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On the Sobolev Stability Threshold for the 2D MHD Equations with Horizontal Magnetic Dissipation

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Abstract
In this article, we consider the stability threshold of the 2D magnetohydrodynamics (MHD) equations near a combination of Couette flow and large constant magnetic field. We study the partial dissipation regime with full viscous and only horizontal magnetic dissipation. In particular, we show that this regime behaves qualitatively differently than both the fully dissipative and the non-resistive setting.

Keywords Magnetohydrodynamics · Partial dissipation · Stability threshold

Mathematics Subject Classification 76E25 · 76E30 · 76E05

Contents

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1 Introduction

The equations of magnetohydrodynamics (MHD)

$$\begin{aligned}\partial_t V + V\cdot\nabla V + \nabla\Pi &= (\nu_x\partial_x^2 + \nu_y\partial_y^2)V + B\cdot\nabla B,\\ \partial_t B + V\cdot\nabla B &= (\kappa_x\partial_x^2 + \kappa_y\partial_y^2)B + B\cdot\nabla V,\\ \nabla\cdot v = \nabla\cdot b &= 0,\\ (t,x,y) &\in \mathbb{R}^+\times\mathbb{T}\times\mathbb{R},\end{aligned} \tag{1}$$

model the evolution of the velocity V of conducting, non-magnetic fluids interacting with a magnetic field B. The MHD equations are commonly used in applications ranging from astrophysics and the description of plasmas to control problems for liquid metals in industrial applications (Davidson 2016). Similarly to the Navier–Stokes and Euler equations, questions of hydrodynamic stability and the behavior for high Reynolds numbers (that is, for ν,κ tending to zero) are a very active area of research both inner-mathematically and in view of applications.

Motivated by stability results for the isotropic full dissipation case ($\nu_x = \nu_y = \kappa_x = \kappa_y > 0$) and instability results for the non-resistive case ($\kappa_x = \kappa_y = 0$), we are interested in the behavior of the two-dimensional magnetohydrodynamic (MHD) equations with partial dissipation, where some of the dissipation coefficients

$$\kappa_y, \kappa_x, \nu_x, \nu_y \geq 0,$$

are allowed to vanish. More specifically, we study the behavior near the stationary solution given by the combination of Couette flow and a (large) constant magnetic field

$$V_s = ye_1, \quad B_s = \alpha e_1, \tag{2}$$

for the case of vanishing vertical resistivity, $\kappa_y = 0$. By the symmetry $B \mapsto -B$, we consider the case $\alpha \geq 0$. For the related case of the Navier–Stokes equations (that is, without any magnetic field), the (in)stability of Couette flow at high Reynolds number is known as the Sommerfeld paradox (Majda and Andrea 2001) and is related to nonlinear instability of the Euler equations (Bedrossian and Masmoudi 2015; Deng and Masmoudi 2018; Deng and Zillinger 2021).

However, for the case of sufficiently small data it was proven in Bedrossian et al. (2018) that (mixing enhanced) dissipation can counteract this instability in the Navier–Stokes equations and that (longtime asymptotic) stability holds in Sobolev spaces for initial data with

$$\|\omega\|_{H^N} \leq \epsilon \ll \nu^\gamma$$

with $\gamma \geq \frac{1}{2}$. Later in Masmoudi and Zhao (2022) this has been improved to $\gamma = \frac{1}{3}$. This is an example of a stability threshold result, which establishes stability for small data and determines suitable (optimal) exponents γ for given norms.

Since the addition of the magnetic field is known to possibly destabilize the dynamics (see the following discussion), our main questions concern the MHD equations (1) in terms of perturbations moving with the underlying shear flow:

$$
\begin{aligned}
v(x, y, t) &= V(x - yt, y, t) - V_s, \\
b(x, y, t) &= B(x - yt, y, t) - B_s.
\end{aligned}
$$

The corresponding perturbed equations in these new variables read

$$
\begin{aligned}
\partial_t v + v_2 e_1 - 2\partial_x \Delta_t^{-1} \nabla_t v_2 &= \nu \cdot \Delta_t v + \alpha \partial_x b + b \nabla_t b - v \nabla_t v - \nabla_t \pi, \\
\partial_t b - b_2 e_1 &= \kappa \cdot \Delta_t b + \alpha \partial_x v + b \nabla_t v - v \nabla_t b, \\
\nabla_t \cdot v = \nabla_t \cdot b &= 0.
\end{aligned} \tag{3}
$$

Here, we introduce the time-dependent derivatives $\partial_y^t = \partial_y - t\partial_x$, $\nabla_t = (\partial_x, \partial_y^t)$ and $\Delta_t = \partial_x^2 + (\partial_y^t)^2$. Furthermore, we use the following short notation for the dissipation operator:

$$
\begin{aligned}
\nu \cdot \Delta_t &= \nu_x \partial_x^2 + \nu_y (\partial_y^t)^2, \\
\kappa \cdot \Delta_t &= \kappa_x \partial_x^2 + \kappa_y (\partial_y^t)^2.
\end{aligned}
$$

In this article, we aim to establish a Sobolev stability threshold for (3) for the specific anisotropic, partial dissipation case

$$
\kappa_y = 0, \ \kappa_x = \nu_x = \nu_y > 0.
$$

In particular, we show that this setting exhibits qualitatively different behavior than the fully dissipative and the non-resistive case.

Following a similar notation as Liss (2020), we make the following definition.

Definition 1.1 (*Stability threshold*) Consider the MHD equations (1) with anisotropic dissipation $0 < \nu_x = \nu_y = \kappa_x =: \mu \ll 1$ and $\kappa_y = 0$ and let X be a Banach space with norm $\|(v, b)\|_X$. We then say that the exponent $\gamma = \gamma(X)$ is a stability threshold for the space X if for initial data with

$$
\|(v_{\text{in}}, b_{\text{in}})\|_X \leq \epsilon \ll \mu^\gamma,
$$

the corresponding solution of (3) remains uniformly bounded for all future times with a quantitative control

$$
\sup_{t>0} \|(v, b)\|_X \lesssim \epsilon.
$$

We remark that this definition does not require optimality (that is, instability for smaller choices of γ). Optimal stability thresholds quantify the appearance of instability in the large Reynolds number limit and are an active area of research for many fluid systems.

In view of the large literature, the interested reader is referred to the following articles for the Navier–Stokes equations (Bedrossian et al. 2018, 2017) and the Boussinesq equations (Zhai and Zhao 2023; Lai et al. 2021; Tao et al. 2020) for a discussion and further references.

For the (isotropic) MHD equations ($\nu := \nu_x = \nu_y$ and $\kappa := \kappa_x = \kappa_y$), there exists several results for non-vanishing magnetic dissipation.

- When considering full isotropic dissipation $\nu = \kappa > 0$, Liss Liss (2020) established a Sobolev threshold in the 3D case. Under a Diophantine condition on the magnetic field, he establishes stability for $\|(v, b)\|_{H^N}$ with $\gamma = 1$. For the 2D case, an improvement to $\gamma = \frac{2}{3}$ is expected due to the lack of lift-up instability. Indeed, in a very recent paper, Dolce (2023), Dolce establishes such a threshold for the regime $0 < C\kappa^3 \leq \nu \leq \kappa$.
- In the 2D inviscid case with isotropic magnetic dissipation, $\nu = 0$ and $\kappa > 0$, in Knobel and Zillinger (2023), the authors established linear instability of nearby (in analytic regularity) so-called traveling wave type solutions in Gevrey 2 regularity. As an (almost) matching nonlinear result, Zhao and Zi (2023) established a stability threshold $\gamma \geq 1$ for Gevrey $2 - \delta$ regularity for any $0 < \delta < 1$.
- The setting with only an underlying magnetic field but without shear flow exhibits qualitatively different behavior and was studied for the case of the whole space in Bardos et al. (1988); Ren et al. (2014) in the full dissipation case and in Cao et al. (2013); Ji et al. (2019) for the partially dissipative case.

To the authors' knowledge, there are no such results in the literature for the non-resistive case $\kappa = 0$ with Couette flow, both for the viscous or inviscid regime $\nu = 0$ or $\nu > 0$, and neither for partial dissipation regimes. In view of linear instability results (Hussain et al. 2018) (see also Proposition 1), for these equations any stability threshold results would need to consider unknowns different from (v, b).

As a step toward understanding this non-resistive regime, in this article we consider the 2D MHD equations with isotropic viscosity but only horizontal resistivity (while (Liss 2020; Dolce 2023) consider full dissipation). In particular, we ask to which extent, as quantified by Sobolev stability thresholds, this partial dissipation regime behaves or does not behave like these extremal cases.

In the (ideal) MHD equations ($\nu = \kappa = 0$), the interaction of shear flows and the magnetic field has been shown to possibly cause instabilities, with arguments both on physical (Chen et al. 1991; Hirota et al. 2005) and mathematical grounds (Hughes and Tobias 2001; Zhai et al. 2021).

As our first result, we show that this instability also persists in the viscous but non-resistive MHD. These equations exhibit norm inflation in H^N for all choices of $\nu > 0$.

Proposition 1 (Instability for the non-resistive MHD equations) *Consider the isotropic equation with $1 \geq \nu > 0$, $\kappa = 0$, $\alpha > \frac{1}{2}$ and $N \geq 3$, then the stationary solution (2) is linearly unstable in H^N. More precisely, there exists initial data $(v, b)_{\text{in}} \in H^N$ such that the solution to the linearized problem satisfies*

$$\|(v, b)\|_{H^N} \approx \langle \nu t \rangle \|(v, b)_{\text{in}}\|_{H^N}$$

as $t \to \infty$. This is optimal in the sense that for all initial data $(v, b)_{\text{in}} \in H^N$ the solution to the linearized problem satisfies

$$\|(v, b)\|_{H^N} \lesssim \langle \nu t \rangle \|(v, b)_{\text{in}}\|_{H^N}.$$

The implicit constants here may depend on α. As a consequence, the nonlinear equations also exhibit arbitrarily large norm inflation in H^N. That is, for any $C = C(\nu) > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there exists initial data $(v, b)_{\text{in}}$ and a time T such that

$$\begin{aligned} \|(v, b)_{\text{in}}\|_{H^N} &= \varepsilon, \\ \|(v, b)|_{t=T}\|_{H^N} &\geq C \|(v, b)_{\text{in}}\|_{H^N}. \end{aligned}$$

In particular, there cannot exist a Sobolev threshold for $\|(v, b)\|_{H^N}$.

We remark that following the same argument also instability in suitable Gevrey spaces can be established.

As mentioned above, the isotropic fully dissipative case is known to be stable in Sobolev regularity (Liss 2020; Dolce 2023). For the associated partial dissipation regimes, in view of the underlying shear dynamics the associated vertical dissipation case is expected to behave similarly as the full dissipation case. The effects of partial dissipation are a very actively studied field of research in other fluid systems, such as the Boussinesq equations (Deng et al. 2021; Cao and Jiahong 2013; Adhikari et al. 2022)), but, to the authors' knowledge, is largely open in the MHD equations near Couette flow.

In the present case of horizontal resistivity, $\kappa_y = 0$ and $\nu_x = \nu_y = \kappa_x$, the lack of vertical dissipation leads to stronger instabilities, requiring finer control and use of the coupling by a strong magnetic field. Our main results are summarized in the following theorem.

Theorem 2 *Consider the MHD equations with horizontal resistivity, $\mu := \nu_x = \nu_y = \kappa_x > 0$ and $\kappa_y = 0$, near the stationary solution (2) with $\alpha > \frac{1}{2}$ and let $N \geq 6$ be given.*

Then there exist constants $c_0 = c(\alpha) > 0$, such that for all initial data $(v, b)_{in}$ which satisfy

$$\|(v, b)_{\text{in}}\|_{H^N} = \varepsilon \leq c_0 \mu^{\frac{3}{2}}$$

the corresponding solution (v, b) of (4) satisfies the estimates

$$\begin{aligned} \|v\|_{L^\infty H^N} + \mu^{\frac{1}{2}} \|\nabla_t v\|_{L^2 H^N} &\lesssim \varepsilon, \\ \|b\|_{L^\infty H^N} + \mu^{\frac{1}{2}} \|\partial_x b\|_{L^2 H^N} &\lesssim \varepsilon. \end{aligned}$$

Let us comment on these results:

- Proposition 1 shows instability in terms of (v, b) for the non-resistive case. Hence, the (horizontal) magnetic dissipation is shown to be necessary for longtime stability results for (v, b).
  However, similarly as in the Boussinesq equations (Bedrossian et al. 2023; Zillinger 2021a), in principle stability results in terms of other unknowns such as the magnetic potential $\phi = (-\Delta_t)^{-1}\nabla_t^{\perp} b$ could hold for longer or even infinite times, which remains an exciting question for future research.
- Theorem 2 establishes a stability threshold $\gamma = \frac{3}{2}$. In particular, we stress that the lack of vertical magnetic dissipation not only poses a key challenge of our analysis but results in a different threshold value than the fully dissipative setting (Liss 2020; Dolce 2023).
  Indeed, the main constraint on our stability threshold is given by the control of the nonlinearity $v \cdot \nabla_t b$ and the reduced decay rates already at the linearized level (see Sect. 2). As we show in Sect. 3.3, our estimates of the so-called reaction terms (28) and (32) require a lower bound on the threshold by $\frac{3}{2}$ and are expected to be optimal for this partial dissipation case.
- Theorem 2 considers the case $\mu := \nu_x = \nu_y = \kappa_x$. As we discuss in Sects. 2 and 3, we expect that instead of equality it suffices to require that $\frac{1}{2\alpha}\nu_y \leq \kappa_x \leq C\nu_y^{\frac{1}{3}}$, similarly as in the full dissipation case studied in Dolce (2023). These constraints naturally arise in the linearized problem studied in Proposition 2.1. Furthermore, we expect that results can be extended to the case of purely vertical viscous dissipation with additional technical effort.
- Due to missing vertical dissipation, we obtain no decay of the x-averaged magnetic field $b_=$ which is forced by the nonlinearity.

To prove our results, it is convenient to work with the unknowns

$$p_1 = \Lambda_t^{-1}\nabla_t^{\perp} \cdot v, \;\; p_2 = \Lambda_t^{-1}\nabla_t^{\perp} \cdot b; \quad \Lambda_t := \sqrt{-\Delta_t}.$$

Similarly to the vorticity and current, the curl operator $\nabla_t^{\perp}$ eliminates the pressure and yields a scalar quantity, while the operator $\Lambda_t^{-1}\nabla_t^{\perp}\cdot$ is of order 0. Moreover, since v and b are divergence-free, similarly to viscosity formulations of the 2D Navier–Stokes equations, it can be shown by integration by parts that

$$\begin{aligned} \|Av\|_{L^2} &= \|Ap_1\|_{L^2}, \\ \|Ab\|_{L^2} &= \|Ap_2\|_{L^2}, \end{aligned}$$

for all Fourier multiplier A which commute with ∇_t and Λ_t. This, in particular, includes $\langle\nabla\rangle^N$ which corresponds to the Sobolev norm $\|\cdot\|_{H^N}$.

In terms of these unknowns our equations read

$$\begin{aligned} \partial_t p_1 - \partial_x \partial_x^t \Delta_t^{-1} p_1 - \alpha\partial_x p_2 &= \mu\Delta_t p_1 + \Lambda_t^{-1}\nabla_t^{\perp}(b\nabla_t b - v\nabla_t v), \\ \partial_t p_2 + \partial_x \partial_x^t \Delta_t^{-1} p_2 - \alpha\partial_x p_1 &= \mu\partial_x^2 p_2 + \Lambda_t^{-1}\nabla_t^{\perp}(b\nabla_t v - v\nabla_t b), \end{aligned} \tag{4}$$

The remainder of the article is structured as follows:

- In Sect. 2, as a first step we establish linear stability of the equations (4). In view of the lack of vertical resistivity, we here crucially rely on the interaction of p_1 and p_2 due to the underlying constant magnetic field. Moreover, we discuss the effects of partial dissipation and the resulting limited (optimal) decay rates in time.
- In Sect. 3, we introduce a bootstrap method for the proof of Theorem 2. Decomposing into low- and high-frequency contributions here yields several error terms, which are handled in different subsections. In particular, we need to distinguish between the evolution of the x-average (which does not experience enhanced dissipation due to the shear) and its L^2-orthogonal complement, as well as different frequency decompositions of the nonlinear terms (called reaction and transport terms in the literature).
- More precisely, in Sect. 3.2 we collect all nonlinear terms which can be estimated in a straightforward way. In view of partial magnetic dissipation, a main challenge is given by the effect of $v\nabla_t b$ on p_2 at high frequencies. Here, we distinguish between terms without x-average in Sect. 3.3 and with average in Sect. 3.4 and perform a decomposition into a transport and a reaction term. The low-frequency regime is discussed in Sect. 3.5 and does not require a very precise analysis.
- As a complementary result, in Sect. 4 we establish instability of the non-resistive, viscous MHD equations and prove Proposition 1. Here we first prove linear algebraic instability and then deduce a nonlinear norm inflation result as a corollary.

1.1 Notations and Conventions

For two real numbers $a, b \in \mathbb{R}$, we denote the minimum and maximum as

$$\min(a, b) = a \wedge b,$$
$$\max(a, b) = a \vee b.$$

We use the notation $f \lesssim g$ if there exists a constant C independent of all relevant parameters such that $|f| \leq C|g|$. Furthermore, we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

Moreover, for any vector or scalar v we define

$$\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}.$$

For a function $f(x, y) \in L^2(\mathbb{T} \times \mathbb{R})$, we denote the x-average and its L^2-orthogonal complement as

$$f_=(y) = \int_{\mathbb{T}} f(x, y)\mathrm{d}x,$$
$$f_{\neq} = f - f_0.$$

Throughout this text, unless noted otherwise, the spatial variables $(x, y) \in \mathbb{T} \times \mathbb{R}$ are periodic in the horizontal direction and the respective Fourier variables are denoted as

$$(k, \xi) \in (\mathbb{Z}, \mathbb{R})$$

or (l, η). The norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^N}$ refer to the standard Lebesgue and Sobolev norms for functions on $\mathbb{T}\times\mathbb{R}$. For time-dependent functions we denote $L^p H^s = L^p_t H^s$ as the space with the norm

$$\|f\|_{L^p H^s} = \left\| \|f\|_{H^s(\mathbb{T}\times\mathbb{R})} \right\|_{L^p(0,T)}.$$

We define the weight A^N and $A^{N'}_\mu$ by the Fourier multipliers

$$A^N = M\langle\nabla\rangle^N,$$
$$A^{N'}_\mu = M\langle\nabla\rangle^{N'} e^{c\mu t \mathbf{1}_{k\neq 0}},$$

for $3 < N' \leq N-2$ and $0 < c < \frac{1}{2}(1-\sqrt{\frac{2}{3}})$. With slight abuse of notation, we identify the multiplier operators with their Fourier symbols. The operator M is a time-dependent Fourier multiplier, introduced in Bedrossian et al. (2018), and is defined to satisfy the following equation:

$$-\frac{\dot{M}}{M} = \frac{|k|}{k^2+|\xi-kt|^2},$$
$$M(0,k,\xi) = 1.$$

That is, M is given as

$$M(t,k,\xi) = \exp\left(-\int_0^t \mathrm{d}\tau \, \frac{|k|}{k^2+(\xi-k\tau)^2}\right).$$

In particular, the operator M is comparable to the identity in the sense that

$$1 \geq M(t,k,\xi) \geq c$$

for some constant c and all $k \neq 0$ (and $M(t,0,\xi) := 1$ for $k = 0$).

The operators A thus define energies comparable to Sobolev (semi)norms:

$$\|A^N \cdot\|_{L^2} \approx \|\cdot\|_{H^N},$$
$$\|A^{N'}_\mu \cdot\|_{L^2} \approx \|e^{c\mu t\mathbf{1}_{k\neq 0}} \cdot\|_{H^N}.$$

In particular, since N is sufficiently large, the norm defined by A^N satisfies an algebra property.

2 Linear Stability

In this section, we study the stability of the linearized version of the equations (4):

$$\begin{aligned} \partial_t p_1 - \partial_x \partial_x^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \cdot \Delta_t p_1, \\ \partial_t p_2 + \partial_x \partial_x^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \kappa \cdot \Delta_t p_2, \\ \nu = (\mu, \mu),\ \kappa &= (\mu, 0). \end{aligned} \tag{5}$$

The ode tools to establish stability of such systems are well known in related systems such as the Boussinesq equations (Lai et al. 2021; Bedrossian et al. 2023; Bianchini et al. 2020; Masmoudi et al. 2023; Zillinger 2021b).

Our main results are summarized in the following proposition.

Proposition 2.1 (Linear stability) *Let $\mu > 0$, $\alpha > \frac{1}{2}$ and $N \geq 6$ be as in Theorem 2. Then the equations (5) are stable in H^N in the sense that for any choice of initial data $p_{\text{in}} \in H^N$ the corresponding solution satisfies*

$$\|p\|_{L^\infty H^N} + \mu^{1/2} \|\nabla_t p_1\|_{L^2 H^N} + \mu^{1/2} \|\partial_x p_2\|_{L^2 H^N} \lesssim e^{-C\mu t} \|p_{\text{in}}\|_{H^N}.$$

As we discuss after the proof, in the case $\frac{1}{2\alpha}\nu \leq \kappa \leq \nu^{1/3}$ the optimal decay rate for large times is given by $\mu = \min(\nu^{1/3}, \kappa)$. In particular, the coupling induced by the underlying magnetic field cannot yield enhanced dissipation rates for both components once the viscous dissipation becomes too large.

Proof of Proposition 2.1 We note that in this linear evolution equation (5) all coefficient functions are independent of both x and y. Therefore the equations decouple after a Fourier transform and we may equivalently consider the ode system

$$\begin{aligned} \partial_t \hat{p}_1 - \tfrac{k(\xi - kt)}{k^2 + (\xi - kt)^2} \hat{p}_1 - \alpha i k \hat{p}_2 &= -\nu(k^2 + (\xi - kt)^2) \hat{p}_1, \\ \partial_t \hat{p}_2 + \tfrac{k(\xi - kt)}{k^2 + (\xi - kt)^2} \hat{p}_2 - \alpha i k \hat{p}_1 &= -\kappa k^2 \hat{p}_2, \end{aligned} \tag{6}$$

for an arbitrary but fixed frequency $(k, \eta) \in \mathbb{Z} \times \mathbb{R}$. Since the equations are trivial for $k = 0$, in the following we further without loss of generality may assume that $k \neq 0$. Furthermore, after shifting t by $\frac{\xi}{k}$, we may assume that $\xi = 0$ and thus obtain a system of the form

$$\partial_t \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\frac{t}{1+t^2} - \nu k^2(1+t^2) & i\alpha k \\ i\alpha k & \frac{t}{1+t^2} - \kappa k^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{7}$$

where we dropped the hats for simplicity of notation.

In a first Naive estimate, we can test this equations with (p_1, p_2) and obtain that

$$\partial_t (|p_1|^2 + |p_2|^2) \leq \left(\frac{|t|}{1+t^2} - \mu k^2 \right) (|p_1|^2 + |p_2|^2),$$

which already yields the desired decay for times $|t| \gg (\mu k^2)^{-1}$. However, a Gronwall-type estimate on the remaining interval would only yield a very rough upper bound on the possible growth by

$$\exp\left(\int_{|t|\lesssim(\mu k^2)^{-1}} \frac{|t|}{1+t^2}\mathrm{d}t\right) \lesssim (1+(\mu k^2)^{-1})^2.$$

In order to improve this estimate, a common trick is to make use of the fact that $|\alpha|$ is relatively large and to consider

$$E = |p_1|^2 + |p_2|^2 - \frac{t}{1+t^2}\Re\left(\frac{1}{i\alpha k} p_1\overline{p_2}\right).$$

Since $|\alpha| > \frac{1}{2}$, this energy is positive definite and comparable to $|p_1|^2 + |p_2|^2$, with a constant which degenerates as $|\alpha| \downarrow \frac{1}{2}$.

Computing the time derivative of the last term, we note that

$$\begin{aligned}
&\frac{t}{1+t^2}\partial_t\Re\left(\frac{1}{i\alpha k}p_1\overline{p_2}\right)\\
&\leq \frac{t}{1+t^2}(|p_1|^2 - |p_2|^2)\\
&\quad + \frac{|t|}{1+t^2}\frac{1}{|\alpha|}\nu k(1+t^2)|p_1||p_2|\\
&\quad + \frac{|t|}{1+t^2}\frac{1}{|\alpha|}\kappa k|p_1||p_2|\\
&\quad + \mathcal{O}(t^{-2})|p_1||p_2|.
\end{aligned}$$

The first term exactly cancels out the possibly large contribution in $\partial_t(|p_1|^2+|p_2|^2)$. For the second and third term, we use the fact that $\frac{1}{\alpha} < 2$ and that these terms can hence be absorbed into the dissipation terms at the cost of a slight loss of constants, provided that

$$\frac{1}{2\alpha}\nu \leq \kappa.$$

We estimate

$$\begin{aligned}
|t|\frac{1}{|\alpha|}\nu k|p_1||p_2| &\leq \tfrac{2}{3}\nu k^2(1+t^2)|p_2|^2 + \tfrac{3}{8}\tfrac{1}{|\alpha|k}\nu k^2|p_2|\\
&\leq \tfrac{2}{3}\nu k^2(1+t^2)|p_2|^2 + \tfrac{3}{4}\kappa k^2|p_2|
\end{aligned}$$

and

$$\frac{|t|}{1+t^2}\frac{1}{|\alpha|}\kappa k|p_1||p_2| \leq \frac{8}{1+t^2}\frac{1}{|\alpha|^2}|p_1|^2 + \tfrac{1}{8}\kappa k^2|p_2|^2$$

Noting that $\partial_t \frac{|t|}{1+t^2} = \mathcal{O}(t^{-2})$ is integrable in time, we thus arrive at

$$\partial_t E \lesssim \mathcal{O}(t^{-2})E - \nu k^2(1+t^2)|p_1|^2 - \kappa k^2|p_2|^2.$$

Further defining

$$\tilde{E} = E \exp\left(\int^t \mathcal{O}(\tau^{-2})\mathrm{d}\tau\right),$$

it follows that $\tilde{E} \approx E$ decays exponentially in time and that the damping terms are integrable in time, which yields the desired result. □

We further remark that for t (corresponding to times $t + \frac{\xi}{k}$) such that $|t| \lesssim |\alpha|(\mu k^2)^{-1/2}$ the system (7) exhibits fast damping in both components (due to the coupling by α). However, for times much larger than this (that is, far away from $\frac{\xi}{k}$) the decay is limited. Indeed, after relabeling $p_1 \mapsto ip_1$ and introducing the energy E to control contributions by $\frac{t}{1+t^2}$, this follows from the fact that the eigenvalues of the matrix

$$\begin{pmatrix} -\mu k^2(1+t^2) & -\alpha \\ \alpha & -\mu k^2 \end{pmatrix}$$

are given by

$$\lambda_{1,2} = -\frac{\mu k^2(2+t^2)}{2} \pm \sqrt{\frac{1}{4}(\mu k^2 t^2)^2 - \alpha^2}.$$

In the first case, the square root is purely imaginary and hence $\Re(\lambda_1) = \Re(\lambda_2)$ is comparable to the stronger dissipation term

$$-\mu k^2(1+t^2).$$

For large times, instead the same eigenvalue computation yields

$$\lambda_1 \approx -\mu k^2 \langle t\rangle^2, \ \lambda_2 \approx -\mu k^2$$

and hence the decay estimates of Proposition 2.1 are not improved to $\mu^{1/3}$. This linear result thus highlights the effects of the coupling induced by the underlying constant magnetic field and shows which optimal decay estimates can be expected. In particular, it clearly illustrates that the loss of vertical magnetic dissipation incurs a change of decay rate compared to the fully dissipative case.

3 Bootstrap Hypotheses and Outline of Proof

We next turn to the full nonlinear problem (4), where we intend to treat the nonlinear contributions as errors and make use of the smallness of our initial data.

Our approach here follows a bootstrap argument, which is by now standard in the field (see, for instance, Bedrossian et al. (2018)). In the notation of Sect. 1.1, we assume that at the initial time

$$\|A^N p\|_{L^2}^2 + \|A_\mu^{N'} p\|_{L^2}^2 \le c_0 \epsilon^2 \tag{8}$$

for $3 < N' \le N - 2$. The constant $c_0 = c_0(\alpha) > 0$ will later be chosen small enough and tends to 0 as $\alpha \to \frac{1}{2}$. Given this estimate at the initial time, our aim in the remainder of this section is to establish the following estimates for the corresponding solution:

- High-frequency estimates

$$\begin{aligned} &\|A^N p_1\|_{L^\infty L^2}^2 + \mu \|A^N \nabla_t p_1\|_{L^2 L^2}^2 + \left\| \sqrt{-\tfrac{\dot{M}}{M}} A^N p_1 \right\|_{L^2 L^2}^2 < \varepsilon^2, \\ &\|A^N p_2\|_{L^\infty L^2}^2 + \mu \|A^N \partial_x p_2\|_{L^2 L^2}^2 + \left\| \sqrt{-\tfrac{\dot{M}}{M}} A^N p_2 \right\|_{L^2 L^2}^2 < \varepsilon^2. \end{aligned} \tag{9}$$

- Low-frequency estimates

$$\begin{aligned} &\|A_\mu^{N'} p_1\|_{L^\infty L^2}^2 + \mu \|A_\mu^{N'} \nabla_t p_1\|_{L^2 L^2}^2 + \left\| \sqrt{-\tfrac{\dot{M}}{M}} A_\mu^{N'} p_1 \right\|_{L^2 L^2}^2 < \varepsilon^2, \\ &\|A_\mu^{N'} p_2\|_{L^\infty L^2}^2 + \mu \|A_\mu^{N'} \partial_x p_2\|_{L^2 L^2}^2 + \left\| \sqrt{-\tfrac{\dot{M}}{M}} A_\mu^{N'} p_2 \right\|_{L^2 L^2}^2 < \varepsilon^2. \end{aligned} \tag{10}$$

By local well-posedness and our assumptions on the initial data, these estimates are satisfied at least on some (small) time interval $(0, T)$. In our bootstrap approach, we assume for the sake of contradiction that the maximal time T with this property is finite. We then show that on that same time interval all estimates hold with improved bounds instead, which however would imply that the estimates could be extended for a small additional time, contradicting the maximality of T.

With this understanding, we suppress T in our notation (see Sect. 1.1) and all L^p norms in time are understood to be norms on $L^p(0, T)$.

The splitting into high and low frequencies is essential to close the estimates in Sects. 3.3 and 3.4. In particular, we need the $e^{-c\mu t}$ decay to bound the so-called reaction error. Moreover, we require strong control of commutators involving A in order to control the so-called transport error. Both error terms impose strong restrictions on the energies and do not allow to close estimates in an easy way. We overcome this difficulty by linking separate energy estimates in the high frequency part and the low-frequency part. On the one hand, we can use the additional $e^{-c\mu t}$ in the low-frequency part to our benefit in the analysis of the high-frequency part. On the other hand, the difference in regularity allows us to control derivatives in the low-frequency estimate by the using high-frequency estimate.

Given a solution (p_1, p_2) of (4) and letting $A = A^N, A_\mu^{N'}$, computing time derivatives we need to control

$$\begin{aligned}
&\partial_t \|Ap_1\|_{L^2}^2 + 2(1-c)\mu\|A\nabla_t p_1\|_{L^2}^2 + 2\left\|\sqrt{-\tfrac{\dot{M}}{M}}Ap_1\right\|_{L^2}^2 \\
&\quad \leq 2\langle A^2 p_1, \partial_x \partial_x^t \Delta_t^{-1} p_1 + \Lambda_t^{-1}\nabla_t^\perp(b\nabla_t b - v\nabla_t v)\rangle =: L[p_1] + NL[p_1], \\
&\partial_t \|Ap_2\|_{L^2}^2 + 2(1-c)\mu\|A\partial_x p_2\|_{L^2}^2 + 2\left\|\sqrt{-\tfrac{\dot{M}}{M}}Ap_2\right\|_{L^2}^2 \\
&\quad \leq 2\langle A^2 p_2, -\partial_x \partial_x^t \Delta_t^{-1} p_2 + \Lambda_t^{-1}\nabla_t^\perp(b\nabla_t v - v\nabla_t b)\rangle =: L[p_2] + NL[p_2].
\end{aligned}$$

Here we have split contributions into linear (that is, quadratic integrals) and nonlinear terms (that is, trilinear integrals). Note that the choice of $0 < c < \frac{1}{2}(1-\sqrt{\frac{2}{3}})$ is made such that $1-c$ is not too small to absorb linear effects for α close to $\frac{1}{2}$. For later reference, we note that the bootstrap assumptions (9) and (10) yield the following estimates:

$$\|\partial_x^2 \Lambda_t^{-1}\Lambda^{-1} p\|_{H^N} \lesssim \tfrac{1}{t}\|p_{\neq}\|_{H^N} \tag{11}$$

and for $A = A^N, A_\mu^{N'}$

$$\begin{aligned}
\|Ap_{1,\neq}\|_{L^2L^2} &\lesssim \mu^{-\frac{1}{2}}\varepsilon, \\
\|Ap_{2,\neq}\|_{L^2L^2} &\lesssim \mu^{-\frac{1}{2}}\varepsilon.
\end{aligned} \tag{12}$$

Furthermore, for the nonlinear terms we will often use the equality

$$\begin{aligned}
\|Av\|_{L^2} &= \|Ap_1\|_{L^2}, \\
\|Ab\|_{L^2} &= \|Ap_2\|_{L^2}.
\end{aligned}$$

Throughout the following sections, we aim to establish smallness of the contributions by the linear terms $L[\cdot]$ and nonlinear terms $NL[\cdot]$. More precisely, we establish the following proposition.

Proposition 3.1 (Control of errors) *Under the assumptions of Theorem 2 suppose that the initial data satisfies the smallness condition (8) and let $T > 0$ be such the high- and low-frequency estimates (9), (10) are satisfied. Then on the same time interval it holds that*

$$\begin{aligned}
\int_0^T L[p_1] + L[p_2] \mathrm{d}t &\leq \tfrac{1}{2\alpha}(c_0+1)\varepsilon^2 + O(\mu^{-1}\varepsilon^3), \\
\int_0^T NL[p_1] + NL[p_2] \mathrm{d}t &\leq \mu^{-\frac{3}{2}}\varepsilon^3.
\end{aligned}$$

As a consequence, supposing that $\alpha > \frac{1}{2}$ and $\epsilon \ll \mu^{3/2}$, this implies that both the high-frequency and low-frequency estimates (9), (10) improve and thus T can only have been maximal if $T = \infty$, which proves Theorem 2. Thus proving Proposition 3.1 is our main concern in this section and our proof is split over the following subsections. The most important estimates, highlighting the effects of partial dissipation, are established in Sects. 3.1, 3.3 and 3.4.

We note that the nonlinear terms

$$\langle Ap_1, \Lambda_t^{-1}\nabla_t^{\perp} A(b\nabla_t b - v\nabla_t v)\rangle = -\langle Av, A(b\nabla_t b - v\nabla_t v)\rangle,$$
$$\langle Ap_2, \Lambda_t^{-1}\nabla_t^{\perp} A(b\nabla_t v - v\nabla_t b)\rangle = -\langle Ab, A(b\nabla_t v - v\nabla_t b)\rangle,$$

for $A = A^N, A_\mu^{N'}$ are all trilinear products involving

$$a^1a^2a^3 \in \{vvv, vbb, bbv, bvb\}$$

and we will use this notation to refer to the specific terms. Since the x-averages do not experience fast (mixing enhanced) decay under the dissipation, we split these products as

$$\begin{aligned}\langle Aa^1, A(a^2\nabla_t a^3)\rangle = &\langle Aa^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{\neq})_{\neq}\rangle\\ &+ \langle Aa^1_{\neq}, A(a^2_{=}\nabla_t a^3_{\neq})\rangle\\ &+ \langle Aa^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{=})\rangle\\ &+ \langle Aa^1_{=}, A(a^2_{\neq}\nabla_t a^3_{\neq})_{=},\end{aligned}$$

where the full splitting is only used for the bvb term.

3.1 Estimate of the Linear Error

In this subsection, we establish the estimate of the linear terms in Proposition 3.1. Here, we use some of the same techniques as in the proof of linear stability in Sect. 2, but instead focus on establishing quantitative bounds on the time integral.

Taking a Fourier transform of (4) yields

$$\begin{aligned}\partial_t \hat{p}_1 - \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\hat{p}_1 - \alpha ik\hat{p}_2 &= -\mu(k^2+(\xi-kt)^2)\hat{p}_1 + \mathcal{F}[NL[p_1]],\\ \partial_t \hat{p}_2 + \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\hat{p}_2 - \alpha ik\hat{p}_1 &= -\mu k^2\hat{p}_2 + \mathcal{F}[NL[p_2]].\end{aligned} \tag{13}$$

Recalling the various contributions, we aim to estimate

$$\begin{aligned}&\langle A^2 p_2, -\partial_x\partial_y^t\Delta_t^{-1}p_2\rangle + \langle A^2 p_1, \partial_x\partial_y^t\Delta_t^{-1}p_1\rangle\\ &= \sum_k \int \mathrm{d}\xi\, A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}(|\hat{p}_1|^2 - |\hat{p}_2|^2).\end{aligned}$$

In the following, with slight abuse of notation, we omit the hat denoting the Fourier transform and only consider $k \neq 0$, since for $k = 0$ this term vanishes.

Similarly as in the linear stability results of Sect. 2, we note that the Fourier multiplier a priori is not integrable in time and cannot easily be estimated by the partial dissipation. Hence, we rely on the coupling induced by the underlying magnetic field to eliminate some of this contribution and to provide better decay. More precisely, multiplying the equations (13) with $\hat{p}_2, \hat{p}_1$ and omitting the hats for simplicity of notation, we obtain the following identity:

$$
\begin{aligned}
&|p_1(k)|^2 - |p_2(k)|^2 \\
&\quad = -\tfrac{1}{i\alpha k}(p_1 \overline{i\alpha k p_1} + i\alpha k p_2 \overline{p_2}) \\
&\quad = -\tfrac{1}{i\alpha k} p_1 (\partial_t \overline{p}_2 + \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\, \overline{p}_2 + \mu k^2 \overline{p}_2 - \overline{\mathcal{F}}[NL[p_2]])) \\
&\qquad - \tfrac{1}{i\alpha k} \overline{p}_2 (\partial_t p_1 - \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} p_1 + (\mu k^2 + \mu(\xi-kt)^2) p_1 - \mathcal{F}[NL[p_1]] \\
&\quad = \tfrac{-1}{i\alpha k}(\partial_t(p_1\overline{p}_2) + \mu(k^2 + (\xi-kt)^2) p_1 \overline{p}_2 + \mu k^2 p_1 \overline{p}_2) \\
&\qquad - \tfrac{1}{\alpha i k}(p_1, p_2) \cdot \mathcal{F}[\Lambda_t^{-1} \nabla_t^{\perp} (b\nabla_t b - v\nabla_t v, b\nabla_t v - v\nabla_t b)].
\end{aligned}
$$

Thus we split L into two linear terms and one nonlinear term:

$$
\begin{aligned}
L &= 2\sum_k \int \mathrm{d}\xi\, A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} \tfrac{-1}{i\alpha k} \partial_t(p_1\overline{p}_2) \\
&\quad + 2\sum_k \int \mathrm{d}\xi\, A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} \tfrac{-1}{i\alpha k} (2\mu k^2 + \mu(\xi-kt)^2) p_1 \overline{p}_2 \\
&\quad - \tfrac{2}{\alpha} \langle A\partial_y^t \Delta_t^{-1}(p_1, p_2)_{\neq}, A\Lambda_t^{-1} \nabla_t^{\perp}(b\nabla_t b - v\nabla_t v, b\nabla_t v - v\nabla_t b)_{\neq}\rangle \\
&= L_1 + L_\mu + ONL.
\end{aligned}
\tag{14}
$$

We estimate L_μ by

$$
\begin{aligned}
L_\mu &= \tfrac{2}{\alpha}\mu \sum_{k\neq 0} \int \mathrm{d}\xi\, A^2 \tfrac{(2k^2+(\xi-kt)^2)(\xi-kt)}{k^2+(\xi-kt)^2} p_1 \overline{p}_2 \\
&= \tfrac{2}{\alpha}\mu \sum_{k\neq 0} \int \mathrm{d}\xi\, A^2 \tfrac{(2k^2+(\xi-kt)^2)(\xi-kt)}{(k^2+(\xi-kt)^2)^{\frac{3}{2}}} p_1 (k^2 + (\xi-kt)^2)^{\frac{1}{2}} \overline{p}_2 \\
&\leq \tfrac{2}{\alpha}\mu \sup_s \left(\tfrac{(2+s^2)s}{(1+s^2)^{\frac{3}{2}}} \right) \|A\partial_x p_2\|_{L^2} \|A\nabla_t p_1\|_{L^2} \\
&\leq \sqrt{\tfrac{2}{3}} \tfrac{1}{\alpha} \mu (\|A\partial_x p_2\|_{L^2}^2 + \|A\nabla_t p_1\|_{L^2}^2),
\end{aligned}
$$

where we used that

$$
\left| \tfrac{(2k^2+(\xi-kt)^2)(\xi-kt)}{(k^2+(\xi-kt)^2)^{\frac{3}{2}}} \right| = \left| \tfrac{(2+(\frac{\xi}{k}-t)^2)(\frac{\xi}{k}-t)}{(1+(\frac{\xi}{k}-t)^2)^{\frac{3}{2}}} \right|
$$

$$\leq \sup_s \left(\frac{(2+s^2)s}{(1+s^2)^{\frac{3}{2}}} \right)$$
$$\leq \sqrt{\frac{2}{3}}.$$

To estimate L_1, we integrate by parts in time to deduce that

$$\begin{aligned}
&\int \mathrm{d}\tau \sum_k \int \mathrm{d}\xi\, A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} \tfrac{-1}{i\alpha k} \partial_t (p_1 \overline{p}_2) \\
&\quad = \left[\tfrac{-1}{i\alpha} \sum_k \int \mathrm{d}\xi\, A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} p_1 \overline{p}_2 \right]_0^t \\
&\qquad + \int \mathrm{d}\tau\, \tfrac{1}{i\alpha} \sum_k \int \mathrm{d}\xi\, p_1 \overline{p}_2 \partial_t (A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2}) \\
&\quad = \left[\tfrac{-1}{i\alpha} \sum_k \int \mathrm{d}\xi\, A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} p_1 \overline{p}_2 \right]_0^t \\
&\qquad + \int \mathrm{d}\tau\, \tfrac{2}{i\alpha} \sum_k \int \mathrm{d}\xi\, p_1 \overline{p}_2 \tfrac{\dot{M}}{M} A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} \\
&\qquad + c\mu \mathbf{1}_{A=A_\mu^{N'}} \int \mathrm{d}\tau\, \tfrac{2}{i\alpha} \sum_k \int \mathrm{d}\xi\, p_1 \overline{p}_2 A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} \\
&\qquad + \int \mathrm{d}\tau\, \tfrac{1}{i\alpha} \sum_k \int \mathrm{d}\xi\, p_1 \overline{p}_2 A^2 \tfrac{k(k^2-(kt-\xi)^2)}{(k^2+(\xi-kt)^2)^2}.
\end{aligned}$$

So we infer by Hölder's inequality that

$$\begin{aligned}
&\int \mathrm{d}\tau \sum_k \int \mathrm{d}\xi\, A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} \tfrac{-1}{i\alpha k} \partial_t (p_1 \overline{p}_2) \\
&\quad \leq \tfrac{1}{\alpha} (\|Ap_1(0)\|_{L^2} \|Ap_2(0)\|_{L^2} + \|Ap_1(t)\|_{L^2} \|Ap_2(t)\|_{L^2}) \\
&\qquad + \mu \|A\partial_x p_1\|_{L^2L^2} \left\| A\sqrt{-\tfrac{\dot{M}}{M}} p_2 \right\|_{L^2L^2} \\
&\qquad + \tfrac{1}{\alpha} \left\| A\sqrt{-\tfrac{\dot{M}}{M}} p_1 \right\|_{L^2L^2} \left\| A\sqrt{-\tfrac{\dot{M}}{M}} p_2 \right\|_{L^2L^2}
\end{aligned}$$

and thus

$$\begin{aligned}
&\int L \mathrm{d}\tau - \int ONL \mathrm{d}\tau \\
&\quad \leq \tfrac{1}{2\alpha} (\|Ap_1(0)\|_{L^2}^2 + \|Ap_2(0)\|_{L^2}^2) \\
&\qquad + \tfrac{1}{2\alpha} (\|Ap_1\|_{L^\infty L^2}^2 + \|Ap_2(t)\|_{L^\infty L^2}^2)
\end{aligned}$$

$$+\frac{1}{\alpha}\left(\mu(1-c)\|\partial_x Ap\|_{L^2}^2+\mu(1-c)\|\partial_y^t Ap\|_{L^2}^2+\left\|\sqrt{-\frac{\dot M}{M}}Ap\right\|_{L^2}^2\right).$$

Using the dissipation estimates (12), we therefore obtain

$$\int L\mathrm{d}\tau \leq \frac{1}{2\alpha}(c_0+1)\varepsilon^2+\int ONL\mathrm{d}\tau, \tag{15}$$

where the ONL part will be estimated at the beginning of the next subsection.

3.2 Immediate Nonlinear Estimates for A^N

In this subsection, we collect some estimates which can be obtained in a straight forward approach using standard techniques (e.g., see (Bedrossian et al. 2018)). In particular, for these terms we are not constrained by the lack of vertical resistivity. For most estimates, we do not aim to establish optimal (mixing enhanced) bounds, since these bounds are in any case better than the ones involving horizontal resistivity and hence do not affect the over all stability threshold. In the following, we write $A=A^N$.

ONL estimate Using integration by parts in space and Hölder's inequality, the nonlinear contribution in (14) can be estimated by

$$\begin{aligned}
ONL &= \frac{2}{\alpha}\langle A\partial_y^t\Delta_t^{-1}v_{\neq}, A(b\nabla_t b - v\nabla_t v)_{\neq}\rangle\\
&\quad+\frac{2}{\alpha}\langle A\partial_y^t\Delta_t^{-1}b_{\neq}, A(b\nabla_t v - v\nabla_t b)_{\neq}\rangle\\
&=\frac{2}{\alpha}\langle A\partial_y^t\Delta_t^{-1}(\nabla_t^\perp\otimes v_{\neq}), A(b\otimes b - v\otimes v)_{\neq}\rangle\\
&\quad+\frac{2}{\alpha}\langle A\partial_y^t\Delta_t^{-1}(\nabla_t^\perp\otimes b_{\neq}), A(b\otimes v - v\otimes b)_{\neq}\rangle\\
&\lesssim \frac{2}{\alpha}\|A(v,b)_{\neq}\|_{L^2}^2\|A(v,b)\|_{L^2}.
\end{aligned}$$

Recalling the bounds (12) and integrating in time, we thus obtain that

$$\int ONL\mathrm{d}\tau \lesssim \mu^{-1}\varepsilon^3. \tag{16}$$

Estimates with an x-average in the second component Let $a^1a^2a^3\in\{vvv, vbb, bbv, bvb\}$, then we need to estimate the trilinear products

$$\begin{aligned}
\langle Aa^1_{\neq}, A(a^2_{=}\nabla_t a^3_{\neq})\rangle &= \langle Aa^1_{\neq}, A(a^2_{=,1}\partial_x a^3_{\neq})\rangle\\
&\lesssim \|Aa^1_{\neq}\|_{L^2}\|Aa^2_{=,1}\|_{L^2}\|A\partial_x a^3_{\neq}\|_{L^2}.
\end{aligned}$$

Integrating in time and again using the bound (12) yields a control by

$$\int \mathrm{d}\tau\langle Aa^1, A(a^2_{=}\nabla_t a^3)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{17}$$

The influence of the underlying x-averaged velocity and magnetic field on the averageless parts can thus be easily controlled by the dissipation, provided $\epsilon \ll \mu$. In the following, we focus on terms involving $a^2_{\neq}$.

vvv estimate We first discuss the velocity nonlinearity and use the algebra property of H^N and the bounds on A to estimate

$$\langle Av, Av_{\neq}\nabla_t v\rangle \leq \|Av\|_{L^2}\|Av_{\neq}\|_{L^2}\|A\nabla_t v\|_{L^2}.$$

Here, the contribution by $\|A\nabla_t v\|_{L^2}$ is square integrable in time due to the dissipation (12), while $\|Av_{\neq}\|_{L^2}$ is square integrable in time by the inviscid damping estimates (11). Integrating in time thus yields a bound by

$$\int \mathrm{d}\tau\, \langle Av, A(v_{\neq}\nabla_t v)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{18}$$

vbb estimate For the contributions by the vbb nonlinearity, we intend to argue similarly, but have to account for the lack of vertical magnetic dissipation (which we compensate for by using the full fluid dissipation). We may split the integral as

$$\begin{aligned}\langle Av, A(b_{\neq}\nabla_t b)\rangle = &\int Av_1 A(b_{1,\neq}\partial_x + b_{2,\neq}\partial_y^t)b_1\\ &+\int Av_2 A(b_{1,\neq}\partial_x + b_{2,\neq}\partial_y^t)b_2.\end{aligned}$$

For the second term we integrate by parts to obtain

$$\int Av_1 A(b_{2,\neq}\partial_y^t b_1) = -\int A\partial_y^t v_1 A(b_{2,\neq} b_1) - \int Av_1 A(\partial_y^t b_{2,\neq} b_1).$$

Furthermore, since b is divergence-free, it holds that $\partial_y^t b_2 = -\partial_x b_1$ and hence

$$\langle Av, A(b_{\neq}\nabla_t b)\rangle \leq \|Av\|_{L^2}\|Ab_{\neq}\|_{L^2}\|A\partial_x b\|_{L^2} + \|\partial_y^t v\|_{L^2}\|Ab_{\neq}\|_{L^2}\|Ab_2\|_{L^2}.$$

We may therefore estimate this term using the full fluid and horizontal magnetic dissipation (12) and integrating in time yields a bound by

$$\int \mathrm{d}\tau\, \langle Av, A(b_{\neq}\nabla_t b)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{19}$$

bbv estimate Finally, for the bbv contribution, we may again use the full fluid dissipation and the algebra property of A (and H^N) to obtain a bound

$$\langle Ab, A(b_{\neq}\nabla_t v)\rangle \lesssim \|Ab\|_{L^2}\|Ab_{\neq}\|_{L^2}\|A\nabla_t v\|_{L^2}.$$

Integrating in time and using (12), we thus obtain a bound by

$$\int \mathrm{d}\tau\, \langle Ab, A(b_{\neq}\nabla_t v)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{20}$$

3.3 High-Frequency *bvb* Term Without *x*-Average

Having established several straightforward estimates using the full fluid dissipation, in this and the following subsections we establish bounds for the high frequency (that is, A^N terms as in (9)) terms involving bvb. For simplicity, we write $A = A^N$ and aim to establish the estimate

$$\langle Ab, A(v_{\neq}\nabla_t b)\rangle \lesssim \mu^{-\frac{3}{2}}\varepsilon^3.$$

We split the bvb term according to (non)vanishing x-averages:

$$\begin{aligned}\langle Ab, A(v_{\neq}\nabla_t b)\rangle = &\langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{\neq})_{\neq}\rangle \\ &+ \langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{=})_{\neq}\rangle \\ &+ \langle Ab_{=}, A(v_{\neq}\nabla_t b_{\neq})_{=}\rangle.\end{aligned}$$

Let us first consider the term without any x-averages, which can be written as

$$\begin{aligned}\langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{\neq})\rangle = &\int Ab_{1,\neq}A((v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{1,\neq}) \\ &+ \int Ab_{2,\neq}A((v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{2,\neq}).\end{aligned}$$

We estimate the second contribution using the algebra property of H^N and that $\partial_y^t b_2 = -\partial_x b_1$, since b is divergence-free:

$$\begin{aligned}&\int d\tau \int Ab_{2,\neq}A(v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{2,\neq} \\ &\leq \int d\tau \|Ab_{2,\neq}\|_{L^2}(\|Av_{1,\neq}\|_{L^2}\|A\partial_x b_{2,\neq}\|_{L^2} + \|Av_{2,\neq}\|_{L^2}\|A\partial_y^t b_{2,\neq}\|_{L^2}) \\ &\leq \int d\tau \|Ab_{2,\neq}\|_{L^2}(\|Av_{1,\neq}\|_{L^2}\|\partial_x b_{2,\neq}\|_{L^2} + \|Av_{2,\neq}\|_{L^2}\|A\partial_x b_{1,\neq}\|_{L^2}).\end{aligned}$$

Employing Hölder's inequality this contribution can thus be estimated as

$$\begin{aligned}&\int d\tau \int Ab_{2,\neq}A((v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{2,\neq}) \\ &\quad\leq \int d\tau \|Ab_{2,\neq}\|_{L^2}\|Av_{\neq}\|_{L^2}\|A\partial_x b_{\neq}\|_{L^2} \\ &\quad\leq \|Ab_{2,\neq}\|_{L^2L^2}\|Av_{\neq}\|_{L^\infty L^2}\|A\partial_x b_{\neq}\|_{L^2L^2} \\ &\lesssim \mu^{-1}\varepsilon^3.\end{aligned} \tag{21}$$

It remains to control the contribution by $b_{1,\neq}$, which in view to the lack of vertical resistivity is the hardest term to control. Since the velocity field v is divergence-free, we observe that

$$\int Ab_{1,\neq}(v_{\neq}\nabla_t Ab_{1,\neq}) = 0.$$

Therefore, we obtain the following cancelations and introduce a splitting in Fourier space:

$$\begin{aligned}\int Ab_{1,\neq}A(v_{\neq}\nabla_t b_{1,\neq}) &= \int Ab_{1,\neq}(A(v\nabla_t b_{1,\neq}) - (v_{\neq}\nabla_t Ab_{1,\neq}))\\ &= \sum_{k,l,k-l\neq 0}\iint \mathrm{d}(\xi,\eta)A(k,\xi)b_1(k,\xi)\frac{(A(k,\xi)-A(l,\eta))(\xi l-\eta k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}\\ &\quad\times\ p_1(k-l,\xi-\eta)b_1(l,\eta)\\ &= T + R + \mathcal{R}.\end{aligned}$$

Here, the Fourier regions

$$\begin{aligned}\Omega_T &= \{|k-l,\xi-\eta| \leq \tfrac{1}{8}|l,\eta|\},\\ \Omega_R &= \{|l,\eta| \leq \tfrac{1}{8}|k-l,\xi-\eta|\},\\ \Omega_{\mathcal{R}} &= \{\tfrac{1}{8}|l,\eta| \leq |k-l,\xi-\eta| \leq 8|l,\eta|\},\end{aligned}$$

correspond to the *transport* (T) or low-high term, *reaction* (R) or high-low term and the *remainder* ($\mathcal{R}$) or high-high term. In the following, we omit the $\neq$ subscripts.

Transport term Since $|k-l,\xi-\eta| \leq \frac{1}{8}|l,\eta|$ we obtain that $|l,\eta| \approx |k,\xi|$. Without loss of generality, we assume that $\xi \leq \eta$, since we can use either of the following splittings

$$\begin{aligned}\xi l - k\eta &= (\xi-\eta)l - (k-l)\eta\\ &= (\xi-\eta)k - \xi(k-l).\end{aligned}$$

Thus using the second equality, we estimate

$$\begin{aligned}T &\leq \|\partial_y\Lambda_t^{-1}p_1\|_{L^\infty}\|Ab_1\|_{L^2}\|\partial_x Ab_1\|_{L^2}\\ &\quad+\sum_{k,l\neq 0}\iint \mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_T}(\mathbf{1}_{2\langle t\rangle(k\vee l)\geq\xi}+\mathbf{1}_{2\langle t\rangle(k\vee l)\leq\xi})\\ &\quad\times A(k,\xi)b_1(k,\xi)\frac{(A(k,\xi)-A(l,\eta))\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}p_1(k-l,\xi-\eta)b_1(l,\eta),\end{aligned} \tag{22}$$

where we distinguished between $2\langle t\rangle(k\vee l) \geq \xi$ and $2\langle t\rangle(k\vee l) \leq \xi$.

The first case is estimated by using the dissipation and (11):

$$\begin{aligned}\sum_{k,l\neq 0}\iint &\mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{\xi\leq 2(k\vee l)\langle t\rangle}A(k,\xi)b_1(k,\xi)\\ &\times \tfrac{(A(k,\xi)-A(l,\eta))\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}p_1(k-l,\xi-\eta)b_1(l,\eta)\\ &\lesssim \langle t\rangle\|Ab_1\|_{L^2}\|\Lambda_t^{-1}\partial_x p_1\|_{L^\infty}\|\partial_x Ab_1\|_{L^2}\\ &\lesssim \|Ab_1\|_{L^2}\|\Lambda\partial_x p_1\|_{L^\infty}\|\partial_x Ab_1\|_{L^2}\\ &\lesssim \|Ab_1\|_{L^2}\|Ap_1\|_{L^2}\|\partial_x Ab_1\|_{L^2}.\end{aligned} \tag{23}$$

For the second case, $2\langle t\rangle(k\vee l)\leq\xi$, we need to estimate

$$\begin{aligned}(A(k,\xi)-A(l,\eta)) &= (M(k,\xi)|k,\xi|^N - M(l,\eta)|l,\eta|^N)\\ &= M(k,\xi)(|k,\xi|^N-|l,\eta|^N)\\ &\quad + M(l,\eta)(\tfrac{M(k,\xi)}{M(l,\eta)}-1)|l,\eta|^N.\end{aligned}$$

By the mean value theorem, we obtain

$$\begin{aligned}|k,\xi|^N-|l,\eta|^N &\leq N|k-\theta l,\xi-\theta\eta|^{N-1}|k-l,\xi-\eta|\\ &\lesssim |k-l,\xi-\eta|(|l,\eta|^{N-1}+|k-l,\xi-\eta|^{N-1})\\ &\lesssim |k-l,\xi-\eta||l,\eta|^{N-1}.\end{aligned} \tag{24}$$

For the differences in M, we use that for $a,b>0$ it holds that $|e^{a-b}-1|\leq e^{a+b}-1$ and hence

$$\begin{aligned}|\tfrac{M(k,\xi)}{M(l,\eta)}-1| &= |\exp\left(\int_0^t \tfrac{|l|}{l^2+(\eta-ls)^2}-\tfrac{|k|}{k^2+(\xi-ks)^2}\mathrm{d}s\right)-1|\\ &\leq |\exp\left(\int_0^t \tfrac{|l|}{l^2+(\eta-ls)^2}+\tfrac{|k|}{k^2+(\xi-ks)^2}\mathrm{d}s\right)-1|.\end{aligned}$$

Thus for $\eta\geq\xi\geq 2t(k\vee l)$ by integrating we obtain that

$$\begin{aligned}|\tfrac{M(k,\xi)}{M(l,\eta)}-1| &\leq \exp\left(\tfrac{1}{|l|}\int_0^t\tfrac{1}{1+(\frac{\eta}{l}-s)^2}\mathrm{d}s+\tfrac{1}{|k|}\int_0^t\tfrac{1}{1+(\frac{\xi}{k}-s)^2}\mathrm{d}s\right)-1\\ &\leq \exp(\tfrac{1}{\eta}+\tfrac{1}{\xi})-1\\ &\lesssim \tfrac{1}{\eta}+\tfrac{1}{\xi}.\end{aligned} \tag{25}$$

With the commutator estimates (24) and (25), we infer

$$\begin{aligned}\left|\tfrac{(A(k,\xi)-A(l,\eta))\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}\right| &\lesssim \left|\tfrac{\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}\right|(\tfrac{1}{\eta}+\tfrac{1}{\xi}+|k-l,\xi-\eta||l,\eta|^{-1})A(l,\eta)\\ &\lesssim \tfrac{1}{\sqrt{1+(\frac{\xi-\eta}{k-l}-t)^2}}|k-l,\xi-\eta|A(l,\eta).\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}&\sum_{k,l,k-l\neq 0}\iint \mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{\xi\geq 2\langle t\rangle(k\vee l)}A(k,\xi)b_1(k,\xi)\frac{(A(k,\xi)-A(l,\eta))\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}\\ &\quad\times\ p_1(k-l,\xi-\eta)b_1(l,\eta)\\ &\lesssim \|Ab_1\|_{L^2}\|\Lambda\Lambda_t^{-1}\partial_x p_1\|_{L^\infty}\|Ab_1\|_{L^2}\\ &\lesssim \langle t\rangle^{-1}\|Ab_1\|_{L^2}\|\Lambda^2\partial_x p_1\|_{L^\infty}\|Ab_1\|_{L^2}\\ &\lesssim \langle t\rangle^{-1}\|Ab_1\|_{L^2}\|Ap_1\|_{L^2}\|Ab_1\|_{L^2},\end{aligned}\tag{26}$$

where we used the estimate (11). Combining (22), (23) and (26), we have derived the following estimate of the transport term:

$$\begin{aligned}\int T\,\mathrm{d}\tau &\lesssim \|Ab_1\|_{L^\infty L^2}\|Ap_1\|_{L^\infty L^2}\|Ab_1\|_{L^2L^2}\\ &\lesssim \mu^{-\frac{1}{2}}\varepsilon^3.\end{aligned}\tag{27}$$

Reaction term Since $|l,\eta|\leq\frac{1}{8}|k-l,\xi-\eta|$, we obtain $|k-l,\xi-\eta|\approx|k,\xi|$. With the identity

$$\xi l-k\eta=l(\xi-\eta-(k-l)t)-(k-l)(\eta-lt)$$

and $A(k,\xi)-A(l,\eta)\lesssim A(k-l,\xi-\eta)$ we infer

$$\begin{aligned}R=&\sum_{k,l,k-l\neq 0}\iint \mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_R}A(k,\xi)b_1(k,\xi)\frac{(A(k,\xi)-A(l,\eta))(l(\xi-\eta-(k-l)t)-(k-l)(\eta-lt))}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}\\ &\times\ p_1(k-l,\xi-\eta)b_1(l,\eta)\\ &\leq \|Ab_1\|_{L^2}\|A\partial_y^t\Lambda_t^{-1}p_1\|_{L^2}\|\partial_x b_1\|_{L^\infty}\\ &\quad+\|Ab_1\|_{L^2}\|A\Lambda_t^{-1}p_1\|_{L^2}\|\partial_y^t\partial_x^2 b_1\|_{L^\infty}\\ &\quad+\|\partial_x Ab_1\|_{L^2}\|A\Lambda_t^{-1}p_1\|_{L^2}\|\partial_y^t\partial_x b_1\|_{L^\infty}.\end{aligned}$$

We split $\partial_y^t=\partial_y-t\partial_x$ and use the definition of the low-frequency multiplier $A_\mu^{N'}$ to estimate

$$\begin{aligned}\|\langle\partial_x\rangle^2\partial_y^t b_1\|_{L^\infty}&\leq\|\langle\partial_x\rangle^2\partial_y b_1\|_{L^\infty}+\|\langle\partial_x\rangle^2 t\partial_x b_1\|_{L^\infty}\\ &\leq t\|\Lambda^{N'}b_1\|_{L^2}\\ &\lesssim te^{-c\mu t}\|A_\mu^{N'}b_1\|_{L^2}\\ &\lesssim \mu^{-1}\|A_\mu^{N'}b_1\|_{L^2}.\end{aligned}$$

Therefore, integrating in time yields the estimate

$$
\begin{aligned}
\int R \mathrm{d}\tau \lesssim \|Ab_1\|_{L^2L^2} \Big(\|A\partial_y^t \Lambda_t^{-1} p_1\|_{L^2L^2} \|Ab_1\|_{L^\infty L^2} \Big) \\
+ \mu^{-1} \|A\partial_x b_1\|_{L^2L^2} \|A\Lambda_t^{-1} p_1\|_{L^2L^2} \|A_\mu^{N'} b_1\|_{L^\infty L^2} \\
\lesssim \varepsilon^3 \mu^{-\frac{3}{2}}.
\end{aligned}
\tag{28}
$$

$\mathcal{R}$ *term* We consider the Fourier region where $\frac{1}{8}|l,\eta| \leq |k-l,\xi-\eta| \leq 8|l,\eta|$. Thus, we have the bounds $|k,\xi| \lesssim |l,\eta|$ and $A(k,\xi) \lesssim A(l,\eta) \approx A(k-l,\xi-\eta)$. Furthermore, we note that

$$
\xi l - \eta k \leq |l,\eta|^2,
$$

and thus estimate the remainder terms as

$$
\begin{aligned}
\mathcal{R} = \sum_{k,l,k-l\neq 0} \iint \mathrm{d}(\xi,\eta) 1_{\Omega_{\mathcal{R}}} A(k,\xi) b_1(k,\xi) \\
\times \frac{(A(k,\xi)-A(l,\eta))(\xi l-\eta k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}} p_1(k-l,\xi-\eta) b_1(l,\eta) \\
\lesssim \|Ab_1\|_{L^2} \|A\Lambda_t^{-1} p_1\|_{L^2} \|\Lambda^2 b_1\|_{L^\infty} \\
\lesssim \|Ab_1\|_{L^2} \|A\Lambda_t^{-1} p_1\|_{L^2} \|Ab_1\|_{L^2}.
\end{aligned}
$$

Hence after integrating in time, we deduce that

$$
\int \mathcal{R} \lesssim \|Ab_1\|_{L^2L^2} \Big\|\sqrt{-\tfrac{\dot{M}}{M}} Ap_1\Big\|_{L^2L^2} \|Ab_1\|_{L^\infty L^2} \lesssim \mu^{-\frac{1}{2}} \varepsilon^3. \tag{29}
$$

Combining the estimates (21), (27), (28) and (29), we finally conclude that

$$
\langle Ab_{\neq}, A(v_{\neq} \nabla_t b_{\neq})_{\neq} \rangle \lesssim \mu^{-\frac{3}{2}} \varepsilon^3. \tag{30}
$$

3.4 High-Frequency Estimates for *bvb* Terms with *x*-Averages

In this subsection, we aim to estimate the remaining terms in the bvb integrals, which involve x-averages. We consider the two terms

$$
\begin{aligned}
\langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{=})_{\neq} \rangle + \langle Ab_{=}, A(v_{\neq}\nabla_t b_{\neq})_{=} \rangle \\
= \langle Ab_{1,\neq}, A(v_{\neq}\nabla_t b_{1,=})_{\neq} \rangle + \langle Ab_{1,=}, A(v_{\neq}\nabla_t b_{1,\neq})_{=} \rangle,
\end{aligned}
$$

where we used that $b_{2,=} = 0$, since b is divergence-free. Using integration by parts and the fact that v is divergence-free, we obtain that

$$
\langle Ab_{1,\neq}, v_{\neq}\nabla_t Ab_{1,=} \rangle + \langle Ab_{1,=}, v_{\neq}\nabla_t Ab_{1,\neq} \rangle
$$

$$= \langle v_{\neq}, \nabla_t (Ab_{1,=} Ab_{1,\neq}) \rangle = 0,$$

and thus

$$\begin{aligned}
&\langle Ab_{1,\neq}, A(v_{\neq} \nabla_t b_{1,=}) \rangle + \langle Ab_{1,=}, A(v_{\neq} \nabla_t b_{1,\neq}) \rangle \\
&\quad = \langle Ab_{1,\neq}, A(v_{\neq} \nabla_t b_{1,=}) - v_{\neq} \nabla_t Ab_{1,=} \rangle + \langle Ab_{1,=}, A(v_{\neq} \nabla_t b_{1,\neq}) - v_{\neq} \nabla_t Ab_{1,\neq} \rangle \\
&\quad = \sum_{k \neq 0} \iint \mathrm{d}(\xi, \eta) A(k, \xi) b_1(k, \xi) \tfrac{(A(k,\xi) - A(0,\eta))(-k\eta)}{\sqrt{k^2 + (\xi - \eta - kt)^2}} p_1(k, \xi - \eta) b_1(0, \eta) \\
&\qquad + \sum_{k \neq 0} \iint \mathrm{d}(\xi, \eta) A(0, \xi) b_1(0, \xi) \tfrac{(A(0,\xi) - A(k,\eta))(-k\xi)}{\sqrt{k^2 + (\xi - \eta - kt)^2}} p_1(k, \xi - \eta) b_1(-k, \eta).
\end{aligned}$$

Again we split this integrals into the transport T, reaction R and remainder terms $\mathcal{R}$ with the same definition of sets in Fourier space:

$$\begin{aligned}
\Omega_T &= \{|\xi - \eta| \leq \tfrac{1}{8}|\eta|\}, \\
\Omega_R &= \{|\eta| \leq \tfrac{1}{8}|\xi - \eta|\}, \\
\Omega_{\mathcal{R}} &= \{\tfrac{1}{8}|\eta| \leq |\xi - \eta| \leq 8|\eta|\}.
\end{aligned}$$

Transport term Since $|\xi - \eta| \leq \frac{1}{8}|\eta|$ we obtain that $|\eta| \approx |\xi|$.

In our estimates, we distinguish the cases $\xi \vee \eta \leq 2k\langle t \rangle$ and $\xi \vee \eta \geq 2k\langle t \rangle$. In the first case, $\xi \vee \eta \leq 2k\langle t \rangle$ we obtain a bound by

$$\begin{aligned}
&\sum_{k \neq 0} \iint \mathrm{d}(\xi, \eta) \mathbf{1}_{\Omega_T} \mathbf{1}_{\xi \vee \eta \leq k\langle t \rangle} A(k, \xi) b_1(k, \xi) \tfrac{(A(k,\xi) - A(0,\eta))k\eta}{\sqrt{k^2 + (\xi - \eta - kt)^2}} p_1(k, \xi - \eta) b_1(0, \eta) \\
&\quad + \sum_{k \neq 0} \iint \mathrm{d}(\xi, \eta) \mathbf{1}_{\Omega_T} \mathbf{1}_{\xi \vee \eta \leq k\langle t \rangle} A(0, \xi) b_1(0, \xi) \tfrac{(A(0,\xi) - A(k,\eta))k\xi}{\sqrt{k^2 + (\xi - \eta - kt)^2}} p_1(k, \xi - \eta) b_1(-k, \eta) \\
&\quad \leq t \|Ab_=\|_{L^2} \|\partial_x^2 \Lambda_t^{-1} p_{1,\neq}\|_{L^\infty} \|Ab_{1,\neq}\|_{L^2} \\
&\quad \lesssim \|Ab_=\|_{L^2} \|Ap_{1,\neq}\|_{L^2} \|Ab_{1,\neq}\|_{L^2}.
\end{aligned}$$

In the case $\xi \vee \eta \geq 2k\langle t \rangle$, we instead estimate

$$\begin{aligned}
A(k, \xi) - A(0, \eta) &\leq M(k, \xi)(\xi^2 + k^2)^{\frac{N}{2}} - \eta^N \\
&= (M(k, \xi) - 1)(\xi^2 + k^2)^{\frac{N}{2}} + ((\xi^2 + k^2)^{\frac{N}{2}} - \eta^N).
\end{aligned}$$

Since $\xi \geq 2k\langle t \rangle$, in the first summand we may bound

$$\begin{aligned}
M(k, \xi) - 1 &= \exp\left(-\int_0^t \tfrac{|k|}{k^2 + (\xi - ks)^2} \mathrm{d}s\right) - 1 \\
&\lesssim \tfrac{1}{\xi} \lesssim \tfrac{1}{\eta}.
\end{aligned}$$

By the mean value theorem, we further infer

$$(\xi^2+k^2)^{\frac{N}{2}}-\eta^N \le ((\xi-\theta\eta)^2+k^2)^{\frac{N-1}{2}}|k,\xi-\eta| \lesssim |k,\xi-\eta|(\xi^2+k^2)^{\frac{N-1}{2}}.$$

Thus, using that $k \le \xi \lesssim \eta$, we deduce that

$$\begin{aligned} A(k,\xi)-A(0,\eta) &\lesssim |k,\xi-\eta|\eta^{N-1},\\ A(k,\eta)-A(0,\xi) &\lesssim |k,\xi-\eta|\eta^{N-1}, \end{aligned}$$

where the proof for $A(k,\eta)-A(0,\xi)$ is analogous. Finally, we obtain

$$\begin{aligned} &\langle Ab_{\neq}, \mathbf{1}_{\Omega_T}\mathbf{1}_{\eta\ge kt}A(v_{\neq}\nabla_t b_{=})\rangle + \langle Ab_{=}, \mathbf{1}_{\Omega_T}1_{\eta\ge kt}A(v_{\neq}\nabla_t b_{\neq})\rangle\\ &\quad\lesssim \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)A(k,\xi)b_1(k,\xi)\tfrac{|k,\xi-\eta|\eta^{N-1}}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(0,\eta)\\ &\qquad+\sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)A(0,\xi)b_1(0,\xi)\tfrac{|k,\xi-\eta|\eta^{N-1}}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(k,\eta)\\ &\quad\lesssim \|Ab_{=}\|_{L^\infty}\|A\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|Ab_{1,\neq}\|_{L^2}\\ &\quad\lesssim \|Ab_{=}\|_{L^\infty}\|A\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|Ab_{1,\neq}\|_{L^2}, \end{aligned}$$

and integrating in time yields the desired bound:

$$\begin{aligned} &\int \langle Ab_{\neq}, \mathbf{1}_{\Omega_T}A(v_{\neq}\nabla_t b_{=})\rangle + \langle Ab_{=}, \mathbf{1}_{\Omega_T}A(v_{\neq}\nabla_t b_{\neq})\rangle \mathrm{d}\tau\\ &\quad\lesssim \mu^{-1}\varepsilon^3. \end{aligned} \tag{31}$$

Reaction term Since $|\eta| \le \frac{1}{8}|\xi-\eta|$, we obtain $|\xi-\eta|\approx|\xi|$ and thus

$$\begin{aligned} R &= \langle Ab_{\neq}, \mathbf{1}_{\Omega_R}A((v_{\neq}\nabla_t b_{=})-v_{\neq}\nabla_t Ab_{=})\rangle + \langle Ab_{=}, \mathbf{1}_{\Omega_R}(A(v_{\neq}\nabla_t b_{\neq})-v_{\neq}\nabla_t Ab_{\neq})_{=}\rangle\\ &\le \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_R}A(k,\xi)b_1(k,\xi)\tfrac{(A(k,\xi)-A(0,\eta))k\eta}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(0,\eta)\\ &\quad+\sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_R}A(0,\xi)b_1(0,\xi)\tfrac{(A(0,\xi)-A(-k,\eta))k\xi}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(-k,\eta)\\ &\lesssim \|Ab_{1,\neq}\|_{L^2}\|A\partial_x\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|\partial_y b_{1,=}\|_{L^\infty}\\ &\quad+\|Ab_{1,=}\|_{L^2}\|A\partial_y\partial_x^{-1}\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|\partial_x^2 b_{1,\neq}\|_{L^\infty}. \end{aligned}$$

Expressing $\partial_y = \partial_y^t + t\partial_x$ in terms of the time-dependent derivatives, at this point we require the splitting into high- and low-frequency estimates. More precisely, using the additional time decay of the low-frequency part, we estimate

$$\begin{aligned} \|A\partial_y\partial_x^{-1}\Lambda_t^{-1}p_{1,\neq}\|_{L^2} &\le \|A\partial_y^t\partial_x^{-1}\Lambda_t^{-1}p_{1,\neq}\|_{L^2} + t\|A\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\\ &\lesssim \|Ap_{1,\neq}\|_{L^2} + t\|A\Lambda_t^{-1}p_{1,\neq}\|_{L^2} \end{aligned}$$

and using the definition of $A_\mu^{N'}$, we can absorb the growth of the factor t at the cost of a power of μ:

$$\begin{aligned}\|\partial_x^2 b_{1,\neq}\|_{L^\infty} &\leq \|\Lambda^{N'} b_{1,\neq}\|_{L^2}\\ &\lesssim e^{-c\mu t}\|A_\mu^{N'} b_{1,\neq}\|_{L^2}\\ &\lesssim \mu^{-1}\langle t\rangle^{-1}\|A_\mu^{N'} b_{1,\neq}\|_{L^2}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}R \lesssim\ &\|A^N p_{1,\neq}\|_{L^2}\|A^N b_{1,=}\|_{L^2}\|A^N b_{1,\neq}\|_{L^2}\\ &+\mu^{-1}\|A^N b_{1,=}\|_{L^2}\|A\Lambda_t^{-1} p_{1,\neq}\|_{L^2}\|A_\mu^{N'} b_{1,\neq}\|_{L^2}.\end{aligned}$$

Integrating in time then yields the estimate

$$\int R \mathrm{d}\tau \lesssim \mu^{-\frac{3}{2}}\varepsilon^3. \tag{32}$$

$\mathcal{R}$ term The remainder term $\mathcal{R}$ can be estimated by the same argument as in the case without x-averages in Sect. 3.3.

Combining the estimates (31), (32) and (30), we conclude that the bvb term can be controlled as

$$\langle Ab, A(v_{\neq}\nabla_t b)\rangle \lesssim \mu^{-\frac{3}{2}}\varepsilon^3. \tag{33}$$

3.5 Low-Frequency Estimates

In this subsection, we establish the estimates on the low-frequency errors. For simplicity of presentation, we present the proof of these estimates for the bvb nonlinearity. The estimates with an x-average in the second component are analogous to the ones in Sect. 3.2. The arguments for the vvv, vbb, bbv or ONL trilinear terms are also analogous.

We aim to establish the bound

$$\langle A_\mu^{N'} b, A_\mu^{N'}(v_{\neq}\nabla_t b)\rangle \lesssim \mu^{-\frac{1}{2}}\varepsilon^3,$$

and, as in the previous section, separately discuss the transport, reaction and remainder term.

For the transport term, we note that

$$\begin{aligned}v_{\neq}\nabla_t &= \nabla_t^{\perp}\Lambda_t^{-1} p_1 \nabla_t\\ &= \nabla^{\perp}\Lambda_t^{-1} p_1 \nabla.\end{aligned}$$

Hence, we may rewrite

$$\langle A_\mu^{N'} b, A_\mu^{N'}(v_{\neq}\nabla_t b)\rangle = \langle A_\mu^{N'} b, A_\mu^{N'}(\nabla^\perp \Lambda_t^{-1} p_{1,\neq}\nabla b)\rangle.$$

In a first step, we estimate the $b_{\neq}$ term by using the algebra property of $A^{N'}$:

$$\begin{aligned}
&\langle A_\mu^{N'} b, A_\mu^{N'}(\nabla^\perp \Lambda_t^{-1} p_{1,\neq}\nabla b_{\neq})\rangle \\
&\quad \leq \|A_\mu^{N'} b\|_{L^2} e^{c\mu_x t}\big(\|A^{N'}\nabla^\perp \Lambda_t^{-1} p_{1,\neq}\|_{L^2}\|\nabla b_{\neq}\|_{L^\infty} \\
&\qquad + \|\nabla^\perp \Lambda_t^{-1} p_{1,\neq}\|_{L^\infty}\|A^{N'}\nabla b_{\neq}\|_{L^2}\big) \\
&\quad \leq \|A_\mu^{N'} b\|_{L^2}\big(\|A^N \Lambda_t^{-1} p_{1,\neq}\|_{L^2}\|A_\mu^{N'} b_{\neq}\|_{L^2} + \|A_\mu^{N'}\Lambda_t^{-1} p_{1,\neq}\|_{L^2}\|A^N b_{\neq}\|_{L^2}\big).
\end{aligned}$$

Integrating in time then yields the estimate

$$\int d\tau \langle A_\mu^{N'} b, A_\mu^{N'}(v_{\neq}\nabla_t b_{\neq})\rangle \lesssim \mu^{-\frac{1}{2}}\varepsilon^3. \tag{34}$$

Furthermore, we estimate the $b_=$ term by partial integration and the algebra property of $A^{N'}$

$$\begin{aligned}
&\langle A_\mu^{N'} b, A_\mu^{N'}(\nabla^\perp \Lambda_t^{-1} p_{1,\neq}\nabla b_=)\rangle \\
&\quad = -\langle A_\mu^{N'} b_{1,\neq}, A_\mu^{N'}(\partial_x \Lambda_t^{-1} p_{1,\neq}\partial_y b_{1,=})\rangle \\
&\quad = \langle \partial_x A_\mu^{N'} b_{1,\neq}, A_\mu^{N'}(\Lambda_t^{-1} p_{1,\neq}\partial_y b_{1,=})\rangle \\
&\quad \leq \|\partial_x A_\mu^{N'} b_{1,\neq}\|_{L^2} e^{c\mu t}\big(\|A^{N'}\Lambda_t^{-1} p_{1,\neq}\|_{L^2}\|\partial_y b_{1,=}\|_{L^\infty} \\
&\qquad + \|\Lambda_t^{-1} p_{1,\neq}\|_{L^\infty}\|\partial_y^{N'+1} b_{1,=}\|_{L^2}\big) \\
&\quad \lesssim \|\partial_x A_\mu^{N'} b_{1,\neq}\|_{L^2}\big(\|A_\mu^{N'}\Lambda_t^{-1} p_{1,\neq}\|_{L^2}\|A^{N'} b_{1,=}\|_{L^2} \\
&\qquad + \|A_\mu^{N'}\Lambda_t^{-1} p_{1,\neq}\|_{L^2}\|A^N b_{1,=}\|_{L^2}\big).
\end{aligned}$$

Integrating in time then yields that

$$\int d\tau \langle A_\mu^{N'} b_{\neq}, A_\mu^{N'}(v_{\neq}\nabla_t b_=)\rangle \lesssim \mu^{-\frac{1}{2}}\varepsilon^3. \tag{35}$$

This concludes our proof of Proposition 3.1 and hence of Theorem 2. More precisely, the claimed estimates for both A^N and $A_\mu^{N'}$ are obtained by combining the respective linear estimate (15), the high- frequency nonlinear estimates (16), (17), (18), (19), (20), (33), and the low-frequency estimates given in (34) and (35).

We emphasize that the stability threshold of $\frac{3}{2}$ is determined by the estimates for the action of the $v \cdot \nabla_t b$ nonlinearity in the estimate (33) and, in particular, by the estimates of the reaction terms (28) and (32). These estimates are expected to be optimal and together with the linear estimates of Sect. 2 highlight the effects of the lack of vertical resistivity.

The partial dissipation case considered in this article

$$\kappa_y = 0,\ \nu_x = \nu_y = \kappa_x > 0,$$

shows the large impact of (partial) magnetic resistivity on the behavior of the MHD equations and the (de)stabilizing role of the magnetic field. As mentioned following Theorem 2, more generally our methods of proof extend to the case where κ_x is bounded below in terms of ν:

$$\nu_y^{1/3} \geq \kappa_x \geq \frac{1}{2\alpha}\nu_y.$$

The complementary regime, where κ_x tends to zero quicker than ν_y, remains an interesting topic for future work. The limiting case, $\kappa_x = 0$, and the associated instability is discussed in the following section.

4 Instability of the Non-resistive MHD System

As a complementary result, in this section we consider the non-resistive MHD equations and establish the instability estimates of Proposition 1.

4.1 Linear Instability

We begin by studying the linearized MHD equations with isotropic viscosity and vanishing resistivity:

$$\begin{aligned} \partial_t p_1 - \partial_x \partial_x^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1, \\ \partial_t p_2 + \partial_x \partial_x^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= 0. \end{aligned} \tag{36}$$

Lemma 3 (Quantitative linear instability of the non-resistive MHD equations) *Under the assumptions of Proposition 1, for the linearized equations (36) there exists initial data p_{in} such that*

$$\begin{aligned} \|p(t)\|_{H^N} &\geq t^{\frac{\nu}{8\alpha^2}} \|p_{\text{in}}\|_{H^N}, \\ \|p(t)\|_{H^{N-1}} &\geq t^{\frac{\nu^2}{32\alpha^4}} \|p_{\text{in}}\|_{H^N}. \end{aligned} \tag{37}$$

Furthermore, for all solutions such that at time τ it holds $p(\tau) \in H^N$, then we obtain

$$\|p(t)\|_{H^N} \lesssim \langle \nu(t-\tau) \rangle \|p(\tau)\|_{H^N} \tag{38}$$

for all $t > \tau$.

Proof of Lemma 3 After a Fourier transform (36) yields

$$\begin{aligned} \partial_t p_1(k) &= -\frac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} p_1(k) + \alpha k p_2(k) - \nu(k^2 + (\xi - kt)^2) p_1(k), \\ \partial_t p_2(k) &= \frac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} p_2(k) - \alpha k p_1(k). \end{aligned} \tag{39}$$

Here, in order to simplify notation we have relabeled $p_2 \mapsto ip_2$ so that we obtain only real-valued coefficient functions.

For the lower bound we fix $k = -1$ and $\xi \geq 3\frac{\alpha^2}{\nu}$ and choose $p_1(0, k, \xi) = 0$, $p_2(0, k, \xi) = 1$. In this case, the Duhamel integral formula yields that

$$p_1 = -\alpha \int_0^t \mathrm{d}\tau_1 \sqrt{\frac{1+(\tau_1+\xi)^2}{1+(t+\xi)^2}} \exp(-\nu(t - \tau + \tfrac{1}{3}((t + \xi)^3 - (\tau_1 + \xi)^3))) p_2(\tau_1),$$

and that

$$\begin{aligned} & p_2 - \sqrt{\frac{1+(t+\xi)^2}{1+\xi^2}} \\ &= -\alpha k \int_0^t \mathrm{d}\tau_2 \sqrt{\frac{1+(t+\xi)^2}{1+(\tau_2+\xi)^2}} p_1(\tau_2) \\ &= -\alpha^2 \int_0^t \mathrm{d}\tau_2 \int_0^{\tau_1} \mathrm{d}\tau_1 \frac{\sqrt{1+(t+\xi)^2}\sqrt{1+(\tau_1+\xi)^2}}{1+(\tau_2+\xi)^2} p_2(\tau_1) \\ &\quad \times \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2 + \xi)^3 - (\tau_1 + \xi)^3))) \end{aligned}$$

Denoting $|p_2|_\infty(t) = \sup_{0 \leq \tau \leq t} |p_2(\tau)|$, the double integral term can be bounded by

$$\alpha^2 |p_2|_\infty \int_0^t \mathrm{d}\tau_1 \int_{\tau_2}^t \mathrm{d}\tau_2 \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2 + \xi)^3 - (\tau_1 + \xi)^3))).$$

Furthermore, we may estimate

$$\begin{aligned} & \int_0^t \mathrm{d}\tau_1 \int_{\tau_2}^t \mathrm{d}\tau_2 \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2 + \xi)^3 - (\tau_1 + \xi)^3))) \\ &= \int_0^t \mathrm{d}\tau_1 \int_{\tau_2}^t \mathrm{d}\tau_2 \frac{1+(\tau_2+\xi)^2}{1+(\tau_2+\xi)^2} \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2 + \xi)^3 - (\tau_1 + \xi)^3))) \\ &\leq \int_0^t \mathrm{d}\tau_1 \int_{\tau_2}^t \mathrm{d}\tau_2 \frac{1+(\tau_2+\xi)^2}{1+(\tau_1+\xi)^2} \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2 + \xi)^3 - (\tau_1 + \xi)^3))) \\ &\leq \frac{1}{\nu} \int_0^t \mathrm{d}\tau_1 \frac{1}{1+(\tau_1+\xi)^2} [\exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2 + \xi)^3 - (\tau_1 + \xi)^3)))]_{\tau_2=\tau_1}^{\tau_2=t} \\ &\leq \frac{1}{\nu\xi}. \end{aligned}$$

Hence, we obtain that

$$|p_2 - \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} p_{2,in}| \leq \tfrac{\alpha^2}{\nu\xi} |p_2|_\infty =: c|p_2|_\infty,$$

For later reference, we note that $c = \frac{\alpha^2}{\nu\xi}$ satisfies $0 < c \leq \frac{1}{3}$ and hence $0 < \frac{c}{1-c} \leq \frac{1}{2}$.

Then it follows that

$$|p_2| \leq c|p_2|_\infty + \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}|,$$

and, since $\xi \geq 0$, the function $\sqrt{\frac{1+(t+\xi)^2}{1+\xi^2}}$ is monotonly increasing in time. This implies that

$$|p_2|_\infty \leq \frac{1}{1-c} \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}|$$

Hence, we infer

$$\begin{aligned} |p_2 - \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} p_{2,in}| &\leq \frac{c}{1-c} \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}| \\ &\leq \frac{1}{2} \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}|. \end{aligned}$$

Since $0 < \frac{c}{1-c} \leq \frac{1}{2}$, p_2 is comparable to $\sqrt{\frac{1+(t+\xi)^2}{1+\xi^2}} p_{2,in}$.

We next keep $k = -1$ fixed but combine this construction for different ξ. More precisely, let $a(\xi)$ be such that $\mathrm{supp}_\xi (a(\xi)) \subset [3\frac{\alpha^2}{\nu}, 4\frac{\alpha^2}{\nu}]$ and $\int (2+\xi^2)^{\frac{N}{2}} a^2(\xi) = 1$. Then for the initial data

$$p_{\text{in}}(k, \xi) = \mathbf{1}_{k=-1} a(\xi)$$

it holds that

$$\begin{aligned} \|p_{\text{in}}\|_{H^N} &= 1, \\ \|p(t)\|_{H^N} &\geq t \tfrac{\nu}{8\alpha^2}, \\ \|p(t)\|_{H^{N-1}} &\geq t \tfrac{\nu^2}{32\alpha^4}, \end{aligned}$$

which proves (37).

We prove the upper bound in three steps

(1) Let $t \geq \tau \geq \nu^{-1} + \frac{\xi}{k}$, then we estimate $|p|(t) \lesssim \langle \nu(t-\tau) \rangle |p|(\tau)$.
(2) Let $\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau \geq -\nu^{-1} + \frac{\xi}{k}$, then we estimate $|p|(t) \lesssim |p|(\tau)$.

(3) Let $-\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau$, then we estimate $|p|(t) \lesssim |p|(\tau)$.

From (1–3), estimate (38) follows directly. In the following, we prove (1–3).

(1) Let $t \geq \tau \geq \nu^{-1} + \frac{\xi}{k}$. Then we obtain

$$\begin{aligned}
\partial_t |p|^2 &\leq 2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}|p_2|^2 \\
&\quad + (-2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} - 2\nu k^2(1+(t-\tfrac{\xi}{k})^2))|p_1|^2 \\
&\leq 2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}|p_2|^2.
\end{aligned}$$

Thus, we obtain by Gronwall's lemma

$$\begin{aligned}
|p|^2(t) &\leq \tfrac{1+(t-\frac{\xi}{k})^2}{1+(\tau-\frac{\xi}{k})^2}|p|^2(\tau) \\
&\leq \tfrac{1+(\tau-\frac{\xi}{k})^2+(t-\frac{\xi}{k})^2-(\tau-\frac{\xi}{k})^2}{1+(\tau-\frac{\xi}{k})^2}|p|^2(\tau) \\
&\leq (1+\tfrac{(t-\frac{\xi}{k})^2-(\tau-\frac{\xi}{k})^2}{1+(\tau-\frac{\xi}{k})^2})|p|^2(\tau) \\
&\leq 2\langle \nu(t-\tau)\rangle^2 |p|^2(\tau).
\end{aligned}$$

(2) Let $\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau \geq -\nu^{-1} + \frac{\xi}{k}$. We define the energy

$$E = |p|^2 + \tfrac{2}{\alpha k}\tfrac{s}{1+s^2}p_1 p_2.$$

As $\alpha > \frac{1}{2}$, E is positive definite with

$$(1-\tfrac{1}{2\alpha})|p|^2 \leq E \leq (1+\tfrac{1}{2\alpha})|p|^2.$$

Then, we derive in time and infer

$$\begin{aligned}
\partial_t E + \nu k^2(1+s^2))p_1^2 &= \tfrac{2}{\alpha k}\tfrac{1-s^2}{(1+s^2)^2}p_1 p_2 \\
&\quad + \tfrac{2}{\alpha}\nu k s p_1 p_2 \\
&\leq \tfrac{2}{\alpha k}\tfrac{1}{1+s^2}p_1 p_2 \\
&\quad + \tfrac{2}{\alpha}\nu p_2^2 + \nu k(1+s^2)p_1^2.
\end{aligned}$$

This further implies that

$$\partial_t E \lesssim (\tfrac{1}{1+s^2} + \nu)E.$$

By Gronwall's lemma we infer

$$\begin{aligned} E(t) &\leq \exp(C\int_{\tau}^{t} \tfrac{1}{1+\tau_1^2} + \nu \mathrm{d}\tau_1) E(\tau) \\ &\leq \exp(C(\pi + 2\nu\nu^{-1}) E(\tau) \\ &\lesssim E(\tau). \end{aligned}$$

Therefore, we obtain that

$$|p|(t) \lesssim |p|(\tau).$$

(3) Let $-\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau$. Then we obtain that

$$\begin{aligned} \partial_t |p|^2 &\leq 2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} |p_2|^2 \\ &\quad + (-2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} - 2\nu k^2(1 + (t - \tfrac{\xi}{k})^2)) |p_1|^2 \\ &\leq 0. \end{aligned}$$

Thus we arrive at the desired estimate

$$|p|(t) \lesssim |p|(\tau).$$

4.2 Nonlinear Norm Inflation

We next consider the nonlinear non-resistive MHD equations in their perturbative form around the stationary solution (2):

$$\begin{aligned} \partial_t p_1 - \partial_x \partial_x^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1 + \nabla_t^{\perp} \Lambda_t^{-1} (b \nabla_t b - v \nabla_t v), \\ \partial_t p_2 + \partial_x \partial_x^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \nabla_t^{\perp} \Lambda_t^{-1} (b \nabla_t v - v \nabla_t b). \end{aligned} \tag{40}$$

The following lemma establishes the norm inflation result of Proposition 1.

Lemma 4 (Nonlinear norm inflation for the non-resistive MHD equations) *Under the same assumptions of Proposition 1, we consider the non-resistive nonlinear MHD equations (40). Then for all $C = C(\nu) > 1$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists initial data p_{in} such that*

$$\|p_{\text{in}}\|_{H^N} = \varepsilon$$

and

$$\|p\|_{L^\infty H^N} \geq \varepsilon C.$$

Proof For the sake of contradiction, we assume that there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ and for any choice of initial data with $\|p_{\text{in}}\|_{H^N} = \epsilon$ it holds that

$$\|p\|_{L^\infty H^N} \le \varepsilon C.$$

Our plan is to choose initial data such that for a choice of ε and t we obtain a contradiction to this bound. In particular, we choose p_{in} as the data of the linear instability result, Lemma 3, such that the associated linear solution p_{lin} satisfies

$$\begin{aligned}\|p_{\text{in}}\|_{H^N} &= \varepsilon,\\ \|p_{\text{lin}}(t)\|_{H^{N-1}} &\ge t\tfrac{\nu^2}{32\alpha^4}.\end{aligned}$$

Let $S(\tau, t)$ be the solution operator for the linearized system. Then in view of (38) we have the estimate

$$\|S(\tau, t)\|_{H^N \to H^N} \lesssim \langle \nu(t-\tau)\rangle \le \langle \nu t\rangle. \tag{41}$$

Thus, since

$$\partial_t(p - p_{\text{lin}}) \le L(p - p_{\text{lin}}) + NL[p],$$

we deduce that

$$\begin{aligned}\|p - p_{\text{lin}}\|_{H^{N-1}}^2 &\le \int_0^t \|S(\tau, t)\|_{H^N \to H^N} \|p - p_{\text{lin}}\|_{H^{N-1}} \|p\|_{H^{N-1}} \|\nabla_t p\|_{H^{N-1}}\\ &\lesssim \|p - p_{\text{lin}}\|_{L^\infty H^{N-1}} \|p\|_{L^\infty H^{N-1}} \|p\|_{L^\infty H^N} 2\int_0^t \tau\langle \nu\tau\rangle\\ &\lesssim t^2\langle \nu t\rangle \varepsilon^2 C^2 \|p - p_{\text{lin}}\|_{L^\infty H^{N-1}}.\end{aligned}$$

This yields the estimate

$$\|p - p_{\text{lin}}\|_{L^\infty H^{N-1}} \le \tilde{C} t^2 \langle \nu t\rangle \varepsilon^2 C^2,$$

for some $\tilde{C}$. Finally, we obtain

$$\begin{aligned}\|p\|_{H^{N-1}} &\ge \|p_{\text{lin}}\|_{H^{N-1}} - \|p - p_{\text{lin}}\|_{L^\infty H^{N-1}}\\ &\ge \|p_{\text{lin}}\|_{H^{N-1}} - t^2\langle \nu t\rangle \varepsilon^2 \tilde{C} C^2\\ &\ge t\varepsilon\left(\tfrac{\nu^2}{32\alpha^4} - t^2\langle \nu t\rangle \varepsilon C^2\right).\end{aligned}$$

This completes our proof by contradiction provided this term is large enough for a given small ε and suitable time. Indeed for the choice $\varepsilon \le \frac{1}{8}\frac{\nu^6}{32^3 C^4 \tilde{C}\alpha^{10}}$ at the time $t = 2C\frac{32\alpha^4}{\nu^2}$ it holds that

$$\|p\|_{H^{N-1}} \ge \tfrac{1}{2} t \tfrac{\nu^2}{32\alpha^4}\varepsilon \ge C\varepsilon.$$

This concludes our proof of the nonlinear norm inflation and hence completes our proof of Proposition 1. □

The behavior of the MHD equations and, in particular, the interaction of shear flows, the magnetic field and dissipation are an area of current active research (Liss 2020; Dolce 2023; Zhao and Zi 2023; Knobel and Zillinger 2023). However, prior works have focused on cases where the resistivity is at least as strong as the fluid viscosity and where thus the behavior is closely related to that of the Navier–Stokes equations. In contrast, the non-resistive MHD equations exhibit additional instability, as for instance shown in Proposition 1.

Motivated by this dichotomy, in this article we have studied the anisotropic, partial dissipation regime

$$\kappa_y = 0, \ \kappa_x = \nu_x = \nu_y$$

and the associated stability threshold in the inviscid limit. As shown in Theorem 2 and highlighted in the estimates of Sects. 2, 3.4 and 3.3, this partial dissipation regime behaves qualitatively differently than both the fully dissipative case and the non-resistive case. Moreover, our analysis crucially used the coupling of the velocity field and magnetic field induced by the underlying magnetic field, which allowed us to obtain improved estimates for the magnetic field despite the lack of the symmetry of the dissipation.

Partial, anisotropic dissipation in the MHD equations is thus shown to give rise to distinct regimes with different (in)stability properties and demonstrates an intricate interplay of shear dynamics, magnetic interaction and anisotropic dissipation. A more complete understanding of all these regimes, the case of resistivity vanishing faster than viscosity and a characterization of the (in)stability properties of the ideal MHD equations remain exciting questions for future research.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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