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# Inequalities characterizing differential operators

Gerd Herzog and Peer Kunstmann

**Abstract.** We show that under certain inequality assumptions an arbitrary linear operator $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ is a differential operator, for example, if $D[f]$ is nonnegative in local minima of $f$.

**Mathematics Subject Classification.** 34A40, 47B92.

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## 1. Introduction

In this note we study linear operators $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ where $C^\infty(\mathbb{R})$ denotes the vector space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $C(\mathbb{R})$ denotes the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$. If $D$ has the form $D[f] = \varphi_1 f' + \varphi_2 f''$ with $\varphi_1, \varphi_2 \in C(\mathbb{R})$ and $\varphi_2 \geq 0$ then $D$ has the following property:

$$\text{If } f \in C^\infty(\mathbb{R}) \text{ has a local minimum at } x_0 \in \mathbb{R} \text{ then } D[f](x_0) \geq 0. \qquad \text{(l-min)}$$

If a linear operator $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ satisfies this property (l-min) then the well known fact that $x_0$ is an (even strict) local minimum of $f$ if $f'(x_0) = 0$ and $f''(x_0) > 0$ shows that $D$ also satisfies the formally weaker property

$$\forall f \in C^\infty(\mathbb{R}) \forall x_0 \in \mathbb{R} : \ f'(x_0) = 0, f''(x_0) > 0 \ \Rightarrow \ D[f](x_0) \geq 0. \qquad \text{(s-min)}$$

As a consequence of Theorem 1 below we will see that, in converse to the observation above, any $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ with this weaker property (s-min) is in fact of the form $D[f] = \varphi_1 f' + \varphi_2 f''$ with $\varphi_1, \varphi_2 \in C(\mathbb{R})$ and $\varphi_2 \geq 0$. Note that this shows that in fact (s-min) and (l-min) are equivalent, since every operator of this form satisfies (l-min).

Gerd Herzog and Peer Kunstmann contributed equally to this work.

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Linear operators with properties like (s-min) have been studied by Feller in a series of papers [3–6]. The motivation of Feller's work has been the characterization of diffusion type linear operators $D : U \to C(\mathbb{R})$ that are local and defined on a linear subspace $U$ of $C(\mathbb{R})$. This has numerous applications, e.g. in the theories of parabolic equations or of stochastic processes. The main emphasis in the mentioned articles is on the non-smooth situation in the sense that the domain $U$ of the operator $D$ does not necessarily contain $C^\infty(\mathbb{R})$ and is rather unspecified. In particular operators $D[f] := (\varphi f')'$ fall into this category where $\varphi$ need not be continuous and might have jumps. If $\varphi$ has a jump at $x_0 \in \mathbb{R}$, then $f'$ cannot be continuous at $x_0$, since $D[f]$ is. For linear operators $D : U \to C(\mathbb{R})$, Feller derived a representation as so-called generalized differential operators of (at most) second order.

Here, we restrict ourselves to the smooth situation where the domain $U$ of $D$ equals $C^\infty(\mathbb{R})$. On the other hand, we generalize condition (s-min) and replace first and second derivative in (s-min) by linear differential operators $L_1, L_2 : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ where the order of $L_1$ is strictly less than the order of $L_2$.

We have been inspired by the characterizations of the first derivative $D[f] = f'$ via algebraic properties such as the product rule or the chain rule by König and Milman [7], and Fechner, Gselmann and Świaatczak [1,2]. These works do not even assume a priori linearity of $D$, which requires additional care in the proofs. Here we assume linearity of $D$ and replace algebraic properties by properties formulated via inequalities.

We have formulated our results for the real line $\mathbb{R}$ for simplicity. Analogous results hold with the same proofs for operators $D : C^\infty(J) \to C(J)$ if $J \subseteq \mathbb{R}$ is an open interval.

## 2. Results

To be precise, we consider two linear differential operators $L_1, L_2 : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ of the form

$$L_1[f] := \sum_{k=0}^{n} a_k f^{(k)}, \quad L_2[f] := f^{(n+1)} + \sum_{k=0}^{n} b_k f^{(k)}$$

with $n \in \mathbb{N}_0$ and $a_0, \ldots, a_n, b_0, \ldots, b_n \in C(\mathbb{R})$ and set out to characterize the linear operators $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ with the property

$$\forall f \in C^\infty(\mathbb{R})\ \forall x_0 \in \mathbb{R} :\ L_1[f](x_0) = 0,\ L_2[f](x_0) > 0 \ \Rightarrow\ D[f](x_0) \geq 0. \quad \text{(P)}$$

Note that (s-min) is property (P) for the operators $L_1[f] = f'$ and $L_2[f] = f''$.

**Theorem 1.** *Let $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ be a linear operator with property (P). Then $D$ is of the form*

$$D[f] = \varphi_1 L_1[f] + \varphi_2 L_2[f] \quad (f \in C^\infty(\mathbb{R}))$$

*for some functions $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ with $\varphi_2 \in C(\mathbb{R})$ nonnegative. Conversely every operator of this form has property (P).*

*Remark 2.* From the representation of $D$ in Theorem 1 we see that $\varphi_2 : \mathbb{R} \to \mathbb{R}$ is uniquely determined, since the leading term is $\varphi_2 f^{(n+1)}$. Clearly we can choose $\varphi_1$ arbitrary if $L_1 = 0$. To get a little more information on $\varphi_1$ in the general case we define $s : \mathbb{R} \to \mathbb{R}$ by

$$s(x) := \sum_{k=0}^{n} (a_k(x))^2.$$

We can see that $\varphi_1$ is uniquely determined and continuous at $x_0 \in \mathbb{R}$ if $s(x_0) > 0$: In this case we can find a function $g \in C^\infty(\mathbb{R})$ such that $L_1[g](x_0) \neq 0$. Since $L_1[g] \in C(\mathbb{R})$ we have $L_1[g] \neq 0$ on an open interval $I$ with $x_0 \in I$, hence

$$\varphi_1(x) = \frac{D[g](x) - \varphi_2(x) L_2[g](x)}{L_1[g](x)} \quad (x \in I).$$

In particular, $\varphi_1 : \mathbb{R} \to \mathbb{R}$ is uniquely determined and $\varphi_1 \in C(\mathbb{R})$ if $s$ has no zeros. As a side note, we note that $D$ is then continuous with respect to the associated Fréchet space topologies on $C^\infty(\mathbb{R})$ and $C(\mathbb{R})$, consistent with the heuristic that positivity implies continuity.

*Example 3.* The following example shows what can happen if $s$ has zeros: Let

$$L_1[f](x) = \sin^2(x) f'(x),\ L_2[f](x) = f''(x) + \sin(x) f'(x),\ D[f](x) = f''(x).$$

Then $D$ has property (P) with respect to $L_1$ and $L_2$, and $D[f] = \varphi_1 L_1[f] + \varphi_2 L_2[f]$ forces $\varphi_2 = 1$ and

$$\varphi_1(x) = -\frac{1}{\sin(x)} \quad (x \notin \pi\mathbb{Z}),$$

whereas $\varphi_1$ can be chosen arbitrary on $\pi\mathbb{Z}$. However, $\varphi_1$ cannot be chosen in $C(\mathbb{R})$.

We state the case $L_1 = 0$ with $L := L_2$ as a separate result.

**Theorem 4.** *Let $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ be a linear operator with property*

$$\forall f \in C^\infty(\mathbb{R})\ \forall x_0 \in \mathbb{R} : L[f](x_0) > 0 \ \Rightarrow\ D[f](x_0) \geq 0. \tag{Q}$$

*Then $D$ is of the form*

$$D[f] = \varphi L[f] \quad (f \in C^\infty(\mathbb{R}))$$

*for a nonnegative function $\varphi \in C(\mathbb{R})$. Conversely every operator of this form has property (Q).*

## 3. Some conclusions

These results have a number of consequences. From Theorem 1 we obtain the following.

**Corollary 5.** *Let $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ be a linear operator with the property (s-min). Then $D$ is of the form $D[f] = \varphi_1 f' + \varphi_2 f''$ with $\varphi_1, \varphi_2 \in C(\mathbb{R})$ and $\varphi_2$ nonnegative.*

*Proof.* Set $L_1[f] = f'$, $L_2[f] = f''$ and apply Theorem 1. □

We also obtain a result for operators satisfying what Feller called the weak minimum property [5].

**Corollary 6.** *Let $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ be a linear operator with the property that, if $x_0 \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$ are such that $f \geq 0$ in a neighborhood of $x_0$ and $f(x_0) = 0$, then $D[f](x_0) \geq 0$. Then $D$ is of the form $D[f] = \varphi_0 f + \varphi_1 f' + \varphi_2 f''$ with $\varphi_0, \varphi_1, \varphi_2 \in C(\mathbb{R})$ and $\varphi_2$ nonnegative.*

*Proof.* We shall apply Corollary 5 to the operator $\widetilde{D} : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ defined by $\widetilde{D}[f] := D[f] - fD[1]$ where 1 denotes the constant 1 function. We work with the property (s-min). So let $f \in C^\infty(\mathbb{R})$ and $x_0 \in \mathbb{R}$ with $f'(x_0) = 0$ and $f''(x_0) > 0$. Then $f'' \geq 0$ in a open interval $I$ containing $x_0$. By Taylor's formula we find, for any $x \in I$ a $\xi \in I$ between $x$ and $x_0$ such that

$$f(x) - f(x_0) = \frac{1}{2} f''(\xi)(x - x_0)^2 \geq 0 \quad (x \in I).$$

Letting $\widetilde{f} := f - f(x_0)1$ we have $\widetilde{f} \in C^\infty(\mathbb{R})$, $\widetilde{f} \geq 0$ on $I$ and $\widetilde{f}(x_0) = 0$. By assumption we thus have $D[\widetilde{f}](x_0) \geq 0$. But

$$D[\widetilde{f}](x_0) = D[f - f(x_0)1](x_0) = D[f](x_0) - f(x_0)D[1](x_0) = \widetilde{D}[f](x_0),$$

which means that we have verified the assumptions of Corollary 5 for $\widetilde{D}$. Hence we find $\varphi_1, \varphi_2 \in C(\mathbb{R})$ with $\varphi_2 \geq 0$ such that $\widetilde{D}[f] = \varphi_1 f' + \varphi_2 f''$. With $\varphi_0 := D[1] \in C(\mathbb{R})$ the assertion follows. □

As a variant of Corollary 5 and further example we consider local minima of $x \mapsto e^{-x^2} f(x)$ and leave the proof of the following corollary to the reader.

**Corollary 7.** *Let $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ be a linear operator with the property that if $f \in C^\infty(\mathbb{R})$ and $x_0 \in \mathbb{R}$ is a local minimum of $x \mapsto e^{-x^2} f(x)$, then $D[f](x_0) \geq 0$. Then $D$ is of the form $D[f] = \varphi_1 L_1[f] + \varphi_2 L_2[f]$, with*

$$L_1[f](x) = f'(x) - 2xf(x), \quad L_2[f](x) = f''(x) - 4xf'(x) + (4x^2 - 2)f(x),$$

*$\varphi_1, \varphi_2 \in C(\mathbb{R})$ and $\varphi_2$ nonnegative.*

By means of Theorem 4 we get the following corollary, for example:

**Corollary 8.** *Let $D : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ be a linear operator with the property that if $I \subseteq \mathbb{R}$ is an open interval and $f \in C^\infty(\mathbb{R})$ is $\begin{bmatrix} \textit{non-decreasing} \\ \textit{convex} \end{bmatrix}$ on $I$, then $D[f](x) \geq 0$ $(x \in I)$. Then $D$ is of the form $\begin{bmatrix} D[f] = \varphi f' \\ D[f] = \varphi f'' \end{bmatrix}$ for some nonnegative function $\varphi \in C(\mathbb{R})$.*

*Proof.* Observe that $D$ has the property

$$f \in C^\infty(\mathbb{R}), \ \begin{bmatrix} f'(x_0) > 0 \\ f''(x_0) > 0 \end{bmatrix} \Rightarrow D[f](x_0) \geq 0,$$

set $\begin{bmatrix} L[f] = f' \\ L[f] = f'' \end{bmatrix}$ and apply Theorem 4. □

## 4. Proof of Theorem 1

In the following we use the notation

$$w_k(x, x_0) := \frac{(x - x_0)^k}{k!} \quad (k \in \mathbb{N}_0),$$

and

$$T_{n+1}(f, x, x_0) := \sum_{k=0}^{n+1} f^{(k)}(x_0) w_k(x, x_0),$$

that is, $T_{n+1}(f, \cdot, x_0)$ is the Taylor polynomial of $f$ centered at $x_0$ of order $n + 1$.

*Proof.* Fix $x_0 \in \mathbb{R}$ and let $g \in C^\infty(\mathbb{R})$ be a function with

$$g^{(k)}(x_0) = 0 \ (k = 0, \ldots, n), \quad g^{(n+1)}(x_0) = 1.$$

Note that $L_1[g](x_0) = 0$, and $L_2[g](x_0) = 1$. Let $f \in C^\infty(\mathbb{R})$. For $\lambda \in \mathbb{R}$ we set

$$h_\lambda(x) := g(x) + \lambda(f(x) - T_{n+1}(f, x, x_0)) \quad (x \in \mathbb{R}).$$

Then $h_\lambda \in C^\infty(\mathbb{R})$ and

$$h_\lambda^{(k)}(x_0) = g^{(k)}(x_0) \quad (k = 0, \ldots, n + 1).$$

Thus $L_1[h_\lambda](x_0) = 0$ and $L_2[h_\lambda](x_0) = 1 > 0$, and (P) yields

$$\forall \lambda \in \mathbb{R} : \ 0 \leq D[h_\lambda](x_0) = D[g](x_0) + \lambda D[f - T_{n+1}(f, \cdot, x_0)](x_0).$$

Therefore $D[f - T_{n+1}(f, \cdot, x_0)](x_0) = 0$, that is

$$D[f](x_0) = D[T_{n+1}(f, \cdot, x_0)](x_0) = \sum_{k=0}^{n+1} h_k(x_0) f^{(k)}(x_0)$$

with

$$h_k(x_0) = D[w_k(\cdot, x_0)](x_0) \quad (k = 0, \dots, n+1).$$

If we specifically take $f = g$, we get

$$0 \leq D[g](x_0) = h_{n+1}(x_0).$$

Since $x_0 \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$ were arbitrary we thus have obtained functions $h_k : \mathbb{R} \to \mathbb{R}$ $(k = 0, \dots, n+1)$ such that

$$\forall f \in C^\infty(\mathbb{R}) : \; D[f] = \sum_{k=0}^{n+1} h_k f^{(k)},$$

and $h_{n+1}$ is nonnegative. Inserting gradually the functions $x \mapsto 1, x, \dots, x^{n+1}$ shows

$$h_k \in C(\mathbb{R}) \quad (k = 0, \dots, n+1).$$

We now set $\varphi_2 := h_{n+1}$. For each $x \in \mathbb{R}$ with $s(x) > 0$ let $m(x) \in \{0, 1, \dots, n\}$ be any index with $a_{m(x)}(x) \neq 0$. If $s(x) = 0$ we set $m(x) := -1$. We define $\varphi_1 : \mathbb{R} \to \mathbb{R}$ as

$$\varphi_1(x) := \begin{cases} (h_{m(x)}(x) - \varphi_2(x) b_{m(x)}(x))/a_{m(x)}(x), & \text{if } s(x) > 0 \\ 0, & \text{if } s(x) = 0 \end{cases},$$

and we consider the linear operator

$$M := D - \varphi_2 L_2 - \varphi_1 L_1$$

on $C^\infty(\mathbb{R})$ (since $\varphi_1$ might be discontinous we have $M : C^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{R}}$). By the definition of $\varphi_2$ the differential operator $M$ has order $n$:

$$M[f] = \sum_{k=0}^{n} c_k f^{(k)}, \quad c_k := h_k - \varphi_2 b_k - \varphi_1 a_k.$$

Again, we fix $x_0 \in \mathbb{R}$ and we want to show that

$$c_k(x_0) = 0 \quad (k = 0, \dots, n).$$

Note that if $s(x_0) > 0$, then

$$c_{m(x_0)}(x_0) = h_{m(x_0)}(x_0) - \varphi_2(x_0) b_{m(x_0)}(x_0) - \varphi_1(x_0) a_{m(x_0)}(x_0) = 0.$$

In case $n = m(x_0) = 0$ we thus have $c_0(x_0) = 0$ and we are done. In any other case let $j \in \{0, \dots, n\} \setminus \{m(x_0)\}$ and let $g_j \in C^\infty(\mathbb{R})$ be a function with

$$g_j^{(k)}(x_0) = \delta_{jk} \quad (k \in \{0, \dots, n\} \setminus \{m(x_0)\}),$$
$$g_j^{(m(x_0))}(x_0) = -\frac{a_j(x_0)}{a_{m(x_0)}(x_0)} \text{ if } s(x_0) > 0, \quad g_j^{(n+1)}(x_0) = \alpha,$$

where $\alpha \in \mathbb{R}$ is variable. Now $L_1[g_j](x_0) = 0$, obviously in case $s(x_0) = 0$, and in case $s(x_0) > 0$ since then

$$L_1[g_j](x_0) = a_j(x_0) - a_{m(x_0)}(x_0)\frac{a_j(x_0)}{a_{m(x_0)}(x_0)} = 0.$$

Inserting $g_j$ in $L_2$ yields

$$\beta := L_2[g_j](x_0) = \begin{cases} b_j(x_0) - \frac{b_{m(x_0)}(x_0)a_j(x_0)}{a_{m(x_0)}(x_0)} + \alpha, & s(x_0) > 0 \\ b_j(x_0) + \alpha, & s(x_0) = 0 \end{cases}.$$

If $\alpha$ is such that $\beta > 0$ then $0 \leq D[g_j](x_0)$, hence

$$c_j(x_0) = M[g_j](x_0) \geq -\varphi_2(x_0)L_2[g_j](x_0) = -\varphi_2(x_0)\beta,$$

and $\beta \to 0+$ yields $c_j(x_0) \geq 0$, and if $\alpha$ is such that $\beta < 0$ then $0 \geq D[g_j](x_0)$, hence

$$c_j(x_0) = M[g_j](x_0) \leq -\varphi_2(x_0)L_2[g_j](x_0) = -\varphi_2(x_0)\beta,$$

and $\beta \to 0-$ yields $c_j(x_0) \leq 0$. Thus $c_j(x_0) = 0$. In case $s(x_0) = 0$ we have $\{0, \dots, n\}\backslash\{m(x_0)\} = \{0, \dots, n\}$ and in case $s(x_0) > 0$ we already know that $c_{m(x_0)}(x_0) = 0$. Summing up $c_k(x_0) = 0$ $(k = 0, \dots, n)$. Since $x_0 \in \mathbb{R}$ was arbitrary we have $M = 0$, that is $D = \varphi_1 L_1 + \varphi_2 L_2$. $\square$

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Gerd Herzog and Peer Kunstmann
Department of Mathematics
Karlsruhe Institute of Technology (KIT)
Englerstraße 2
76131 Karlsruhe
Germany
e-mail: gerd.herzog2@kit.edu

Peer Kunstmann
e-mail: peer.kunstmann@kit.edu



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