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# Natural convection in the horizontal annulus: Critical Rayleigh number for the steady problem

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For the 2D Oberbeck–Boussinesq system in an annulus, we are looking for the critical Rayleigh number for which the (non-zero) basic flow loses stability. For this, we consider the corresponding Euler–Lagrange equations and construct a precise functional analytical frame for the Laplace and the Stokes problem as well as the Bilaplacian operator in this domain. With this frame and the right set of basis functions, it is then possible to construct and apply a numerical scheme providing the critical Raleigh number.

## 1 | INTRODUCTION

Consider the flow of an incompressible Newtonian fluid between two horizontal coaxial cylinders with radii $0 < R_i < R_o$ (cf. Figure 1). The flow is driven by a gravitational field perpendicular to the cylindrical axis and the temperature difference between $T_i$ on the inner and $T_o$ on the outer jacket (where $T_o < T_i$).

This setting is a model to study very diverse phenomena, such as thermal energy storage systems, aircraft cabin insulation, cooling of electronic components, electrical power cable and thin films.

In this configuration, the flow is mainly characterized by two non-dimensional parameters: the "thinness" of the gap between the cylinders, which we measure as the *inverse relative gap width*

$$\mathcal{A} := \frac{2R_i}{R_o - R_i} \quad \text{and} \quad \mathrm{Ra} := \frac{\alpha g}{\nu k}(T_i - T_o)(R_o - R_i)^3 , \tag{1.1}$$

the *Rayleigh number*, which classifies the heat transfer regime in the flow. More precisely, it measures the ratio between conduction and convection since in the definition, we have $\alpha$ as the volumetric expansion coefficient, $g$ the gravity acceleration, $\nu$ the kinematic viscosity and $k$ stands for the thermal diffusivity.

The study of convective flows between horizontal coaxial cylinders inside a gravitational field shows (see references in [8]) that only for an intermediate range of the parameter $\mathcal{A}$ it is interesting to look at three-dimensional behaviour, and for $\mathcal{A} < 2,8$ (wide gap) as well as $\mathcal{A} > 8,5$ (small gap) everything interesting happens in the cross-section. For that in

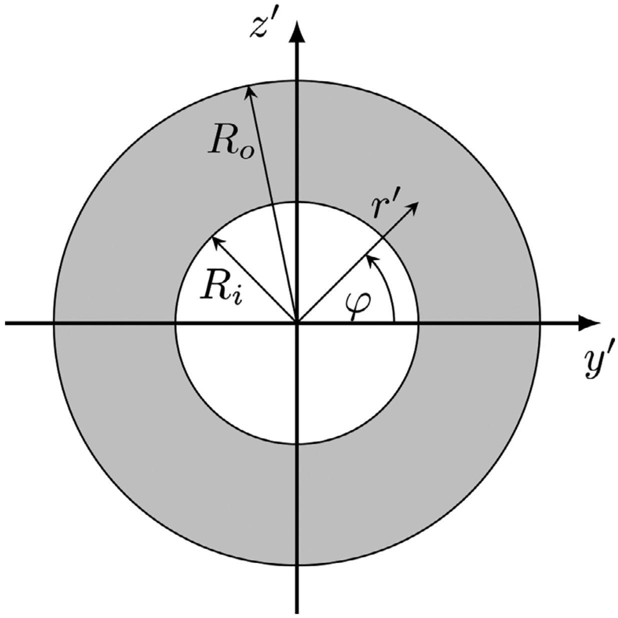


**FIGURE 1** Two-dimensional flow region – temperature boundary values $T_i > T_o$.

this paper, we study the problem in the two-dimensional annulus (i.e. the cross-section of the above mentioned geometry) knowing that our study will also give information for the three-dimensional flow problem in the small and wide gap cases.

In contrast to the classical Bénard problem in our geometry, even for very small Rayleigh numbers, there is never a zero velocity solution. Instead there is a so-called basic flow which solves our equations for any Ra but for which we do not know an analytical expression. For that, the simplest methods which work for the Bénard problem cannot be adapted for the annulus. But we can still work with the characterisation of the critical Rayleigh number $\mathrm{Ra}_c$ as the inverse of the supremum of the ratio between functionals for the kinetic energy and the heat transfer in Equation (2.3) below (see, e.g., Straughan [21]). In order to calculate our $\mathrm{Ra}_c$ with this supremum, we first derive the corresponding Euler–Lagrange equations and later on transform them to an eigenvalue problem for a compact self-adjoint operator. For this, we have to make sure that the abstract procedure translates into a precisely known frame for our operators. Here our paper becomes a bit technical since we need a few convenient properties of all involved operators (which partially are almost obvious and partially need a few lines of proof and functional analytical results). In particular, we prove that the supremum exists and is indeed a maximum in Lemma 2.1. Moreover, it is finite and thus, the critical Rayleigh number as defined in Equation (2.3) is finite as well.

Unfortunately, it is not straightforward to use standard eigenvalue software in order to numerically solve the found eigenvalue problem for the critical Rayleigh number since it turns out not to be stable. Instead we use special sets of bases of eigenfunctions of the Laplace and the Stokes problem in the annulus (which contain Bessel functions) to formulate and apply a numerical scheme which is stable and approximates the wanted $\mathrm{Ra}_c$ without having to refer to determinants which are too big to handle comfortably. Here we have to translate the representation of operators in the existence proof to a (slightly) different set of operators which make the numerical scheme stable.

Finally, in two-dimensional flow, people often prefer to work with the streamfunction formulation. For that, we include how to formulate the functional analytical frame with the streamfunction and the Bilaplace operator in the annulus.

Let us introduce the necessary *notation*.

**General notation A**. Let $\mathbb{R}^2$ be endowed with the usual Euclidian norm $\|.\|$ and elements of $\mathbb{R}^2$ be denoted by underlined small letters. The unit circle is $\omega := \{\underline{x} \in \mathbb{R}^2 : \|\underline{x}\| = 1\}$ and closed circles around the origin with radius $r$ are $\omega_r := \{\underline{x} \in \mathbb{R}^2 : \|\underline{x}\| = r\} = r\omega$ for all $r \in (0, \infty)$.

**Annulus domains**. It is useful to study our domain without dimensions. As usual, we pick the annulus with fixed gap width 1 using our non-dimensional parameter $\mathcal{A}$ as follows: For any $\mathcal{A} \in (0, \infty)$, we denote by

$$\Omega_{\mathcal{A}} := \{\underline{x} \in \mathbb{R}^2 : \mathcal{A}/2 < \|\underline{x}\| < 1 + \mathcal{A}/2\}.$$

Its boundary $\partial\Omega_{\mathcal{A}}$ consists of two parts, namely the inner and outer boundary $\omega_{\mathcal{A}/2}$ and $\omega_{1+\mathcal{A}/2}$, respectively.

**General notation B**. Let $\Omega$ be any of the domains introduced above. In what follows, we will use by way of an abbreviation (.) for ($\Omega$) everywhere. We consider the usual Lebesgue and Sobolev spaces $\mathbb{L}_2(.)$ and $\mathbb{W}_2^k(.)$ of scalar functions and the Lebesgue and Sobolev spaces $\underline{\mathbb{L}}_2(.) = (\mathbb{L}_2(.))^2$ and $\underline{\mathbb{W}}_2^k(.) = (\mathbb{W}_2^k(.))^2$ of vector functions. The scalar product in $\mathbb{L}_2(.)$ and $\underline{\mathbb{L}}_2(.)$ is written as $(.,.)_2 := (.,.)$ and the norms are denoted by $\|.\|_2$. The notation $\overset{o}{\mathbb{W}}{}_2^k(.)$ is taken for the closure of $C_o^\infty(.)$ in $\mathbb{W}_2^k(.)$. All solenoidal vector functions belonging to $\underline{C}_o^\infty(.)$ form $\underline{\mathcal{V}}(.)$. The closures of $\underline{\mathcal{V}}(.)$ in $\underline{\mathbb{L}}_2(.)$ and $\underline{\mathbb{W}}_2^1(.)$, respectively, are denoted by $\underline{\mathbb{H}}(.)$ and $\underline{\mathbb{V}}(.)$, respectively. We suppose for all function written in polar coordinates $(r, \varphi)$ the general periodicity in the angular coordinate $\varphi$. We define the dual spaces $\mathbb{W}_2^{-2}(.) := (\overset{o}{\mathbb{W}}{}_2^2(.))'$, $\mathbb{W}_2^{-1}(.) := (\overset{o}{\mathbb{W}}{}_2^1(.))'$ and $\underline{\mathbb{V}}'(.) := (\underline{\mathbb{V}}(.))'$ and use the evolution (Gelfand) triples

$$[\overset{o}{\mathbb{W}}{}_2^2(.), \mathbb{L}_2(.), \mathbb{W}_2^{-2}(.)], \quad [\overset{o}{\mathbb{W}}{}_2^1(.), \mathbb{L}_2(.), \mathbb{W}_2^{-1}(.)] \quad \text{and} \quad [\underline{\mathbb{V}}(.), \underline{\mathbb{H}}(.), \underline{\mathbb{V}}'(.)] . \tag{1.2}$$

**General notation C**. Due to the shape of our domain together with Cartesian, we also use polar coordinates – whichever makes more sense. In particular, $\underline{e}_r$ is the unit vector in direction $r$, $\underline{e}_3 = \sin\varphi\,\underline{e}_r + \cos\varphi\,\underline{e}_\varphi$ the unit vector in direction of $z$, and we collect all arising gradients in $\Pi$ (thus, it is not the thermodynamical pressure).

$$b := \ln \frac{R_o}{R_i} = \ln\left(1 + \frac{2}{\mathcal{A}}\right), \tag{1.3}$$

is a purely geometric parameter, unbounded as $\mathcal{A}$ tends to zero. Moreover, we have a number for material properties, the Prandtl number $\mathrm{Pr} = \nu/k$. Instead of the temperature $T$, we treat the *excess temperature*

$$\tau := \frac{T}{T_i - T_o} - T^* = \frac{T}{T_i - T_o} - \frac{T_i}{T_i - T_o} + \frac{1}{b}\left(\ln r - \ln \frac{R_i}{R_o - R_i}\right) = \frac{T}{T_i - T_o} - \frac{T_i}{T_i - T_o} + \frac{1}{b}\ln\frac{2r}{\mathcal{A}} . \tag{1.4}$$

Here, the scalar field $T^*$ is the conductive solution – namely, $T^*$ solves $\Delta T^* = 0$ with boundary conditions $T^*(\frac{\mathcal{A}}{2}, \varphi) = \frac{T_i}{T_i - T_o}$ and $T^*(\frac{\mathcal{A}}{2} + 1, \varphi) = \frac{T_o}{T_i - T_o}$.

**Equations**. The full non-dimensional Oberbeck–Boussinesq system in polar coordinates on $(0, \infty) \times \Omega_{\mathcal{A}}$ is

$$\begin{aligned} \frac{1}{\mathrm{Pr}}(\partial_t \underline{v} + (\underline{\nabla}^T \cdot \underline{v})^T \cdot \underline{v}) - \Delta \underline{v} + \underline{\nabla}\Pi &= \frac{\mathrm{Ra}}{b}\sin\varphi\,\underline{e}_r + \mathrm{Ra}\,\tau\underline{e}_3 , \\ \operatorname{div}\underline{v} &= 0 , \\ \partial_t\tau + \underline{v}^T \cdot \underline{\nabla}\tau - \Delta\tau &= \frac{1}{r\,b}\underline{v}^T \cdot \underline{e}_r \end{aligned} \tag{1.5}$$

(see, e.g. Ferrario et al. [8]) endowed with the boundary and initial conditions

$$\underline{v} = \underline{0}, \quad \tau = 0 \quad \text{on} \quad \partial\Omega_{\mathcal{A}} \quad \text{and} \quad \underline{v} = \underline{v}_0, \quad \tau = \tau_0 \quad \text{for } t = 0 . \tag{1.6}$$

In Ferrario et al. [8], it is outlined in detail that the stability analysis of Equation (1.5) is closely related to the investigation of the stability of the basic flow the following linear homogeneous system

$$\begin{aligned} \underline{\nabla}^T \cdot \underline{w} &= 0 , \\ \frac{1}{\mathrm{Pr}}\partial_t \underline{w} - \Delta\underline{w} + \underline{\nabla}p &= \mathrm{Ra}\,\theta\,\underline{e}_3 , \\ \partial_t\theta - \Delta\theta &= \frac{w_r}{r\,b} \end{aligned} \tag{1.7}$$

with $\underline{w}$ and $\theta$ as unknowns and zero Dirichlet boundary conditions. In particular, it is shown in Ferrario et al. [8, Sec. 4] that the number $Ra_c$ (defined below) is a good approximation of the critical Raleigh number for the asymptotic non-linear stability of steady flows for small $\mathcal{A}$. In this regard we also refer to [26]. We study Equation (1.7) in detail.

## 2 | DESCRIPTION BY FUNCTIONALS AND EULER–LAGRANGE EQUATIONS

The validity of *Poincaré inequalities* ensures that the spaces $\overset{o}{\mathbb{W}}{}^1_2(.)$ and $\underline{\mathbb{V}}(.)$ can be equipped additionally with equivalent norms generated by the Dirichlet norms (which otherwise would only be semi-norms). We use Cartesian coordinates to introduce the Dirichlet scalar products and Dirichlet norms here.

$$(u,v)_D := \sum_{k=1}^{2}\left(\frac{\partial u}{\partial x_k},\frac{\partial v}{\partial x_k}\right)_2 \;\forall u,v \in \overset{o}{\mathbb{W}}{}^1_2(.),\quad (\underline{u},\underline{v})_D := \sum_{j,k=1}^{2}\left(\frac{\partial u_j}{\partial x_k},\frac{\partial v_j}{\partial x_k}\right)_2 \quad \forall \underline{u},\underline{v} \in \underline{\mathbb{V}}(.)$$
$$\|u\|_D := \sqrt{(u,u)_D} \quad \forall u \in \overset{o}{\mathbb{W}}{}^1_2(.),\qquad \|\underline{u}\|_D := \sqrt{(\underline{u},\underline{u})_D} \qquad \forall \underline{u} \in \underline{\mathbb{V}}(.), \tag{2.1}$$

where the so-called Frobenius inner product is involved in the definition.

In this paper, we give a procedure to compute the critical Rayleigh number $\mathrm{Ra}_c$ starting with a standard method, for example, from Straughan [21]. So, for fixed $\mathcal{A}$ and given functions $(\underline{w},\theta) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}^1_2$, we define the following functionals, as well as a constant denoted by $\mathrm{Ra}_c$:

$$D(\underline{w},\theta) := \|\underline{w}\|_D^2 + \|\theta\|_D^2\,,$$
$$\mathcal{F}(\underline{w},\theta) := (\theta, w_z) + \frac{1}{b}\left(\theta,\frac{w_r}{r}\right), \qquad \text{where } w_z := \underline{w}^T\cdot\underline{\mathfrak{e}}_3\,, \tag{2.2}$$
$$\frac{1}{\mathrm{Ra}_c(\mathcal{A})} := \sup\frac{\mathcal{F}(\underline{w},\theta)}{D(\underline{w},\theta)}\,, \tag{2.3}$$

where the supremum is taken over all non-trivial couples of functions $(\underline{w},\theta) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}^1_2$.

**Lemma 2.1.** *The supremum in Equation (2.3) is attained as maximum by a pair of functions fulfilling the Euler–Lagrange equations. This means in particular that a critical point of $F/D$ exists.*

*Proof.* Let $\underline{\mathfrak{w}} := (\underline{w},\theta) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}^1_2 := X$. We define

$$J(\underline{\mathfrak{w}}) := \frac{\mathcal{F}(\underline{w},\theta)}{D(\underline{w},\theta)} \quad \text{and} \quad J_0 := \sup J\,,$$

where the supremum is taken as in Equation (2.3). We can directly check that $J$ is homogeneous of degree zero in real $\lambda \neq 0$, or explicitly $J(\underline{\mathfrak{w}}) = J(\lambda\underline{\mathfrak{w}})\ \forall\lambda\neq 0$ and thus values of $J$ are determined by values on the unit sphere $\|\underline{\mathfrak{w}}\|_X = \sqrt{D(\underline{w},\theta)} = 1$. $J$ is not infinitely large, thus $J_0 < \infty$. To see this, we use the boundedness of $\mathcal{F}(.,.)$ (2.2) for $\underline{\mathfrak{w}} := (\underline{w},\theta) \in \underline{\mathbb{H}} \times \mathbb{L}_2 := X_1$ and the compact embedding of $X$ in $X_1$ as well as $\|\underline{\mathfrak{w}}\|_X = 1$. Therefore, the set $\mathcal{F}(\{\underline{\mathfrak{w}} : \|\underline{\mathfrak{w}}\|_X = 1\})$ is compact. Due to compactness, $\lim \mathcal{F}(\underline{\mathfrak{w}}_n) = \mathcal{F}(\underline{\mathfrak{w}}_0)$ and it is easy to see that $\mathcal{F}(\underline{\mathfrak{w}}_0)$ is bounded from above by a constant. For that, there exists a sequence $\{\underline{\mathfrak{w}}_n\}_{n=1}^{\infty}$ ($\mathfrak{w}_n \in X$ for all elements of the sequence) for which $J(\underline{\mathfrak{w}}_n)$ converges to $J_0$. Its projection to the unit sphere converges to $J_0$ as well. Then there is a subsequence which converges weakly in $X$ to some value $\underline{\mathfrak{w}}_0 \in X$, strongly in $X_1 = \underline{\mathbb{H}} \times \mathbb{L}_2$ and almost everywhere. $J(\underline{\mathfrak{w}}_n) \to J(\underline{\mathfrak{w}}_0)$ and thus, the supremum in Equation (2.3) exists. Next we check due to lower semicontinuity and since $\mathcal{D}(\underline{\mathfrak{w}}_n)$ exists, that

$$J_0 = \lim J(\underline{\mathfrak{w}}_n) = \frac{\lim \mathcal{F}(\underline{\mathfrak{w}}_n)}{\lim \mathcal{D}(\underline{\mathfrak{w}}_n)} \leq \frac{\mathcal{F}(\underline{\mathfrak{w}}_0)}{\mathcal{D}(\underline{\mathfrak{w}}_0)} \leq J_0.$$

$\Rightarrow \quad J(\underline{\mathfrak{w}}_0) = \sup_{\underline{\mathfrak{w}}\in X} J(\underline{\mathfrak{w}})$. If the supremum in Equation (2.3) is attained for some[1] $(\underline{\widetilde{w}},\widetilde{\theta})$, then $(\underline{\widetilde{w}},\widetilde{\theta})$ solve the corresponding Euler–Lagrange equations, which we derive now. To this end, we consider for an arbitrary real parameter $\eta$ and

[1] The notation $(\underline{\widetilde{w}},\widetilde{\theta})$ with tilde means that the stationary point has nothing to do with the solutions of Equation (1.7); as we will see, it is solution of a different system of equations.

arbitrary *variations* $(\underline{\widetilde{u}},\widetilde{\sigma}) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$ the real function

$$\frac{\mathcal{F}}{D} = \frac{\mathcal{F}(\underline{\widetilde{w}} + \eta\underline{\widetilde{u}}, \widetilde{\theta} + \eta\widetilde{\sigma})}{D(\underline{\widetilde{w}} + \eta\underline{\widetilde{u}}, \widetilde{\theta} + \eta\widetilde{\sigma})}.$$

Since $(\underline{\widetilde{w}},\widetilde{\theta})$ is a maximum the Gateaux derivative $\frac{d}{d\eta}|_{\eta=0}$ has to vanish. Thus,

$$0 = \frac{d}{d\eta}\left(\frac{\mathcal{F}}{D}\right)\bigg|_{\eta=0} = \left[\frac{1}{D^2}\left(D\frac{d}{d\eta}\mathcal{F} - \mathcal{F}\frac{d}{d\eta}D\right)\right]_{\eta=0} = \frac{\frac{d\mathcal{F}}{d\eta}}{D}\Bigg|_{\eta=0} - \frac{1}{\mathrm{Ra}_c}\frac{\frac{dD}{d\eta}}{D}\Bigg|_{\eta=0}. \tag{2.4}$$

Inserting the expressions

$$\frac{d\mathcal{F}}{d\eta}\bigg|_{\eta=0} = (\widetilde{\sigma},\widetilde{w}_z) + (\widetilde{\theta},\widetilde{u}_z) + \frac{1}{b}\left(\left(\widetilde{\theta},\frac{\widetilde{u}_r}{r}\right) + \left(\widetilde{\sigma},\frac{\widetilde{w}_r}{r}\right)\right),$$

$$\frac{dD}{d\eta}\bigg|_{\eta=0} = 2(\underline{\widetilde{u}},\underline{\widetilde{w}})_D + 2(\widetilde{\theta},\widetilde{\sigma})_D$$

into Equation (2.4), we conclude that for all variations $(\underline{\widetilde{u}},\widetilde{\sigma}) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$, it holds

$$(\widetilde{\sigma},\widetilde{w}_z) + (\widetilde{\theta},\widetilde{u}_z) + \frac{1}{b}\left(\left(\widetilde{\theta},\frac{\widetilde{u}_r}{r}\right) + \left(\widetilde{\sigma},\frac{\widetilde{w}_r}{r}\right)\right) = \frac{2}{\mathrm{Ra}_c}\left((\underline{\widetilde{u}},\underline{\widetilde{w}})_D + (\widetilde{\theta},\widetilde{\sigma})_D\right). \tag{2.5}$$

□

If we assume higher regularity of $(\underline{\widetilde{w}},\widetilde{\theta})$, we get by integrating by parts

$$(\widetilde{\sigma},\widetilde{w}_z) + (\widetilde{\theta},\widetilde{u}_z) + \frac{1}{b}\left(\left(\widetilde{\theta},\frac{\widetilde{u}_r}{r}\right) + \left(\widetilde{\sigma},\frac{\widetilde{w}_r}{r}\right)\right) = -\frac{2}{\mathrm{Ra}_c}\left((\Delta\underline{\widetilde{w}},\underline{\widetilde{u}}) + (\Delta\widetilde{\theta},\widetilde{\sigma})\right).$$

We have the freedom to choose either (a) arbitrary $\underline{\widetilde{u}}$ and $\widetilde{\sigma} = 0$; or (b) $\underline{\widetilde{u}} = 0$ and arbitrary $\widetilde{\sigma}$. In the first case, we obtain

$$\int_{\Omega_\mathcal{A}}\left(\widetilde{\theta}\underline{\mathfrak{e}}_3 + \frac{\widetilde{\theta}}{br}\underline{\mathfrak{e}}_r + \frac{2}{\mathrm{Ra}_c}\Delta\underline{\widetilde{w}}\right)^T \cdot \underline{\widetilde{u}}\,d\underline{x} = 0 \quad \Rightarrow \quad \widetilde{\theta}\underline{\mathfrak{e}}_3 + \frac{\widetilde{\theta}}{br}\underline{\mathfrak{e}}_r + \frac{2}{\mathrm{Ra}_c}\Delta\underline{\widetilde{w}} + \underline{\nabla}\,\widetilde{p} = 0.$$

Here we used that $\underline{\widetilde{u}}$ is arbitrary but divergence free, which implies that the term in brackets belongs to the orthogonal complement with respect to Helmholtz's decomposition. That is why $\underline{\nabla}\,\widetilde{p}$ is added. In the second case, we get

$$\int_{\Omega_\mathcal{A}}\left(\widetilde{w}_z + \frac{\widetilde{w}_r}{br} + \frac{2}{\mathrm{Ra}_c}\Delta\widetilde{\theta}\right)\widetilde{\sigma}d\underline{x} = 0 \quad \Rightarrow \quad \widetilde{w}_z + \frac{\widetilde{w}_r}{br} + \frac{2}{\mathrm{Ra}_c}\Delta\widetilde{\theta} = 0.$$

Moreover, if we set

$$\underline{\mathfrak{e}}_3 + \frac{\underline{\mathfrak{e}}_r}{br} = \underline{\nabla}S, \qquad \text{where} \quad S := r\sin\varphi + \frac{1}{b}\ln r, \tag{2.6}$$

the strong form of the Euler–Lagrange equations for the maximum $(\underline{\widetilde{w}},\widetilde{\theta})$ of the functional $\mathcal{F}/D$ read as

$$\widetilde{\theta}\underline{\nabla}S + \lambda\Delta\underline{\widetilde{w}} = -\underline{\nabla}\,\widetilde{p}, \quad \underline{\nabla}^T \cdot \underline{\widetilde{w}} = 0, \tag{2.7}$$

$$\underline{\widetilde{w}}^T \cdot \underline{\nabla}S + \lambda\Delta\widetilde{\theta} = 0 \qquad \text{with} \qquad \lambda = \frac{2}{\mathrm{Ra}_c}. \tag{2.8}$$

In what follows, we regard the problem (2.7)–(2.8) in the form of

*Problem A* (Euler–Lagrange equations of the functional $\mathcal{F}/D$). We search for non-trivial solutions $(\lambda, \underline{w}, \theta) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}^1_2$ of the equations

$$\theta \underline{\nabla} S + \lambda \Delta \underline{w} = -\underline{\nabla}\, p\,, \quad \underline{\nabla}^T \cdot \underline{\widetilde{w}} = 0\,, \tag{2.9}$$

$$\underline{w}^T \cdot \underline{\nabla} S + \lambda \Delta \theta = 0 \qquad \text{with} \qquad \lambda \in \mathbb{C}\,, \tag{2.10}$$

where Equations (2.9) and (2.10) are equations in $\underline{\mathbb{V}}'(.)$ resp. $\mathbb{W}_2^{-1}(.)$. □

We will show that Problem A can be equivalently formulated as an eigenvalue problem, which possesses a solution for all non-trivial eigenvalues $\lambda$ with the corresponding eigenvectors. We then define the critical Rayleigh number $\mathrm{Ra}_c$ via the largest eigenvalue $\lambda_{\max}$ through $\mathrm{Ra}_c := \frac{2}{\lambda_{\max}}$.

We now prepare the framework to re-formulate the Problem A as an eigenvalue problem.

## 3 | STANDARD DIFFERENTIAL OPERATORS AND THE SYMMETRIC OPERATOR $\widetilde{A}$

In what follows, we take $\Omega := \Omega_{\mathcal{A}}$ as symbol for any of the annulus domains. The differential operators we will introduce here are the Laplacian, the Stokes operator and the Bilaplacian in the sense of functional analysis. The domains of definition of these operators are dense in the Hilbert spaces $\mathbb{L}_2(\Omega)$, $\underline{\mathbb{H}}(\Omega)$ and $\mathbb{L}_2(\Omega)$.

**Definition A.** The Laplace operator is defined as

$$\boldsymbol{L}^\circ\, v := -\Delta_{\underline{x}} v \qquad \forall\, v \in D(\boldsymbol{L}^\circ) = C_o^\infty(\Omega)\,.$$

We denote the Friedrichs' extension of $\boldsymbol{L}^\circ$ by $\boldsymbol{L} := \overline{\boldsymbol{L}^\circ}$, where $\boldsymbol{L}$ works on $D(\boldsymbol{L}) := \overset{o}{\mathbb{W}}{}^1_2(\Omega) \cap \mathbb{W}^2_2(\Omega)$. □

*Remark.* The range of $\boldsymbol{L}$ is $R(\boldsymbol{L}) = \mathbb{L}_2(\Omega)$. In this sense, we write: $\boldsymbol{L} = -\Delta_{\underline{x}} : D(\boldsymbol{L}) \subseteq \mathbb{L}_2(\Omega) \longmapsto \mathbb{L}_2(\Omega)$.

We need some preparations to define the Stokes operator. The Leray–Helmholtz projector $\Upsilon$ is the well-defined projector of $\underline{\mathbb{L}}_2(.)$ onto its subspace $\underline{\mathbb{H}}(.)$ of generalised solenoidal fields with vanishing generalised traces in the normal direction on the boundary. We note that the Leray–Helmholtz projector $\Upsilon$ is also used in the sense of: $\Upsilon : \underline{\mathbb{W}}^1_2(\Omega) \longmapsto \underline{\mathbb{V}}(\Omega)$.

**Definition B.** The Stokes operator is defined as $\boldsymbol{S}^\circ\, \underline{v} := -\Delta_{\underline{x}} \underline{v} \quad \forall\, \underline{v} \in D(\boldsymbol{S}^\circ) = \underline{\mathcal{V}}(\Omega)$. We denote the Friedrichs' extension of $\boldsymbol{S}^\circ$ by $\boldsymbol{S} := \overline{\boldsymbol{S}^\circ}$, where $\boldsymbol{S}$ is defined on its domain $D(\boldsymbol{S}) = \underline{\mathbb{S}}^2(\Omega) = \underline{\mathbb{V}}^2(\Omega) := \underline{\mathbb{W}}^2_2(\Omega) \cap \underline{\mathbb{V}}(\Omega)$.

*Remark.* The range of $\boldsymbol{S}$ is $R(\boldsymbol{S}) = \underline{\mathbb{H}}(.)$. In this context, we use $\boldsymbol{S} = -\Upsilon \Delta_{\underline{x}} : \underline{\mathbb{S}}^2(.) \subseteq \underline{\mathbb{H}}(.) \longmapsto \underline{\mathbb{H}}(.)$.

Finally, we give the definition of the Bilaplacian:

**Definition C.** The Bilaplacian (biharmonic operator) is defined as

$$\boldsymbol{B}^\circ\, v := \Delta^2_{\underline{x}} v \qquad \forall\, v \in D(\boldsymbol{B}^\circ) = C_o^\infty(\Omega)\,.$$

We denote the Friedrichs' extension of $\boldsymbol{B}^\circ$ by $\boldsymbol{B} := \overline{\boldsymbol{B}^\circ}$, where the domain of the Friedrichs' extension $\boldsymbol{B}$ is $D(\boldsymbol{B}) := \overset{o}{\mathbb{W}}{}^2_2(\Omega) \cap \mathbb{W}^4_2(\Omega)$.

*Remark.* The range of $\boldsymbol{B}$ is $R(\boldsymbol{B}) = \mathbb{L}_2(\Omega)$. We write here: $\boldsymbol{B} = \Delta^2_{\underline{x}} : D(\boldsymbol{B}) \subseteq \mathbb{L}_2(.) \longmapsto \mathbb{L}_2(\Omega)$ also.

We recall the essential properties of the operators $\boldsymbol{L}$, $\boldsymbol{S}$ as well as $\boldsymbol{B}$ using the Stokes operator $\boldsymbol{S}$ as example:

**Theorem 3.1.** *The Stokes operator $\boldsymbol{S}$ is positive and self-adjoint. Its inverse $\boldsymbol{S}^{-1}$ is injective, self-adjoint and compact.*

The proof of Theorem 3.1 is a simple modification of Theorems 4.3 and 4.4 in Constantin and Foias [2]. The essential tools are the Rellich theorem and Lax–Milgram lemma. The well-known theorem of Hilbert and regularity results like Temam [22, Prop. I.2.2] lead to more detailed results.

**Lemma 3.2.** *The Stokes operator is an operator with a pure point spectrum. All eigenvalues $\lambda_j^* = (\kappa_j)^2$ of $\boldsymbol{S}$ are real and of finite multiplicity. The associated eigenfunctions $\{\underline{v}_j(\underline{x})\}_{j=1}^{\infty}$ of the Stokes operator $\boldsymbol{S}$ (counted in multiplicity) are an orthogonal basis of $\underline{\mathbb{H}}(.)$ and $\underline{\mathbb{V}}(.)$. We obtain that*

$$(a)\quad \boldsymbol{S}\underline{v}_j = \lambda_j^* \underline{v}_j \quad \text{for} \quad \underline{v}_j \in D(\boldsymbol{S}) \quad \forall\, j = 1, 2, \dots\,;$$

$$(b)\quad 0 < \lambda_1^* \le \lambda_2^* \le \cdots \le \lambda_j^* \le \cdots \quad \text{and} \quad \lim_{j\to\infty} \lambda_j^* = \infty\,;$$

$$(c)\quad \|\underline{v}_j\|_{\underline{\mathbb{H}}} = 1 \quad \forall\, j = 1, 2, \dots\,.$$

Notation 3.3. We write the eigenpair-systems of the Laplacian $\boldsymbol{L}$, the Stokes operator $\boldsymbol{S}$ (cf. Lemma 3.2) and of the Bilaplacian $\boldsymbol{B}$ as:

$$\{(\omega_j)^2, \chi_j\}_{j=1}^{\infty}, \qquad \{(\kappa_j)^2, \underline{v}_j\}_{j=1}^{\infty} = \{\lambda_j^*, \underline{v}_j\}_{j=1}^{\infty} \quad \text{and} \quad \{\mu_j^2, \psi_j\}_{j=1}^{\infty}. \tag{3.1}$$

We choose the systems of eigenfunctions of $\boldsymbol{L}$: $\{\chi_j\}_{j=1}^{\infty}$, of $\boldsymbol{S}$: $\{\underline{v}_j\}_{j=1}^{\infty}$ and of $\boldsymbol{B}$: $\{\chi_j\}_{j=1}^{\infty}$ ordered by increasing eigenvalues and counted in multiplicity as an orthonormal basis in each case, such that:

$$\mathbb{L}_2 = \overline{\operatorname{span}\{\chi_j\}_{j=1}^{\infty}}^{\mathbb{L}_2}, \qquad \underline{\mathbb{H}} = \overline{\operatorname{span}\{\underline{v}_j\}_{j=1}^{\infty}}^{\underline{\mathbb{H}}} \quad \text{and} \quad \mathbb{L}_2 = \overline{\operatorname{span}\{\psi_j\}_{j=1}^{\infty}}^{\mathbb{L}_2}.$$

*Remark.* We refer to Lee and Rummler [12] for formulas defining the complete sets of Laplace and Stokes eigenfunctions on the circular annuli $\Omega_\sigma^*$, with

$$\Omega_\sigma^* := \{\underline{x} \in \mathbb{R}^2 \,:\, 0 < \sigma < \|\underline{x}\| < 1\}.$$

The (obvious) transformation rules from $\Omega_\sigma^*$ to $\Omega_{\mathcal{A}}$ are given in Rummler et al. [19].

Notation 3.4. Let

$$\widetilde{\boldsymbol{A}} : \underline{\mathbb{V}}^2(\Omega) \times (\overset{o}{\mathbb{W}}{}_2^1(\Omega) \cap \mathbb{W}_2^2(\Omega)) = D(\boldsymbol{S}) \times D(\boldsymbol{L}) \subseteq \underline{\mathbb{H}}(\Omega) \times \mathbb{L}_2(\Omega) \longrightarrow \underline{\mathbb{H}}(\Omega) \times \mathbb{L}_2(\Omega)$$

$$\widetilde{\boldsymbol{A}} = (A_1, A_2) \quad \text{with} \quad \begin{cases} A_1 \,:\, \underline{\mathbb{V}}^2(.) \times \overset{o}{\mathbb{W}}{}_2^1(\Omega) \cap \mathbb{W}_2^2(\Omega) \subseteq \underline{\mathbb{H}}(\Omega) \times \mathbb{L}_2(\Omega) \longrightarrow \underline{\mathbb{H}}(\Omega) \\ A_2 \,:\, \underline{\mathbb{V}}^2(.) \times \overset{o}{\mathbb{W}}{}_2^1(\Omega) \cap \mathbb{W}_2^2(\Omega) \subseteq \underline{\mathbb{H}}(\Omega) \times \mathbb{L}_2(\Omega) \longrightarrow \mathbb{L}_2(\Omega) \end{cases}$$

$$A_i^{\underline{\mathbb{V}}} := A_i|_{D(\boldsymbol{S})\times\{0\}}; \quad A_i^{\mathbb{W}} := A_i|_{\{\underline{0}\}\times D(\boldsymbol{L})}, \quad i = 1, 2,$$

be the operators defined by density $\forall\, \underline{\Psi} \in D(\boldsymbol{S})$ and $\forall\, \varphi \in D(\boldsymbol{L})$ via

$$A_i = A_i^{\underline{\mathbb{V}}} + A_i^{\mathbb{W}}, \quad i = 1, 2 \;:$$

$$(\underline{\Psi}, A_1^{\underline{\mathbb{V}}}\underline{w} + A_1^{\mathbb{W}}\theta) = -\lambda\big(\underline{\Psi}, \underline{w}\big)_D + \big(\underline{\Psi}, \theta\underline{\nabla} S\big)$$

$$(\varphi, A_2^{\underline{\mathbb{V}}}\underline{w} + A_2^{\mathbb{W}}\theta) = -\lambda(\varphi, \theta)_D + \big(\varphi, (\underline{\nabla} S)^T \underline{w}\big)$$

and

$$\widetilde{\boldsymbol{A}}((\underline{w}, \theta)) = \begin{pmatrix} A_1^{\underline{\mathbb{V}}} & A_1^{\mathbb{W}} \\ \\ A_2^{\underline{\mathbb{V}}} & A_2^{\mathbb{W}} \end{pmatrix} \begin{pmatrix} \underline{w} \\ \\ \theta \end{pmatrix} = \begin{pmatrix} -\lambda\,\boldsymbol{S} & \underline{\nabla} S \\ \\ (\underline{\nabla} S)^T & -\lambda\,\boldsymbol{L} \end{pmatrix} \begin{pmatrix} \underline{w} \\ \\ \theta \end{pmatrix}. \tag{3.2}$$

Using $\Upsilon$ as the projection onto the divergence-free vector space, we receive also

$$\left(A_1 = A_1^{\mathbb{V}} + A_1^{\mathbb{W}}\right)(\underline{w}, \theta) = -\lambda\, \boldsymbol{S}\underline{w} + \Upsilon\mathcal{L}\theta \quad \text{and} \quad \left(A_2 = A_2^{\mathbb{V}} + A_2^{\mathbb{W}}\right)(\underline{w}, \theta) = (\underline{\nabla} S)^T \underline{w} - \lambda\, \boldsymbol{L}\theta$$

(cf. Theorem 3.1 and Equation 2.6) where

$$\mathcal{L}(\theta) = \theta\underline{\nabla}S.$$

**Lemma 3.5.** *$\widetilde{\boldsymbol{A}}$ is symmetric in the Hilbert space $\underline{\mathbb{H}}(.) \times \mathbb{L}_2(.)$.*

*Proof.* We use the definition and our equations to calculate

$$\begin{aligned}(\underline{\Psi}, A_1^{\mathbb{V}}\underline{w}) &= -\lambda \cdot (\underline{\Psi}, \underline{w})_D = (A_1^{\mathbb{V}}\underline{\Psi}, \underline{w}),\\ (\varphi, A_2^{\mathbb{W}}\theta) &= -\lambda \cdot (\varphi, \theta)_D \quad = (A_2^{\mathbb{W}}\varphi, \theta),\\ (\underline{\Psi}, A_1^{\mathbb{W}}\theta) &= (\underline{\Psi}, \theta\underline{\nabla}S) = (\theta, (\underline{\nabla}S)^T \cdot \underline{\Psi}) = (\theta, A_2^{\mathbb{V}}\underline{\Psi})\end{aligned}$$

or $\left(A_2^{\mathbb{V}}\right) = \left(A_1^{\mathbb{W}}\right)^T$. □

Notice that the eigenvalue, which is equivalent to Problem A, is concerned with the symmetric operator given by the blocks out of the diagonal, while the diagonal blocks multiplied with $(-1)$ give a symmetric metric tensor (by definition positive defined for any $\lambda > 0$).

We note that the energetic extension $\boldsymbol{S}_{en} : \underline{\mathbb{V}} \longrightarrow \underline{\mathbb{V}}'$ of the Stokes operator $\boldsymbol{S}$ and the energetic extension $\boldsymbol{L}_{en} : \overset{o}{\mathbb{W}}{}_2^1 \longrightarrow \mathbb{W}_2^{-1}$ of $\boldsymbol{L}$ are symmetric operators in the Hilbert spaces $\underline{\mathbb{V}}'$ and $\mathbb{W}_2^{-1}$ as well. One can use the structure of Equation (3.2) to make sure that the extension of $\widetilde{\boldsymbol{A}}$ on $\underline{\mathbb{V}}' \times \mathbb{W}_2^{-1}$ is symmetric.

## 4 | INVESTIGATION OF PROBLEM A

Our main theoretical result to the Problem A is the following:

**Theorem 4.1.** *Problem A (cf. Equations 2.9 and 2.10) is for all $\mathcal{A}$ equivalent to the eigenvalue problem of the compact self-adjoint operator $\tilde{C} \in \mathfrak{L}(\mathbb{l}_2 \times \mathbb{l}_2, \mathbb{l}_2 \times \mathbb{l}_2)$ (see below). All the eigenvalues $\lambda$ are real and $\lambda = 0$ is not an eigenvalue of the operator $\widetilde{C}$, but is an accumulation point of eigenvalues, which is the only element in the continuous spectrum of $\tilde{C}$. The critical Rayleigh number $\mathrm{Ra}_c$ is fixed through*

$$\frac{2}{\mathrm{Ra}_c} := |\lambda_{\max}| = \|\tilde{C}\|_{\mathfrak{L}}\ .$$

*The set $\{\lambda\}$ of the $\tilde{C}$-eigenvalues consists of a countably infinite number of (finite multiplicity) eigenvalues.*

*The eigen-triples $(\lambda, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}) \in \mathbb{C} \times \mathbb{l}_2 \times \mathbb{l}_2$ correspond to a solution $(\lambda, \underline{\tilde{w}}, \tilde{\theta}) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$ of Problem A (2.9)–(2.10).*

We will prove Theorem 4.1 by tools of functional analysis. The Problem A (2.9)–(2.10), that is, the Euler–Lagrange equations of the functional $\mathcal{F}/D$, is reduced to an "algebraic one" by the use of the complete system of the eigenfunctions of the Laplacian $\boldsymbol{L}$ and the Stokes operator $\boldsymbol{S}$. In what follows, we are going to make some arrangements for the proof of Theorem 4.1.

**Lemma 4.2.** *The system (cf. Notation 3.3)*

$$\left\{(\underline{v}_j, \chi_k)\right\}_{j,k\in\mathbb{N}}$$

*is a complete basis for the space herein defined, namely $\underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$.*

*Proof.* We apply the properties of positive self-adjoint operators with a pure point spectrum (cf. Theorem 3.1 and Lemma 3.2). ☐

In what follows, we are going to use the Fourier representations for any velocity $\underline{w} \in \underline{\mathbb{V}}$ and any temperature $\theta \in \overset{o}{\mathbb{W}}{}_2^1$ as expansions in the systems of eigenfunctions $\{\underline{v}_j\}_{j=1}^{\infty}$ and $\{\chi_j\}_{j=1}^{\infty}$ in the sense of $\underline{\mathbb{H}}$ and $\mathbb{L}_2$, respectively (cf. Lemma 3.2 and Equation 3.1):

$$\underline{w} = \sum_{j=1}^{\infty} c_j \underline{v}_j \qquad \text{and} \qquad \theta = \sum_{j=1}^{\infty} d_j \chi_j \tag{4.1}$$

The sequences $\{c_j\}_{j=1}^{\infty}$ and $\{d_j\}_{j=1}^{\infty}$ – as sequences of real or complex numbers – elements of *Hilbert space of sequences* $\mathbb{l}_2$ because of the properties of the orthonormal systems $\{\underline{v}_j\}_{j=1}^{\infty}$ and $\{\chi_j\}_{j=1}^{\infty}$ (in $\underline{\mathbb{H}}$ and $\mathbb{L}_2$). The well-known Hilbertian space of sequences $\mathbb{l}_2$ is defined by $\mathbb{l}_2 := (l_2, \|.\|_{2,(l)})$, with the linear vector space $l_2 := \{\{a_j\}_{j=1}^{\infty} ; a_j \in \mathbb{R}(\text{ resp. } \in \mathbb{C}) \forall j \in \mathbb{N} : \sum_{j=1}^{\infty} |a_j|^2 < \infty\}$ and the norm $\|.\|_{2,(l)}$.

The norm $\|.\|_{2,(l)}$ is given by the square root of the scalar product $(\{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty})_{2,(l)} := \sum_{j=1}^{\infty} a_j \overline{b}_j$, $\forall \{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty} \in l_2$:

$$\|\{a_j\}_{j=1}^{\infty}\|_{2,(l)} := \sqrt{\sum_{j=1}^{\infty} a_j \overline{a}_j} \quad \forall \{a_j\}_{j=1}^{\infty} \in l_2 .$$

We choose subspaces of $\mathbb{l}_2$ regarding any $\underline{w} \in \underline{\mathbb{V}}$ or any $\theta \in \overset{o}{\mathbb{W}}{}_2^1$ as weighted sequences, where the eigenpairs (cf. Equation 3.1) will be especially highlighted.

Now we will frame special sequence spaces of coefficients in relation to the spaces $\underline{\mathbb{V}}$ and $\overset{o}{\mathbb{W}}{}_2^1$.

Notation 4.3 (Special sequence spaces of coefficients). Let us regard the sequences of eigenvalues from the eigenpairs-system for the Laplacian $\boldsymbol{L}$ and the Stokes operator $\boldsymbol{S}$ (cf. Notation 3.3) ordered by increasing eigenvalues taking in account their multiplicities (and the correspondent eigenfunctions):

$$\{(\omega_j)^2 = |\omega_j|^2\}_{j=1}^{\infty}, \quad \{\lambda_j^* = |\kappa_j|^2\}_{j=1}^{\infty} \tag{4.2}$$

and the correspondent sequences of positive roots:

$$\{\omega_j\}_{j=1}^{\infty}, \quad \{\sqrt{\lambda_j^*} = \kappa_j\}_{j=1}^{\infty}.$$

We declare the spaces $\mathbb{l}_2^{\mathbb{W}}, \mathbb{l}_2^{\underline{\mathbb{V}}}$ with respect to spectral operators in the following way:

$\mathbb{l}_2^{\mathbb{W}} := (l_2^{\mathbb{W}}, \|.\|_{2,(\mathbb{W})})$ is defined by

$$l_2^{\mathbb{W}} := \left\{ \{a_j\}_{j=1}^{\infty} \in l_2 : \sum_{j=1}^{\infty} \omega_j^2 |a_j|^2 < \infty \right\}$$

$$\text{with the norm and the scalar product } \forall \{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty} \in l_2^{\mathbb{W}} : \tag{4.3}$$

$$\|\{a_j\}_{j=1}^{\infty}\|_{2,(\mathbb{W})} := \sqrt{(\{a_j\}_{j=1}^{\infty}, \{a_j\}_{j=1}^{\infty})_{2,(\mathbb{W})}}, \quad (\{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty})_{2,(\mathbb{W})} := \sum_{j=1}^{\infty} \omega_j^2 a_j \overline{b}_j .$$

$\mathbb{l}_2^{\mathbb{V}} := (l_2^{\mathbb{V}}, \|.\|_{2,(\mathbb{V})})$ is taken as

$$l_2^{\mathbb{V}} := \left\{ \{a_j\}_{j=1}^{\infty} \in l_2 : \sum_{j=1}^{\infty} \kappa_j^2 |a_j|^2 < \infty \right\}$$

$$\text{with the norm and the scalar product } \forall \{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty} \in l_2^{\mathbb{V}} : \tag{4.4}$$

$$\|\{a_j\}_{j=1}^{\infty}\|_{2,(\mathbb{V})} := \sqrt{(\{a_j\}_{j=1}^{\infty}, \{a_j\}_{j=1}^{\infty})_{2,(\mathbb{V})}}, \quad (\{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty})_{2,(\mathbb{V})} := \sum_{j=1}^{\infty} \kappa_j^2 a_j \overline{b}_j .$$

**Lemma 4.4.** *Let the Fourier coefficients of $\underline{w} \in \underline{\mathbb{V}}$ with respect to the $\underline{\mathbb{H}}$ orthonormal system $\{\underline{v}_j\}_{j=1}^{\infty}$ given by $\{c_j\}_{j=1}^{\infty}$ and the Fourier coefficients of $\theta \in \overset{o}{\mathbb{W}}{}_2^1$ with respect to the $\mathbb{L}_2$ orthonormal system $\{\chi_j\}_{j=1}^{\infty}$ given by $\{d_j\}_{j=1}^{\infty}$. Then the following statements are true:*

$$(i)\quad \{c_j\}_{j=1}^{\infty} \in \mathit{l}_2^{\mathbb{V}} \Leftrightarrow \underline{w} \in \underline{\mathbb{V}}\,, \qquad \|\underline{w}\|_D = \|\{c_j\}_{j=1}^{\infty}\|_{2,(\mathbb{V})}\,\forall\,\underline{w} \in \underline{\mathbb{V}}$$
$$(ii)\quad \{d_j\}_{j=1}^{\infty} \in \mathit{l}_2^{\mathbb{W}} \Leftrightarrow \theta \in \overset{o}{\mathbb{W}}{}_2^1, \quad \|\theta\|_D = \|\{d_j\}_{j=1}^{\infty}\|_{2,(\mathbb{W})}\,\forall\,\theta \in \overset{o}{\mathbb{W}}{}_2^1,$$

*where we reference to the Dirichlet norms in Equation (2.1).*

*Proof.* Using our definitions, one easily sees that Equations (4.3) and (4.4) are the properties of the Fourier coefficients for the energy spaces $\overset{o}{\mathbb{W}}{}_2^1$ (for $\boldsymbol{L}$) and $\underline{\mathbb{V}}$ (for $\boldsymbol{S}$) (cf. Triebel [23] (4.1.8) and (4.1.9)). □

The Cartesian product space $\mathit{l}_2^{\mathbb{V}} \times \mathit{l}_2^{\mathbb{W}}$ is naturally equipped with the following scalar product and norm

$$[(\mathbf{a},\mathbf{b}),\,(\mathbf{c},\mathbf{d})]_{\mathit{l}_2^{\mathbb{V}}\times \mathit{l}_2^{\mathbb{W}}} := \Big((\mathbf{a},\mathbf{c})_{2,(\mathbb{V})} + (\mathbf{b},\mathbf{d})_{2,(\mathbb{W})}\Big) \quad \forall\,(\mathbf{a},\mathbf{b}),\,(\mathbf{c},\mathbf{d}) \in \mathit{l}_2^{\mathbb{V}} \times \mathit{l}_2^{\mathbb{W}},$$
$$\|(\mathbf{c},\mathbf{d})\|_{\mathit{l}_2^{\mathbb{V}}\times \mathit{l}_2^{\mathbb{W}}} := \sqrt{[(\mathbf{c},\mathbf{d}),(\mathbf{c},\mathbf{d})]_{\mathit{l}_2^{\mathbb{V}}\times \mathit{l}_2^{\mathbb{W}}}}$$

(cf. also Notation 7.3). After these suitable preparations, we are now in the position to carry out the proof of Theorem 4.1.

*Proof* (Theorem 4.1). We re-write the Problem A (2.9)–(2.10) in the weak formulation, where we use the potential $S$ (cf. Equation (2.6)):

We are looking for $(\lambda, \underline{w}, \theta) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$ fulfilling for arbitrary $\underline{u} \in \underline{\mathbb{V}}$ and $\sigma \in \overset{o}{\mathbb{W}}{}_2^1$:

$$\big(\theta, (\underline{\nabla} S)^T \underline{u}\big) = \lambda\,\big(\underline{u}, \underline{w}\big)_D, \tag{4.5}$$
$$\big(\sigma, (\underline{\nabla} S)^T \underline{w}\big) = \lambda\,(\theta, \sigma)_D. \tag{4.6}$$

By taking the sum of Equations (4.5) and (4.6), we get:

$$\big(\sigma, (\underline{\nabla} S)^T \underline{w}\big) + \big(\theta, (\underline{\nabla} S)^T \underline{u}\big) = \lambda\big((\underline{u}, \underline{w})_D + (\theta, \sigma)_D\big), \tag{4.7}$$

which corresponds to Equation (2.5) with $\lambda = \mathrm{Ra}_c$.

We express the elements $\underline{u}, \underline{w}, \sigma$ and $\theta$ as Fourier series in the systems of eigenfunctions $\{\underline{v}_j\}_{j=1}^{\infty}$ and $\{\chi_j\}_{j=1}^{\infty}$ with respect to $\underline{\mathbb{H}}$ and $\mathbb{L}_2(.)$, respectively, that is,

$$\underline{w} = \sum_{j=1}^{\infty} c_j \underline{v}_j, \quad \underline{u} = \sum_{j=1}^{\infty} a_j \underline{v}_j \tag{4.8}$$

$$\theta = \sum_{j=1}^{\infty} d_j \chi_j, \quad \sigma = \sum_{j=1}^{\infty} b_j \chi_j. \tag{4.9}$$

Moreover, we set

$$\mathbf{a} = \{a_j\}_{j=1}^{\infty}, \ \mathbf{b} = \{b_j\}_{j=1}^{\infty}, \ \mathbf{c} = \{c_j\}_{j=1}^{\infty} \text{ and } \mathbf{d} = \{d_j\}_{j=1}^{\infty} \tag{4.10}$$

and regard them as sequences of Fourier coefficients. For sequences $\mathbf{a}$ considered as a row, we write in this sense $\mathbf{a}^T$. In what follows, we are going to use the common writing of double series like products of matrices. Such method was employed, for example, by Schmidt [20] and is a standard method of functional analysis. The background for this method is the re-arrangement theorem for double series. Inserting Equations (4.8) and (4.9) in Equations (4.5) and (4.6) results in:

$$\mathbf{a}^T\big(C \cdot \mathbf{d} - \lambda\,[\mathrm{diag}(\kappa_k^2)] \cdot \mathbf{c}\big) = 0\,, \tag{4.11}$$

$$\mathbf{b}^T\big(C^T \cdot \mathbf{c} - \lambda\,[\mathrm{diag}(\omega_k^2)] \cdot \mathbf{d}\big) = 0, \tag{4.12}$$

where

$$C_{j,k} := (\chi_k, (\underline{\nabla} S)^T \underline{v}_j). \tag{4.13}$$

One easily sees that $|C_{j,k}| \leq \sqrt{2} \cdot \gamma_S$ with $\gamma_S := \max_{\underline{x}\in\Omega_A} \left( |\frac{\partial S}{\partial r}| + |\frac{\partial S}{r\partial\varphi}| \right)$, using that $\|\underline{v}_j\|_{\underline{\mathbb{H}}} = 1$ and $\|\chi_k\|_{\mathbb{L}_2} = 1 \forall j,k \in \mathbb{N}\times\mathbb{N}$.

The sum of the Equations (4.11) and (4.12) can be written in $\ell_2 \times \ell_2$ as:

$$(\mathbf{a}^T, \mathbf{b}^T) \left[ \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix} - \lambda \begin{pmatrix} \operatorname{diag}(\kappa_k^2) & 0 \\ 0 & \operatorname{diag}(\omega_k^2) \end{pmatrix} \right] \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = 0. \tag{4.14}$$

The sequences of Fourier coefficients corresponding to $\underline{u}$, $\underline{w}$, $\sigma$ and $\theta$ belong to the spaces $\ell_2^{\mathbb{V}}$ and $\ell_2^{\mathbb{W}}$. From these, it follows that $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}, \tilde{\mathbf{d}} \in \ell_2$, with

$$\tilde{\mathbf{a}} = \{\tilde{a}_j := \kappa_j a_j\}_{j=1}^{\infty}, \ \tilde{\mathbf{b}} = \{\tilde{b}_j := \omega_j b_j\}_{j=1}^{\infty}, \ \tilde{\mathbf{c}} = \{\tilde{c}_j := \kappa_j c_j\}_{j=1}^{\infty} \text{ and } \tilde{\mathbf{d}} = \{\tilde{d}_j := \omega_j d_j\}_{j=1}^{\infty}, \tag{4.15}$$

because of $\underline{u}, \underline{w} \in \underline{\mathbb{V}}$ and $\sigma, \theta \in \overset{o}{\mathbb{W}}{}_2^1$. Additionally, we define the matrix $\tilde{C}$ by

$$\tilde{C}_{j,k} := \frac{1}{\kappa_j \omega_k} (\chi_k, (\underline{\nabla} S)^T \underline{v}_j) = \frac{1}{\kappa_j \omega_k} C_{j,k} \quad \forall j,k \in \mathbb{N}\times\mathbb{N}. \tag{4.16}$$

So we are able now to re-write the problem (4.14) in the $\ell_2 \times \ell_2$ -setting through:

$$(\tilde{\mathbf{a}}^T, \tilde{\mathbf{b}}^T) \left[ \begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^T & 0 \end{pmatrix} - \lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right] \begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = 0, \tag{4.17}$$

where $I$ denotes the identity in $\ell_2$. We interpret Equation (4.17) as an operator equation in $\ell_2 \times \ell_2$. This is possible since the relation (4.17) is valid for arbitrary $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) \in \ell_2 \times \ell_2$.

We define the linear and bounded operators $\tilde{\mathcal{C}}$ and $\mathcal{I}$: $\tilde{\mathcal{C}}, \mathcal{I} : \ell_2 \times \ell_2 \longrightarrow \ell_2 \times \ell_2$ by:

$$\tilde{\mathcal{C}} := \begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^T & 0 \end{pmatrix} \quad , \quad \mathcal{I} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.18}$$

Finally, Equation (4.17) is here equivalent to the eigenvalue problem:

$$\tilde{\mathcal{C}} \begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = \lambda \mathcal{I} \begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix}. \tag{4.19}$$

The operators $\tilde{\mathcal{C}}$ and $\mathcal{I}$ are elements of the Banach space $\mathcal{L}(\ell_2 \times \ell_2, \ell_2 \times \ell_2)$ of linear and bounded operators. We are now interested in the properties and the operator norms of $\tilde{\mathcal{C}}$ and $\mathcal{I}$.

It is well known that the identity $\mathcal{I}$ in $\ell_2 \times \ell_2$ is a self-adjoint operator in $\mathcal{L}(\ell_2 \times \ell_2, \ell_2 \times \ell_2)$ and $\|\mathcal{I}\|_{\mathcal{L}} = 1$.

The boundedness of the operator $\tilde{\mathcal{C}}$ is obvious (cf. Equations 4.13 and 4.16).

To show the compactness of $\tilde{\mathcal{C}}$, we use that it can be approximated by a sequence of finite operators $\{\tilde{\mathcal{C}}_\ell\}_{\ell\in\mathbb{N}}$, $\{\widetilde{\mathcal{C}_\ell}\}_{\ell\in\mathbb{N}} \in \mathcal{L}(\ell_2 \times \ell_2, \ell_2 \times \ell_2)$, with $\{\widetilde{\mathcal{C}_\ell}\}_{\ell\in\mathbb{N}} \underset{\mathcal{L}}{\rightarrow} \tilde{\mathcal{C}}$, where we use $\forall \ell \in \mathbb{N}$ the matrices $\tilde{C}_\ell$:

$$\tilde{C}_{\ell,j,k} := \begin{cases} \tilde{C}_{j,k} \ \forall j,k \in \mathbb{N}\times\mathbb{N}: & j,k \leq \ell \\ 0 \quad \forall j,k \in \mathbb{N}\times\mathbb{N}: & j > \ell \vee k > \ell \end{cases} \tag{4.20}$$

and $\forall \ell \in \mathbb{N}$ the definitions:

$$\tilde{\mathcal{C}}_\ell := \begin{pmatrix} 0 & \tilde{C}_\ell \\ \tilde{C}_\ell^T & 0 \end{pmatrix}. \tag{4.21}$$

The convergence $\{\widetilde{\mathcal{C}_\ell}\}_{\ell\in\mathbb{N}} \underset{\mathcal{L}}{\rightarrow} \tilde{\mathcal{C}}$ follows from Equations (4.13), (4.16), $\lim_{j\to\infty} \kappa_j = \infty$ and $\lim_{k\to\infty} \omega_k = \infty$.

Since $\tilde{C}$ is evidently symmetric, one has just to show that it is self-adjoint. This follows since $D(\tilde{C}) = \mathbb{L}_2 \times \mathbb{L}_2$ for the symmetric operator $\tilde{C}$ (cf. Triebel [23, Th.1 in 4.1.6]).

The existence of all the real eigenvalues $\lambda$ (and all the eigen-elements $(\underline{w}, \theta) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}^1_2$) is a simple consequence of Triebel [23], Th.1 in 4.2.6] regarding to the properties of $\tilde{C}$ and the problem (4.19). One has to note that $\tilde{C}$ as a linear and compact operator on the separable infinite-dimensional space $\mathbb{L}_2 \times \mathbb{L}_2$ has at most a countably infinite number of eigenvalues, where the non-zero eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ (counted in multiplicity) can be ordered by their absolute values (cf. Triebel [23] (Thm.1, 4.2.6)). There we have to take into account the finite multiplicity of the non-zero eigenvalues. The number zero: $0 \in \mathbb{C}$ is an element of the continuous spectrum in either case as the only accumulation point of the eigenvalues of $\tilde{C}$, but not an eigenvalue of $\tilde{C}$, in what follows, that $\tilde{C}^{-1}$ exists as a linear but unbounded operator. So we finish the proof by showing that $\lambda = 0$ is not an eigenvalue of $\tilde{C}$ in the following Lemma 4.5. □

**Lemma 4.5.** *$\lambda = 0$ is not an eigenvalue of $\tilde{C}$, but as an accumulation point of eigenvalues, the only one element of the continuous spectrum of $\tilde{C} \in \mathbb{L}(\mathbb{L}_2 \times \mathbb{L}_2, \mathbb{L}_2 \times \mathbb{L}_2)$.*

*Proof.* Assuming that $\lambda = 0$ is an eigenvalue of $\tilde{C}$, resp. of problem (2.9)–(2.10), then Equation (2.9) becomes

$$\theta \underline{\nabla} S = -\underline{\nabla} p\,.$$

By applying the curl operator to both sides, we obtain

$$\underline{\nabla} \times (\theta \underline{\nabla} S) = \underline{\nabla}\theta \times \underline{\nabla} S = \underline{0}$$

but, as it is easy to check, the condition that $\underline{\nabla}\theta$ and $\underline{\nabla} S$ are parallel vectors requires that

$$\theta = \frac{1}{b} \ln r + r \sin\varphi$$

(modulo some constants), which does not obey the boundary conditions for the temperature. Finally, we get by application of Triebel [23] (Thm.1, 4.2.6) the property of $\lambda = 0$ to be the only one element of the continuous spectrum of $\tilde{C}$. □

## 5 | THE NUMERICAL APPROXIMATION OF THE CRITICAL CONSTANT

Taking into account that the maximum of our functional corresponds to the largest positive eigenvalue $\lambda = \lambda_{max} = \frac{2}{\mathrm{Ra}_c}$ of problem (2.9)–(2.10), respectively to the operator norm of $\tilde{C}$, we are going to create an approximation method enabling this aim. We use determinants of square matrices of finite-dimensional matrix formats (regarded as elements of the sequence of finite operator) to create approximations for $\lambda$. Following ideas and methods of Schmidt [20], it is easy to prove that our sequence (or subsequence) of approximations $\{\lambda_{\mathcal{A}}^{\ell_{\tilde{j}}}\}_{\tilde{j}=1}^{\infty} = \{\lambda^{\ell_{\tilde{j}}}\}_{\tilde{j}=1}^{\infty}$ is monotonically increasing with the limit $\lambda = \|\tilde{C}\|_{\mathbb{L}}$.

One could regard the eigenvalue equation for the eigenvalues $\lambda$ of $\tilde{C}$ also in the meaning of a limes of finite-dimensional $2\ell \times 2\ell$-matrices $\{\Gamma_\ell\}_{\ell\in\mathbb{N}}$, $\{\tilde{\mathcal{J}}_\ell\}_{\ell\in\mathbb{N}}$, with $\Gamma_\ell\ \mathcal{J}_\ell, \forall\, \ell \in \mathbb{N}$ according to description (5.6):

$$\lim_{\ell\to\infty} \det[\tilde{\Gamma}_\ell - \lambda\,\mathcal{J}_\ell] =: \det\big[\tilde{C} - \lambda\,\mathcal{I}\big] = \det\left[\begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^T & 0 \end{pmatrix} - \lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] = 0 \tag{5.1}$$

*Remark.* We choose a number $\ell \in \mathbb{N}$ in such a way that the truncated eigenpair-systems of the Laplacian and the Stokes operator written as

$$\{(\omega_j)^2, \chi_j\}_{j=1}^{\ell} \qquad \text{and} \qquad \{(\kappa_j)^2, \underline{v}_j\}_{j=1}^{\ell} \tag{5.2}$$

have the following properties:

1. The spaces spanned by the systems $\{\chi_j\}_{j=1}^{\ell}$ and $\{\underline{v}_j\}_{j=1}^{\ell}$ contain the entire eigenspaces of all eigenvalues $\{(\omega_j)^2\}_{j=1}^{\ell}$ resp. $\{(\kappa_j)^2\}_{j=1}^{\ell}$, particularly of $(\omega_\ell)^2$ and $(\kappa_\ell)^2$.
2. The number $\ell$ is specified by explicite determination of the eigenpair-systems until a number $m \gg \ell$.

**Lemma 5.1.** *There exists a strictly monotonic increasing sequence* $\{\ell_{\tilde{j}}\}_{\tilde{j}=1}^{\infty} \longrightarrow \infty$ *fulfilling with Equation (5.2) as:*

$$\{(\omega_j)^2, \chi_j\}_{j=1}^{\ell_{\tilde{j}}} \qquad \text{and} \qquad \{(\kappa_j)^2, \underline{v}_j\}_{j=1}^{\ell_{\tilde{j}}} \tag{5.3}$$

*the properties of the remark above for any* $\mathcal{A} \in (0, \infty)$.

*Proof.* We use the intermeshing property of eigenvalues $\{(\omega_j)^2\}_{j=1}^{\infty}$ resp. $\{(\kappa_j)^2\}_{j=1}^{\infty}$, which grow in the exact same manner in regard to their multiplicities. There one has to note that we have $\Omega_{\mathcal{A}}$ as a 2d-domain and that the divergence div $\underline{v}_j = 0$ works like a one-dimensional restriction for 2d-vector fields on 2d-domains. □

We will use for the approximations of $\lambda = \|\tilde{C}\|_{\mathcal{L}}$ the Courant minimax principle.

Let us introduce step by step notations for sets of square block matrices $\forall \ell \in \{\ell_{\tilde{j}}\}_{\tilde{j}=1}^{\infty} \subset \mathbb{N}$:

We denote by $\hat{C}_{\ell}$ the $\ell \times \ell$-matrices $\hat{C}_{\ell}$:

$$\hat{C}_{\ell,j,k} := \tilde{C}_{j,k} \ \forall j, k \in \mathbb{N} \times \mathbb{N} : j, k \leq \ell, \tag{5.4}$$

where we regard to Equation (4.16) for the definition of the $\tilde{C}_{j,k}$.

The identity matrix of size $\ell$ is termed by $\hat{I}_{\ell}$. $\hat{I}_{\ell}$ is the known square matrix with

$$\hat{I}_{\ell,j,k} := \delta_{j,k} \ \forall j, k \in \mathbb{N} \times \mathbb{N} : j, k \leq \ell, \tag{5.5}$$

where $\delta_{j,k}$ is the Kronecker delta. ($\forall \ell \in \mathbb{N}$)

We explain for all $\ell \in \{\ell_{\tilde{j}}\}_{\tilde{j}=1}^{\infty}$ the block matrices $\Gamma_{\ell}$ and $J_{\ell}$ by:

$$\Gamma_{\ell} := \begin{pmatrix} 0 & \hat{C}_{\ell} \\ \hat{C}_{\ell}^T & 0 \end{pmatrix}, \quad J_{\ell} := \hat{I}_{2\ell} = \begin{pmatrix} \hat{I}_{\ell} & 0 \\ 0 & \hat{I}_{\ell} \end{pmatrix}. \tag{5.6}$$

One can also introduce a sequence of block matrices $\{\Gamma_{\ell}\}_{\ell \in \{\ell_{\tilde{j}}\}_{\tilde{j}=1}^{\infty}, \ell_{\tilde{j}} < \ell_{\tilde{j}+1}}$ from a theoretical point of view.

It is worth to note that the numerical methods and computer algebra systems for the evaluation of all eigenvalues are restricted to small matrix formats. So we skip any tries to compute all eigenvalues of the matrices $\{\Gamma_{\ell}\}_{\ell \in \{\ell_{\tilde{j}}\}_{\tilde{j}=1}^{\infty}, \ell_{\tilde{j}} < \ell_{\tilde{j}+1}}$ to focus on approximations of $\lambda = \|\tilde{C}\|_{\mathcal{L}}$. The Courant minimax principle for $\lambda = \|\tilde{C}\|_{\mathcal{L}}$ is here the standard:

$$\lambda = \|\tilde{C}\|_{\mathcal{L}} := \max_{\|(\tilde{\mathbf{c}}, \tilde{\mathbf{d}})\|_{\mathfrak{l}_2 \times \mathfrak{l}_2} = 1} \left[ (\tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \, \tilde{C} \begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} \right], \tag{5.7}$$

or again with Courant minimax written as a simple limiting value like in the proof of [23, Thm. in Sec. 4.2.5]

$$\lambda = \|\tilde{C}\|_{\mathcal{L}} = \lim_{\tilde{j} \to \infty} \max_{\underline{z} \in \mathbb{R}^{2\ell_{\tilde{j}}} : \|\underline{z}\| = 1} (\underline{z}^T \cdot \tilde{\Gamma}_{\ell_{\tilde{j}}} \cdot \underline{z}) \ , \tag{5.8}$$

where $\|.\|$ denotes the Euclidian norm of the $\mathbb{R}^{2\ell_{\tilde{j}}}$.

One can use straightforward method of the evaluation of all of $\tilde{\Gamma}_{\ell_{\tilde{j}}}$ for $\tilde{j} = 1, 2, 3$ to calculate

$$\lambda^{\ell_{\tilde{j}}} := \max_{\underline{z} \in \mathbb{R}^{2\ell_{\tilde{j}}} : \|\underline{z}\| = 1} (\underline{z}^T \cdot \tilde{\Gamma}_{\ell_{\tilde{j}}} \cdot \underline{z}) \qquad \forall \tilde{j} \in \mathbb{N} \tag{5.9}$$

in a first step. The start process sector is accurately described by the order $2\ell_{\tilde{j}}$ of the square matrices through $2\ell_{\tilde{j}} \approx 40$.

We get the inequalities: $\lambda^{\ell_1} \leq \lambda^{\ell_2} \leq \lambda^{\ell_3} \leq \ldots$ using Lemma 5.1 and the definition (5.9) of $\lambda^{\ell_{\tilde{j}}}$, where we recognize the sequence $\{\lambda^{\ell_{\tilde{j}}}\}_{\tilde{j}=1}^{\infty}$ to be monotonically increasing with the limit $\lambda$ as a simple consequence. So is our method a method to approach the eigenvalue $\lambda$ from below.

We take the square (block) matrices of order $2\ell_{\tilde{j}}$: $\Gamma_{\ell_{\tilde{j}}}$ and $J_{\ell_{\tilde{j}}}$ (cf. Equation 5.6) to explain the functions

$$F_{\tilde{j}}(\lambda) := \det\left[\Gamma_{\ell_{\tilde{j}}} - \lambda J_{\ell_{\tilde{j}}}\right] \quad \forall \tilde{j} \in \mathbb{N}. \tag{5.10}$$

**TABLE 1** Numerical results for the critical Rayleigh number for the case $\mathcal{A} = 1$.

| $\ell_j$ | $\lambda_1^{\ell_j}$ | $\mathrm{Ra}_c(\mathcal{A} = 1)$ |
|---|---|---|
| 7 | 0.065275715709889022434641241556310 | 30.639265739938989496469183142650 |
| 12 | 0.067433400164287488107980906050821 | 29.658892998535072667362035625305 |
| 22 | 0.070024363834146514766687200657022 | 28.561487609327265913546978946005 |
| 49 | 0.070752337753538312200877678082614 | 28.267617205340757028944998021332 |
| 92 | 0.070843842842887568707945584675261 | 28.231105481325421889287981461511 |
| 300 | 0.070926398413629912903178212704811 | 28.198245571928834452591903765447 |

**TABLE 2** Numerical results for the critical Rayleigh number for the case $\mathcal{A} = 10$.

| $\ell_j$ | $\lambda_1^{\ell_j}$ | $\mathrm{Ra}_c(\mathcal{A} = 10)$ |
|---|---|---|
| 11 | 0.036922959760029763204380893713543 | 54.166838547029519643082315254994 |
| 21 | 0.036922959760029763204380893713509 | 54.166838547029519643082315255044 |
| 39 | 0.036988540331501778832593839394502 | 54.070800904156620330758041435409 |
| 62 | 0.036988763473199326352547015240593 | 54.070474711843912995766624606994 |
| 90 | 0.036988784077179802197566851096866 | 54.070444592794772101553849052213 |

We evaluate $\lambda^{\ell_{\tilde{j}}}$ by calculating the zeros of $F_{\tilde{j}}(\lambda)$ in the interval $[\lambda^{\ell_{\tilde{j}-1}}, \lambda^{\ell_{\tilde{j}-1}} + \delta(\mathcal{A})]$. There we use the bisection method as the root-finding method. The initial approximations (start interval) are chosen by means of $\lambda^{\ell_{\tilde{j}-1}}$ to be $[\lambda^{\ell_{\tilde{j}-1}}, \lambda^{\ell_{\tilde{j}-1}} + \delta(\mathcal{A})]$, where one can take the constant $\delta(\mathcal{A}) \approx 1$. The initial approximations are taken like in the complete induction method with $\tilde{j} - 1 \geq 1$.

This method for the approach of $\lambda$ is also restricted by the matrix formats. Because of the very long calculating time for determinants for large $\ell_{\tilde{j}}$, we make a rough estimate for the border area around $\ell_{\tilde{j}} \approx 400$. That applies to other representations of $F_{\tilde{j}}(\lambda)$ (cf. Equation 5.10) like

$$F_{\tilde{j}}(\lambda) := \det\left[\lambda^2 \cdot \hat{I}_{\ell_{\tilde{j}}} - \hat{C}_{\tilde{j}} \cdot \hat{C}_{\tilde{j}}^T\right] \quad \forall \tilde{j} \in \mathbb{N}. \tag{5.11}$$

too.

Now we present some results of calculations for the numerical approximation of critical Rayleigh numbers. The calculations of the critical Rayleigh numbers or $\lambda_1 = \lambda = \|\tilde{C}\|_{\mathbb{L}}$ were implemented and performed in Maple 2021 (and Maple 2022) to ensure tight tolerances for the eigenfunctions, eigenvalues and the items of $\Gamma_{\ell_{\tilde{j}}}$. There we use especially some tools for the investigation of Besselfunctions (cf., e.g. Refs. [12,19]).

We present numerical results of approximations of $\mathrm{Ra}_c(\mathcal{A})$ for $\mathcal{A} = 1$ and $\mathcal{A} = 10$ (cf. Markert [13]). The numbers $\ell_j$ are chosen in the way described in Lemma 5.1.

Here one can observe that the sequences $\{\lambda_1^{\ell_{\tilde{j}}}\}_{\tilde{j} \geq 1}$ are monotonically increasing. This increasing behaviour of the $\lambda_1^{\ell_{\tilde{j}}}(\mathcal{A})$ is justified in the growing dimension of the variation-sets (the matrix formats of the $\Gamma_{\ell_{\tilde{j}}}$) in Tables 1 and 2.

## 6 | ALTERNATIVE PROOF VIA THE STREAM-FUNCTION

We will sketch another way for a proof of a modified version of Theorem 4.1 using streamfunctions. For this reason, we re-phrase the propositions of Theorem 4.1 along the lines of Theorem 4.1 in the following Theorem 6.1:

**Theorem 6.1.** *Problem A (cf. Equations 2.9 and 2.10) is for all $\mathcal{A}$ equivalent to the eigenvalue problem of the compact self-adjoint operator $\tilde{\mathcal{B}} \in \mathbb{L}(\mathbb{l}_2 \times \mathbb{l}_2, \mathbb{l}_2 \times \mathbb{l}_2)$ (see below). All the eigenvalues $\lambda$ are real and $\lambda = 0$ is not an eigenvalue of the operator $\widetilde{\mathcal{B}}$, but is as an accumulation point of eigenvalues the only one element of the continuous spectrum of $\tilde{\mathcal{B}}$. The critical Rayleigh number $\mathrm{Ra}_c$ is fixed through*

$$\frac{2}{\mathrm{Ra}_c} := |\lambda_{\max}| = \|\tilde{\mathcal{B}}\|_{\mathbb{L}} .$$

*The set $\{\lambda\}$ of the $\tilde{\mathcal{B}}$-eigenvalues consists of an countably infinite number of (finite multiplicity) eigenvalues.*

*The eigen-triples* $(\lambda, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}) \in \mathbb{C} \times \mathbb{l}_2 \times \mathbb{l}_2$ *correspond to a solution* $(\lambda, \underline{\tilde{w}}, \tilde{\theta}) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$ *of Problem A (2.9)–(2.10).*

*Remark.* The use of streamfunction will get to an additional effort in the treatment of the Problem A (2.9)–(2.10). Especially we get worse estimations (6.17) in relation to the proof of Theorem 4.1 and an deteriorate convergence of the sequence of finite operators to $\widetilde{\mathcal{B}}$ (at showing of the compactness of $\widetilde{\mathcal{B}}$) also.

In a first step, we are going to make some arrangements for the proof of Theorem 6.1. So we regard Problem A (2.9)–(2.10) again. We will use Fourier representations for any streamfunction $\Psi \in \overset{o}{\mathbb{W}}{}_2^2$ as expansions in the systems of eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ for the Bilaplacian $\boldsymbol{B}$.

**Lemma 6.2.** *The system (cf. Notation 3.3)*

$$\left\{(\psi_j, \chi_k)\right\}_{j,k\in\mathbb{N}}$$

*is a complete basis for the space herein defined, namely* $\overset{o}{\mathbb{W}}{}_2^2 \times \overset{o}{\mathbb{W}}{}_2^1$.

*Proof.* We have to apply the properties of positive self-adjoint operators with a pure point spectrum (cf. Theorem 3.1 and Lemma 3.2, or [23, Th. in 4.5.1]). □

In what follows, we are going to use the Fourier representations for any streamfunction $\Psi \in \overset{o}{\mathbb{W}}{}_2^2$ and any temperature $\theta \in \overset{o}{\mathbb{W}}{}_2^1$ as expansions in the systems of eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ and $\{\chi_j\}_{j=1}^{\infty}$ in the sense of $\mathbb{L}_2$, respectively (cf. Lemma 3.2 and Equation 3.1):

$$\Psi = \sum_{j=1}^{\infty} c_j \psi_j \quad \text{and} \quad \theta = \sum_{j=1}^{\infty} d_j \chi_j \tag{6.1}$$

The sequences $\{c_j\}_{j=1}^{\infty}$ and $\{d_j\}_{j=1}^{\infty}$ are – as sequences of real or complex numbers – elements of *Hilbertian space of sequences* $\mathbb{l}_2$ [23, Lemma 2. in 2.1.3]. Parseval's equation is fulfilled because of the properties of the complete orthonormal systems $\{\psi_j\}_{j=1}^{\infty}$ and $\{\chi_j\}_{j=1}^{\infty}$ (in $\mathbb{L}_2$). (We note that the system $\{\psi_j\}_{j=1}^{\infty}$ is an orthogonal system in the space $\overset{o*}{\mathbb{W}}{}_2^2 := (\overset{o}{W}{}_2^2, \|.\|_\Delta)$. Here is the linear vectorspace $\overset{o}{W}{}_2^2$ endowed with the norm to the $\Delta$-(quasi-)scalar product (cf. Notation 6.3). The system $\{\chi_j\}_{j=1}^{\infty}$ is a complete orthogonal system in $\overset{o}{\mathbb{W}}{}_2^1$ by construction.

Notation 6.3. We denote $\forall \Psi, \Phi \in \overset{o}{\mathbb{W}}{}_2^2$ by

$$(\Phi, \Psi)_\Delta := (\Delta\Phi, \Delta\Psi) \tag{6.2}$$

the $\Delta$-(quasi-)scalar product on the linear vectorspace $\overset{o}{W}{}_2^2$.

*Remark.* The $\Delta$-(quasi-)scalar product generates an equivalent norm to the standard norm of $\mathbb{W}_2^2$ on the linear vectorspace $\overset{o}{W}{}_2^2$.

Notation 6.4 (Special sequence space of coefficients). Let us regard the sequence of eigenvalues from the eigenpairs-system for the Bilaplacian $\boldsymbol{B}$ (cf. Notation 3.3) ordered by increasing eigenvalues taking in account their multiplicities (and the correspondent eigenfunctions):

$$\{(\mu_j)^2 = |\mu_j|^2\}_{j=1}^{\infty} \qquad \text{and the correspondent sequences of positive roots:} \quad \{\mu_j\}_{j=1}^{\infty}. \tag{6.3}$$

We declare the space $\mathbb{l}_2^{\mathbb{W}2}$, with respect to spectral operators in the following way:
$\mathbb{l}_2^{\mathbb{W}2} := (l_2^{\mathbb{W}2}, \|.\|_{2,(\mathbb{W})})$ is defined by

$$l_2^{\mathbb{W}2} := \{\{a_j\}_{j=1}^{\infty} \in l_2 : \sum_{j=1}^{\infty} \mu_j^2 |a_j|^2 < \infty\}$$

$$\text{with the norm and the scalar product } \forall \{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty} \in l_2^{\mathbb{W}2} : \tag{6.4}$$

$$\|\{a_j\}_{j=1}^{\infty}\|_{2,(\mathbb{W}2)} := \sqrt{(\{a_j\}_{j=1}^{\infty}, \{a_j\}_{j=1}^{\infty})_{2,(\mathbb{W}2)}} \quad , \quad (\{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty})_{2,(\mathbb{W}2)} := \sum_{j=1}^{\infty} \mu_j^2 a_j \overline{b}_j \,.$$

We get like in Section 4, Lemma 4.4 the following:

**Lemma 6.5.** *Let the Fourier coefficients of* $\Psi \in \overset{o}{\mathbb{W}}{}_2^2$ *with respect to the* $\mathbb{L}_2$ *orthonormal system* $\{\psi_j\}_{j=1}^{\infty}$ *given by* $\{c_j\}_{j=1}^{\infty}$*. Then the following statements are true:*

$$(iii) \quad \{c_j\}_{j=1}^{\infty} \in \mathbb{l}_2^{\mathbb{W}2} \Leftrightarrow \Psi \in \overset{o}{\mathbb{W}}{}_2^2 \quad and \; \sqrt{(\Psi,\Psi)\Delta} = \|\{c_j\}_{j=1}^{\infty}\|_{2,(\mathbb{W}2)} \, \forall \Psi \in \overset{o}{\mathbb{W}}{}_2^2,$$

*where we reference to the scalar product in Equation (6.2).*

*Proof.* One has only to re-write our definitions. □

Let us now consider the representation of two-dimensional (respectively of three-dimensional) vector fields by means of the streamfunction $\Psi = \Psi(r, \varphi)$, or, by considering the reference frame of toroidal vector fields here chosen:

$$\underline{w} = \underline{\nabla}\,\Psi \times \underline{\mathfrak{e}}_1 \,,$$

where $\underline{\mathfrak{e}}_1$ denotes the unit vector in the horizontal direction. We are going to employ the common vector product of the three-dimensional space to avoid additional notations as the wedge-product. So we understand our terms in the sense of three-dimensional expression and our results also as three-dimensional vector fields with non-vanishing coefficients of $\underline{\mathfrak{e}}_1$, respectively, in the following passage.

One can calculate the curl of all terms in Equations (2.9) and (2.10) by assuming higher regularity

$$\underline{\nabla} \times (\theta \underline{\nabla} S) = \underline{\nabla}\theta \times (\underline{\nabla} S)$$
$$\underline{\nabla} \times (\lambda \Delta \underline{w}) = -\lambda \Delta^2 \Psi \underline{\mathfrak{e}}_1$$
$$(\underline{\nabla} \times \underline{\nabla})\, p = \underline{0} \,.$$

Hence, Problem A provides by calculation and obtaining simple results from the nabla calculus:

$$\underline{\mathfrak{e}}_1^T \cdot (\underline{\nabla}\theta \times \underline{\nabla} S) - \lambda \Delta^2 \Psi = 0 \tag{6.5}$$
$$\underline{\mathfrak{e}}_1^T \cdot (\underline{\nabla}\Psi \times \underline{\nabla} S) - \lambda \Delta \theta = 0 \,. \tag{6.6}$$

We are going to investigate Equations (6.5) and (6.6) as bilinear forms again. So we seek for $(\lambda, \underline{w} = \underline{\nabla}\,\Psi \times \underline{\mathfrak{e}}_1, \theta) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$ fulfilling for arbitrary $\underline{u} = \underline{\nabla}\,\Phi \times \underline{\mathfrak{e}}_1$, and $\sigma$, the equations (written with $\Psi, \Phi \in \overset{o}{\mathbb{W}}{}_2^2$):

$$\left(\Phi, \underline{\mathfrak{e}}_1^T \cdot (\underline{\nabla}\theta \times \underline{\nabla} S)\right) = \lambda\, (\Phi, \Psi)_{\Delta} \,, \tag{6.7}$$

$$\left(\sigma, \underline{\mathfrak{e}}_1^T \cdot (\underline{\nabla}\Psi \times \underline{\nabla} S)\right) = -\lambda\, (\sigma, \theta)_D, \tag{6.8}$$

where we have used $\underline{w} = \underline{\nabla}\,\Psi \times \underline{\mathfrak{e}}_1 \in \underline{\mathbb{V}}$ (cf. Problem A) and the $\Delta$-(quasi-)scalar product in Equation (6.7).

**Lemma 6.6.** *$\forall \psi, \chi \in D(\mathbf{L})$, (resp. $\forall \psi, \chi \in \overset{o}{\mathbb{W}}{}_2^1$) we have*

$$\Big(\psi, \underline{\mathbf{e}}_1^T \cdot (\underline{\nabla}\chi \times \underline{\nabla}S)\Big) = \int_{\Omega_{\mathcal{A}}} (\underline{\mathbf{e}}_1^T \cdot \underline{\nabla}\chi \times \underline{\nabla}S)\psi d\underline{x} = -\int_{\Omega_{\mathcal{A}}} (\underline{\mathbf{e}}_1^T \cdot \underline{\nabla}\psi \times \underline{\nabla}S)\chi d\underline{x} = -\Big(\chi, \underline{\mathbf{e}}_1^T \cdot (\underline{\nabla}\psi \times \underline{\nabla}S)\Big). \tag{6.9}$$

*Proof.* Since $\underline{\nabla} \times \underline{\nabla}S = \underline{0}$, the left-hand side is equal to

$$\int_{\Omega_{\mathcal{A}}} \underline{\mathbf{e}}_1^T \cdot (\psi\underline{\nabla} \times (\chi\underline{\nabla}S))d\underline{x} = \int_{\Omega_{\mathcal{A}}} \underline{\mathbf{e}}_1^T \cdot \underline{\nabla} \times (\psi\chi\underline{\nabla}S)d\underline{x} - \int_{\Omega_{\mathcal{A}}} \underline{\mathbf{e}}_1^T \cdot (\underline{\nabla}\psi \times (\chi\underline{\nabla}S))d\underline{x}$$

and, on the other hand, here the first integral on the right-hand side vanishes, because of the Gauss theorem and of the boundary condition for $\psi$:

$$\int_{\Omega_{\mathcal{A}}} \underline{\mathbf{e}}_1^T \cdot \underline{\nabla} \times (\psi\chi\underline{\nabla}S)d\underline{x} = \int_{\partial\Omega_{\mathcal{A}}} \psi\chi(\underline{\nabla}S)^T d\underline{x} = 0.$$

□

We are now in the position to do give a shortened proof of Theorem 6.1:

*Proof* (Theorem 6.1). We start with an appropriate weak formulation of the Problem A (2.9)–(2.10) in the streamfunction formulation. By taking the sum of Equations (6.7) and (6.8), we get:

$$\Big(\Phi, \underline{\mathbf{e}}_1^T \cdot (\underline{\nabla}\theta \times \underline{\nabla}S)\Big) + \Big(\sigma, \underline{\mathbf{e}}_1^T \cdot (\underline{\nabla}\Psi \times \underline{\nabla}S)\Big) = \lambda\left((\Phi, \Psi)_\Delta - (\theta, \sigma)_D\right), \tag{6.10}$$

Now we consider the elements $\Phi, \Psi \in \overset{o}{W}{}_2^2$ and $\sigma, \theta \in \overset{o}{\mathbb{W}}{}_2^1$ as Fourier series in the systems of eigenfunctions $\{\psi_j\}_{j=1}^\infty$ and $\{\chi_j\}_{j=1}^\infty$ in the sense of $\mathbb{L}_2$ and $\mathbb{L}_2$, respectively. We set

$$\Psi = \sum_{j=1}^\infty c_j\psi_j, \quad \Phi = \sum_{j=1}^\infty a_j\psi_j \tag{6.11}$$

$$\theta = \sum_{j=1}^\infty d_j\chi_j, \quad \sigma = \sum_{j=1}^\infty b_j\chi_j. \tag{6.12}$$

The sequences of Fourier coefficients are denoted by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ as in Equation (4.10). The sequences are regarded as columns and for a sequences $\mathbf{a}$ considered as a row we write in this sense $\mathbf{a}^T$ again.

The Fourier series are substituted in Equations (6.7) and (6.8) and we get:

$$\mathbf{a}^T\left(B\mathbf{d} - \lambda\,[\mathrm{diag}(\mu_k^2)]\mathbf{c}\right) = 0\,, \tag{6.13}$$

$$\mathbf{b}^T\left(B^T\mathbf{c} - \lambda\,[\mathrm{diag}(\omega_k^2)]\mathbf{d}\right) = 0, \tag{6.14}$$

where

$$B_{j,k} := (\psi_j, \underline{\mathbf{e}}_1^T \cdot (\underline{\nabla}\chi_k \times \underline{\nabla}S))\,, \qquad \forall j,k \in \mathbb{N} \times \mathbb{N}. \tag{6.15}$$

The sum of the Equations (6.13) and (6.14) (the equivalent to Equation (6.10)) can be written in $\mathbb{l}_2 \times \mathbb{l}_2$ as:

$$(\mathbf{a}^T, \mathbf{b}^T)\left[\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} - \lambda\begin{pmatrix} \mathrm{diag}(\mu_k^2) & 0 \\ 0 & \mathrm{diag}(\omega_k^2) \end{pmatrix}\right]\begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = 0\,. \tag{6.16}$$

With respect to Lemma 6.6, we get the structure and the symmetry of the following linear operator written as a "block matrix" on $\mathbb{l}_2 \times \mathbb{l}_2$.

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

But here are the elements $B_{j,k}$ in Equation (6.15) not uniformly bounded in contrast to the uniform boundedness of the elements $C_{j,k}$ of $C$ in Section 4 in Equation (4.13). We need for estimation of the $B_{j,k}$ the equivalent norms on $\overset{o}{W}{}_2^2$, respectively, on $\overset{o}{\mathbb{W}}{}_2^1$.

This results in $\forall\, j,k \in \mathbb{N}\times\mathbb{N}$:

$$|B_{j,k}| = |(\psi_j, \underline{e}_1^T \cdot (\underline{\nabla}\chi_k \times \underline{\nabla}S))| = |(\chi_k, \underline{e}_1^T \cdot (\underline{\nabla}\psi_j \times \underline{\nabla}S))| \le \gamma_* \gamma_S^{II} \ \max\left(\mu_j, \omega_k\right) \tag{6.17}$$

where we have used $\gamma_*$ as constant for the both equivalent norms and $\gamma_S^{II} := \max_{\underline{x}\in\Omega_{\mathcal{A}}} \sqrt{\left(|\frac{\partial S}{\partial r}|^2 + |\frac{\partial S}{r\partial\varphi}|^2\right)}$, $\|\psi_j\|_{\mathbb{L}_2} = 1$ and $\|\chi_k\|_{\mathbb{L}_2} = 1\ \forall\, j,k \in \mathbb{N}\times\mathbb{N}$.

Now the sequences of Fourier coefficients corresponding to $\Phi$, $\Psi$, $\sigma$, and $\theta$ belong to the spaces $\mathbb{l}_2^{\mathbb{W}2}$ and $\mathbb{l}_2^{\mathbb{W}}$ (cf. Lemma 4.4 and in this context, the Notation 6.4 also). From these, it follows that $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}, \tilde{\mathbf{d}} \in \mathbb{l}_2$, with

$$\tilde{\mathbf{a}} = \{\tilde{a}_j := \mu_j a_j\}_{j=1}^{\infty},\ \tilde{\mathbf{b}} = \{\tilde{b}_j := \omega_j b_j\}_{j=1}^{\infty},\ \tilde{\mathbf{c}} = \{\tilde{c}_j := \mu_j c_j\}_{j=1}^{\infty} \text{ and } \tilde{\mathbf{d}} = \{\tilde{d}_j := \omega_j d_j\}_{j=1}^{\infty}. \tag{6.18}$$

Additionally, we explain the matrix $\tilde{B}$ by

$$\tilde{B}_{j,k} := \frac{1}{\mu_j \omega_k} B_{j,k} \quad \forall\, j,k \in \mathbb{N}\times\mathbb{N}. \tag{6.19}$$

So we are able now to formulate the problem (6.16) equivalent in the $\mathbb{l}_2 \times \mathbb{l}_2$-sense through:

$$(\tilde{\mathbf{a}}^T, \tilde{\mathbf{b}}^T)\left[\begin{pmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{pmatrix} - \lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]\begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = 0, \tag{6.20}$$

where $I$ denotes the identity in $\mathbb{l}_2$ again. We interpret Equation (6.20) as an operator equation in $\mathbb{l}_2 \times \mathbb{l}_2$. This is possible since the relation (6.20) is valid for arbitrary $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) \in \mathbb{l}_2 \times \mathbb{l}_2$.

We define the linear and bounded operators $\tilde{\mathcal{B}}$ and $\mathcal{I}$; $\tilde{\mathcal{B}}, \mathcal{I} : \mathbb{l}_2 \times \mathbb{l}_2 \longrightarrow \mathbb{l}_2 \times \mathbb{l}_2$ by:

$$\tilde{\mathcal{B}} := \begin{pmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{pmatrix}, \quad \mathcal{I} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{6.21}$$

The operators $\tilde{\mathcal{B}}$ and $\mathcal{I}$ are elements of the Banach space $\mathbb{L}(\mathbb{l}_2 \times \mathbb{l}_2, \mathbb{l}_2 \times \mathbb{l}_2)$ of linear and bounded operators. Now we have made all preparations to follow the line of argument of section 4.

To show the compactness of $\tilde{\mathcal{B}}$, we use that $\tilde{\mathcal{B}}$ can be approximated by a sequence of finite operators $\{\tilde{\mathcal{B}}_\ell\}_{\ell\in\mathbb{N}}$, $\{\widetilde{\mathcal{B}_\ell}\}_{\ell\in\mathbb{N}} \in \mathbb{L}(\mathbb{l}_2 \times \mathbb{l}_2, \mathbb{l}_2 \times \mathbb{l}_2)$, with $\{\widetilde{\mathcal{B}_\ell}\}_{\ell\in\mathbb{N}} \underset{\mathbb{L}}{\rightarrow} \tilde{\mathcal{B}}$, where we use $\forall\,\ell \in \mathbb{N}$ matrices $\tilde{B}_\ell$ defined along the lines of Equation (4.20) for $\tilde{C}$. The convergence $\{\widetilde{\mathcal{B}_\ell}\}_{\ell\in\mathbb{N}} \underset{\mathbb{L}}{\rightarrow} \tilde{\mathcal{B}}$ follows from Equations (6.17) and (6.19), $\lim_{j\to\infty}\mu_j = \infty$ <u>or</u> $\lim_{k\to\infty}\omega_k = \infty$. Finally, we use Lemma 4.5 again in a modified formulation: $\lambda = 0$ is not an eigenvalue of $\tilde{\mathcal{B}}$ resp. of problem (2.9)–(2.10) and we set for the proof $\underline{w} = \underline{\nabla}\,\Psi \times \underline{e}_1$.

## 7 | ORTHOGONALITY RELATIONS

At first, it is reasonable to feature the multiplicity of the eigenvalues. We choose the usual way to explain it by the dimension of a null space:

Notation 7.1. Let $\tilde{C} \in \mathcal{L}(l_2 \times l_2, l_2 \times l_2)$ be the operator stated in Theorem 4.1 (equivalent to the Problem A ((2.9) and (2.10)) for all $\mathcal{A}$). We call $N(\tilde{C} - \lambda\,\mathcal{I})$ as a subspace of $l_2 \times l_2$ the null space of the operator $\tilde{C} - \lambda\,\mathcal{I}$ for $\lambda \in \mathbb{C}$:

$$N(\tilde{C} - \lambda\,\mathcal{I}) \;:=\; \{\tilde{\mathbf{x}} \in l_2 \times l_2 \,:\, (\tilde{C} - \lambda\,\mathcal{I})\tilde{\mathbf{x}} \,=\, \tilde{\mathbf{o}}\}\,, \tag{7.1}$$

where $\tilde{\mathbf{x}} \,=\, (\tilde{\mathbf{c}}, \tilde{\mathbf{d}})\,, \tilde{\mathbf{o}} \in l_2 \times l_2$, with $\tilde{\mathbf{o}}$ as the zero element of the space $l_2 \times l_2$.

The multiplicity of the eigenvalue $\lambda$ of $\tilde{C}$ is specified by

$$\mathfrak{N}(\lambda) \;:=\; \dim N(\tilde{C} - \lambda\,\mathcal{I})\,. \tag{7.2}$$

We will prove the following orthogonality relations in addition to Theorem 4.1 regarding the eigen-triples of Problem A:

**Lemma 7.2.** *For any eigenvalue $\lambda$ of $\tilde{C}$ (resp. of the Problem A (cf. (2.9) and (2.10)), there exists the eigenvalue $(-\lambda)$ also. We have that to every eigen-triple $(\lambda, \underline{w}, \theta) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}^1_2$ exists a correspondent eigen-triple $(-\lambda, \underline{w}^-, \theta^-) \;:=\; (-\lambda, -\underline{w}, \theta) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}^1_2$. The eigenvalues $\lambda$ and $-\lambda$ have the same multiplicity: $\mathfrak{N}(\lambda) \,=\, \mathfrak{N}(-\lambda)$.*

*Proof.* To delineate the idea of the proof, let us assume that $(\lambda, \underline{w}, \theta) \in \mathbb{C} \times \underline{\mathbb{V}}^2 \times \overset{o}{\mathbb{W}}{}^1_2 \cap \mathbb{W}^2_2$ is a fixed solution of Equations (2.9) and (2.10). Then we have also by inserting $(-\lambda, -\underline{w}, \theta)$ in Equations (2.9) and (2.10) (resp. multiplying the second Equation 2.10 with $-1$):

$$\theta \underline{\nabla} S - \lambda \Delta(-\underline{w}) = -\underline{\nabla}\, p\,, \quad \underline{\nabla}^T \cdot \underline{\widetilde{w}} = 0\,,$$
$$-\underline{w}^T \cdot \underline{\nabla} S - \lambda \Delta \theta = 0\,.$$

It is hence evident that the problem is fulfilled by

$$(-\underline{w}, \theta),$$

where the elements $(\underline{w}, \theta)$ and $(-\underline{w}, \theta)$ are linear independent in the Cartesian product.

We use now the linear and bounded operators $\tilde{C}$ and $\mathcal{I}$; $\tilde{C}, \mathcal{I} \,:\, l_2 \times l_2 \longrightarrow l_2 \times l_2$ (4.18) and the equivalent eigenvalue problem (cf. Equation 4.19). The sequences of Fourier coefficients $\mathbf{c}$ and $d$ corresponding to $\underline{w}$ and $\theta$ belong to the spaces $l_2^{\mathbb{V}}$ and $l_2^{\mathbb{W}}$. So we apply the notations $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$, $(\tilde{\mathbf{c}}, \tilde{\mathbf{d}} \in l_2)$ compare Equation (4.15) for the correspondent eigen-triple $(\lambda, \tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \cong (\lambda, \underline{w}, \theta)$. Let us suppose that the eigen-triple $(\lambda, \tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \cong (\lambda, \underline{w}, \theta)$ fulfills the equation:

$$\tilde{C}\begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^T & 0 \end{pmatrix}\begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = \begin{pmatrix} \tilde{C}\tilde{\mathbf{d}} \\ \tilde{C}^T\tilde{\mathbf{c}} \end{pmatrix} = \lambda\,\mathcal{I}\begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = \lambda\begin{pmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix}. \tag{7.3}$$

It is obvious that also $(-\lambda, -\tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \cong (-\lambda, -\underline{w}, \theta)$ satisfies as an eigen-triple the “ruling” equation:

$$\tilde{C}\begin{pmatrix} -\tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^T & 0 \end{pmatrix}\begin{pmatrix} -\tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = \begin{pmatrix} \tilde{C}\tilde{\mathbf{d}} \\ -\tilde{C}^T\tilde{\mathbf{c}} \end{pmatrix} = -\lambda\,\mathcal{I}\begin{pmatrix} -\tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} = -\lambda\begin{pmatrix} -\tilde{\mathbf{c}} \\ \tilde{\mathbf{d}} \end{pmatrix} \tag{7.4}$$

The multiplicities of $\lambda$ and $-\lambda$ are equal. To see this, we use Equations (7.3) and (7.4) and $\mathfrak{N}(\lambda) \,<\, \infty$ from Theorem 4.1. Then we choose $(\tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \in N(\tilde{C} - \lambda\,\mathcal{I})$ as a fixed element of an $l_2 \times l_2$ -orthonormal basis of $N(\tilde{C} - \lambda\,\mathcal{I})$ to get, that $(-\tilde{\mathbf{c}}, \tilde{\mathbf{d}})$ is an element of an $l_2 \times l_2$ -orthonormal basis of $N(\tilde{C} + \lambda\,\mathcal{I})$. The assertion is showed by interchanging the roles of $\lambda$ and $-\lambda$. □

We write down the statement of the scalar product and orthogonality on the Cartesian product of Hilbert space to illustrate the use of an orthonormal basis on $l_2 \times l_2$ (cf. Section 4, Equation 5.7).

Notation 7.3. Let us use the abbreviations from Equation (4.10) for elements of the Hilbertian sequence space $\mathbb{l}_2$. Then the scalar product on the Cartesian product is defined via

$$[(\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d})]_{\mathbb{l}_2\times\mathbb{l}_2} := \frac{1}{2}\left((\mathbf{a},\mathbf{c})_{2,(l)} + (\mathbf{b},\mathbf{d})_{2,(l)}\right)$$
$$\forall\,(\mathbf{a},\mathbf{b}),\ (\mathbf{c},\mathbf{d}) \in \mathbb{l}_2\times\mathbb{l}_2$$

The norm on $\mathbb{l}_2\times\mathbb{l}_2$ (cf. Equation (5.7) also) is taken in the usual way as:

$$\|(\mathbf{c},\mathbf{d})\|_{\mathbb{l}_2\times\mathbb{l}_2} = \sqrt{[(\mathbf{c},\mathbf{d}),(\mathbf{c},\mathbf{d})]_{\mathbb{l}_2\times\mathbb{l}_2}}$$

**Lemma 7.4.** *The inner product $[(\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d})]_{\mathbb{l}_2\times\mathbb{l}_2}$ given in Notation 7.3 is a scalar product on $l_2\times l_2$ and the Hilbert space $\mathbb{l}_2\times\mathbb{l}_2$ is defined by $\mathbb{l}_2\times\mathbb{l}_2 := (l_2\times l_2, [(.,.),(.,.)]_{\mathbb{l}_2\times\mathbb{l}_2})$.*

*Let the orthonormal basis of the $\mathbb{l}_2$ be taken as follows:*

$$\{\boldsymbol{\delta}_j\}_{j=1}^{\infty} \quad \textit{with } \boldsymbol{\delta}_j := \{\delta_{j,k}\}_{k=1}^{\infty} \quad \forall\, j\in\mathbb{N}, \tag{7.5}$$

*where $\delta_{j,k}$ denote the Kronecker $\delta$ (as function of the variables $j$ and $k$).*

*Then one gets an orthonormal basis of $\mathbb{l}_2\times\mathbb{l}_2$ by choosing the system:*

$$\{\boldsymbol{\gamma}_j = (\boldsymbol{\delta}_j,\boldsymbol{\delta}_j)\}_{j=1}^{\infty} \quad\cup\quad \{\boldsymbol{\eta}_j = (-\boldsymbol{\delta}_j,\boldsymbol{\delta}_j)\}_{j=1}^{\infty} \tag{7.6}$$

*Proof.* The axioms of a scalar product on $\mathbb{l}_2\times\mathbb{l}_2$ are to check easily. The only critical point in showing that we have an orthonormal basis of the $\mathbb{l}_2\times\mathbb{l}_2$ by the choosing system (7.6) is the simple calculation:

$$[\boldsymbol{\gamma}_j,\boldsymbol{\eta}_j]_{\mathbb{l}_2\times\mathbb{l}_2} = [(\boldsymbol{\delta}_j,\boldsymbol{\delta}_j),(-\boldsymbol{\delta}_j,\boldsymbol{\delta}_j)]_{\mathbb{l}_2\times\mathbb{l}_2} = 0 \quad \forall\, j\in\mathbb{N}.$$

The completeness follows, since the zero element of $\mathbb{l}_2\times\mathbb{l}_2$ is the only element $\boldsymbol{\xi}\in\mathbb{l}_2\times\mathbb{l}_2$, which is orthogonal to the system (7.6). □

*Remark.* Important properties of the eigenvalues $\lambda$ of $\tilde{C}$ are written down in Theorem 4.1: The eigenvalues $\lambda$ of $\tilde{C}$ are real and $\lambda = 0$ is not an eigenvalue of $\tilde{C}$ but the only accumulation point of the eigenvalues. The set of all eigenvalues $\{\lambda\}$ consists of an countably infinite number of (finite multiplicity: $\mathfrak{N}(\lambda) < \infty$) eigenvalues. It is to be considered that the features of the construction of an orthonormal basis in $\mathbb{l}_2\times\mathbb{l}_2$ can be transferred to the spaces $\underline{\mathbb{H}}\times\mathbb{L}_2$ and $\underline{\mathbb{V}}\times\overset{o}{\mathbb{W}}{}_2^1$ in one to one manner. We achieve in doing so: The connections of the factors in the Cartesian products are conserved for the Cartesian products together with the features: $\underline{\mathbb{V}} := \boldsymbol{S}^{-\frac{1}{2}}(\underline{\mathbb{H}})$ and $\overset{o}{\mathbb{W}}{}_2^1 := \boldsymbol{L}^{-\frac{1}{2}}(\mathbb{L}_2)$. There we have considered the Stokes operator and the Laplacian as positive definite operators in the context of spectral operators.

**Proposition 7.5** (Orthogonality). *Let $\lambda_1,\lambda_2$ be eigenvalues of $\tilde{C}\in\mathcal{L}(\mathbb{l}_2\times\mathbb{l}_2,\mathbb{l}_2\times\mathbb{l}_2)$, with $\lambda_1\neq\lambda_2$, then there are the correspondent eigenspaces $N(\tilde{C}-\lambda_1\,\mathcal{I})$ and $N(\tilde{C}-\lambda_2\,\mathcal{I})$ orthogonal subspaces in $\mathbb{l}_2\times\mathbb{l}_2$.*

*Proof.* According to Theorem 4.1 is $\tilde{C}$ a self-adjoint operator on $\mathbb{l}_2\times\mathbb{l}_2$ and $\lambda = 0$ is not an eigenvalue of $\tilde{C}$. The statement is obtained by the appliance of the Theorem (sect. 4.2.3 [23]). □

**Theorem 7.6** (Completeness). *The set of orthonormalized eigenfunctions of $\tilde{C}$ generates a complete basis for the Cartesian product space $\mathbb{l}_2\times\mathbb{l}_2$. If we denote by $\{\lambda_\ell\}_{\ell=1}^{\infty}$ the set of eigenvalues of $\tilde{C}$, than we have:*

$$\mathbb{l}_2\times\mathbb{l}_2 = \overline{\bigcup_{\ell=1}^{\infty} N(\tilde{C}-\lambda_\ell\,\mathcal{I})}^{\mathbb{l}_2\times\mathbb{l}_2}. \tag{7.7}$$

*Proof.* For every $\lambda_\ell$, there exists a finite orthonormal basis in $N(\tilde{C}-\lambda_\ell\,\mathcal{I})$. According to Proposition 7.5, the eigenspaces for different $\lambda$ are orthogonal to each other. So we have only to show Equation (7.7). We use that $\lambda = 0$ is not an eigenvalue of $\tilde{C}$ and that the operator $\tilde{C}$ is self-adjoint and compact. So we apply the orthogonal decomposition theorem (sect. 2.4.2, [23]) and the theorem to the spectrum of compact operators (sect. 2.4.3, [23]) to get the justification of Equation (7.7). □

**Proposition 7.7.** *Let $\widetilde{\mathfrak{A}} : D(\boldsymbol{S}) \times D(\boldsymbol{L}) \to \underline{\mathbb{H}} \times \mathbb{L}_2$ be regarded as in Equation (7.11). Then is $\widetilde{\mathfrak{A}}$ a self-adjoint positive definite operator. $\widetilde{\mathfrak{A}}$ is in the sense of Equation (7.11) and of the matrix representation on product spaces a component-by-component combination of the self-adjoint positive definite operators $\boldsymbol{S}$ and $\boldsymbol{L}$ with pure point spectrum. We get for the "diagonal matrices:"*

$$\widetilde{\mathfrak{A}}^{-1} = \begin{pmatrix} \boldsymbol{S}^{-1} & \underline{0} \\ (\underline{0})^T & \boldsymbol{L}^{-1} \end{pmatrix}, \quad \widetilde{\mathfrak{A}}^{-\frac{1}{2}} = \begin{pmatrix} \boldsymbol{S}^{-\frac{1}{2}} & \underline{0} \\ (\underline{0})^T & \boldsymbol{L}^{-\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad \widetilde{\mathfrak{A}}^{\frac{1}{2}} = \begin{pmatrix} \boldsymbol{S}^{\frac{1}{2}} & \underline{0} \\ (\underline{0})^T & \boldsymbol{L}^{\frac{1}{2}} \end{pmatrix}, \tag{7.8}$$

*where one gets with $\underline{\mathfrak{u}} := (\underline{\Psi}, \varphi) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$:*

$$\left(\widetilde{\mathfrak{A}}^{\frac{1}{2}} \underline{\mathfrak{u}}, \widetilde{\mathfrak{A}}^{\frac{1}{2}} \underline{\mathfrak{w}}\right) = (\boldsymbol{S}^{\frac{1}{2}} \underline{\Psi}, \boldsymbol{S}^{\frac{1}{2}} \underline{w}) + (\boldsymbol{L}^{\frac{1}{2}} \varphi, \boldsymbol{L}^{\frac{1}{2}} \theta) = \left(\underline{\Psi}, \underline{w}\right)_D + (\varphi, \theta)_D. \tag{7.9}$$

*The energetic space of $\widetilde{\mathfrak{A}}$ is $\underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1 = \widetilde{\mathfrak{A}}^{-\frac{1}{2}}(\underline{\mathbb{H}} \times \mathbb{L}_2)$. The self-adjoint operators $\widetilde{\mathfrak{A}}^{-1}$ and $\widetilde{\mathfrak{A}}^{-\frac{1}{2}}$ are compact in $\mathfrak{L}(\mathbb{L}(\underline{\mathbb{H}} \times \mathbb{L}_2, \underline{\mathbb{H}} \times \mathbb{L}_2)$. This is also true for $\widetilde{\mathfrak{A}}^{-1}$ regarded as: $\widetilde{\mathfrak{A}}^{-1} : \underline{\mathbb{H}} \times \mathbb{L}_2 \to \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$.*

*Proof.* It is a one to one transference of theorem (sect. 4.5.2, [23]) on product spaces.) □

**Theorem 7.8.** *Problem A (2.9)–(2.10) is for all $\mathcal{A}$ equivalent to the eigenvalue problem of the compact self-adjoint operator $\widetilde{\mathfrak{C}}$ on the Cartesian product space $\underline{\mathbb{H}} \times \mathbb{L}_2$.*

*Proof.* Let us use Notation 3.4 and Equation (3.2). So we write

$$\widetilde{A}((\underline{w}, \theta)) = (\widetilde{\mathfrak{S}} - \lambda \widetilde{\mathfrak{A}})((\underline{w}, \theta)) = \left( \begin{pmatrix} 0 & \underline{\nabla} S \\ (\underline{\nabla} S)^T & 0 \end{pmatrix} - \lambda \begin{pmatrix} \boldsymbol{S} & \underline{0} \\ (\underline{0})^T & \boldsymbol{L} \end{pmatrix} \right) \begin{pmatrix} \underline{w} \\ \theta \end{pmatrix}, \tag{7.10}$$

where one can understand the zeros in $\widetilde{\mathfrak{S}}$ also as the zero elements of $\mathfrak{L}(\underline{\mathbb{H}}, \underline{\mathbb{H}})$ resp. $\mathfrak{L}(\mathbb{L}_2, \mathbb{L}_2)$.

The operators $\widetilde{\mathfrak{A}} : D(\boldsymbol{S}) \times D(\boldsymbol{L}) \to \underline{\mathbb{H}} \times \mathbb{L}_2$ and $\widetilde{\mathfrak{S}} : \underline{\mathbb{H}} \times \mathbb{L}_2 \to \underline{\mathbb{H}} \times \mathbb{L}_2$ in Equation (7.10) are also defined through the bilinearforms at $(\underline{\Psi}, \varphi) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$:

$$\begin{aligned} ((\underline{\Psi}, \varphi), \widetilde{\mathfrak{A}}(\underline{w}, \theta)) &:= \left(\underline{\Psi}, \underline{w}\right)_D + (\varphi, \theta)_D \\ ((\underline{\Psi}, \varphi), \widetilde{\mathfrak{S}}(\underline{w}, \theta)) &:= \left(\underline{\Psi}, \theta \, \underline{\nabla} S\right) + \left(\varphi, (\underline{\nabla} S)^T \underline{w}\right). \end{aligned} \tag{7.11}$$

There is especially Equation (7.11) to understand in sense on dense definition. We investigate first the properties of $\widetilde{\mathfrak{A}}$ to establish a relationship to the task of Problem A (2.9)–(2.10): the non-trivial solutions $(\lambda, \underline{w}, \theta) \in \mathbb{C} \times \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$ of (2.9) and (2.10), where we have there only equations in $\underline{\mathbb{V}}'$ resp. $\mathbb{W}_2^{-1}$.

Using the notation $\widetilde{\mathfrak{A}}_{en}$ for the energetic expansion of $\widetilde{\mathfrak{A}}$ in the sense of $\widetilde{\mathfrak{A}}_{en} : \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1 \to \underline{\mathbb{V}}' \times \mathbb{W}_2^{-1}$ and $\underline{\mathfrak{w}} := (\underline{w}, \theta) \in \underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$ we may write

$$\left(\widetilde{\mathfrak{S}} - \lambda \widetilde{\mathfrak{A}}_{en}\right) \underline{\mathfrak{w}} = \underline{0}. \tag{7.12}$$

The properties of $\widetilde{\mathfrak{A}}$ are summarized in:

We get the boundedness of $\widetilde{\mathfrak{S}}$ using $\gamma_S$ (cf. Equation 4.13). The crucial tool for our argumentation is the mapping property: By an appropriate norm on $\underline{\mathbb{V}} \times \overset{o}{\mathbb{W}}{}_2^1$, we have $\widetilde{\mathfrak{A}}^{\frac{1}{2}}$ as an unitary mapping on $\underline{\mathbb{H}} \times \mathbb{L}_2$ and so forth. The idea of one to one mappings result in $\underline{\mathfrak{u}} = \widetilde{\mathfrak{A}}^{-\frac{1}{2}} \underline{\mathfrak{v}}$ and $\underline{\mathfrak{w}} = \widetilde{\mathfrak{A}}^{-\frac{1}{2}} \underline{\mathfrak{y}}$. We get for Problem A (2.9)–(2.10) using Equations (7.10) and (7.11):

$$(\widetilde{\mathfrak{A}}^{-\frac{1}{2}}\underline{\mathfrak{v}}, \widetilde{\mathfrak{S}}\widetilde{\mathfrak{A}}^{-\frac{1}{2}}\underline{\mathfrak{y}}) - \lambda(\widetilde{\mathfrak{A}}^{\frac{1}{2}}\widetilde{\mathfrak{A}}^{-\frac{1}{2}}\underline{\mathfrak{v}}, \widetilde{\mathfrak{A}}^{\frac{1}{2}}\widetilde{\mathfrak{A}}^{-\frac{1}{2}}\underline{\mathfrak{y}})$$

$$= (\underline{\mathfrak{v}}, \underbrace{\widetilde{\mathfrak{A}}^{-\frac{1}{2}}\widetilde{\mathfrak{S}}\widetilde{\mathfrak{A}}^{-\frac{1}{2}}}_{=:\widetilde{\mathfrak{C}}}\underline{\mathfrak{y}}) - \lambda(\underline{\mathfrak{v}}, \underline{\mathfrak{y}}) = 0\,. \tag{7.13}$$

Finally, we use the identity $\widetilde{\mathfrak{I}}$ in $\underline{\mathbb{H}} \times \mathbb{L}_2$ to get:

$$(\widetilde{\mathfrak{C}} - \lambda\widetilde{\mathfrak{I}})\underline{\mathfrak{y}} = \underline{0} \tag{7.14}$$

as the equivalent eigenvalue problem in $\underline{\mathbb{H}} \times \mathbb{L}_2$ for the self-adjoint compact operator $\widetilde{\mathfrak{C}}$. □

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