Gromov-Hausdorff limits of length surfaces

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Für meine kleine Tochter, meine ganze Welt.

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Prior publications

The majority of this thesis is based on the following publications and preprints:

[A] T. Dott. Gromov-Hausdorff limits of closed surfaces. *Anal. Geom. Metr. Spaces*, 12:20240003, 2024.

[B] T. Dott. On the Gromov-Hausdorff limits of compact surfaces with boundary. *Ann. Global Anal. Geom.*, 66(3):15, 2024.

[C] T. Dott. Convergence of Riemannian 2-manifolds under a uniform curvature and contractibility bound. Preprint, arXiv:2501.06351, 2025.

Essentially, the relation between the content of the chapters and [A]-[C] can be described as follows:

- Chapters 2-5 and 8 are largely based on the preliminaries of [A]-[C].
- Chapter 6 is based on Sections 1 and 3-5 of [A].
- Chapter 7 is based on Sections 1, 2.4 and 3-4 of [B].
- Chapter 9 is based on Sections 1 and 3-5 of [C].

Direct and indirect quotations from [C] and the accepted manuscripts corresponding to [A] and [B] appear throughout this thesis. For the sake of readability, these quotations are not marked as such. Figures quoted from the accepted manuscripts may differ slightly from their originals.

Contents

1 Introduction

We start with a brief introduction to the central question of this thesis.

1.1 The Gromov-Hausdorff distance

When studying metric spaces, it can be helpful to consider a given space as the "limit" of less complex spaces and then exploit continuity properties. The Gromov-Hausdorff distance is one way to realize such a notion of convergence:
For a metric space X the *Hausdorff distance* is defined by

$$d_{\mathcal{H}}^{X}(A,B) \coloneqq \inf\{\varepsilon > 0\colon A \subset U_\varepsilon(B), B \subset U_\varepsilon(A)\}$$

for all $A, B \subset X$ where $U_\varepsilon(\cdot)$ denotes the open ε-neighborhood. An example regarding the Hausdorff distance in the plane is shown in Figure 1.
Using isometric embeddings, a natural notion of distance for metric spaces arises:
The *Gromov-Hausdorff distance* is defined by

$$d_{\mathcal{GH}}(X,Y) \coloneqq \inf\Big\{d_{\mathcal{H}}^{Z}(f_1(X_1), f_2(X_2))\Big\}$$

for all metric spaces X_1 and X_2 where the infinum is taken over all metric spaces Z and isometric embeddings $f_i\colon X_i \to Z$.
Indeed, this function defines a semi-metric on the class of compact metric spaces and induces a metric on their isometry classes (cf. [6, p. 259]).
We say that a sequence of compact metric spaces $(X_n)_{n\in\mathbb{N}}$ *Gromov-Hausdorff converges* to some compact metric space X provided $d_{\mathcal{GH}}(X_n, X) \longrightarrow 0$. Up to isometries, such a limit space X is unique. We present an example of Gromov-Hausdorff convergence:

Example 1.1.1. Let X be the round 1-sphere and X_n be the metric product $X \times [0, 2^{-n}]$. Then X is isometric to the subset $A_n \coloneqq X \times \{0\} \subset X_n$ and we have $d_{\mathcal{H}}^{X_n}(A_n, X_n) \leq 2^{-n}$. As a consequence, we derive $d_{\mathcal{GH}}(X, X_n) \leq 2^{-n}$ and hence $d_{\mathcal{GH}}(X_n, X) \longrightarrow 0$. We conclude that $(X_n)_{n\in\mathbb{N}}$ Gromov-Hausdorff converges to X.

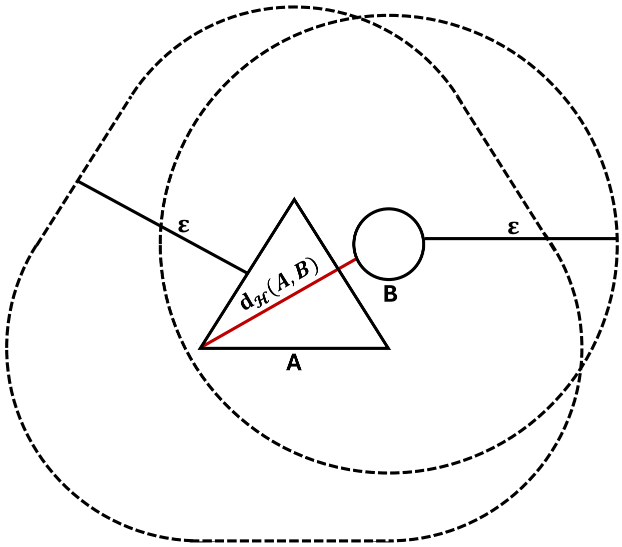

Figure 1: The Hausdorff distance in the plane. The triangle A lies in the open ε-neighborhood of the circle B. Vice versa, we have $B \subset U_\varepsilon(A)$. The Hausdorff distance between A and B is the infinum of all such $\varepsilon > 0$. This is equal to the length of the line segment shown in red.

In Chapter 2 we will discuss Gromov-Hausdorff convergence in more detail.

1.2 Length spaces

Motivated by Riemannian manifolds, length spaces play an important role in metric geometry:
We recall that the *length* of a curve $\gamma\colon I \to X$ in a metric space is defined by

$$length(\gamma) := \inf \left\{ \sum_{i=1}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \right\}$$

where the infinum is taken over all partitions $t_1 < \ldots < t_n \in I$.
Moreover the *induced length metric* on X is defined by

$$\hat{d}(x,y) := \inf\{length(\gamma)\}$$

for all $x, y \in X$ where the infinum is taken over all paths γ connecting x and y in X. We denote X as a *length space* provided its metric coincides with $\hat{d}$. In particular, X equipped with $\hat{d}$ is a length space if $\hat{d}$ is finite (cf. [6, pp. 28, 37]). We discuss a brief example regarding length spaces:

Example 1.2.1. The subspace $X := \mathbb{R}^2 \setminus \{0\}$ is a length space: If the line segment l between two points $x_1, x_2 \in X$ intersects the origin, we replace $\bar{B}_r(0) \cap l$ with one half of $\partial B_r(0)$. For small $r > 0$ this yields paths γ_r connecting x and y such that $length(\gamma_r) \longrightarrow \|x_1 - x_2\|_{\mathbb{R}^2}$.
On the other hand, the subspace $Y := \mathbb{R}^2 \setminus (\{0\} \times [-a, a])$ is not a length space: The length of a shortest path between the points $y_1 := (-1, 0)$ and $y_2 := (1, 0)$ is equal to $2\sqrt{1 + a^2} > \|y_1 - y_2\|_{\mathbb{R}^2}$.

We call a metric space X *geodesic* if for all $x, y \in X$ there is a path γ connecting x and y in X such that $length(\gamma) = d(x, y)$. Geodesic metric spaces are special cases of length spaces. Furthermore every compact length space is geodesic (cf. [6, p. 50]).
With regard to Gromov-Hausdorff convergence we note that the Gromov-Hausdorff limit of compact length spaces is again a compact length space (cf. [6, p. 265]).
In the following we call a length space that is homeomorphic to a surface a *length surface*.

1.3 Limits of length surfaces

Generally speaking, topological properties are not stable under Gromov-Hausdorff convergence. In particular, the limit of manifolds does not have to be a manifold again. The situation is even more dramatic as the following two results show:
From a topological point of view there is a wide range of possible limit spaces in the case of surfaces. This is emphasized by a result due to Cassorla from the 1990s:

Theorem (Cassorla). (cf. [9, p. 505]) Every compact length space can be obtained as the Gromov-Hausdorff limit of closed length surfaces.

Indeed, every metric space that is compact, connected and locally connected is homeomorphic to a compact length space (cf. [4, p. 1109]).
Even if we increase the dimension and bound the topology of the converging manifolds, there is still a wide range of possible limit spaces. The following result by Ferry and Okun from the same decade supports this statement:

Theorem (Ferry, Okun). (cf. [11, p. 1866]) Let M be a closed connected smooth manifold of dimension at least three and X be a simply connected compact absolute neighborhood retract (ANR) carrying a length metric. Then X can be obtained as the Gromov-Hausdorff limit of length spaces that are homeomorphic to M.

We note that a simply connected compact ANR carrying a length metric may be infinite dimensional and that every finite dimensional locally contractible metric space is an ANR (cf. [4, p. 1109], [34, p. 392]).
The aforementioned observations lead to the central question of this thesis:

Key question:

What do the Gromov-Hausdorff limits of compact length surfaces with uniformly bounded Euler characteristic look like?

We will study this question in the following three scenarios. In particular, we will give an intrinsic description of the possible limit spaces for each scenario.

Scenarios:

I) Sequences of closed surfaces.

II) Sequences of compact surfaces with boundary.

III) Uniformly semi-locally 1-connected sequences of closed Riemannian 2-manifolds with uniformly bounded total absolute curvature.

The notion of a uniformly semi-locally 1-connected sequence is defined in Section 2.3.3.

1.4 Main results

In this section we summarize the most important results of this thesis. For an extended and more detailed overview we refer the reader to the introductions of Chapters 6-7 and 9.

1.4.1 Scenarios I & II

We start with Scenarios I and II.
The first result addresses the topological properties of the limit spaces:

Corollary A. Let S be a compact surface. Moreover let X be a space that can be obtained as the Gromov-Hausdorff limit of length spaces that are homeomorphic to S. Then the following statements apply:

1) X is at most 2-dimensional.

2) X is locally simply connected.

3) There are finitely many compact surfaces $S_1, \ldots, S_n$ and $k \in \mathbb{N}_0$ such that $\pi_1(X)$ is isomorphic to the free product

$$\pi_1(S_1) * \ldots * \pi_1(S_n) * \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{k\text{-times}}.$$

If S is a closed surface, then $S_1, \ldots, S_n$ can be chosen as closed surfaces.

The aforementioned result summarizes Theorems 6.1.1 and 7.1.1.
Our next result completely describes the limit spaces for Scenarios I and II.
Before we state the result, we refer to some definitions: The connectivity number of a compact surface is introduced in Section 5.2. For the definition of a metric 2-point identification we refer to Section 6.1.1. The concept of a generalized cactoid and its connectivity number is defined in Section 7.1.1. There we also give the definition of a boundary identification.

Corollary B. Let $c \in \mathbb{N}_0$ and X be a compact length space. Then the following statements are equivalent:

1) X can be obtained as the Gromov-Hausdorff limit of compact length surfaces whose connectivity number is equal to c.

2) There are $k, k_0 \in \mathbb{N}_0$ and a geodesic generalized cactoid Y such that the following statements apply:

   a) X can be obtained by a successive application of k metric 2-point identifications to Y such that k_0 of them are boundary identifications.

   b) We have $c_0 - k_0 + 2k \leq c$ where c_0 denotes the connectivity number of Y.

Moreover the equivalence remains true if we restrict the first statement to closed surfaces and the second to generalized cactoids with empty boundary (in particular, $k_0 = 0$).

The aforementioned result summarizes Main Theorem I and Main Theorem II.
Roughly speaking, the space Y in the second statement is a compact length space whose "non-degenerate parts" are compact surfaces. The space X is isometric to a space obtained by gluing finitely many points in Y. The quantitative formula in b) further restricts the topology of X.

1.4.2 Scenario III

Now we turn our attention to Scenario III.
Maximal cyclic subsets and cut points of a compact length space X are defined in Section 3.1.
Let S be a closed surface. We say that X *satisfies property* $(*)$ *for* S if the following statements apply:

1) If S is homeomorphic to $\mathbb{S}^2$, then all maximal cyclic subsets of X are homeomorphic to $\mathbb{S}^2$.

2) If S is not homeomorphic to $\mathbb{S}^2$, then one maximal cyclic subset of X is homeomorphic to S and all others are homeomorphic to $\mathbb{S}^2$.

3) Every maximal cyclic subset of X contains only finitely many cut points of X. Only finitely many of them contain more than two cut points.

The last result of this section provides a description of the limit spaces for Scenario III. Before we state the result, we refer to some defintions: The concept of a surface with bounded curvature M, its absolute curvature measure $|\omega_M|$ and its total angle function θ_M are introduced in Section 8.1. For the definition of the subsets Cut_X and $Sing_X$ and the index function ind_X we refer to Section 9.1.1 .
In the following we write $\mathcal{R}(S, C)$ for the class of all Riemannian 2-manifolds R such that R is homeomorphic to S and the total absolute curvature of R is at most C.

Corollary C. Let S be a closed surface and X be a compact length space. Then the following statements are equivalent:

1) X can be obtained as the Gromov-Hausdorff limit of a uniformly semi-locally 1-connected sequence $X_n \in \mathcal{R}(S, C_n)$ such that $\lim\limits_{n\to\infty} C_n \leq C$.

2) For X the following statements apply:
   a) X satisfies property $(*)$ for S.
   b) Every maximal cyclic subset of X is a closed surface with bounded curvature.
   c) Let $(T_n)_{n\in\mathbb{N}}$ be an enumeration of the maximal cyclic subsets of X. Then the following inequality holds:

$$\sum_{n=1}^{\infty}|\omega_{T_n}|(T_n\setminus Cut_X)+\sum_{p\in Sing_X}2\pi\,|ind_X(p)-2|+\sum_{p\in Cut_X}\sum_{n=1}^{\infty}\mathbb{1}_{T_n}(p)\,\theta_{T_n}(p)\leq C.$$

The aforementioned result summarizes Main Theorem III (A) and Main Theorem III (B).
Compared to Corollary B, the topology of the space X from the second statement is much more restricted. In particular, no gluings are carried out and at most one of the "non-degenerate parts" is not homoemorphic to $\mathbb{S}^2$. The geometric structure of these parts is also much richer: Surfaces with bounded curvature are generalizations of connected Riemannian 2-manifolds.

1.5 Organisation

This thesis is organized as follows:
Chapters 2-5 and 8 are introductory in nature: In the second chapter we discuss Gromov-Hausdorff convergence in more detail. Afterwards we give an introduction to Whyburn's cyclic element theory which provides the language for the main results of this thesis. Topological tools related to the Seifert-Van Kampen theorem are presented in Chapter 4. It follows a chapter on the theory of surfaces. In Chapter 8 we give a brief overview of surfaces with bounded curvature in the sense of Alexandrov which play an essential role in the study of Scenario III.
Scenarios I-III are investigated in Chapters 6-7 and 9.

2 Gromov-Hausdorff Convergence

This chapter is devoted to Gromov-Hausdorff convergence. In the first two sections we discuss the basics of the topic. Afterwards we present more advanced results for later purposes.

2.1 The Gromov-Hausdorff space

As already mentioned in the introduction, the Gromov-Hausdorff distance naturally induces a metric on the isometry classes of compact metric spaces. We call the corresponding metric space the *Gromov-Hausdorff space* and denote it by $\mathcal{GH}$. In this section we want to summarize important properties of this space.
We start with the following theorem:

Theorem 2.1.1. (cf. [30, pp. 50-51]) The space $\mathcal{GH}$ is complete, separable and geodesic.

The completeness gives us a first tool to prove convergence.
The diameter map induces a continuous map on $\mathcal{GH}$ (cf. [30, p. 38]). Hence the sequence $([0,n])_{n\in\mathbb{N}}$ has no convergent subsequence in $\mathcal{GH}$. In particular, the space $\mathcal{GH}$ is non-compact.
Next we will state *Gromov's precompactness theorem* which yields a criterion for the existence of convergent subsequences. Before that, we introduce a definition: Let $\mathcal{X} \subset \mathcal{GH}$. We denote $\mathcal{X}$ as *uniformly totally bounded* if there is some $D \in \mathbb{R}$ and a function $N\colon \mathbb{R} \to \mathbb{R}$ such that for all $X \in \mathcal{X}$ the following statements apply:

1) The diameter of X is at most D.

2) For every $\varepsilon > 0$ there is an ε-net in X of cardinality at most $N(\varepsilon)$.

We state the precompactness theorem

Theorem 2.1.2. (cf. [30, p. 50]) A subclass $\mathcal{X} \subset \mathcal{GH}$ is uniformly totally bounded if and only if $\mathcal{X}$ is precompact in $\mathcal{GH}$.

A well-known application of the precompactness theorem is given as follows:

Example 2.1.3. (cf. [13, 197, 206], [29, p. 404]) Let $n \in \mathbb{N}_{\geq 2}$ and $\mathcal{X}(n, D, K)$ be the class of all closed connected Riemannian n-manifolds X such that the following statements apply:

1) The diameter of X is at most D.

2) The Ricci curvature of X is at least $(n-1)K$.

Then $\mathcal{X}(n, D, K)$ is precompact in $\mathcal{GH}$. Indeed, we only need to show that the second condition of Gromov's precompactness theorem is satisfied: There is an ε-net $\{x_1, \ldots, x_k\} \subset X$ such that the corresponding open balls $B_{\frac{\varepsilon}{2}}(x_i)$ are pairwise disjoint (cf. [29, p. 403]). Then we derive

$$\sum_{i=1}^{k} \frac{\mathcal{H}_X^n\Big(B_{\frac{\varepsilon}{2}}(x_i)\Big)}{\mathcal{H}_X^n(X)} \leq 1$$

where $\mathcal{H}_X^n$ denotes the n-dimensional Hausdorff measure of X.
Due to the Gromov-Bishop inequality (cf. [29, p. 279]) we further have

$$\frac{\mathcal{H}_X^n(X)}{\mathcal{H}_X^n\Big(B_{\frac{\varepsilon}{2}}(x_i)\Big)} \leq \frac{\mathcal{H}_M^n(M)}{\mathcal{H}_M^n\Big(B_{\frac{\varepsilon}{2}}(p)\Big)} =: N(\varepsilon)$$

where M is any complete simply connected Riemannian n-manifold of constant sectional curvature K and p is any point of M.
This finally yields $k \leq N(\varepsilon)$ and hence the second condition of the precompactness theorem is satisfied.

Now we present some density results:

Proposition 2.1.4. (cf. [6, pp. 261, 266-267])

1) The class of finite metric spaces is dense in $\mathcal{GH}$.

2) The class of finite metric graphs is dense in the class of compact length spaces.

3) The class of finite metric trees is dense in the class of compact metric trees.

The separability statement in Theorem 2.1.1 can be shown using the first statement. Throughout this work we denote a simple closed curve as a *Jordan curve*. We note that a compact length space is a metric tree if and only if the space does not contain Jordan curves.

2.2 A characterization of Gromov-Hausdorff convergence

In this section we discuss equivalent definitions of Gromov-Hausdorff convergence.
For this we introduce the concept of almost isometries: A map $f\colon X \to Y$ between metric spaces is called an *ε-isometry* if the following statements apply:

1) $dis(f) := \sup_{x_1,x_2 \in X}\{|d_X(x_1,x_2) - d_Y(f(x_1), f(x_2))|\} \leq \varepsilon$.

2) $f(X)$ is an ε-net in Y.

The value ε can be interpreted as an error term that measures how much the map f differs from an isometry. By [6, p. 258] the existence of an ε-isometry $f\colon X \to Y$ implies the existence of a 4ε-isometry $g\colon Y \to X$.
The next theorem provides a characterization of Gromov-Hausdorff convergence.
Before we state the result, we note that the Hausdorff distance in X is a metric on the set of all compact subsets of X (cf. [6, p. 252]). We write $A_n \xrightarrow{\mathcal{H}} A$ for Hausdorff convergence and $X_n \xrightarrow{\mathcal{GH}} X$ for Gromov-Hausdorff convergence.

Theorem 2.2.1. (cf. [16, pp. 64-65], [6, p. 260]) Let X_n and X be compact metric spaces. Then the following statements are equivalent:

1) $X_n \xrightarrow{\mathcal{GH}} X$.

2) There is a compact metric space Z and isometric embeddings $f_n\colon X_n \to Z$ and $f\colon X \to Z$ such that $f_n(X_n) \xrightarrow{\mathcal{H}} f(X)$.

3) There are ε_n-isometries $f_n\colon X_n \to X$ such that $\varepsilon_n \longrightarrow 0$.

Due to the second statement it makes sense to talk about Hausdorff convergence of subsets $A_n \subset X_n$ to some subset $A \subset X$ whenever $X_n \xrightarrow{\mathcal{GH}} X$. We will use this observation frequently without mentioning the choice of Z.
In the third statement the spaces X and X_n may be interchanged. In particular, the statement provides a useful tool to prove convergence and makes convergence more tangible: The sequence converges to X if the spaces become more and more isometric to X.
The following example demonstrates this new tool:

Example 2.2.2. We remove the two open balls $B_{2^{-(n+1)}}(0, \pm 2^{-n})$ from $\mathbb{D}^2$. Moreover we equip the subset with its induced length metric and write D_n for the constructed space.

In D_n we finally glue the boundaries of the balls along orientation preserving isometries. We denote this gluing by X_n and the corresponding projection map by $\pi_n\colon D_n \to X_n$. From a topological point of view X_n is a 2-torus in which the interior of some closed topological 2-disc has been removed.
Our goal is to show that $X_n \xrightarrow{\mathcal{GH}} \mathbb{D}^2$:
First we prove that the map π_n is a suitable almost isometry:
The map π_n is surjective and 1-lipschitz. For two points x and y from the boundaries of the balls we have

$$d_{D_n}(x,y) \leq 2 \cdot 2^{-(n+1)}\pi + 2 \cdot \left(2^{-n} - 2^{-(n+1)}\right) = 2^{-n}(\pi+1) =: \varepsilon_n.$$

Therefore we derive

$$d_{D_n}(x,y) \leq d_{X_n}(\pi_n(x), \pi_n(y)) + \varepsilon_n$$

for all $x, y \in D_n$.
We conclude that π_n is an ε_n-isometry.
As a second step, we show that the inclusion map $i_n\colon D_n \to \mathbb{D}^2$ is a suitable almost isometry: The map i_n is 1-lipschitz. Moreover its image is a $2^{-(n+1)}$-net in $\mathbb{D}^2$ and we have

$$d_{D_n}(i_n(x), i_n(y)) \leq d_{\mathbb{D}^2}(x,y) + 2 \cdot 2^{-(n+1)}\pi = d_{\mathbb{D}^2}(x,y) + 2^{-n}\pi.$$

We derive that i_n is also an ε_n-isometry.
This implies the existence of a $4\varepsilon_n$-isometry $g_n\colon \mathbb{D}^2 \to D_n$. We derive that $f_n \coloneqq \pi_n \circ g_n$ is a $6\varepsilon_n$-isometry. Hence Theorem 2.2.1 yields $X_n \xrightarrow{\mathcal{GH}} \mathbb{D}^2$.

A topological illustration of the example is shown in Figure 2.

2.3 Controlled Gromov-Hausdorff convergence

As already pointed out in the introduction, in general topological properties are not stable under Gromov-Hausdorff convergence. However, sometimes some sort of stability can be achieved by considering geometrically controlled versions of topological properties. In this subsection we present three examples of this kind.

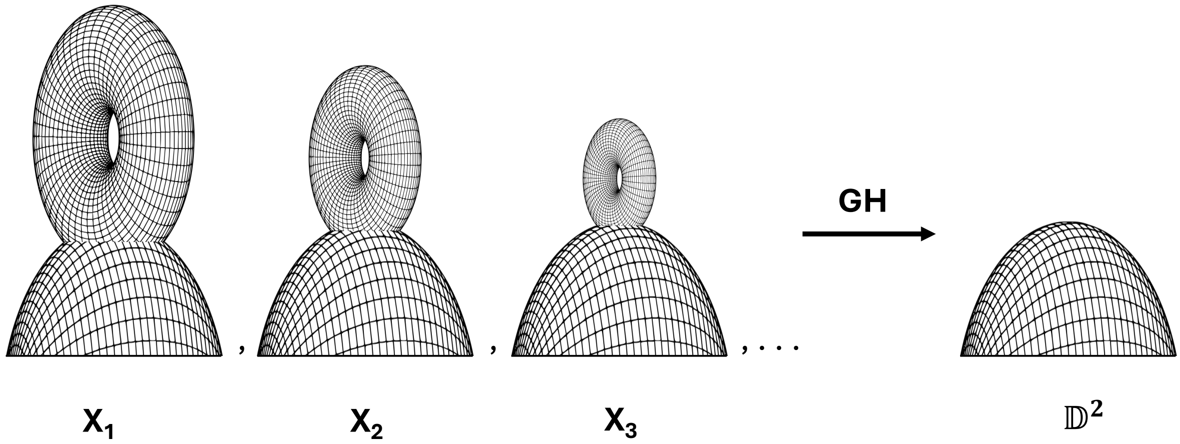

Figure 2: A topological illustration of Example 2.2.2. A sequence of topological 2-tori in which the interior of a closed topological 2-disc has been removed Gromov-Hausdorff converges to a closed topological 2-disc.

2.3.1 Uniformly locally connected sequences

A sequence of metric spaces $(X_n)_{n\in\mathbb{N}}$ is called *uniformly locally connected* if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $n \in \mathbb{N}$ the following statement applies: Every pair of points $x, y \in X_n$ of distance less than δ lies in some continuum in X_n of diameter less than ε.
We note that sequences of length spaces always satisfy this uniform bound.
Both results presented in this subsection are due to Whyburn.
Under the uniform bound decompositions display a nice behavior under convergence:

Lemma 2.3.1. (cf. [6, p. 253], [38, p. 412-413]) Let $(X_n)_{n\in\mathbb{N}}$ be a uniformly locally connected sequence of compact metric spaces and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover let $A_n, B_n \subset X_n$ be a decomposition of X_n into closed subsets. Then, after passing to a subsequence, there are subsets $A, B \subset X$ such that $A_n \xrightarrow{\mathcal{H}} A$, $B_n \xrightarrow{\mathcal{H}} B$ and $A_n \cap B_n \xrightarrow{\mathcal{H}} A \cap B$. If the sequence $(A_n \cap B_n)_{n\in\mathbb{N}}$ is uniformly locally connected, then the same applies to $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$.

The second result is about the limits of closed topological 2-discs:

Proposition 2.3.2. (cf. [38, p. 421-422]) Let $(X_n)_{n\in\mathbb{N}}$ be a uniformly locally connected sequence of metric spaces that are homeomorphic to the closed 2-disc and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover we assume that $(\partial D_n)_{n\in\mathbb{N}}$ is uniformly locally connected and $\partial D_n \xrightarrow{\mathcal{H}} J$ where $diam(J) > 0$. Then one maximal cyclic subset $T \subset X$ (see Definition 3.1.1) is homeomorphic to the closed 2-disc and all others are homeomorphic to the 2-sphere. Moreover we have $\partial T = J$.

2.3.2 Uniformly locally k-connected sequences

Let $k \in \mathbb{N}_0$. We denote a sequence of metric spaces $(X_n)_{n\in\mathbb{N}}$ as *uniformly locally k-connected* provided there is a non-decreasing function $p\colon [0, R) \to [0, \infty]$ such that the following statements apply:

1) $p(0) = 0$ and $p(t) \geq t$ for all $t \in [0, R)$.

2) p is continuous at 0.

3) For every $t \in [0, R)$, $n \in \mathbb{N}$ and $x \in X_n$ all continuous maps $f\colon \mathbb{S}^k \to B_t(x)$ are null-homotopic in $B_{p(t)}(x)$.

Petersen showed the following stability result:

Theorem 2.3.3. (cf. [28, p. 501]) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces and $X_n \xrightarrow{\mathcal{GH}} X$. If the sequence is uniformly k-connected for all $k \leq m$, then the same applies to $\{X\} \cup \{X_n\}_{n\in\mathbb{N}}$.

The next result is also due to Petersen and deals with a geometrically controlled version of homotopy equivalences: Let $f\colon X \to Y$ be a continuous map between metric spaces that satisfies the following properties:

1) There is a continuous map $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to id_X and id_Y respectively.

2) There are corresponding homotopies H_X and H_Y such that

$$d_Y(f(x), f \circ H_X(x,t)) < \varepsilon \quad \text{and} \quad d_Y(y, H_Y(y,t)) < \varepsilon$$

for all $x \in X$, $y \in Y$ and $t \in [0, 1]$.

Then f is called an *ε-equivalence*.

Theorem 2.3.4. (cf. [28, p. 501]) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces and $X_n \xrightarrow{\mathcal{GH}} X$. We further assume that the dimension of the spaces in $X \cup \{X_n\}_{n\in\mathbb{N}}$ is at most m and that the sequence $(X_n)_{n\in\mathbb{N}}$ is uniformly k-connected for all $k \leq m$. Then, after passing to a subsequence, there are ε_n-equivalences $f_n\colon X \to X_n$ such that $\varepsilon_n \longrightarrow 0$ and $d(f_n, id_X) \longrightarrow 0$.

2.3.3 Uniformly semi-locally 1-connected sequences

We say that a sequence of metric spaces $(X_n)_{n\in\mathbb{N}}$ is *uniformly semi-locally 1-connected* if there is some $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ the following statement applies: There is no non-contractible loop in X_n whose diameter is less than ε.
This uniform bound yields a stability result for fundamental groups:

Theorem 2.3.5. (cf. [36, p. 3588]) Let X be a compact semi-locally 1-connected metric space and $(X_n)_{n\in\mathbb{N}}$ be a uniformly semi-locally 1-connected sequence of compact length spaces such that $X_n \xrightarrow{\mathcal{GH}} X$. Then $\pi_1(X)$ is isometric to $\pi_1(X_n)$ for all but finitely many $n \in \mathbb{N}$.

The proof of the aforementioned theorem is essentially contained in [37, pp. 209-210]. If the uniform bound does not apply but X is still semi-locally 1-connected, then there are at least epimorphisms $f_n\colon \pi_1(X_n) \to \pi_1(X)$ for all but finitely many $n \in \mathbb{N}$ (cf. [36, p. 3588]). Provided the converging spaces X_n are simply connected, it follows that the same applies to X.

2.4 Related notions of convergence

Now we introduce two more notions of convergence for metric spaces and relate them to Gromov-Hausdorff convergence.

2.4.1 Measured Gromov-Hausdorff convergence

With regard to metric measure spaces the definition of Gromov-Hausdorff convergence can be extended in the following sense: Let (X_n, μ_n) and (X, μ) be compact metric spaces together with finite Borell measures over them. We say that (X_n, μ_n) *Gromov-Hausdorff converges* to (X, μ) if $X_n \xrightarrow{\mathcal{GH}} X$ and there is a choice of the common ambient space Z from Theorem 2.2.1 such that the measures μ_n converge weakly to μ as measures over

Z.

We consider an example regarding Hausdorff measures:

Example 2.4.1. Let X be a compact metric space such that the k-dimensional Hausdorff measure satisfies $0 \neq \mathcal{H}^k_X < \infty$ where $k > 0$. From Proposition 2.1.4 we already know that there is a sequence of finite metric spaces such that $X_n \xrightarrow{\mathcal{GH}} X$. Due to the finiteness we have $\mathcal{H}^k_{X_n} = 0$. Hence $\mathcal{H}^k_{X_n}$ converges weakly to the 0-measure for all choices of the common ambient space Z. In particular, $\left(X_n, \mathcal{H}^k_{X_n}\right)$ does not Gromov-Hausdorff converge to $\left(X, \mathcal{H}^k_X\right)$.

The following result states a condition for the existence of a convergent subsequence:

Proposition 2.4.2. Let (X_n, μ_n) be compact metric measure spaces together with finite Borell measures over them and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover we assume that the measures are uniformly bounded. Then, after passing to a subsequence, there is a finite Borell measure μ over X such that $(X_n, \mu_n) \xrightarrow{\mathcal{GH}} (X, \mu)$.

The aforementioned result directly follows from the Banach-Alaoglu theorem.

2.4.2 Uniform convergence

We have already seen the characterization of Gromov-Hausdorff convergence via almost isometries. In general, these maps do not even have to be continuous. Our next notion of convergence requires these maps to be homeomorphisms: Let X_n and X be metric spaces. We say that X_n *converges uniformly* to X if there are homeomorphims $f_n \colon X \to X_n$ such that $dis(f_n) \longrightarrow 0$. In the following we write $X_n \xrightarrow{uni.} X$ for uniform convergence. If the topological embeddings are only defined on some subset $U \subset X$, we say that the sequence *converges uniformly* on U.

As a direct consequence of the definition, uniform convergence is stronger than Gromov-Hausdorff convergence. This is also true if we assume that X is already homeomorphic to the spaces X_n:

Example 2.4.3. We consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} A_n^+ &:= \partial B_1\Big(0, 1-2^{-n}\Big) \cap (\mathbb{R} \times \mathbb{R}_{\geq 0}), \\ A_n^- &:= \partial B_1\Big(0, -1+2^{-n}\Big) \cap (\mathbb{R} \times \mathbb{R}_{\leq 0}), \\ B_n &:= B_{2^{-n}}(0, 2), \\ X_n &:= B_n \cup A_n^+ \cup A_n^-, \\ X &:= \partial B_1(0, 1) \cup \partial B_1(0, -1). \end{aligned}$$

The subsets X and X_n are homeomorphic to a wedge sum of two circles. Estimating the Hausdorff distance in $\mathbb{R}^2$, we further derive that $X_n \xrightarrow{\mathcal{GH}} X$.
For the sake of contradiction, we assume that the convergence is uniform. Let f_n be the homeomorphisms corresponding to the uniform convergence. Then, after passing to a subsequence, we may assume that $dis(f_n) \leq 2^{-1}$ and $f_n(0,0), f_n(0,1) \in B_n$. But this yields

$$2^{-1} \leq \|f_n(0,0) - f_n(0,1)\|_{\mathbb{R}^2} \leq 2^{-n+1}.$$

A contradiction.

The aforementioned example is illustrated in Figure 3.
Next we state a criterion for uniform convergence in the case of closed surfaces. It is a direct consequence of Jakobsche's 2-dimensional α-approximation theorem in [20, p. 2]:

Theorem 2.4.4. Let S be a closed surface. Moreover let X and X_n be metric spaces that are homeomorphic to S. We assume the existence of ε_n-equivalences $f_n\colon X \to X_n$ such that $\varepsilon_n \longrightarrow 0$ and $dis(f_n) \longrightarrow 0$. Then we have $X_n \xrightarrow{uni.} X$ and the homeomorphisms $\varphi_n\colon X \to X_n$ corresponding to the uniform convergence can be chosen such that $d(\varphi_n, f_n) \longrightarrow 0$.

2.5 Gromov-Hausdorff convergence of maps

In this section we introduce a notion of convergence for maps between compact metric spaces.
Let $f_n\colon X_n \to Y_n$ and $f\colon X \to Y$ be maps between compact metric spaces. We say that the sequence f_n *Gromov-Hausdorff converges* to f if the following statements apply:

1) $X_n \xrightarrow{\mathcal{GH}} X$ and $Y_n \xrightarrow{\mathcal{GH}} Y$.

2) There is a choice of common ambient spaces as in Theorem 2.2.1 such that for every sequence $x_n \in X_n$ converging to some $x \in X$ we have $f_n(x_n) \longrightarrow f(x)$.

The sequence f_n is called *equicontinuous* if for every $\varepsilon > 0$ and sequence $x_n \in X_n$ there is some $\delta > 0$ such that

$$f_n(B_\delta(x_n)) \subset B_\varepsilon(f_n(x_n))$$

for all $n \in \mathbb{N}$.
For example the sequence f_n is equicontinuous if there is a uniform constant $L \geq 0$ such that f_n is L-lipschitz.
The following convergence criterion is related to the Arzela-Ascoli theorem:

Proposition 2.5.1. (cf. [6, p. 402]) Let $f_n \colon X_n \to Y_n$ be maps between compact metric spaces. Moreover let $X_n \xrightarrow{\mathcal{GH}} X$ and $Y_n \xrightarrow{\mathcal{GH}} Y$. If the sequence f_n is equicontinuous, then there is a subsequence that Gromov-Hausdorff converges to some map $f \colon X \to Y$.

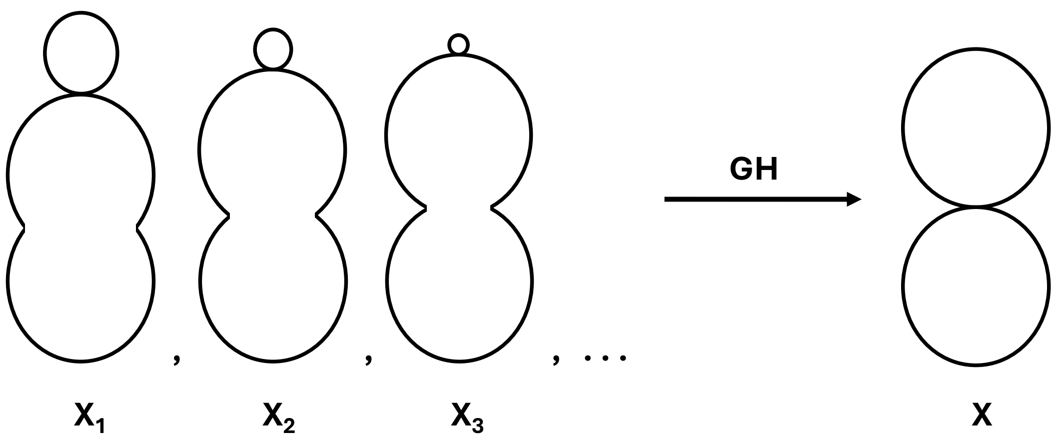

Figure 3: Illustration of Example 2.4.3. The space X is homeomorphic to the spaces X_n and $X_n \xrightarrow{\mathcal{GH}} X$. However, the convergence is not uniform.

3 Cyclic element theory

We recall that a compact connected metric space is called a *continuum*. Moreover we denote a locally connected continuum as a *Peano space*.
As a consequence of [4, p. 1109], Peano spaces can be characterized as follows:

Theorem 3.0.1. A metric space is a Peano space if and only if it is homeomorphic to a compact length space.

The so called *cyclic element theory* was founded by Whyburn in the 1920s. Its subject of investigation is a special decomposition of Peano spaces. We will use the language from this theory to describe the Gromov-Hausdorff limits of length surfaces. Indeed, the limit spaces are compact length spaces and hence Peano spaces.

3.1 Decomposing Peano Spaces

The decomposition investigated in the theory uses three types of subsets. They are presented in the following definition:

Definition 3.1.1. Let X be a Peano space, $p \in X$ and $A \subset X$.

1) The point p is called a *cut point* of X if $X \setminus \{p\}$ is disconnected.
2) The point p is called an *endpoint* of X if it admits arbitrarily small open neighborhoods $U \subset X$ such that $|\partial U| = 1$.
3) The subset A is called *cyclicly connected* if every pair of points in A can be connected by a Jordan curve in A. Provided A is maximal with this property and $|A| > 1$, we denote A as a *maximal cyclic subset.*

The name of the theory comes from the fact that the singletons consisting of cut or endpoints and the maximal cyclic subsets are called the *cyclic elements* of X. However, we will not use this term further. We denote the set of all cut points by Cut_X and its union with the set of all endpoints by $Sing_X$.
Whyburn showed the following structure theorem for Peano spaces:

Theorem 3.1.2. (cf. [39, pp. 64, 79]) Let X be a Peano space and $p \in X$. Then p is either a cut point, an endpoint or there is a unique maximal cyclic subset containing p.

We discuss a simple example:

Example 3.1.3. Let S_n be the round 2-sphere of diameter 2^{-n}. We consider the space X obtained by gluing the north pole of S_n with the south pole of S_{n+1} for all $n \in \mathbb{N}$ and denote its completion by $\bar{X}$. Then $\bar{X}$ is a Peano space and the subsets S_n are its maximal cyclic subsets. Furthermore the cut points of $\bar{X}$ are given by the gluing points in X and $\bar{X} \setminus X$ is a singleton consisting of the only endpoint $e \in \bar{X}$.
If we glue $\bar{X}$ along the free south pole of S_1 and e, we obtain a new Peano space Y. This space has only one maximal cyclic subset that is given by the space itself. Moreover Y is free of cut and endpoints.

The space $\bar{X}$ is illustrated in Figure 4.

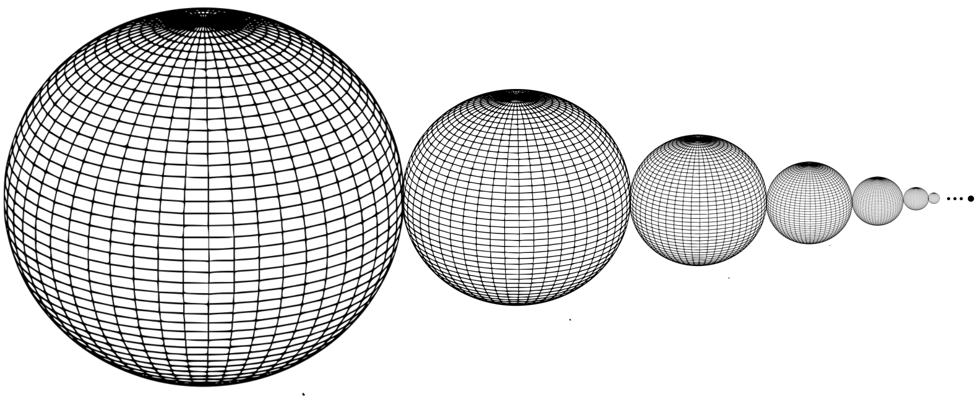

Figure 4: An illustration of the space $\bar{X}$ from Example 3.1.3. The maximal cyclic subsets of the space are given by the topological 2-spheres. Moreover the intersection points of the 2-spheres are the cut points of $\bar{X}$ and the indicated limit point is the only endpoint of the space.

Now we summarize basic properties of the decomposition:

Lemma 3.1.4. (cf. [39, 65-71, 79, 143]) Let X be a Peano space and $T \subset X$ be a maximal cyclic subset.

1) X has only countably many maximal cyclic subsets.

2) $X \setminus T$ consists of only countably many connected components.

3) T contains only countably many cut points of X.

4) For every sequence of pairwise distinct connected components $C_n \subset X \setminus T$ we have $diam(C_n) \longrightarrow 0$.

5) For every sequence of pairwise distinct maximal cyclic subsets $T_n \subset X$ we have $diam(T_n) \longrightarrow 0$.

6) For every connected component $C \subset X \setminus T$ there is some $p_C \in T$ with $\partial C = \{p_C\}$.

7) The map $r\colon X \to T$ defined by $r(x) = x$ on T and $r(x) = p_C$ on every connected component $C \subset X \setminus T$ is continuous.

8) For every pair of distinct maximal cyclic subsets $T_1, T_2 \subset X$ we have $|T_1 \cap T_2| \leq 1$.

3.2 Conjugate points and cyclic connectivity

In a continuum X we call two distinct points $x, y \in X$ *conjugate to each other* if there is no point $p \in X$ such that x and y lie in distinct connected components of $X \setminus \{p\}$. We say that a subset $A \subset X$ *separates* x and y in X provided the points lie in distinct connected components of $X \setminus A$.
For example two distinct points in the space $\bar{X}$ from Example 3.1.3 are conjugate to each other if and only if they lie in the same topological 2-sphere.
We start with some results on conjugate points and cyclicly connected subsets:

Lemma 3.2.1. (cf. [39, pp. 65, 67, 79]) Let X be a Peano space and $T \subset X$ be a maximal cyclic subset.

1) For every pair of conjugate points there is a Jordan curve in X connecting them.

2) For every cyclicly connected subset that contains more than one point there is a unique maximal cyclic subset containing it.

3) Provided a subset $A \subset T$ separates two points in T, it also separates these points in X.

Whyburn's so called *cyclic connectivity theorem* provides a useful criterion:

Theorem 3.2.2. (cf. [39, p. 79]) A Peano space is cyclicly connected if and only if the space is free of cut points.

A Peano space such that all maximal cyclic subsets are homeomorphic to the 1-sphere is called a *1-cactoid*. This special type of Peano spaces can be charcaterized as follows:

Proposition 3.2.3. (cf. [38, p. 417]) Let X be a continuum. Then the following statements are equivalent:

1) X is a 1-cactoid.

2) For every pair of conjugate points $x, y \in X$ the subset $\{x, y\}$ is separating in X.

4 Fundamental groups of gluings

This brief chapter provides two results related to the famous Seifert-Van Kampen theorem.
The first one deals with wedge sums:

Proposition 4.0.1. (cf. [15, p. 176]) Let X and Y be locally simply connected and path connected metric spaces. Moreover let Z be a wedge sum of X and Y. Then the fundamental group $\pi_1(Z)$ is isomorphic to the free product $\pi_1(X) * \pi_1(Y)$.

In Section 3.1 we already introduced the concept of cut points. The definition completely carries over to connected metric spaces X. Moreover we call a point $p \in X$ a *local cut point* provided there is a connected open neighborhood U of p such that p is a cut point of U. For example the local cut points of the space Y in Example 3.1.3 are given by the intersection points of the 2-spheres.
From the HNN-Seifert-Van Kampen theorem in [12, p. 1688] we obtain the second result:

Proposition 4.0.2. Let X be a locally simply connected and path connected metric space. Then the following statements apply:

1) If Y is a topological space obtained by gluing exactly two points in X, then the fundamental group $\pi_1(Y)$ is isomorphic to the free product $\pi_1(X) * \mathbb{Z}$.

2) If there is a local cut point in X that is no cut point, then there is a group G such that the fundamental group $\pi_1(X)$ is isomomorphic to the free product $G * \mathbb{Z}$.

5 Theory of surfaces

By a *surface*, we mean a connected topological 2-manifold with boundary. A compact surface without boundary is denoted as a *closed surface*. In this chapter we present results from the theory of surfaces that will be important for our investigation.

5.1 Classification of surfaces

One of the central observations in the theory of surfaces is that the range of compact surfaces is comparable small.
For closed surfaces we have the following classification theorem:

Theorem 5.1.1. (cf. [19, p. 122]) Every closed surface is homeomorphic to the 2-sphere, a connected sum of 2-tori or a connected sum of real projective planes.

In Figure 5 the aforementioned classification result is illustrated. For this and later figures parametrizations from [14, p. 334] and [33, p. 141] are used to represent the real projective plane, the Klein bottle or parts of them.
The classification of compact surfaces builds on that of closed surfaces:

Theorem 5.1.2. (cf. [19, p. 129]) Every compact surface is homeomorphic to a space obtained by removing the interior of finitely many disjoint closed topological 2-discs from a closed surface.

Up to homeomorphisms, there is exactly one simply connected non-compact surface:

Theorem 5.1.3. (cf. [10, p. 85]) Every non-compact simply connected surface is homoemorphic to the plane.

5.2 The connectivity number

With regard to compact surfaces we will work with the connectivity number rather than the Euler characteristic:

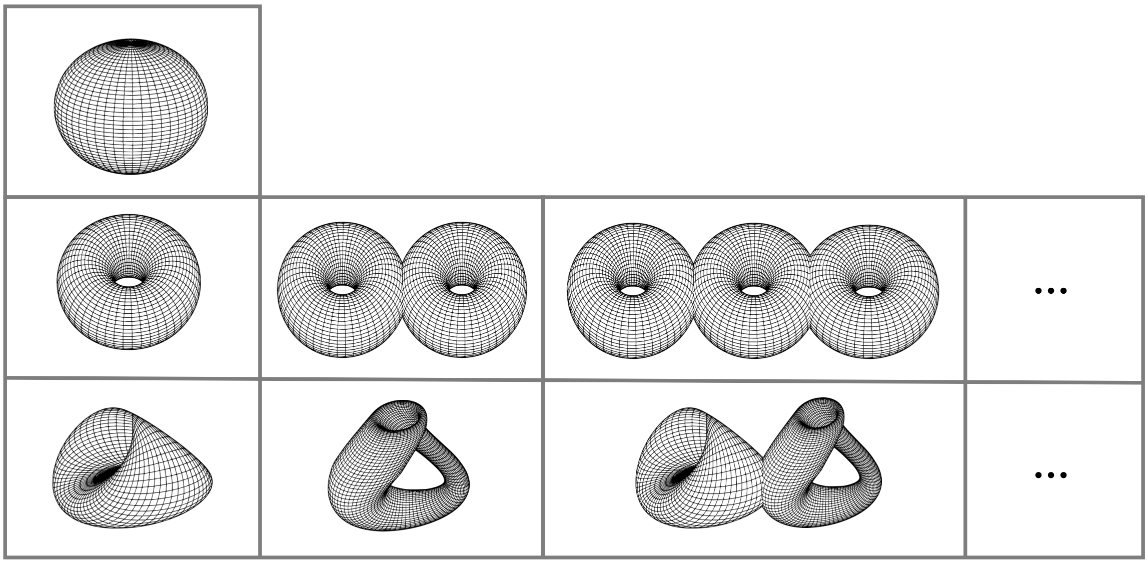

Figure 5: The classification of closed surfaces.

The *connectivity number* of a compact surface S is defined as $2-\chi(S)$. Roughly speaking, this quantity can be calculated by adding the number of boundary components, the number of "cross-caps" and twice the number of "holes". If we subtract the number of boundary components, we get the definition of the *reduced connectivity number* of S.
We consider an example regarding the connectivity number:

Example 5.2.1. Let S be a surface that can be obtained by removing the interior of two disjoint closed topological 2-discs from the Klein bottle. Then S has two boundary components, two "cross-caps" and no "holes". Hence the connectivity number of S is equal to four and its reduced connectivity number is equal to two.

5.3 Curves in compact surfaces

Now we discuss classification results for several types of curves in compact surfaces.

5.3.1 Simple arcs

In the following we denote the boundary of a compact surface S by ∂S and its interior by S^0. An arc in S is called *simple* if its endpoints lie in ∂S and its interior lies in S^0.
Together the upcoming two results classify simple arcs in compact surfaces:

Proposition 5.3.1. (cf. [25, pp. 54-55]) Let S be a compact surface of connectivity number c. Further let $\gamma \subset S$ be a separating simple arc which does not form a contractible Jordan curve together with a subarc of some boundary component.
Then there are $c_1, c_2 \in \mathbb{N}_{\geq 2}$ with $c_1 + c_2 = c + 1$ and a compact surface S_i of connectivity number c_i such that the topological quotient S/γ is a wedge sum of S_1 and S_2. Moreover the wedge point lies in $\partial S_1 \cap \partial S_2$.
At least one of the surfaces is non-orientable if and only if S is non-orientable.

Proposition 5.3.2. (cf. [25, pp. 54-55]) Let S be a compact surface of connectivity number c and $\gamma \subset S$ be a non-separating simple arc.
Then there is a compact surface S_1 of connectivity number $c - 1$ such that S/γ is a topological 2-point identification of S_1. Moreover the glued points lie in ∂S_1.
If S is orientable, then S_1 is orientable.

5.3.2 Jordan curves

In a compact surface contractible Jordan curves can be characterized as follows:

Theorem 5.3.3. (cf. [10, p. 85]) Let S be a compact surface and $J \subset S$ be a Jordan curve. Then J is contractible if and only if J bounds a closed topological 2-disc in S.

We say that a Jordan curve $J \subset S$ is *simple* if its intersection with ∂S contains at most one point.
Non-contractible simple Jordan curves can be classified as follows:

Proposition 5.3.4. (cf. [25, pp. 54-55]) Let S be a compact surface of connectivity number c and $J \subset S$ be a non-contractible simple Jordan curve. Then the topological quotient $X := S/J$ can be described in one of the following ways:

1) There are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$ and a compact surface S_i of connectivity number c_i such that X is a wedge sum of S_1 and S_2. Moreover at least one of the surfaces is non-orientable if and only if S is non-orientable.

2) There is a compact surface of connectivity number $c-2$ such that X is a topological 2-point identification of it. Moreover the surface is orientable if S is orientable.

3) X is a compact surface of connectivity number $c - 1$ and S is non-orientable.

If S is a closed surface, then all other surfaces occurring in the aforementioned cases are also closed surfaces.

The classification of non-contractible Jordan curves in closed surfaces is illustrated in Figure 6.

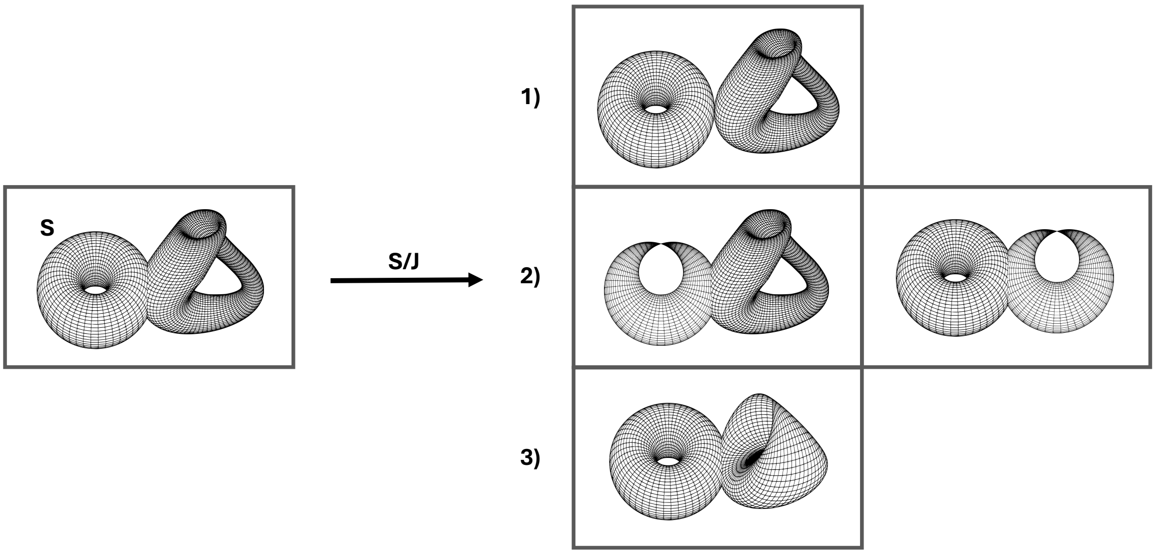

Figure 6: A classification of the non-contractible Jordan curves in a non-orientable closed surface of connectivity number four. The surface S is situated on the left hand side. On the opposite side we see all possible quotient spaces S/J where J is a non-contractible Jordan curve in S. The numbering corresponds to the cases described in Proposition 5.3.4.

5.4 Surface groups and local cut points

By a *surface group*, we mean a group that is isomorphic to the fundamental group of some closed surface. We note that the classification of closed surfaces directly yields a classification of surface groups.
Surfaces are free of local cut points. In this section we will see that this property also holds for certain metric spaces whose fundamental group is a surface group.
We state a property of surface groups:

Proposition 5.4.1. (cf. [21, pp. 141, 264], [3, pp. 1480-1481]) Let G be a surface group. Then G is not isomorphic to a free product of non-trivial groups.

As a consequence of the aforementioned result and Proposition 4.0.2, we obtain:

Proposition 5.4.2. Let X be a locally simply connected and path connected metric space that is free of cut points. If the fundamental group $\pi_1(X)$ is a surface group, then X is free of local cut points.

6 Scenario I: Closed surfaces

In this chapter we completely describe the Gromov-Hausdorff closure of the class of length spaces that are homeomorphic to a fixed closed surface. As a corollary, we derive a 2-dimensional version of Theorem (Ferry, Okun).

6.1 Introduction

6.1.1 Key results

We will see that the spaces of the closure have the following topological properties:

Theorem 6.1.1. Let X be the Gromov-Hausdorff limit of a convergent sequence of length spaces that are homeomorphic to a fixed closed surface. Then the following statements apply:

1) X is at most 2-dimensional.

2) X is locally simply connected.

3) There are finitely many closed surfaces $S_1, \ldots, S_n$ and $k \in \mathbb{N}_0$ such that $\pi_1(X)$ is isomorphic to the free product $\pi_1(S_1) * \ldots * \pi_1(S_n) * \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{k\text{-times}}$.

Before we state the main result of this chapter, we introduce the following definitions:

Definition.

1) We say that a Peano space is a *generalized cactoid* if all its maximal cyclic subsets are closed surfaces and only finitely many of them are not homeomorphic to the 2-sphere.

2) Let X be a metric space. A space that is isometric to a metric quotient of X whose underlying equivalence relation identifies exactly two points is referred to as a *metric 2-point-indentification* of X.

A Peano space whose maximal cyclic subsets are all homeomorphic to the 2-sphere is just called a *cactoid*. In Figure 7 an example of a generalized cactoid is shown.

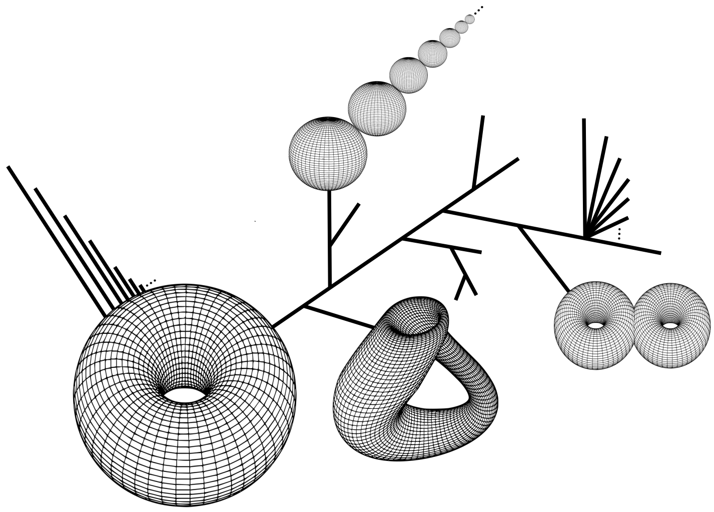

Figure 7: A generalized cactoid. All maximal cyclic subsets are closed surfaces and only three of them are not homeomorphic to the 2-sphere.

The following main result of this chapter completely describes the Gromov-Hausdorff closure of the class of closed length surfaces whose connectivity number is fixed:

Main Theorem I. Let $c \in \mathbb{N}_0$ and X be a compact length space. Then the following statements are equivalent:

1) X can be obtained as the Gromov-Hausdorff limit of closed length surfaces whose connectivity number is equal to c.

2) X can be obtained by a successive application of k metric 2-point identifications to a geodesic generalized cactoid such that the sum of the connectivity numbers of its maximal cyclic subsets is less or equal to $c - 2k$.

This result was partly conjectured by Young (cf. [40, p. 348], [32, p. 854]). Further the equivalence of the statements remains true even if we restrict the first statement to smooth Riemannian or polyhedral 2-manifolds (cf. [27, p. 1674], [31, p. 77]). We will also see how the result changes if we restrict the first statement to orientable or non-orientable surfaces.

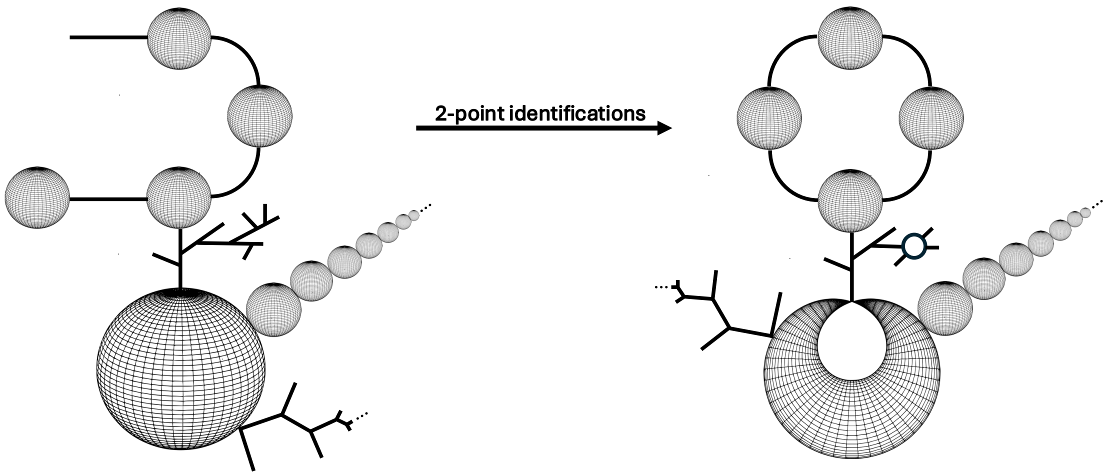

Figure 8: The space on the right hand side can be obtained by a successive application of three metric 2-point identifications to the geodesic cactoid situated on the left hand side. Since the connectivity number of the 2-sphere is equal to zero, Main Theorem I implies that the space on the right hand side can be obtained as the limit of closed length surfaces whose connectivity number is equal to six.

Main Theorem I is illustrated in Figure 8.
In the 1930s, Whyburn already proved the following result:

Theorem A (Whyburn). (cf. [38, p. 419]) A space that can be obtained as the limit of length spaces that are homeomorphic to the 2-sphere is a cactoid.

Moreover there is a related result about closed Riemannian 2-manifolds with uniformly bounded total absolute curvature by Shioya (cf. [35, p. 1767]) and a sketch of a local description of the limit spaces by Gromov (cf. [17, p. 102]).
The following corollary is a 2-dimensional version of Theorem (Ferry, Okun):

Corollary 6.1.2. Let X be a metric space and S be a closed surface. Then the following statements are equivalent:

1) X is a simply connected ANR that can be obtained as the Gromov-Hausdorff limit of length spaces that are homeomorphic to S.

2) X is a geodesic cactoid having only finitely many maximal cyclic subsets.

6.1.2 Organisation

This chapter is organized as follows:
The second section is devoted to the topological connection between maximal cyclic subsets and their ambient space. In particular, we derive a fundamental group formula for locally simply connected Peano spaces in terms of their maximal cyclic subsets. The section also provides first topological properties of generalized cactoids.
In Section 6.3 we show that the first statement of Main Theorem I implies the second. For this we begin with a consideration of uniformly semi-locally 1-connected sequences. At the end of the section we give a proof of Theorem 6.1.1.
The aim of the last section is to show the remaining direction of Main Theorem I. After this we prove Corollary 6.1.2.
The final results of Section 6.3 and 6.4 refine their corresponding statement of Main Theorem I. Together they completely describe the Gromov-Hausdorff closure of the class of length spaces that are homeomorphic to a fixed closed surface.

6.1.3 Notation

In this chapter we use the following notations:

$\mathcal{M}$	The class of compact metric spaces.
$\mathcal{S}(c)$	The class of closed length surfaces whose connectivity number is equal to c.
$\mathcal{S}(c,\varepsilon)$	The class of spaces in $\mathcal{S}(c)$ that do not contain non-contractible loops of diameter less than 2ε.
$\mathcal{G}(c)$	The class of geodesic generalized cactoids such that the connectivity numbers of their maximal cyclic subsets sum up to c.
$\mathcal{W}$	The class of successive metric wedge sums of non-degenerate cyclicly connected compact length spaces and finite metric trees.
$\mathcal{W}_0$	The class of successive metric wedge sums of closed length surfaces such that every wedge point is only shared by two of their surfaces.

We note that we allow a change of the wedge point in every construction step of a successive metric wedge sum. In Figure 9 we see a space in $\mathcal{W}_0$.

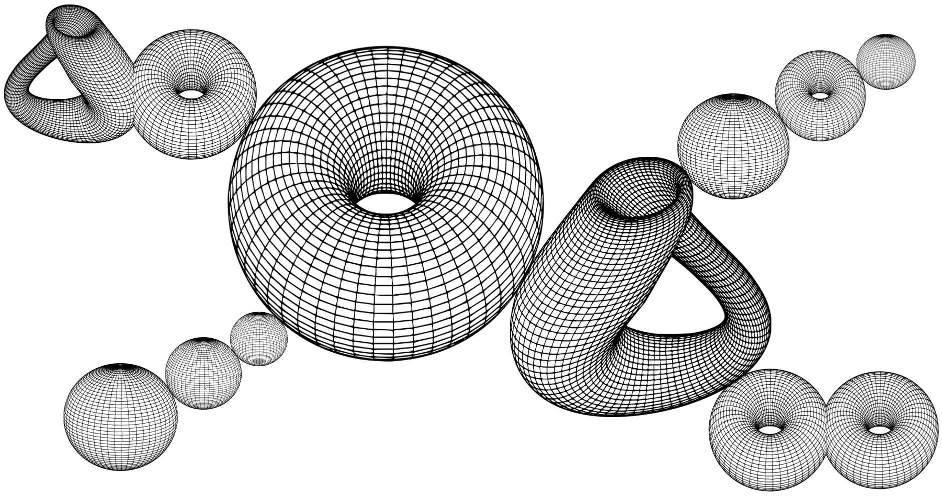

Figure 9: A space in W_0. The space is a successive metric wedge sum of ten closed length surfaces and every wedge point is only shared by two of the surfaces.

6.2 Topology via maximal cyclic subsets

In this section we investigate the topological connection between maximal cyclic subsets and their ambient space. From the results we derive first topological properties of generalized cactoids.

6.2.1 A dimension bound

First we consider the dimension of Peano spaces.
There are many properties which are satisfied by the whole Peano space provided they are shared by all its maximal cyclic subsets (cf. [39, pp. 81-83]). The next result contains an example of such a property:

Proposition 6.2.1. Let X be a Peano space. Then the following statements apply:

1) If all maximal cyclic subsets of X are at most n-dimensional, then X is at most n-dimensional. (cf. [39, p. 82])

2) If X is at most n-dimensional and Y is a metric 2-point identification of X, then Y is at most n-dimensional (cf. [34, pp. 266, 271]).

As a consequence, we derive the following result about generalized cactoids:

Corollary 6.2.2. Let X be a space that can be obtained by a successive application of metric 2-point identifications to a generalized cactoid. Then X is at most 2-dimensional.

6.2.2 Local contractibility

Now we consider the property of local contractibility.
A metric space X is called *locally strongly contractible* provided every $x \in X$ has arbitrarily small open neighborhoods such that $\{x\}$ is a strong deformation retract of them. For example X is locally strongly contractible if it is a manifold.
The following observation motivates the next result: We denote the circle of radius $1/n$ around the point $(1/n, 0) \in \mathbb{R}^2$ by C_n. Then the subset $C := \cup_{n \in \mathbb{N}} C_n$ is a Peano space whose maximal cyclic subsets are homeomorphic to the 1-sphere (see Figure 10). Hence the maximal cyclic subsets are locally strongly contractible. But C is also the Hawaiian earring and it is well-known that this space is not even semi-locally 1-connected.
On the other hand, the union of finitely many circles in $\mathbb{R}^2$ is always locally strongly contractible.

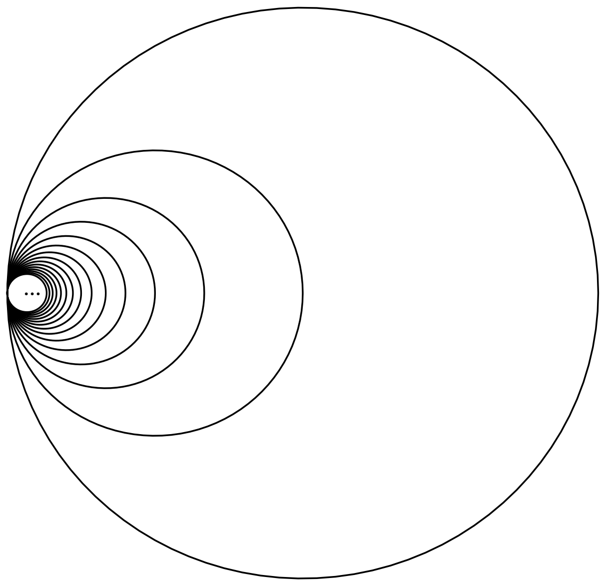

Figure 10: A Peano space whose maximal cyclic subsets are locally strongly contractible but the space itself is not even semi-locally 1-connected. The space is also called the *Hawaiian earring*.

Proposition 6.2.3. Let X be a Peano space. Then the following statements apply:

1) If X contains infinitely many non-contractible maximal cyclic subsets, then X is not locally contractible.

2) If X has only finitely many maximal cyclic subsets and all of them are locally strongly contractible, then X is locally strongly contractible.

Proof. 1) There is a sequence of pairwise distinct non-contractible maximal cyclic subsets in X. From Lemma 3.1.4 follows that there is a subsequence converging to some $p \in X$. Hence an arbitrarily small open neighborhood of p contains a non-contractible maximal cyclic subset of X.
For the sake of contradiction we assume that X is locally contractible. Then there is a contractible open neighborhood U of p. This neighborhood contains some non-contractible maximal cyclic subset T of X. By Lemma 3.1.4 we have that T is a retract of U. Therefore T is also contractible. A contradiction.
2) We denote the number of maximal cyclic subsets in X by n.
If $n = 0$, then X is homeomorphic to a compact metric tree and hence locally strongly contractible (cf. [1, p. 20]).
If $n = 1$, we consider $p \in X$ and $\varepsilon > 0$. We may assume that p is contained in the only maximal cyclic subset T of X. Then there is an open neighborhood U_0 of p in T with diameter less than ε such that $\{p\}$ is a strong deformation retract of U_0. Further we may assume that U_0 contains infinitely many cut points of X.
Let $(c_k)_{k\in\mathbb{N}}$ be an enumeration of these cut points. We write C_k for the closure of the union of all connected components of $X \setminus T$ whose boundaries are given by $\{c_k\}$. Then C_k is a compact metric tree. Hence there is an open neighborhood U_k of c_k in C_k with diameter less than ε such that $\{c_k\}$ is a strong deformation retract of U_k. In particular, there is a homotopy H_k between the identity map on U_k and the map sending every point of U_k to c_k such that $H_k(c_k, t) = c_k$ for every $t \in [0, 1]$.
Now we set $U := \cup_{k\in\mathbb{N}_0} U_k$ and define a map $H\colon U \times [0, 1] \to U$ by:

$$H(x,t) = \begin{cases} H_k(x,t), & x \in U_k; \\ x, & x \in U_0. \end{cases}$$

The diameter of U is less than 2ε. From Lemma 3.1.4 we derive that U is an open neighborhood of p in X and H is continuous. Hence U_0 is a strong deformation retract

of U and it follows that $\{p\}$ is a strong deformation retract of U. We conclude that X is locally strongly contractible.
If $n \geq 2$, then X is homeomorphic to a wedge sum of Peano spaces with less than n maximal cyclic subsets. If both spaces are locally strongly contractible, then so is X. Hence the claim follows by induction. □

A metric space whose dimension is finite is an ANR if and only if it is locally contractible (cf. [34, pp. 347, 392]). Hence Corollary 6.2.2 and the last proposition imply the following statement:

Corollary 6.2.4. Let X be a generalized cactoid. Then the following statements are equivalent:

1) X is an ANR.

2) X has only finitely many maximal cyclic subsets.

6.2.3 A fundamental group formula

In this subsection we present a formula for the fundamental group of a locally simply connected Peano space in terms of its maximal cyclic subsets. For this we first reduce the complexity of the problem:

Lemma 6.2.5. Let X be a compact length space and $(T_k)_{k=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then X can be obtained as the limit of compact length spaces having only finitely many maximal cyclic subsets. Further the maximal cyclic subsets of the space with index n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^{n}$ for every $n \in \mathbb{N}$.

Proof. Let $n, k \in \mathbb{N}$. We define an equivalence relation $\sim$ on X as follows: $x \sim y$ if and only if x and y lie in the same connected component of $\cup_{m=n+1}^{n+k} T_m$. Further we define $X_{n,k}$ as the metric quotient ${}^{X}\!/\!_{\sim}$ and denote the corresponding projection map by $p_{n,k}$. Then $p_{n,k}$ is surjective and 1-lipschitz. Hence we may assume that there is a space $X_n \in \mathcal{M}$ and a map $p_n \colon X \to X_n$ such that $X_{n,k} \longrightarrow X_n$ and $p_{n,k} \longrightarrow p_n$ uniformly. Since the map $p_{n,k}$ is monotone and its restriction to T_m is distance preserving for every $m \in \{1, \ldots, n\}$, the same applies to p_n (cf. [39, p. 174]).
Let T be a maximal cyclic subset of X_n. Then we find some $k \in \mathbb{N}$ such that $T \subset p_n(T_k)$ (cf. [39, pp. 145-146]). Because p_n is constant on T_m for every $m \in \mathbb{N}$ with $m > n$, we have $k \leq n$. Hence $p_n(T_k)$ is cyclicly connected and we derive that the inclusion is an

equality.
Due to the fact that for every non-degenerate cyclicly connected subset of X_n there is a maximal cyclic subset containing it (cf. [39, p. 79]), we derive that $\{p_n(T_k)\}_{k=1}^n$ is the set of maximal cyclic subsets of X_n and has cardinality n.
Since p_n is surjective and 1-lipschitz, we may assume that there is $\tilde{X} \in \mathcal{M}$ with $X_n \longrightarrow \tilde{X}$. Choosing a diagonal sequence, we may assume that $X_{n,n} \longrightarrow \tilde{X}$ and $(p_{n,n})_{n\in\mathbb{N}}$ is uniformly convergent. Finally the limit map is an isometry between X and $\tilde{X}$. □

Lemma 6.2.6. Let X be a compact length space having only finitely many maximal cyclic subsets. Then X can be obtained as the limit of spaces in $\mathcal{W}$. Further the maximal cyclic subsets of the spaces of the sequence are in isometric one-to-one correspondence with those of X.

Proof. Let ε be the minimum of the diameters of the maximal cyclic subsets of X. If $n \in \mathbb{N}$ and T is a maximal cyclic subset of X, we denote the set of connected components of $X \setminus T$ having diameter less than ε/n by $\mathcal{C}_T$. Moreover we set $\mathcal{T}$ as the set of maximal cyclic subsets of X and $X_n \coloneqq X \setminus (\cup_{T\in\mathcal{T}} \cup_{C\in\mathcal{C}_T} C)$.
The space X_n is a compact length space which has the same maximal cyclic subsets as X. Further it is an ε/n-net in X and it follows that the inclusion map from X_n to X is an ε/n-isometry. Hence we have $X_n \longrightarrow X$.
Moreover X_n is a successive metric wedge sum of its maximal cyclic subsets and finitely many compact metric trees. Let D be such a tree. We denote the wedge points lying in D by $p_1, \ldots, p_N$. There is a sequence $(D_k)_{k\in\mathbb{N}}$ of finite metric trees converging to D. Further we may assume the existence of a sequence $(f_k)_{k\in\mathbb{N}}$ such that f_k is a $1/k$-isometry from D to D_k. We define $X_{n,k}$ as the successive metric wedge sum created by replacing D with D_k in X_n and the corresponding wedge points with $f_k(p_1), \ldots, f_k(p_N)$.
This new space is again a compact length space such that its maximal cyclic subsets are in isometric one-to-one correspondence with those of X. In particular, we may assume that $X_{n,k} \in \mathcal{W}$. Otherwise we repeat the argument above until every tree is replaced by a finite one. Moreover we have that $X_{n,k} \longrightarrow X_n$ since f_k defines a $1/k$-isometry between X_n and $X_{n,k}$. Choosing a diagonal sequence, we may assume that $X_{n,n} \longrightarrow X$. □

Corollary 6.2.7. Let X be a compact length space and $(T_k)_{k=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then X can be obtained as the limit of spaces in $\mathcal{W}$. Further the maximal cyclic subsets of the space with index n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^n$ for every $n \in \mathbb{N}$.

Every compact interval or point can be obtained as the limit of spaces in $\mathcal{S}(0)$. Hence we can proceed as in the second half of the last proof to get rid of the finite metric trees first and then the cut points lying in more than one maximal cyclic subset. From this we derive the following statement for later purposes:

Corollary 6.2.8. Let X be geodesic generalized cactoid and $(T_k)_{k=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then X can be obtained as the limit of spaces in $\mathcal{W}_0$. Further the maximal cyclic subsets of the space with index n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^{n}$ and finitely many spaces in $\mathcal{S}(0)$ for every $n \in \mathbb{N}$.

Now we are able to state the formula:

Proposition 6.2.9. Let X be a locally simply connected Peano space and $(T_n)_{n=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then $\pi_1(X)$ is isomorphic to the free product $\pi_1(T_1) * \ldots * \pi_1(T_n)$ for all but finitely many $n \in \mathbb{N}$.

Proof. Since X is locally simply connected, the same applies to its maximal cyclic subsets. Moreover there is some $\varepsilon > 0$ such that every loop in X of diameter less than ε lies in some simply connected subset of X. If T is a maximal cyclic subset of X, then also every loop in T of diameter less than ε lies in some simply connected subset of T. By Theorem 3.0.1 we may assume that X is geodesic. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence as in Corollary 6.2.7. An application of Proposition 4.0.1 yields that the sequence is uniformly semi-locally 1-connected. Hence $\pi_1(X)$ is isomorphic to $\pi_1(X_n)$ for all but finitely many $n \in \mathbb{N}$. From the same proposition we also derive that $\pi_1(X_n)$ is isomorphic to $\pi_1(T_1) * \ldots * \pi_1(T_n)$ for every $n \in \mathbb{N}$. This closes the proof. □

6.2.4 Cactoids

As it will turn out, limits of spaces in $\mathcal{S}(c)$ locally look like cactoids (see Corollary 6.3.7). Hence we are interested in the topology of cactoids. Again we first reduce the complexity of the problem:

Lemma 6.2.10. Let $X \in \mathcal{G}(0)$. Then X is homeomorphic to a space that can be obtained as the limit of compact length spaces whose maximal cyclic subsets are isometric to round 2-spheres.

Proof. First we may assume that there are infinitely many maximal cyclic subsets in X. Let $(T_n)_{n=1}^{\infty}$ be an enumeration of them. There is a homeomorphism f_n from T_n to the round 2-sphere of diameter $1/2^n$. We denote this 2-sphere by S_n.

The following condition naturally induces an equivalence relation $\sim$ on the disjoint union of the closures of the connected components of $X \setminus T_1$ and S_1: $x \sim y$ if $f_1(x) = y$. We equip the disjoint union with its induced length metric and denote the corresponding metric quotient by Y_1. Analogously we construct the space Y_2 using the disjoint union of the closures of the connected components of $Y_1 \backslash T_2$ and S_2. We continue this construction and derive a sequence $(Y_n)_{n\in\mathbb{N}}$ of metric spaces.
We note that there is an induced enumeration of the maximal cyclic subsets of Y_n. If $k \in \mathbb{N}$, we define the metric space $Y_{n,k}$ in the same way for Y_n as $X_{n,k}$ is defined for X in the proof of Lemma 6.2.5.
The maps $f_1, \dots, f_n$ naturally induce a homeomorphism from X to Y_n. We denote the composition of this homeomorphism with the projection map from Y_n to $Y_{n,k}$ by $g_{n,k}$. Moreover we find a map $p_{n,k} \colon Y_{n+1,k} \to Y_{n,k+1}$ such that the diagram together with the maps $g_{n+1,k}$ and $g_{n,k+1}$ commutes. This map is 1-lipschitz and has a distortion less than $1/2^n$. Furthermore the sequence $(g_{n,k})_{k\in\mathbb{N}}$ is equicontinuous. We may assume that there is a space $\tilde{Y}_n$ such that $Y_{n,k} \longrightarrow \tilde{Y}_n$. In addition we may assume the existence of maps $p_n \colon \tilde{Y}_{n+1} \to \tilde{Y}_n$ and $g_n \colon X \to \tilde{Y}_n$ such that $p_{n,k} \longrightarrow p_n$ and $g_{n,k} \longrightarrow g_n$ uniformly.
Using the proof of Lemma 6.2.5, we see that every maximal cyclic subset of $\tilde{Y}_n$ is isometric to a round 2-sphere. Further the diagram consisting of p_n, g_n and g_{n+1} commutes and p_n is a 1-lipschitz map that has a distortion less or equal to $1/2^n$. If we set $\varepsilon_k := \sum_{m=k}^{\infty} 1/2^m$, then $p_k \circ \ldots \circ p_{n-1}$ is an ε_k-isometry between $\tilde{Y}_n$ and $\tilde{Y}_k$ for every $n \in \mathbb{N}$ with $n > k$. Hence $\left(\tilde{Y}_n\right)_{n\in\mathbb{N}}$ is convergent and we denote its limit space by Y. Moreover $(g_n)_{n\in\mathbb{N}}$ is equicontinuous and we may assume the existence of a map $g \colon X \to Y$ such that $g_n \longrightarrow g$ uniformly. Finally g is bijective and hence a homeomorphism. □

Corollary 6.2.11. Cactoids are locally simply connected and simply connected.

Proof. In a successive metric wedge sum of round 2-spheres and finite metric trees every open ball is simply connected. A theorem by Petersen in [28, p. 501] implies that this property is stable under Gromov-Hausdorff convergence. Hence Theorem 3.0.1, Corollary 6.2.7 and Lemma 6.2.10 close the proof. □

6.3 The limit spaces

The goal of this chapter is to show that the first statement of Main Theorem I implies the second. As a consequence, we derive Theorem 6.1.1.

6.3.1 Controlled convergence

First we investigate the limit spaces of uniformly semi-locally 1-connected sequences.
A metric space X is called a *local cactoid* if every point in X has an open neighborhood that is homeomorphic to an open subset of a cactoid.

Lemma 6.3.1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c,\varepsilon)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then X is a local cactoid.

Proof. First we may assume that all surfaces of the sequence are not homeomorphic to the 2-sphere. Otherwise the claim follows by Theorem A (Whyburn).
Let $x \in X$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ such that $x_n \longrightarrow x$. We define $\mathcal{D}$ as the set of all closed topological 2-discs in X_n that are bounded by a Jordan curve in $B_\varepsilon(x_n)$. Moreover we set A as the union of $B_\varepsilon(x_n)$ and the discs in $\mathcal{D}$.
It follows that A is open and connected. If J is a Jordan curve in $B_\varepsilon(x_n)$, then J is contractible in X_n since $X_n \in \mathcal{S}(c,\varepsilon)$. By Theorem 5.3.3 the curve J bounds a closed topological 2-disc in X_n and we derive that J is contractible in A. This already yields that every loop in $B_\varepsilon(x_n)$ is contractible in A (cf. [24, p. 626]).
Let $\gamma\colon [0,1] \to A$ be a loop. The set consisting of $B_\varepsilon(x_n)$ and the interiors of all discs in $\mathcal{D}$ is an open cover of $\gamma([0,1])$. Since γ is uniformly continuous and its image is compact, there is a finite subdivision $t_0 \coloneqq 0 < t_1 < \ldots < t_k \coloneqq 1$ of the unit interval such that $\gamma([t_i,t_{i+1}]) \subset B_\varepsilon(x_n)$ or we find some $D \in \mathcal{D}$ whose interior contains $\gamma([t_i,t_{i+1}])$. An induction over the number of subcurves not lying in $B_\varepsilon(x_n)$ finally shows that γ is homotopic to some loop in $B_\varepsilon(x_n)$.
We deduce that A is simply connected and with Section 5.1 we conclude that A is homeomorphic to the plane or the 2-sphere. Because X_n is not homeomorphic to the 2-sphere, the first case applies. We derive that the metric quotient $Y_n \coloneqq {X_n}/{A^c}$ is homeomorphic to the 2-sphere. Especially we have $Y_n \in \mathcal{S}(0)$.
Since the natural projection $p_n\colon X_n \to Y_n$ is surjective and 1-lipschitz, we may assume that there is a space $Y \in \mathcal{M}$ and a map $p\colon X \to Y$ such that $Y_n \longrightarrow Y$ and $p_n \longrightarrow p$.
From Theorem A (Whyburn) we get that Y is a cactoid. Further we may assume that the sequences $\left(\bar{B}_{\frac{\varepsilon}{2}}(x_n)\right)_{n\in\mathbb{N}}$ and $\left(\bar{B}_{\frac{\varepsilon}{2}}(p_n(x_n))\right)_{n\in\mathbb{N}}$ are convergent. Their limits are given by $\bar{B}_{\frac{\varepsilon}{2}}(x)$ and $\bar{B}_{\frac{\varepsilon}{2}}(p(x))$. Because p_n defines an isometry between $\bar{B}_{\frac{\varepsilon}{2}}(x_n)$ and $\bar{B}_{\frac{\varepsilon}{2}}(p_n(x_n))$, the same applies to p with respect to $\bar{B}_{\frac{\varepsilon}{2}}(x)$ and $\bar{B}_{\frac{\varepsilon}{2}}(p(x))$. In particular, this also holds for the corresponding open balls. We finally conclude that X is a local cactoid. $\square$

Corollary 6.3.2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c,\varepsilon)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then the following statements apply:

1) X is locally simply connected.

2) If $c = 0$, then the maximal cyclic subsets of X are simply connected.

3) If $c > 0$, then there is a closed surface S of connectivity number c such that the fundamental group of one maximal cyclic subset of X is isomorphic to $\pi_1(S)$ and all other maximal cyclic subsets are simply connected. Moreover S is orientable if and only if X_n is orientable for infinitely many $n \in \mathbb{N}$.

Proof. Combining Corollary 6.2.11 and Lemma 6.3.1, we derive the first statement. Since the sequence is uniformly semi-locally 1-connected, Theorem 2.3.5 implies that $\pi_1(X)$ isomorphic to $\pi_1(X_n)$ for all but finitely many $n \in \mathbb{N}$. Hence Proposition 5.4.1 and Proposition 6.2.9 close the proof. □

From the upcoming lemma follows that limits of spaces in $\mathcal{S}(c, \varepsilon)$ are generalized cactoids:

Lemma 6.3.3. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c, \varepsilon)$, $X \in \mathcal{M}$ with $X_n \longrightarrow X$ and T be a maximal cyclic subset of X. Then T is a closed surface.

Proof. First we show that T is free of local cut points: For the sake of contradiction we assume that T contains a local cut point. Due to Corollary 6.3.2 the space X is locally simply connected. From Proposition 4.0.2 follows that there is a group G such that $\pi_1(T)$ is isomorphic to $G * \mathbb{Z}$. Moreover Corollary 6.3.2 yields that $\pi_1(T)$ is isomorphic to the fundamental group of some closed surface. This contradicts Proposition 5.4.1.
Let $p \in T$. Then Lemma 6.3.1 implies that there is a connected open neighborhood V of p in X and a homeomorphism f from V to an open subset of some cactoid C. Further Lemma 3.1.4 yields that $T \cap V$ is connected. Since T is free of local cut points, there is a Jordan curve J in $V \cap T$. In particular, $f(J)$ is contained in some maximal cyclic subset S of C.
The subset $f(V) \cap S$ is connected and S is free of local cut points. Therefore we derive that $f(V \cap T) = f(V) \cap S$. Hence $V \cap T$ is homeomorphic to an open subset of the 2-sphere.
We conclude that T is a surface. Especially T is a closed surface because $\pi_1(T)$ is isomorphic to the fundamental group of some closed surface. □

Combining the last lemma and Corollary 6.3.2, we get the following result:

Corollary 6.3.4. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c, \varepsilon)$, where $c > 0$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then one maximal cyclic subset T of X is a closed surface of connectivity number c and all other maximal cyclic subsets are homeomorphic to the 2-sphere. Moreover T is orientable if and only if X_n is orientable for infinitely many $n \in \mathbb{N}$.

6.3.2 The general case

Next we see what happens if we omit the additional topological control.
If a sequence in $\mathcal{S}(c)$ has no uniformly semi-locally 1-connected subsequence, then there is a subsequence and a sequence of Jordan curves as in the following lemma:

Lemma 6.3.5. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Further let $(J_n)_{n\in\mathbb{N}}$ be a sequence such that J_n is a non-contractible Jordan curve in X_n for every $n \in \mathbb{N}$ and $diam(J_n) \longrightarrow 0$. Then one of the following cases applies:

1) There are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$, a convergent sequence of spaces in $\mathcal{S}(c_1)$ and a convergent sequence of spaces in $\mathcal{S}(c_2)$ such that X is a metric wedge sum of their limits. If X_n is non-orientable for infinitely many $n \in \mathbb{N}$, then the surfaces of at least one of the sequences may be chosen to be non-orientable.

2) There is a convergent sequence of spaces in $\mathcal{S}(c-2)$ such that X is its limit or a metric 2-point identification of it.

3) There is a sequence of spaces in $\mathcal{S}(c-1)$ converging to X.

If X_n is orientable for infinitely many $n \in \mathbb{N}$, then always one of the first two cases applies and the surfaces of the corresponding sequences may be chosen to be orientable.

Proof. First we may assume that all surfaces of the sequence are orientable or all are non-orientable. Moreover we may assume that the Jordan curves all belong to the same class in the sense of Proposition 5.3.4. In particular, we have that the corresponding sequence $(Y_n)_{n\in\mathbb{N}}$ of metric quotients is convergent with limit X. We just go through the cases of the proposition:
In the first case there are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$ such that Y_n is a metric wedge sum of a space in $\mathcal{S}(c_1)$ and a space in $\mathcal{S}(c_2)$. Especially at least one of the surfaces is non-orientable if and only if X_n is non-orientable. Moreover we may assume that the sequence of the wedge points and the sequences of the surfaces considered as subsets of the wedge sums are convergent. From Lemma 2.3.1 we derive that X is a metric wedge sum of the limits.
Now we consider the second case: Then there is a space Z_n in $\mathcal{S}(c-2)$ such that Y_n is a metric 2-point identification of Z_n. In particular, Z_n is orientable if X_n is orientable. Further we may assume the sequence $(Z_n)_{n\in\mathbb{N}}$ to be convergent. It follows that X is the limit or a metric 2-point identification of it.
If we look at the third case, then we have that Y_n is a space in $\mathcal{S}(c-1)$ and X_n is non-orientable. □

The final result of this section refines a statement of Main Theorem I:

Theorem 6.3.6. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then X can be obtained by a successive application of k metric 2-point identifications to some space $Y \in \mathcal{G}(c_0)$ where $c_0 \leq c - 2k$. Moreover the following statements apply:

1) If X_n is orientable for infinitely many $n \in \mathbb{N}$, then the maximal cyclic subsets of Y are orientable.

2) If X_n is non-orientable for infinitely many $n \in \mathbb{N}$ and the maximal cyclic subsets of Y are orientable, then $c_0 < c$.

Proof. The proof proceeds by induction over the connectivity number:
In the case $c = 0$ the claim directly follows by Theorem A (Whyburn).
Now we consider the case $c > 0$. Moreover we assume that the claim is true, if the connectivity number is less than c. Provided the sequence is uniformly semi-locally 1-connected, the claim directly follows by Corollary 6.3.4. Otherwise we may assume that there is a sequence $(J_n)_{n\in\mathbb{N}}$ such that J_n is a non-contractible Jordan curve in X_n and $diam(J_n) \longrightarrow 0$. Hence one of the cases of Lemma 6.3.5 applies. We note that the surfaces of the sequences occurring there have a connectivity number less than c.
Finally an application of the induction hypothesis and the following observation yield the claim: Let Y_1 and Y_2 be metric spaces. Further let Z_i be a space that can be obtained by a successive application of k_i metric 2-point identifications to Y_i. Then every metric wedge sum of Z_1 and Z_2 is a space that can be obtained by a successive application of $k_1 + k_2$ metric 2-point identifications to a metric wedge sum of Y_1 and Y_2. Moreover every metric 2-point identification of Z_1 is a space that can be obtained by a successive application of $k_1 + 1$ metric 2-point identifications to Y_1. □

The property of being a local cactoid is stable under applications of metric wedge sums and metric 2-point identifications. Hence the induction above also yields the following result:

Corollary 6.3.7. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then X is a local cactoid.

Finally we derive Theorem 6.1.1:

Proof of Theorem 6.1.1. From Corollary 6.2.2 follows that X is at most 2-dimensional. Further Corollary 6.2.11 and Corollary 6.3.7 imply that X is locally simply connected. Finally Proposition 4.0.2 and Proposition 6.2.9 yield the desired representation of $\pi_1(X)$. □

6.4 Approximation of generalized cactoids

The aim of this chapter is to prove that the second statement of Main Theorem I implies the first. At the end we show Corollary 6.1.2.
We start by showing that metric wedge sums of closed length surfaces can be approximated by closed length surfaces:

Lemma 6.4.1. Let S_1 and S_2 be closed length surfaces and X be a metric wedge sum of them. Then X can be obtained as the limit of length spaces that are homeomorphic to a connected sum of S_1 and S_2.

Proof. We denote the wedged points by p_1 and p_2. For every $i \in \{1, 2\}$ and $n \in \mathbb{N}$ there is a closed topological 2-disc $D_{i,n}$ of diameter less than $1/n$ in S_i that contains p_i in its interior. Moreover we may assume $D_{i,n}$ to be bounded by a piecewise geodesic Jordan curve $J_{i,n} \colon [0, b_{i,n}] \to X$ (cf. [35, p. 1794], [38, pp. 413-415]).
Using the Kuratowski embedding, we identify S_i with a subset of $l^\infty(X)$. Further we define $\tilde{D}_{i,n}$ to be the union of all linear segments from p_i to a point of $\partial D_{i,n}$. By direct calculation using the embedding we see that linear segments from p_i to distinct points of S_i only intersect in p_i. Therefore we get that $F_{i,n} \coloneqq \tilde{D}_{i,n} \cup (S_i \setminus D_{i,n})$ is homeomorphic to S_i. Now we equip $F_{i,n}$ with its induced length metric and denote the obtained space by $S_{i,n}$. It follows that the identity map is a homeomorphism between $F_{i,n}$ and $S_{i,n}$. Moreover we have $S_{i,n} \longrightarrow S_i$.
For every $\lambda \in (0, 1]$ the map $\gamma_{i,\lambda}(t) \coloneqq \lambda J_{i,n}\left(\frac{t}{\lambda}\right) + (1 - \lambda)p_i$ is a piecewise geodesic Jordan curve in $S_{i,n}$. Moreover the curve bounds a closed topological 2-disc $B_{i,\lambda}$ that contains p_i in its interior. We have that $l_{i,\lambda} \coloneqq length(\gamma_{i,\lambda}) \longrightarrow 0$, if $\lambda \longrightarrow 0$, and there are sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ in $(0, 1]$ converging to 0 such that $l_{1,\alpha_k} = l_{2,\beta_k}$.
We assume the subsets in the upcoming construction to be equipped with their induced length metric. There is a natural equivalence relation $\sim$ on $\left(S_{1,n} \setminus \mathring{B}_{1,\alpha_k}\right) \sqcup \left(S_{2,n} \setminus \mathring{B}_{2,\beta_k}\right)$ defined by the following condition: $x \sim y$ if there is some $t \in [0, l_{1,\alpha_k}]$ such that $x = \gamma_{1,\alpha_k}(t)$ and $y = \gamma_{2,\beta_k}(t)$. We denote the corresponding metric quotient by $\tilde{S}_{n,k}$ and note that this space is homeomorphic to a connected sum of S_1 and S_2 (cf. [5, p. 69]).
Now $\left(\tilde{S}_{n,k}\right)_{k\in\mathbb{N}}$ converges to a metric wedge sum W_n of $S_{1,n}$ and $S_{2,n}$ along p_1 and p_2. Since $W_n \longrightarrow X$, we may close the proof. □

From Corollary 6.2.8 and the last lemma we inductively get the following result:

Corollary 6.4.2. Let $X \in \mathcal{G}(c)$. Then there is a sequence of spaces in $\mathcal{S}(c)$ converging to X. Moreover the following statements apply:

1) If all maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.

2) If there is a non-orientable maximal cyclic subset in X, then the surfaces of the sequence may be chosen to be non-orientable.

A similar argument to that used in the proof of Lemma 6.4.1 shows the following result concerning metric 2-point identifications:

Lemma 6.4.3. Let $S \in \mathcal{S}(c)$ and X be a metric 2-point identification of S. Then there is a sequence of spaces in $\mathcal{S}(c+2)$ converging to X. Moreover the following statements apply:

1) The surfaces of the sequence may be chosen to be non-orientable.

2) If S is orientable, then the surfaces of the sequence may be chosen to be orientable.

Now we combine the last two results:

Lemma 6.4.4. Let Y be a space that can be obtained by a successive application of k metric 2-point identifications to a space $X \in \mathcal{G}(c)$. Then there is a sequence of spaces in $\mathcal{S}(c+2k)$ converging to Y. Moreover the following statements apply:

1) If there is a non-orientable maximal cyclic subset in X or $k > 0$, then the surfaces of the sequence may be chosen to be non-orientable.

2) If all maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.

Proof. The proof proceeds by induction over k:
In the case $k = 0$ the claim directly follows by Corollary 6.4.2.
Now we consider the case $k > 0$. Furthermore we assume that the claim is true for every $k_0 \in \mathbb{N}$ with $k_0 < k$. There is a space Z that can be obtained by a successive application of $k-1$ metric 2-point identifications to X such that Y is a metric 2-point identification of Z. By the induction hypothesis there is a sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c+2(k-1))$ with $Z_n \longrightarrow Z$ as in the claim.
Let $z_1, z_2 \in Z$ be the points that are glued to construct Y. There is a sequence $(z_{i,n})_{n\in\mathbb{N}}$ with $z_{i,n} \in Z_n$ converging to z_i. In particular, we may assume that $z_{1,n}$ and $z_{2,n}$ are distinct. If Y_n denotes a metric 2-point identification of Z_n along $z_{1,n}$ and $z_{2,n}$, then we have $Y_n \longrightarrow Y$.
Choosing a diagonal sequence, the claim follows by Lemma 6.4.3. □

Using a metric wedge sum with a vanishing sequence of length spaces that are all homeomorphic to the 2-torus or all homeomorphic to the real projective plane, we derive a further corollary of Lemma 6.4.1:

Corollary 6.4.5. Let $S \in \mathcal{S}(c)$. Then the following statements apply:

1) S can be obtained as the limit of non-orientable closed length surfaces whose connectivity number is equal to $c+1$.

2) If S is orientable, then S can be obtained as the limit of orientable closed length surfaces whose connectivity number is equal to $c+2$.

Now the last two results provide all tools to prove the final result of this section which refines the remaining direction of Main Theorem I:

Theorem 6.4.6. Let $c \in \mathbb{N}_0$ and Y be space that can be obtained by a successive application of k metric 2-point identifications to a space $X \in \mathcal{G}(c_0)$ where $c_0 \leq c - 2k$. Then there is a sequence in $\mathcal{S}(c)$ converging to Y. Moreover the following statements apply:

1) If all maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.

2) If there is a non-orientable maximal cyclic subset in X or $c_0 < c$, then the surfaces of the sequence may be chosen to be non-orientable.

We note that the last theorem and Theorem 6.3.6 completely describe the Gromov-Hausdorff closure of the class of length spaces that are homeomorphic to a fixed closed surface.
Finally we are able to prove Corollary 6.1.2:

Proof of Corollary 6.1.2. Let Y be a space that can be obtained by a successive application of metric 2-point identifications to a generalized cactoid. By Theorem 6.1.1 and Main Theorem I all such spaces are locally simply connected. Hence Proposition 4.0.2 and Proposition 6.2.9 imply that Y is simply connected if and only if Y is a cactoid.
From Theorem 6.4.6 follows that every geodesic cactoid can be obtained as the limit of length spaces that are homeomorphic to S. Now the claim follows by Main Theorem I and Corollary 6.2.4. $\square$

7 Scenario II: Compact surfaces with boundary

7.1 Introduction

In this chapter we completely describe the Gromov-Hausdorff limits of length spaces that are homeomorphic to compact surfaces of fixed connectivity number. In particular, we allow the possibility that the surfaces have non-empty boundary and thus generalize Main Theorem I. Our investigation builds on the results of Chapter 6 and extends the central concept of a generalized cactoid. As an additional technical difficulty, the limit of the boundaries may display a rather wild behavior. Even the statement of the main result (see Main Theorem II) is more complicated since new topological quantities appear.

7.1.1 Key results

It will turn out that the limit spaces satisfy the following topological properties:

Theorem 7.1.1. Let X be a space that can be obtained as the Gromov-Hausdorff limit of length spaces that are homeomorphic to a fixed compact surface. Then the following statements apply:

1) X is at most 2-dimensional.

2) X is locally simply connected.

3) There are finitely many compact surfaces $S_1, \dots, S_n$ and $k \in \mathbb{N}_0$ such that $\pi_1(X)$ is isomorphic to the free product $\pi_1(S_1) * \dots * \pi_1(S_n) * \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k\text{-times}}$.

Moreover we have the following local description of the limit spaces:

Theorem 7.1.2. Let X be a space that can be obtained as the Gromov-Hausdorff limit of length spaces that are homeomorphic to a fixed compact surface. Then every point of X admits an open neighborhood that is homeomorphic to an open subset of some Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc.

For the global description we extend the definition of a generalized cactoid:
Let X be a Peano space whose maximal cyclic subsets are compact surfaces and $C \subset X$ be a subcontinuum. Then C is denoted as *admissible* in X provided $T \cap C$ is a point or a boundary component of T for every maximal cyclic subset $T \subset X$.

Definition 7.1.3. Let X be a Peano space. Then X is called a *generalized cactoid (with boundary)* if the following statements apply:

1) All maximal cyclic subsets are compact surfaces and only finitely many of them are not homeomorphic to the 2-sphere or the closed 2-disc.

2) There are finitely many disjoint admissible subcontinua $C_1, \ldots, C_n \subset X$ such that the boundary components of the maximal cyclic subsets of X are covered by the subcontinua.

There exists a natural choice $C_1, \ldots, C_n$ of the admissible subcontinua as above which is uniquely defined by the following property: The number n is minimal and the union $\cup_{i=1}^{n} C_i$ is maximal among all choices with n admissible subcontinua (see Lemma 7.2.4). We define the *boundary* of X as $\cup_{i=1}^{n} C_i$ and denote it by ∂X. Further we say that C_i is a *boundary component* of X.
Especially we will see that the boundary components are 1-cactoids (see Lemma 7.2.1). According to the original definition of a generalized cactoid, all maximal cyclic subsets are supposed to be closed surfaces (cf. [32, p. 854]). The extension presented here, including the definition of the boundary, is completely new.
Compact surfaces are simple examples of generalized cactoids. For compact surfaces, the usual definition of the boundary coincides with the definition for generalized cactoids. A more advanced example of a generalized cactoid is shown in Figure 11.
The limit spaces we study are closely related to generalized cactoids. Our description makes use of the following concept: If we consider a space that can be obtained by a successive application of $k > 0$ metric 2-point identifications to some generalized cactoid X and $p_1, \ldots, p_k$ denotes a choice of the corresponding projection maps, then p_i is called a *boundary identification* provided it identifies two points of $(p_{i-1} \circ \ldots \circ p_0)(\partial X)$ where

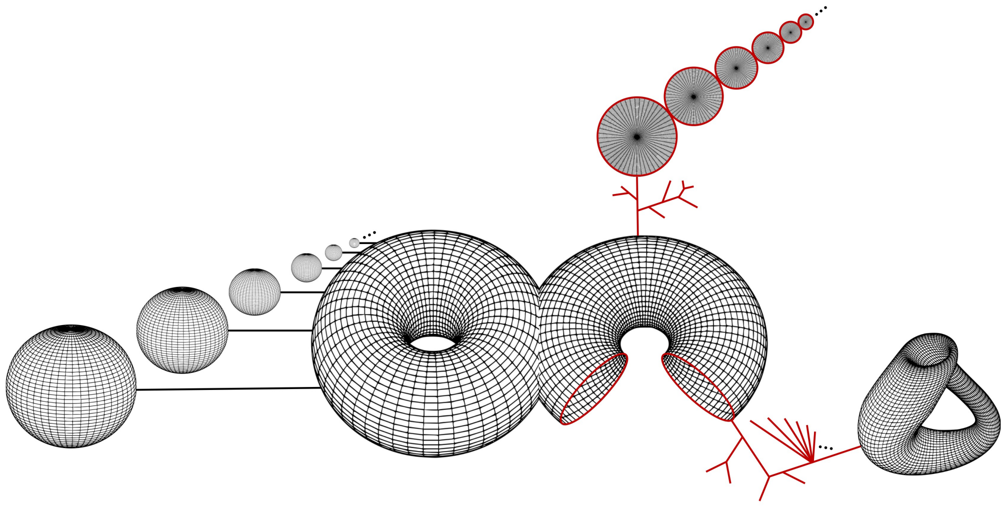

Figure 11: A generalized cactoid with three boundary components. All maximal cyclic subsets are compact surfaces and only two of them are not homeomorphic to the 2-sphere or the closed 2-disc. The boundary of the generalized cactoid is shown in red.

$p_0 := id_X$.

Finally we want to assign a topological quantity to each generalized cactoid: We define the *connectivity number* of a generalized cactoid as the sum of the reduced connectivity numbers of its maximal cyclic subsets and the number of its boundary components. For compact surfaces, the definition of the connectivity number from Section 5.2 coincides with the definition for generalized cactoids. The generalized cactoid in Figure 11 has a connectivity number of seven.

The main result of this chapter completely describes the Gromov-Hausdorff closure of the class of compact length surfaces whose connectivity number is fixed:

Main Theorem II. Let $c \in \mathbb{N}_0$ and X be a compact length space. Then the following statements are equivalent:

1) X can be obtained as the Gromov-Hausdorff limit of compact length surfaces whose connectivity number is equal to c.

2) There are $k, k_0 \in \mathbb{N}_0$ and a geodesic generalized cactoid Y such that the following statements apply:

a) X can be obtained by a successive application of k metric 2-point identifications to Y such that k_0 of them are boundary identifications.

b) We have $c_0 - k_0 + 2k \leq c$ where c_0 denotes the connectivity number of Y.

In general the choice of the generalized cactoid Y in the second statement is not unique and the quantity $c_0 - k_0 + 2k$ highly depends on this choice. Provided the converging surfaces are closed, Main Theorem I implies that the boundary of Y is always empty and hence $k_0 = 0$. The possible appearance of a non-empty boundary in Y marks the key difference between the main result of this chapter and that of Chapter 6.
For a better understanding of Main Theorem II, we want to discuss a simple example:

Example 7.1.4. Let $D \subset \mathbb{R}^2$ be the closed 2-disc and X_n be a metric quotient obtained by identifying $n \in \mathbb{N}$ distinct points of the boundary with the center of D. Then D is the only choice for the generalized cactoid Y. In particular, we have

$$c_0 - k_0 + 2k = 1 - (n-1) + 2n = n + 2.$$

As a consequence of Main Theorem II, X_n can be obtained as the Gromov-Hausdorff limit of compact length surfaces whose connectivity number is equal to $n+2$. Moreover it follows that $n+2$ is the smallest value for which the statement is true.

A more advanced example is illustrated in Figure 12.
Also some modifications of Main Theorem II are possible: Restricting the first statement of Main Theorem II to smooth Riemannian or polyhedral 2-manifolds, does not effect the validity of the equivalence. This is a direct consequence of the fact that every compact length surface S can be obtained as the limit of smooth Riemannian 2-manifolds that are homeomorphic to S and also of polyhedral surfaces that are homeomorphic to S (cf. [27, p. 1674], [31, p. 77]). Furthermore we will investigate how Main Theorem II changes if we restrict the first statement to orientable or non-orientable surfaces (see Theorem 7.3.12 and Theorem 7.4.10).
Beyond Main Theorem I, we are only aware of the following predecessors: In the 1930s, Whyburn showed the following result:

Theorem B (Whyburn). (cf. [38, p. 422]) Let X be a space that can be obtained as the limit of length spaces X_n that are homeomorphic to the closed 2-disc. Moreover let $\partial X_n \xrightarrow{\mathcal{H}} J$. Then the following statements apply:

1) The maximal cyclic subsets of X are homeomorphic to the 2-sphere or the closed 2-disc.

2) X is a generalized cactoid with at most one boundary component.

3) J is a subset of ∂X.

Furthermore Gromov states the first statement of Theorem 7.1.1 for orientable surfaces without proof and attributes it to Ivanov (cf. [17, p. 103]).

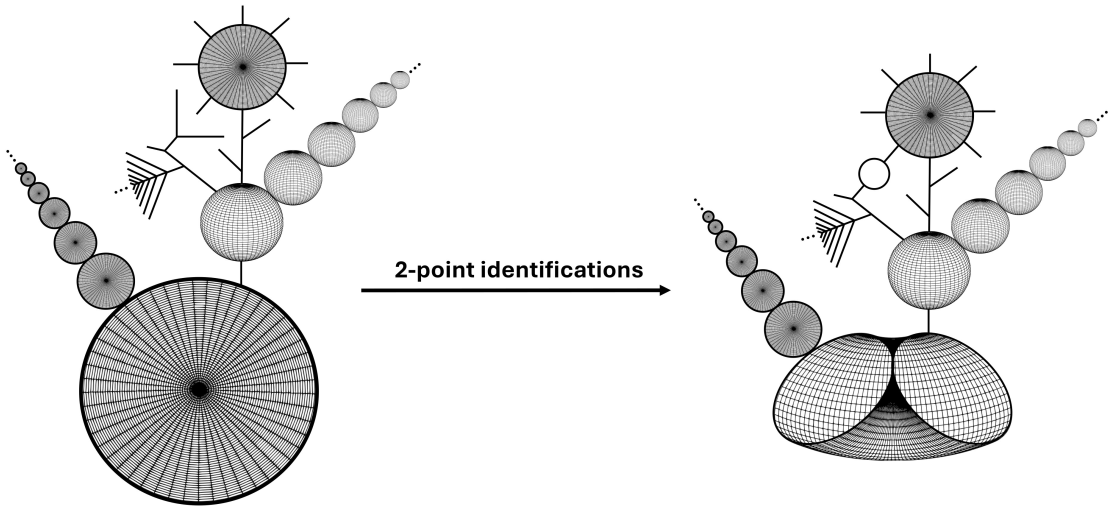

Figure 12: An illustration of Main Theorem II. On the left hand side we see a geodesic generalized cactoid with two boundary components whose maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc. We note that it has a connectivity number of two. A successive application of three metric 2-point identifications yields the space shown on the right hand side. In particular, we can do this using only one boundary identification. As a consequence of Main Theorem II, the space on the right hand side can be obtained as the Gromov-Hausdorff limit of compact length surfaces whose connectivity number is equal to seven.

7.1.2 Organisation

This chapter is organized as follows:
In the second section we show that the boundary of a generalized cactoid is well-defined and describe the topology of the boundary.
The aim of the third section is to show that the first statement of Main Theorem II

implies the second. To do this, we first look at sequences with additional topological control. We also prove the first two statements of Theorem 7.1.1 and give a proof of Theorem 7.1.2.
In Section 7.4 we treat the remaining direction of Main Theorem II. At the end of the section we show the third statement of Theorem 7.1.1.
We note that the final results of Section 7.3 and 7.4 particularly describe how Main Theorem II changes if we restrict the first statement to orientable or non-orientable surfaces.

7.1.3 Notation

We introduce the central notations of this chapter:

$\mathcal{M}$ The class of compact metric spaces.

$\mathcal{S}(c)$ The class of compact length surfaces whose connectivity is equal to c.

$\mathcal{S}(c, b)$ The class of compact length surfaces with b boundary components whose reduced connectivity number is equal to c.

$\mathcal{G}(c)$ The class of geodesic generalized cactoids whose connectivity number is equal c.

$\mathcal{G}(c, b)$ The class of geodesic generalized cactoids with b boundary components such that the reduced connectivity numbers of their maximal cyclic subsets sum up to c.

$\mathcal{W}$ The class of successive metric wedge sums of non-degenerate cyclicly connected compact length spaces and finite metric trees.

$\mathcal{W}_0$ The class of successive metric wedge sums of compact length surfaces such that every wedge point is only shared by two of their surfaces.

With regard to the last two notations, we note the following: In every construction step of a successive metric wedge sum we allow a change of the wedge point. An example of a space in $\mathcal{W}_0$ is illustrated in Figure 13.

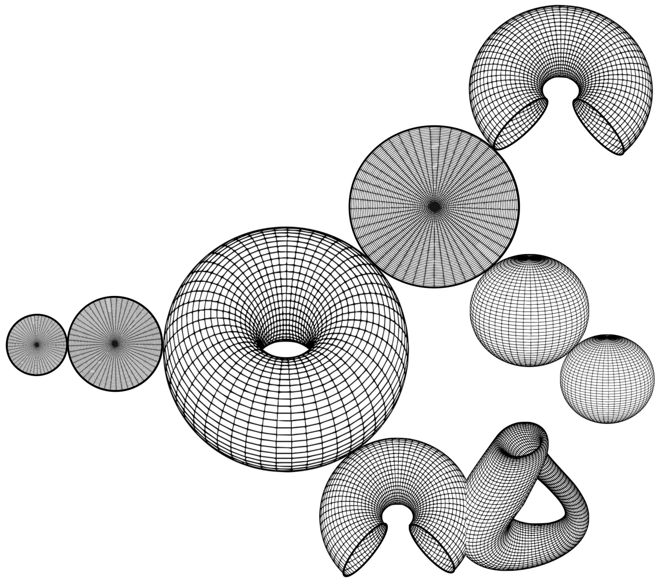

Figure 13: A successive metric wedge sum of eight compact length surfaces. Since every wedge point is only shared by two of the surfaces, the space lies in $\mathcal{W}_0$.

7.2 Generalized cactoids

In this section we deal with generalized cactoids.
First we describe the topology of admissible subcontinua:

Lemma 7.2.1. Let X be a Peano space whose maximal cyclic subsets are compact surfaces and $C \subset X$ be an admissible subcontinuum. Then C is a 1-cactoid.

Proof. Let $x, y \in C$ be conjugate. Then the points are also conjugate in X. Due to Lemma 3.2.1 the points lie in some maximal cyclic subset $T \subset X$. Since C is admissible, there is some boundary component $b \subset T$ such that $T \cap C = b$. Further we find an arc $\gamma \subset T$ satisfying the following property: The intersection of γ with b is given by $\{x, y\}$ and γ separates two points of b in T.
By Lemma 3.2.1 the arc also separates these points in X. It follows that $C \setminus \{x, y\}$ is disconnected. From Proposition 3.2.3 we derive that C is a 1-cactoid. □

We have the following technical lemma:

Lemma 7.2.2. Let X be a Peano space whose maximal cyclic subsets are compact surfaces. Then the following statements apply:

1) If $C_1, C_2 \subset X$ are admissible subcontinua intersecting exactly once, then $C := C_1 \cup C_2$ is an admissible subcontinuum.

2) Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of subcontinua in X which intersect every maximal cyclic subset at most once. If the sequence Hausdorff converges to some $C \subset X$, then C is a subcontinuum which intersects every maximal cyclic subset at most once.

Proof. 1) It directly follows that C is a continuum. Let $T \subset X$ be a maximal cyclic subset and $x, y \in T \cap C$ be distinct points. Then we may assume that $x \in C_1 \setminus C_2$.
For the sake of contradiction we further assume that $y \in C_2 \setminus C_1$. By Lemma 7.2.1 the subsets C_1 and C_2 are arcwise connected. Hence there is an arc $\gamma \subset C$ connecting x and y which is the union of non-degenerate arcs $\gamma_1 \subset C_1$ and $\gamma_2 \subset C_2$. From Lemma 3.1.4 we get $\gamma \subset T$. Because C_1 and C_2 are admissible, there are boundary component $b_1, b_2 \subset T$ with $T \cap C_1 = b_1$ and $T \cap C_2 = b_2$. Since γ is connected, we get $b_1 = b_2$. This contradicts the fact that $|C_1 \cap C_2| = 1$.
We conclude that $T \cap C = T \cap C_1$. Due to the fact that C_1 is admissible, it follows that $T \cap C$ is a boundary component of T. Therefore we derive that C is admissible.
2) We directly get that C is a continuum. For the sake of contradiction we assume the existence of a maximal cyclic subset $T \subset X$ and distinct points $x_1, x_2 \in T \cap C$. Then there is a sequence $(x_{i,n})_{n\in\mathbb{N}}$ converging to x_i such that $x_{i,n} \in C_n$ and $x_{1,n} \neq x_{2,n}$. By Proposition 3.0.1 we may assume that X is geodesic and we find a geodesic $\gamma_{i,n} \subset X$ connecting $x_{i,n}$ and x_i. Since $x_1 \neq x_2$, we may assume that the geodesics do not intersect. From Lemma 7.2.1 we derive that C_n is arcwise connected. Hence there is a non-degenerate arc $\alpha_n \subset C_n$ connecting $x_{1,n}$ and $x_{2,n}$. Due to the fact that $\gamma_{1,n}$ and $\gamma_{2,n}$ do not intersect, we may assume that α_n intersects $\gamma_{1,n} \cup \gamma_{2,n}$ only twice. Then $\gamma_n := \gamma_{1,n} \cup \alpha_n \cup \gamma_{2,n}$ is an arc and Lemma 3.1.4 yields $\gamma_n \subset T$. This contradicts the fact that C_n intersects every maximal cyclic subset at most once. □

Next we introduce some definitions: Let X be a Peano space whose maximal cyclic subsets are compact surfaces. Further let $C_1, \ldots, C_n \subset X$ be disjoint admissible subcontinua such that the boundary components of the maximal cyclic subsets are covered by the subcontinua. If every C_i contains a boundary component of some maximal cyclic subset, then we denote $\cup_{i=1}^n C_i$ as a *pre-boundary* of X. Provided the number n is minimal among all pre-boundaries of X, we say that the pre-boundary is *minimal*.
We note that the second property of Definition 7.1.3 can be restated as the existence of a pre-boundary. The following example illustrates this property:

Example 7.2.3. We consider a subset $X \subset \mathbb{R}^3$ which is the union of the 2-sphere and disjoint subsets $(D_n)_{n\in\mathbb{N}}$ that are homeomorphic to the closed 2-disc. Further we assume that $diam(D_n) \longrightarrow 0$ and that D_n intersects the 2-sphere exactly once for every $n \in \mathbb{N}$. Then X is a Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc. Hence the first property of Definition 7.1.3 is satisfied. But X is not a generalized cactoid since there is no pre-boundary in X.
For our second example we replace the 2-sphere in the construction above with a further subset $D \subset \mathbb{R}^3$ that is homeomorphic to the closed 2-disc. We denote this new subset by Y. Then Y is a Peano space whose maximal cyclic subsets are homeomorphic to the closed 2-disc. Hence the first property of Definition 7.1.3 is again satisfied. Moreover Y is a generalized cactoid if and only if ∂D_n intersects ∂D for all but finitely many $n \in \mathbb{N}$.

Finally we show that the boundary of a generalized cactoid is well-defined:

Lemma 7.2.4. Let X be a generalized cactoid. Then the union of all minimal pre-boundaries is a minimal pre-boundary.

Proof. Let $P \subset X$ be a minimal pre-boundary. We denote the set of all subcontinua in X which intersect every maximal cyclic subset at most once by $\mathcal{T}$. Further we define P_0 as the union of P with all subcontinua in $\mathcal{T}$ which intersect P exactly once.
We show that P_0 is a minimal pre-boundary: Since P covers the boundary components of the maximal cyclic subsets, the same applies to P_0. Moreover P_0 has at most as many connected components as P.
We prove that every connected component $C \subset P_0$ is compact: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in P_0 and $x \in X$ with $x_n \longrightarrow x$. By construction there is a subcontinuum $A_n \in \mathcal{T}$ which contains x_n and intersects P exactly once. After passing to a subsequence, we may assume that the subcontinua $(A_n)_{n\in\mathbb{N}}$ Hausdorff converge to some subcontinuum $A \subset X$.
Since P is compact, the intersection of A and P is non-empty. From Lemma 7.2.2 it follows that A intersects every maximal cyclic subset at most once. In particular, A is admissible and hence arcwise connected by Lemma 7.2.1. Therefore we find an arc $\gamma \subset A$ connecting x and P which intersects P exactly once. Especially we have $\gamma \in \mathcal{T}$ and we deduce $\gamma \subset P_0$. This yields $x \in P_0$.
We conclude that P_0 is closed. Because X is compact, we get that P_0 is compact. Hence C is also compact.
Due to the fact that the subcontinua in $\mathcal{T}$ and the connected components of P are admissible, Lemma 7.2.2 yields that C is admissible. We conclude that P_0 is a minimal

pre-boundary.

Next we show that every pre-boundary $A \subset X$ lies in P_0: For the sake of contradiction we assume that A is not a subset of P_0. Then we find some $p \in A \setminus P_0$. We denote the connected component of A containing p by C. The subset C contains a boundary component of some maximal cyclic subset. Moreover P covers the boundary components of the maximal cyclic subsets. Therefore C intersects P. Since C is admissible, it is arcwise connected. Hence there is an arc $\gamma \subset C$ starting in p whose endpoint e is the only intersection point of γ with P.

We note that $\gamma \setminus \{e\}$ does not intersect boundary components of maximal cyclic subsets. Due to the fact that C is admissible, we have $\gamma \in \mathcal{T}$. Finally we deduce $\gamma \subset P_0$ and therefore $p \in P_0$. A contradiction. □

We remark that the boundary components of a generalized cactoid are 1-cactoids by Lemma 7.2.1.

7.3 The limit spaces

In this section we prove that the first statement of Main Theorem II implies the second. We also show the first two statements of Theorem 7.1.1 and give a proof of Theorem 7.1.2.

7.3.1 Topological properties

First we prove that the limit spaces are at most 2-dimensional and locally simply connected. For this we introduce some notations:

Notation 7.3.1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c, b)$, where $b > 0$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$.

We denote the metric gluing of $X_n \sqcup X_n$ along ∂X_n by $2X_n$. Then the sequence $(2X_n)_{n\in\mathbb{N}}$ is convergent and we denote its limit by $2X$. Especially there are subsets $X^\pm \subset 2X$ and maps $\tau^\pm\colon 2X \to X^\pm$ such that the following statements apply:

1) $X^+ \cup X^- = 2X$.

2) $X^\pm$ is isometric to X.

3) The restriction of $\tau^\pm$ to $X^\pm$ is the identity map and the restriction to $X^\mp$ is an isometry.

4) The restriction of $\tau^{\pm} \circ \tau^{\mp}$ to $X^{\pm}$ is the identity map.

We fix some isometries as in the second statement. For every $A \subset X$ we denote its corresponding subset of $X^{\pm}$ by $A^{\pm}$. After passing to a subsequence, we may and will assume that $(\partial X_n)_{n\in\mathbb{N}}$ is convergent and we denote its limit by $\partial^{\infty} X$. Finally also the following property is satisfied:

5) $X^{+} \cap X^{-} = (\partial^{\infty} X)^{+} = (\partial^{\infty} X)^{-}$.

Now we show that the limit spaces fulfill the two topological properties:

Proof of Theorem 7.1.1 (Part I). There is a sequence $(X_n)_{n\in\mathbb{N}}$ of length spaces that are homeomorphic to a fixed compact surface such that $X_n \longrightarrow X$. If X_n is a closed surface, then X is at most 2-dimensional and locally simply connected by Proposition 6.1.1. Hence we may assume that the boundary of X_n is non-empty.
1) We note that $2X_n$ is a closed surface. Hence $2X$ is at most 2-dimensional. Since we have $X^{+} \subset 2X$, we derive that X^{+} is also at most 2-dimensional (cf. [34, p. 266]). Because X and X^{+} are isometric, we deduce that X is at most 2-dimensional.
2) The statements listed in Notation 7.3.1 imply the following: Let $V \subset 2X$ be an open and simply connected subset. Since V is open, the same applies to $\tau^{+}(V)$. Moreover every loop in $\tau^{+}(V)$ is the composition of τ^{+} with some loop in V. The map τ^{+} is continuous. Due to the fact that V is simply connected, we hence get that $\tau^{+}(V)$ is simply connected. We further note that the diameter of $\tau^{+}(V)$ does not exceed that of V.
Because $2X_n$ is a closed surface, the space $2X$ is locally simply connected. It follows that X^{+} is locally simply connected. Therefore the same applies to X. □

7.3.2 Regular convergence

In this subsection we consider sequences in $\mathcal{S}(c, b)$ with additional topological control:

Definition 7.3.2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c, b)$. Then the sequence is called *regular* if one of the following cases applies:

1) $b = 0$ and $(X_n)_{n\in\mathbb{N}}$ is uniformly semi-locally 1-connected.

2) $b > 0$ and $(2X_n)_{n\in\mathbb{N}}$ is uniformly semi-locally 1-connected.

Provided the sequence $(X_n)_{n\in\mathbb{N}}$ converges to some $X \in \mathcal{M}$, the definition directly implies that $\partial^{\infty} X$ has b connected components.

The next result states a property of non-regular sequences. We remark that Notation 7.3.1 can also be applied to the constant sequence $(X_n, X_n, \ldots)$. In the upcoming proof we use this notation and denote the corresponding maps by $\tau_n^{\pm}$.

Lemma 7.3.3. Let $(X_n)_{n\in\mathbb{N}}$ be a non-regular sequence in $\mathcal{S}(c,b)$ where $b > 0$. After passing to a subsequence, we may assume that one of the following cases applies:

1) There is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that γ_n is a non-separating simple arc in X_n. Moreover we have $diam(\gamma_n) \longrightarrow 0$.

2) There is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that γ_n is a separating simple arc in X_n which does not form a contractible Jordan curve together with a subarc of some boundary component. Moreover we have $diam(\gamma_n) \longrightarrow 0$.

3) There is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that γ_n is a non-contractible simple Jordan curve in X_n. Moreover we have $diam(\gamma_n) \longrightarrow 0$.

Proof. We consider the case that the first two statements of the lemma do not apply. By non-regularity we may assume the existence of a sequence $(J_n)_{n\in\mathbb{N}}$ such that J_n is a non-contractible Jordan curve in $2X_n$ and $diam(J_n) \longrightarrow 0$. Since the first statement of the lemma does not apply, we may assume that J_n intersects exactly one boundary component $b_n \subset X_n^+$ and the intersection is non-degenerate.
There is a homeomorphism $f_n\colon 2X_n \to Y_n$ such that Y_n is a Riemannian manifold and $f_n(b_n)$ is a piecewise geodesic Jordan curve. We note that the Jordan curve $f_n(J_n)$ can be obtained as the Hausdorff limit of piecewise geodesic Jordan curves that are homotopic to $f_n(J_n)$ (cf. [35, p. 1794], [38, pp. 413-415]). Since Y_n is a compact Riemannian manifold, there is some $\varepsilon_n > 0$ such that every pair of points in Y_n with distance less than ε_n can be connected by a unique geodesic. It follows that two distinct geodesic Jordan curves in Y_n intersect at most finitely many times.
By the observations above we may assume that J_n and b_n intersect only finitely many times. Then there is a finite subdivision of J_n into simple arcs in X_n^+ and X_n^- and arcs in b_n. Because the second statement of the lemma does not apply, we may assume that each of the arcs is homotopic to some arc in b_n. This yields that J_n is homotopic to some loop in b_n.
We derive that $\gamma_n := \tau_n^+(J_n)$ is a non-contractible loop in X_n^+ and $diam(\gamma_n) \longrightarrow 0$. In particular, we may assume that γ_n does not intersect ∂X_n^+. Moreover we find a Jordan curve in γ_n which is non-contractible in X_n^+ (cf. [24, p. 626]). Therefore we finally may assume that γ_n is a Jordan curve. □

Limits of boundary components

As a first step we investigate the limits of boundary components for regular sequences. In the next three results we extend Whyburn's proof ideas regarding the limits of discs (cf. [38, pp. 421-424]):

Proposition 7.3.4. Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c,q)$, where $q > 0$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. If b is a connected component of $\partial^\infty X$, then b is a 1-cactoid.

Proof. By regularity there is a sequence $(b_n)_{n\in\mathbb{N}}$ such that b_n is a boundary component of X_n and $b_n \longrightarrow b$. Since b can be obtained as the Hausdorff limit of continua, it is also a continuum.
For the sake of contradiction we assume that b is not a 1-cactoid. From Proposition 3.2.3 it follows the existence of conjugate points $x, y \in b$ such that $b \setminus \{x, y\}$ is connected. There are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with $x_n, y_n \in b_n$ and $x_n \neq y_n$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$. We denote the subarcs of b_n connecting x_n and y_n by α_n and β_n. Moreover we may assume that there are $\alpha, \beta \subset b$ such that $\alpha_n \longrightarrow \alpha$ and $\beta_n \longrightarrow \beta$.
Then we have $\alpha \cup \beta = b$ and there exists $z \in \alpha \cap \beta \setminus \{x, y\}$. Further we choose sequences $(z_n)_{n\in\mathbb{N}}$ and $(\tilde{z}_n)_{n\in\mathbb{N}}$ with $z_n \in \alpha_n$ and $\tilde{z}_n \in \beta_n$ such that $z_n \longrightarrow z$ and $\tilde{z}_n \longrightarrow z$.
Let $\gamma_n \subset X_n$ be a geodesic between z_n and $\tilde{z}_n$. After passing to a subsequence and subarcs of the geodesics, we may assume γ_n to be a simple arc. By regularity and $diam(\gamma_n) \longrightarrow 0$ we also may assume that γ_n is separating.
Now there are compact surfaces $U_n, V_n \subset X_n$ such that $x_n \in U_n$, $y_n \in V_n$, $U_n \cup V_n = X_n$ and $U_n \cap V_n = \gamma_n$. We may assume the corresponding sequences to be convergent with limits U and V. By Lemma 2.3.1 this leads to $U \cup V = X$ and $U \cap V = \{z\}$. Finally we derive that z separates x and y in X and therefore also in b. A contradiction. □

The proof above also demonstrates the following lemma:

Lemma 7.3.5. Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c,q)$, where $q > 0$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. If b is a connected component of $\partial^\infty X$ and $x, y, z \in b$ are such that z separates x and y in b, then z separates x and y in X.

In the next two results we study the intersections with maximal cyclic subsets of $2X$:

Lemma 7.3.6. Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c,q)$, where $q > 0$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. If b is a connected component of $\partial^\infty X$ and T is a maximal cyclic subset of $2X$ with $|T \cap b^+| > 1$, then the intersection is homeomorphic to the 1-sphere.

Proof. Let $x, y \in T \cap b^+$ with $x \neq y$. Since $x, y \in T$, the points lie on some Jordan curve in $2X$. If $\gamma \subset 2X$ is a path between x and y which does not contain a certain point of b^+, then $\tau^+ \circ \gamma$ is also such a path. Therefore Lemma 7.3.5 implies that x and y are conjugate in b^+.
By Proposition 7.3.4 the subset b^+ is a 1-cactoid. From Lemma 3.2.1 we derive that x and y are contained in some maximal cyclic subset $S \subset b^+$. We note that S is homeomorphic to the 1-sphere. Using Lemma 3.1.4 and Lemma 3.2.1, we get $T \cap b^+ \subset S$ and $S \subset T$. Especially we have $S \subset T \cap b^+$ and therefore $S = T \cap b^+$. □

Lemma 7.3.7. Let $(X_n)_{n \in \mathbb{N}}$ be a regular sequence in $\mathcal{S}(c, q)$, where $q > 1$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. If b is a connected component of $\partial^\infty X$, then there is a maximal cyclic subset of $2X$ which intersects b^+ and a further connected component of $(\partial^\infty X)^+$.

Proof. By regularity there is a sequence $(b_n)_{n \in \mathbb{N}}$ such that b_n is a boundary component of X_n and $b_n \longrightarrow b$. We choose a geodesic $\gamma_n \subset X_n$ between some point of b_n and some point of $\partial X_n \setminus b_n$. Then we may assume the existence of a geodesic $\gamma \subset X$ such that $\gamma_n \longrightarrow \gamma$.
Due to regularity γ connects some point of b with some point of $\partial^\infty X \setminus b$. After passing to a subarc, we may assume that the interior of γ does not intersect $\partial^\infty X$. It follows that $J := \gamma^+ \cup \gamma^-$ is a non-degenerate Jordan curve in $2X$. By Lemma 3.2.1 there is a maximal cyclic subset $T \subset 2X$ containing J. We conclude that T intersects b^+ and a further connected component of $(\partial^\infty X)^+$. □

Regular limit spaces

Now we describe the limits of regular sequences:

Lemma 7.3.8. Let $(X_n)_{n \in \mathbb{N}}$ be a regular sequence in $\mathcal{S}(c, b)$, where $b > 0$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Further let T be a maximal cyclic subset of X^+. Then one of the following cases applies:

1) T is a maximal cyclic subset of $2X$ and a closed surface. Moreover we have $\left|T \cap (\partial^\infty X)^+\right| \leq 1$.

2) $T \cup \tau^-(T)$ is a maximal cyclic subset of $2X$ and T is a compact surface with non-empty boundary. Moreover we have $\partial T = T \cap (\partial^\infty X)^+$.

Also the following statement applies: Every maximal cyclic subset of $2X$ which is not a maximal cyclic subset of X^+ or X^- can be obtained as in the second case.

Proof. Since the sequence is regular, Theorem A (Whyburn) and Proposition 6.3.4 imply that every maximal cyclic subset of $2X$ is a closed surface.
First we consider the case that T is a maximal cyclic subset of $2X$: For the sake of contradiction we assume $\left|T \cap (\partial^\infty X)^+\right| > 1$. Then Proposition 3.2.2 implies that $T \cup \tau^-(T)$ is cyclicly connected. Further we have $T \neq \tau^-(T)$ by Proposition 7.3.4. Hence T is a proper subset of some cyclicly connected subset in $2X$. A contradiction.
Now we consider the case that T is not a maximal cyclic subset of $2X$: By Lemma 3.2.1 there is a maximal cyclic subset $S \subset 2X$ containing T.
Let V be the closure of a connected component of $S \setminus (\partial^\infty X)^+$. Then we may assume $V \subset X^+$. As a consequence of Lemma 7.3.6, the subset V is a compact surface. It also follows that $V \cap (\partial^\infty X)^+$ is a disjoint union of ∂V and k points. In particular, ∂V is non-empty since S is a closed surface. Due to the fact that $W := V \cup \tau^-(V)$ is cyclicly connected, Lemma 3.1.4 and Lemma 3.2.1 yield $W \subset S$. Because S is a closed surface, we derive $k = 0$ and $S = W$. This implies $V = X^+ \cap S$ and therefore $T \subset V$. We note that V is cyclicly connected. Hence we get $T = V$.
Using Lemma 3.1.4 and Lemma 3.2.1, the paragraph above also implies the last statement of our result. □

It follows the main result of this subsection:

Proposition 7.3.9. Let $(X_n)_{n\in\mathbb{N}}$ be a regular sequence in $\mathcal{S}(c, b)$ where $b > 0$ and $c + b > 1$. Further let $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then X is a compact length space satisfying the following properties:

1) All but one maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc and one maximal cyclic subset is homeomorphic to X_n for all but finitely many $n \in \mathbb{N}$.

2) X is a generalized cactoid with b boundary components and $\partial^\infty X \subset \partial X$.

Proof. From Proposition 6.3.4 and Lemma 7.3.8 we get the following: All but one maximal cyclic subsets of X are homeomorphic to the 2-sphere or the closed 2-disc. Moreover one maximal cyclic subset $T \subset X$ is a compact surface with non-empty boundary whose connectivity number is equal to $c + b$. We also have that T is orientable if and only if X_n is orientable for all but finitely many $n \in \mathbb{N}$.
By regularity $\partial^\infty X$ has b connected components. Combining Lemma 7.3.7 and Lemma 7.3.8, we derive that T has b boundary components. Hence the reduced connectivity number of T is equal to c. This yields that T is homeomorphic to X_n for all but finitely

many $n \in \mathbb{N}$.
Moreover the connected components of $\partial^\infty X$ are disjoint subcontinua of X. Due to Lemma 7.3.6 and Lemma 7.3.8 the subcontinua are admissible and they cover the maximal cyclic subsets of X. Therefore X is a generalized cactoid.
Since T has b boundary components, the pre-boundary $\partial^\infty X$ is minimal. We conclude that X has b boundary components and $\partial^\infty X \subset \partial X$. □

7.3.3 The general case

We already described the limits of closed length surfaces and regular sequences. Now we investigate non-regular sequences: Let $(X_n)_{n\in\mathbb{N}}$ be a non-regular sequence in $\mathcal{S}(c,b)$ where $b > 0$. Further let $X \in \mathcal{M}$ with $X_n \longrightarrow X$.
By non-regularity we may assume that there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of simple arcs or simple Jordan curves as in Lemma 7.3.3. Since the diameters of the curves vanish, the metric quotients X_n/γ_n converge to X. From Proposition 5.3.1, Proposition 5.3.2 and Proposition 5.3.4 we get a topological description of these quotient spaces. Using this description, we derive the following two results:

Lemma 7.3.10. If the curves of the sequence $(\gamma_n)_{n\in\mathbb{N}}$ are simple arcs, then one of the following cases applies:

1) There are $c_1, c_2 \in \mathbb{N}_{\geq 2}$ with $c_1 + c_2 = c + 1$ and a sequence $(Y_{i,n})_{n\in\mathbb{N}}$ in $\mathcal{S}(c_i)$ converging to some $Y_i \in \mathcal{M}$ such that X is isometric to a metric wedge sum of Y_1 and Y_2. Furthermore the wedge point lies in $\partial^\infty Y_1 \cap \partial^\infty Y_2$ and we find a corresponding isometry p such that $p(\partial^\infty Y_1 \cup \partial^\infty Y_2) = \partial^\infty X$.
Provided X_n is non-orientable for infinitely many $n \in \mathbb{N}$, the surfaces of at least one of the sequences may be chosen to be non-orientable.

2) There is a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c-1)$ converging to some $Y \in \mathcal{M}$ such that X is isometric to Y or a metric 2-point identification of it. Furthermore the glued points lie in $\partial^\infty Y$ and we find a corresponding isometry or projection map p such that $p(\partial^\infty Y) = \partial^\infty X$.

If X_n is orientable for infinitely many $n \in \mathbb{N}$, then the surfaces of the sequences above may be chosen to be orientable.

Lemma 7.3.11. If the curves of the sequence $(\gamma_n)_{n\in\mathbb{N}}$ are simple Jordan curves, then one of the following cases applies:

1) There are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$ and a sequence $(Y_{i,n})_{n\in\mathbb{N}}$ in $\mathcal{S}(c_i)$ converging to some $Y_i \in \mathcal{M}$ such that X is isometric to a metric wedge sum of Y_1 and Y_2. Furthermore we find a corresponding isometry p such that $p(\partial^\infty Y_1 \cup \partial^\infty Y_2) = \partial^\infty X$.

   Provided X_n is non-orientable for infinitely many $n \in \mathbb{N}$, the surfaces of at least one of the sequences may be chosen to be non-orientable.

2) There is a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c-2)$ converging to some $Y \in \mathcal{M}$ such that X is isometric to Y or a metric 2-point identification of it. Furthermore we find a corresponding isometry or projection map p such that $p(\partial^\infty Y) = \partial^\infty X$.

3) There is a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c-1)$ converging to some $Y \in \mathcal{M}$ such that X is isometric to Y. Furthermore we find a corresponding isometry p such that $p(\partial^\infty Y) = \partial^\infty X$.

If X_n is orientable for infinitely many $n \in \mathbb{N}$, then always one of the first two cases applies and the surfaces of the corresponding sequences may be chosen to be orientable.

Now we prove that the first statement of Main Theorem II implies the second. In particular, we describe what happens if we restrict ourselves to orientable or non-orientable surfaces:

Theorem 7.3.12. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then there are $k, k_0 \in \mathbb{N}_0$ and a space $Y \in \mathcal{G}(c_0)$, where $c_0 - k_0 + 2k \leq c$, such that the following statements apply:

1) X can be obtained by a successive application of k metric 2-point identifications to Y such that k_0 of them are boundary identifications.

2) If X_n is orientable for infinitely many $n \in \mathbb{N}$, then the maximal cyclic subsets of Y are orientable.

3) If X_n is non-orientable for infinitely many $n \in \mathbb{N}$ and the maximal cyclic subsets of Y are orientable, then $c_0 < c$.

Proof. First we add a statement to the claim: There is a choice $p_1, \ldots, p_k$ of the corresponding projection maps such that $\partial^\infty X \subset (p_k \circ \ldots \circ p_0)(\partial Y)$ where $p_0 := id_Y$.
We show the claim using an induction over the connectivity number:
The case $c = 0$ is a direct consequence of Theorem A (Whyburn).

Now let $c > 0$. For the sake of induction, we assume that the claim is true for connectivity numbers less than c. If the sequence $(X_n)_{n\in\mathbb{N}}$ contains infinitely many closed surfaces or is regular, then the claim follows from Theorem B (Whyburn), Theorem 6.3.6 or Proposition 7.3.9. Therefore we may assume that one of the cases in Lemma 7.3.10 or Lemma 7.3.11 applies.
The surfaces of the sequences appearing there have a connectivity number less than c. Hence we can apply the induction hypothesis and derive the claim. □

Finally we are able to prove the local description of the limit spaces:

Proof of Theorem 7.1.2. Due to Theorem 7.3.12 there is some generalized cactoid Y such that X is homeomorphic to a topological quotient of Y whose underlying equivalence relation identifies only finitely many points. We denote the number of maximal cyclic subsets in Y that are not homeomorphic to the 2-sphere or the closed 2-disc by k.
First we show the following claim: Every $y \in Y$ admits an open neighborhood that is homeomorphic to an open subset of some Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc.
In the case $k = 0$ the claim follows directly.
If $k = 1$, then we may assume that y lies in the maximal cyclic subset $T \subset Y$ that is not homeomorphic to the 2-sphere or the closed 2-disc. There is a neighborhood D of y in T that is homeomorphic to the closed 2-disc. We denote the union of the connected components of $Y \setminus T$ whose closures intersect D by A. It follows that $Z := D \cup A$ is a Peano space whose maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc.
Moreover there is an open neighborhood V of y in T that is contained in D. We denote the union of the connected components of $Y \setminus T$ whose closures intersect V by B. Then $V \cup B$ is an open neighborhood of y in Y and Z. This yields the claim.
If $k \geq 2$, then Y is a wedge sum of Peano spaces satisfying the following property: All maximal cyclic subsets are compact surfaces and less than k of them are not homeomorphic to the 2-sphere or the closed 2-disc.
Provided both spaces locally look like Peano spaces whose maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc, the same applies to Y. Hence the claim follows by induction.
By the claim X locally looks like a successive wedge sum of Peano spaces whose maximal cyclic subsets are homeomorphic to the 2-sphere or the closed 2-disc. We note that such a successive wedge sum is then also a Peano space of this kind. □

7.4 Approximation of generalized cactoids

In this chapter we show that the second statement of Main Theorem II implies the first. For this we successively reduce the complexity of the problem. We also prove the third statement of Theorem 7.1.1.

7.4.1 Approximation by surface gluings

The goal of this subsection is to approximate generalized cactoids by suitable spaces in $\mathcal{W}_0$. Our construction extends over the next three results:

Lemma 7.4.1. Let $X \in \mathcal{G}(c,b)$ and $(T_k)_{k\in\mathbb{N}}$ be an enumeration of its maximal cyclic subsets. Then X can be obtained as the limit of compact length spaces $(X_n)_{n\in\mathbb{N}}$ satisfying the following properties:

1) X_n has only finitely many maximal cyclic subsets. Moreover the maximal cyclic subsets of X_n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^n$.

2) After passing to a subsequence, we may assume the existence of a pre-boundary $\partial^* X_n \subset X_n$ with b connected components such that $\partial^* X_n \longrightarrow \partial X$.

Proof. We extend the proof of Lemma 6.2.5: First we define an equivalence relation on X by: $x \sim y$ if and only if x and y lie in the same connected component of $\cup_{m=n+1}^{n+k} T_m$. Moreover we denote the corresponding metric quotient by $X_{n,k}$ and the corresponding projection map by $p_{n,k}$. As shown in the reference, after passing to a subsequence, we may assume that there is a space $X_n \in \mathcal{M}$ and a map $p_n \colon X \to X_n$ such that $X_{n,k} \longrightarrow X_n$ and $p_{n,k} \longrightarrow p_n$ uniformly.
From the reference we also already know the following: The maximal cyclic subsets of X_n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^n$ via the map p_n. Further we have $X_n \longrightarrow X$.
Now we show that $p_n(\partial X)$ is a pre-boundary for infinitely many $n \in \mathbb{N}$: Let C be a connected component of ∂X. Since p_n is continuous, $p_n(C)$ is a subcontinuum of X_n. Further let $m \in \{1, \dots, n\}$ and $x_1, x_2 \in p_n(C) \cap p_n(T_m)$ with $x_1 \neq x_2$. Then there are $c_i \in C$ and $t_i \in T$ such that $p_n(c_i) = p_n(t_i) = x_i$. Moreover we choose a geodesic $\gamma_i \subset X$ between c_i and t_i and derive $p_n(\gamma_i) = \{x_i\}$. Since $x_1 \neq x_2$, the geodesics do not intersect. From Lemma 7.2.1 it follows that C is arcwise connected and we find a non-degenerate arc $\alpha \subset C$ connecting c_1 and c_2. Because γ_1 and γ_2 do not intersect, we may assume that α intersects $\gamma_1 \cup \gamma_2$ only twice. We derive that $\gamma \coloneqq \gamma_1 \cup \alpha \cup \gamma_2$ is an arc and Lemma

3.1.4 implies $\gamma \subset T$. Finally we conclude $x_i \in p_n(T \cap C)$.
The observation above implies the following: If the subset $p_n(C) \cap p_n(T_m)$ is non-degenerate, then it equals $p_n(C \cap T_m)$. Because C is admissible and p_n is an isometry on T_m, we deduce that $p_n(C)$ is admissible.
Due to the fact that C contains a boundary component of some maximal cyclic subset, we may assume that the same applies to $p_n(C)$. Moreover $p_n(\partial X)$ covers the boundary components of the maximal cyclic subsets as ∂X does.
The map p_n is 1-lipschitz. After passing to a subsequence, we hence may assume $(p_n)_{n\in\mathbb{N}}$ to be convergent. We denote its limit by p and it follows that p is an isometry. Further we may assume the sequence $(p_n(\partial X))_{n\in\mathbb{N}}$ to be convergent. Since $p(\partial X) = \partial X$, we have $p_n(\partial X) \longrightarrow \partial X$. Therefore we may assume that ∂X has as many connected components as $p_n(\partial X)$. We finally deduce that $p_n(\partial X)$ is a pre-boundary with b connected components. □

Lemma 7.4.2. Let X be a geodesic generalized cactoid having only finitely many maximal cyclic subsets. Further let $\partial^* X \subset X$ be a pre-boundary with b connected components. Then X can be obtained as the Hausdorff limit of compact subsets $(X_n)_{n\in\mathbb{N}}$ satisfying the following properties:

1) We have $X_n \in \mathcal{W}$ and the maximal cyclic subsets of X_n are equal to those of X.

2) There is a pre-boundary $\partial^* X_n \subset X_n$ with b connected components.

3) The sequence $(\partial^* X_n)_{n\in\mathbb{N}}$ Hausdorff converges to $\partial^* X$.

Proof. We extend the proof of Lemma 6.2.6: First we define ε as the minimum of the diameters of the maximal cyclic subsets in X. For every maximal cyclic subset $T \subset X$ we remove the connected components of $X \setminus T$ whose diameters are less than ε/n. We denote the constructed subset by Y_n.
The subset Y_n is a compact length space. Furthermore it is a successive metric wedge sum of its maximal cyclic subsets and compact metric trees $D_1, \dots, D_k$. For every metric tree D_i there is a finite metric tree $F_i \subset D_i$ whose Hausdorff distance to D_i is less than ε/n (cf. [6, p. 267]). In particular, we may assume that F_i intersects the same maximal cyclic subsets as D_i. Next we define X_n as the union of the maximal cyclic subsets of Y_n and the finite metric trees. Moreover we set $\partial^* X_n := \partial^* X \cap X_n$.
By construction we have $X_n \in \mathcal{W}$. Further the maximal cyclic subsets of X_n are equal to those of X and $\partial^* X_n$ is a pre-boundary of X_n with b connected components. Finally the sequences $(X_n)_{n\in\mathbb{N}}$ and $(\partial^* X_n)_{n\in\mathbb{N}}$ Hausdorff converge to X and $\partial^* X$. □

Lemma 7.4.3. Let X be a generalized cactoid in $\mathcal{W}$. Further let $\partial^* X \subset X$ be a pre-boundary with b connected components. Then there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{W}_0$ satisfying the following properties:

1) The maximal cyclic subsets of X_n are in isometric one-to-one correspondence with the maximal cyclic subsets of X and finitely many length spaces that are homeomorphic to the 2-sphere or the closed 2-disc.

2) X_n has b boundary components.

3) There is an ε_n-isometry $f_n \colon X_n \to X$ such that $f_n(\partial X_n) = \partial^* X$ and $\varepsilon_n \longrightarrow 0$.

Proof. The space X is a successive metric wedge sum of its maximal cyclic subsets and compact intervals. In particular, we may assume that the wedge points do not lie in the interior of the intervals and that every interval whose interior intersects $\partial^* X$ lies in $\partial^* X$.

We consider the following construction: Let e be one of the wedged intervals. Then there is a $1/n$-isometry $f \colon D \to e$ satisfying the following properties: The preimages of the endpoints of e contain exactly one point. If e lies in $\partial^* X$, then D is a length space that is homeomorphic to the closed 2-disc and $f(\partial D) = e$. Otherwise D is a length space that is homeomorphic to the 2-sphere.

Now we remove the interior of e from X and paste D along f. This yields a compact length space Y. Especially the maximal cyclic subsets of Y are in isometric one-to-one correspondence with the set consisting of D and the maximal cyclic subsets of X.

If e lies in $\partial^* X$, then we set $\partial^* Y := (\partial^* X \setminus e) \cup \partial D$. Otherwise we set $\partial^* Y := \partial^* X$. We note that $\partial^* Y$ is a pre-boundary of Y with b connected components. Provided Y is a successive metric wedge sum of its maximal cyclic subsets, we have $\partial^* Y = \partial Y$.

Furthermore the map f naturally induces $1/n$-isometry $g \colon Y \to X_n$ with $g(\partial^* Y) = \partial^* X$.

We successively repeat this construction until there is no wedged interval left and denote the constructed space by X_n.

The space X_n is a successive metric wedge sum of its maximal cyclic subsets. For every wedge point $p \in X_n$ which lies in more than one maximal cyclic subset there is a $1/n$-isometry $f \colon D \to \{p\}$ satisfying the following properties: If p lies in ∂X_n, then D is a length space that is homeomorphic to the closed 2-disc. Otherwise D is a length space that is homeomorphic to the 2-sphere.

Using a similar construction as above, we finally may assume that $X_n \in \mathcal{W}_0$. □

We note that a sequence of ε_n-isometries between converging spaces has a convergent subsequence provided $\varepsilon_n \longrightarrow 0$. In particular, its limit is an isometry between the limit spaces. Morever the boundary of a generalized cactoid is invariant under self-isometries. Combining Proposition 2.2.1 and the last three results, we hence get the desired approximating sequence:

Corollary 7.4.4. Let $X \in \mathcal{G}(c,b)$ and $(T_k)_{k\in\mathbb{N}}$ be an enumeration of its maximal cyclic subsets. Then X can be obtained as the limit of spaces $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{W}_0$ satisfying the following properties:

1) The maximal cyclic subsets of X_n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^n$ and finitely many length spaces that are homeomorphic to the 2-sphere or the closed 2-disc.

2) After passing to a subsequence, we may assume that X_n has b boundary components and $\partial X_n \longrightarrow \partial X$.

7.4.2 Elementary surface gluings

Now we provide useful tools concerning the approximation of elementary surface gluings. We start with wedge sums:

Lemma 7.4.5. Let $S_1 \in \mathcal{S}(c_1,b_1)$, $S_2 \in \mathcal{S}(c_2,b_2)$ and X be a metric wedge sum of S_1 and S_2. Then the following statements apply:

1) There is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c_1+c_2,b_1+b_2)$ and an ε_n-isometry $f_n\colon X_n \to X$ such that $f_n(\partial X_n) = \partial X$ and $\varepsilon_n \longrightarrow 0$.

2) If the wedge point is contained in $\partial S_1 \cap \partial S_2$, then there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c_1+c_2,b_1+b_2-1)$ and an ε_n-isometry $f_n\colon X_n \to X$ such that $f_n(\partial X_n) = \partial X$ and $\varepsilon_n \longrightarrow 0$.

If S_1 and S_2 are orientable, then the surfaces of the sequence may be chosen to be orientable. Provided at least one of the wedged surfaces is non-orientable, the surfaces of the sequence may be chosen to be non-orientable.

Proof. 1) We may assume the wedge points not to lie in $\partial S_1 \cup \partial S_2$. In Lemma 6.4.1 we already showed the statement for closed surfaces. The corresponding proof does not depend on the fact that the wedged surfaces are closed and it gives rise to a proof for the general case.

2) Let a_1 and a_2 be the intersecting boundary components of the surfaces and p be the wedge point. We choose an arc $\gamma_{i,n} \subset a_i$ containing p in its interior. In particular, we may assume that the arc is a geodesic of length $1/n$ such that p is its midpoint. Next we define X_n as the metric gluing of X along $\gamma_{1,n}$ and $\gamma_{2,n}$. Moreover we denote the corresponding projection map by g_n and find a map $f_n\colon X_n \to X$ such that $f_n \circ g_n$ is the identity map on $X_n \setminus (\gamma_{1,n} \cup \gamma_{2,n})$ and $(f_n \circ g_n)(\gamma_{1,n}) = \gamma_{1,n} \cup \gamma_{2,n}$. Finally we deduce that the space X_n and the map f_n satisfy the desired properties. □

Using similar arguments as above, we derive the following result concerning metric 2-point identifications:

Lemma 7.4.6. Let S be a space in $\mathcal{S}(c)$ and X be a metric 2-point identification of S. Further let p be a corresponding projection map. Then the following statements apply:

1) There is sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c+2)$ and an ε_n-isometry $f_n\colon X_n \to X$ with $f_n(\partial X_n) = p(\partial S)$ and $\varepsilon_n \longrightarrow 0$.

2) If p is a boundary identification, then there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c+1)$ and an ε_n-isometry $f_n\colon X_n \to X$ with $f_n(\partial X_n) = p(\partial S)$ and $\varepsilon_n \longrightarrow 0$.

The surfaces of the sequence may be chosen to be non-orientable. If S is orientable, then the surfaces of the sequence may be chosen to be orientable.

7.4.3 Gluings of generalized cactoids

In this subsection we approximate spaces that can be obtained by a successive application of k metric 2-point identifications to some generalized cactoid.
Using an induction, Corollary 7.4.4 and Lemma 7.4.5 yield the case $k = 0$:

Corollary 7.4.7. Let $X \in \mathcal{G}(c, b)$. Then X can be obtained as the limit of spaces $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c, b)$. Moreover the following statements apply:

1) The sequence may be chosen such that $\partial X_n \longrightarrow \partial X$.

2) If the maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.

3) If there is a non-orientable maximal cyclic subset in X, then the surfaces of the sequence may be chosen to be non-orientable.

Now we show the general case:

Lemma 7.4.8. Let $k, k_0 \in \mathbb{N}_0$ and $X \in \mathcal{G}(c)$. Further let Y be a space that can be obtained by a successive application of k metric 2-point identifications to X such that k_0 of them are boundary identifications. Then Y can be obtained as the limit of spaces $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c - k_0 + 2k)$. Moreover the following statements apply:

1) If the maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.

2) If there is a non-orientable maximal cyclic subset in X or $k > 0$, then the surfaces of the sequence may be chosen to be non-orientable.

Proof. First we add a statement to the claim: There is a choice $p_1, \ldots, p_k$ of the corresponding projection maps such that $\partial Y_n \longrightarrow (p_k \circ \ldots \circ p_0)(\partial X)$ where $p_0 := id_X$.
We show the claim using an induction over k:
The case $k = 0$ is a direct consequence of Corollary 7.4.7.
Now let $k > 0$. For the sake of induction, we assume that the claim is true if the number of identifications is less than k. Let $p_1, \ldots, p_k$ be a choice of the corresponding projection maps. We set $Z := (p_{k-1} \circ \ldots \circ p_0)(X)$ and denote the number of boundary identifications in $\{p_{k-1}, \ldots, p_1\}$ by $\tilde{k}_0$. Then Z is a space that can be obtained by a successive application of $k - 1$ metric 2-point identifications to X such that $\tilde{k}_0$ of them are boundary identifications. Hence we can apply the induction hypothesis and derive a corresponding sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathcal{S}\big(c - \tilde{k}_0 + 2(k-1)\big)$. In particular, we may assume that $\partial Z_n \longrightarrow (p_{k-1} \circ \ldots \circ p_0)(\partial X)$.
Let $z_1, z_2 \in Z$ be distinct points with $p_k(z_1) = p_k(z_2)$. Then there is a sequence $(z_{i,n})_{n\in\mathbb{N}}$ with $z_{i,n} \in Z_n$ and $z_{1,n} \neq z_{2,n}$ such that $z_{i,n} \longrightarrow z_i$. Provided p_k is a boundary identification, we may assume $z_{i,n} \in \partial Z_n$. Further we define W_n as the metric gluing of Z_n along $z_{1,n}$ and $z_{2,n}$. We denote the corresponding projection map by $p_{k,n}$.
By construction $p_{k,n}$ is a boundary identification if p_k is. Moreover it follows $W_n \longrightarrow Y$ and we may assume that $(p_{k,n})_{n\in\mathbb{N}}$ converges to some map q_k. We note that $q_k, p_{k-1}, \ldots, p_1$ is also a possible choice of the projection maps corresponding to the construction of Y. Hence we may assume that $p_{k,n}(\partial Z_n) \longrightarrow (p_k \circ \ldots \circ p_0)(\partial X)$.
Finally we apply Lemma 7.4.6 to W_n and denote the corresponding sequence of surfaces by $(Y_{n,m})_{m\in\mathbb{N}}$ and the corresponding sequence of almost isometries by $(f_{n,m})_{m\in\mathbb{N}}$. Then we have $f_{n,m}(\partial Y_{n,m}) = p_{k,n}(\partial Z_n)$. Choosing a diagonal sequence, we may assume that $Y_n := Y_{n,n} \longrightarrow Y$ and that $(f_{n,n})_{n\in\mathbb{N}}$ converges to an isometry f. We note that $(f^{-1} \circ p_k), p_{k-1}, \ldots, p_1$ is also a possible choice of the projection maps corresponding to the construction of Y. Hence we finally may assume that $\partial Y_n \longrightarrow (p_k \circ \ldots \circ p_0)(\partial X)$. $\square$

Using sequences of closed 2-discs or real projective planes whose diameters tend to zero, we get the following corollary of Lemma 7.4.5:

Corollary 7.4.9. Let $X \in \mathcal{S}(c)$. Then X can be obtained as the limit of spaces in $\mathcal{S}(c+1)$. Moreover the following statements apply:

1) If S is orientable, then the surfaces of the sequence may be chosen to be orientable.

2) The surfaces of the sequence may be chosen to be non-orientable.

As a direct consequence of the last two results, we derive that the second statement of Main Theorem II implies the first. In particular, we are able to describe under which conditions the approximating surfaces may be chosen to be orientable or non-orientable:

Theorem 7.4.10. Let $c, k, k_0 \in \mathbb{N}_0$ and $X \in \mathcal{G}(c_0)$ where $c_0 - k_0 + 2k \leq c$. Further let Y be a space that can be obtained by a successive application of k metric 2-point identifications to X such that k_0 of them are boundary identifications. Then Y can be obtained as the limit of spaces $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c)$. Moreover the following statements apply:

1) If the maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.

2) If there is a non-orientable maximal cyclic subset in X or $c_0 < c$, then the surfaces of the sequence may be chosen to be non-orientable.

Finally we show the third statement of Theorem 7.1.1:

Proof of Theorem 7.1.1 (Part II). If Y is a space that can be obtained by a successive application of metric 2-point identifications to some geodesic generalized cactoid, then the second statement of Theorem 7.1.1 and Theorem 7.4.10 imply that Y is locally simply connected. Due to Theorem 7.3.12 the space X can be obtained in this way. Hence Proposition 6.2.9 and Proposition 4.0.2 finally yield the desired fundamental group formula for X. □

8 Surfaces with bounded curvature

In the 1940s, Alexandrov introduced surfaces with bounded curvature as a generalization of connected Riemannian 2-manifolds. As it will turn out, the maximal cyclic subsets of the limit spaces in Scenario III are surfaces with bounded curvature. In preparation, we give a brief introduction to the theory and discuss useful results.

8.1 Basic definitions

We recall that the *total curvature measure* over a Riemannian 2-manifold R is the signed measure defined by

$$\omega_R(A) := \int_A K \, d\mathcal{H}^2$$

for all Borel sets $A \subset R$ where K denotes the Gaussian curvature on R.
In the following we write $|\mu|$ for the total variation measure of a signed measure μ. We state the central definition of this chapter:

Definition 8.1.1. A length surface X without boundary is called a *surface with bounded curvature* if there is a sequence of Riemannian 2-manifolds R_n satisfying the following properties:

1) $R_n \xrightarrow{uni.} X$.

2) The measures $|\omega_{R_n}|$ are uniformly bounded.

A basic example of a surface with bounded curvature is illustrated in Figure 14.
Now we introduce a notion of curvature for surfaces with bounded curvature: Let φ_n be the homeomorphisms corresponding to the uniform convergence in the aforementioned definition. Due to the Banach-Alaoglu theorem, after passing to a subsequence, the measures $\omega_{R_n} \circ \varphi_n$ weakly converge to some signed measure over X. Indeed, the limit measure does not depend on the choice of the spaces R_n (cf. [2, pp. 230-241]) and we

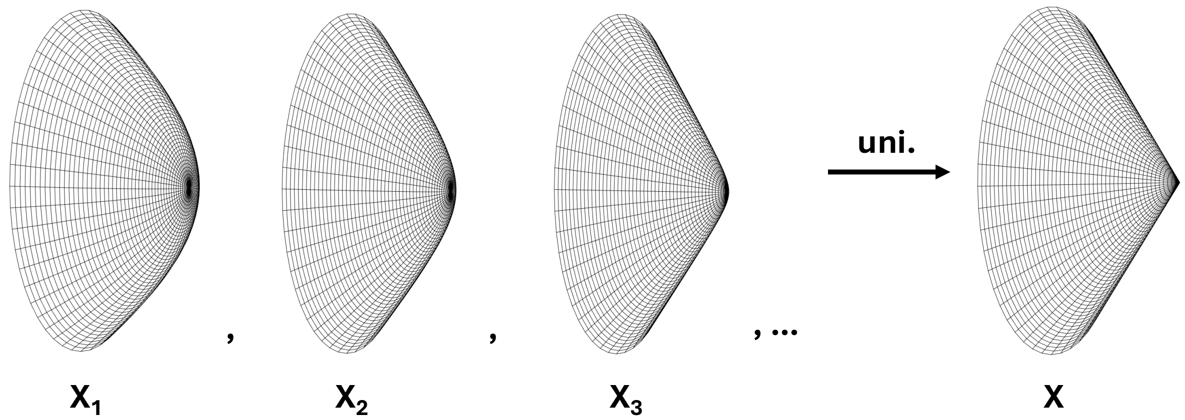

Figure 14: A surface with bounded curvature. The spaces X and X_n are the surfaces of revolution of the functions $f(x) = |x|$ and $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $(-1, 1)$ together with their induced length metrics. In particular, the spaces X_n are Riemannian 2-manifolds with uniformly bounded total absolute curvature such that $X_n \xrightarrow{uni.} X$. By definition X is a surface with bounded curvature.

refer to it as the *total curvature measure* ω_X over X. The measure $|\omega_X|$ is denoted as the *total absolute curvature measure* over X.
For a point $p \in X$ the quantity

$$\theta_X(p) := 2\pi - \omega_X(p)$$

is called the *total angle* of p. Furthermore p is denoted as a *peak point* of X if $\theta_X(p) = 0$. We note that the concept of a surface with bounded curvature and its total curvature measure can be defined purely geometrically via the behavior of triangles in the length surface X (cf. [2, pp. 6, 156]). In particular, it can be defined without approximating Riemannian 2-manifolds.

8.2 Stability results

In this section we discuss stability results for surfaces with bounded curvature.

8.2.1 Local uniform convergence

The first result deals with stability under local uniform convergence:

Theorem 8.2.1. (cf. [2, pp. 88, 141, 240-241, 269]) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of surfaces with bounded curvature such that the measures $|\omega_{X_n}|$ are uniformly bounded by C, X be a length space and $p \in X$. Moreover let $X_n \xrightarrow{uni.} X$ on some open neighborhood of p. We denote the topological embeddings corresponding to the uniform convergence by φ_n. Then, after passing to a subsequence, there is some $R > 0$ such that the following statements apply:

1) $B_R(p)$ is isometric to an open subset of some surface with bounded curvature Y.

2) The following inequalities hold:

   a) $|\omega_Y|(B_r(p)) \leq \liminf_{n\to\infty} |\omega_{X_n}|(\varphi_n(B_r(p)))$ for every $r \leq R$.

   b) $\mathcal{H}^2_Y(B_r(p)) \leq \liminf_{n\to\infty} \mathcal{H}^2_{X_n}(\varphi_n(B_r(p)))$ for every $r \leq R$.

8.2.2 Completions

Generally speaking, the property of being a surface with bounded curvature is not stable under taking completions. In this subsection we present a positive example.
We say that the *area growth of open balls* in a metric space X *is quadratic* if there are $A, R > 0$ such that

$$\mathcal{H}^2(B_r(x)) \leq Ar^2$$

for all $x \in X$ and $r < R$.
For a closed Riemannian 2-manifold such constants A and R can be calculated on basis of its absolute curvature and diameter:

Proposition 8.2.2. (cf. [35, p. 1773]) Let R be a closed Riemannian 2-manifold such that the measure $|\omega_R|$ is bounded by C. Then we have

$$\mathcal{H}^2(B_r(x)) \leq \Big(\frac{2\pi + C}{2}\Big) r^2$$

for all $x \in R$ and $r < 2^{-1} diam(R)$.

Following Lytchak's arguments in [24, pp. 630-631] and [23, pp. 7-8], we derive:

Theorem 8.2.3. Let X be a length space that is homeomorphic to some closed surface and $F \subset X$ be a finite subset. Moreover let the area growth of open balls in X be at most quadratic. Then the following statements apply:

1) $X \setminus F$ is a length space.

2) If $X \setminus F$ is a surface with bounded curvature such that the measure $|\omega_{X\setminus F}|$ is finite, then X is also a surface with bounded curvature.

Proof. For the sake of simplicity, we assume that $F = \{x\}$.
1) We follow the arguments in [24, pp. 630-631]: First we choose some closed topological 2-disc $D \subset X$ with $x \in D^0$. Moreover let $y \in J := \partial D$.
For sufficiently small $r > 0$ we have $S_r := \partial S_r \subset D^0$ and that S_r separates x and y in D. Using the bound on the area growth, the coarea formula (cf. [24, p. 603]) implies that $\liminf_{r\to 0} \mathcal{H}^1(S_r) = 0$. Hence there is a sequence $r_n \longrightarrow 0$ and Jordan curves $J_n \subset S_{r_n}$ with $\lim_{n\to\infty} length(J_n) = 0$ such that J_n separates x and y in D (cf. [24, p. 623]). In particular, J_n bounds a closed topological 2-disc D_n with $x \in D_n^0$.
We conclude that $X \setminus F$ is a length space.
2) We follow the arguments in [23, pp. 7-8]: There is some $\varepsilon > 0$ such that for every non-contractible loop γ in $D \setminus \{p\}$ of length at most ε the intersection with J is empty. We may assume that $length(J_n) \leq \frac{\varepsilon}{3}$ and choose a shortest non-contractible loop γ_n in the topological cylinder bounded by J and J_n.
Then γ_n is a Jordan curve bounding a closed topological 2-disc $A_n \subset D$ with $p \in A_n^0$. Moreover A_n is convex and we have $\lim_{n\to\infty} diam(A_n) = 0$ and $\lim_{n\to\infty} l_n := length(\gamma_n) = 0$. Due to the finiteness of the measure $|\omega_{X\setminus F}|$, after passing to a subsequence, γ_n is free of peak points and the absolute rotation of γ_n, measured from the side of $X \setminus A_n$, (cf. [2, pp. 272, 308]) is bounded by some constant that does not depend on n.
We equip $X \setminus A_n^0$ with its induced length metric and glue this new space and the round hemisphere of length l_n along some length-preserving homeomorphism between the boundary components. Then the gluing theorem in [2, p. 289] implies that this gluing X_n is a surface with bounded curvature such that the measure $|\omega_{X_n}|$ is bounded by some constant that does not depend on n.
By construction we have $X_n \xrightarrow{uni.} X$. Hence the space X is a surface with bounded curvature (cf. [2, pp. 88, 141]). □

8.3 Angles

In the upcoming two sections X denotes a surface with bounded curvature.
For every choice of three distinct points $p, x, y \in X$ there is an isometric embedding $f\colon \{p, x, y\} \to \mathbb{R}^2$. We consider the plane triangle whose vertices are given by the image of f and denote its angle at $f(p)$ by $\angle_p(x, y)$.
The upcoming result is needed to define the angle between two geodesics:

Lemma 8.3.1. (cf. [2, pp. 115-116]) Let $\alpha, \beta \subset X$ be two geodesics emanating from p such that p is their only intersection point. Then the quantity

$$\angle(\alpha, \beta) \coloneqq \lim_{a,b \to p} \angle_p(a, b),$$

where $a \in \alpha$ and $b \in \beta$, is well-defined.

We call this quantity the *angle* between α and β.
We introduce a further related notion of angle for geodesics:
Due to the Jordan curve theorem it makes sense to talk about a left- and a right-hand side of the "hinge" defined by α and β. We write $\mathcal{A}_l$ for the set of all geodesics on the left-hand side that emanate from p.
Moreover we define the following quantity:

$$\tilde{\angle}_l(\alpha, \beta) \coloneqq \sup\left\{\sum_{i=0}^{n} \angle(\gamma_i, \gamma_{i+1})\right\}$$

where $\gamma_0 \coloneqq \alpha$, $\gamma_{n+1} \coloneqq \beta$ and the supremum is taken over all choices $\gamma_1, \ldots, \gamma_n \in \mathcal{A}_l$ such that the geodesics are sorted by their successive order around p from α to β and p is their only intersection point. This quantity is denoted as the *left sector angle* between α and β. Analogously we define their *right sector angle* $\tilde{\angle}_r(\alpha, \beta)$.
The sector angles take values in $[0, \infty)$ (cf. [2, p. 124]). Moreover they can be used to give the total angle a geometric meaning:

Lemma 8.3.2. (cf. [2, p. 120]) For every choice of the geodesics α and β we have

$$\theta_X(p) = \tilde{\angle}_l(\alpha, \beta) + \tilde{\angle}_r(\alpha, \beta).$$

8.4 The rotation of a Jordan curve

Let $J \subset X$ be a two-sided Jordan curve (i.e. J admits an open neighborhood homeomorphic to $\mathbb{S}^1 \times (0,1)$). Then it makes sense to talk about a left- and a right-hand side of J. Moreover J can be obtained as the uniform limit of piecewise geodesic Jordan curves J_n on the left-hand side of J. In particular, every curve J_n is a concatenation $\gamma_{n,1} * \ldots * \gamma_{n,k_n}$ of finitely many geodesics.
In order to define a notion of curvature for Jordan curves we present the following result:

Lemma 8.4.1. (cf. [2, pp. 191-192]) The quantity

$$\tau_l(J) \coloneqq \lim_{n\to\infty} \sum_{i=1}^{k_n} \pi - \tilde{\angle}_l(\gamma_{n,i}, \gamma_{n,i+1}),$$

where $\gamma_{n,k_n+1} \coloneqq \gamma_{n,1}$, is well-defined.

We refer to this quantity as the *left rotation* of J. Analogously we define its *right rotation* $\tau_r(J)$.
The famous Gauss-Bonnet theorem relates the geometry of a compact surface in X to its topology:

Theorem 8.4.2. (cf. [2, p. 192]) Let $S \subset X$ be a compact surface and $J_1, \ldots, J_n$ be its boundary components. Then the following equality holds:

$$2\pi\chi(S) = \omega_X\left(S^0\right) + \sum_{i=1}^{n} \tau(J_i)$$

where the rotation is measured from the side of S.

A metric space is called a *topological cylinder* provided it is homeomorphic to $\mathbb{S}^1 \times [0,1]$.
Finally we state the following convergence result which is strongly based on a result by Burago in [8, p. 14]:

Theorem 8.4.3. Let X and $(X_n)_{n\in\mathbb{N}}$ be surfaces with bounded curvature such that the measures $|\omega_{X_n}|$ are uniformly bounded by C and $X_n \xrightarrow{uni.} X$ on some topological cylinder $Z \subset X$. We assume that there are homeomorphisms $\varphi_n \colon Z \to Z_n$ corresponding to the uniform convergence such that $d(\varphi_n, id_X) \longrightarrow 0$. Moreover let Z be free of peak points and $J \subset Z^0$ be a non-contractible Jordan curve. Then for every $\varepsilon > 0$, after passing to a subsequence, the following statements apply:

1) There is a non-contractible Jordan curve $J_\varepsilon \subset Z^0$ on the left-hand side of J such that $|\tau_l(J) - \tau_l(J_\varepsilon)| \leq \varepsilon$.

2) There are homeomorphisms $\psi_{\varepsilon,n} \colon Z \to Z_n$ with $d(\psi_{\varepsilon,n}, \varphi_n) \longrightarrow 0$ such that the following inequality holds:

$$|\tau_l(J_\varepsilon) - \lim_{n\to\infty} \tau_l(\psi_{\varepsilon,n} \circ J_\varepsilon)| \leq \lim_{n\to\infty} |\omega_{X_n}|(\psi_{\varepsilon,n} \circ J_\varepsilon).$$

Proof. Using the definition of the rotation and a similar construction as in [38, pp. 413-415], after passing to a subsequence, we obtain a piecewise geodesic curve J_ε as in statement 1) and piecewise geodesic Jordan curves $J_{\varepsilon,n} \subset Z_n^0$ such that the number of edges is uniformly bounded and $J_{\varepsilon,n} \xrightarrow{\mathcal{H}} J_\varepsilon$.
In the case that, after passing to a subsequence, the absolute left rotation of the curves $\varphi_n(J_\varepsilon)$ (cf. [2, pp. 272, 308]) is uniformly bounded statement 2) is covered by [8, p. 14]. Hence we only need to find homeomorphisms $\psi_{\varepsilon,n}$ corresponding to the uniform convergence such that this boundedness property is satisfied and $d(\psi_{\varepsilon,n}, \varphi_n) \longrightarrow 0$:
The length of the curves $J_{\varepsilon,n}$ is uniformly bounded. By [2, p. 226] it follows that, after passing to a subsequence, the curves $\varphi_n^{-1} \circ J_{\varepsilon,n}$ converge uniformly to J_ε as maps. A similar procedure as in [2, pp. 227-229] yields that, after passing to a subsequence, there are homeomorphisms $\psi_{\varepsilon,n} \colon Z \to Z_n$ corresponding to the uniform convergence $X_n \xrightarrow{uni.} X$ on Z such that $\psi_{\varepsilon,n}(J_\varepsilon) = J_{\varepsilon,n}$ and $d(\psi_{\varepsilon,n}, \varphi_n) \longrightarrow 0$. Finally we note that the absolute left rotation of the curves $J_{\varepsilon,n}$ is uniformly bounded. □

9 Scenario III: Riemannian 2-manifolds with uniformly bounded total absolute curvature

9.1 Introduction

In the middle of the 20th century, Alexandrov developed the theory of surfaces with bounded curvature. One of the central results states that every surface with bounded curvature can be obtained as the uniform limit of Riemannian 2-manifolds with uniformly bounded total absolute curvature (cf. [31, p. 77]).
From this result a natural question arises: Given a closed surface S and $C > 0$. What can we say about the Gromov-Hausdorff limits of Riemannian 2-manifolds R_n satisfying the following properties:

1) The space R_n is homeomorphic to S.

2) The total absolute curvature of R_n is at most C.

Burago was the first to address this question: From results in [7, pp. 143-145] a geometric description for the case that S is homeomorphic to $\mathbb{S}^2$ can be deduced. However, there are no proofs given in the paper and we note that our investigation is completely independent from this work. In the 1990s, Shioya gave a purely topological description of the limit spaces (cf. [35, p. 1767]). In Chapter 6 we answered the curvature-free formulation of the question.
The aim of the present chapter is a description of the limit spaces under the following additional assumption:

3) The sequence R_n is uniformly semi-locally 1-connected.

We write $\mathcal{R}(S, C)$ for the class of all Riemannian 2-manifolds satisfying the first two conditions. The third assumption is only used in order to simplify the presentation of

the results. The procedure developed in Chapter 6 suggests a way to describe the spaces in the entire closure of $\mathcal{R}(S, C)$ on basis of Main Theorem III (A).
In particular, we will address the geometric properties of the limit spaces.

9.1.1 Key results

The *index* of a point p in a Peano space X is defined as the number of connected components in $X \setminus \{p\}$ and we denote it by $ind_X(p)$.
Combining Shioya's description in [35, p. 1767] with Theorem A (Whyburn) and Corollary 6.3.4, we directly obtain the following result about the topological structure of the limit spaces:

Theorem 9.1.1. Let X be a space that can be obtained as the Gromov-Hausdorff limit of a uniformly semi-locally 1-connected sequence in $\mathcal{R}(S, C)$. Then the following statements apply:

1) If S is homeomorphic to $\mathbb{S}^2$, then all maximal cyclic subsets of X are homeomorphic to $\mathbb{S}^2$.
2) If S is not homeomorphic to $\mathbb{S}^2$, then one maximal cyclic subset of X is homeomorphic to S and all others are homeomorphic to $\mathbb{S}^2$.
3) Every maximal cyclic subset of X contains only finitely many cut points of X. Only finitely many of them contain more than two cut points.
4) The following inequality holds:

$$\sum_{p \in X} \max\{ind_X(p) - 2, 0\} \leq \frac{C}{2\pi}.$$

The following curvature-free result will create the needed connection between Gromov-Hausdorff and uniform convergence. In particular, it marks the starting point of our investigation.

Theorem 9.1.2. Let X be a space that can be obtained as the Gromov-Hausdorff limit of a uniformly semi-locally 1-connected sequence of length spaces that are homeomorphic to S. Then for every $p \in X$ we either have $p \in \overline{Cut_X}$ or the sequence X_n converges uniformly to X on some open neighborhood of p.

Now we state the two main results of this chapter:

Main Theorem III (A). Let X be a compact length space. If X can be obtained as the Gromov-Hausdorff limit of a uniformly semi-locally 1-connected sequence in $\mathcal{R}(S,C)$, then the following statements apply:

1) The topological structure of X is given as in Theorem 9.1.1.

2) Every maximal cyclic subset of X is a closed surface with bounded curvature.

3) Let $(T_n)_{n\in\mathbb{N}}$ be an enumeration of the maximal cyclic subsets of X. Then the following inequality holds:

$$\sum_{n=1}^{\infty}|\omega_{T_n}|(T_n \setminus Cut_X) + \sum_{p\in Sing_X} 2\pi\,|ind_X(p)-2| + \sum_{p\in Cut_X}\sum_{n=1}^{\infty}\mathbb{1}_{T_n}(p)\,\theta_{T_n}(p) \leq C.$$

We denote the class of all compact length spaces meeting the aforementioned description by $\mathcal{L}(S,C)$. In Figure 15 we see a space in $\mathcal{L}(S,C)$.

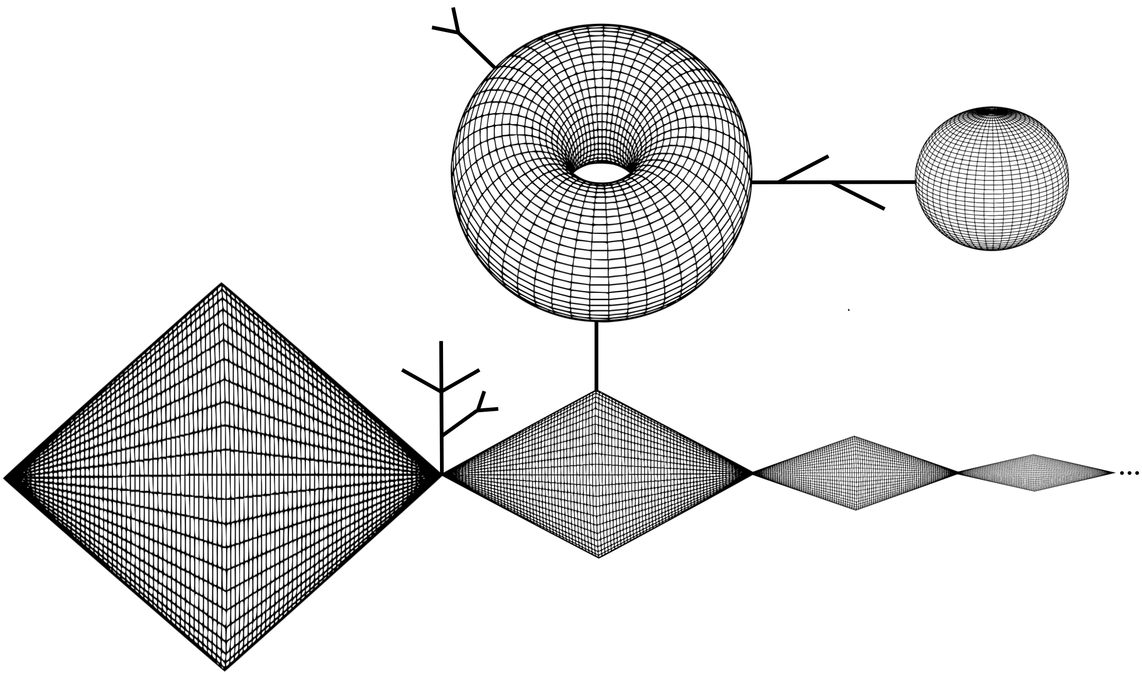

Figure 15: A space in $\mathcal{L}(\mathbb{T}^2,C)$ for some $C>0$.

Also the following converse statement holds:

Main Theorem III (B). Let X be a space in $\mathcal{L}(S,C)$. Then X can be obtained as the Gromov-Hausdorff limit of a uniformly semi-locally 1-connected sequence $X_n \in \mathcal{R}(S,C_n)$ where $\lim\limits_{n\to\infty} C_n \leq C$.

9.1.2 Organisation

This chapter is organized as follows:
The second section is devoted to the proof of Theorem 9.1.2.
Main Theorem III (A) is discussed in Section 9.3. The corresponding proof is divided into two parts: We start by bounding the first summand of the quantitative formula. Afterwards we bound the sum of the second and the third summand.
In the last section of this chapter we show Main Theorem III (B).

9.1.3 Notation

$\mathcal{R}(S, C)$	The class of all Riemannian 2-manifolds R such that R is homeomorphic to the closed surface S and the measure $\lvert\omega_R\rvert$ is bounded by C.
$\mathcal{L}(S, C)$	The class of compact length spaces satisfying the three properties stated in Main Theorem III (A).
$\mathcal{M}(S, \varepsilon)$	The class of all length spaces X such that X is homeomorphic to the closed surface S and every loop in X of diameter at most ε is contractible.
$\mathcal{M}(S, \varepsilon, C)$	The class of all surfaces with bounded curvature X in $\mathcal{M}(S, \varepsilon)$ such that the measure $\lvert\omega_X\rvert$ is bounded by C.

9.2 Local uniform approximations

In this section we show Theorem 9.1.2.
First we state the curvature-free formulation of Theorem 9.1.1 which is a direct consequence of Theorem A (Whyburn) and Corollary 6.3.4:

Theorem 9.2.1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S, \varepsilon)$ and $X_n \xrightarrow{\mathcal{GH}} X$. Then the following statements apply:

1) If S is homeomorphic to $\mathbb{S}^2$, then all maximal cyclic subset of X are homeomorphic to $\mathbb{S}^2$.

2) If S is not homeomorphic to $\mathbb{S}^2$, then one maximal cyclic subset of X is homeomorphic to S and all others are homeomorphic to $\mathbb{S}^2$.

Now we approximate closed topological 2-discs in X by closed topological 2-discs in X_n:

Lemma 9.2.2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon)$ and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover let $D \subset X$ be a closed topological 2-disc such that $Cut_X \cap D = \emptyset$ and $diam(\partial D) < \varepsilon$. Then, after passing to a subsequence, there are closed topological 2-discs $D_n \subset X_n$ with $D_n \xrightarrow{\mathcal{H}} D$ and $\partial D_n \xrightarrow{\mathcal{H}} \partial D$.

Proof. By [38, p. 413] there is a uniformly locally connected sequence of Jordan curves $J_n \subset X_n$ with $J_n \xrightarrow{\mathcal{H}} \partial D$. Since $X_n \in \mathcal{M}(S,\varepsilon)$ and $diam(\partial D) < \varepsilon$, we further may assume that J_n bounds a closed topological 2-disc in X_n. We denote the closures of the two connected components of $X_n \setminus J_n$ by $A_{1,n}$ and $A_{2,n}$. After passing to a subsequence, we may assume the sequences $A_{i,n}$ to be Hausdorff convergent. By Lemma 2.3.1 we may assume that $A_{1,n} \xrightarrow{\mathcal{H}} D$.
For the sake of contradiction, we assume that $A_{1,n}$ is not a closed topological 2-disc for infinitely many $n \in \mathbb{N}$. Then $A_{2,n}$ is a closed topological 2-disc. Therefore Lemma 2.3.1 and Proposition 2.3.2 imply that the maximal cyclic subsets of X are homeomorphic to $\mathbb{S}^2$. From Theorem 9.2.1 we derive that X_n is homeomorphic to $\mathbb{S}^2$. It follows that $A_{1,n}$ is a closed topological 2-disc. A contradiction. □

We take a closer look at the approximating discs:

Lemma 9.2.3. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon)$ and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover let $D \subset X$ be a closed topological 2-disc such that $Cut_X \cap D = \emptyset$ and $D_n \subset X_n$ be closed topological 2-discs with $D_n \xrightarrow{\mathcal{H}} D$ and $\partial D_n \xrightarrow{\mathcal{H}} \partial D$. Then $(D_n)_{n\in\mathbb{N}}$ is uniformly locally 1-connected.

Proof. For the sake of contradiction, we assume that the sequence is not uniformly locally 1-connected. After passing to a subsequence, we then may assume the existence of $\delta > 0$, $p \in D$ and Jordan curves $J_n \subset D_n$ such that J_n is non-contractible in $U_\delta(J_n) \subset D_n$ and $J_n \xrightarrow{\mathcal{H}} \{p\}$. We note that J_n bounds a closed topological 2-disc in X_n. Moreover we denote the closures of the two connected components of $X_n \setminus J_n$ by $A_{1,n}$ and $A_{2,n}$.
We may assume that $A_{1,n}$ is a closed topological 2-disc and that $\partial D_n \subset A_{2,n}$. Due to $\partial D_n \xrightarrow{\mathcal{H}} \partial D$ and the fact that J_n is non-contractible in $U_\delta(J_n)$ it follows that $diam(A_{i,n} \cap D_n)$ is uniformly bounded from below by a positive constant.
By Lemma 2.3.1, after passing to a subsequence, we find $A_i \subset X$ such that $A_{i,n} \xrightarrow{\mathcal{H}} A_i$, $A_1 \cup A_2 = X$ and $A_1 \cap A_2 = \{p\}$. Because $diam(A_i \cap D) > 0$, it follows that $p \in Cut_X$. A contradiction. □

As a consequence of the aforementioned lemma, we derive the following corollary:

Corollary 9.2.4. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon)$ and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover let $D \subset X$ be a closed topological 2-disc such that $Cut_X \cap D = \emptyset$ and $D_n \subset X_n$ be closed topological 2-discs with $D_n \xrightarrow{\mathcal{H}} D$ and $\partial D_n \xrightarrow{\mathcal{H}} \partial D$. Then for every closed topological 2-disc $D' \subset D^0$ there are closed topological 2-discs $D'_n \subset D^0_n$ such that $D'_n \xrightarrow{uni.} D'$. Moreover the homeomorphisms $\varphi_n \colon D' \to D'_n$ corresponding to the uniform convergence can be chosen such that $d(\varphi_n, id_X) \longrightarrow 0$.

Proof. First we double the chosen ambient space Y in which $D_n \xrightarrow{\mathcal{H}} D$ along the boundaries of the discs. For every $A \subset Y$ we write $2A$ for the corresponding doubled subset of this new ambient space.
Then we have $2D_n \xrightarrow{\mathcal{H}} 2D$ and the subsets are homeomorphic to $\mathbb{S}^2$. By Lemma 9.2.3 the sequence $(D_n)_{n\in\mathbb{N}}$ is uniformly locally 1-connected. Hence also $(2D_n)_{n\in\mathbb{N}}$ is uniformly locally 1-connected. Since non-compact surfaces are $K(G,1)$ spaces (cf. [18, p. 88]), it even follows that the sequence $(2D_n)_{n\in\mathbb{N}}$ is uniformly k-connected for all $k \in \mathbb{N}_0$.
From Theorem 2.3.4 we derive the existence of ε_n-equivalences $f_n \colon 2D \to 2D_n$ such that $\varepsilon_n \longrightarrow 0$ and $d(f_n, id_{2X}) \longrightarrow 0$. Hence Theorem 2.4.4 closes the proof. □

Now Theorem 9.1.2 follows from Theorem 9.2.1, Lemma 9.2.2 and Corollary 9.2.4.
In a similar manner we obtain the following related result for cylinders:

Proposition 9.2.5. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon)$ and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover let $Z \subset X$ be a topological cylinder satisfying the following properties:

1) $X \setminus Z$ is disconnected.

2) $Cut_X \cap Z = \emptyset$.

3) The diameter of the boundary components of Z is less than ε.

Then for every topological cylinder $Z' \subset Z^0$, after passing to a subsequence, there are topological cylinders $Z'_n \subset X_n$ such that $Z'_n \xrightarrow{uni.} Z'$. Furthermore the homeomorphisms $\varphi_n \colon Z' \to Z'_n$ corresponding to the uniform convergence can be chosen such that $d(\varphi_n, id_X) \longrightarrow 0$.

Proof. The procedure is analogues to the previous proofs of this section. Only the following changes are made:
We denote the boundary components of Z by J_1 and J_2. Then there is a uniformly locally connected sequence of Jordan curves $J_{i,n} \subset X_n$ with $J_{i,n} \xrightarrow{\mathcal{H}} J_i$. In particular, we may assume that $J_{i,n}$ bounds a closed topological 2-disc in X_n.

We denote the subsurface of X_n bounded by J_1 and J_2 by Z_n. Using the same arguments as in the proof of Lemma 9.2.2, we conclude that, after passing to a subsequence, we have that $Z_n \xrightarrow{\mathcal{H}} Z$ and Z_n is a topological cylinder.
For the sake of contradiction, we assume that the sequence $(Z_n)_{n\in\mathbb{N}}$ is not uniformly locally 1-connected.
After passing to a subsequence, we find $\delta > 0$, $p \in Z$, Jordan curves $\gamma_n \subset Z_n$ and compact subsurfaces $A_{1,n}, A_{2,n} \subset X_n$ as in the proof of Lemma 9.2.3. Especially, we may assume that $A_{1,n}$ contains some boundary component of Z_n. Depending on the contractibility of γ_n in Z_n, the subsurface $A_{2,n}$ is either a closed topological 2-disc in Z_n or it also contains some boundary component of Z_n. Due to $\partial Z_n \xrightarrow{\mathcal{H}} \partial Z$ and the fact that γ_n is non-contractible in $U_\varepsilon(\gamma_n)$ it follows that $diam(A_{i,n} \cap Z_n)$ is uniformly bounded from below by a positive constant.
The contradiction follows exactly as in the aforementioned proof.
Along the same lines as in the proof of Corollary 9.2.4, we find the desired topological cylinders and homeomorphisms. □

9.3 Description of the limit spaces

This section is devoted to the proof that all limit spaces meet the description in Main Theorem III (A).

9.3.1 Bounded curvature of maximal cyclic subsets

As a first step, we show that the maximal cyclic subsets of the limit space are surfaces with bounded curvature.
In the following we write X^{int} for a metric space X equipped with its induced length metric.
By Theorem 9.1.1, Theorem 8.2.1 and Proposition 9.2.5 we have the following result:

Corollary 9.3.1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon,C)$, $(X_n, |\omega_{X_n}|) \xrightarrow{\mathcal{GH}} (X, |\omega|_\infty)$ and $\left(X_n, \mathcal{H}^2_{X_n}\right) \xrightarrow{\mathcal{GH}} (X, \mathcal{H}^2_\infty)$. Moreover let $T \subset X$ be a maximal cyclic subset and $U := T \setminus Cut_X$. Then the following statements apply:

1) U^{int} is a surface with bounded curvature and $|\omega_{U^{int}}|(V) \leq |\omega|_\infty(V)$ for every connected open subset $V \subset U^{int}$.

2) $\mathcal{H}^2_X(V) \leq \mathcal{H}^2_\infty(V)$ for every connected open subset $V \subset T$.

From the aforementioned corollary, Proposition 8.2.2 and the fact that X_n can be obtained as the uniform limit of spaces in $\mathcal{R}(S, 2C)$ (cf. [31, p. 144]) we derive a bound on the area growth of open balls:

Proposition 9.3.2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S, \varepsilon, C)$ and $X_n \xrightarrow{\mathcal{GH}} X$. Then the area growth of open balls in X is at most quadratic.

Due to Theorem 9.1.1 every maximal cyclic subset contains only finitely many cut points of X. Using Theorem 8.2.3 and the last two results, we conclude that the maximal cyclic subsets are surfaces with bounded curvature. Moreover we obtain a bound on the total absolute curvature of these surfaces.

Corollary 9.3.3. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S, \varepsilon, C)$ and $(X_n, |\omega_{X_n}|) \xrightarrow{\mathcal{GH}} (X, |\omega|_\infty)$. Then the following statements apply:

1) Every maximal cyclic subset of X is a surface with bounded curvature.
2) Let $(T_n)_{n\in\mathbb{N}}$ be an enumeration of the maximal cyclic subsets of X. Then the following inequality holds:

$$\sum_{n=1}^{\infty} |\omega_{T_n}|(T_n \setminus Cut_X) \leq |\omega|_\infty(X \setminus Sing_X).$$

9.3.2 Geometry of singular points

In this subsection we study the geometry of the singular points in the limit space. More precisely, we show the following inequality:

Proposition 9.3.4. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S, \varepsilon, C)$ and $(X_n, |\omega_{X_n}|) \xrightarrow{\mathcal{GH}} (X, |\omega|_\infty)$. Moreover let $p \in Sing_X$ and $(T_n)_{n\in\mathbb{N}}$ be an enumeration of the maximal cyclic subsets of X. Then the following inequality holds:

$$2\pi|ind(p) - 2| + \sum_{n=1}^{\infty} \mathbb{1}_{T_n}(p)\theta_{T_n}(p) \leq |\omega|_\infty(p).$$

If p is an endpoint, then we have $ind(p) = 1$ and p lies in no maximal cyclic subset. Hence the inequality becomes $2\pi \leq |\omega|_\infty(p)$.
For cut points we have $ind(p) \geq 2$ and p lies in at most finitely many maximal cyclic subsets $T_{n_1}, \ldots, T_{n_k}$. In this case the inequality becomes

$$2\pi(ind(p) - 2) + \sum_{i=1}^{k} \theta_{T_{n_i}}(p) \leq |\omega|_\infty(p).$$

We will conclude the aforementioned proposition from the following lemma:

Lemma 9.3.5. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon,C)$ and $(X_n,|\omega_{X_n}|)\xrightarrow{\mathcal{GH}}(X,|\omega|_\infty)$. Moreover let $p\in Sing_X$ and $p_n\in X_n$ with $p_n\longrightarrow p$. We denote the maximal cyclic subsets containing p by $T_1,\dots,T_N$. Then for every $\delta>0$, after passing to a subsequence, there are compact subsurfaces $S_n\subset X_n$ satisfying the following properties:

1) We have $p_n\in S_n^0$ and $S_n\xrightarrow{\mathcal{H}}\{p\}$.

2) S_n has $ind(p)$ boundary components $J_{1,n},\dots,J_{ind(p),n}$ and $\chi(S_n)=2-ind(p)$.

3) If p is a cut point, then the following estimates apply:

   a) $\tau(J_{i,n})\geq\theta_{T_i}(p)-2\delta$ for all $1\leq i\leq N$

   b) $\tau(J_{i,n})\geq -2\delta$ for all $N+1\leq i\leq ind(p)$

   where the rotation is measured from the side of p_n.

4) If p is an endpoint, then $\tau(J_{1,n})\leq 2\delta$ where the rotation is measured from the side of p_n.

First we explain how to derive the proposition from the lemma:
We start with the case that p is a cut point. Since $S_n\xrightarrow{\mathcal{H}}\{p\}$, after passing to a subsequence, we may assume that

$$|\omega_{X_n}|(S_n)\leq|\omega|_\infty(p)+\delta.$$

From the identity for $\chi(S_n)$ and the Gauss-Bonnet theorem we derive

$$\sum_{i=1}^{ind(p)}\tau(J_{i,n})\leq 2\pi(2-ind(p))+|\omega|_\infty(p)+\delta$$

where the rotation is measured from the side of p_n. Due to the estimates for the rotation of the boundary components we derive

$$2\pi(ind(p)-2)+\sum_{i=1}^{N}\theta_{T_i}(p)\leq|\omega|_\infty(p)+(2\cdot ind(p)+1)\delta.$$

Taking the limit $\delta\longrightarrow 0$, we get the desired inequality.
If p is an endpoint, then we have $ind(p)=1$ and the Gauss-Bonnet theorem implies

$$2\pi\leq\tau(J_{1,n})+|\omega|_\infty(p)+\delta.$$

By the estimate for the rotation of the boundary component we get

$$2\pi \leq |\omega|_\infty(p) + 3\delta.$$

As before, taking the limit $\delta \longrightarrow 0$, we obtain the desired inequality.
Now we turn to the proof of the lemma:

Approximation of curves

First we prove that certain Jordan curves J in the limit space admit Hausdorff approximations by Jordan curves whose rotations converge to the rotation of J:

Lemma 9.3.6. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon,C)$ and $(X_n, |\omega_{X_n}|) \xrightarrow{\mathcal{GH}} (X, |\omega|_\infty)$. Moreover let $D \subset X$ be an open topological 2-disc, $p \in D$ with $Cut_X \cap (D \setminus \{p\}) = \emptyset$ and $p_n \in X_n$ with $p_n \longrightarrow p$. Then for every $\delta > 0$, after passing to a subsequence, the following statements apply:

1) There is a Jordan curve $J \subset D$ such that J bounds a closed topological 2-disc in D containing p. Moreover we have $\tau(J) \geq -\delta$ where the rotation is measured from the side of p.

2) There are separating Jordan curves $J_n \subset X_n$ with $p_n \notin J_n$, $J_n \xrightarrow{\mathcal{H}} J$, $|\omega_{X_n}|(J_n) \leq \delta$ and $\tau(J_n) \geq \tau(J) - \delta$ where the rotation is measured from the side of p_n and p respectively.

Proof. There is some $R \in \left(0, \frac{\varepsilon}{2}\right]$ such that for all $0 < r < R$ the ball $\bar{B}_r(p)$ is a closed topological 2-disc in D whose surface boundary coincides with $\partial B_r(p)$ and the rotation of the boundary, measured from the side of p, is larger than $-\delta$ (cf. [31, p. 151-152]). Since $|\omega|_\infty$ is a finite measure, there is some $r_0 \in (0, R)$ such that $J := \partial B_{r_0}(p)$ satisfies $|\omega|_\infty(J) = 0$.
Let $\tilde{\delta} > 0$ with $r_0 \pm \tilde{\delta} \in (0, R)$. Then the curves $\partial B_{r_0 \pm \tilde{\delta}}(p)$ bound a topological cylinder $Z_{\tilde{\delta}} \subset D$ such that J is a non-contractible curve in its interior. From Theorem 9.2.1 follows that $Z_{\tilde{\delta}}$ separates X. Because $\lim_{\tilde{\delta}\to 0} |\omega|_\infty(Z_{\tilde{\delta}}) = |\omega|_\infty(J)$, we may assume that $|\omega|_\infty(Z_{\tilde{\delta}}) < \min\{2\pi, \delta\}$. In particular, it follows that $Z_{\tilde{\delta}}$ is free of peak points.
By Proposition 9.2.5, after passing to a subsequence, we may assume the existence of topological cylinders $Z_{\tilde{\delta},n} \subset X_n$ such that $Z_{\tilde{\delta},n} \xrightarrow{uni.} Z_{\tilde{\delta}}$. Moreover the homeomorphisms $\varphi_n \colon Z_{\tilde{\delta}} \to Z_{\tilde{\delta},n}$ corresponding to the uniform convergence can be chosen such that $d(\varphi_n, id_X) \longrightarrow 0$. Since $|\omega|_\infty(Z_{\tilde{\delta}}) < \delta$, after passing to subsequence, we may assume

that $|\omega_{X_n}|\left(Z_{\tilde{\delta},n}\right) \leq \delta$.
We define $J_{\tilde{\delta},n} \coloneqq \varphi_n \circ J$. Then we have $J_{\tilde{\delta},n} \xrightarrow{\mathcal{H}} J$, $|\omega_{X_n}|\left(J_{\tilde{\delta},n}\right) \leq \delta$ and we may assume that $p_n \notin J_{\tilde{\delta},n}$. Since $diam(J) < \varepsilon$, we further may assume that $J_{\tilde{\delta},n}$ is separating in X_n. By Lemma 2.3.1 and Theorem 8.4.3, after passing to a subsequence, we may assume that

$$|\tau(J) - \lim_{n\to\infty} \tau\left(J_{\tilde{\delta},n}\right)| \leq |\omega|_\infty(Z_{\tilde{\delta}})$$

where the rotation is measured from the side of p_n and p respectively. Hence for sufficiently small $\tilde{\delta}$, after passing to a subsequence, we have $\tau\left(J_{\tilde{\delta},n}\right) \geq \tau(J) - \delta$. □

Approximation of points

As a next step, we approximate certain points of the limit space by Jordan curves such that the rotation is uniformly bounded from below:

Lemma 9.3.7. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon,C)$ and $(X_n, |\omega_{X_n}|) \xrightarrow{\mathcal{GH}} (X, |\omega|_\infty)$. Moreover let $I \subset X$ be an open subset that is isometric to some open interval. Then, after passing to a subsequence, there are topological cylinders $Z_n \subset X_n$ and $J \subset I$ such that $Z_n \xrightarrow{\mathcal{H}} J$ and $\partial Z_n \xrightarrow{\mathcal{H}} \partial J$.

Proof. We may assume that the midpoint p of the interval satisfies $|\omega|_\infty(p) = 0$. Moreover we choose $0 < r \leq \frac{\varepsilon}{2}$ such that $R \coloneqq 100r < 2^{-1} length(I)$.
Let $p_n \in X_n$ such that $p_n \longrightarrow p$. By [31, p. 77, 144] and [35, p. 1772] we may assume that X_n is a Riemannian 2-manifold, r and R are non-exceptional with respect to p_n in the sense of the latter reference and $\bar{B}_r(p_n)$ is a compact surface whose surface boundary is given by $\partial B_r(p_n)$. We write Z_n for the union of $\bar{B}_r(p_n)$ with all closed topological 2-discs in $B_R(p_n) \setminus B_r(p_n)$ that are bounded by some connected component of $\partial B_r(p_n)$. Choosing r small enough, after passing to a subsequence, we may assume the inequality $|\omega_{X_n}|(B_R(p_n)) \leq 1$. Then Z_n is a compact surface whose boundary is given by connected components of $\partial B_r(p_n)$ and, choosing r even smaller, the last reference implies that $\chi(Z_n) \geq 0$. Since $X \setminus \bar{B}_R(p)$ consists of two connected components, Lemma 2.3.1 implies that, after passing to a subsequence, Z_n has at least two boundary components. As a consequence, Z_n is a topological cylinder.
Finally, after passing to a subsequence, there is some $J \subset I$ such that $Z_n \xrightarrow{\mathcal{H}} J$ and $\partial Z_n \xrightarrow{\mathcal{H}} \partial J$. □

We define the *length* of a topological cylinder Z as the distance between its boundary components and denote it by $length(Z)$.

Lemma 9.3.8. Let X_n be closed surfaces with bounded curvature and $Z_n \subset X_n$ be topological cylinders. Moreover we assume that $(Z_n)_{n\in\mathbb{N}}$ Gromov-Hausdorff converges to some compact interval $[0,l] \subset \mathbb{R}$ and $\partial Z_n \xrightarrow{\mathcal{H}} \{0,l\}$. Then for every $\delta > 0$, after passing to a subsequence, there are topological cylinders $Z_n' \subset Z_n$ satisfying the following properties:

1) The boundary components of Z_n' are piecewise geodesic.

2) The boundary components of Z_n' are non-contractible in Z_n.

3) We have $length(Z_n') \geq \frac{l}{2} - 2\delta$ and $d(\partial Z_n', \partial Z_n) \geq \frac{l}{4} - 2\delta$.

4) The length of the boundary components of Z_n' is at most δ.

Proof. Let $\gamma \subset Z_n$ be a geodesic with $length(\gamma) = length(Z_n)$ that connects the boundary components of Z_n. We denote the midpoint of γ by p and its endpoints by x_1 and x_2.
Due to the convergence of the cylinders, after passing to a subsequence, we obtain that $length(\gamma) \geq l - \delta$, $S := \partial B_{\frac{l}{4}}(p) \subset Z_n^0$ and that S separates x_1 and x_2 in Z_n. Moreover we may assume the existence of some Jordan curve $J_1 \subset S$ with $diam(J_1) \leq \frac{\delta}{3}$ that separates x_1 and x_2 in Z_n (cf. [24, p. 623]). In particular, J_1 is non-contractible in Z_n and we may assume that J_1 separates x_1 and p in Z_n.
Using a similar argument, we also may assume that there is a second Jordan curve $J_2 \subset S$ with $J_1 \cap J_2 = \emptyset$ and $diam(J_2) \leq \frac{\delta}{3}$ that is non-contractible in Z_n and separates x_2 and p in Z_n.
There is some $\varepsilon_n > 0$ such that every loop in Z_n of diameter at most ε_n is contractible in Z_n. Let y be the intersection point of J_i with γ and $0 < \tilde{\delta} \leq \min\left\{\frac{\varepsilon_n}{2}, \frac{\delta}{3}\right\}$. We may assume that $length(J_i) > \tilde{\delta}$ and choose a subdivision $t_0 := 0 < t_1 < \ldots < t_k := 1$ of the domain of J_i such that $length\left(J_i|_{[t_j,t_{j+1}]}\right) \leq \tilde{\delta}$. After passing to a subsequence, there is some geodesic $\alpha_j \subset Z$ connecting y with $J_i(t_j)$ and we may assume that $\alpha_j \cap \gamma$ and $\alpha_j \cap \alpha_{j+1}$ are connected.
By construction there is some $t_0 < t_j < t_k$ such that α_j is homotopic to $J_i|_{[0,t_j]}$ and α_{j+1} is homotopic to $J_i|_{[t_{j+1},1]}$. After passing to a subsequence, there is some geodesic $\xi \subset Z$ connecting $J_i(t_j)$ with $J_i(t_{j+1})$ such that $\xi \cap \alpha_j$ and $\xi \cap \alpha_{j+1}$ are connected. In particular, ξ is homotopic to $J_i|_{[t_j,t_{j+1}]}$. We derive that the concatenation $\alpha_j * \xi * \alpha_{j+1}$ is homotopic

to J_i.
Up to parametrization there is exactly one Jordan curve β_i contained in this concatenation. The curve β_i is non-contractible in Z_n and piecewise geodesic. After passing to a subsequence, we have $\beta_1 \cap \beta_2 = \emptyset$ and the curves bound a topological cylinder $Z'_n \subset Z_n$. In particular, we get $length(Z'_n) \geq \frac{l}{2} - 2\delta$, $d(\partial Z'_n, \partial Z_n) \geq \frac{l}{4} - 2\delta$ and the length of the boundary components of Z'_n is at most δ. $\square$

Next we show a general result about surfaces with bounded curvature. The exact estimates in the statement do not play an essential role in the further discussion. It is only important that $\tau(J)$ can be chosen close to zero if $\frac{c}{a}$ and $|\omega_X|(Z)$ are close to zero.

Lemma 9.3.9. Let X be a closed surface with bounded curvature and $Z' \subset Z \subset X$ be topological cylinders with piecewise geodesic boundary components. Moreover we assume that the boundary components of Z' are non-contractible in Z and that $\delta := |\omega_X|(Z) < \frac{\pi}{5}$. We introduce the following notation:

- $a := length(Z')$
- $b := d(\partial Z', \partial Z)$
- $c :=$ The length of a longest boundary component of Z'.
- $\alpha := \frac{c}{a} + 7\delta$

If $c < b$ and $\arccos(\alpha) \geq \frac{\pi}{2} - \delta$, then there is some Jordan curve $J \subset Z$ such that $|\tau(J)| \leq 4\delta$. The estimate does not depend on the direction from which the rotation is measured.

Proof. First we choose a geodesic $\gamma \subset Z$ with $length(\gamma) = length(Z)$ that connects the boundary components of Z. Then we cut Z^{int} along γ. The resulting space Q is a surface with bounded curvature in the sense of [31, p. 141] that is homeomorphic to the closed 2-disc. Let $\pi\colon Q \to Z^{int}$ be the projection map corresponding to the cutting and γ_1 and γ_2 be the two connected components of $\pi^{-1}(\gamma)$. We note that these curves are geodesics in Q.
Let $x_1, x_2 \in \gamma_2$ with $x_1 \neq x_2$. Moreover let $\xi_i \subset Q$ be a geodesic from x_i to some closest point of γ_1 such that ξ_i does not intersect $\pi^{-1}(\partial Z)$. We further assume that the subsets $\xi_i \cap \gamma_2$ and $\xi_1 \cap \xi_2$ are connected. For the sake of simplicity, we only consider the case that $\xi_i \cap \gamma_2$ is a singleton and $\xi_1 \cap \xi_2$ is empty. All remaining cases can be obtained similarly or follow from this case.

The first variation formula (cf. [22, pp. 1034, 1036]) shows that the two sector angles between γ_1 and ξ_i are at least $\frac{\pi}{2}$ and it follows that they lie in $\left[\frac{\pi}{2}, \frac{\pi}{2} + \delta\right]$. We note that the sum of the two sector angles between ξ_i and γ_2 lies in $[\pi, \pi + \delta]$.

There is a unique closed topological 2-disc $R \subset Q$ that is bounded by a subset of $\gamma_1 \cup \xi_1 \cup \gamma_2 \cup \xi_2$. The Gauss-Bonnet theorem implies that the sum of the sector angle between ξ_1 and γ_2 and the sector angle between ξ_2 and γ_2, both measured from the side of R, lies in $[\pi - 3\delta, \pi + \delta]$. We conclude that the sector angle between ξ_1 and γ_2, measured from the side of R, differs from the sector angle between ξ_2 and γ_2, measured from outside of R, by at most 5δ.

We may assume that γ_2 is parametrized by arc length and denote its domain by I. Furthermore let $I_0 \subset I$ be the maximal subinterval such that $\pi^{-1}(\gamma \cap Z') \subset \gamma_2(I_0)$ and the following property is satisfied: For every $t \in I_0$ and every geodesic connecting $\gamma_2(t)$ with γ_1 of length $d_{\gamma_1}(\gamma_2(t)) \coloneqq d(\gamma_1, \gamma_2(t))$ the intersection with $\pi^{-1}(\partial Z)$ is empty.

Using the aforementioned estimate for the difference of the sector angles, the first variation formula yields

$$|\partial^+ d_{\gamma_1}(\gamma_2(s)) - \partial^+ d_{\gamma_1}(\gamma_2(t))| \leq 7\delta$$

for all $s, t \in I_0$.

For the sake of contradiction, we assume that $|\partial^+ d_{\gamma_1}(\gamma_2(t))| > \alpha$ for all $t \in I_0$. Then we may assume that $\partial^+ d_{\gamma_1}(\gamma_2(t)) > \frac{c}{a}$. By the mean value theorem for one-sided differentiable functions in [26, p. 473] we have

$$d_{\gamma_1}(\gamma_2(s)) > \frac{c}{a}(s - t)$$

for all $s \geq t$. There is a choice of the parameters such that $d_{\gamma_1}(s) \leq c$ and $s - t \geq a$. This yields $c > 2c$. A contradiction.

It follows the existence of some $t_0 \in I_0$ with $|\partial^+ d_{\gamma_1}(\gamma_2(t_0))| \leq \alpha$. Due to the first variation formula there is a geodesic $\xi \subset Q$ connecting $\gamma_2(t_0)$ with γ_1 of length $d_{\gamma_1}(\gamma_2(t_0))$ such that $\xi \cap \gamma_2$ is a singleton and the sector angles between ξ and γ_2 lie in $[\arccos(\alpha), \pi - \arccos(\alpha)]$. We connect the endpoints of $\pi(\xi)$ by some subarc of γ and denote the constructed Jordan curve in Z by J. Removing the endpoints of $\pi(\xi)$ from J, yields two local geodesics. By the estimates for the sector angles we finally derive

$$|\tau(J)| \leq \frac{\pi}{2} - \arccos(\alpha) + 3\delta \leq 4\delta.$$

In particular, the estimate does not depend on the direction from which the rotation is measured. □

Since $|\omega|_\infty$ is a finite measure, every open subset contains some point with measure zero. Combining the last three results, we therefore obtain the following corollary:

Corollary 9.3.10. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(S,\varepsilon,C)$ and $X_n \xrightarrow{\mathcal{GH}} X$. Moreover let $I \subset X$ be an open subset that is isometric to some open interval. Then for every $\delta > 0$, after passing to a subsequence, there is a point $p \in I$ and Jordan curves $J_n \subset X_n$ such that $J_n \xrightarrow{\mathcal{H}} \{p\}$ and $|\tau(J_n)| \leq \delta$. The estimate does not depend on the direction from which the rotation is measured.

Finally we are able to show Lemma 9.3.5. As already discussed, this is the last step to prove Proposition 9.3.4.

Proof of Lemma 9.3.5. We write $A_1, \dots, A_{ind(p)}$ for the connected components of $B_r(p) \setminus \{p\}$.
By Lemma 9.3.6 and Corollary 9.3.10, after passing to a subsequence, we find disjoint Jordan curves $J_{i,n}$ bounding closed topological 2-discs in X_n such that $J_{i,n}$ Hausdorff converges to some subset $J_i \subset A_i$ and $\tau(J_{i,n}) \geq -2\delta$ where the rotation is measured from the side of p_n.
For $1 \leq i \leq N$ we may sharpen the construction further: By Lemma 9.3.6 we may assume that the subset J_i is the boundary of a closed topological 2-disc $D_i \subset A_i \cup \{p\}$ with $p \in D_i^0$ and $\tau(J_{i,n}) \geq \tau(J_i) - \delta$ where the rotation is measured from the side of p_n and p respectively. Choosing r small enough, we may assume that $|\omega_{T_i}|(D_i \setminus \{p\}) \leq \delta$. Then the Gauss-Bonnet theorem yields $\tau(J_i) \geq \theta_{T_i}(p) - \delta$ and hence $\tau(J_{i,n}) \geq \theta_{T_i}(p) - 2\delta$.
Moreover we may assume that all curves $J_{k,n}$ with $k \neq i$ lie on the side of $J_{i,n}$ that contains p_n. This yields that the curves bound a compact subsurface $S_n \subset X_n$ with $\partial S_n = \cup_{i=1}^{ind(p)} J_{i,n}$ and $p_n \in S_n^0$.
In the following we denote the Jordan curves constructed via the radius r by $J_{i,n}(r)$ and the corresponding compact subsurfaces by $S_n(r)$. Choosing a diagonal sequence, we may assume that the subsets $S_n\left(\frac{1}{n}\right)$ Hausdorff converge to $\{p\}$. Moreover due to Theorem 9.2.1 we may assume that $\chi(S_n) = 2 - ind(p)$.
This gives the construction in the case that p is a cut point.
The proof for endpoints proceeds similar: For arbitrarily small r the subset A_1 is isometric to some open interval or A_1 contains infinitely many maximal cyclic subsets of X.

We start with the first case: From Corollary 9.3.10, after passing to a subsequence, we obtain Jordan curves $J_{1,n} \subset X_n$ bounding closed topological 2-discs such that $J_{1,n}$ Hausdorff converges to some point in A_1 and $\tau(J_{1,n}) \leq 2\delta$ where the rotation is measured from the side of p_n.

In the second case by Lemma 9.3.6 we may assume the existence of such curves such that $J_{1,n}$ Hausdorff converges to the boundary of a closed topological 2-disc in A_1. In addition we may assume that $\omega_{X_n}(J_{1,n}) \leq \delta$ and $\tau(J_{1,n}) \geq -\delta$ where the rotation is measured from the side opposite to p_n. Then it follows that the rotation measured from the side of p_n is at most 2δ.

The rest of the proof proceeds as before. $\square$

Because the measure $|\omega|_\infty$ is bounded by C, Main Theorem III (A) is a direct consequence of Corollary 9.3.3 and Proposition 9.3.4.

9.4 Approximation of spaces in L(S,C)

This section is devoted to the proof of Main Theorem III (B).

First we note that all spaces $X \in \mathcal{L}(S, C)$ are locally simply connected by Theorem 6.1.1 and Main Theorem I.

A deeper analysis of the construction in the proof of Lemma 6.2.5 yields our first result. We give a brief sketch of the construction:

Let $(T_n)_{n\in\mathbb{N}}$ be an enumeration of the maximal cyclic subsets in X. We shrink each connected component of $\cup_{i=n+1}^{n+k} T_i$ to a point and denote the constructed space by $X_{n,k}$. After passing to a subsequence, the spaces $X_{n,k}$ Gromov-Hausdorff converge to some space X_n. The maximal cyclic subsets of X_n are isometric to the first n maximal cyclic subsets of X and we have $X_n \xrightarrow{\mathcal{GH}} X$.

From the construction we deduce the following lemma:

Lemma 9.4.1. Let $X \in \mathcal{L}(S, C)$. Then X can be obtained as the Gromov-Hausdorff limit of a uniformly semi-locally 1-connected sequence $X_n \in \mathcal{L}(S, C)$ such that X_n has only finitely many maximal cyclic subsets.

In the following we write $L_{k,n}$ for the metric space obtained by gluing k disjoint copies of $[0, 2^{-n}]$ along 0.

A space $X \in \mathcal{L}(S, C)$ having only finitely many maximal cyclic subsets is a successive metric wedge sum of its maximal cyclic subsets and finitely many compact intervals.

Replacing every wedge point $p \in X$ with $L_{ind(p),n}$ and then slightly moving the intervals, we derive the following result:

Lemma 9.4.2. Let X be a space in $\mathcal{L}(S, C)$ having only finitely many maximal cyclic subsets. Then X can be obtained as the Gromov-Hausdorff limit of spaces $X_n \in \mathcal{L}(S, C)$ satisfying the following properties:

1) X_n is a successive metic wedge sum of the maximal cyclic subsets of X and finitely many compact intervals.

2) Every wedge point is the endpoint of some of the intervals. Moreover it lies in some maximal cyclic subset of X_n or the interior of some of the intervals.

3) Every wedge point lies in exactly two of the wedged spaces.

4) The maximal cyclic subsets of X_n do not intersect.

The upcoming proposition provides a tool to simplify the approximating spaces further.

Proposition 9.4.3. Let S_1 and S_2 be closed surfaces with bounded curvature and X be a metric wedge sum of them. We further assume that every loop in S_1 of diameter at most ε is contractible in S_1 and that S_2 is homeomorphic to $\mathbb{S}^2$. Moreover let $F := \{p_1, \ldots, p_k\} \subset X \setminus \{p\}$ where p denotes the wedge point of X. Then X can be obtained as the Gromov-Hausdorff limit of surfaces with bounded curvature X_n satisfying the following properties:

1) X_n is homeomorphic to S_1.

2) We have

$$\lim_{n \to \infty} |\omega_{X_n}|(X_n) \leq \sum_{i=1}^{2} |\omega_{S_i}|(S_i \setminus \{p\}) + \theta_{S_i}(p).$$

3) There are isometries $f_i \colon S_i \to S_i$ and points $p_{j,n} \in X_n$ such that $p_{j,n} \longrightarrow f_i(p_j)$ and $\theta_{X_n}(p_{j,n}) = \theta_{S_i}(p_j)$ for every $p_j \in S_i$.

4) Every loop in X_n of diameter at most $\frac{\varepsilon}{2}$ is contractible in X_n.

Proof. Proceeding as in the proof of Theorem 8.2.3, we find convex closed topological 2-discs $D_{i,n} \subset S_i \setminus F$ satisfying the following properties: The point p lies in the interior of the disc, $J_{i,n} := \partial D_{i,n}$ is free of peak points and the absolute rotation of $J_{i,n}$, measured

from the side of $S_i \setminus D_{i,n}$, (cf. [2, pp. 272, 308]) is bounded by some constant that does not depend on n. Moreover we have $\lim_{n\to\infty} l_{i,n} := length(J_{i,n}) = 0$, $r_n := l_{2,n} - l_{1,n} > 0$ and $\lim_{n\to\infty} diam(D_{i,n}) = 0$.
From the Gauss-Bonnet theorem we derive $\tau(J_{i,n}) \leq \theta_{S_i}(p) + |\omega_{S_i}|\Big(D^0_{i,n} \setminus \{p\}\Big)$ where the rotation is measured from the side of p.
Let now $T_n \subset \mathbb{R}^2$ be an isosceles trapezoid such that its basis has the lengths $l_{1,n}$ and $l_{2,n}$ and its hight is equal to $\sqrt{r_n}$. Moreover let Z_n be the topological cylinder obtained by gluing the legs of T_n along an isometry. We denote the boundary components of this cylinder by $b_{1,n}$ and $b_{2,n}$ where $length(b_{i,n}) = l_{i,n}$.
Furthermore we equip $X_{i,n} := S_i \setminus D^0_{i,n}$ with its induced length metric and glue the disjoint union $X_{1,n} \sqcup Z_n \sqcup X_{2,n}$ along length preserving homeomorphisms $f_i \colon J_{i,n} \to b_{i,n}$. We denote the constructed space by X_n.
It follows $X_n \xrightarrow{\mathcal{GH}} X$ and that X_n is homeomorphic to S_1.
From the gluing theorem in [2, p. 289] we derive that X_n is a surface with bounded curvature. In addition the theorem implies that, after passing to a subsequence, the desired inequality for $\lim_{n\to\infty} |\omega_{X_n}|(X_n)$ holds.
Moreover we may assume the points $p_{j,n} := p_j \in X_n$ to be convergent and we find isometries $f_i \colon S_i \to S_i$ such that $p_{j,n} \longrightarrow f_i(p_j)$ for every $p_j \in S_i$. In particular, we have $\theta_{X_n}(p_{j,n}) = \theta_{S_i}(p_j)$.
As in the proof of Lemma 7.3.3, we see that every Jordan curve in X_n is arbitrarily Hausdorff close and homotopic to some Jordan curve that intersects $J_{1,n}$ only finitely many times. Since every loop in S_1 of diameter at most ε is contractible in S_1, after passing to a subsequence, the same applies to X_n with the constant $\frac{\varepsilon}{2}$. □

The construction in our next proof uses flat cylinders in $\mathbb{R}^3$. Therefore we introduce the following notation:

$$\bar{Z}_{n,l} := \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 2^{-n},\ 0 \leq |z| \leq \frac{l}{2}\right\},$$
$$D_{n,l} := \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 \leq 2^{-n},\ |z| = \frac{l}{2}\right\}.$$

Moreover we equip $Z_{n,l} := \bar{Z}_{n,l} \cup D_{n,l}$ with its induced length metric.
We state the final approximation step:

Lemma 9.4.4. Let X be a space as in the approximating sequence from Lemma 9.4.2 such that every loop in X of diameter at most ε is contractible in X. Then X can be obtained as the Gromov-Hausdorff limit of spaces $X_n \in \mathcal{M}\Big(S, \frac{\varepsilon}{2}, C_n\Big)$ where $\lim\limits_{n\to\infty} C_n \le C$.

Proof. We denote the wedged intervals by $I_1, \dots, I_k$ and define $l_q := length(I_q)$. Next we cut out every I_q and glue in the flat cylinder Z_{n,l_q} instead. The gluing proceeds along the points $\Big(0, 0, \pm\frac{l_q}{2}\Big)$. We denote the constructed space by Y_n.
Now we successively "repair" the wedge points $p \in Y_n$ by applying the construction from the proof of Proposition 9.4.3. However, we have one more rule to follow: If p lies in the "lid" of some glued-in cylinder, we choose the curve $J_{i,n}$ from the aforementioned construction as the boundary of the "lid".
After each wedge point has been eliminated, we end up with a new space X_n. In particular, we have $X_n \xrightarrow{\mathcal{GH}} X$. Finally the proof of Proposition 9.4.3 yields that, after passing to a subsequence, we have $X_n \in \mathcal{M}\Big(S, \frac{\varepsilon}{2}, C_n\Big)$ where $\lim\limits_{n\to\infty} C_n \le C$. □

We note that X_n can be obtained as the uniform limit of spaces in $\mathcal{R}(S, C_{n,k})$ where $\lim\limits_{k\to\infty} C_{n,k} \le C_n$ (cf. [31, p. 144]). Hence Main Theorem III (B) is a direct consequence of the lemmas in this subsection.

Bibliography

[1] S. Alexander, V. Kapovitch, and A. Petrunin. *An invitation to Alexandrov geometry. CAT(0) spaces.* SpringerBriefs Math. Cham: Springer, 2019.

[2] A. D. Alexandrov and V. A. Zalgaller. *Intrinsic geometry of surfaces*, volume 15 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1967.

[3] L. Babai. Automorphism groups, isomorphism, reconstruction. In *Handbook of combinatorics. Volume 2*, pages 1447–1540. Amsterdam: Elsevier (North-Holland); Cambridge, MA: MIT Press, 1995.

[4] R. H. Bing. Partitioning a set. *Bull. Amer. Math. Soc.*, 55(12):1101–1110, 1949.

[5] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren Math. Wiss.* Berlin: Springer, 1999.

[6] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society, 2001.

[7] Y. Burago. Closure of the class of manifolds of bounded curvature. Proceedings of the Steklov Institute of Mathematics 76, 175-183 (1967); translation from Trudy Matematicheskogo Instituta Imeni V. A. Steklova 76, 141-147 (1965).

[8] Y. Burago. Curves in converging spaces. *Proc. Steklov Inst. Math.*, 76:3–29, 1965.

[9] M. Cassorla. Approximating compact inner metric spaces by surfaces. *Indiana Univ. Math. J.*, 41(2):505–513, 1992.

[10] D. B. A. Epstein. Curves on 2-manifolds and isotopies. *Acta Math.*, 115:83–107, 1966.

[11] S. C. Ferry and B. L. Okun. Approximating topological metrics by Riemannian metrics. *Proc. Amer. Math. Soc.*, 123(6):1865–1872, 1995.

[12] S. Friedl. Topology. Lecture notes, University of Regensburg, 2016-2024. `https://friedl.app.uni-regensburg.de/papers/1t-total-public-october-7-2024.pdf`. Accessed: 03-05-2025.

[13] K. Fukaya. Metric Riemannian geometry. In *Handbook of differential geometry. Vol. II*, pages 189–313. Amsterdam: Elsevier, 2006.

[14] A. Gray. *Modern differential geometry of curves and surfaces with Mathematica.* Boca Raton, FL: CRC Press, 2nd edition, 1998.

[15] H. B. Griffiths. The fundamental group of two spaces with a common point. *Q. J. Math.*, 5:175–190, 1954.

[16] M. Gromov. Groups of polynomial growth and expanding maps. Appendix by Jacques Tits. *Publ. Math. Inst. Hautes Études Sci.*, 53:53–78, 1981.

[17] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* Mod. Birkhäuser Class. Boston, MA: Birkhäuser, 2007.

[18] A. Hatcher. *Algebraic topology.* Cambridge: Cambridge University Press, 2002.

[19] M. Henle. *A combinatorial introduction to topology.* New York, NY: Dover Publications, Inc., corrected republication of the 1979 original edition, 1994.

[20] W. Jakobsche. Approximating homotopy equivalences of surfaces by homeomorphisms. *Fundam. Math.*, 118:1–9, 1983.

[21] C. Löh. *Geometric group theory. An introduction.* Universitext. Cham: Springer, 2017.

[22] A. Lytchak. Differentiation in metric spaces. *St. Petersburg Math. J.*, 16(6):1017–1041, 2005.

[23] A. Lytchak. On conformal planes of finite area. *Math. Z.*, 305(3):50, 2023.

[24] A. Lytchak and S. Wenger. Intrinsic structure of minimal discs in metric spaces. *Geom. Topol.*, 22(1):591–644, 2018.

[25] J. Matoušek, E. Sedgwick, M. Tancer, and U. Wagner. Untangling two systems of noncrossing curves. *Israel J. Math.*, 212(1):37–79, 2016.

[26] A. D. Miller and R. Výborný. Some remarks on functions with one-sided derivatives. *Amer. Math. Monthly*, 93(6):471–475, 1986.

[27] D. Ntalampekos and M. Romney. Polyhedral approximation of metric surfaces and applications to uniformization. *Duke Math. J.*, 172(9):1673–1734, 2023.

[28] P. V. Petersen. Gromov-Hausdorff convergence of metric spaces. In *Differential geometry: Riemannian geometry*, volume 54 part 3 of *Proc. Sympos. Pure Math.*, pages 489–504. Providence, RI: American Mathematical Society, 1993.

[29] P. V. Petersen. *Riemannian geometry*, volume 171 of *Grad. Texts in Math.* Cham: Springer, 3rd edition, 2016.

[30] A. Petrunin. *Pure metric geometry.* SpringerBriefs Math. Cham: Springer, 2023.

[31] Y. G. Reshetnyak and R. V. Gamkrelidze, editors. *Geometry IV. Non-regular Riemannian geometry. Transl. from the Russian by E. Primrose*, volume 70 of *Encyclopaedia of Mathematical Sciences.* Berlin: Springer, 1993.

[32] J. H. Roberts and N. E. Steenrod. Monotone transformations of two-dimensional manifolds. *Ann. of Math. (2)*, 39:851–862, 1938.

[33] H. Ruskeepaa. *Mathematica navigator. Mathematics, statistics and graphics. With CD-ROM.* Amsterdam: Elsevier/Academic Press, 3rd edition, 2009.

[34] K. Sakai. *Geometric aspects of general topology.* Springer Monogr. Math. Tokyo: Springer, 2013.

[35] T. Shioya. The limit spaces of two-dimensional manifolds with uniformly bounded integral curvature. *Trans. Amer. Math. Soc.*, 351(5):1765–1801, 1999.

[36] C. Sormani and G. Wei. Hausdorff convergence and universal covers. *Trans. Amer. Math. Soc.*, 353(9):3585–3602, 2001.

[37] W. Tuschmann. Hausdorff convergence and the fundamental group. *Math. Z.*, 218(2):207–211, 1995.

[38] G. T. Whyburn. On sequences and limiting sets. *Fund. Math.*, 25:408–426, 1935.

[39] G. T. Whyburn. *Analytic topology*, volume 28 of *Amer. Math. Soc. Colloq. Publ.* American Mathematical Society, Providence, RI, 1942.

[40] G. S. j. Young. On 1-regular convergence of sequences of 2-manifolds. *Amer. J. Math.*, 71(2):339–348, 1949.