conceived in two main steps: In the first step, carried out in section 3, we employ the fact that each automorphism of a finite-dimensional vector space has a primary decomposition to get the following result:

Theorem A. Let $\mathbb{F}$ be a field. Then we have an equivalence of spectra

$$K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}) \simeq \bigoplus_{\substack{(t)\neq \mathfrak{m}\subset \mathbb{F}[t] \\ \text{maximal ideal}}} K^{\oplus}(\mathbf{Mod}_{\mathbb{F}[t]/\mathfrak{m}}^{\mathrm{nil}}).$$

Here $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}}$ is the abelian category of finite-dimensional vector spaces together with a nilpotent endomorphism. By admitting the formerly excluded maximal ideal (t), the equivalence of theorem A can be extended to

$$K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{end}}) \simeq \bigoplus_{\substack{\mathfrak{m}\subset \mathbb{F}[t] \\ \text{maximal ideal}}} K^{\oplus}(\mathbf{Mod}_{\mathbb{F}[t]/\mathfrak{m}}^{\mathrm{nil}}),$$

where $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{end}} = \mathrm{Map}(\mathrm{B}\mathbb{N}, \mathbf{Mod}_{\mathbb{F}})$ is the abelian category of vector spaces over $\mathbb{F}$ together with an endomorphism, see remark 3.11 for details.

Under the equivalence of theorem A, the summand $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}})$ is a retract of the summand $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}[t]/(t-1)}^{\mathrm{nil}}) \simeq K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}})$; this retract is obtained by combining the functors $\mathbf{Mod}_{\mathbb{F}} \to \mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}}$, endowing a given vector space with the zero endomorphism, and $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}} \to \mathbf{Mod}_{\mathbb{F}}$, forgetting the endomorphism.

In the second step, carried out in section 4, we study the additive K-theory of $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}}$. Although the forgetful functor $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}} \to \mathbf{Mod}_{\mathbb{F}}$ induces an equivalence on Quillen K-theory, the same is far from being true for additive K-theory. We instead show the following:

Theorem B. Let $\mathbb{F}$ be a field. Then we have an equivalence of spectra

$$K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}}) \simeq \bigoplus_{i=1}^{\infty} K(\mathbb{F}).$$

Roughly speaking, the index set corresponds to the fact that nilpotent endomorphisms can be decomposed into Jordan blocks of arbitrary size. Our proof of theorem B relies on an explicit description of the fibre of the comparison map between $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}})$ and $K(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}})$ in terms of Quillen K-theory of a certain auxiliary abelian category; this description is due to Auslander and Sherman [37] in the connective setting, and to Schlichting [35] in the non-connective setting used in this work. The main equivalence (1) then follows by combining theorems A and B.

Remark 1.1. If the ground field $\mathbb{F}$ is assumed to be perfect, then the equivalences in theorems A and B are in fact equivalences of $K(\mathbb{F})$-module spectra: this follows at once by observing that all functors that are used for establishing the equivalences are $\mathbb{F}$-linear and hence tensored over $\mathbf{Mod}_{\mathbb{F}}$. See remark 3.8 for details.

Remark 1.2. The symmetric monoidal structure on $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}$ given by tensor product over $\mathbb{F}$ gives rise to an E_∞-ring structure on $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})$. Given two maximal ideals $\mathfrak{m}, \mathfrak{m}' \in \mathrm{Spec}(\mathbb{F}[t^{\pm 1}]) = \mathbb{G}_m(\mathbb{F})$, it is easy to see that the product of the two direct summands of $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})$ corresponding to $\mathfrak{m}, \mathfrak{m}'$ lands inside the (finitely many) summands corresponding to maximal ideals in the image of the finite scheme

$$\mathrm{Spec}(\mathbb{F}[t^{\pm 1}]/\mathfrak{m} \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]/\mathfrak{m}') \subset \mathbb{G}_m(\mathbb{F}) \times_{\mathbb{F}} \mathbb{G}_m(\mathbb{F})$$

along the multiplication map of $\mathbb{G}_m(\mathbb{F})$. It would be interesting to derive a complete description of the E_∞-ring product in the light of the joint decomposition of the spectrum $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})$ given by theorems A and B.

In the final section 5, we discuss the topological cases of $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$, where the natural topology on the automorphism groups is taken into account. More precisely, let $\mathbf{Mod}^{\simeq}_{\mathbb{C},\mathrm{top}}$ and $\mathbf{Mod}^{\simeq}_{\mathbb{R},\mathrm{top}}$ be the topologically enriched symmetric monoidal groupoids of finite-dimensional vector spaces over $\mathbb{C}$ and $\mathbb{R}$, respectively, with isomorphisms as morphisms, i.e. we have the two homotopy equivalences

$$\mathbf{Mod}^{\simeq}_{\mathbb{C},\mathrm{top}} \simeq \coprod_{d \geqslant 0} \mathrm{BU}(d), \qquad \mathbf{Mod}^{\simeq}_{\mathbb{R},\mathrm{top}} \simeq \coprod_{d \geqslant 0} \mathrm{BO}(d).$$

Very generally, for any E_∞-algebra A and any space X, there is a canonical map of spectra $\mathrm{B}^\infty\mathrm{Map}(X, A) \to \mathrm{Map}(X, \mathrm{B}^\infty A)$. Note that the source of the map is always connective, whereas the target may be non-connective. For $A = \mathbf{Mod}^{\simeq}_{\mathbb{F},\mathrm{top}}$, we show:

Theorem C. For $\mathbb{F} = \mathbb{C}, \mathbb{R}$ and for any finite cell complex (We will prove the corresponding ∞-categorical statement, which works for X any compact anima.) X, the canonical map of spectra

$$\mathrm{B}^\infty\mathrm{Map}(X, \mathbf{Mod}^{\simeq}_{\mathbb{F},\mathrm{top}})) \to \mathrm{Map}(X, \mathrm{B}^\infty\mathbf{Mod}^{\simeq}_{\mathbb{F},\mathrm{top}})$$

is an equivalence after passing to connective covers. Specialising to the case $X = S^1$, we have equivalences of spectra

$$\mathrm{B}^\infty\mathrm{Map}(S^1, \mathbf{Mod}^{\simeq}_{\mathbb{C},\mathrm{top}}) \simeq \Sigma^\infty_+ S^1 \wedge \mathrm{ku},$$

$$\mathrm{B}^\infty\mathrm{Map}(S^1, \mathbf{Mod}^{\simeq}_{\mathbb{R},\mathrm{top}}) \simeq \mathrm{colim}\left(\mathrm{ko} \xleftarrow{r} \mathrm{ku} \xrightarrow{r} \mathrm{ko}\right).$$

Here ku and ko denote the connective complex and real K-theory spectra, respectively, and $r\colon \mathrm{ku} \to \mathrm{ko}$ is the 'realification', induced by restriction of scalars $\mathbb{C} \to \mathbb{R}$. We point out that the infinite loop space of the left hand side $\mathrm{B}^\infty\mathrm{Map}(X, \mathbf{Mod}^{\simeq}_{\mathbb{F},\mathrm{top}})$ is exactly the group completion of the E_∞-algebra of vector bundles over X, recovering Atiyah's original definition of topological K-theory of X. In the case of $X = S^1$, this spectrum can be regarded as Segal's K-theory of the topologically enriched category of finite-dimensional vector spaces over $\mathbb{F}$ together with an automorphism.

The first statement of theorem C, specialised to $X = S^1$, is in contrast with theorems A and B, which imply that, for a discrete field $\mathbb{F}$, the canonical map

$$K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}) \to \mathrm{Map}(S^1, K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}))$$

is far from being an equivalence (even after passing to connective covers) as the left-hand side is identified with the direct sum $\bigoplus_{(t)\neq\mathfrak{m}\subset\mathbb{F}[t]}\bigoplus_{i\geqslant 1}K(\mathbb{F}[t]/\mathfrak{m})$, whereas the right-hand side has the rather 'smaller' homotopy type $K(\mathbb{F})\oplus\Sigma^{-1}K(\mathbb{F})$.

There is, however, also a striking similarity between theorem C and theorem A: We observe that the maximal ideals in $\mathbb{C}[t]$ different from (t) naturally form a *space*, namely $\mathbb{C}\setminus\{0\}\simeq S^1$, and every maximal ideal $\mathfrak{m}$ yields the 'same' residue field $\mathbb{C}\cong\mathbb{C}[t]/\mathfrak{m}$. Similarly, maximal ideals in $\mathbb{R}[t]$ different from (t) are parametrised by the space $\{z\in\mathbb{C}\setminus\{0\}:\Im(z)\geqslant 0\}\simeq[0,1]$. This space contains the subspace $\mathbb{R}\setminus\{0\}\simeq\{0,1\}$, and the residue field $\mathbb{R}[t]/\mathfrak{m}$ is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$, depending on whether the parameter of $\mathfrak{m}$ lies in $\mathbb{R}\setminus\{0\}$ or not. Finally, the lack of terms corresponding to Jordan blocks of size at least 2 is, intuitively, due to the fact that in the topological setting every automorphism of a finite-dimensional vector space can be canonically homotoped to a unitary or orthogonal one, which is semisimple.

1.3 Related work

The *Quillen* K-theory of modules with an automorphisms, more precisely the kernel of $K_i(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})\to K_i(\mathbf{Mod}_{\mathbb{F}})$, has been studied in [15, 16]. Furthermore, [1], describes K_0 of the category of modules with an *endomorphism*; the higher K-groups have been also discussed in [2, 39]. In particular, we point out the similarity between the decomposition of [2, Thm. 5.2] for Quillen K-theory and our theorem A. A more modern treatment of Quillen K-theory of endomorphisms can be found in [7,§ 3].

The additive K-theory of modules with an automorphism is a key ingredient in a description of a weight filtration of Quillen K-theory of rings, due to [17]. More precisely, Grayson considers, for a given ring R, a simplicial ring $R\mathbb{A}^\bullet$ whose 0-simplices are R itself. The n^{th} filtration component on $K(R)$ is built out of the Segal K-theories of $R\mathbb{A}^\bullet$-modules together with at most n mutually commuting automorphisms. In that context, we calculated the 0-simplices of the first filtration step in the case where R is a field.

The current article can also be regarded as an instance of the following problem: Replacing E_∞ by E_1 in the above discussion, the free loop space $\mathrm{Map}(S^1,A)$ of an E_1-algebra A has a pointwise E_1-algebra structure, admitting a group-completion $\Omega B\mathrm{Map}(S^1,A)$. (We focus on maps from S^1 for simplicity; the same problem can be reformulated with respect to maps from an arbitrary space.) We might also first group-complete A and then take the free loop space, obtaining the space $\mathrm{Map}(S^1,\Omega BA)$. As before, there is a canonical E_1-map

$$\Omega B\mathrm{Map}(S^1,A)\to\mathrm{Map}(S^1,\Omega BA)$$

and we are broadly interested in understanding, in concrete examples, whether this map is an equivalence, or rather how much this map is *not* an equivalence. In [5], we determined, in a slightly different setting, the homotopy type of the group-completion of $\mathrm{Map}(S^1,A)$, where A is the E_1-algebra given by the disjoint union of classifying spaces of either of the following three sequences of groups: mapping class groups of oriented surfaces of genus $g\geqslant 0$ with one boundary curve, Artin braid groups, and symmetric groups. In all these cases, the above map is far from being an equivalence. This article studies the case of A being the disjoint union of classifying spaces of general linear groups $\mathrm{GL}_d(\mathbb{F})$.

2 Preliminaries

2.1 Segal K-theory

We start by recalling the definition of Segal K-theory for symmetric monoidal categories and some of its properties. While the most part of the paper can be read and understood with Segal's original construction [36] in mind, we decided to use the more modern language of ∞-categories from [25, 26].

Notation 2.1. Let $\mathbf{Cat}_\infty$ and $\mathbf{S}$ denote the (large) ∞-categories of small ∞-categories and small ∞-groupoids (a.k.a. *spaces*), respectively. The ∞-categories of small symmetric monoidal ∞-categories and small symmetric monoidal ∞-groupoids (a.k.a. E_∞-algebras in spaces) are denoted by $\mathbf{CMon}(\mathbf{Cat}_\infty)$ and $\mathbf{CMon}(\mathbf{S})$, respectively.

Reminder 2.2. The inclusion $\mathbf{S}\hookrightarrow\mathbf{Cat}_\infty$ has a right-adjoint $(-)^\simeq\colon\mathbf{Cat}_\infty\to\mathbf{S}$, called the *core groupoid functor* [23, Cor. 5.1.17]. Since $(-)^\simeq$ preserves products, it induces a functor between categories of commutative monoid objects, which we shall still call $(-)^{\simeq}\colon \mathbf{CMon}(\mathbf{Cat}_\infty) \to \mathbf{CMon}(\mathbf{S})$. It is again right-adjoint to the forgetful functor $\mathbf{CMon}(\mathbf{S}) \to \mathbf{CMon}(\mathbf{Cat}_\infty)$, see [12, Lem. 6.1].

Reminder 2.3. The ∞-category of spectra is denoted by $\mathbf{Sp}$, and the full subcategory of connective spectra is denoted by by $\mathbf{Sp}^{\mathrm{cn}}$. The latter is equivalent to the full subcategory of $\mathbf{CMon}(\mathbf{S})$ spanned by group-like objects [26,5.2.6.26], via the *infinite loop space* functor $\Omega^\infty\colon \mathbf{Sp}^{\mathrm{cn}} \hookrightarrow \mathbf{CMon}(\mathbf{S})$. The functor Ω^∞ has a left-adjoint $\mathrm{B}^\infty\colon \mathbf{CMon}(\mathbf{S}) \to \mathbf{Sp}^{\mathrm{cn}}$, see [27] and [12, Rmk. 4.5].

Now we can give the definiton of Segal K-theory as in [12, Def. 8.3]:

Definition 2.4. We define *Segal K-theory* to be the composition of functors

$$K^S\colon \mathbf{CMon}(\mathbf{Cat}_\infty) \xrightarrow{(-)^{\simeq}} \mathbf{CMon}(\mathbf{S}) \xrightarrow{\mathrm{B}^\infty} \mathbf{Sp}^{\mathrm{cn}}.$$

2.2 Mapping spaces and representations

In this subsection, we make the aforementioned connection between categories of automorphisms and free loop spaces precise.

Construction 2.5. Let $\mathbf{C}$ be a symmetric monoidal ∞-category and let $\mathbf{I}$ be any small ∞-category. Then the ∞-category $\mathrm{Map}(\mathbf{I}, \mathbf{C})$ of functors from $\mathbf{I}$ to $\mathbf{C}$ is again symmetric monoidal by pointwise operations, and it is an ∞-groupoid if $\mathbf{C}$ is.

In the case where both $\mathbf{I}$ and $\mathbf{C}$ are small ∞-groupoids, i.e. spaces, the ∞-groupoid $\mathrm{Map}(\mathbf{I}, \mathbf{C})$ can be regarded as a classical mapping space between spaces, which we also denote, more traditionally, as $\mathrm{Map}(\mathbf{I}, \mathbf{C})$.

Example 2.6. For a discrete monoid G, we denote by $\mathrm{B}G$ the 1-category with a single object that has G as its endomorphism monoid.

If $\mathbf{C}$ is a (classical) category, then $\mathrm{Map}(\mathrm{B}G, \mathbf{C})$ is the category of G-representations in $\mathbf{C}$; and if, additionally, $\mathbf{C}$ is symmetric monoidal, then the monoidal structure on $\mathrm{Map}(\mathrm{B}G, \mathbf{C})$ is given by taking monoidal products of G-representations.

For example, $\mathrm{Map}(\mathrm{B}\mathbb{Z}, \mathbf{C})$ is the category whose objects are pairs (X, κ) where X is an object of $\mathbf{C}$ and κ is an automorphism of X, and whose arrows $(X, \kappa) \to (X', \kappa')$ are morphisms $\phi\colon X \to X'$ with $\kappa' \circ \phi = \phi \circ \kappa$.

Notation 2.7. For any ∞-category $\mathbf{C}$, we write $\mathbf{C}^{\mathrm{aut}} = \mathrm{Map}(\mathrm{B}\mathbb{Z}, \mathbf{C})$.

Lemma 2.8. For any small symmetric monoidal ∞-category $\mathbf{C}$, we have an equivalence of connective spectra $K^S(\mathbf{C}^{\mathrm{aut}}) \simeq \mathrm{B}^\infty\mathrm{Map}(S^1, \mathbf{C}^{\simeq})$. In particular, for a field $\mathbb{F}$, we have an equivalence of connective spectra

$$K^S(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}) \simeq \mathrm{B}^\infty\mathrm{Map}\left(S^1, \coprod_{d \geqslant 0} \mathrm{BGL}_d(\mathbb{F})\right).$$

Proof. Since $\mathrm{B}\mathbb{Z} \simeq S^1$ is an ∞-groupoid, the canonical functor of symmetric monoidal ∞-groupoids $\mathrm{Map}(S^1, \mathbf{C}^{\simeq}) \to \mathrm{Map}(S^1, \mathbf{C})^{\simeq}$ is an equivalence by [23, Rmk. 5.1.18]. We hence have an equivalence of spectra

$$K^S(\mathbf{C}^{\mathrm{aut}}) = \mathrm{B}^\infty\mathrm{Map}(S^1, \mathbf{C})^{\simeq} \simeq \mathrm{B}^\infty\mathrm{Map}(S^1, \mathbf{C}^{\simeq}) = \mathrm{B}^\infty\mathrm{Map}(S^1, \mathbf{C}^{\simeq}).$$

In the special case of $\mathbf{C} = \mathbf{Mod}_{\mathbb{F}}$, we use that the E_∞-algebra $\mathbf{Mod}_{\mathbb{F}}^{\simeq}$ is equivalent to the disjoint union of classifying spaces $\coprod_{d \geqslant 0} \mathrm{BGL}_d(\mathbb{F})$. ■

Thus, our original question can be phrased geometrically as follows: We want to study, for a field $\mathbb{F}$, the group-completion of the free loop space of $\coprod_{d \geqslant 0} \mathrm{BGL}_d(\mathbb{F})$.

2.3 Quillen K-theory and additive K-theory

Reminder 2.9. An *exact structure* on an abelian category $\mathbf{A}$ is a class $\mathbf{E}$ of short sequences $A \to A' \to A''$ which satisfies a list axioms [33,§2.1]. The pair $(\mathbf{A}, \mathbf{E})$ is then called an *exact category*. An *exact functor* $(\mathbf{A}, \mathbf{E}) \to (\mathbf{A}', \mathbf{E}')$ is an additive functor $F\colon \mathbf{A} \to \mathbf{A}'$ with $F(\mathbf{E}) \subset \mathbf{E}'$. We refrain from recalling all axioms, as we will only be concerned with two classes of exact structures:

- The collection of all *split* short exact sequences $A \to A \oplus A'' \to A''$ is an exact structure on $\mathbf{A}$; it is actually the smallest one satisfying Quillen's axioms. We refer to this as the *split* exact structure on $\mathbf{A}$.
- The collection of *all* short exact sequences is an exact structure on $\mathbf{A}$, which we refer to as the *maximal* exact structure on $\mathbf{A}$. If not stated otherwise, this is the canonical exact structure that we consider on an abelian category.

For an exact category $(\mathbf{A}, \mathbf{E})$, its Quillen K-theory has been defined as a loop space in [33,§ 2.2], further deloopings have been studied in [13, 22, 38, 41]. In this work, the *Quillen K-theory* $K(\mathbf{A}, \mathbf{E})$ is a (not necessarily connective) spectrum, using the definition from [35,§ 12.1].

Notation 2.10. For an abelian category $\mathbf{A}$ we denote by $K^{\oplus}(\mathbf{A})$ and $K(\mathbf{A})$ Quillen's K-theories of $\mathbf{A}$ with respect to the maximal and the split exact structures, respectively. We refer to the first as *additive K-theory* of $\mathbf{A}$, and to the second, by abuse of terminology, simply as *Quillen's K-theory* of $\mathbf{A}$.

Reminder 2.11. Any abelian category $\mathbf{A}$ can be regarded as a symmetric monoidal category via its (co-)cartesian structure. Then it is a classical result that $K^{S}(\mathbf{A})$ is naturally equivalent to the connective cover of $K^{\oplus}(\mathbf{A})$, see [14] for a proof on the level of spaces and [8, Thm. 10.2] and [6, Rmk. 9.33] for a proof on the level of spectra.

Reminder 2.12. We call an abelian category $\mathbf{A}$ *noetherian* if for each object A of $\mathbf{A}$, each ascending chain of subobjects $A_0 \subset A_1 \subset \cdots \subset A$ eventually stops. It is shown in [35, Thm. 7] that if $\mathbf{A}$ is noetherian, then $K(\mathbf{A})$ is connective.

Example 2.13. If $\mathbb{F}$ is any field, then the maximal and the split exact structure on the (small) abelian category $\mathbf{Mod}_{\mathbb{F}}$ of finite-dimensional vector spaces over $\mathbb{F}$ are equal. Therefore, the spectra $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}})$ and $K(\mathbf{Mod}_{\mathbb{F}})$ are equivalent to each other. As $\mathbf{Mod}_{\mathbb{F}}$ is noetherian, these spectra are connective, and hence agree with $K^{S}(\mathbf{Mod}_{\mathbb{F}})$. We shall denote their homotopy type simply by $K(\mathbb{F})$.

Reminder 2.14. For a general abelian category $\mathbf{A}$, Quillen K-theory differs from additive K-theory, but there is a comparison map $\omega_A\colon K^{\oplus}(\mathbf{A}) \to K(\mathbf{A})$, since the identity of $\mathbf{A}$ is an exact functor from the split to the maximal exact structure.

2.4 Semiadditivity

Remark 2.15. The ∞-category of (small) abelian categories and additive functors, and the ∞-category of spectra, are *semiadditive*, i.e. initial and terminal objects exist and coincide, and for each finite collection $(X_i)_{i\in J}$ of objects, the canonical morphism from the coproduct $\coprod_{i\in J} X_i$ to the product $\prod_{i\in J} X_i$ is an equivalence, see e.g. [12, Prop. 2.3+8].

Notation 2.16. For an arbitrary set I and a family $(X_i)_{i\in I}$ of either abelian categories or spectra, we denote by $\bigoplus_{i\in I} X_i = \coprod_{i\in I} X_i$ the categorical *coproduct*, and we will refer to it as a 'direct sum'.

Remark 2.17. In any ambient category, a coproduct $\coprod_{i\in I} X_i$ over an arbitrary set I is the filtered colimit of the coproducts $\coprod_{i\in J} X_i$ over the poset of finite subset $J \subseteq I$. If, additionally, our ambient category is semiadditive, then we can replace each of the latter finite coproducts by a finite product and, accordingly, identify $\coprod_{i\in I} X_i$ with the filtered colimit of the finite *products* $\prod_{i\in J} X_i$.

Lemma 2.18. Let $(\mathbf{A}_i)_{i\in I}$ be a family of abelian categories. Then the following canonical map of spectra is an equivalence:

$$\bigoplus_{i\in I} K^{\oplus}(\mathbf{A}_i) \to K^{\oplus}\left(\bigoplus_{i\in I} \mathbf{A}_i\right).$$

Proof. By [35, Cor. 4+5], the functor $(\mathbf{A}, \mathbf{E}) \mapsto K(\mathbf{A}, \mathbf{E})$ from the category of exact categories to spectra commutes with finite products and filtered colimits. Hence the statement follows from the observation that taking finite products and filtered colimits of split exact categories (in the category of exact categories) agrees with taking finite products and filtered colimits of the underlying abelian categories and endowing the result with its split exact structure. ■

3 Primary decompositions of automorphisms

The goal of this section is to prove theorem A.

Notation 3.1. Let R be a commutative ring. For each $a \in R$, we denote by $Ra = (a)$ the ideal generated by a, and we write $R/a = R/(a)$ for the quotient ring.

Reminder 3.2. (Primary decomposition). Let R be a principal ideal domain and let M be a finitely generated torsion module over R. Then there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset R$ and positive integers $r_1, \ldots, r_n > 0$, unique up to permutation, with

$$M \cong \bigoplus_{i=1}^{n} R/\mathfrak{m}_i^{r_i}.$$

Example 3.3. For a field $\mathbb{F}$, we identify the category $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{end}}$ of finite-dimensional vector spaces over $\mathbb{F}$ together with an endomorphism, with the full abelian subcategory $\mathbf{Mod}_{\mathbb{F}[t]}^{\mathrm{tors}} \subseteq \mathbf{Mod}_{\mathbb{F}[t]}$ spanned by those $\mathbb{F}[t]$-modules M with $\mathbf{dim}_{\mathbb{F}} M < \infty$: these are precisely the finitely generated torsion modules. In this description, the endomorphism of a vector space is given by multiplication by t on a module.

By the above primary decomposition theorem, we obtain a decomposition of each $M \in \mathbf{Mod}_{\mathbb{F}[t]}^{\mathrm{tors}}$ into summands of the form $\mathbb{F}[t]/\mathfrak{m}_i^{r_i}$, where each $\mathfrak{m}_i$ is generated by an irreducible polynomial. The endomorphism of M, given by multiplication by t, is an automorphism if and only if none of these ideals $\mathfrak{m}_i$ equals (t).

Definition 3.4. For a maximal ideal $\mathfrak{m} \subset \mathbb{F}[t]$, we let $\mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}}$ be the full abelian subcategory of $\mathbf{Mod}_{\mathbb{F}[t]}^{\mathrm{tors}}$ containing all modules isomorphic to $\bigoplus_{i=1}^{n} \mathbb{F}[t]/\mathfrak{m}^{r_i}$ for some $n \geqslant 0$ and $r_1, \ldots, r_n > 0$.

Example 3.5. The category $\mathbf{Mod}_{\mathbb{F}[t]}^{(t)}$ coincides with the abelian category of finite-dimensional vector spaces over $\mathbb{F}$ together with a nilpotent endomorphism, also denoted by $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}}$. This abelian category will play a central rôle in section 4.

Lemma 3.6. We have an equivalence of abelian categories

$$\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}} \simeq \bigoplus_{\substack{(t)\neq\mathfrak{m}\subset\mathbb{F}[t] \\ \text{maximal ideal}}} \mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}}.$$

Proof. We have an additive functor $\bigoplus_{\mathfrak{m}} \mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}} \to \mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}$ given by taking a finitely supported family $(M_{\mathfrak{m}})_{\mathfrak{m}}$ to the direct sum $\bigoplus_{\mathfrak{m}} M_{\mathfrak{m}}$. This functor is essentially surjective by the primary decomposition theorem, and it is clearly faithful. It is also full, since for two different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, the powers $\mathfrak{m}_1^{r_1}$ and $\mathfrak{m}_2^{r_2}$ are generated by coprime polynomials q_1 and q_2, and in such a situation, there are no non-zero $\mathbb{F}[t]$-linear maps $\mathbb{F}[t]/q_1 \to \mathbb{F}[t]/q_2$. ■

Note that under the equivalence of lemma 3.6, the retract $\mathbf{Mod}_{\mathbb{F}} \hookrightarrow \mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}$ from the introduction, given by endowing a given vector space V with the identity automorphism, is actually a retract of the summand $\mathbf{Mod}_{\mathbb{F}[t]}^{(t-1)}$, as the primary decomposition of (V, id_V) is given by $\bigoplus_{i=1}^{\dim(V)} \mathbb{F}[t]/(t-1)$. We can now study each summand $\mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}}$ separately.

Lemma 3.7. Let $\mathbb{F}$ be a field and let $\mathfrak{m} \subset \mathbb{F}[t]$ be any maximal ideal. Then there is an equivalence of abelian categories

$$\mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}} \simeq \mathbf{Mod}_{\mathbb{F}[t]/\mathfrak{m}}^{\mathrm{nil}}.$$

We thank the referee for suggesting the following argument, proving lemma 3.7 without our original assumption that $\mathfrak{m}$ is generated by a separable polynomial.

Proof. We denote by $\widehat{\mathbb{F}[t]}^{\mathfrak{m}} := \lim_{n\in\mathbb{N}} \mathbb{F}[t]/\mathfrak{m}^n$ the completion of $\mathbb{F}[t]$ at the ideal $\mathfrak{m}$. Then $\mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}}$ is isomorphic to the abelian category of $\mathfrak{m}$-nilpotent and finitely generated modules over $\widehat{\mathbb{F}[t]}^{\mathfrak{m}}$. Similarly, $\mathbf{Mod}_{\mathbb{F}[t]/\mathfrak{m}}^{\mathrm{nil}}$ is isomorphic to the category of (x)-nilpotent, finitely generated modules over $(\mathbb{F}[t]/\mathfrak{m})[\![x]\!]$, the ring of power series over $\mathbb{F}[t]/\mathfrak{m}$. It is then a classical theorem of Cohen [10] that $\widehat{\mathbb{F}[t]}^{\mathfrak{m}}$, being a complete discrete valuation ring of equal characteristic, is isomorphic *as a commutative ring* to $(\mathbb{F}[t]/\mathfrak{m})[\![x]\!]$. This isomorphism lets the (unique) maximal ideals correspond to each other, and hence gives rise to an isomorphism between categories of finitely generated modules that are nilpotent with respect to the maximal ideal. ■

Remark 3.8. If the maximal ideal $\mathfrak{m}$ from lemma 3.7 is generated by a separable polynomial q, then the desired equivalence can be made very explicit: Let α be some root q in the algebraic closure of $\mathbb{F}$ and let $\mathbb{L} = \mathbb{F}(\alpha)$ be the field extension of $\mathbb{F}$ by α, which is isomorphic to the residue field $\mathbb{F}[t]/\mathfrak{m}$. If $\mathbf{Mod}_{\mathbb{L}[t]}^{\mathfrak{m}}$ denotes the full subcategory of $\mathbf{Mod}_{\mathbb{L}[t]}^{\mathrm{tors}}$ spanned by those modules M for which there exists an $r \geqslant 0$ with $q^r M = 0$, then the composition of additive functors

$$\mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}} \xrightarrow{M \mapsto \mathbb{L}[t] \otimes_{\mathbb{F}[t]} M} \mathbf{Mod}_{\mathbb{L}[t]}^{\mathfrak{m}} \xrightarrow{M \mapsto M_{t-\alpha}} \mathbf{Mod}_{\mathbb{L}[t]}^{(t-\alpha)}$$

is an equivalence, where $M_{t-\alpha} \subseteq M$ denotes the $(t-\alpha)$-primary part of M. The target category, however, is equivalent to $\mathbf{Mod}_{\mathbb{L}[t]}^{(t)} = \mathbf{Mod}_{\mathbb{F}[t]/\mathfrak{m}}^{\mathrm{nil}}$ by a simple shift of coordinates. Since all these functors are $\mathbb{F}$-linear, the equivalence from lemma 3.7 does induce an equivalence of $K(\mathbb{F})$-module spectra on $K^{\oplus}$.

However, if $\mathfrak{m}$ is not generated by a separable polynomial, then there exists in general no isomorphism between $\widehat{\mathbb{F}[t]}^{\mathfrak{m}}$ and $(\mathbb{F}[t]/\mathfrak{m})[\![x]\!]$ *as* $\mathbb{F}$*-algebras*. For instance, let p be a prime number, let $\mathbb{F} = \mathbb{F}_p(a)$ be the field of rational functions in a variable a, and let $\mathfrak{m} = (t^p - a) \subset \mathbb{F}[t]$. Then $\widehat{\mathbb{F}[t]}^{\mathfrak{m}}$ is not isomorphic as an $\mathbb{F}$-algebra to $(\mathbb{F}[t]/\mathfrak{m})[\![x]\!]$, since the element $a \in \mathbb{F}$ admits no p^{th} root in the first ring $\widehat{\mathbb{F}[t]}^{\mathfrak{m}}$: in fact a has no p^{th} root even in the quotient ring $\mathbb{F}[t]/\mathfrak{m}^2$; yet the 'same' element a admits a p^{th} root in $(\mathbb{F}[t]/\mathfrak{m})[\![x]\!]$. We also remark that the isomorphism of rings $\widehat{\mathbb{F}[t]}^{\mathfrak{m}} \simeq (\mathbb{F}[t]/\mathfrak{m})[\![x]\!]$ proved by Cohen is non-canonical, i.e. it depends on certain choices, when $\mathbb{F}$ is non-perfect and $\mathfrak{m}$ is generated by a non-separable polynomial; on the contrary, if $\mathbb{F}$ is perfect, there exists a unique (hence canonical) isomorphism of $\mathbb{F}$-algebras $\widehat{\mathbb{F}[t]}^{\mathfrak{m}} \simeq (\mathbb{F}[t]/\mathfrak{m})[\![x]\!]$.

As a consequence of these remarks, we will have the following: if $\mathbb{F}$ is a perfect field, then the isomorphism from theorem, A will be a canonical isomorphism of $K(\mathbb{F})$-modules; but if $\mathbb{F}$ is non-perfect, then we will only obtain a non-canonical isomorphism of spectra.

Proof of theorem A. We use from lemma 2.18 that $K^{\oplus}$ is compatible with direct sums of abelian categories, and combine lemma 3.6 and lemma 3.7 to obtain

$$K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}) \simeq \bigoplus_{\substack{(t)\neq\mathfrak{m}\subset\mathbb{F}[t] \\ \text{maximal ideal}}} K^{\oplus}(\mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}}) \simeq \bigoplus_{\substack{(t)\neq\mathfrak{m}\subset\mathbb{F}[t] \\ \text{maximal ideal}}} K^{\oplus}(\mathbf{Mod}_{\mathbb{F}[t]/\mathfrak{m}}^{\mathrm{nil}}).$$

■

Remark 3.9. The argument of the proof of theorem A also works with respect to the maximal exact structure, recovering [2, Thm. 5.2].

Example 3.10. If $\mathbb{F}$ is algebraically closed, then all maximal ideals are of the form $(t-\alpha)$ with $\alpha \in \mathbb{F}$, whence in this case theorem A yields an equivalence of spectra

$$K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}}) \simeq \bigoplus_{\alpha \in \mathbb{F}\setminus\{0\}} K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}}).$$

Remark 3.11. Recall from example 3.3 that the absence of primary summands of the form $\mathbb{F}[t]/(t^r)$ is exactly what tells automorphisms from general endomorphisms. The equivalence of lemma 3.6 can hence be extended to an equivalence between $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{end}} = \mathrm{Map}(B\mathbb{N}, \mathbf{Mod}_{\mathbb{F}})$, the abelian category of vector spaces with an endomorphism, and $\bigoplus_{\mathfrak{m}\subset\mathbb{F}[t]} \mathbf{Mod}_{\mathbb{F}[t]}^{\mathfrak{m}}$, where the maximal ideal (t) is now part of the indexing set. Since lemma 3.7 can also applied in this situation, the equivalence of theorem A can be generalised to the following equivalence of spectra

$$K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{end}}) \simeq \bigoplus_{\substack{\mathfrak{m}\subseteq\mathbb{F}[t] \\ \text{maximal ideal}}} K^{\oplus}(\mathbf{Mod}_{\mathbb{F}[t]/\mathfrak{m}}^{\mathrm{nil}}).$$

4 Segal K-theory of nilpotent endomorphisms

The goal of this section is to prove theorem B, i.e. to determine the homotopy type of the spectrum $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}})$. Throughout this section, we will abbreviate the abelian category $\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}}$ by $\mathbf{N}_{\mathbb{F}}$ to save space. In order to prove theorem B, we once again employ the primary decomposition theorem from reminder 3.2.

Reminder 4.1. As in example 3.5, we identify $\mathbf{N}_{\mathbb{F}}$ with the full abelian subcategory of $\mathbf{Mod}_{\mathbb{F}[t]}$ containing all $\mathbb{F}[t]$-modules M which satisfy $\mathbf{dim}_{\mathbb{F}} M < \infty$ and $t^r M = 0$ for some integer $r \geqslant 0$.

For each integer $r \geqslant 0$, the $\mathbb{F}[t]$-module $M_r = \mathbb{F}[t]/t^r$ is called *Jordan block of size* r; we have $\mathbf{dim}_{\mathbb{F}} M_r = r$, and the $\mathbb{F}[t]$-linear morphisms $M_s \to M_r$ are all of the form $\phi_q^{r,s}: x \mapsto q\cdot x$ for some polynomial $q \in \mathbb{F}[t]$, with the additional requirement that t^{r-s} must divide q if $s < r$.

The primary decomposition theorem tells us that each object in $\mathbf{N}_{\mathbb{F}}$ is isomorphic to a direct sum of the form $M_{r_1} \oplus \cdots \oplus M_{r_n}$ with $r_1, \dots, r_n \geqslant 1$.

We first recall the *Quillen* K-theory of $\mathbf{N}_{\mathbb{F}}$, which can easily be determined by *dévissage*, a method due to Quillen that will turn out to be useful later as well:

Reminder 4.2. (Dévissage). Let $\mathbf{A}$ be an abelian category and let $\mathbf{A}' \subset \mathbf{A}$ be a full subcategory, which contains 0 and is closed under isomorphisms, as well as taking subobjects, quotients and finite products in $\mathbf{A}$. Then $\mathbf{A}'$ is itself an abelian category and the inclusion functor $\mathbf{A}' \hookrightarrow \mathbf{A}$ is exact.

We say that $\mathbf{A}'$ *filters* $\mathbf{A}$ if for each object A of $\mathbf{A}$ there is a filtration

$$0 = A^0 \subset \cdots \subset A^l = A$$

such that all quotients A^{k+1}/A^k lie in $\mathbf{A}'$. In this case, the dévissage theorem tells us that the induced map $K(\mathbf{A}') \to K(\mathbf{A})$ is an equivalence, see [33, Thm. 4], as well as [30, Cor. 5.4.6] for a version for (not necessarily connective) spectra.

Corollary 4.3. The functor $U\colon \mathbf{N}_{\mathbb{F}} \to \mathbf{Mod}_{\mathbb{F}}$ that forgets the nilpotent endomorphism induces an equivalence on Quillen K-theory.

This result is known a special case of Grayson's fundamental theorem [14, p. 236]. In our setting, it is a consequence of the primary decomposition theorem:

Proof. The functor U has an exact section S by identifying $\mathbf{Mod}_{\mathbb{F}}$ with the subcategory of $\mathbf{N}_{\mathbb{F}}$ spanned by vector spaces with the zero endomorphism; we prove that this section induces an equivalence on K. The subcategory of trivial endomorphisms is closed under isomorphisms, subobjects, quotients, and finite products, so we are left to show that it filters $\mathbf{N}_{\mathbb{F}}$. By the primary decomposition theorem, it suffices to show that each M_i admits such a filtration. This follows inductively, using that M_1 carries the trivial endomorphism and using the short exact sequences

$$0 \longrightarrow M_{r-1} \xrightarrow{\phi_t^{r,r-1}} M_r \xrightarrow{\phi_1^{1,r}} M_1 \longrightarrow 0.$$

∎

In order to understand $K^{\oplus}(\mathbf{N}_{\mathbb{F}})$, we shall study the comparison map between additive K-theory and Quillen K-theory from reminder 2.14:

Reminder 4.4. Let $\mathbf{A}$ be an abelian category. In [37], the fibre of the comparison map $\omega_{\mathbf{A}}\colon K^{\oplus}(\mathbf{A}) \to K(\mathbf{A})$ from reminder 2.14 has been identified with $K(\mathbf{A}^{\#})$, where $\mathbf{A}^{\#}$ is the following abelian category: We denote the category of abelian groups by $\mathbf{Ab}$ and let $[-,-]\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Ab}$ be the Hom-functor. Then $\mathbf{A}^{\#}$ is the full subcategory of the abelian category of additive functors $\mathbf{A}^{\mathrm{op}} \to \mathbf{Ab}$, containing those functors F such that there is an epimorphism $\beta\colon B \twoheadrightarrow B'$ in $\mathbf{A}$ with

$$F \cong \operatorname{coker}([-,\beta]\colon [-,B] \to [-,B']).$$

It is shown in [3, Prop. 2.1] that $\mathbf{A}^{\#}$ is closed under taking biproducts, kernels, and cokernels in the functor category, and hence is an abelian subcategory. We note that a sequence $0 \to F \to F' \to F'' \to 0$ in $\mathbf{A}^{\#}$ is exact if and only if $0 \to FA \to F'A \to F''A \to 0$ is an exact sequence of abelian groups for all $A \in \mathbf{A}$.

In [37] the category $\mathbf{A}^{\#}$ is denoted as $\hat{\mathbf{A}}_0$, as it is conceived in several steps. We point out that [37] only describes the underlying space of the fibre; an identification on the level of spectra can be found in [35, Thm. 9].

Lemma 4.5. We have an equivalence of spectra $K^{\oplus}(\mathbf{N}_{\mathbb{F}}) \simeq K(\mathbb{F}) \oplus K(\mathbf{N}_{\mathbb{F}}^{\#})$.

Proof. By naturality of the comparison maps $\omega_\bullet$ from reminder 2.14, we obtain a commuting square of spectra

$$\begin{array}{ccc} K^{\oplus}(\mathbf{N}_{\mathbb{F}}) & \xrightarrow{\omega_{\mathbf{N}_{\mathbb{F}}}} & K(\mathbf{N}_{\mathbb{F}}) \\ {\scriptstyle K^{\oplus}(U)}\downarrow & & \simeq \downarrow {\scriptstyle K(U)} \\ K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}) & \xrightarrow[\simeq]{\omega_{\mathbf{Mod}_{\mathbb{F}}}} & K(\mathbf{Mod}_{\mathbb{F}}), \end{array}$$

where U is the exact functor from corollary 4.3, having a section S. By functoriality, $K^{\oplus}(S)$ is a section of the left vertical map $K^{\oplus}(U)$; since the right vertical map $K(U)$ is an equivalence, we have that $K^{\oplus}(\mathbf{N}_{\mathbb{F}})$ splits, up to equivalence, into $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}) \simeq K(\mathbb{F})$ and the fibre of $\omega_{\mathbf{N}_{\mathbb{F}}}$, which is $K(\mathbf{N}_{\mathbb{F}}^{\#})$. ∎

It therefore remains to study K of the abelian category $\mathbf{N}_{\mathbb{F}}^{\#}$.

Lemma 4.6. Let F be an object of $\mathbf{N}_{\mathbb{F}}^{\#}$. Then $\mathbf{dim}_{\mathbb{F}}(FM_r) < \infty$ holds for each $r \geqslant 0$ and $FM_r = 0$ for all but finitely many r. In particular, $\mathbf{N}_{\mathbb{F}}^{\#}$ is noetherian.

Proof. There is an epimorphism $\beta\colon N = \bigoplus_{i=1}^{n} M_{r_i} \twoheadrightarrow N' = \bigoplus_{j=1}^{m} M_{s_j}$ such that F is isomorphic to the cokernel of $[-,\beta]$. In particular, FM_r is a quotient of $[M_r, N']$ and therefore finite-dimensional. Moreover, if r is larger than each of the r_i, then the map $[M_r,\beta]\colon [M_r,N] \twoheadrightarrow [M_r,N']$ is again surjective, showing that $FM_r = 0$.

To show that each ascending filtration of each object F becomes stationary, we note that for each subobject $F' \subset F$, either $F' = F$ holds or we have a strict inequality $\sum_{r\geqslant 0} \mathbf{dim}(F'M_r) < \sum_{r\geqslant 0} \mathbf{dim}(FM_r)$ of non-negative integers. ■

Corollary 4.7. The spectra $K^{\oplus}(\mathbf{N}_{\mathbb{F}}) = K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{nil}})$ and $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})$ are connective.

Proof. Since $\mathbf{N}_{\mathbb{F}}^{\#}$ is noetherian, the spectrum $K(\mathbf{N}_{\mathbb{F}}^{\#})$ is connective by reminder 2.12. Using lemma 4.5 (or just the fibre sequence from reminder 4.4), it follows that also $K^{\oplus}(\mathbf{N}_{\mathbb{F}})$ is connective. For $K^{\oplus}(\mathbf{Mod}_{\mathbb{F}}^{\mathrm{aut}})$, we now employ theorem A. ■

In order to finally determine the homotopy type of $K(\mathbf{N}_{\mathbb{F}}^{\#})$, we use the following consequence of dévissage, as in [33, Cor. 1]:

Reminder 4.8. Let $\mathbf{A}$ be an abelian category. We call an object $S \in \mathbf{A}$ *simple* if 0 and S are the only subobjects of S. Assume that $\{S_i\}_{i\in I}$ is a set of representatives for the isomorphism classes of simple objects of $\mathbf{A}$. If every object A of $\mathbf{A}$ has *finite length*, i.e. it admits a filtration $0 = A^0 \subset \cdots \subset A^l = A$ such that each quotient A^{k+1}/A^k is simple, then the dévissage theorem provides an equivalence of spectra

$$K(\mathbf{A}) \simeq \bigoplus_{i\in I} K(\mathbf{Mod}_{\mathrm{End}(S_i)^{\mathrm{op}}}),$$

where $\mathrm{End}(S_i)$ is the ring of endomorphisms of S_i (which, by Schur's lemma, is actually a skew field).

It is hence our aim to classify the simple objects of $\mathbf{N}_{\mathbb{F}}^{\#}$ and to understand their endomorphism rings. We start with a general observation.

Remark 4.9. Each additive functor $F\colon (\mathbf{N}_{\mathbb{F}})^{\mathrm{op}} \to \mathbf{Ab}$ has a canonical lift to an $\mathbb{F}[t]$-linear functor $\tilde{F}\colon (\mathbf{N}_{\mathbb{F}})^{\mathrm{op}} \to \mathbf{Mod}_{\mathbb{F}[t]}$ along the forgetful map $\mathbf{Mod}_{\mathbb{F}[t]} \to \mathbf{Ab}$ as follows: Given M in $\mathbf{N}_{\mathbb{F}}$, we define an $\mathbb{F}[t]$-module structure on FM by defining the scalar multiplication by $q \in \mathbb{F}[t]$ to be the map $F(M \to M, x \mapsto qx)$. Similarly, a natural transformation $\lambda\colon F \to F'$ between additive functors F and F' gives a natural transformation $\tilde{\lambda}\colon \tilde{F} \to \tilde{F}'$ between the corresponding lifts $\tilde{F}$ and $\tilde{F}'$.
By abuse of notation, we will henceforth regard $\mathbf{N}_{\mathbb{F}}^{\#}$ as a subcategory of the category of $\mathbb{F}[t]$-linear functors $\mathbf{N}_{\mathbb{F}}^{\mathrm{op}} \to \mathbf{Mod}_{\mathbb{F}[t]}$.

Definition 4.10. For $r \geqslant 1$ we define F_r to be the cokernel of $[-, \beta_r]$, where

$$\beta_r\colon M_{r-1} \oplus M_{r+1} \twoheadrightarrow M_r, \qquad (x, y) \mapsto tx + y,$$

Lemma 4.11. For all $r, s \geqslant 1$, we have isomorphisms of $\mathbb{F}[t]$-modules

$$F_r M_s \cong \begin{cases} M_1 & \text{for } r = s, \\ 0 & \text{else.} \end{cases}$$

Proof. We note that $\beta_r|_{M_{r-1}} = \phi_t^{r,r-1}$ and $\beta_r|_{M_{r+1}} = \phi_1^{r,r+1}$, and moreover:

- if $s < r$, then $[-, \phi_t^{r,r-1}]\colon [M_s, M_{r-1}] \to [M_s, M_r]$ is surjective,
- if $s > r$, then $[-, \phi_1^{r,r+1}]\colon [M_s, M_{r+1}] \to [M_s, M_r]$ is surjective.

Finally, the evaluation $\mathrm{ev}\colon [M_r, M_r] \to M_r$ with $\mathrm{ev}(\phi) = \phi(1)$ is an isomorphism, and we have $\mathrm{ev}(\mathrm{im}([M_r, \beta_r])) = tM_r$, and hence $M_r/tM_r \cong M_1$. ■

Lemma 4.12. For each $F \neq 0$ in $\mathbf{N}_{\mathbb{F}}^{\#}$, there is an $r \geqslant 1$ and a monomorphism $F_r \hookrightarrow F$.

Proof. Let $r \geqslant 1$ be minimal such that $F\phi_1^{r,r+1}\colon FM_r \to FM_{r+1}$ is not injective: such an r exists because $F \neq 0$ and because of lemma 4.6. Moreover, we pick an element $v \in \ker(F\phi_1^{r,r+1}) \subseteq FM_r$ with $v \neq 0$. We can additionally achieve that $t \cdot v = 0$, by repeatedly replacing v by $t \cdot v$ as long as this condition is not satisfied.

The Yoneda lemma gives us a natural transformation $Y(v)\colon [-, M_r] \to F$ that sends $\mathrm{id}_{M_r} \in [M_r, M_r]$ to v. The composition $Y(v) \circ [-, \phi_1^{r,r+1}]\colon [-, M_{r+1}] \to F$ is the map $Y((F\phi_1^{r,r+1})(v))$, which is trivial since v lies in the kernel of $F\phi_1^{r,r+1}$. In a similar way, the composition $Y(v) \circ [-, \phi_t^{r,r-1}]\colon [-, M_{r-1}] \to F$ is the map $Y((F\phi_t^{r,r-1})(v))$; and again we have $(F\phi_t^{r,r-1})(v) = 0$, as can be checked by composing with the injective map $F\phi_1^{r-1,r}$ to get $F(\phi_1^{r-1,r} \circ \phi_t^{r,r-1})(v) = F(t \cdot \mathrm{id}_{M_r})(v) = t \cdot v = 0$.

Combining the two previous observations, we obtain that $Y(v) \circ [-, \beta_r]$ vanishes, whence we get an induced map $\lambda_v\colon F_r \to F$. This natural transformation is injective when evaluated on M_r, as it gives the map $M_1 \to FM_r$ sending the standard generator to v; it is also trivially injective when evaluated on the other objects M_s, and by additivity it is injective on every object, i.e. it is a monomorphism in $\mathbf{N}_{\mathbb{F}}^{\#}$. ■

Corollary 4.13. An object $F \in \mathbf{N}_{\mathbb{F}}^{\#}$ is simple if and only if there is an $r \geqslant 1$ with $F \cong F_r$.

Proof. The 'if'-part is implied by lemma 4.11 and the fact that M_1 has no non-trivial submodules, while the 'only if'-part is implied by lemma 4.12. ■

Now we have everything together to prove theorem B:

Proof of theorem B. The combination of lemma 4.12 and lemma 4.6 shows that each F in $\mathbf{N}_{\mathbb{F}}^{\#}$ has finite length. Moreover, by corollary 4.13, $\{F_r\}_{r\geqslant 1}$ is a system of representatives for the isomorphism classes of simple objects of $\mathbf{N}_{\mathbb{F}}^{\#}$. Finally, we note that endomorphisms $\eta : F_r \to F_r$ are uniquely determined by their component $\eta_{M_r} : FM_r \to FM_r$, which is an $\mathbb{F}$-linear map $\mathbb{F} \to \mathbb{F}$. Thus, the endomorphism ring of each F_r is isomorphic to $\mathbb{F} = \mathbb{F}^{\mathrm{op}}$. By applying reminder 4.8, we get

$$K(\mathbf{N}_{\mathbb{F}}^{\#}) \simeq \bigoplus_{r\geqslant 1} K(\mathbb{F}).$$

Now theorem B follows from lemma 4.5. ■

5 The topological case

The goal of this section is to prove theorem C. We start by recalling the Quillen plus-construction and the group-completion theorem in the language of ∞-categories.

Reminder 5.1. ([21], [19, Prop. III.13], [20, Constr. 3.2.18]). A discrete group G is said to be *hypoabelian* if for every non-trivial subgroup $H \subset G$, the abelianisation of H is non-trivial. A space X is *hypoabelian* if the fundamental group $\pi_1(X, *)$ is hypoabelian for every choice of basepoint $* \in X$.

We have a full subcategory $\mathbf{S}_{\mathrm{hypo}} \subset \mathbf{S}$ spanned by hypoabelian spaces. Moreover, the inclusion functor $\iota\colon \mathbf{S}_{\mathrm{hypo}} \hookrightarrow \mathbf{S}$ admits a left-adjoint functor $(-)^+\colon \mathbf{S} \to \mathbf{S}_{\mathrm{hypo}}$, called the *Quillen plus-construction*. If $X \in \mathbf{S}$ is a hypoabelian space, then the unit of the adjunction $X \to \iota(X^+)$ is an equivalence.

Reminder 5.2. The ∞-category $\mathbf{CMon}(\mathbf{S})$ of E_∞-algebras in spaces contains a full subcategory $\mathbf{CMon}(\mathbf{S})^{\mathrm{gp}}$ that is spanned by group-like objects, i.e. objects A such that the commutative monoid $\pi_0(A)$ is in fact an abelian group. The inclusion of $\mathbf{CMon}(\mathbf{S})^{\mathrm{gp}}$ into $\mathbf{CMon}(\mathbf{S})$ has a left-adjoint, the *group-completion* functor, which agrees with the composite $\Omega^\infty B^\infty$. For each object A in $\mathbf{CMon}(\mathbf{S})$, the underlying space of $\Omega^\infty B^\infty A$ has abelian fundamental group for each choice of basepoint, and is in particular hypoabelian.

For any A in $\mathbf{CMon}(\mathbf{S})$ and any $x \in A$, we define the *mapping telescope* $\mathrm{Tel}_x A$ to be the colimit in $\mathbf{S}$ of the diagram $(A \xrightarrow{-\cdot x} A \xrightarrow{-\cdot x} A \xrightarrow{-\cdot x} \ldots b)$. We have a canonical map of spaces $\mathrm{Tel}_x A \to \Omega^\infty B^\infty A$, and since the target is hypoabelian, this induces a canonical map of hypoabelian spaces $(\mathrm{Tel}_x A)^+ \to \Omega^\infty B^\infty A$.

We say that $x \in A$ is a *propagator* if, for every $y \in A$, there exists $z \in A$ and $k \geqslant 0$ such that x^k and yz are in the same path-component of A. The group-completion theorem [29, 31, 34] asserts that if x is a propagator for A, then the canonical map $(\mathrm{Tel}_x A)^+ \to \Omega^\infty B^\infty A$ is an equivalence of spaces.

Proof of theorem C. In the first part of the proof, we use the symbols "$O_{\mathbb{C}}$" and "$O_{\mathbb{R}}$" as aliases of "U" and "O", respectively, and we let $\mathbb{F}$ be $\mathbb{C}$ or $\mathbb{R}$. We let $X \in \mathbf{S}^\omega$ be a compact anima, e.g. the underlying homotopy type of a finite cell complex.

We note that $A_{\mathbb{F}} = \mathbf{Mod}^{\simeq}_{\mathbb{F},\mathrm{top}}$ is equivalent, as a space, to the disjoint union of classifying spaces $\coprod_{d\geqslant 0} BO_{\mathbb{F}}(d)$. Let $x \in A_{\mathbb{F}}$ be a point in $BO_{\mathbb{F}}(1)$ and let $c_x \colon X \to A_{\mathbb{F}}$ be the constant map with value x. Then we have a commutative diagram of spaces

$$\begin{array}{ccccc} \mathrm{Tel}_{c_x}\mathrm{Map}(X, A_{\mathbb{F}}) & \longrightarrow & (\mathrm{Tel}_{c_x}\mathrm{Map}(X, A_{\mathbb{F}}))^+ & \longrightarrow & \Omega^\infty B^\infty \mathrm{Map}(X, A_{\mathbb{F}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(X, \mathrm{Tel}_x A_{\mathbb{F}}) & \longrightarrow & \mathrm{Map}(X, (\mathrm{Tel}_x A_{\mathbb{F}})^+) & \longrightarrow & \mathrm{Map}(X, \Omega^\infty B^\infty A_{\mathbb{F}}). \end{array}$$

Our goal is to prove that the right vertical map is an equivalence of spaces, and we will do so by showing that every horizontal map and the left vertical map are equivalences. For the left vertical map, we use that X is a compact space, whence $\mathrm{Map}(X, -)$ commutes with filtered colimits.

For the rightmost horizontal maps, we invoke the group-completion theorem, using that x and c_x are propagators in $A_{\mathbb{F}}$ and in $\mathrm{Map}(X, A_{\mathbb{F}})$, respectively; to justify the latter, note that a point $y \in \mathrm{Map}(X, A_{\mathbb{F}})$, i.e. a map $y \colon X \to A_{\mathbb{F}} \simeq \coprod_{d\geqslant 0} BO_{\mathbb{F}}(d)$, classifies a $\mathbb{F}$-vector bundle over X of some rank d; if we let $z \colon X \to BO_{\mathbb{F}}(d')$ be a map classifying a complementary vector bundle over X (i.e. the direct sum of the two vector bundles is a trivial vector bundle of rank $d + d'$), then we have $yz \simeq c_x^{d+d'}$.

For the left horizontal maps, we use that $\mathrm{Tel}_x A_{\mathbb{F}}$ and $\mathrm{Tel}_{c_x}\mathrm{Map}(X, A_{\mathbb{F}})$ are already hypoabelian, which we see as follows: First, $\mathrm{Tel}_x A_{\mathbb{F}}$ is equivalent to $\mathbb{Z} \times BO_{\mathbb{F}}$, and this space has an abelian fundamental group for each choice of basepoint; and second, we see $\mathrm{Tel}_{c_x}\mathrm{Map}(X, A_{\mathbb{F}}) \simeq \mathrm{Map}(X, \mathrm{Tel}_x A_{\mathbb{F}}) \simeq \mathbf{lim}_{x\in X} \mathrm{Tel}_x A_{\mathbb{F}}$, and the full subcategory $\mathbf{S}_{\mathrm{hypo}} \subset \mathbf{S}$ is closed under limits.

This altogether shows that the right vertical map is an equivalence of spaces, and since it additionally is a morphism of E_∞-algebras, this concludes the proof of the first statement of theorem C.

For the second statements about the case of $X = S^1$, let us denote by $E\langle k\rangle$ the k-connective cover of a given spectrum E. Then we use Bott periodicity for complex K-theory to obtain and equivalence of spectra

$$\mathrm{Map}(S^1, \mathbb{Z} \times BU)\langle 0\rangle \simeq \mathrm{ku} \oplus (\Sigma^{-1}\mathrm{ku})\langle 0\rangle \simeq \mathrm{ku} \oplus \Sigma\mathrm{ku} \simeq \Sigma^\infty_+ S^1 \wedge \mathrm{ku},$$

and we use the Wood cofibre sequence for the 'realification' [43], which can be written as $\mathrm{ko}\langle 1\rangle \to \Sigma^2\mathrm{ku} \xrightarrow{\Sigma^2 r} \Sigma^2\mathrm{ko}$, implying that $(\Sigma^{-1}\mathrm{ko})\langle 0\rangle \simeq \Sigma^{-1}(\mathrm{ko}\langle 1\rangle)$ is the cofibre of r, to obtain an equivalence of spectra

$$\begin{aligned} \mathrm{Map}(S^1, \mathbb{Z} \times BO)\langle 0\rangle &\simeq \mathrm{ko} \oplus (\Sigma^{-1}\mathrm{ko})\langle 0\rangle \\ &\simeq \mathrm{ko} \oplus \mathrm{cofib}(r) \\ &\simeq \mathrm{colim}\big(\mathrm{ko} \xleftarrow{r} \mathrm{ku} \xrightarrow{r} \mathrm{ko}\big). \end{aligned}$$
■

Funding

A. B. was partially supported by the *European Research Council* under the *European Union's Horizon 2020 research and innovation programme* (grant agreement No. 772960), by the *Danish National Research Foundation* through the *Copenhagen Centre for Geometry and Topology* (DNRF151), and by the Max Planck Institute for Mathematics in Bonn.

Acknowledgments

We would like to thank Søren Galatius and Jens Reinhold for sparking our interest in mapping spaces into E_1-algebras and their group-completion, and for drawing our attention to the work of Grayson. We furthermore owe thanks to Lars Hesselholt and Bjørn Ian Dundas for making us aware of Auslander and Sherman's results. In addition to that, we are grateful to Simon Gritschacher, Kaif Hilman, Manuel Krannich, and Christian Mauz for helpful conversations on the subject. Finally, we would like to thank the anonymous referee for many inspiring comments, which significantly improved the paper, including a way to get around a superfluous assumption in section 3.

References

1. Almkvist, G. "The Grothendieck ring of the category of endomorphisms." *J. Algebra* **28** (1974): 375–88. https://doi.org/10.1016/0021-8693(74)90047-7.
2. Almkvist, G. "K-theory of endomorphisms." *J. Algebra* **55**, no. 2 (1978): 308–40. https://doi.org/10.1016/0021-8693(78)90224-7.
3. Auslander, M. "Coherent functors." *Proceedings of the Conference on Categorical Algebra*, edited by Eilenberg S., Harrison D. K., MacLane S., and Röhrl H. 189–231. Berlin, Heidelberg: Springer, 1966.
4. Barwick, C. "On exact ∞-categories and the Theorem of the Heart." *Comp. Math.* **151**, no. 11 (2015): 2160–86. https://doi.org/10.1112/S0010437X15007447.
5. Bianchi, A., F. Kranhold, and J. Reinhold. "Parametrised moduli spaces of surfaces as infinite loop spaces." *Forum Math. Sigma* **10** (2022). https://doi.org/10.1017/fms.2022.29.
6. Blumberg, A. J., D. Gepner, and G. Tabuada. "A universal characterizsation of higher algebraic k-theory." *Geom. Topol.* **17** (2013): 733–838. https://doi.org/10.2140/gt.2013.17.733.
7. Blumberg, A. J., D. Gepner, and G. Tabuada. "K-theory of endomorphisms via noncommutative motives." *Trans. Amer. Math. Soc.* **368**, no. 2 (2016): 1435–65. https://doi.org/10.1090/tran/6507.
8. Bohmann, A. M. and A. M. Osorno. "A multiplicative comparison of Segal and Waldhausen K-theory." *Math. Z.* **295** (2020): 1205–43. https://doi.org/10.1007/s00209-019-02394-7.
9. Borel, A. "Stable real cohomology of arithmetic groups. II." *Manifolds and Lie Groups, number 14 in Progress in Mathematics*, edited by Hano J., Morimoto A., Murakami S., Okamoto K., and Ozeki H. 21–55. Boston: Birkhäuser, 1981.
10. Cohen, I. "On the structure and ideal theory of complete local rings." *Trans. Amer. Math. Soc.* **59** (1946): 54–106. https://doi.org/10.1090/S0002-9947-1946-0016094-3.
11. Elmanto, E. and V. Sosnilo. "On nilpotent extensions of ∞-categories and the cyclotomic trace." *Internat. Math. Res. Notices* **2022** (2022): 16569–633. https://doi.org/10.1093/imrn/rnab179.
12. Gepner, D., M. Groth, and T. Nikolaus. "Universality of multiplicative infinite loop space machines." *Algebraic & Geometric Topology* **15**, no. 6 (2015): 3107–53. https://doi.org/10.2140/agt.2015.15.3107.
13. Gillet, H. and D. R. Grayson. "The loop space of the Q-construction." *Illinois J. Math.* **31**, no. 4 (1987): 574–97. https://doi.org/10.1215/ijm/1256063571.
14. Grayson, D. R. "Higher algebraic K-theory: II." *Algebraic K-theory, 551 in Lecture Notes in Mathematics*, edited by Stein M. R. 217–40. Berlin, Heidelberg: Springer, 1976.
15. Grayson, D. R. "The K-theory of endomorphisms." *J. Algebra* **48** (1976): 439–46. https://doi.org/10.1016/0021-8693(77)90320-9.
16. Grayson, D. R. "K_2 and the K-theory of automorphisms." *J. Algebra* **58** (1979): 12–30.
17. Grayson, D. R. "Weight filtrations via commuting automorphisms." *K-Theory* **9**, no. 2 (1995): 139–72. https://doi.org/10.1007/BF00961457.
18. Hebestreit, F. and W. Steimle. *Stable moduli spaces of hermitian forms*, 2021.
19. Hebestreit, F. and F. Wagner. Algebraic and hermitian K-theory, 2021. Lecture notes available athttps://sites.google.com/view/fabian-hebestreit/home/lecture-notes.
20. Hilman, K. and J. McCandless. Algebraic K-theory, 2024. Lecture notes available athttps://sites.google.com/view/jonasmccandless/introduction-to-algebraic-k-theory.
21. Hoyois, M. *On Quillen's plus construction*, 2019. Available athttps://hoyois.app.uni-regensburg.de/papers/acyclic.pdf.
22. Jardine, J. F. "The multiple Q-construction." *Canad. J. Math.* **39** (1987): 1174–209. https://doi.org/10.4153/CJM-1987-060-0.
23. Land, M. "Introduction to Infinity-Categories." *Compact Textbooks in Mathematics*. Birkhäuser: Cham, 2021.
24. Land, M., A. Mathew, L. Meier, and G. Tamme. "Purity in chromatically localized algebraic K-theory." *J. Amer. Math. Soc.* **37** (2024): 1011–40. https://doi.org/10.1090/jams/1043.
25. Lurie, J. *Higher Topos Theory*. *Number 170 in Annals of Mathematics Studies*. Princeton: Princeton University Press, 2009.
26. Lurie, J. *Higher Algebra*. Cambridge, Massachusetts: Harvard University, 2017.

27. May, J. P. "E_∞ spaces, group completions, and permutative categories." *New Developments in Topology, number 11 in London Mathematical Society Lecture Note Series*, edited by Segal G. B. 61–94. Cambridge: Cambridge University Press, 1974.
28. May, J. P. "The spectra associated to permutative categories." *Topol.* **17**, no. 3 (1978): 225–8. https://doi.org/10.1016/0040-9383(78)90027-7.
29. McDuff, D. and G. B. Segal. "Homology fibrations and the "group-completion" theorem." *Invent. Math.* **31** (1976): 279–84. https://doi.org/10.1007/BF01403148.
30. Mochizuki, S. "A dèvissage theorem of non-connective K-theory." (2019).
31. Nikolaus, T. *The group completion theorem via localizations of ring spectra*, 2017. Available athttps://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Group_completion.pdf.
32. Quillen, D. "On the cohomology and K-theory of the general linear groups over a finite field." *Ann. of Math. (2)* **96**, no. 3 (1972): 552–86. https://doi.org/10.2307/1970825.
33. Quillen, D. "Higher algebraic K-theory: I." *Higher K-theories, 341 in Lecture Notes in Mathematics*, edited by Bass H. 85–147. Berlin, Heidelberg: Springer, 1973.
34. Randal-Williams, O. ""Group-Completion", local coefficient systems, and perfection." *Quart. J. Math. Oxford Ser. (2)* **64** (2013): 795–803. https://doi.org/10.1093/qmath/hat024.
35. Schlichting, M. "Negative K-theory of derived categories." *Math. Z.* **253** (2006): 97–134. https://doi.org/10.1007/s00209-005-0889-3.
36. Segal, G. "Categories and cohomology theories." *Topol* **13** (1974): 293–312. https://doi.org/10.1016/0040-9383(74)90022-6.
37. Sherman, C. "On the homotopy fiber of the map $BQA^{\oplus} \to BQA$ (after M. Auslander)." *Contemp. Math.* **83** (1989): 343–8.
38. Shimakawa, K. "Multiple categories and algebraic K-theory." *J. Pure Appl. Algebra* **41** (1986): 285–304. https://doi.org/10.1016/0022-4049(86)90114-3.
39. Stienstra, J. "Operations in the higher K-theory of endomorphisms." *Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., volume 2*. 59–115. Amer. Math. Soc, 1982.
40. Thomason, R. W. "Homotopy colimits in the category of small categories." *Math. Proc. Cambr. Phil. Soc.* **85**, no. 1 (1979): 91–109. https://doi.org/10.1017/S0305004100055535.
41. Waldhausen, F. "Algebraic K-theory of spaces." *Algebraic and Geometric Topology, 1126 in Lecture Notes in Mathematics*, edited by Ranicki A., Levitt N., and Quinn F. 318–419. Berlin, Heidelberg: Springer, 1985.
42. Weibel, C. A. *The K-book*. An Introduction to Algebraic K-theory, Number 145 in Graduate Studies in Mathematics. American Mathematical Society, 2013.
43. Wolbert, J. J. "Classifying modules over K-theory spectra." **124** (1998): 289–323.

International Mathematics Research Notices, 2025, 1–15
https://doi.org/10.1093/imrn/rnaf028
Article