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On the singularities of the exponential function of a semidirect product

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Abstract

We show that the Fréchet–Lie groups of the form $C^\infty(M) \rtimes \mathbb{R}$ resulting from smooth flows on compact manifolds M fail to be locally exponential in several cases: when at least one non-periodic orbit is locally closed, or when the flow restricts to a linear one on an orbit closure diffeomorphic to a torus. As an application, we prove that the Bondi–Metzner–Sachs group of symmetries of an asymptotically flat spacetime is not locally exponential.

Keywords: infinite-dimensional Lie group, locally exponential, BMS group, asymptotically flat spacetime, Liouville number

1. Introduction

For infinite-dimensional Lie groups beyond the Banach space setting, the Lie group exponential in general does not induce a local diffeomorphism near the unit, as singular points might exist arbitrarily close to the unit. Groups exhibiting this pathology are called *non-locally exponential*. Examples include diffeomorphism groups and the formal diffeomorphisms of \mathbb{R}^n , cf [Nee06]. In the present note we investigate when the Lie group exponential of a semidirect product exhibits singularities making the resulting Lie groups non-locally exponential.

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Concretely, we are interested in semidirect products arising as the lift of a right Lie group action $\sigma: M \times G \rightarrow M$ on a compact manifold M to the space of smooth functions $C^\infty(M)$. These give rise to our main examples of semidirect products

$$C^\infty(M) \rtimes_\alpha G, \quad \text{where } \alpha: G \times C^\infty(M) \rightarrow C^\infty(M), \quad \alpha_g(F) = F \circ \sigma_g.$$

Note that to study singularities in these semidirect products it suffices to study singularities of the exponential function by restricting to the action of 1-parameter subgroups of G . Every 1-parameter subgroup of G gives rise to a subgroup $C^\infty(M) \rtimes \mathbb{R}$ of $C^\infty(M) \rtimes G$. Then singularities of the exponential function of the subgroup are inherited by the exponential function of the semidirect product by naturality.

We can and will therefore focus our investigation on semidirect products involving 1-parameter groups. Several sufficient criteria for the existence of singularities arbitrarily close to the unit in these semidirect products are established. These allow us to treat important classes of examples such as the Bondi–Metzner–Sachs (BMS) group for asymptotically flat spacetimes from general relativity, [PS22, AA25]. An asymptotically flat spacetime models an isolated gravitational system which approximates Minkowski spacetime ‘at infinity’. These spacetimes admit a unique conformal compactification. While the BMS-group was originally conceived via the construction of a Lie algebra of vector fields satisfying certain decay conditions towards infinity, the compactification of spacetime allows to identify them with certain diffeomorphisms of a regular part at the boundary. In particular, this identifies the BMS group as a semidirect product of the Poincaré group and an infinite dimensional vector space of ‘supertranslations’. Hence our techniques allow us to prove the following

Theorem 1.1. *The BMS-group is not locally exponential.*

This settles the non-exponentiality conjecture on the BMS groups from [PS22, conjecture 3.14] in the affirmative. A more detailed discussion of the relevance of the non-local exponentiality of the BMS group and our other results will be given in the section at the end of this introduction. While the results for semidirect products of Lie group actions on spaces of smooth functions are new and yield interesting results, we were not able to completely characterize non-local exponentiality of the resulting groups. A summary and aggregate of theorem 3.10 and corollary 5.11 reads as follows.

Theorem 1.2. *Let (M, σ) be a smooth flow on a compact manifold. The Lie group $C^\infty(M) \rtimes_\alpha \mathbb{R}$ is not locally exponential if any of the following conditions holds:*

- (1) *The flow has at least one non-periodic, locally closed orbit.*
- (2) *The flow restricts to a linear non-trivial one on a torus embedded in M .*
- (3) *M admits an equivariant submersion onto a fixed-point-free flow on \mathbb{S}^1 .*
- (4) *M admits an equivariant submersion onto a linear flow on a torus.*

Thus we have reason to conjecture the following general statement.

Conjecture 1. *For every smooth non-trivial flow $\sigma: \mathbb{R} \times M \rightarrow M$ on a compact manifold M , the Lie group $C^\infty(M, \mathbb{R}) \rtimes_\alpha \mathbb{R}$, specified by $\alpha_t(f) := f \circ \sigma_t$, is not locally exponential. More specifically, there is a sequence $t_n \in \mathbb{R}_+$ with $t_n \rightarrow 0$ such that the points $(0, t_n)$ are singular for the exponential function.*

Finally, we collect in section 5 several results on the related case of semidirect products involving a Lie group acting on Banach spaces of functions of finite order differentiability. Also in this case the exponential map fails to be a local diffeomorphism. The physical relevance of these results is discussed in the next section.

Interpretation and consequences of our results for physics

The BMS group has been proposed as a candidate for ‘the symmetry group’ of asymptotically flat spacetime. The original derivation of this group in [BBM62, Sac62] was achieved via a gauge fixing approach. Later it was discovered that the BMS group can also be viewed as a group of conformal diffeomorphisms of a certain part of the boundary of the conformal compactification of spacetime. In [PS22] the basic infinite-dimensional Lie theory for the BMS group and some related groups was described. There it was uncovered that the BMS and related groups are not analytic Lie groups. This, together with the pathology uncovered in theorem A of the present paper, shows that the interplay between the BMS group and its Lie algebra is more intricate than usually in finite dimensions. In particular, there is no sensible way to construct exponential coordinates for the BMS group.

Since the BMS group is in a certain sense a minimal candidate for a symmetry group of asymptotically flat spacetimes, the complications in the correspondence of Lie algebra and associated Lie group should also be expected for the other groups proposed as candidates for asymptotic symmetry groups. More specifically, several extensions, such as the Newman–Unti (NU) group, the generalised BMS group (gBMS), the extended BMS group (eBMS) (see e.g. [PS22] for a summary) and the conformal BMS group (cBMS, [HHPB17]) have been proposed (arising e.g. through different choices in the gauge fixing procedure). The NU, gBMS and eBMS groups are constructed as pairs of global group objects and associated Lie algebras. These are also semidirect products. Beyond differences in gauge fixing, the symmetry groups discussed so far consider only the (future) null infinity part of the boundary and excise the corner parts of the conformal compactification (see e.g. [AR92]). Generalizations of the BMS group taking these parts of the compactification into account, can be expected to exhibit a semidirect product structure, whence we expect them not to be locally exponential.

Another problem is exemplified by the cBMS group. In [HHPB17], it is described via the generators of an infinite-dimensional Lie algebra. For this example and related proposed generalizations of the BMS group, our results imply that the passage to a global object will require a serious effort, as a ‘simple attempt to exponentiate the Lie algebra’ will most probably not work. Refined tools, such as criteria as to when extensions of infinite-dimensional Lie algebras integrate to Lie group extensions might help to resolve the problem in the passage from infinitesimal to global level for asymptotic symmetry groups [Nee02, Nee04, Nee07].

Finally, we would like to point out that the results in section 5 are not only of use to derive theorem B, but also have significance for the physical theory. It has been discussed in the literature whether the BMS group should not be a semidirect product of the Poincaré group with a space $C^k(M)$ of k -times continuously differentiable functions for $2 \leq k < \infty$. While [PS22] shows that in this case no Lie group structure is obtained, one might still wonder whether the topological group obtained via the construction still exhibits good mathematical properties useful for the physical theory, for instance as a half Lie group as in [MN18]. As our results in section 5 show, there is no hope to recover a well behaved analogue of the Lie group exponential in this setting. Hence we conclude that these groups can not be used to repair the pathology of the BMS group uncovered in theorem A of the present paper.

2. Groups acting on vector spaces

In this section we consider first a general setup for a simple class of semidirect products arising from a group action on a locally convex vector space. Here and in the following we write *locally convex space* as a shorthand for Hausdorff *locally convex topological vector space*.

Definition 2.1. Let G be a Lie group [Nee06, definition II.1.1], E a locally convex space, and $\alpha: G \rightarrow \mathrm{GL}(E), g \mapsto \alpha_g$ a homomorphism for which the corresponding action map

$$\alpha: G \times E \rightarrow E, \quad (g, v) \mapsto \alpha_g(v)$$

(denoted abusively by the same symbol) is smooth in the Bastiani sense. A map $f: E \supseteq U \rightarrow F$ on an open subset U of a locally convex space E is (Bastiani-)smooth if for every $x \in U, v_1, \dots, v_k \in E, k \in \mathbb{N}_0$ the iterated k -fold directional derivatives $d^k f(x)(v_1, \dots, v_k)$ exist and yield continuous mappings $d^k f: U \times e^k \rightarrow F$, see [Sch23]. The chain rule holds for Bastiani-smooth maps, hence manifolds, smooth functions on them and Lie groups can be defined in the usual way using charts.

Interpreting the vector space E as an abelian Lie group, we construct the semidirect product Lie group $E \rtimes_\alpha G$. Its Lie algebra is $\mathbf{L}(E \rtimes_\alpha G) = E \rtimes_{\mathbf{L}(\alpha)} \mathbf{L}(G)$, the semidirect product induced by the derived action, given for $x \in \mathbf{L}(G)$ by

$$\mathbf{L}(\alpha)(x)v = T_e(\alpha^v)x \quad \text{for} \quad \alpha^v(g) = \alpha_g(v).$$

For a Lie group G we call a smooth function $\exp: \mathbf{L}(G) \rightarrow G$ an *exponential function* if all curves $\gamma_x: \mathbb{R} \rightarrow G, \gamma_x(t) := \exp(tx)$, are one-parameter groups and $\gamma'_x(0) = x$. The existence of an exponential function is standard for Banach Lie groups, but there exist infinite-dimensional Lie groups without exponential function, see [Nee06, FP6]. We investigate the exponential function of the semidirect product $E \rtimes_\alpha G$. Let us recall the general form of the exponential function of such groups (see [Nee06, example II.5.9]).

Recall [Trè06, post definition 5.2] that *sequentially complete* topological vector spaces are those whose Cauchy sequences converge. This ensures the existence of integral operators such as (2.3) below (see the discussion following [Nee06, definition I.1.4]). We assume locally convex spaces carrying actions sequentially complete unless explicitly stating otherwise.

Lemma 2.2. Suppose that G has an exponential function $\exp_G: \mathbf{L}(G) \rightarrow G$, and that E is sequentially complete, so that for $x \in \mathbf{L}(G)$ the integrals $\beta_x := \int_0^1 \alpha(\exp_G(sx)) ds$ exist pointwise and define continuous linear maps on E . Then the exponential map of $E \rtimes_\alpha G$ is

$$\exp(v, x) = (\beta_x v, \exp_G(x)) \quad (2.1)$$

This is verified easily by showing that the curves $t \mapsto \exp(tv, tx)$ are one-parameter groups with tangent vector (v, x) in $t = 0$. In particular, if $G = \mathbb{R}$, we obtain as a special case the exponential map of $E \rtimes_\alpha \mathbb{R}$:

$$\exp_{E \rtimes_\alpha \mathbb{R}}: E \rtimes \mathbb{R} \rightarrow E \rtimes \mathbb{R}, \quad (v, t) \mapsto (\beta_t v, t) \quad (2.2)$$

$$\text{with } \beta_t = \int_0^1 \alpha(st) ds = \frac{1}{t} \int_0^t \alpha(s) ds. \quad (2.3)$$

In this case it suffices to study the integral operators β_t to deduce whether the group $E \rtimes_\alpha \mathbb{R}$ is locally exponential or not. We remark that [Nee06, example II.5.9 (a)] gives an easy example in which the group $E \rtimes_\alpha \mathbb{R}$ fails to be locally exponential for an action on a Fréchet space E (i.e. a locally convex completely metrizable space, [Trè06, section 10]); we shall see more below. We fix some notation.

Definition 2.3. Let $\alpha: \mathbb{R} \times E \rightarrow E, \alpha_t(v) := \alpha^v(t) := \alpha(t, v)$ be a left Lie group action on the locally convex vector space E . For the infinitesimal generator of $(\alpha_t)_{t \in \mathbb{R}}$ we write

$$D := \left. \frac{d}{dt} \right|_{t=0} \alpha_t. \quad (2.4)$$

In the following we will study the invertibility of the operators β_t . In this context, the relation

$$\frac{\alpha_t - \mathbf{1}}{t} = D \circ \beta_t$$

will prove useful. It follows immediately from

$$\frac{\alpha_t - \mathbf{1}}{t} = \frac{1}{t} \int_0^t \frac{d}{ds} \alpha_s ds = \frac{1}{t} \int_0^t D \alpha_s ds = D \circ \beta_t. \quad (2.5)$$

This discussion leads to:

Proposition 2.4. *For $T > 0$, the operator β_T is not injective if and only if there exist T -periodic vectors which are not fixed by $(\alpha_t)_{t \in \mathbb{R}}$.*

Proof. Suppose first that β_T is not injective. In view of (2.5), $\alpha_T - \mathbf{1}$ vanishes on $\ker \beta_T$. Therefore the flow on the closed subspace $\ker \beta_T$ is T -periodic, so that we obtain a representation of the compact group $\mathbb{T}_T := \mathbb{R}/T\mathbb{Z}$. The characters of this group are of the form $\chi_m: \mathbb{T}_T \rightarrow \mathbb{T}$, $\chi_m([t]) = e^{2\pi i m t/T}$, $m \in \mathbb{Z}$. For an action on a sequentially complete real locally convex space E , applying the general Peter–Weyl theorem [HM23, theorem 3.51] to the representation on the complexified space $E_{\mathbb{C}}$, shows that $F := \ker \beta_T$ is topologically generated by the eigenspaces

$$F_m := \ker \left(D^2 + \frac{4\pi^2 m^2}{T^2} \mathbf{1} \right), \quad m \in \mathbb{Z} \setminus \{0\}. \quad (2.6)$$

Actually, the Peter–Weyl theorem first implies that the eigenvectors span a dense subspace, but the eigenspaces $F_{\mathbb{C}}(\chi_m)$ combine to the corresponding real D -eigenspace

$$F_m = F \cap (F_{\mathbb{C}}(\chi_m) + F_{\mathbb{C}}(\chi_{-m})).$$

In view of this relation, the inclusion $F_m \subseteq \ker \beta_T$ follows from the fact that every χ_m -eigenvector is killed by β_T :

$$\beta_T v = \frac{1}{T} \int_0^T \alpha_s v ds = \frac{1}{T} \int_0^T e^{2\pi i m s/T} ds \cdot v = 0 \cdot v = 0.$$

Further $\beta_T|_{\ker D} = \text{id}$ implies that some F_m is non-zero. We thus obtain T -periodic vectors that are not fixed.

Conversely, if T -periodic vector exists which is not fixed by the action, then one of the eigenspaces $E_m, m \neq 0$, as in (2.6) is non-trivial by the Peter–Weyl theorem [HM23, theorem 3.51]. In view of (2.5) and the injectivity of D on each E_m , these eigenspaces are contained in $\ker(\beta_T)$. \square

Proposition 2.5. *If $i\lambda \in i\mathbb{R}^{\times}$, is an eigenvalue of D on $E_{\mathbb{C}}$, then β_T is not injective for $T \in \frac{2\pi}{\lambda}\mathbb{Z}$.*

Proof. Let $v = v_1 + i v_2 \in E_{\mathbb{C}}$ be a corresponding eigenvector. Then

$$\beta_T v = \int_0^T e^{i\lambda s} v ds = \frac{1}{i\lambda} (e^{i\lambda T} - 1) v, \quad (2.7)$$

which vanishes for $0 \neq T \in \frac{2\pi}{\lambda}\mathbb{Z}$, so that $\beta_T(v_1) = \beta_T(v_2) = 0$. \square

Now applying the observation of the previous proposition to 1-parameter groups obtained from an arbitrary Lie group action we deduce the following.

Corollary 2.6. *Let G be a Lie group acting from the left on a locally convex vector space E . If there exists a 1-parameter group in G such that the infinitesimal generator D of its action on $E_{\mathbb{C}}$ admits an unbounded sequence of purely imaginary eigenvalues, then $E \rtimes_{\alpha} G$ is not locally exponential.*

Proposition 2.7. *If there is a continuous linear eigenfunctional $\nu: E_{\mathbb{C}} \rightarrow \mathbb{C}$ which is T -periodic and not fixed by $(\alpha_t)_t$, then β_T is not surjective.*

Proof. Indeed, any such eigenfunctional will annihilate the range of β_T by (2.7): choose $n \in \mathbb{Z}$ such that $\nu \circ \alpha_t = e^{2\pi i n t / T} \nu$ for $t \in \mathbb{R}$. By assumption, $n \neq 0$, so that

$$\nu(\beta_T(v)) = \int_0^T e^{2\pi i n t / T} dt \cdot \nu(v) = 0$$

implies the assertion. \square

Remark 2.8. If E is a complex algebra and D a derivation, then $Dv = i\lambda v$, and thus

$$Dv^n = ni\lambda v^n.$$

If all powers v^n are non-zero, then all numbers $ni\lambda$ are eigenvalues of D , and therefore $E \rtimes_{\alpha} G$ is not locally exponential.

3. Semidirect products with spaces of smooth functions

We work with spaces $C^k(M) := C^k(M, \mathbb{R})$, $0 \leq k \leq \infty$ of functions on smooth (i.e. C^{∞}) finite-dimensional manifolds M , with or without a boundary (see remark 3.1 (1)). Recall that for a finite-dimensional manifold N , the compact-open topology on $C^0(N, \mathbb{R}^n)$ is the locally convex topology induced by the seminorms $\|f\|_K := \sup_{x \in K} \|f(x)\|$, where $K \subseteq N$ is compact. Then the compact-open C^k -topology, is the locally convex topology turning the map

$$C^k(M) \rightarrow \prod_{0 \leq p \leq k} C^0(T^p M, T^p \mathbb{R}) \cong \prod_{0 \leq p \leq k} C^0(T^p M, \mathbb{R}^{2^p}), \quad f \mapsto (T^p f)_{0 \leq p \leq k} \quad (3.1)$$

into a topological embedding, where the spaces on the right are endowed with the compact-open topology, cf [Nee13, 5.1]. Localizing with the help of charts as in [Sch23, lemma 2.5], we exploit that, on open subsets of euclidean space, the topology coincides with the topology of uniform convergence on compact sets of functions and derivatives up to order k . By [Trè06, example 10.1] $C^k(M)$ is a Fréchet space.

For compact M , the space $C^k(M)$ with finite k , will be Banach spaces with respect to the compact open C^k -topology. In particular, $\|\cdot\|_0$ will denote the supremum norm on M . For $k > 0$ we note that norms inducing this topology measure the supremum of a function and its derivatives up to order k . Write $\|\cdot\|_r$ for a norm inducing the compact open C^r -topology on $C^k(M)$, $0 \leq r \leq k$. Adjusting choices, we may assume that

$$\|f\|_r \leq \|f\|_s \text{ for all } r \leq s, f \in C^k(M).$$

Moreover, $C^{\infty}(M)$ is in any case ([Rud91, section 1.46] or [Trè06, proposition 34.4]) a Fréchet and Montel space (an (FM) -space as in [Köt69, section 27.2]) in the sense of [Trè06, definition 34.2], i.e. it is a barreled Fréchet space [Trè06, definition 33.1] such that closed bounded subsets are compact.

Consider a smooth action (or flow)

$$\mathbb{R} \times M \ni (t, p) \longmapsto \sigma_t(p) =: \sigma^p(t) \in M$$

on a compact smooth manifold M , with associated *infinitesimal generator* [Lee13, theorem 9.12] X (a smooth vector field on M):

$$X_p = \left. \frac{d}{dt} \right|_{t=0} \sigma_t(p) \in T_p M, \quad p \in M. \quad (3.2)$$

The orbits $\sigma_{\mathbb{R}}(p)$ are precisely the *integral curves* of X [Lee13, pre example 9.1].

The action of \mathbb{R} on M induces one on each $C^k(M)$, $0 \leq k \leq \infty$ defined by

$$\mathbb{R} \ni t \mapsto \alpha_t \in \text{Aut}(C^k(M)), \quad \alpha_t f := f \circ \sigma_t. \quad (3.3)$$

The material in section 2 then applies to $E := C^k(M)$ (we will be mostly interested in C^∞) carrying the \mathbb{R} -action α .

Remark 3.1. (1) Whether smooth manifolds M are allowed a boundary ∂M will make little difference to the discussion, so we will not revisit the issue: flows on M will leave the boundary invariant by the latter's topological invariance [Lee13, theorem 1.37], whence one can always complete such a flow to one on the boundary-less *double* [Lee13, example 9.32] $M \cup_{\partial M} M$.

See also [Lee13, theorem 9.34] and surrounding discussion for remarks to the effect that the theory of flows and integral curves holds good in the presence of a boundary so long as one restricts attention to vector fields whose restriction to ∂M is tangent to it.

(2) The \mathbb{R} -actions on $C^k(M)$ resulting from flows on M extend to the space $L_c^1(\mathbb{R})$ of compactly-supported integrable functions with respect to the usual Lebesgue (= Haar) measure in the familiar fashion ([HR79, theorems 19.18, 22.3]), by

$$\alpha_\varphi f := \int_{\mathbb{R}} \varphi(s) \alpha_s f \, ds, \quad \varphi \in L_c^1(\mathbb{R}), \quad f \in C^k(M). \quad (3.4)$$

In terms of that extension, we have

$$\beta_s = \frac{1}{s} \alpha_{\chi_{[0,s]}}, \quad \chi_A := \text{the characteristic function of the subset } A \subset \mathbb{R}. \quad (3.5)$$

The typical setup for (3.4) is that of actions on *Banach* spaces, but the extension does make sense here: for any compact interval $I \subseteq \mathbb{R}$ the partial orbit $\alpha^f(I)$ is compact, hence has compact closed convex hulls [Köt69, section 20.6(3)], so (3.4) falls within the scope of standard vector-valued integration machinery [Rud91, theorem 3.27].

Moreover, (3.4) makes sense for arbitrary $\varphi \in L^1$ whenever the orbit map α^f is bounded: Simply extend the continuous linear map $C_c(\mathbb{R}) \rightarrow E, \varphi \mapsto \alpha_\varphi f$ continuously to E .

In preparation for results concerning actions on arbitrary manifolds, we first consider operators on $C^\infty(\mathbb{R})$.

Definition 3.2 (The period s -difference and integral operators). For $s \in \mathbb{R}^\times$ we define the operators

$$\Delta_s: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad \Delta_s f(x) := f(x+s) - f(x).$$

and similarly, adapting (2.2) with $\alpha_t(f)(x) = f(x+t)$,

$$\beta_s: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad \beta_s f(x) = \int_0^1 f(x+st) \, dt.$$

Lemma 3.3. *The kernel of the operator β_s consists of all derivatives of s -periodic functions.*

Proof. The kernel of Δ_s is the closed subspace $C_s^\infty(\mathbb{R})$ of s -periodic functions. If $F \in C^\infty(\mathbb{R})$ and $f = F'$, then the function $\beta_s f$ (with β_s as in the previous section) satisfies the relation

$$\beta_s f(x) = \int_0^1 f(x+st) dt = \frac{1}{s} \int_0^s F'(x+t) dt = \frac{1}{s} \Delta_s F(x). \quad (3.6)$$

□

Setup 3.4 (The image of both operators contains the compactly supported smooth functions). Consider $g \in C_c^\infty(\mathbb{R})$, then we exploit the compact support of g to define for $s \neq 0$ a smooth function

$$f(x) := \sum_{k=1}^{\infty} g(x-ks) = g(x-s) + g(x-2s) + \cdots \in C^\infty(\mathbb{R})$$

and we note that $\Delta_s f = g$. Likewise, the compact support of g' allows us to define a smooth function

$$h(x) := s \cdot \sum_{k=1}^{\infty} g'(x-ks) = s(g'(x-s) + g'(x-2s) + \cdots)$$

such that $\beta_s h = g$.

Lemma 3.5. *We consider the space*

$$C_{\text{el}}^\infty(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) \mid \lim_{x \rightarrow +\infty} f(x) \text{ and } \lim_{x \rightarrow -\infty} f(x) \text{ exist in } \mathbb{R} \right\}.$$

The map β_s is injective on $C_{\text{el}}^\infty(\mathbb{R})$, but $\beta_s(C_{\text{el}}^\infty(\mathbb{R}))$ does not contain $C_c^\infty(\mathbb{R})$.

Proof. Injectivity is easy to see: by lemma 3.3 the kernel of β_s consists of s -periodic functions, which must thus be constant if they have limits at $\pm\infty$. But such a function is also required by lemma 3.3 to have a periodic antiderivative, so it must in fact vanish.

To show that

$$\beta_s(C_{\text{el}}^\infty(\mathbb{R})) \cap C_c^\infty(\mathbb{R}) \neq C_c^\infty(\mathbb{R}),$$

let $g \in C_c^\infty(\mathbb{R})$ and let $f \in C^\infty(\mathbb{R})$ be a smooth function with $\beta_s f = g$. By setup 3.4

$$f_0(x) := s \sum_{k=1}^{\infty} g'(x-ks) = s(g'(x-s) + g'(x-2s) + \cdots)$$

satisfies $\beta_s f_0 = g$, so that

$$f = f_0 + p, \quad \text{with } p \in \ker \beta_s$$

with p the derivative of an s -periodic function.

For all $x < \min(\text{supp } g)$, we have that $f(x) = p(x)$. If $f \in C_{\text{el}}^\infty(\mathbb{R})$, then the limit for $x \rightarrow -\infty$ exists, this implies that $p = 0$ since a periodic function having a limit has to be constant and the only constant which is the derivative of a periodic function is zero. Now, take the limit for $x \rightarrow \infty$ and we see that for $M := \max(\text{supp } g), f_0|_{[M, \infty)}$ is s -periodic. More concretely, the s -periodization

$$f_1(x) := s \cdot \text{per}_s(g')(x) := s \sum_{k \in \mathbb{Z}} g'(x-ks) = s \cdot (\text{per}_s g)'(x)$$

coincides with f_0 for large x . By the preceding argument, $\beta_s(f_0 - f_1) = g$ implies $f_1 = s \cdot \text{per}_s(g') = 0$. Note that $\text{per}_s(g') = 0$ is not satisfied for all $g \in C_c^\infty(\mathbb{R})$, proving that $\beta_s(C_{\text{el}}^\infty(\mathbb{R}))$ does not contain $C_c^\infty(\mathbb{R})$. \square

Lemma 3.5 permits us to treat certain aperiodic \mathbb{R} -actions. We illustrate this for the case where M is a circle.

Example 3.6 (Vector fields on the circle with zeros). Let \mathbb{S}^1 be the unit circle identified with $\mathbb{R}/2\pi\mathbb{Z}$ by $\mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{S}^1, \theta \mapsto e^{i\theta}$. Then 2π -periodic functions in $C^\infty(\mathbb{R}, \mathbb{R})$ correspond to vector fields on the circle. Let $0 \neq Z$ be a vector field on \mathbb{S}^1 with a zero θ_0 . The flow of Z is an \mathbb{R} -action $\alpha_Z := \text{Fl}^Z: \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$ on \mathbb{S}^1 . Choosing some $e^{i\theta_1} \in \mathbb{S}^1$ in which Z does not vanish as the initial point of the flow curve, we pull back the functions in $C^\infty(\mathbb{S}^1)$ along the resulting flow curve $\gamma(t) := \alpha_Z(t, e^{i\theta_1})$. In both cases, due to the flow approaching the stationary points of the flow, we thus obtain subspaces of $C_{\text{el}}^\infty(\mathbb{R})$ which clearly contain the space $C_c^\infty(\mathbb{R})$. Thus lemma 3.5 implies that β_s is not surjective. We deduce that $C^\infty(\mathbb{S}^1) \rtimes_{\alpha_Z} \mathbb{R}$ is not locally exponential.

Example 3.7 (Translations on \mathbb{R}). In this example we connect the integral operator to the discussion in section 2. The key is to consider the translation action on the reals. We consider the algebra $\mathcal{A} = C_c^\infty(\mathbb{R}, \mathbb{C})$ and the flow

$$\alpha_t(f)(x) = f(x+t), \quad x, t \in \mathbb{R}$$

generated by the vector field $Xf = f'$. Note that the integral operator for the exponential in the semidirect product $\mathcal{A} \rtimes_\alpha \mathbb{R}$ constructed from α is simply β_1 discussed above. The discussion shows that β_1 is injective (this also follows from proposition 2.4) but not surjective. Alternatively, one can use proposition 2.7 to derive that β_1 is not surjective: For each $p \in \mathbb{R}$, the functional

$$\lambda_p(f) := \widehat{f}(p) = \int_{\mathbb{R}} e^{ipx} f(x) \, dx$$

is continuous and satisfies

$$\lambda_p \circ \alpha_t = e^{-ipt} \lambda_p.$$

So it is 1-periodic for $p \in 2\pi\mathbb{Z}$.

Setup 3.8. Consider now a smooth flow $(\sigma_t)_t$ on a compact manifold M , T -periodic for some $T > 0$ in the sense that $\sigma_T = \text{id}$, with the induced action α of (3.3) and resulting semidirect product $E \rtimes_\alpha \mathbb{R}$.

The vector field X associated to α as in (3.2) then also produces a derivation

$$D: C^\infty(M) \rightarrow C^\infty(M), \quad D(f) = \left. \frac{d}{dt} \right|_{t=0} \alpha_t f = Xf. \quad (3.7)$$

on the algebra $C^\infty(M)$ (non-zero, because $X \neq 0$).

We will now apply the results from section 2 to prove under some conditions that $E \rtimes_\alpha \mathbb{R}$ is not locally exponential. Since D is a derivation, remark 2.8 implies that it suffices to find a non-zero imaginary eigenvalue of D on $E_{\mathbb{C}}$. Since α is T -periodic, all eigenvalues of D are contained in $\frac{2\pi i}{T}\mathbb{Z}$ and, by the Peter–Weyl theorem [HM23], the eigenvectors span a dense

subspace of $E_{\mathbb{C}}$. As $D \neq 0$, D has a non-zero eigenvalue $\lambda = 2\pi i \frac{m}{T}$ for an $m \in \mathbb{Z} \setminus \{0\}$. Hence proposition 2.5 implies that, for

$$t_n := \frac{2\pi i}{n\lambda} = \frac{T}{nm} \neq 0$$

the operators β_{t_n} are not surjective, whence not invertible. Consequently, $E \rtimes_{\alpha} \mathbb{R}$ is not locally exponential.

Example 3.9 (Linear flows on tori). Let $M := \mathbb{T}^d, d \geq 2$, $0 \neq \theta = (\theta_1, \theta_2, \dots, \theta_d) \in \mathbb{R}^d$, and consider on M the smooth flow

$$\sigma = (\sigma_t)_t, \quad \sigma_t(\mathbf{z}) := (e^{it\theta_1} z_1, \dots, e^{it\theta_d} z_d).$$

For the action on $E = C^{\infty}(\mathbb{T}^d)$ defined by $\alpha_t f := f \circ \sigma_t$, we claim that the operators β_s of (2.3) will be non-injective as well as non-surjective for a dense set of parameters $s \in \mathbb{R}_{>0}$. Indeed, the characters $\chi_{\mathbf{m}}(\mathbf{z}) := z_1^{m_1} \cdots z_d^{m_d}$ are α -eigenvectors with

$$D\chi_{\mathbf{m}} = i(m_1\theta_1 + \cdots + m_d\theta_d)\chi_{\mathbf{m}}.$$

Because $\theta \neq 0$, at least one θ_j is non-zero, and $\mathbf{m} = m_j \mathbf{e}_j$ yields a non-zero imaginary eigenvalue $i\lambda$. As no power of $\chi_{\mathbf{m}}$ vanishes, all numbers $\mathbb{Z}i\lambda$ are eigenvalues. But then:

- β_s will fail to be injective for $s \in \frac{2\pi i}{\lambda} \mathbb{Q}_{>0}$ by proposition 2.5;
- and similarly, it will also fail to be surjective for the same values of s as a consequence of proposition 2.7, upon noting that the flow $(\sigma_t)_t$ preserves the Haar measure on the torus and hence the characters we have identified as eigenvectors can also be cast as eigenfunctionals

$$C^{\infty}(\mathbb{T}) \ni f \mapsto \chi \int_{\mathbb{T}} \chi \cdot f d\mu_{\mathbb{T}} \in \mathbb{C}.$$

In particular, the Fréchet–Lie group $E \rtimes_{\alpha} \mathbb{R}$ is not locally exponential.

The following will address a broad class of cases of the introductions conjecture 1.

Theorem 3.10. *Let $\sigma = (\sigma_t)_{t \in \mathbb{R}}$ be a non-trivial smooth flow on a compact manifold M . The operators β_s on $C^{\infty}(M)$ are non-surjective in either of the following cases.*

- (1) *For almost all s if there is a non-periodic locally closed orbit $\sigma_{\mathbb{R}}(p)$.*
- (2) *For a dense set of $s \in \mathbb{R}_{>0}$ if there exists an $p \in M$, such that $T := \sigma_{\mathbb{R}}(p)$ is a torus on which the flow is linear.*

Remark 3.11. Recall [Bou66, section I.3.3, definition 2, proposition 5] that the *locally closed* subspaces of a topological space are those open in their closures.

That hypothesis of theorem 3.10(1) on $\sigma_{\mathbb{R}}(p)$ can be recast as asking that the orbit map σ^p be non-periodic and the orbit $\sigma_{\mathbb{R}}(p)$ be *locally connected* under its subspace topology has a basis consisting of connected open subsets ([Wil04, definition 27.7]).

Before going into the proof, we note that [Nee06, example II.5.9] shows that exponential surjectivity locally at the origin is equivalent to β_s being a diffeomorphism for all sufficiently small $s \in \mathbb{R}_{>0}$. Hence theorem 3.10 proves the following.

Corollary 3.12. *For non-trivial flows $(\sigma_t)_{t \in \mathbb{R}}$ as in theorem 3.10 the exponential of the corresponding Fréchet–Lie group $C^{\infty}(M) \rtimes_{\alpha} \mathbb{R}$ is not a diffeomorphism locally at the origin.*

Proof of theorem 3.10. (1) Fix an integral curve $\iota := \sigma^p$ as in the statement. Setting

$$A := \iota^* C^\infty(M) \leq C^\infty(\mathbb{R}), \quad \iota^*(f) := f \circ \iota,$$

note first that $C^\infty_{[u,v]}(\mathbb{R}) \leq A$ (functions supported in $[u, v]$) for some $u < v$. We claim that for almost all s the image $\beta_s(A)$ cannot contain all of $C^\infty_c(\mathbb{R})$.

To see this, observe first that the $[u, v]$ -supported $g \in C^\infty_{[u,v]}(\mathbb{R})$ are precisely the functions of the form $\Delta_s G$ for arbitrary

$$G := \sum_{k \geq 1} g(\bullet - ks)$$

vanishing on $(-\infty, u]$ and s -periodic on $[v, \infty)$. By (3.6) we have

$$\Delta_s(g) = s\beta_s(g'), \quad g \in C^\infty(\mathbb{R}),$$

each $g \in C^\infty_c(\mathbb{R})$ is the image through β_s of a unique function $\tilde{g} \in C^\infty(\mathbb{R})$ which

- vanishes on $(-\infty, u]$
- and has s -periodic antiderivative on $[v, \infty)$

The last condition is equivalent to \tilde{g} being s -periodic and having vanishing integral over its length- s intervals in $[v, \infty)$.

Suppose $g = \beta_s f$ for some $f \in A$. The latter function will be s -periodic on both $(-\infty, u]$ and $[v, \infty)$ and hence, for almost all s , eventually constant at $\pm\infty$ by lemma 3.13 below (which addresses the restriction $f|_{[v, \infty)}$ directly; the other case is analogous). Because

$$\beta_s(f - \tilde{g}) = 0 \implies f - \tilde{g} \text{ is globally } s\text{-periodic}$$

and \tilde{g} vanishes close to $-\infty$, it too must be constant close to both $\pm\infty$ (and indeed, vanish: not only is it periodic, but it has periodic antiderivative at $\pm\infty$). This, though, contradicts the freedom to choose the s -periodic-antiderivative $\tilde{g}|_{[v, \infty)}$ arbitrarily.

- (2) The restriction map $C^\infty(M) \rightarrow C^\infty(T)$ is onto by standard smooth-function extension theory [Lee13, lemma 2.26], so the linear-flow non-surjectivity of β_s for densely many s (example 3.9) entails non-surjectivity for $C^\infty(M)$. □

Lemma 3.13. *Let σ^p be an integral curve meeting the hypothesis of theorem 3.10(1) and fix some $a > 0$. For almost all s , the eventually- s -periodic continuous functions on (a, ∞) in $(\sigma^p)^* C^\infty(M)$ are eventually constant.*

Proof. Let $f \in (\sigma^p)^* C^\infty(M)$. The assumptions on $f|_{(a, \infty)}$ are its s -periodicity and the fact that $f \in (\sigma^p)^* C^\infty(M)$. As σ^p takes values in the compact manifold M , there is a convergent subsequence

$$\sigma^p(n_k) \xrightarrow[k]{} x \in M \implies f(n_k) \xrightarrow[k]{} f(x), \quad (n_1 < n_2 < \dots) \subset \mathbb{Z}_{>0}. \quad (3.8)$$

The crucial ingredient, next, is Weyl's result [KN74, theorem 1.4.1] to the effect that for almost all $s > 0$ the sequence $\left(\frac{n_k}{s}\right)_k$ is *uniformly distributed modulo 1* in the sense of [KN74, chapter 1, definition 1.1]:

$$\lim_{n \rightarrow \infty} \frac{\#\{1 \leq k \leq n : x_k \in J\}}{n} = \text{length of } J, \quad \text{for every subinterval } J \subseteq [0, 1].$$

This implies in particular that the set of fractional parts $\{\frac{n_k}{s}\} := \frac{n_k}{s} - \lfloor \frac{n_k}{s} \rfloor$ is dense in $[0, 1]$ so, rescaling by s ,

$$n_k = t_k N_k s + (1 - t_k)(N_k + 1)s, \quad N_k \in \mathbb{Z}_{>0} \quad \text{with} \quad \{t_k\}_k \subset [0, 1] \text{ dense.}$$

The s -periodicity of f and the convergence (3.8) then implies that f is constant (for the almost-all s in question), as claimed. \square

Corollary 3.14. *If α is a non-trivial flow on $M = \mathbb{S}^1$, then $C^\infty(\mathbb{S}^1) \rtimes_\alpha \mathbb{R}$ is not locally exponential.*

Proof. Let X be the generating vector field of α . If X has no zeros, then α is periodic, hence a linear flow on the circle with respect to suitable coordinates. Therefore theorem 3.10(1) applies. If X has a zero, then theorem 3.10(1) applies to every non-fixed point $m \in \mathbb{S}^1$. \square

The criteria above imply in particular that periodic flows on compact manifolds lead to semidirect product groups which are not locally exponential. Unfortunately, we were not able to prove an analogous criterion for the non-periodic case, but we find it very likely that the groups $C^\infty(M) \rtimes_\alpha \mathbb{R}$ are **never** locally exponential. theorem 3.10 provides sufficient criteria that can be applied in concrete cases. We demonstrate this in the following example relevant to physics.

4. Conformal actions on the sphere

In this section we first explore the actions of the conformal group on the unit sphere. Our presentation here follows [NÓ14, section 5]. As a special case, we then explain how our results solve the non-local exponentiality conjecture from [PS22, conjecture 3.14] for the BMS group.

Setup 4.1. For $d \in \mathbb{N}$ we let \mathbb{S}^d be the unit-sphere in \mathbb{R}^{d+1} , realized as the subset

$$S := \{(1, x) \in \mathbb{R}^{1, d+1} : \|x\| = 1\}$$

in $(d+2)$ -dimensional Minkowski space. The isotropic (lightlike) vectors in $\mathbb{R}^{1, d+1}$ with positive first component are of the form $\tilde{x} := \begin{pmatrix} \|x\| \\ x \end{pmatrix}$ and the (unit component of the) Lorentz group $G := \text{SO}_{1, d+1}(\mathbb{R})_0$ acts in the obvious fashion on this set by

$$\begin{pmatrix} a & b \\ c & m \end{pmatrix} \cdot \begin{pmatrix} \|x\| \\ x \end{pmatrix} = \begin{pmatrix} a\|x\| + bx \\ c\|x\| + mx \end{pmatrix}.$$

So $g * x := c\|x\| + mx$ defines an action of G on $\mathbb{R}^{d+1} \setminus \{0\}$ satisfying $g * (\lambda x) = \lambda(g * x)$ for $\lambda > 0$. The action on the unit sphere $\mathbb{S}^d \subseteq \mathbb{R}^{1, d+1}$ is then given by

$$g.x := \frac{g * x}{\|g * x\|}.$$

For unit vectors we have $a + bx = \|c + mx\| = \|g * x\|$, which leads to the cocycle relation

$$\|(g_1 g_2) * x\| = \|g_1 * (g_2 * x)\| = \|g_2 * x\| \cdot \|g_1 * (g_2.x)\|.$$

Therefore the *conformal factors* $J_g(x) := (1 + bx)^{-1}$ on \mathbb{S}^d satisfy the cocycle relation

$$J_{g_1 g_2}(x) = J_{g_1}(g_2.x) J_{g_2}(x),$$

so that

$$(\sigma_{g^{-1}} f)(x) := J_g(x) f(g.x) \tag{4.1}$$

defines a smooth left action of G on $C^\infty(\mathbb{S}^d)$. Note that the action on \mathbb{S}^d can also be written as

$$g \cdot x = (a + b \cdot x)^{-1} (c + m \cdot x) \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & m \end{bmatrix}, b \in M_{1,d}(\mathbb{R}), m \in M_d(\mathbb{R}) \quad (4.2)$$

(cf [NÓ14, equation (27)]). This action is faithful because $-1 \notin G$, which follows from

$$G \subseteq \mathrm{O}_{1,d}^+(\mathbb{R}) = \left\{ g = \begin{bmatrix} a & e \\ c & d \end{bmatrix} \in \mathrm{O}_{1,d+1}(\mathbb{R}) \mid a > 0 \right\}.$$

Lemma 4.2. *Let V be a finite-dimensional real normed space, $S \subseteq V$ its unit sphere and $A \in \mathrm{End}(V)$ with real spectrum. Then, for every $v \in S$, the limit of $e^{tA} \cdot v = \frac{e^{tA} v}{\|e^{tA} v\|}$ for $t \rightarrow \infty$ exists and is a fixed point of e^{tA} for $t \in \mathbb{R}$, acting on S .*

Proof. By theorem A.1 we have the Jordan decomposition $A = A_s + A_n$ of A , so that A_s is diagonalizable and A_n is nilpotent. Write $v = v_0 + v_1$, where v_0 is an A_s -eigenvector for the largest eigenvalue λ occurring in v , and v_1 is a sum of eigenvectors for smaller eigenvalues. Further, let $d \in \mathbb{N}_0$ be maximal with $A_n^d v_0 \neq 0$. Then

$$e^{tA} v = e^{t\lambda} e^{tA_n} v_0 + e^{tA} v_1$$

with $e^{-t\lambda} e^{tA} v_1 \rightarrow 0$ for $t \rightarrow \infty$. Moreover, $e^{tA_n} v_0$ is a polynomial of degree d . Therefore

$$t^{-d} e^{-t\lambda} e^{tA} v \rightarrow \frac{1}{d!} A_n^d v_0,$$

and thus $e^{tA} \cdot v$ converges to $\frac{A_n^d v_0}{\|A_n^d v_0\|}$ in S . □

Proposition 4.3. *Consider the conformal action of $\mathrm{SO}_{1,d+1}(\mathbb{R})$ on the sphere \mathbb{S}^d and for some non-zero $x \in \mathfrak{so}_{1,d+1}(\mathbb{R})$. Then, for $\gamma_x(t) := \sigma_{\exp(tx)}$, the group $C^\infty(\mathbb{S}^d, \mathbb{R}) \rtimes_{\gamma_x} \mathbb{R}$ is not locally exponential and, in particular, $C^\infty(\mathbb{S}^d, \mathbb{R}) \rtimes \mathrm{SO}_{1,d}(\mathbb{R})$ is not locally exponential.*

Proof. Recall, e.g. from [Kna02, I.8] that $\mathfrak{so}_{1,d+1}(\mathbb{R})$ is a semisimple Lie algebra. We can thus apply theorem A.2 to obtain a real Jordan decomposition $x = x_e + x_h + x_n$ of $x \in \mathfrak{so}_{1,n}(\mathbb{R})$, where x_e is elliptic, x_h is hyperbolic and x_n is nilpotent and these summands commute pairwise.

Case 1: If $x_e = 0$, then lemma 4.2 implies that flow lines on the sphere have limits for $t \rightarrow \pm\infty$, so that theorem 3.10(1) applies.

Case 2: Now we assume $x_e \neq 0$. Using the preceding lemma, we see that the flow generated by $x_h + x_n$ has fixed points, and the set F of these fixed points is invariant under the flow generated by x_e . As $x \neq 0$, there exists $m \in F$ not fixed by x_e . Then the orbit closure $N := \overline{\exp(\mathbb{R}x_e) \cdot m}$ is a torus on which x_e generates a linear flow with a dense orbit. We multiply the invariant measure μ_0 on N with a non-constant eigenfunction f for the x_e -action on N , say $x_e f = i\lambda f$. Then the measure $\mu := f\mu_0$ on the torus N is an eigenvector for the action of $\exp(\mathbb{R}x)$ in the space of measures, corresponding to a non-zero imaginary eigenvalue $i\lambda$ of x . This implies the existence of arbitrarily small t with β_t not surjective. □

Non-local exponentiality of the BMS group

The BMS group models symmetries of an asymptotically flat spacetime in general relativity (see e.g. [BBM62, Sac62]). From a Lie theoretic point of view this group and related examples have been discussed in [PS22]. It was conjectured that the BMS-group is not locally exponential. Recall first the definition of the BMS-group.

Definition 4.4. For $d \in \mathbb{N}, d \geq 2$, the d -dimensional BMS_d group is the Lie group

$$BMS_d := C^\infty(\mathbb{S}^d) \rtimes_\sigma SO_{1,d+1}(\mathbb{R}),$$

where the right action σ is given by (4.1). For $d = 2$, the resulting group is simply known as the BMS group.

The presentation of the BMS group here differs somewhat from the one in [PS22]. However, see e.g. [Töt02, section 1.2] for a discussion connecting it with the Möbius transformation picture used in [PS22]. Due to our previous results, the proof of the following result is now easy.

Theorem 4.5. *The BMS_d -group is not locally exponential. In particular, the BMS-group is not locally exponential.*

Proof. Let $0 \neq x \in \mathfrak{so}_d(\mathbb{R}) \subseteq \mathfrak{so}_{1,d+1}(\mathbb{R})$. Then the linear action of $\exp(\mathbb{R}x) \subseteq SO_d(\mathbb{R})$ leaves the sphere $\mathbb{S}^d \subseteq \mathbb{R}^{1,d+1}$ invariant, so that $J_{\exp tx} = 1$ for all $t \in \mathbb{R}$. Therefore the subgroup $C^\infty(\mathbb{S}^d) \rtimes_\sigma \exp(\mathbb{R}x)$ of the BMS_d group is not locally exponential by proposition 4.3 and this implies that the BMS_d group is not locally exponential. Specializing to the case $d = 2$ yields the non local exponentiality of the BMS group. \square

5. Complements: C^k spaces, periodic flows, and topological embeddings

The present section contains a number of results complementing those of section 3. In first instance, note that the operators β_s of (3.5) are much more easily proven non-isomorphic on C^k spaces for *finite* k . To help justify the claim, we record the following simple remark on representations of locally compact abelian groups on Banach spaces. The essence of the discussion is that Banach norms afford the full force of the theory of *Arveson spectra* introduced in [Arv74] (see also [Tak03, section XI.1] or [Arv82, section 2]).

Proposition 5.1. *Let G be a locally compact abelian group and $G \xrightarrow{\alpha} GL(V)$ a representation on a Banach space V with continuous orbit maps and $\sup_{g \in G} \|\alpha_g\| < \infty$. If α is not norm-continuous, then none of the bounded operators*

$$\alpha_f := \int_G f(g) \alpha_g d\mu_G(g), \quad f \in L^1(G),$$

on V are invertible.

Proof. By [Arv82, theorem 2.13] (also [Tak03, corollary XI.1.16]) the *Arveson spectrum* [Tak03, definition XI.1.2] $\sigma(\alpha)$ of the representation is non-compact; the spectrum of any operator $\alpha_f, f \in L^1(G)$ on V thus contains 0 by [Arv82, equation (2.7), p 217], so cannot be invertible. \square

Corollary 5.2. *For a non-trivial flow $(\sigma_t)_{t \in \mathbb{R}}$ on a compact smooth manifold M inducing a bounded family $(\alpha_t)_{t \in \mathbb{R}}$ on C^k for some $0 \leq k < \infty$, the operators $\alpha_f, f \in L^1(\mathbb{R})$ are never automorphisms on $C^k(M)$ for $0 \leq k < \infty$.*

In particular, the operators $\beta_s = \frac{1}{s} \alpha_{\chi_{[0,s]}}$, $s > 0$ of (3.5) are never invertible.

Proof. This will be an immediate consequence of theorem 3.10, once we confirm that result's hypothesis: the map

$$\mathbb{R} \ni t \mapsto \alpha_t \in \mathcal{L}(C^k(M)) := \{\text{bounded operators on } C^k(M)\}$$

is not norm-continuous for $0 \leq k < \infty$. Indeed, norm continuity is equivalent to the C^k -norm continuity of the infinitesimal generator $Xf = \frac{d}{dt} \big|_0 f \circ \sigma_t$ of $(\alpha_t)_t$ ([Rud91, theorem 13.36]). That

continuity fails is easily seen by restricting to a small portion of an integral curve in a coordinate patch: the differentiation operator is not continuous on $C^\infty(\mathbb{R})$ for the C^k -norm, for small functions can have large derivatives. \square

Returning to the setup of theorem 3.10, recall that, by the *Open Mapping theorem* [Trè06, theorem 17.1], a morphism $E \xrightarrow{f} F$ of Fréchet spaces is onto precisely when it induces a topological and linear isomorphism $E/\ker f \cong F$. The appropriate dual to surjectivity, for a linear map $E \xrightarrow{f} F$, is that it be a *topological embedding*: a homeomorphism onto its image.

Remark 5.3. Equivalently, a morphism $E \xrightarrow{f} F$ of locally convex topological vector spaces is a topological embedding precisely when for every continuous seminorm p on E there is some continuous seminorm q on F with

$$q(f(v)) \geq p(v), \quad \forall v \in E.$$

For morphisms of Fréchet spaces this is easily seen to also be equivalent to injectivity and having a closed range:

If the range $f(F_1)$ is closed then the map is an isomorphism of its domain onto its image by the Open Mapping theorem (applicable because the range, being closed in a Fréchet space, is itself Fréchet).

Conversely, the topological embedding condition implies the continuity of the inverse $f^{-1}: f(F_1) \rightarrow F_1$, so the range must be complete (because the domain was) and hence closed in F_2 .

A first general remark concerning topological embeddings:

Proposition 5.4. *Let $M \rightarrow \pi N$ be a smooth submersion, equivariant for two flows σ^M and σ^N on M and N respectively. If $f \in L^1(\mathbb{R})$ and α_f^N is not a topological embedding on $C^\infty(N)$, then neither is α_f^M on $C^\infty(M)$.*

Proof. If α_f were a topological embedding on $C^\infty(M)$, then its restriction to the closed subspace $C^\infty(N) \cong \pi^* C^\infty(N) \subseteq C^\infty(M)$ would also have this property. \square

Nothing like theorem 3.10(1) can hold in that form assuming only the existence of *periodic* orbits. Noting that such an orbit carries an action of the quotient $\mathbb{R} \rightarrow \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, we will focus, for the purpose of elucidating matters, on circle actions. theorem 5.8 says effectively that, by contrast to theorem 3.10(1), *most* operators α_{χ_I} associated to smooth circle flows (M, σ) and segments $I \subset \mathbb{S}^1$ are invertible.

To make sense of ‘most’, recall first ([Bak22, p 2], [Bug12, definition E.6], [Oxt80, pre Thm. 2.3]) that a *Liouville number* $x \in \mathbb{R} \setminus \mathbb{Q}$ is one such that

$$(\forall N \in \mathbb{Z}_{>0}) (\exists p \in \mathbb{Z}, q \in \mathbb{Z}_{>1}) : \left| x - \frac{p}{q} \right| < \frac{1}{q^N}.$$

The following alternative characterization of (non-)Liouville numbers will be more directly useful below; the statement employs the familiar *big O notation* (e.g. [Ten15, Notation, p xxiii])

$$f = O(g) \quad \text{meaning} \quad |f| \leq C|g| \text{ for some constant } C > 0.$$

Lemma 5.5. *The irrational non-Liouville numbers are precisely those real numbers x for which*

$$\frac{1}{|e^{2\pi i k x} - 1|} = O(k^N) \text{ for some } N \in \mathbb{Z}_{>0}. \quad (5.1)$$

Proof. Irrationality is of course equivalent to the left-hand side of (5.1) being finite, so we henceforth take it for granted. As to the balance of the claim, observe first that

$$|e^{2\pi i kx} - 1| = 2|\sin \pi kx| \asymp \min(\{kx\}, 1 - \{kx\}),$$

as a function of k , where $\{\cdot\}$ indicates fractional parts and ‘ \asymp ’ means being of the same order (i.e. each function is O (the other); [Ten15, p xxiii] again).

That last quantity always dominates $\frac{C}{k^N}$ for some $C > 0$ and positive integer N precisely when

$$|kx - p| > \frac{C}{k^N} \iff \left|x - \frac{p}{k}\right| > \frac{C}{k^{N+1}}, \quad \forall p \in \mathbb{Z}, k \in \mathbb{Z}_{>0} :$$

the negation of the defining property of Liouville numbers, in other words. \square

Motivated by lemma 5.5, we discuss a multi-dimensional version of the Liouville property.

Definition 5.6. For d -tuples $\mathbf{k} = (k_j)_{j=1}^d$ and $\mathbf{k}' = (k'_j)_{j=1}^d$ of real numbers, write

$$\mathbf{k} \cdot \mathbf{k}' := \sum_j k_j k'_j, \quad |\mathbf{k}| := (\mathbf{k} \cdot \mathbf{k})^{1/2}.$$

for d -tuples of real numbers. We might occasionally write $\mathbf{k}^2 := \mathbf{k} \cdot \mathbf{k}$.

A tuple $\boldsymbol{\theta} = (\theta_j)_{j=1}^d \in \mathbb{R}^d$ is

- (1) *rationally dependent* if $\mathbf{k} \cdot \boldsymbol{\theta} \in \mathbb{Z}$ for some non-zero $\mathbf{k} \in \mathbb{Z}^d$ (or: the θ_j and $1 \in \mathbb{R}$ are linearly independent in the \mathbb{Q} -vector space \mathbb{R});
- (2) *rationally independent* otherwise;
- (3) and *Liouville* if rationally independent and

$$\forall N \in \mathbb{Z} \quad : \quad \frac{1}{|e^{2\pi i \mathbf{k} \cdot \boldsymbol{\theta}} - 1|} \neq O(|\mathbf{k}|^N) \quad \text{for } 0 \neq \mathbf{k} \in \mathbb{Z}^d.$$

For $d = 1$ rational (in)dependence specializes back to being (ir)rational respectively and similarly, the Liouville property for tuples specializes (by lemma 5.5) to that for numbers.

Theorem 5.7. Let $\sigma = (\sigma_t)_t$ be a linear flow on the torus $\mathbb{T}^d \cong (\mathbb{S}^1)^d$ in the sense of example 3.9, given by

$$\sigma_t(z_1, \dots, z_d) := (e^{2\pi i t \theta_1} z_1, \dots, e^{2\pi i t \theta_d} z_d).$$

The following conditions on an interval $I \subset \mathbb{R}$ of length ℓ are equivalent.

- (a) The operator $\alpha_{\chi_I} := \int_I \alpha_s ds$ is invertible on $C^\infty(\mathbb{T}^d)$.
- (b) α_{χ_I} is a topological embedding.
- (c) α_{χ_I} is surjective.
- (d) The tuple $(\ell \theta_j)_{j=1}^d$ is rationally independent and non-Liouville.

Proof. The implication (a) \Rightarrow (b) & (c) is trivial, so the substantial claims are

$$(b) \quad \text{or} \quad (c) \implies (d) \implies (a). \tag{5.2}$$

An application of the *Fourier transform* [Rud90, section 1.2]

$$f \longmapsto \hat{f}, \quad \hat{f}(\mathbf{k}) := \int_0^1 e^{-2\pi i \mathbf{k} \cdot \mathbf{t}} f(\mathbf{t}) d\mu_{\mathbb{T}^d}(\mathbf{t}), \quad \mathbf{k} \in \mathbb{Z}^d$$

for the Pontryagin dual pair $(\mathbb{T}^d, \mathbb{Z}^d)$ will map $C^\infty(\mathbb{T}^d)$ precisely onto the space of (*rapidly decreasing*) sequences $(x_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ for which the norms

$$\|(x_{\mathbf{k}})_{\mathbf{k}}\|_N := \left(\sum_{\mathbf{k} \in \mathbb{Z}^d} (1 + \mathbf{k} \cdot \mathbf{k})^N \cdot |x_{\mathbf{k}}|^2 \right)^{1/2} \quad (5.3)$$

are all finite, topologized as a Fréchet space by that family of norms (see e.g. [Rud91, chapter 7, Exercise 22]). The translation action by

$$\mathbf{z} = e^{2\pi i \mathbf{s}} := (e^{2\pi i s_j})_{j=1}^d \in \mathbb{T}^d, \quad \mathbf{s} = (s_i)_{i=1}^d$$

transports over to

$$(x_{\mathbf{k}})_{\mathbf{k}} \xrightarrow{\mathbf{z}} (\mathbf{z}^{\mathbf{k}} x_{\mathbf{k}})_{\mathbf{k}} = (e^{2\pi i \mathbf{k} \cdot \mathbf{s}} x_{\mathbf{k}})_{\mathbf{k}},$$

so that

$$(x_{\mathbf{k}})_{\mathbf{k}} \xrightarrow{\alpha_{\chi_{[0, \ell]}}} \left(\int_0^\ell e^{2\pi i s \mathbf{k} \cdot \boldsymbol{\theta}} x_{\mathbf{k}} ds \right)_{\mathbf{k}} = \left(\frac{e^{2\pi i \ell \mathbf{k} \cdot \boldsymbol{\theta}} - 1}{2\pi i \mathbf{k} \cdot \boldsymbol{\theta}} \cdot x_{\mathbf{k}} \right)_{\mathbf{k}} \quad (5.4)$$

(the last parenthetic scalar $\frac{e^{2\pi i \ell \mathbf{k} \cdot \boldsymbol{\theta}} - 1}{2\pi i \mathbf{k} \cdot \boldsymbol{\theta}}$ being understood to be ℓ when $\mathbf{k} \cdot \boldsymbol{\theta} = 0$). That operator is plainly injective precisely when $\ell \boldsymbol{\theta}$ is rationally independent, so we need not consider rationally dependent tuples $\ell \boldsymbol{\theta}$ past this point. We can now return to (5.2).

(d) \Rightarrow (a): Assuming $\ell \boldsymbol{\theta}$ rationally independent non-Liouville, consider the only possible candidate

$$(x_{\mathbf{k}})_{\mathbf{k}} \xrightarrow{\alpha_{\chi_{[0, \ell]}}} \left(\frac{2\pi i \mathbf{k} \cdot \boldsymbol{\theta}}{e^{2\pi i \ell \mathbf{k} \cdot \boldsymbol{\theta}} - 1} \cdot x_{\mathbf{k}} \right)_{\mathbf{k}} \quad (5.5)$$

for an inverse to (5.4). The hypothesis ensures that the scalars $\frac{2\pi i \mathbf{k} \cdot \boldsymbol{\theta}}{e^{2\pi i \ell \mathbf{k} \cdot \boldsymbol{\theta}} - 1}$ increase at most polynomially in k , so this is indeed a continuous operator on the Fréchet space defined by the norms (5.3).

(b) \Rightarrow (d): To prove the contrapositive, assume this time that $\ell \boldsymbol{\theta}$ is Liouville. We can thus find tuples \mathbf{k}_N in \mathbb{Z}^d with

$$\left| \frac{e^{2\pi i \ell \mathbf{k}_N \cdot \boldsymbol{\theta}} - 1}{2\pi i \mathbf{k}_N \cdot \boldsymbol{\theta}} \right|^2 < \frac{1}{2^{2N} (1 + \mathbf{k}_N^2)^N}, \quad \forall N \in \mathbb{Z}_{>0}. \quad (5.6)$$

Writing $\delta_{\mathbf{k}_N}$ for the \mathbb{Z}^d -indexed sequence with a single entry of 1 in position \mathbf{k}_N and 0's elsewhere, (5.3) and (5.4) imply that

$$\forall N \in \mathbb{Z}_{>0} \quad : \quad \|\delta_{\mathbf{k}_N}\|_0 = 1 \quad \text{and} \quad \|\alpha_{\chi_I} \delta_{\mathbf{k}_N}\|_N < \frac{1}{2^N}.$$

This, of course, violates the topological embedding requirement for α_{χ_I} by remark~5.3.

(c) \Rightarrow (d): We again address the contrapositive claim, once more having fixed a sequence $(\mathbf{k}_N)_N$ satisfying (5.6). To prove α_{χ_I} non-surjective, simply note that the \mathbb{Z}^d -sequence

$$x_{\mathbf{k}} := \begin{cases} \left| \frac{e^{2\pi i \ell_{\mathbf{k}_N} \cdot \boldsymbol{\theta}} - 1}{2\pi i \mathbf{k}_N \cdot \boldsymbol{\theta}} \right|^{\frac{1}{2}} & \text{if } \mathbf{k} = \mathbf{k}_N \text{ for some } N \\ 0 & \text{otherwise} \end{cases}$$

cannot belong to its range.

□

Theorem 5.7 has some consequences pertaining to periodic flows.

Theorem 5.8. *Let $\sigma = (\sigma_t)_{t \in \mathbb{R}}$ be a non-trivial smooth flow of $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ on a compact manifold M . The following conditions on an arc $I \subseteq \mathbb{S}^1$ of length ℓ are equivalent.*

(a) *The operator*

$$\alpha_{\chi_I} := \int_I \alpha_s \, ds$$

is invertible on $C^\infty(M)$.

(b) *α_{χ_I} is a topological embedding.*

(c) *α_{χ_I} is surjective.*

(d) *The length ℓ is irrational non-Liouville.*

In particular, the complement of the set of lengths ℓ producing invertible α_{χ_I} has Lebesgue measure 0.

This Theorem will be proved after proposition 5.10.

Remark 5.9. Theorem 5.8 contrasts strikingly with corollary 5.2 (which may as well be placed in the same setup of \mathbb{S}^1 -flows, for it applies to those as a particular case): the ‘same’ operators are frequently invertible on C^∞ , but rarely so on C^k for finite k .

Such phenomena will be familiar from the literature on pathological spectral behavior for operators on non-normable Fréchet spaces: see [Shi74, p 291], for instance, for a recollection of an empty-spectrum operator on a space of smooth functions; there too, an abundance of differentiability translates to an abundance of invertibility.

In the following, we work with the spectrum $\text{Sp}(\alpha)$ of a separately continuous action $\alpha : G \times E \rightarrow E$ of a compact abelian group G on a Montel space E . For complex E this is

$$\text{Sp}(\alpha) = \left\{ \gamma \in \widehat{G} = \text{Hom}(G, \mathbb{T}) : E_\gamma \neq \{0\} \right\}, \quad \text{where}$$

$$E_\gamma = E_{\alpha, \gamma} := \{v \in E : (\forall g \in G) \alpha_g v = \gamma(g) v\} \quad (\text{the } \gamma\text{-eigenspace of } \alpha).$$

As for *real* Montel spaces, those can always be complexified so as to recover the analogous description of the spectrum. Note also that the material of section 2 is available, for Montel spaces are automatically sequentially complete (indeed, even *quasi-complete* [Trè06, definition 34.1]: closed bounded subsets are complete, being compact).

Proposition 5.10. *Let $\mathbb{S}^1 \times E \xrightarrow{\alpha} E$ be a continuous action of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ on a sequentially complete locally convex space and $I \subseteq \mathbb{S}^1$ an arc of length $\ell \in (0, 1)$. Then the following assertions hold:*

- (1) The operator α_{χ_I} on E is non-injective if and only if ℓ is an integral multiple of some number $\frac{1}{n}$, $0 \neq n \in \text{Sp}(\alpha)$. In particular $\ell \in \mathbb{Q}$.
- (2) α_{χ_I} is injective if and only if it has dense image.
- (3) If E is Fréchet, then α_{χ_I} is surjective if and only if it is a topological isomorphism. This is the case if α_{χ_I} is a topological embedding.

Proof. There is no harm in once more complexifying and thus working over \mathbb{C} . We will also identify $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, with $e^{2\pi i t}$ corresponding to (the image of) $t \in \mathbb{R}$.

- (1) The kernel of α_{χ_I} is an invariant subspace, hence non-zero if and only if it contains an α -eigenvector. On an eigenvector $v \in E_\gamma$, $\gamma(t) = e^{2\pi i n t}$, $n \neq 0$, we have

$$\alpha_{\chi_I} v = \frac{e^{2\pi i n \ell} - 1}{2\pi i n} v.$$

Therefore the kernel of α_{χ_I} is non-zero if and only if there exists $0 \neq n \in \text{Spec}(\alpha)$ with $n\ell \in \mathbb{Z}$, i.e. $\ell \in \mathbb{Z}\frac{1}{n}$.

- (2) From (1) we know that α_{χ_I} is injective if and only if, for all non-zero n with $E_n \neq \{0\}$, we have $\ell \notin \mathbb{Z}\frac{1}{n}$. If this is the case, then $\text{im}(\alpha_{\chi_I})$ contains all eigenspaces E_n because on E_0 it acts by multiplication with ℓ . By the Peter–Weyl Theorem, $\sum_{n \in \mathbb{Z}} E_n$ is dense in E . So that the image of α_{χ_I} is dense.

If, conversely, α_{χ_I} has dense range and $0 \neq n$ with $E_n \neq \{0\}$, then $P_n \circ \alpha_{\chi_I} \neq 0$, where $P_n : E \rightarrow E_n$ denotes the canonical projection. As this operator kills all eigenspaces E_m , $m \neq n$, it must be non-zero on E_n , on which it multiplies by a multiple of $e^{2\pi i n \ell} - 1$. Thus $e^{2\pi i n \ell} \neq 1$, so that α_{χ_I} is injective.

- (3) Because bijective morphisms between Fréchet spaces are isomorphisms [Trè06, theorem 17.1], the first statement follows from the fact that, in view of (2), surjectivity implies injectivity. Now suppose that α_{χ_I} is a topological embedding. Then it is injective, and by (2) it has dense range. Because the range is also complete, α_{χ_I} is surjective.

□

Proof of theorem 5.8. The second statement follows from the first given that the set of Liouville numbers has measure 0 by [Oxt80, theorem 2.4].

Theorem 5.7 resolves the case $M \cong \mathbb{S}^1$, so it is enough to prove the following local-to-global principle for \mathbb{S}^1 -actions (and a non-trivial arc $I \subseteq \mathbb{S}^1$):

$$\alpha_{\chi_I} \text{ iso on } C^\infty(M) \iff \alpha_{\chi_I} \text{ iso on } C^\infty(\sigma_{\mathbb{R}}(p)), \quad \forall p \in M. \quad (5.7)$$

Bijectivity being equivalent to surjectivity by proposition 5.10(3), the forward implication is immediate: if α_{χ_I} is surjective on $C^\infty(M)$, it is so on its \mathbb{S}^1 -equivariant quotient space

$$C^\infty(M) \xrightarrow{\text{restriction}} C^\infty(\sigma_{\mathbb{R}}(p)).$$

The interesting implication is the converse, which we now turn to.

If the action is trivial there is nothing to do: α_{χ_I} is the scalar ℓ (length of I). On the other hand, if the action is non-trivial, then ℓ must be irrational non-Liouville (by the $M \cong \mathbb{S}^1$ case of the theorem) and hence α_{χ_I} is invertible for the standard rotation action on \mathbb{S}^1 .

Write M as a twisted product [Bre72, section I.6(A)]

$$M \cong \mathbb{S}^1 \times_{\mathbb{S}^1} M := \mathbb{S}^1 \times M / ((z, p) \sim (zg, \sigma_{g^{-1}}(p)), \quad z, g \in \mathbb{S}^1, p \in M). \quad (5.8)$$

This is a quotient of $\mathbb{S}^1 \times M$ by the diagonal \mathbb{S}^1 -action

$$(z, p) \triangleleft g := (zg, \sigma_{g^{-1}}(p)),$$

so that the quotient map $\mathbb{S}^1 \times M \rightarrow M$ coincides with the action map. That the diagonal action commutes with the obvious left-hand-factor translation action, and the latter induces the original σ on M upon making the identification (5.8). Because α_{χ_I} is an isomorphism on $C^\infty(\mathbb{S}^1)$ under standard translation, it also operates isomorphically as $\alpha_{\chi_I} \hat{\otimes} \text{id}$ on

$$C^\infty(\mathbb{S}^1 \times M) \cong C^\infty(\mathbb{S}^1) \hat{\otimes} C^\infty(M)$$

(complete tensor product of *nuclear* topological vector spaces [Trè06, theorem 51.6]). The realization of M as a quotient (5.8) by a free action then gives

$$C^\infty(M) \cong (C^\infty(\mathbb{S}^1) \hat{\otimes} C^\infty(M))^{\triangleleft}_{\text{closed}} \leq C^\infty(\mathbb{S}^1) \hat{\otimes} C^\infty(M), \quad (5.9)$$

the superscript denoting invariant functions under the diagonal action \triangleleft . Because the diagonal action and translation on the left-hand tensor commute, the isomorphism $\alpha_{\chi_I} \hat{\otimes} \text{id}$ leaves invariant the closed subspace (5.9) on which it must thus act as a topological embedding (being the restriction of one). The conclusion that

$$\alpha_{\chi_I}|_{C^\infty(M)} = \alpha_{\chi_I} \text{ restricted to (5.9)}$$

is an isomorphism follows from proposition 5.10(3). \square

Finally, some consequences of the preceding discussion:

Corollary 5.11. *Let (M, σ) be a smooth flow on a compact manifold admitting an equivariant submersion onto either*

- (1) *A fixed-point-free flow on \mathbb{S}^1 ;*
- (2) *Or a non-trivial linear flow on a torus.*

Then the operators β_s of (3.5) fail to be topological embeddings on $C^\infty(M)$ for a dense set of parameters $s \in \mathbb{R}_{>0}$. In particular, the Lie groups $C^\infty \rtimes_{\alpha} \mathbb{R}$ attached to such flows are not locally exponential.

Proof. This follows from proposition 5.4, given that the claim holds for periodic flows on \mathbb{S}^1 and linear flows on tori by theorem 5.7. \square

Data availability statement

No new data were created or analyzed in this study.

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Conflict of interest

All authors declare that there is no conflict of interest or competing interests associated to this work.

Appendix. Real Jordan decomposition in semisimple Lie algebras

Since it seems to be difficult to cite a sufficiently complete statement of the real Jordan decomposition in semisimple real Lie algebras from the literature, we provide a proof in this appendix.

Theorem A.1 (Real Jordan decomposition). *Let V be a finite-dimensional real vector space and $A \in \text{End}(V)$. Then there exist uniquely determined endomorphisms A_n, A_s, A_h, A_e such that*

- (a) A_n is nilpotent, A_h is diagonalizable, A_s is semisimple, A_e is semisimple with purely imaginary spectrum,
- (b) $A = A_n + A_s = A_n + A_h + A_e$,
- (c) A_n, A_h, A_e commute pairwise.
- (d) $B \in \text{End}(V)$ commutes with A if and only if it commutes with A_n, A_h and A_e .

Proof. Let $V_{\mathbb{C}}$ be the complexification of V , $V_{\mathbb{C}}^{\lambda}$ the generalized λ -eigenspace of A on $V_{\mathbb{C}}$, and $V_{\mathbb{C}, \lambda}$ the λ -eigenspace of A on $V_{\mathbb{C}}$. We put

$$V_{[\lambda]} := V \cap \left(V_{\mathbb{C}}^{\lambda} + V_{\mathbb{C}}^{\bar{\lambda}} \right)$$

and observe that

$$V = \bigoplus_{\text{Im} \lambda \geq 0} V_{[\lambda]}.$$

Let $A = A_s + A_n$ be the Jordan decomposition of A into nilpotent and semisimple component. Then A_s and A_n commute and $V_{\mathbb{C}}^{\lambda}$ is the λ -eigenspace of A_s on $V_{\mathbb{C}}$. We define A_h by $A_h v = \text{Re} \lambda \cdot v$ for $v \in V_{[\lambda]}$. Then A_h is diagonalizable over \mathbb{R} and commutes with A_n and A_s , hence it also commutes with $A_e := A_s - A_h$, and A_e acts on $V_{\mathbb{C}}^{\lambda}$ by multiplication with $i \text{Im} \lambda$. This proves (a)-(c). To verify (d), we assume that B commutes with A . Then B preserves the generalized eigenspaces of A on $V_{\mathbb{C}}$, hence commutes with A_s and therefore also with A_n . It also preserves the eigenspaces of A_h , hence commutes with A_h , and finally also with $A_e = A_s - A_h$. \square

Theorem A.2. *Let L be a real semisimple Lie algebra and $x \in L$. Then there exist pairwise commuting uniquely determined elements $x_n, x_h, x_e \in L$ such that $x = x_n + x_h + x_e$ and $\text{ad} x = \text{ad} x_n + \text{ad} x_h + \text{ad} x_e$ is the real Jordan decomposition of $\text{ad} x$.*

Proof. Since $\text{ad}: L \rightarrow \text{der}(L)$ is a linear isomorphism, it suffices to verify that, for a derivation $D \in \text{der} L$, the real Jordan components are contained in $\text{der} L$. For D_s this follows from the fact that

$$[L_{\mathbb{C}}^{\lambda}, L_{\mathbb{C}}^{\mu}] \subseteq L_{\mathbb{C}}^{\lambda+\mu}, \quad \lambda, \mu \in \mathbb{C}.$$

This in turn implies that the real eigenspaces

$$L_{\mu}(D_h) = L \cap \left(\sum_{\text{Re} \lambda = \mu} L_{\mathbb{C}}^{\lambda} \right)$$

in L satisfy

$$[L_{\mu}(D_h), L_{\nu}(D_h)] \subseteq L_{\mu+\nu}(D_h).$$

Therefore D_h is a derivation of D . As $D_n = D - D_s$ is a derivation, the same holds for $D_e = D_s - D_h$ [AA25]. \square

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References

- [AR92] Ashtekar A and Romano J D 1992 Spatial infinity as a boundary of spacetime *Class. Quantum Grav.* **9** 1069–100
- [Arv74] Arveson W 1974 On groups of automorphisms of operator algebras *J. Funct. Anal.* **15** 217–43
- [Arv82] Arveson W 1980 The harmonic analysis of automorphism groups *Operator Algebras and Applications, Part I (Proc. Sympos. Pure Math)* vol 38 (Amer. Math. Soc) pp 199–269
- [Bak22] Baker A 2022 Transcendental number theory. Cambridge Mathematical Library *With an Introduction by David Masser* (Cambridge University Press) Reprint of the 1975 original (0422171)
- [BBM62] Bondi H, Van der Burg M G J and Metzner A W K 1962 Gravitational waves in general relativity. VII: waves from axi-symmetric isolated systems *Proc. R. Soc. Ser. A* **269** 21–52
- [Bou66] Bourbaki N 1966 Elements of mathematics. General topology. Part 1. Translation of the French original *Actualites Scientifiques et Industrielles Hermann Adiwes International Series in Mathematics. Hermann, Editeurs des Sciences et des Arts* vol VII (Addison-Wesley Publishing Company) p 436
- [Bre72] Bredon G E 1972 Introduction to compact transformation groups *Pure Appl. Math., Academic* vol 46 (Academic)
- [Bug12] Bugeaud Y 2012 Distribution modulo one and Diophantine approximation *Camb. Tracts Math* vol 193 (Cambridge University Press)
- [AA25] de Aguiar Alves N 2025 Lectures on the Bondi–Metzner–Sachs group and related topics in infrared physics (arXiv:2504.12521v1)
- [HHPB17] Haco S J, Hawking S W, Perry M J and Bourjaily J L 2017 The conformal BMS group *J. High Energy Phys.* **JHEP11(2017)012**
- [HM23] Hofmann K H and Morris S A 2023 The structure of compact groups. A primer for the student. A handbook for the expert *De Gruyter Stud. Math* vol 25, 5th edn (De Gruyter)
- [HR79] Hewitt E and Ross K A 1979 Abstract harmonic analysis *Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences)* vol I, 2nd edn (Springer) Structure of topological groups, integration theory, group representations
- [KN74] Kuipers L and Niederreiter H 1974 Uniform distribution of sequences *Pure and Applied Mathematics* (Wiley)
- [Kna02] Knapp A W 2002 *Lie groups beyond an introduction* vol 140, 2nd edn (Birkhäuser)
- [Köt69] Köthe G 1969 Topological vector spaces. I *Die Grundlehren der Mathematischen Wissenschaften, Band 159* and D J H Garling (Springer) Translated from the German
- [Lee13] Lee J M 2013 Introduction to smooth manifolds *Graduate Texts in Mathematics* vol 218, 2nd edn (Springer)
- [MN18] Marquis T and Neeb K-H 2018 Half-Lie groups *Transform. Groups* **23** 801–40
- [Nee02] Neeb K-H 2002 Central extensions of infinite-dimensional Lie groups *Ann. Inst. Fourier (Grenoble)* **52** 1365–442
- [Nee04] Neeb K-H 2004 Abelian extensions of infinite-dimensional Lie groups *Travaux Mathématiques. Fasc. XV (Trav. Math.)* vol 15 (Univ. Luxemb.) pp 69–194
- [Nee06] Neeb K-H 2006 Towards a Lie theory of locally convex groups *Jpn. J. Math.* **1** 291–468
- [Nee07] Neeb K-H 2007 Non-abelian extensions of infinite-dimensional Lie groups *Ann. Inst. Fourier (Grenoble)* **57** 209–271

- [Nee13] Neeb K-H 2013 Holomorphic realization of unitary representations of Banach-Lie groups *ie Groups: Structure, Actions and Representations (Honor of Joseph A. Wolf on the Occasion of his 75th Birthday)* (Birkhäuser/Springer) pp 185–223
- [NÓ14] Neeb K-H and Ólafsson G 2014 Reflection positivity and conformal symmetry *J. Funct. Anal.* **266** 2174–224
- [Oxt80] Oxtoby J C 1980 Measure and category a survey of the analogies between topological and measure spaces *Grad. Texts Math* vol 2, 2nd edn (Springer)
- [PS22] Prinz D and Schmeding A 2022 Lie theory for asymptotic symmetries in general relativity: the BMS group *Class. Quantum Grav.* **39** 22
- [Rud90] Rudin W 1990 *Fourier Analysis on Groups* (Wiley)
- [Rud91] Rudin W 1991 Functional analysis *International Series in Pure and Applied Mathematics* 2nd edn (Mcgraw-Hill, Inc.)
- [Sac62] Sachs R 1962 Asymptotic symmetries in gravitational theory *Phys. Rev. II. Ser.* **128** 2851–64
- [Sch23] Schmeding A 2023 An introduction to infinite-dimensional differential geometry *Camb. Stud. Adv. Math* vol 202 (Cambridge University Press)
- [Shi74] Shields A L 1974 The spectrum of an operator on an F-space *Proc. R. Ir. Acad., A* **74** 291–2
- [Tak03] Takesaki M 2003 Theory of operator algebras. II *Encyclopaedia of Mathematical Sciences* vol 125 (Springer)
- [Ten15] Tenenbaum G 2015 *Introduction to Analytic and Probabilistic Number Theory* 3rd expanded edn
- [Tót02] Tóth G 2002 Finite Möbius groups, minimal immersions of spheres and moduli *Universitext* (Springer)
- [Trè06] Trèves F 2006 *Topological Vector Spaces, Distributions and Kernels* (Dover Publications, Inc.) Unabridged republication of the 1967 original
- [Wil04] Willard S 2004 *General Topology* (Dover Publications, Inc.) Reprint of the 1970 original (Addison-Wesley, Reading, MA; MR0264581)