Dehn functions of groups with filiform subgroups

Zur Erlangung des akademischen Grades eines
DOKTORS DER NATURWISSENSCHAFTEN
von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)

genehmigte

DISSERTATION
von

Jerónimo García Mejía

Tag der mündlichen Prüfung:	11. September 2024
1. Referent:	JProf. Dr. Claudio Llosa Isenrich
2. Referent:	Prof. Dr. Roman Sauer

A mi madre Icela y a mi padre Victor,
quienes aún con la distancia se han hecho
presentes.

Acknowledgments

My deepest gratitude goes first and foremost to my supervisor, Claudio Llosa Isenrich, for his never-ending support, patience, and words of encouragement. I am thankful for all the mathematics and advice I learned from you. Your wisdom and kindness have been a crucial part in this project. I am happy and honoured for not only having your advice regarding mathematics, research, and academia, but also for all the conversations, time spent together, and, of course, for the lead climbing lessons. Thank you for this great experience. It has been without a doubt a great pleasure to have done my Ph.D. with you.

Second, I would also like to thank Roman Sauer for agreeing to be a referee for this thesis. Thank you for your support and for creating such a nice atmosphere at the Mathemathical Department at the KIT. I am grateful to my present and former colleagues at the Mathematical Department.

Special thanks to Gabriel Pallier for providing an ideal working atmosphere while collaborating with him, for his useful comments on an early version of this thesis, and for all the knowledge generously shared with me about geometry, groups, and maths in general. Thanks to Matteo for all the interesting mathematical and non-mathematical discussions; to Philippe for making sure laughs were always present with his quick jokes and funny remarks; to Marek for his humour, encouragement, and cool polish jazz; to João for his friendship and conversations about being an international Ph.D. student, maths, academia, and more; to Julia for her tips on writing and for her always present will to help; to Manuel for his support and advice regarding interviews and life as a Ph.D. student; and to Ines for her help with all the bureaucracy, paperwork, and, of course, for taking care of my plants while I was away. I am also deeply grateful to the rest of my colleagues: Kevin, Laura, Sebastian, David, Leonid, Gangotry, Florian, (and of course also) Marta, Merik, and Fernando for making sure that my stay at the KIT was a pleasant one! Thank you Bence for the shared moments in Bonn and Cologne, I am happy we both went on this path and even more that we both made it.

I am grateful for all the mathematical friends I made throughout this journey, I am happy to have met such a nice community of people. I am also deeply grateful with my current and former flatmates from the Marien-WG in Karlsruhe.

Although short, your presence made a big impact on my life in Karlsruhe and Germany. You immediately made me feel at home. Thank you for your support and understanding during the last phase of this journey. I will miss being your flatmate!

Thank you to all my friends and family here and on the other side of the ocean. Thank you for the immeasurable support you have knowingly and unknowingly provided me since the beginning of this journey. Your friendship is truly invaluable and has provided me with happiness and hope. Gracias a mis amistades en México quienes con todo y un océano de diferencia se acuerdan de mí. Max, Ivan, Lucía, Abi, Guille, Marco, Samuel, gracias infinitas.

I gratefully acknowledge the financial support received by the DFG Research Training Group (RTG) 2229 Asymptotic Invariants and Limits of Groups and Spaces.

Gracias a mi madre y padre por su apoyo infinito. Sin ustedes nada de esto sería posible. Gracias a mi hermano, quien siempre me ha dado palabras de aliento.

Last, but must certainly not least, I would like to thank my partner and friend Christine for all her warm support and encouragement during a major part of this journey. Thank you for bringing joy and meaning to it. I am sure that without your love, encouragement, and support this process would have been completely different.

Abstract

This thesis provides new insights on the asymptotic geometry of locally compact groups with a particular focus on their Dehn functions.

In this work, we focus on the study of the Dehn functions of compact and finitely generated groups that contain so-called *filiform groups* as subgroups. Filiform groups play a central role in understanding the asymptotic geometry of the groups we investigate. We present several classes of groups where the Dehn functions are determined by the Dehn function of a maximal filiform subgroup. This leads to the question of whether this is a general principle. We answer this question negatively by identifying an interesting phenomenon in the context of nilpotent groups. Specifically, we observe that in certain central products of nilpotent groups, where filiform groups are used as building blocks, their Dehn functions are smaller than those of the maximal filiform subgroup.

The main results of this work consist of determining the precise Dehn functions of a large class of *nilpotent groups* and of natural families of *mapping tori of right-angled Artin groups*, *RAAGs* for short. The study of Dehn functions of nilpotent groups can be understood in the context of the conjectural quasi-isometry classification of nilpotent groups (see Conjecture 1); whereas the study of the Dehn functions of mapping tori of RAAGs is motivated by a question of Vogtmann [PR19, p.2] motivated by the work in [BP94, BG96, BG10].

In the first part of this work, we establish *upper bounds on the Dehn functions of an uncountable family of nilpotent groups arising as central products of nilpotent groups* (see Theorems A and B). These results and the methods used generalise those appearing in [LIPT23]. Moreover, these results provide further supporting evidence for a conjecture made by the authors in [LIPT23, Conjecture 11.13] (see Conjecture 3). The results presented here combined with the lower bounds on Dehn functions obtained in [GMLIP23] (see Section 3.6 for an overview) provide us with the precise Dehn functions of these central products. This has the consequence of significantly extending the class of pairs of groups with bilipschitz equivalent asymptotic cones and different Dehn functions (see Corollary C). Since the Dehn function is a quasi-isometry invariant we distinguish such groups from each other up to quasi-isometry. An important feature is that it is possible to quantify the failure of the existence of a quasi-isometry in Corollary C by means of a computable algebraic invariant defined by Cornulier [Cor17]. As consequence of Theorems A and B we show that this invariant is asymptotically optimal for a particular class of simply connected nilpotent Lie groups (see Theorem D).

In the second part of this work we *determine Dehn functions of natural families of mapping tori of RAAGs*, namely semidirect products of a right-angled Artin group G by $\mathbb{Z}$. We list our main results obtained in this part:

1. A novel result in this work is the full characterisation of the Dehn functions of mapping tori of the RAAG $\mathbb{Z}^2 * \mathbb{Z}^2$ (see Theorem E). To the best of our knowledge, this provides a full description of the Dehn functions for an entire class of mapping tori of RAAGs, for which their Dehn functions have not been studied or known before. Moreover, the methods we use to obtain these Dehn functions, modulo a fairly reasonable Conjecture 5, which holds for $\mathbb{Z}^2 * \mathbb{Z}^2$ (see Lemma 4.3), have the advantage to be generalisable to obtain the Dehn functions of mapping tori of $\mathbb{Z}^m * \mathbb{Z}^n$ with $m, n \geqslant 2$.
2. We obtain a full characterisation of the Dehn functions of mapping tori of the direct product of finitely many finite rank free groups: $F_{m_1} \times \ldots \times F_{m_k}$ with $k \geqslant 2$ (see Theorem F and Theorem 4.7). This generalises the result in [PR19, Pue16, Thereom A] for the case $k = 2$.
3. Finally, we present an alternative new proof to the one in [PR19, Section 6] for the mapping torus of $\mathbb{Z}^2 * \mathbb{Z}$ with quadratic Dehn function (see Section 4.4 and Theorem 4.10). We anticipate that the strategy used to prove this case can serve as a general approach to fully characterise the Dehn functions of the mapping tori of right-angled Artin groups $\mathbb{Z}^m * \mathbb{Z}$ with $m \geqslant 2$.

We would like to mention that together the results and methods from 1 and 3, mentioned above, outline a promising path to obtaining a full characterisation of the Dehn functions of the mapping tori of the RAAGs $\mathbb{Z}^m * \mathbb{Z}^n$ with $m, n \geqslant 1$.

Statement of originality

I hereby declare that the content of this thesis is, to the best of my knowledge, original and solely my own work, except where explicitly stated otherwise. I further declare that this thesis has not been submitted, either in whole or in part, for any degree or qualification at this institution or any other institution.

Chapter 2 contains known results from the literature. Unless explicitly stated otherwise, these results are not my original work. Additionally, we provide proofs for results that, while commonly known to experts, are not easily found in the literature.

Chapter 3 is based on my joint work with my advisor, Claudio Llosa Isenrich, and my colleague, Gabriel Pallier [GMLIP23]. It first explains in detail the derivation of the upper bounds on Dehn functions of the central products of a filiform group with a simply connected nilpotent Lie group of lower nilpotency class. This corresponds to the part of [GMLIP23] for which I was a main contributor and played the leading role in the writing process (see Sections 4 and 5, and Theorems II and III in [GMLIP23]). I then place these upper bounds in context by explaining the overall results of [GMLIP23, Section 3] (see Section 3.6); I also made at least partial contributions to the contents and writing of all other parts of [GMLIP23].

Finally, Chapter 4 is my own original work, unless indicated otherwise.

Jerónimo García Mejía

Karlsruhe, 17.07.2024

Contents

List of Figures

Chapter 1

Introduction

"Seremos imperfectos porque la perfección seguirá siendo el aburrido privilegio de los dioses"

Eduardo Galeano, *El derecho al delirio*

"Wir werden siegen, wenn wir zu lernen nicht verlernt haben"

Rosa Luxemburg

The research presented in this thesis focuses on the study of the asymptotic geometry of compactly presented groups. These are locally compact topological groups given by a compact presentation, a generalisation of finite presentation in the realm of locally compact groups. Specifically, we study these groups through their Dehn functions. This places our work in the intersection of the areas of geometric group theory and combinatorial group theory.

In this work, we examine the Dehn functions of compactly and finitely generated groups that include *filiform groups* as subgroups. Filiform groups are thus central to our understanding of the asymptotic geometry of the groups we study. We present classes of groups where the Dehn functions are determined by the Dehn function of a maximal filiform subgroup (see Theorem E). This raises the question of whether this is always the case. We answer this question negatively by observing a remarkable phenomenon within the context of nilpotent groups. Specifically, we show that for an infinite family of central products of nilpotent groups, using filiform groups as building blocks, their Dehn functions are smaller than those of the maximal filiform subgroup (see Theorems A and B).

The primary classes of groups whose Dehn functions we study have geometric origins, providing us with geometric intuition to construct combinatorial and geometric arguments. Specifically, in Chapter 3, we study the Dehn functions of nilpotent groups, with a particular focus on central products of nilpotent groups, which naturally arise in the context of Lie groups. Additionally, in Chapter 4, we study Dehn functions of mapping tori of RAAGs. These groups appear naturally as fundamental groups of topological mapping tori of complexes.

1.1 Overview

In this section, we provide a concise overview of the work presented in this thesis. We contextualise our results and highlight their significance. We start by recalling the definition of Dehn function, which is the main quasi-isometry invariant studied in this work, and then proceed to elaborate on the main results obtained in Chapters 3 and 4. Before continuing, let us point out, that the objects and results mentioned in this overview are properly defined in Chapter 2.

The Dehn function is a fundamental quasi-isometry invariant of finitely presented groups. Its study has attracted significant attention in geometric group theory over the past few decades. More recently there has been a growing interest in studying this quasi-isometry invariant within the broader class of compactly presented groups, see for instance [dCT10, CT17, LB17, LIPT23].

Algorithmically, the Dehn function of a group measures the complexity of a direct attack on the *word problem*, one of the three decision problems in group theory published by Dehn in 1911 [Deh11]. It asks whether there exists an algorithm that can determine if two given words in the generators of a finitely generated group represent the same group element. Equivalently if there exists an algorithm that can determine if a given word in the generators of a finitely generated group represents the neutral element in the group.

Geometrically, the Dehn function of a group $G = \pi_1(M)$ quantifies simple connectivity of the universal cover $\widetilde{M}$ of a closed Riemannian manifold M: every closed loop in $\widetilde{M}$ bounds a disc, the Dehn function provides an upper bound on the area of this filling disc in terms of the length of the loop.

Given a a locally compact group G with compact presentation $\mathcal{P} \coloneqq \langle S \mid R \rangle$, the *area* of a word $w \in F_S$ representing the neutral element in G is the number $\mathrm{Area}_{\mathcal{P}}(w)$ defined as the minimal number of conjugates of relators $r^{\pm 1} \in R$ whose product is freely equal to w, see Section 2.6 for details. The *Dehn function of G*

is the function $\delta_G \colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ such that

$$\delta_G \colon n \mapsto \sup \{\mathrm{Area}_{\mathcal{P}}(w) \mid w =_G 1,\ |w|_S \leqslant n\}.$$

As defined the Dehn function δ_G depends on the choice of presentation for G. To remedy this, it is customary to consider Dehn functions up to an equivalence relation of functions which we denote by $\asymp$-equivalence, see Section 2.1.2. It turns out that the Dehn functions defined using two different presentations are $\asymp$-equivalent. More generally the Dehn functions of two quasi-isometric groups are $\asymp$-equivalent, see Proposition 2.39. The definition of the Dehn function of a finitely presented group is a special case of the definition of the Dehn function for compactly presented groups. We encourage readers not familiar with compactly presented locally compact groups to think about Dehn functions in the more familiar setting of finitely presented groups.

Note that under the $\asymp$-equivalence relation, all polynomial functions of the same degree d are $\asymp$-equivalent to one another and that the functions $n \mapsto a^n$ and $n \mapsto b^n$ are $\asymp$-equivalent for all $a, b > 1$. Thus, it is appropriate to say that a group has *linear* Dehn function, *quadratic* Dehn function, *polynomial* Dehn function, *exponential* Dehn function, etc.

A remarkable feature about Dehn functions is that they characterise word hyperbolic groups. Namely, a finitely presented group is *word hyperbolic* if and only if its Dehn function is linear [Gro87]. Section 2.6.1 contains a discussion on the possible Dehn functions that finitely presented groups can exhibit and important examples.

The two main classes studied in this work are: nilpotent groups and mapping tori of right-angled Artin groups, which are semidirect products of RAAGs with $\mathbb{Z}$. We now introduce them and present the results we obtained related to them.

1.1.1 Dehn functions of nilpotent groups

An important discovery that turned nilpotent groups into natural objects of study in geometric group theory is the celebrated *Polynomial Growth Theorem* by Gromov [Gro81]. It characterises finitely generated groups Γ having finite index nilpotent subgroups as precisely those with polynomial growth. In particular, this shows that groups quasi-isometric to nilpotent groups are (up to finite index) nilpotent.

Nilpotent groups are a significant class of groups that have attracted considerable attention, with numerous unresolved questions and open problems that continue to motivate current research. An important open problem is the conjectural quasi-isometry classification of nilpotent groups:

Conjecture 1 (see [Cor18, Conjecture 19.114])**.** Two simply connected nilpotent Lie groups are quasi-isometric if and only if they are isomorphic.

It is clear that two isomorphic nilpotent Lie groups are quasi-isometric, but the converse remains open. A well-known result by Mal'cev is that every finitely generated torsion-free nilpotent group Γ embeds as a cocompact lattice in a simply connected Lie group G, called its *real Mal'cev completion* [Mal51]. Thus, in the realm of finitely generated nilpotent groups Conjecture 1 can be stated as follows.

Conjecture 2 ([FM00, p.5], [Cor18, p.335], and [Lüc08, Conjecture 3.6])**.** Two finitely generated torsion-free nilpotent groups are quasi-isometric if and only if they embed as cocompact lattices in the same simply connected nilpotent Lie group.

It should be noted, that Conjecture 1 is stronger than Conjecture 2, since there exist nilpotent groups without lattices, see for instance [Rag72, Remark II.2.14].

In terms of the Mal'cev completion, Conjecture 2 states that two finitely generated torsion-free nilpotent groups Γ_1 and Γ_2 are quasi-isometric if and only if their real Mal'cev completions G_1 and G_2 are isomorphic.

Some evidence for Conjecture 2 was provided by Pansu in the 80's [Pan89, Pan83]. Pansu proved that if two simply connected nilpotent Lie groups are quasi-isometric, then their associated *Carnot graded groups* are isomorphic, see Section 2.3.2 for the precise definition of the Carnot graded group.

The evidence provided by Pansu can be stated as follows.

Theorem 1.1 ([Pan89, Théorème 3])**.** *Let Γ_1 and Γ_2 be two finitely generated torsion-free nilpotent groups and let G_1 and G_2 be their corresponding real Mal'cev completions. If Γ_1 and Γ_2 are quasi-isometric, then $gr(G_1)$ and $gr(G_2)$ are isomorphic as Lie groups.*

One key step in Pansu's proof of Theorem 1.1 was to show that all the *asymptotic cones* of a finitely generated torsion-free nilpotent group Γ are bilipschitz equivalent to the Carnot graded group $\mathrm{gr}(G)$ associated to G. In view of this result, the problem of classifying finitely generated torsion-free nilpotent groups, reduces to the problem of classifying finitely generated torsion-free nilpotent groups having bilipschitz equivalent asymptotic cones.

In contrast to Pansu's result, Benoist discovered a pair of non-isomorphic nilpotent groups with equivalent asymptotic cones [Sha04, p.121] and Shalom [Sha04] showed that they are not quasi-isometric. Concretely, Shalom proved that the *Betti numbers* of quasi-isometric finitely generated nilpotent groups coincide and by computing the second Betti numbers of Benoist's pair and noting they differ concluded that the groups are not quasi-isomorphic.

Strengthening Shalom's work, Sauer [Sau06] showed that the *real cohomology algebra* is a quasi-isometry invariant of a finitely generated nilpotent group, see also [GK21]. Sauer's results extend the class of cone equivalent pairs of nilpotent groups that can be distinguished up to quasi-isometry.

Besides the real cohomology algebra, in [LIPT23] Llosa Isenrich, Pallier, and Tessera proved that the Dehn function of a simply connected nilpotent Lie group is a quasi-isometry invariant that distinguishes between groups with equivalent asymptotic cones. This highlights the interest of Dehn functions in the context of the conjectural quasi-isometry classification of nilpotent groups.

In this context sit the results of the first part of this thesis. In Chapter 3 we compute upper bounds for Dehn functions of nilpotent groups (see Theorems A and B). Our results in the first part of the work, Chapter 3, can be further understood in the context of the $(c+1)$-*conjecture* [Gro93, $5A_5'$] which was settled by Gersten, Riley, and Holt [GHR03]: a finitely generated torsion-free nilpotent group N of nilpotency class c has Dehn function $\delta_N(n) \preccurlyeq n^{c+1}$. This upper bound is sharp for the 3-Heisenberg group, which has nilpotency class 2 and Dehn function n^3 [ECH$^+$92]. In contrast, for the 5-Heisenberg group the upper bound is *not* sharp: it has nilpotency class 2 and Dehn function n^2 [All98] and [OS99]. The results we obtained in Chapter 3 show that this behaviour is not coincidental. In particular, we show that for a large family of *central products of nilpotent groups* this behaviour appears.

Let K and L be groups, and let $\theta : Z(K) \to Z(L)$ be an isomorphism between their centres. The *central product* of K and L is the group

$$K \times_\theta L = (K \times L)/R,$$

where $R = \{(g,h) \in Z(K) \times Z(L) : h = \theta(g)\}$. If there is no ambiguity in θ we simply write $K \times_Z L$. The central product of two nilpotent groups is again a nilpotent group. A basic example is the 5-dimensional integral Heisenberg group H_5. This group can be seen as the central product of the 3-Heisenberg group with itself (see Figure 1.1).

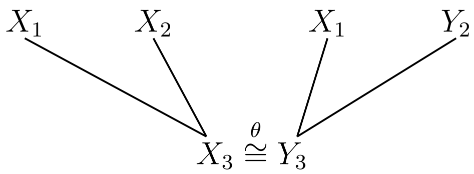

Figure 1.1: A presentation for the central product can be drawn using this kind of diagrams. For instance, for the 5-Heisenberg group as the central product of two copies of the 3-Heisenberg group: one copy is generated by X_1, X_2, and X_3, the other by Y_1, Y_2 and Y_3, and the identification can be chosen to be $\theta : X_3 \mapsto Y_3$. We denote $Z := X_3 = \theta(X_3)$. In both copies Z is the central element so for example, in the diagram, the line that goes from X_1 to X_2 passing through Z corresponds to the relation $[X_1, X_2] = Z$, no line between elements means they commute.

It is this decomposition that leads to the reduction in Dehn function of the 5-Heisenberg group [OS99, You13a]. It is conjectured that this behaviour happens in general:

Conjecture 3 ([LIPT23, Conjecture 11.3])**.** Let $2 \leqslant \ell \leqslant k$ be two integers. Consider a central product $G = K \times_\theta L$ where K and L are simply connected nilpotent Lie groups with one-dimensional centres, and class k and ℓ respectively. Then, $n^k \preccurlyeq \delta_G(n) \prec n^{k+1}$.

We present a large family of central products for which Conjecture 3 holds. The elements in this large class of nilpotent groups are constructed via central products of filiform groups. A *model filiform nilpotent Lie group* L_{k+1} is the nilpotent Lie group of class k whose associated Lie algebra has Lie algebra presentation

$$\langle X_1, \ldots, X_{k+1} \mid [X_1, X_i] = X_{i+1},\ [X_i, X_j] = 0 \text{ for all } 2 \leqslant i, j \leqslant k \rangle.$$

In particular, they have one-dimensional centres. Note that the 3-Heisenberg group is L_3. The group with additional non-trivial relation $[X_2, X_3] = X_{k+1}$ (replacing the relation $[X_2, X_3] = 0$ in L_{k+1}) is denoted by $L^{\lrcorner}_{k+1}$.

In this work we provide a detailed computation of the upper bounds on Dehn functions for this large class of central products. In particular, we obtain the following two results.

Theorem A (One filiform factor of highest nilpotency class)**.** *Let $k > \ell \geqslant 2$ be integers. Let K be either the group L_{k+1} or the group $L^{\lrcorner}_{k+1}$.*[1] *Let L be a simply connected nilpotent Lie group with one-dimensional centre of nilpotency class ℓ. Let $G = K \times_Z L$. Then $\delta_G(n) \preccurlyeq n^k$.*

[1] The group $L^{\lrcorner}_{k+1}$ is only defined for $k \geqslant 4$. So for $k = 3$ (resp. $\ell \leqslant 3$) the only allowed choice for K (resp. K or L) in Theorem A (resp. Theorem B) is L_4 (resp. L_3 or L_4).

Theorem B (Two filiform factors)**.** *Let $k \geqslant \ell \geqslant 2$ be integers. Let K be either the group L_{k+1} or $L^{\lrcorner}_{k+1}$. Let L be either the group $L_{\ell+1}$ or $L^{\lrcorner}_{\ell+1}$. Let $G = K \times_Z L$. Then $\delta_G(n) \preccurlyeq n^k$.*

In Section 3.6 we present a concise presentation on lower bounds based on our work in [GMLIP23]. For a detailed treatment we refer the reader to [GMLIP23, Section 3]. With both estimates at hand we conclude that for the infinite family of central products of nilpotent groups in Theorems A and B, Conjecture 3 holds.

We would like to mention that our arguments and techniques in [GMLIP23] to compute Dehn functions are a generalisation of the methods and techniques developed in [LIPT23]. Let us briefly explain to what extent Theorems A and B generalise the results obtained in [LIPT23]. In [LIPT23], the authors compute the Dehn functions of central products of model filiform groups, but in contrast with Theorem A, they only treat the cases $L_{k+1} \times_Z L_{k+1}$ and $L_{k+1} \times_Z L_k$ for $k \geqslant 2$. In particular, their results only deal with: (1) model filiform groups and (2) the nilpotency class of the factors differ by at most one. In contrast, in Theorem A we allow (1) the second factor to be a general simply connected nilpotent Lie group, and (2) a bigger difference between the nilpotency class of the factors.

In terms of Conjecture 1, one of the consequences of Theorem A is that the central product of the groups satisfying the assumptions is not quasi-isometric to its asymptotic cone. Namely, we obtain the following result.

Corollary C. *Let K and L be simply connected nilpotent Lie groups of classes k and ℓ with $k > \ell$ satisfying the assumption in Theorem A or Theorem B. Then, the group $G = K \times_Z L$ is not quasi-isometric to its asymptotic cone.*

The proof of Corollary C follows directly from the upper bounds on the Dehn functions of $G = K \times_Z L$ as we explain in Section 3.8. As mentioned before, the first examples of nilpotent groups with equivalent asymptotic cones and different Dehn functions were presented by the authors in [LIPT23]. Our results considerably expand the family of pairs of groups with equivalent asymptotic cones and different Dehn functions.

As mentioned before, to the best of our knowledge, the only known quasi-isometry invariants of nilpotent groups that can distinguish between groups with cone equivalent asymptotic cones are: Dehn functions [LIPT23] and the real cohomology algebra [Sau06]. One key difference between Dehn functions and the real cohomology algebra is that the nature of Dehn functions offers the possibility to have quantitative statements. Concretely, one can quantify the failure of existence of a quasi-isometry in Corollary C using so-called *sublinear bilipschitz equivalences*. Recall that two metric spaces X and Y are called *$O(r^e)$-bilipschitz equivalent*, for $e \in [0,1)$, if there exist $L > 1$, $c \geqslant 0$, a pair of maps $(f\colon X \to Y,\ g\colon Y \to X)$, and some $x_0 \in X$ and $y_0 \in Y$, such that for all $r > 0$ the map f (respectively g) is a (L, cr^e)-quasi-isometric embedding when restricted to $B(x_0, r)$ (respectively to $B(y_0, r)$), and for all $x \in X,\ y \in Y$

$$d(g(f(x)), x) \leqslant c(1 + d(x, x_0)^e), \quad d(f(g(y)), y) \leqslant c(1 + d(y, y_0)^e).$$

This notion does not depend on the choice of x_0 and y_0, up to changing c. In [Cor17, §6] Cornulier introduced a computable algebraic invariant $e_G \in [0,1)$ for every simply connected nilpotent Lie group such that G is $O(r^{e_G})$-bilipschitz equivalent to its asymptotic cone. He asked in [Cor17, Question 1.23] whether this invariant is optimal (for a given fixed group). In Section 3.7 we show that for the groups $L_{k+1} \times_Z L_3$ Cornulier's invariant is asymptotically optimal as the nilpotency class k goes to infinity. This emphasises one of the key improvements from our work in [GMLIP23] compared to [LIPT23], see Section 3.7.

Theorem D. *Let $k \geqslant 2$. Let $\underline{e}$ be the infimum of the exponents e such that there exists a $O(r^e)$-sublinear bilipschitz equivalence between $G = L_{k+1} \times_Z L_3$ and its asymptotic cone. Then*

$$\frac{1}{2k+2} \leqslant \underline{e} \leqslant \frac{2}{k} = e_G.$$

The upper bound is due to Cornulier [Cor17] and the lower bound follows from our work (see Section 3.7). Note that asymptotically as $k \to \infty$ the two bounds coincide up to a ratio of 4, showing that Cornulier's general upper bound is optimal in this asymptotic sense. To our knowledge these are the first examples of this kind. In [LIPT23] the authors produced non-zero lower bounds on $\underline{e}$ for all groups $L_{k+1} \times_Z L_k$ with $k \geqslant 3$, but in this case the asymptotics of their lower bound did not coincide with those of Cornulier's upper bound as $k \to \infty$.

We end this part of the introduction with the following remarks.

First, aside from Theorems A and B we are also able, using tailored methods treating case by case, to compute the precise Dehn functions of all central products of nilpotent Lie groups of dimension at most 5 with one-dimensional centre (see [GMLIP23, Theorem III]). To maintain the presentation concise we omit these results from this work, we refer the interested reader to [GMLIP23, Section 5].

Second, it would be interesting to know if with this new streamlined version of the techniques from [LIPT23] we can compute the Dehn functions of general central products of nilpotent groups. Thus, obtaining a full understanding of Conjecture 3.

Finally, a natural question that arises from Corollary C is the following:

Question 1.2 ([GMLIP23, Question 1.5])**.** Are there uncountably many possible growth types for the Dehn functions of compactly presented groups?

1.1.2 Dehn functions of mapping tori of RAAGs

Mapping tori of non-positively curved groups played a key role in the study of 3-manifolds. From Agol's proof of the virtual Haken conjecture [Ago13] it follows that a large class of 3-manifolds are (up to passing to a finite cover) obtained by taking the *topological mapping torus* M_f of a surface S_g with respect to a homomorphism $f \in \mathrm{Hom}(S_G)$ (see Figure 1.2):

$$M_f := \frac{S_g \times [0,1]}{(x,0) \sim (f(x),1)}.$$

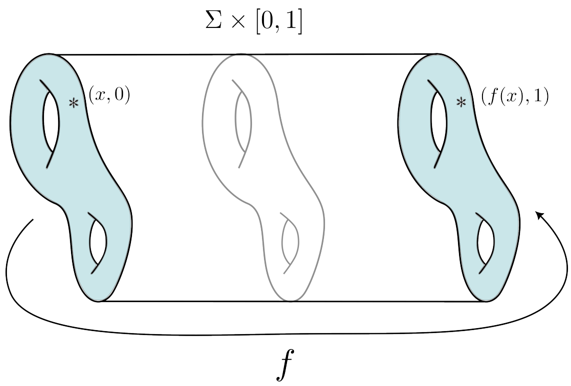

Figure 1.2: The topological mapping torus of a surface

Such a 3-manifold M_f is then said to *fibre over the circle* S^1, since by construction one has a fibre bundle

$$S_g \longrightarrow M_f \longrightarrow S^1.$$

On the group theoretic side, this corresponds to having the following split short exact sequence:

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(M_f) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

which exhibits the fundamental group of M_f as a semidirect product

$$\pi_1(M_f) \cong \pi_1(S_g) \rtimes_{\pi_1(f)} \mathbb{Z},$$

where $\pi_1(f)$ is the automorphism of $\pi_1(S_g)$ induced by $f \in \mathrm{Hom}(S_g)$.

The asymptotic geometry of 3-manifold groups has been well-studied, leading to a comprehensive understanding of their Dehn functions [ECH+92, BP94, BG96, BG10]. Notably, these groups include lattices in the geometries *Nil* and *Sol*, which are (up to commensurability) of the form $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ and certain groups of the form $F_m \rtimes_\psi \mathbb{Z}$.

More generally, the study of the large scale geometry of groups arising as *semi-direct products* $G \rtimes \mathbb{Z}$ has received a lot of attention. For general *free-by-cyclic* groups $F_m \rtimes_\psi \mathbb{Z}$ [BF92, Bri00, BG10, Gho23] and for *free abelian-by-cyclic* groups $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ [BP94, BG96] we have a full understanding of their Dehn functions.

Free groups and free abelian groups belong to a rich class of non-positively curved groups, namely the class of *right-angled Artin groups* (RAAGs for short). RAAGs are usually defined via a (finite) presentation. A natural way to encode this presentation is via a (finite) simplical graph L, the so-called *defining graph.*

The *right-angled Artin group* A_L *associated to* L is the group defined as

$$A_L = \langle v \in V(L) \mid [u, w] = 1 \text{ if and only if } \{u, w\} \in E(L)\rangle$$

for $V(L)$ the *vertex set of* L and $E(L) \subseteq V(L) \times V(L)$ the *edge set of* L. For instance, in a finite rank free group none of the generators commute, while for the free abelian group of finite rank all the generators commute, see Figure 1.3.

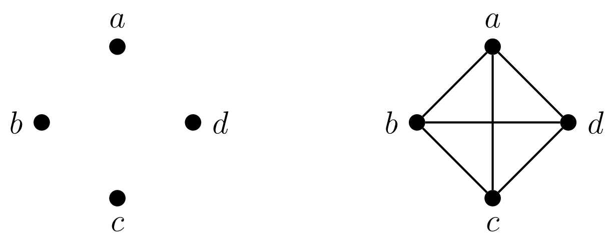

Figure 1.3: From left to right: the graph with no edges corresponds to the free group of rank 4 on the generators $\{a, b, c, d\}$. The complete graph on four vertices corresponds to the free abelian group of rank 4 given by the presentation $\langle a, b, c, d\rangle$.

This gives us a way to assign a finitely presented group to a finite simplicial graph:

$$L \mapsto A_L.$$

An important result exhibiting the interplay between the algebra of A_L and the combinatorics of L is that two RAAGs are isomorphic if and only if their underlying graphs are isomorphic [Dro87].

From this perspective, we say that RAAGs extrapolate between free groups and free abelian groups: free groups correspond to graphs with no edges, while free abelian groups correspond to *complete graphs*, i.e. graphs where each vertex is connected to every other vertex by an edge. See Figure 1.4 for an example of a RAAG which is neither a free group nor a free abelian group.

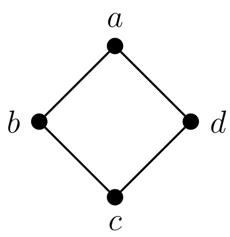

Figure 1.4: The defining graph of the RAAG which is the direct product of two free groups of rank 2: one generated by $\{a, c\}$ and the other by $\{b, d\}$.

Since RAAGs extrapolate between free abelian and free groups, it is a natural problem to understand which properties that free and free abelian groups have, are also common to general RAAGs. In this context, Vogtmann suggested studying the Dehn functions of mapping tori of RAAGs (see [PR19, p.2]).

Aside from the Dehn functions of mapping tori of free and free abelian groups, further results for certain RAAGs were obtained by Pueschel and Riley [PR19]. In particular, they computed the Dehn functions of the mapping tori of the RAAGs:

- $F_2 \times \mathbb{Z}$ [PR19, Theorem A],
- some cases $F_k \times \mathbb{Z}$ with $k \geqslant 3$ [PR19, Section 8],
- $F_k \times F_l$ where $k, l \geqslant 2$ [PR19, Theorem C], and
- $\mathbb{Z}^2 * \mathbb{Z}$ [PR19, Theorem B].

Moreover, Pueschel presented in her thesis [Pue16, Chapter 5] some partial results to obtain the Dehn functions of certain mapping tori of $\mathbb{Z}^m * \mathbb{Z}$. In particular, Pueschel classified the Dehn functions of the mapping tori of $\mathbb{Z}^3 * \mathbb{Z}$, and presented some arguments to obtain the Dehn function of certain mapping tori of $\mathbb{Z}^m * \mathbb{Z}$ with $m \geqslant 2$, namely those where the action of the automorphism on $\mathbb{Z}^m$ consists of a matrix with a single Jordan block [Pue16, p.75]. Nevertheless, her strategy does not seem to carry over for the general case.

A relevant observation regarding the above mapping tori, is that if A_L is $\mathbb{Z}^m, F_k \times F_l$ or $\mathbb{Z}^2 * \mathbb{Z}$, then the precise asymptotics of the Dehn functions of the respective mapping tori can be read off from the so-called *(cyclic) growth* of the corresponding automorphism, see Definition 2.50. In particular, the Dehn functions of the respective mapping tori all exhibit a dichotomy: they are either exponential or polynomial with integral exponent.

Our results regarding the Dehn functions of mapping tori of RAAGs also present this dichotomy, namely the Dehn functions of the mapping tori of RAAGs that we compute are either exponential or polynomial with integer degree, see Theorems E and F.

The second part of this work, Chapter 4, expands our knowledge of the Dehn functions of mapping tori of RAAGs. We now explain the three main results concerning the Dehn functions of these groups.

First, we obtain a full classification of the Dehn functions of the mapping tori of $\mathbb{Z}^2 * \mathbb{Z}^2$ showing that the Dehn function of the mapping torus $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Psi \mathbb{Z}$ is either quadratic, cubic, or exponential. As we explain in Section 4.2 these cases are mutually exclusive and exhaustive. Moreover, we can read off the Dehn function of the mapping torus $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Psi \mathbb{Z}$ from the asymptotics of a well-chosen representative of $[\Psi] \in \mathrm{Out}(\mathbb{Z}^2 * \mathbb{Z}^2)$.

Theorem E. *For every $\Psi \in \mathrm{Aut}(\mathbb{Z}^2 * \mathbb{Z}^2)$ there exists $\Phi \in \mathrm{Aut}(\mathbb{Z}^2 * \mathbb{Z}^2)$ such that $[\Phi] = [\Psi^2] \in \mathrm{Out}(\mathbb{Z}^2 * \mathbb{Z}^2)$ and Φ restricted to the free factors $\mathbb{Z}^2$ induces $\phi_1, \phi_2 \in \mathrm{Aut}(\mathbb{Z}^2)$. In particular, the Dehn functions of the associated mapping tori M_Ψ and M_Φ are equivalent and δ_{M_Φ} satisfies the following. If there exists $i \in \{1, 2\}$ such that the Dehn function of $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ is exponential, then the Dehn function of M_Φ is exponential. Else, precisely one of the following holds:*

1. *There exists $i \in \{1, 2\}$ such that the Dehn function of $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ is cubic, in which case the Dehn function of M_Φ is cubic.*
2. *For every $i \in \{1, 2\}$ the Dehn function of $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ is quadratic, in which case the Dehn function M_Φ is quadratic.*

To the best of our knowledge this result has not appeared in the literature before.

As shown by [BP94, BG96] the Dehn functions of the mapping tori $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ appearing in Theorem E can be read off from the asymptotics of $\phi_i \in \mathrm{Aut}(\mathbb{Z}^2)$, see Theorem 4.1. Therefore, Theorem E tells us that we can read off the Dehn function of the mapping torus $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Psi \mathbb{Z}$ from the asymptotics of a well-chosen representative of $[\Psi] \in \mathrm{Out}(\mathbb{Z}^2 * \mathbb{Z}^2)$.

A key result used in the proof of Theorem E is Lemma 4.3. We conjecture that this result holds in larger generality (see Conjecture 5). Thus, modulo Conjecture 5, our proof can be adapted to obtain the Dehn functions of mapping tori of $\mathbb{Z}^m * \mathbb{Z}^n$ with $m, n \geqslant 2$.

Second, building on [PR19, Theorem C], we obtain a precise description of the Dehn functions of the mapping tori of the RAAG $F_{m_1} \times \ldots \times F_{m_k}$ with $k \geqslant 2$ and $m_i \geqslant 3$. We prove the following result.

Theorem F. *Let* $G = F_{m_1} \times \ldots \times F_{m_k}$*, with* $k \geqslant 2, m_i \geqslant 2$ *and* $\Psi \in \mathrm{Aut}(G)$*. There exists* $\Phi \in \mathrm{Aut}(G)$ *such that* $[\Phi] = [\Psi^p] \in \mathrm{Out}(G)$ *and for every* $i \in \{1, \ldots, k\}$ $\Phi|_{F_{m_i}} = \phi_i \in \mathrm{Aut}(F_{m_i})$ *and the Dehn functions of the associated mapping tori* M_Ψ *and* M_Φ *are equivalent. Moreover, the Dehn function of the mapping torus* $G \rtimes_\Phi \mathbb{Z}$ *is either polynomial (with integer degree) or exponential, and its asymptotics only depend on the growth of the automorphisms* $\phi_1, \ldots \phi_k$*.*

For a statement of Theorem F in which the asymptotics of the Dehn functions are precisely described we refer the reader to Theorem 4.7.

Finally, we provide an alternative proof to the one in [PR19, Section 6] and [Pue16, Chapter 4] for the case of the quadratic Dehn function of the mapping torus of $\mathbb{Z}^2 * \mathbb{Z}$, see Theorem 4.10. Moreover, we expect this proof to serve as an outline for a general strategy for obtaining the Dehn functions of the mapping tori of $\mathbb{Z}^m * \mathbb{Z}$ with $m \geqslant 2$.

We finish this introduction by pointing out that the Dehn functions of all the mapping tori of the RAAGs obtained in this work also exhibit the dichotomy mentioned above: they are all either exponential or polynomial with integral exponents. It turns out, as Corollary 2.65 shows, that the Dehn functions of mapping tori of RAAGs are at most exponential. This naturally raises the following question.

Question 1.3. Let A_L be a RAAG and $\Phi \in \mathrm{Aut}(A_L)$. If the Dehn function of the associated mapping torus M_Φ is not exponential, then is it true that there exists $d \in \mathbb{N}$ such that $\delta_{M_\Phi}(n) \asymp n^d$?

1.2 Structure

The core of this work is concentrated in Chapters 3 and 4, where the results obtained on Dehn functions of nilpotent groups and mapping tori of RAAGs are treated.

In Chapter 2, we provide the necessary framework for this thesis, where we set up the main definitions and notation that we use in the consecutive chapters and that were mentioned in the above introduction.

In Chapter 3 we establish the upper bounds on the Dehn functions corresponding to Theorems A and B. We then conclude with the proof of Theorem D and Corollary C.

In Chapter 4 we first present the proof of Theorem E. Then, we restate Theorem F in full detail (see Theorem 4.7) and present a proof. We conclude Chapter 4 by presenting an alternative new proof for the mapping torus of $\mathbb{Z}^2 * \mathbb{Z}$ with quadratic Dehn function.

Chapter 2

Preliminaries

We start this section by establishing the notation that we use throughout this work.

In Section 2.2 we recall the basics about nilpotent groups. In Section 2.3 we first present basic and well-known facts regarding Lie groups and Lie algebras. We then introduce the important class of so-called Carnot groups and filiform groups. After this, we state Pansu's Theorem and give a short account on asymptotic cones.

In Section 2.6 we define Dehn functions of compactly presented groups, where we also present a short account on the state of the art regarding Dehn functions.

We then restrict ourselves to the world of finitely generated (respectively presented groups). In Section 2.7 we recall the definition of van Kampen diagrams, which are a common tool for obtaining bounds (both lower and upper) for Dehn functions. In Section 2.8 we recall two well-known results regarding automorphisms of direct products and free products, which we use in Chapter 4 to compute Dehn functions.

In Section 2.9 we present the definition of the growth function of an automorphism and some results about it. Finally, in Section 2.10 we present general well-known bounds on Dehn functions that we use in Chapters 3 and 4.

2.1 Notation and conventions

2.1.1 Groups

In this section we set the basic notation and some basic definitions that we use in the consecutive sections and chapters.

2.1.1.1 Words, alphabets, and presentations

Given a set S the *free monoid* S^* *generated by* S is the set of all finite tuples of elements in S, usually written as a string, where the empty string corresponds to 1 and the operation is concatenation.

We denote the *free group on a set* S by F_S. By a *word* w *in* S we mean an element of the free monoid $(S \cup S^{-1})^*$, and we denote it by $w := w(S)$. We call the set S an *alphabet*. By a slight abuse of notation, we denote elements of the free group on F_S as words in $(S \cup S^{-1})^*$. Strictly speaking elements in F_S are equivalence classes of words in $(S \cup S^{-1})^*$, which we denote by $[w]$ for $w \in (S \cup S^{-1})^*$, whenever we need to make a distinction.

We often consider alphabets that can be expressed as the union of different alphabets, for instance $S \cup \{a\}$; in this situation we write $w(S, a)$ and if we want to emphasise that a word $w := w(S, a)$ does not contain the letter a we write $w(S, 1)$.

Given a group G and a subset $X \subseteq G$ we denote by $\langle X \rangle$ the subgroup of G generated by X. Recall that a group G is *generated by a set* $S \subseteq G$ if $\langle S \rangle = G$, we call S a *generating set for* G and say that S *generates* G. There is a canonical map (of monoids) $(S \cup S^{-1})^* \to G$ that maps every word w to the element it represents which we denote by $\overline{w}$. Moreover, an alphabet S comes with a canonical epimorphism $\pi_S \colon F_S \twoheadrightarrow G$.

A *presentation* $\langle S|R \rangle$ for a group G consists of a generating set $S \subset G$ and a set $R \subset K := \ker(\pi_S)$ such that K is the normal closure (in F_S) of R. Thus,

$$G \cong \langle S|R \rangle := F_S/K$$

We call the elements of R *relators* and the identities $r = 1$ with $r \in R$ *relations*.

We say that a word $w \in F_S$ is *null-homotopic in* G or that it *represents the trivial element in* G if $w \in \ker \pi_S$. We usually omit the epimorphism π_s and just write $w =_G 1$ or $w =_{\mathcal{P}} 1$.

When manipulating words with respect to a presentation $\mathcal{P} := \langle S|R \rangle$ of a group G it is useful to distinguish between identities that hold in the free group F_S, called *free identities* and denoted by $\overset{\text{free}}{=}$, and identities that hold in $\mathcal{P}$, which we refer to as *identities holding in* G which we denote by $=_{\mathcal{P}}$ or $=_G$. When an equality follows from using a definition we simply write $=_{\text{def}}$.

Given a group G and $S \subset G$ we denote by $|\cdot| : G \to \mathbb{R} \cup \{\infty\}$ the (possibly infinite) *word metric* in G with respect to the subset S; if S generates G the function $|\cdot|$ takes only finite values.

Given a letter $s \in S$ we denote by $\ell_s(\cdot)$ the number of occurrences of letters $s^{\pm 1}$'s appearing in a word in the alphabet S and by $\epsilon_s(w)$ we denote the exponent sum in $s^{\pm 1}$'s of w. Note that $\epsilon_s(w) \leqslant \ell_s(w)$.

For $g, h \in G$, we use the conventions $g^h := h^{-1}gh$ and $[g, h] := g^{-1}h^{-1}gh$. Similarly, we use this convention for letters and words in an alphabet.

2.1.1.2 Group constructions

We now set the notation of the basic group constructions that are relevant for Chapters 3 and 4, namely central products and mapping tori of groups.

Definition 2.1 (Central product)**.** Let K and L be two groups, and let $\theta : Z(K) \to Z(L)$ be an isomorphism between their centres. The *central product* of K and L is the group

$$K \times_\theta L = (K \times L)/R,$$

where $R = \{(g, h) \in Z(K) \times Z(L) : h = \theta(g)\}$. When there is no ambiguity on θ we simply write $K \times_Z L$. Unless otherwise stated, by *factor* of a central product $G = K \times_Z L$ we mean the canonical subgroups K or L of G. When we consider direct factors of groups splitting as direct products instead, this will be explicitly specified.

As mentioned in Chapter 1, an important example is the 5-Heisenberg group, which can be seen as the central product of the 3-Heisenberg group with itself. For convenience of the reader we recall the pictorial representation that encloses a presentation for is viewed as the central product of two 3-Hesienberg groups, see Figure 2.1.

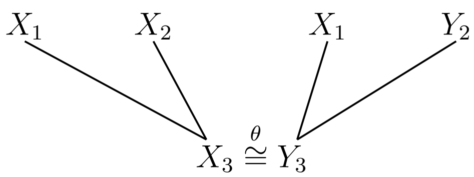

Figure 2.1: Recall from the introduction that this diagram depicts the presentation for the 5-Hesenberg group viewed as the central product of two 3-Heisenberg groups, one generated by X_1 and X_2, and the other by Y_1 and Y_2.

We encourage the reader to have in mind Figure 2.1 and pictures of the kind when reading Chapter 3. These pictures are a useful way to encode a presentation for certain nilpotent groups and central products between them. For instance, for

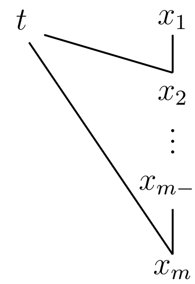

Figure 2.2: A presentation for the uniform lattice $\mathbb{Z}^m \rtimes_\phi \langle t \rangle$ of the model filiform group L_{m+1}.

the semidirect product $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ with $m \geqslant 1$ and $\phi(x_i) = x_i x_{i+1}$ and $\phi(x_m) = x_m$ a presentation for the group can be encoded using these pictures, see Figure 2.2.

We can think of this particular semidirect product $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ in two ways. On one hand, it is a nilpotent group, more precisely a lattice in a model filiform group (see Definition 2.32 below). Note that if $p = 3$ it is the integral 3-Heisenberg group. On the other hand, it is the mapping torus of $\mathbb{Z}^{p-1}$, as we now define.

Definition 2.2 (Mapping tori)**.** Let G be a group with presentation $\langle S|R \rangle$ and $\phi\colon G \to G$ an injective endomorphism, the *(algebraic) mapping torus of G with respect to ϕ* is the HNN-extension

$$M_\phi := \langle S, t \mid R, t^{-1}st = \phi(s) \text{ for all } s \in S\} \rangle \tag{2.1}$$

we call t *the stable letter*. It is clear that if G is finitely presented, then M_ϕ is also finitely presented.

In this work, ϕ is always an automorphism, therefore $M_\phi \cong G \rtimes_\phi \mathbb{Z}$ where $\mathbb{Z} \cong \langle t \rangle$. We refer to the presentation for $G \rtimes_\phi \mathbb{Z}$ given by (2.1) as the *standard finite presentation of M_ϕ as HNN-extension.*

2.1.1.3 Automorphism groups

Given a group G and $\iota \in \mathrm{Aut}(G)$, we say that ι is an *inner automorphism* if there exists $h \in G$ such that for all $g \in G$ we have that $\iota(g) = h^{-1}gh$, and say that ι acts by conjugation by h in G. We denote by $\mathrm{Inn}(G)$ the normal subgroup of $\mathrm{Aut}(G)$ consisting of inner automorphism. We denote by $\mathrm{Out}(G)$ the quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ called the *outer automorphisms of G*. For an element $\phi \in \mathrm{Aut}(G)$ we denote by $[\phi] \in \mathrm{Out}(G)$ its corresponding image under the canonical projection $\mathrm{Aut}(G) \twoheadrightarrow \mathrm{Out}(G)$ and call it its *outer class.*

2.1.1.4 Right-angled Artin groups

Throughout this section Γ denotes a finite simplicial graph. We recall the definition of a right-angled Artin group given in the introduction.

Definition 2.3. The *right-angled Artin group A_L with underlying graph L* as the group defined as

$$A_L = \langle v \in V(L) \mid [u, w] = 1 \text{ if and only if } \{u, w\} \in E(L)\rangle.$$

It is clear that if L is finite, then A_L is finitely presented. For an introduction to RAAGs we refer the reader to [Cha07].

2.1.2 Asymptotic equivalences

When computing Dehn functions we ultimately care about the asymptotic behaviour of them. For this we introduce the following partial orders on the set of functions.

Definition 2.4. Two functions $f, g \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ are said to be $\asymp$*-equivalent* if $f \preccurlyeq g$ and $g \preccurlyeq f$, by $f \preccurlyeq g$ we mean that there exist constants $A, B > 0$ and $C, D, E \geqslant 0$ such that

$$f(n) \leqslant Ag(Bn + C) + Dn + E \tag{2.2}$$

holds for all $n \geqslant 0$.

By a slight abuse of notation we usually write $f(n) \preccurlyeq g(n)$ and $f(n) \asymp g(n)$ to denote $f \preccurlyeq g$ and $f \asymp g$, respectively.

Another equivalence relation that we use is that of $\sim$*-equivalence.*

Definition 2.5. Two functions $f, g \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ are said to be $\sim$*-equivalent* if $f \preceq g$ and $g \preceq f$, by $f \preceq g$ we mean that there exist $A > 0$ and $B \geqslant 0$ such that $f(n) \leqslant A \cdot g(n) + B$ for all $n \geqslant 0$.

Similarly, as for the previous equivalence, by an abuse of notation we write usually write $f(n) \preceq g(n)$ and $f(n) \sim g(n)$, to denote $f \preceq g$ and $f \sim g$, respectively.

Remark 2.6. Note that $f \preceq g$ implies $f \preccurlyeq g$, and consequently $f \sim g$ implies $f \asymp g$. Moreover, for $B = 0$, the notion of $\sim$-equivalence is the more familiar notion of *Lipschitz equivalence.*

Finally, besides the asymptotic comparisons of functions defined above, when manipulating identities and keeping track of the number of relations used to go from one word to the other, we make use of the following comparison.

Notation 2.7. Given $A, B \geqslant 0$ and a parameter $a > 0$ we use the notation $A \lesssim_a B$ to mean that there exists some constant $C > 0$ that only depends on a such that $A \leqslant C \cdot B$. And $A \simeq_a B$ if $A \lesssim_a B$ and $B \lesssim_a A$ hold. We also say that A is in $O_a(B)$ if $A \lesssim_a B$ and $A = O_a(B)$ if $A \simeq_a B$.

2.2 Nilpotent groups

In this section we present some basic definitions and well-known facts about nilpotent groups.

Definition 2.8 (Lower central series)**.** Let G be a group. We define the *lower central series of* G recursively as

$$\gamma_1(G) \coloneqq G, \quad \text{and} \quad \gamma_{i+1}(G) \coloneqq [G, \gamma_i(G)]$$

for each $i \geqslant 1$.

Definition 2.9 (Nilpotent group)**.** A group G is said to be *nilpotent of class* c or c*-step nilpotent* if $\gamma_{c+1}(G) = \{1\}$ but $\gamma_c(G) \neq \{1\}$ for some $c \geqslant 1$. The number c is called the *nilpotency class of* G.

The class of nilpotent groups is closed under taking subgroups, quotients, and finite direct products. In particular, the central product of two nilpotent groups is again a nilpotent group.

A basic example is the 3-dimensional Heisenberg group. For R a ring the *3-dimensional Heisenberg group (over R)* or simply the *3-Hesienberg group* is defined as

$$H_3(R) \coloneqq \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in R \right\}.$$

Let us recall from Chapter 1 that one of the motivations to study nilpotent groups is the celebrated *Polynomial Growth Theorem* by Gromov. This landmark result made nilpotent groups a natural class to study within Geometric Group Theory. Moreover, it can be seen as the starting point of Gromov's idea to classify finitely generated groups up to quasi-isometries.

Theorem 2.10 (Gromov's Polynomial Growth Theorem, [Gro81]). *Let Γ be a finitely generated group. Γ has polynomial growth if and only if Γ is virtually nilpotent.*

Gromov actually proved the "only if" direction of Theorem 2.10. It was Wolf [Wol68] who first showed that finitely generated nilpotent groups have polynomial growth. Later on, independently Bass [Bas72] and Guivarc'h [Gui73] computed the exact degree of the polynomial. Recall that the *growth of a finitely generated group* is a quasi-isometry invariant that, roughly speaking, measures the asympotic growth of balls centred at the identity of the group by giving a bound on the number of elements in such a ball. The bound is given as a function of the radius of the ball.

A consequence of Theorem 2.10 is that the class of finitely generated virtually nilpotent groups is *closed under quasi-isometries*:

Corollary 2.11 (Gromov). *Let Γ_1 and Γ_2 be two finitely generated quasi-isometric groups. If Γ_1 is virtually nilpotent, then Γ_2 is also virtually nilpotent.*

In contrast with the free group of rank 2, every subgroup of a finitely generated nilpotent group is finitely generated. Moreover, we have the following well-known and basic result.

Lemma 2.12. *Let G be a finitely generated nilpotent group, then G is finitely presented.*

The proof of this result is a simple exercise. For the readers familiar with geometric group theory but not with nilpotent groups we provide a quick sketch.

Sketch of proof. The proof is done by induction on the nilpotency class and uses the fact that subgroups of a finitely generated nilpotent group are finitely generated. The base of induction is clear, since abelian groups are finitely presented. For the induction step we use the following two facts. First, recall that finitely generated nilpotent groups are Noetherian, namely subgroups are finitely generated, to see this one can proceed by induction on the nilpotency class, see for instance [Bro73, Lemma 2]. Second, being finitely presented is stable under extensions, namely if $N \trianglelefteq G$ is a normal subgroup such that N and G/N are finitely presented, then G is finitely presented. □

The following result is assumed in Chapter 3. It allows us to work with torsion-free nilpotent groups when estimating their Dehn functions.

Proposition 2.13. *Every finitely generated nilpotent group G has a subgroup $G' \trianglelefteq G$ of finite index which is torsion-free.*

Proof. For a proof we refer the reader to [Seg83, Proposition 2, p. 2] or [CSD21, Lemma 2.42]. □

Since finite index subgroups of a finitely generated group have the same geometry as the ambient group, more precisely they are quasi-isometric, in view of Proposition 2.13 we can restrict to the class of torsion-free nilpotent groups without loosing geometric information. Moreover, another reason we restrict to torsion-free nilpotent groups comes from the result of Mal'cev [Mal51], who established a way to embed a torsion-free nilpotent group as a co-compact lattice in a simply connected nilpotent Lie group. For instance, the integral 3-dimensional Heisenberg group $H_3(\mathbb{Z})$ embeds as a cocompact lattice in the real 3-dimensional Heisenberg group $H_3(\mathbb{R})$.

We recall from Chapter 1 the conjectural classification of finitely generated torsion-free nilpotent groups.

Conjecture ([FM00, p.5] and [Cor18, p.335])**.** Two finitely generated torsion-free nilpotent groups are quasi-isometric if and only if they embed as cocompact lattices in the same simply connected nilpotent Lie group.

Therefore, the study of the Dehn function of finitely generated nilpotent groups reduces to the study of Dehn functions of simply connected nilpotent Lie groups. This makes the class of simply connected nilpotent Lie groups a natural class of study for us.

We now establish some basic notions of Lie groups and Lie algebras.

2.3 Lie groups and algebras

In the first part of this section we present standard and well-known results concerning Lie groups and algebras. For a detailed treatment of these topics we refer the reader to [FH91] and [Hel78]. We then present the so-called Carnot-graded groups, see Definition 2.26. Finally, we define the important classes of model filiform groups (see Definition 2.32 below).

Unless otherwise stated, all the Lie groups and Lie algebras are assumed to be finite-dimensional and either over the real or complex numbers. Lie groups are an important family of topological groups.

Definition 2.14 (Topological group)**.** A *topological group* consists of a group G and a topology on G such that the binary operation $G \times G \to G$ and the inversion map $G \to G$ such that $g \mapsto g^{-1}$ are continuous with respect to the topology.

Definition 2.15 (Lie group)**.** A *Lie group* is a topological group G endowed with the structure of a smooth manifold, so that the binary operation $G \times G \to G$ and the inversion map $G \to G$, are smooth maps. We say that G is *finite-dimensional* if as a differentiable manifold it is finite-dimensional.

Recall, that a homomorphism of Lie groups is a group homomorphism which is also a smooth map. Every countable discrete group (a topological group with the discrete topology) is a Lie group. The real 3-dimensional Heisenberg group defined in Section 2.2, is a Lie group.

The manifold structure on a Lie group allows us to exploit the notions of Riemannian geometry. Concretely, every Lie group can be equipped with a left-invariant Riemannian metric, i.e. a Riemannian metric that is invariant under left multiplication by elements of G, namely if for every $g \in G$ the map

$$\begin{aligned} L_g \colon G &\to G, \\ h &\mapsto gh. \end{aligned}$$

is an isometry. To construct such a metric consider a positive definite inner product $\langle \cdot, \cdot \rangle_e$ on T_eM the tangent space at the identity. Note that the map L_g is a diffeomorphism and that the action of G on itself via left multiplication is simply transitive. We define an inner product $\langle \cdot, \cdot \rangle_g$ on T_gM as the push-forward of $\langle \cdot, \cdot \rangle_e$ under the differential $D_eL \colon T_eG \to T_gG$ of L_g at the identity e, namely

$$\begin{aligned} \langle \cdot, \cdot \rangle_g \colon T_gG \times T_gG &\to \mathbb{R} \\ (X, Y) &\mapsto \langle D_e(L_g)^{-1}(X), D_e(L_g)^{-1}(Y) \rangle_e. \end{aligned}$$

Moreover, every Lie group G acts on itself from the left: for $g \in G$ let $\rho_g \colon G \to G$ be the map defined by $h \mapsto ghg^{-1}$. The map ρ_g is a smooth map and fixes the identity $e \in G$. Therefore, it induces an action on T_eG:

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(T_eG), \\ g &\mapsto D_e(\rho_g) \end{aligned}$$

called the *adjoint representation of* G. The differential of the adjoint representation at the identity is the map

$$\mathrm{ad}\colon T_eG \to \mathrm{End}(T_eG),$$

since $\mathrm{Aut}(T_eG)$ is an open subset of $\mathrm{End}(T_eG)$ its tangent space at the identity can be identified with $\mathrm{End}(T_eG)$.

Definition 2.16 (Lie algebra)**.** Let $R \in \{\mathbb{R}, \mathbb{C}\}$. A *Lie algebra over* R is an R-vector space $\mathfrak{g}$ equipped with a binary operation

$$[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the *Lie bracket*, satisfying the following axioms:

(L1) (Bilinearity) For every $\lambda \in R$ and $X, Y, Z \in \mathfrak{g}$ we have

$$[\lambda X, Y] = \lambda[X, Y] \text{ and } [X + Y, Z] = [X, Z] + [Y, Z].$$

(L2) (Skew-symmetry) For every $X, Y \in \mathfrak{g}$ we have $[X, Y] = -[Y, X]$.
(L3) (Jacobi identity) For every $X, Y, Z \in \mathfrak{g}$ we have

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

This makes $[\cdot,\cdot]$ a skew-symmetric bilinear map satisfying the Jacobi identity.

There is a strong relation between Lie groups and Lie algebras with many consequences for the structure of both objects.

2.3.1 Correspondence between Lie groups and Lie algebras

For every (connected) Lie group G there is a Lie algebra called the *associated Lie algebra* $\mathfrak{g}$. This is defined as follows: define $\mathfrak{g}$ as the vector space T_eG together with the Lie bracket $[\cdot,\cdot]$ defined as:

$$[X, Y] \coloneqq \mathrm{ad}(X)(Y) \quad \text{for all } X, Y \in \mathfrak{g}.$$

The exponential map $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism. Thus by the implicit function theorem there is an open neighbourhood of the identity U that is mapped diffeomorphically an open neighbourhood $\exp(U)$ of the identity in G. The inverse map is the logarithm map $\log : \exp(U) \to U$.

A fundamental result about the structure of Lie algebras and Lie groups is the following result.

Theorem 2.17 (Ado's Theorem). *Every finite-dimensional Lie algebra $\mathfrak{g}$ over a field F of characteristic zero (for instance $\mathbb{R}$) is linear, that is the subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional vector space V over F. In particular, every Lie group locally embeds in $GL(V)$ for some finite-dimensional real vector space V .*

Proof. For a proof of this result we refer the reader to [FH91, Thereom E.4]. □

The correspondence $G \mapsto \mathfrak{g} := T_eG$ has strong implications, one of them is the following result.

Proposition 2.18. *Let G and H Lie groups with $\mathfrak{g}$ and $\mathfrak{h}$ the associated Lie algebras respectively. Let $\Phi\colon \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras. If G is simply connected, then there exists a unique homomorphism $\phi\colon G \to H$ such that $D_e(\phi) = \Phi$.*

Proof. For a proof we refer the reader to see [FH91, p.119] or [War83, Theorem 3.27]. □

A consequence of Proposition 2.18 is that if two simply connected Lie group G and H have isomorphic Lie algebras, then G and H are isomorphic.

In fact, the correspondence between (connected) Lie groups and Lie algebras defines a functor from the category of (connected) Lie groups to the category of Lie algebras.

Theorem 2.19 (Lie's Third Theorem). *For every finite-dimensional real Lie algebra $\mathfrak{g}$ there exists a unique (up to isomorphism) connected simply connected Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$.*

Proof. For a proof of Theorem 2.19 that uses Ado's Theorem 2.17 we refer the reader to [War83, p. 101] or [Hal15, Section 5.10]. □

Said differently, Theorem 2.19 states that there is a one-to-one correspondence between isomorphism classes of Lie algebras and isomorphism classes of simply connected Lie groups.

2.3.2 Carnot groups and algebras

A particular class of nilpotent Lie groups and Lie algebras relevant to our work in Chapter 3 is that of Carnot-graded groups and Carnot-graded algebras. Before defining this classes we recall the definition of a nilpotent Lie algebra.

Definition 2.20 (Nilpotent Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. The lower central series of $\mathfrak{g}$ is defined similarly as for a group:

$$\gamma_1\mathfrak{g} \coloneqq \mathfrak{g} \quad \text{and} \quad \gamma_{i+1}\mathfrak{g} \coloneqq [\mathfrak{g}, \gamma_i\mathfrak{g}],$$

for $i \geqslant 1$. Observe, that in this case $[\cdot,\cdot]$ denotes the Lie bracket of $\mathfrak{g}$. Moreover, we say that the Lie algebra $\mathfrak{g}$ is *c-nilpotent or nilpotent of class c* if $\gamma_{c+1}\mathfrak{g} = 0$ but $\gamma_c\mathfrak{g} \neq 0$ for some $c \geqslant 1$.

In this section we explain how to construct for every simply connected nilpotent Lie group a Lie algebra admitting a specific kind of grading. We present some important examples. We refer the reader to [Cor16, 3.2], [LD17], and [Mon02, Chapters 8 and 9] for further reading.

We begin with the following basic definition.

Definition 2.21 (Lie algebra grading)**.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. A *positive grading* of $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{i \geqslant 1} \mathfrak{m}_i,$$

such that for every $i, j \geqslant 1$ we have that $\mathfrak{m}_i \subset \mathfrak{g}$ is an $\mathbb{R}$-linear subspace and

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j}.$$

Definition 2.22 (Graded Lie algebra)**.** We say that the Lie algebra $\mathfrak{g}$ is a *positively graded Lie algebra* or simply *graded Lie algebra* if it is endowed with a positive grading.

Remark 2.23. If $\mathfrak{g}$ is a positively graded Lie algebra, then $\gamma_j\mathfrak{g} \subset \oplus_{i \geqslant j}\mathfrak{m}_i$. Therefore, $\mathfrak{g}$ is nilpotent [LD17, Lemma 2.16]. However, the converse is not true. Namely, there exists a nilpotent Lie algebra that does not admit a positive grading [LD17, Example 2.8].

The nilpotent Lie algebras that admit a positive grading have an important role in Chapter 3. Moreover, we focus on a particular kind of grading, namely those where $\mathfrak{m}_1$ generates $\mathfrak{g}$ as a Lie algebra.

Definition 2.24 (Carnot grading, Carnot algebra, and Carnot-graded algebra I)**.** Let $\mathfrak{g}$ be a graded Lie algebra over $\mathbb{R}$ with positive grading $\oplus_{i \geqslant 1}\mathfrak{m}_i$. We say that the grading is a *Carnot grading* if $\mathfrak{m}_1$ generates $\mathfrak{g}$ as a $\mathbb{R}$-algebra. Moreover,

- we call a Lie algebra $\mathfrak{g}$ *Carnot gradable Lie algebra* or simply *Carnot* if it admits a Carnot grading.
- we say that $\mathfrak{g}$ is a *Carnot-graded Lie algebra* if it is endowed with a Carnot grading.

At the level of Lie groups we have the following definition.

Definition 2.25 (Carnot gradable Lie group)**.** We say that a simply connected nilpotent Lie group G is *Carnot gradable* if its associated Lie algebra $\mathfrak{g}$ is Carnot gradable.

There is an equivalent way to define a Carnot-graded Lie algebra as we now explain. Consider a nilpotent Lie group G of class c and let $\mathfrak{g}$ be the Lie algebra associated to G. Since G is c-nilpotent, we have that the lower central series of $\mathfrak{g}$ is finite and $\gamma_{c+1}(\mathfrak{g}) = 0$, i.e. $\mathfrak{g}$ is nilpotent. In fact, the lower central series defines a filtration

$$\gamma_{c+1}\mathfrak{g} \subseteq \gamma_c\mathfrak{g} \subseteq \ldots \subseteq \gamma_1\mathfrak{g} = \mathfrak{g} \tag{2.3}$$

which is compatible with the Lie bracket. In particular,

$$[\gamma_{i+1}\mathfrak{g}, \gamma_j\mathfrak{g}], [\gamma_i\mathfrak{g}, \gamma_{j+1}\mathfrak{g}] \subset \gamma_{i+j+1}\mathfrak{g}.$$

From this it follows that the Lie bracket defines a linear map

$$\gamma_i\mathfrak{g}/\gamma_{i+1}\mathfrak{g} \otimes \gamma_j\mathfrak{g}/\gamma_{j+1}\mathfrak{g} \to \gamma_{i+j}\mathfrak{g}/\gamma_{i+j+1}\mathfrak{g},$$

which induces a Lie bracket $[\cdot,\cdot]_{\mathrm{gr}}$ in the vector space

$$\mathrm{gr}(\mathfrak{g}) \coloneqq \bigoplus_{i\geqslant 1} \gamma_i\mathfrak{g}/\gamma_{i+1}\mathfrak{g}$$

turning it into a Lie algebra.

Definition 2.26 (Associated Carnot-graded)**.** We call the direct sum

$$\mathrm{gr}(\mathfrak{g}) \coloneqq \bigoplus_{i\geqslant 1} \gamma_i\mathfrak{g}/\gamma_{i+1}\mathfrak{g}$$

equipped with the Lie bracket $[\cdot,\cdot]_{\mathrm{gr}}$ the *associated Carnot-graded Lie algebra.* We call its associated simply connected Lie group the *associated Carnot graded Lie group* and denote it by $\mathrm{gr}(G)$.

For Lie groups we define the following.

Definition 2.27 (Carnot group)**.** Let G be a simply connected nilpotent Lie group. We say that G is a *Carnot group* if $G \cong \mathrm{gr}(G)$ as Lie groups.

Proposition 2.28. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{R}$. Then, $\mathfrak{g}$ is Carnot if and only if it is isomorphic (as $\mathbb{R}$-algebra) to its associated Carnot-graded algebra gr($\mathfrak{g}$). Moreover, for any Carnot grading of $\mathfrak{g}$, the graded algebras $\mathfrak{g}$ and gr($\mathfrak{g}$) are isomorphic as $\mathbb{R}$-algebras.*

Proof. For a proof we refer the reader to [Cor16, Proposition 3.5]. □

Remark 2.29. Every nilpotent Lie algebra of nilpotency class 2 is Carnot [Cor16, Example 3.12]. However, in general, not every nilpotent Lie algebra is Carnot. For instance, from Magnin's classification of nilpotent Lie algebras in dimension $\leqslant 6$ [Mag10] we can see that the following example from [Cor16, Section 1.2.2] is not Carnot: The 5-dimensional Lie algebra generated by $\{X_1, \dots, X_5\}$ with only non-trivial relations being

$$[X_1, X_i] = X_{i+1} \ \text{ for } \ 2 \leqslant i \leqslant 4, \quad \text{and} \quad [X_2, X_3] = X_5$$

is a nilpotent Lie algebra of nilpotency class 4. Its Carnot-graded Lie algebra is the 5-dimensional Lie algebra generated by $\{X_1, \dots, X_5\}$ with only non-trivial relations being $[X_1, X_i] = X_{i+1}$ for all $2 \leqslant i \leqslant 4$.

Remark 2.30. In this work, in particular in Chapter 3, the Lie algebra from Remark 2.29 is denoted by $\mathfrak{l}_5^{\lrcorner}$ and its associated Lie group by $L_5^{\lrcorner}$. These are examples of filiform algebras and groups respectively, see Section 2.3.3 below for the definition. The simply connected Lie group $L_5^{\lrcorner}$ played a significant role in gaining intuition behind the proofs presented in Section 3.4.

Beyond Remark 2.29 there are more examples of nilpotent Lie algebras which are not Carnot. It turns out, that one way of obtaining interesting examples of nilpotent Lie algebras which are not Carnot is by means of central products of nilpotent Lie algebras (see Remark 2.35 below).

In view of Proposition 2.28, for a simply connected group G being Carnot is equivalent to saying that its associated Lie algebra is Carnot-graded.

We now turn to define two important classes of graded nilpotent Lie algebras and graded Lie groups.

2.3.3 Filiform groups and algebras

In this section we define the classes of filiform groups (algebras) and filiform groups (algebras). These classes are the main object in Chapter 3, where we consider central products of (model) filiform groups. Moreover, in Chapter 4 they have a substantial role (e.g. Lemma 4.3) since they admit lattices that are mapping tori of $\mathbb{Z}^m$ (see Remark 2.33).

Definition 2.31 (Model filiform Lie algebra)**.** The *model filiform nilpotent Lie algebra* $\mathfrak{l}_{m+1}$ *of dimension* $m+1$ is the m-nilpotent Lie algebra of dimension m with basis $\{X_1, \ldots, X_{m+1}\}$ satisfying $[X_1, X_i] = X_{i+1}$ for all $2 \leqslant i \leqslant m$ and $[X_i, X_j] = 0$ for $1 < i \leqslant j \leqslant m+1$ or if $(i, j) = (1, m+1)$.

Note that $\{X_{m+1}\}$ is a basis for $\gamma_m\mathfrak{g}$, $\{X_m, X_{m+1}\}$ is a basis for $\gamma_{m-1}\mathfrak{g}$, and, in general, $\{X_i, \ldots, X_{m+1}\}$ for $\gamma_{i-1}\mathfrak{g}$ and $i > 2$. In this sense we could think of the lower central series of a filiform group as being threaded or threadlike, which explains the French term *filiform* introduced by Vergne [Ver70]. In particular, the model filiform algebra $\mathfrak{l}_{m+1}$ has 1-dimensional centre $\mathfrak{z}$ generated by X_{m+1} [GJMNnV98, Theorem 1].

Definition 2.32 (Model filiform group)**.** A *model filiform nilpotent Lie group* L_{m+1} is the simply connected nilpotent Lie group of class m whose associated Lie algebra is $\mathfrak{l}_{m+1}$.

In general, a simply connected nilpotent Lie group is called *filiform of class* k if it is of minimal possible dimension, namely $k+1$, among all nilpotent groups of class k. We call the corresponding Lie algebra *filiform algebra*. In fact, a filiform algebra is generated by a particular basis, called an *adapted basis*, $\{X_1, \ldots, X_{k+1}\}$, such that $[X_1, X_i] = X_{i+1}$ for $2 \leqslant i \leqslant k$, with possibly other brackets being nonzero.

Remark 2.33. The semi-direct product $\mathbb{Z}^m \rtimes_\phi \langle t \rangle$ with finite presentation

$$\langle x_1, \ldots, x_m, t \mid [x_i, x_j] = [x_m, t] = 1 \text{ for } 1 \leqslant i, j \leqslant m,\ [x_i, t] = x_{i+1} \text{ for } 1 \leqslant i \leqslant m-1 \rangle$$

defines a lattice in the model filiform group L_{m+1} (see Section 2.1.1.2).

In Remark 2.29 we saw an example of a nilpotent Lie algebra that was not Carnot. We also mentioned that central products are a way to obtain interesting examples of these kind. The central product for Lie algebras, can be defined in a completely analogous way as for groups (see Definition 2.1):

Definition 2.34 (Central product of Lie algebras)**.** Let $\mathfrak{k}$ and $\mathfrak{l}$ be two Lie algebras with central subspaces $\mathfrak{z} \subset \mathfrak{k}$ and $\mathfrak{z}' \subset \mathfrak{l}$. Suppose that there is an isomorphism $\theta \colon \mathfrak{z} \to \mathfrak{z}'$. The *central product of* $\mathfrak{k}$ *with* $\mathfrak{l}$ *with respect to* θ is the Lie algebra

$$\mathfrak{g} := \mathfrak{k} \times_\theta \mathfrak{l}$$

defined as the quotient of $\mathfrak{k} \times \mathfrak{l}$ by the ideal generated by $\{(z, -\theta(z)) \mid z \in \mathfrak{z}\}$.

Remark 2.35 ([LIPT23, p. 708])**.** For $3 \leqslant q \leqslant p$ we can define the Lie algebra $\mathfrak{g}_{p,q}$ as the central product of the Lie algebras $\mathfrak{l}_p$ and $\mathfrak{l}_q$. It turns out that the Lie algebras $\mathfrak{g}_{p,q}$ are Carnot gradable if and only if $p = q$. In fact, $\operatorname{gr}(\mathfrak{g}_{p,q})$ is isomorphic to the direct product $\mathfrak{l}_p \times \mathfrak{l}_{q-1}$.

Finally, a particular family of filiform groups and algebras that is the main object in Chapter 3 (e.g. Theorems A and B) consists of the filiform algebras (and their associated Lie groups) which we denote by $\mathfrak{l}^{\lrcorner}_{m+1}$: it is generated by $\{X_1, \ldots, X_{m+1}\}$ with only non-trivial relations $[X_1, X_i] = X_{i+1}$ for all $1 \leqslant i \leqslant m$ and $[X_2, X_3] = X_{m+1}$. The associated Lie groups are denoted by $L^{\lrcorner}_{m+1}$.

2.4 Pansu's Theorem

As mentioned in the introduction, Pansu provided evidence for the conjectural quasi-isometry classification of torsion-free finitely generated nilpotent groups (see Conjecture 2).

Theorem 2.36 ([Pan89])**.** *Let G and G' be simply connected nilpotent Lie groups. If G and G' are quasi-isometric, then* $\operatorname{gr}(G)$ *and* $\operatorname{gr}(G')$ *are isomorphic.*

One of the steps in the proof of Pansu's theorem consists of identifying the so-called asymptotic cone of G with $\operatorname{gr}(G)$ equipped a particular metric (called the Carnot-Carathéodory metric) [Pan89]. Therefore, Theorem 2.36 restricts the problem of classifying nilpotent groups up to quasi-isometries to that of discerning nilpotent groups with the same asymptotic cones (or equivalently, isomorphic associated Carnot-graded groups) up to quasi-isometries.

As we explain in Section 3.8, this identification done by Pansu, between the associated Carnot-graded group and the asymptotic cone is a key observation that allows us to conclude Corollary C from Theorems A and B (see Section 3.8 for details).

2.5 Asymptotic cones

In this section we recall the general idea of what the asymptotic cone of a group is. Due to the technical nature of the definition, we decide to keep its presentation short, and rather focus on providing an intuitive idea by presenting some examples. We refer the reader to [BRS07, Part II, Chapter 4] and [DK18, Chapter 10] for a further discussion.

The concept of asymptotic cones was used by Gromov [Gro81] and then formally introduced by van den Dries and Wilkie [vdDW84]. The idea is to construct, for a given metric space (X, d), a new space that captures X as seen from "infinitely far". More precisely, the idea is to define the notion of a limit of a sequence of metric spaces $(X, \frac{d}{\epsilon})$ as $\epsilon \to \infty$, where $\epsilon > 0$ is a "scaling factor".

The *asymptotic cone* $(\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}), d_\omega)$ of a metric space (X, d) is a metric space defined using the following information:

- A sequence of *base points* $\mathbf{e} = (e_i)_{i\in\mathbb{N}}$ in X.
- A *scaling family* $\mathbf{s}$, that is, a sequence of strictly positive real numbers $\mathbf{s} = (s_i)_{i\in\mathbb{N}}$ such that $s_i \to \infty$ as $i \to \infty$.
- A *nonprincipal ultrafilter*[1] ω *on* $\mathbb{N}$, which is a finite probability measure on $\mathbb{N}$, i.e. it is an additive map $2^{\mathbb{N}} \to \{0, 1\}$ such that $\omega(\mathbb{N}) = 1$ and for every finite subset $F \subset \mathbb{N}$ we have $\omega(F) = 0$.

We use these to construct an asymptotic cone as follows. Let $X_b^{\mathbb{N}}$ be the set of sequences $\{y_i\}_{i\in\mathbb{N}}$ in X such that $\frac{d(e_i,y_i)}{s_i}$ is bounded and let $\sim$ be the equivalence relation in $X_b^{\mathbb{N}}$ defined by

$$\{x_i\}_{i\in\mathbb{N}} \sim \{y_i\}_{i\in\mathbb{N}} \iff \lim_\omega \frac{d(x_i, y_i)}{s_i} = 0.$$

We define the *asymptotic cone* $(\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}), d_\omega)$ as the set

$$\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}) = X_b^{\mathbb{N}}/\sim .$$

with metric $d_\omega(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}}) = \lim_\omega \frac{d(x_i,y_i)}{s_i}$.

An asymptotic cone of a finitely generated group G with generating set S is the asymptotic cone of the metric space (G, d_S) where d_S denotes the word metric. By homogeneity of G the sequence of points $\mathbf{e} = (e_i)_{i\in\mathbb{N}}$ can be taken to be the

[1] A nonprincipal ultrafilter on $\mathbb{N}$ should be thought of as a gadget that guarantees convergence: it is a way to coherently choose limit points of bounded sequences, see [DK18, Chapter 10] for a detailed discussion.

constant sequence the identity element in G. So to avoid heavy notation it is common to simply denote the asymptotic cone of G by $\mathrm{Cone}_\omega(G)$.

We now list a series of examples and properties of asymptotic cones.

1. Asymptotic cones of quasi-isometric spaces are bilipschitz equivalent, see [DK18, Lemma 10.83].
2. A finitely generated group G is word hyperbolic if and only if all of its asymptotic cones are $\mathbb{R}$-trees. This was first observed by Gromov [Gro93] and Drutu gave a relatively elementary proof in [Dt02], see also [BRS07, Theorem 4.2.2].
3. A finitely generated group is virtually abelian if and only if its asymptotic cones are isometric to the Euclidean space $\mathbb{R}^n$, see [Gro81] and [Pan83].

Some results where asymptotic cones play a key role are for instance in the quasi-isometric rigidity of symmetric spaces and Euclidean buildings due to Kleiner and Leeb [KL97] or in the proof of the quasi-isometric rigidity of the mapping class group by Behrstock, Kleiner, Minsky, and Mosher [BKMM12] and Hamenstädt [Ham06].

We would like to emphasise that the asymptotic cone depends on the choice of ω and $\mathbf{s}$. Nevertheless, a finitely generated group is virtually nilpotent if and only if all its asymptotic cones are locally compact metric spaces. In [Gro81] Gromov proved that they are proper, and Drutu in [Dt02] proved the converse and added that proper can be replaced by locally compact, see [BRS07, p. 130]. Moreover, in view of Pansus's work, given a simply connected nilpotent Lie group G the asymptotic cones of G are bilipschitz equivalent to the associated Carnot-graded group $\mathrm{gr}(G)$ equipped with a left-invariant metric, called the Carnot–Carathéodory metric. In contrast to the asymptotic cones, this space has a pretty concrete description.

Finally, as a way to motivate the next section, we would like to mention two facts relating asymptotic cones and Dehn functions (see Section 2.6 below for a definition) (i) asymptotic cones of groups with quadratic Dehn function are all simply connected [Pap96] and (ii) if all the asymptotic cones of a finitely generated group are simply connected, then the group has Dehn function bounded above by a polynomial (with exponent not necessarily an integer), see [Gro93] and [BRS07, Part II, Theorem 4.3.3]. Having said this, we now turn to define the main quasi-isometry invariant of this work, namely Dehn functions.

2.6 Dehn functions of compactly presented groups

A lot of research in geometric group theory has been devoted to the study of finitely generated and finitely presented groups. Both classes of groups have natural generalisations to the realm of locally compact groups as compactly generated and compactly presented groups, as introduced by Kneser [Kne64]. For us locally compact groups are always assumed to be second countable and Hausdorff topological groups.

The idea of generalising quantitative quasi-isometry invariants such as Dehn functions to the class of locally compact groups is a fairly recent trend, see for instance [CT17, CM18, CdlH16a] for recent progress on the topic[2].

We start this section by defining compactly generated (respectively presented). We then define the Dehn function of a compactly presented group.

Generalising finite generation, we say that a locally compact group G is *compactly generated* if it admits a compact generating set S. Moreover, generalising finite presentability for discrete groups we say that G is *compactly presented* if, in addition to being compactly generated, it admits a presentation $\langle S|R\rangle$ with a set $R \subset S$ of null-homotopic words with word length bounded by a constant. In this case we say that G is *compactly presented with compact presentation* $\mathcal{P} := \langle S|R\rangle$.

To be precise, if no compact (respectively finite) presentation is given we should just say that G is *compactly (respectively finitely) presentable*; by an abuse of notation we call it compactly (finitely) presented, respectively generated.

Examples 2.37. We list some basic examples of compactly presented groups that are treated in this work.

1. Finitely presented discrete groups are compactly presented.
2. Simply connected (nilpotent) Lie groups are compactly presented, see [Tes18, Theorem 2.6] or [CdlH16b, Corollary 8.A.9].

We now define the main quasi-isometry invariant of this work.

Definition 2.38 (Dehn function)**.** Let G be a compactly presented locally compact group with compact presentation $\mathcal{P} := \langle S|R\rangle$. The *area* of a null-homotopic word w in G with respect to $\mathcal{P}$ is the number defined as

$$\mathrm{Area}_{\mathcal{P}}(w) := \min\Big\{k \in \mathbb{N} \mid w \overset{\text{free}}{=} \prod_{i=1}^{k} u_i^{-1} r_i^{\pm 1} u_i, \text{ with } u_i \in F_S,\ r_i \in R\Big\}.$$

[2] It should be noted that Abels already in the 70's used compact presentability to establish finite presentability for some S-arithmetic lattices [Abe87]

The *Dehn function of the presentation* $\mathcal{P}$ is the function $\delta_{\mathcal{P}}\colon \mathbb{N} \to \mathbb{N}\cup\{\infty\}$ defined via

$$\delta_{\mathcal{P}}(n) = \sup\left\{\mathrm{Area}_{\mathcal{P}}(w) \mid w =_G 1,\ |w|_S \leqslant n\right\}.$$

Proposition 2.39 ([CT17, §2.B])**.** *Let G be a compactly presented locally compact group. The following properties hold:*

1. *G admits a presentation $\mathcal{P} := \langle S|R_d\rangle$, where R_d is defined as the set of all null-homotopic words w of length at most d. Moreover, $\delta_{\mathcal{P}}(n) < \infty$ for all n.*
2. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two compact presentations for G such that $\delta_{\mathcal{P}_1}$ and $\delta_{\mathcal{P}_2}$ take values in $\mathbb{N}$. Then, $\delta_{\mathcal{P}_1}(n) \asymp \delta_{\mathcal{P}_2}(n)$ for all n.*
3. *The $\asymp$-behaviour of the Dehn function of G is a quasi-isometry invariant of G.*

This shows, as in the case of finite presentations [Alo90], that the Dehn function is a well-defined quasi-isometry invariant of compactly presented groups up to $\asymp$-equivalence.

Notation 2.40. Strictly speaking, in view of Proposition 2.39, we define the *Dehn function of G*, denoted by δ_G, as the $\asymp$-class of $\delta_{\mathcal{P}}$. By an abuse of notation, we treat the Dehn function of a compactly presentable group G and the Dehn function of a compact presentation $\mathcal{P}$ for G indistinctly. For example, we say that the Dehn function of G is $\asymp$-equivalent to n (or simply linear) if there is a compact presentation $\mathcal{P}$ such that $\delta_{\mathcal{P}}(n) \asymp n$ for all n.

2.6.1 Some known Dehn functions

In this section we make a short account on what is known about the Dehn functions of finitely presented groups.

As mentioned in the introduction word hyperbolic groups are precisely the groups with linear Dehn function. Moreover, if the Dehn function of a group G satisfies $\delta_G(n) \prec n^2$, then $\delta_G(n) \asymp n$ [Bow95, Gro87, Ols91]. This means that the class of groups having subquadratic Dehn function is precisely the one of hyperbolic groups. In contrast, the class of groups with Dehn function $\asymp n^2$ seems to be very broad, see for instance [Sap11, Section 4], [Bri02, Section 6] and [Ril17, Section 5].

To put this behaviour into perspective it is natural to consider the *isoperimetric spectrum*:

$$\mathrm{IP} := \{\rho \in [1, +\infty) \mid f(n) = n^\rho \text{ is } \asymp\text{-equivalent to a Dehn function}\}.$$

The behaviour mentioned above that subquadratic Dehn functions exhibit implies that there is a *gap* in IP, namely the interval $(1,2)$. This begs the question if there is another gap. Bridson constructed an infinite family of groups with polynomial Dehn functions with *rational* exponents [Bri99]. In particular, he used mapping tori of $\mathbb{Z}^m$. Nevertheless, this set of fraction is far from being dense, so by the time Bridson constructed this family, there were still many gaps in IP. Later on, Brady and Bridson [BB00] settled this question by constructing examples of finitely presented groups with Dehn functions $\asymp n^\alpha$ where the set of such exponents forms a (countable) dense subset in $[2,+\infty)$; it is worth mentioning that in a previous work [SBR02] the authors provided much finer information regarding the interval $[4,+\infty)$.

The Dehn function of a group can be polynomial or exponential (see for instance [BP94]). Moreover, we know that the Dehn function of a finitely presented group can grow faster than any iterated exponential [Pla04].

For a detailed survey on Dehn functions of finitely presented groups, we refer the reader to [Bri02] and [BRS07]. It is important to note that since the publication of Bridson's survey and the 2005 mini-course that culminated in the book [BRS07], there have been significant advances in the field. Here, we briefly mention a few of these important contributions, while acknowledging that this summary may not be exhaustive.

- The Dehn function of *Thompson's group F* is quadratic [Gub06].[3]
- The finitely presented *Stallings–Bieri* groups have quadratic Dehn functions [CF17, DERY09].[4]
- The Dehn functions of finitely presented *Bestvina–Brady groups*[5] are all bounded by a polynomial of degree 4 [Dis08]. Some specific examples of

[3]Thompson's group F consists of all piecewise-linear orientation-preserving homeomorphisms of the unit interval $[0,1]$ whose derivatives have finitely many dyadic break points and the slopes of linear pieces are powers of 2. It is finitely presented, and it has a presentation with two generators and two relations, see for instance [CFP96]. It was the first example of a torsion-free group which is of type FP_∞ [BG84].

[4]These groups arise as the kernel of epimorphisms $F_2 \times \ldots \times F_2 \to \mathbb{Z}$ sending each generator to 1. These groups were the first examples of groups with exotic finiteness conditions. They form a a family of groups of type $\mathcal{F}_n$ but not of type $\mathcal{F}_{n+1}$ [Bie81]. Here G is of type $\mathcal{F}_1$ if it is finitely generated, $\mathcal{F}_2$ if it is finitely presented, and more generally $\mathcal{F}_n$ when it admits an Eilenberg-–Maclane space or $K(G,1)$ with finite n-skeleton.

[5]These groups are generalisations of Stallings–Bieri groups. More precisely, they are subgroups of RAAGs arising as kernels of epimorphisms $A_L \to \mathbb{Z}$ that send every generator to the identity. Bestvina and Brady used them to answer a long-standing open problem: the existence of groups of type $FP(\mathbb{Z})$ but not finitely presented [BB97]. In particular, a great deal of information about the structure of these kernels, such as finitely presentability, is encoded in the defining graph of the corresponding RAAG L.

Bestvina–Brady groups with quadratic, cubic, and quartic Dehn functions were known by Brady. See also [Cha21] for further examples. In 2025, after a physical copy of this work was submitted and after my PhD defence, together with M. Migliorini and Y.-C. Chang we completely classified the Dehn functions of Bestvina–Brady groups, see [CGMM25].

- For all $n \geqslant 4$ the Dehn function of $SL_n(\mathbb{Z})$ is quadratic, the case $n \geqslant 5$ was proven in [You13a] and the case $n = 4$ follows from [LY21] where it was proven that lattices in higher rank Lie groups have quadratic Dehn functions.
- There exists a finitely presented simple group whose Dehn function is at least exponential [Zar24]. All the previously known examples of Dehn functions of finitely presented simple groups have fairly small Dehn functions, e.g. Thompson's group F.

We finish this section by observing that there are countably many $\asymp$-classes of Dehn functions of finite presentations of groups; this follows since there are countably many finite presentations of groups. Therefore, the isoperimetric spectrum for finitely presented groups is a countable (and dense) subset of $\{1\} \cup [2, +\infty)$. In contrast, going from finitely presented groups to compactly presented ones, one is led to consider uncountably many groups, see for instance [BKS20] and Corollary C, which raises Question 1.2.

2.7 Van Kampen diagrams

A standard technique for estimating Dehn functions of finitely presented groups is using diagrams that encode the process of applying relations to reduce a word. These diagrams, called van Kampen diagrams are the topic of this section. This section follows closely [Bri02] and [BRS07, Part III] to which we refer for a thorough discussion and treatment of van Kampen diagrams and their relation with Dehn functions and isoperimetric inequalities. Van Kampen diagrams were first introduced by van Kampen [Kam33] but went unnoticed until Lyndon rediscovered them [Lyn66] (see also [LS01]).

In this section we fix a finitely presented group G and a finite presentation $\mathcal{P} := \langle S|R\rangle$. We can assume without loss of generality that the set R consists of *cyclically reduced* words, that is, words for which all cyclic permutations are reduced words. Indeed, if $\widetilde{R}$ is the smallest set containing R, all the cyclically reduces words of R and their inverses, then $\langle S|R\rangle \cong \langle S|\widetilde{R}\rangle$. For a null-homotopic word w in G, there is a geometric way to describe the free identity

$$w \overset{\text{free}}{=} \prod_{i=1}^{k} u_i^{-1} r_i^{\pm 1} u_i, \tag{2.4}$$

and thus encoding the number k of relators needed. This consists of a planar compact contractible combinatorial 2-complex D called a van Kampen diagram. These are meant to give geometrical and topological meaning to the identity (2.4). We define them following [BRS07, Part III, Chapter II].

Definition 2.41 (van Kampen diagram)**.** Let D be a finite, connected, oriented, pointed, labelled, planar graph where each oriented edge is labelled by an element of S. The base point lies on the boundary of the unbounded region of $\mathbb{R}^2 \setminus D$. Suppose in addition that for each bounded region F of $\mathbb{R}^2 \setminus D$, called *face*, the topological boundary ∂F of F is labelled by a word in R. This word is obtained by reading the labels on the edges as they are traversed, starting from some vertex on the boundary of F, in one of the two possible directions. Each label on the edge traversed is given a ± 1 exponent according to whether the direction of traversal coincides with, or is opposite to, the orientation of the edge. The choice of direction and starting point alters the word read by inversion and/or cyclic conjugation. The boundary word of the diagram D is the word w read on the boundary of the unbounded region of $\mathbb{R}^2 \setminus D$, starting from the base vertex. In this case we say that D is a *van Kampen diagram for w over the presentation $\mathcal{P}$*.

We could also to think of a van Kampen diagram D as a planar 2-complex: the 0-skeleton corresponds to the vertices of D, the 1-skeleton corresponds to the edges of D, and for each face F of D we attach a 2-cell: given a relator r the attaching map sends a polygon with $|r|$ many 1-cells to the edge-loop in D with label r.

To illustrate Definition 2.41 let us consider the following example.

Example 2.42. Consider the group $\mathbb{Z}^2$ with finite presentation

$$\mathcal{P} := \langle x, y | [x, y] = 1 \rangle.$$

The word $w := [x^2, y^2]$ is a null-homotopic word in $\mathcal{P}$. A van Kampen diagram for w over $\mathcal{P}$ with four faces can be drawn as shown in Figure 2.3. Each face has boundary word the relator $[x, y]$. Recall our convention is $[x, y] = x^{-1}y^{-1}xy$.

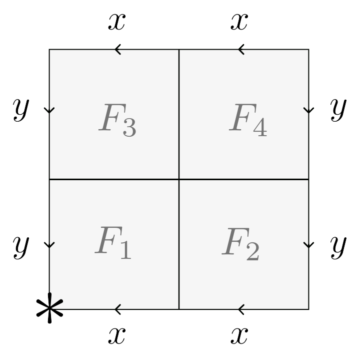

Figure 2.3: A van Kampen diagram for the null-homotopic word $w := [x^2, y^2]$. The word can be read off by travelling in the counter clock-wise direction starting from the base point $*$.

Observe that we can express w as a product of conjugates of the single defining relator $[x, y]$ in $\mathcal{P}$ as follows:

$$w \overset{\text{free}}{=} [x,y] \cdot [x,y]^{y^{-1}xy} \cdot [x,y]^{xy} \cdot [x,y]^{y}. \tag{2.5}$$

Since this is a free identity, the word on the right-hand side has the same area as w. Pictorially, the free identity Equation (2.5) can be visualized by means of "pulling apart" the van Kampen diagram in Figure 2.3 as depicted in Figure 2.4. Each boundary word of the faces is again labelled by the relator $[x, y]$ and the

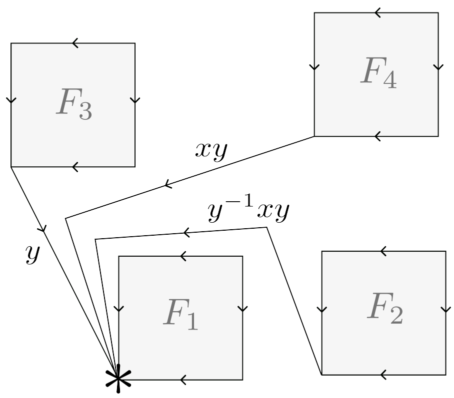

Figure 2.4: A van Kampen diagram for the null-homotopic word $[x,y] \cdot [x,y]^{y^{-1}xy} \cdot [x,y]^{xy} \cdot [x,y]^{y}$ over the given presentation $\mathcal{P}$. For readability we omit the labels of the boundary words of the faces.

"whiskers" consisting of the paths connecting the faces and the base point $*$ are labelled by the conjugating elements $y^{-1}xy$, xy, and y, respectively.

The key point is that this process can be done in the reverse way. Namely, suppose we are given a product of conjugates of relators together with a van Kampen diagram for it. Then, we can "fold" this diagram to obtain a diagram for a word w which would be freely equal to the product of conjugates of relators we were given. This idea lies behind van Kampen's Lemma, see Theorem 2.43.

Continuing our discussion on van Kampen diagrams, note that given two paths in a van Kampen diagram, if they have the same endpoints, then the words corresponding to their labels represent the same element in the group.

Van Kampen diagrams offer us a geometric interpretation of the area of a null-homotopic word. The number of faces (or 2-cells) in a van Kampen diagram for the null-homotopic word w over the presentation $\mathcal{P}$ is called the *area of* D and denoted by $\mathrm{Area}_{\mathcal{P}}(D)$. The relation between the area of D and the area of w was initially realized by van Kampen [Kam33]. The modern way to state this relation is as follows.

Theorem 2.43 (van Kampen's Lemma)**.** *Let G be a finitely presented group given by a finite presentation $\mathcal{P} := \langle S|R\rangle$.*

1. *A word $w := w(S)$ is null-homotopic in G if and only if there exists a van Kampen diagram D over $\mathcal{P}$ with boundary word w.*
2. *If a word $w := w(S)$ is null-homotopic in G, then*

$$\mathrm{Area}_{\mathcal{P}}(w) = \min\{\mathrm{Area}_{\mathcal{P}}(D) \mid D \text{ is a van Kampen diagram for } w \text{ in } \mathcal{P}\}$$

Proof. For a proof we refer the reader to [Bri02, Theorem 4.2.2] or [BRS07, Part III, Prop. 2.3]. □

A consequence of Theorem 2.43 is the existence of a van Kampen diagram for every null-homotopic word w whose number of faces is precisely $\mathrm{Area}_{\mathcal{P}}(w)$. We call these diagrams *minimal area van Kampen diagrams.*

In view of van Kampen's Lemma, Theorem 2.43, the Dehn function of G can be interpreted via isoperimetric inequalities for van Kampen diagrams over $\mathcal{P}$. The use of van Kampen diagrams is crucial in the so-called *Filling Theorem* [Bri02, 2.1.2 Filling Theorem], which states that the Dehn function of a finitely presented group G, arising as the fundamental group of a closed Riemannian manifold M, is $\asymp$-equivalent to the so-called filling function of the universal cover of M. Here the filling function is the smallest function bounding the area of minimal area discs in terms of their boundary length.

2.7.1 Corridors

As mentioned at the beginning, van Kampen diagrams and their structure are a powerful tool in geometric group theory. In particular, when estimating precise Dehn functions. Certain features of the presentations are closely related to the geometry and structure of the corresponding van Kampen diagrams. A notable feature is the existence of so-called *corridors*. Corridors appear in van Kampen diagrams over a presentation $\langle S, t | R \rangle$ with $t \notin S$ such that all the relators in R involving $t^{\pm 1}$ are of the form

$$t^{\pm 1} s t^{\mp 1} w(S).$$

This sort of presentations, as we saw in Chapter 2, arise naturally for HNN-extensions where t is the stable letter. In particular, they arise naturally in presentations for mapping tori of groups. To define corridors in a van Kampen diagram we first need the definition of a subdiagram. Informally speaking, a subdiagram of a van Kampen diagram is a cut out part of it.

Definition 2.44 (Subdiagram)**.** Let $\langle S | R \rangle$ be a finite presentation for a group G. Let w be a null homotopic in G and D a van Kampen diagram for w over the presentation $\mathcal{P}$. We say that a van Kampen diagram E for a null-homotopic word w' is a *subdiagram of* D if there exists words u and v such that the identity $w =_{\mathcal{P}} uw'v$ holds in G and E is a connected subcomplex of D with boundary label w'.

Given a null-homotopic word a and a van Kampen diagram D for w over$\mathcal{P}$, the *thin* part of ∂D is the topological closure of those 1-cells which do not lie in the boundary of any 2-cell in D. The *thick* part of ∂D is the topological closure of the complement of the thin part.

Definition 2.45 (Corridor and annulus)**.** Let D be a van Kampen diagram over the presentation $\langle S, t | R \rangle$ and suppose that all the relators in R involving $t^{\pm 1}$ are of the form $t^{\pm 1} s t^{\mp 1} w(S)$. An edge in D is called a *t-edge* if it is labelled by t. Let $D^{\star}$ be the dual graph to the 1-skeleton of D, let v_{∞} a vertex dual to the unbounded region of $D \subset \mathbb{R}^2$, and λ a loop in $D^{\star}$ whose edges are all dual to t-edges. The subdiagram K of D, consisting of all the closed 1-cells and 2-cells of D dual to $\lambda \setminus \{v_{\infty}\}$, is called a *$t$-corridor* if λ includes v_{∞}, and a *t-annulus* if it does not.

Observe that if K is a t-corridor in D, then it is a subdiagram K of D such that all the 2-cells in K are labeled by the relators $t^{-1} s t w(S)$ or a single 1-cell e in D with label t belonging to the thin part of D. In both cases we require that $D \setminus K$ consists of two disjoint components. See Figure 2.5.

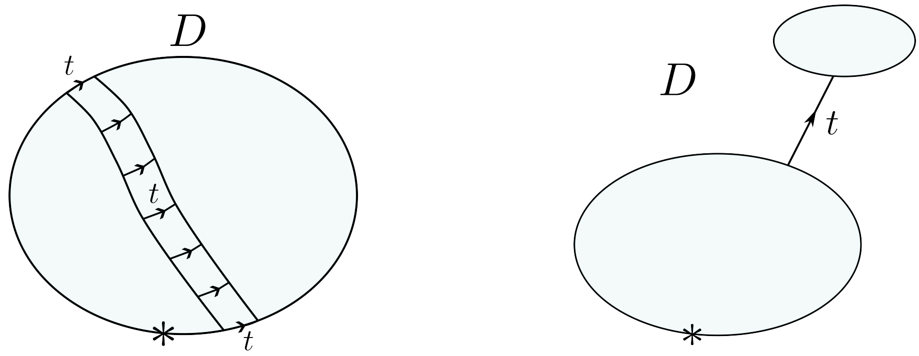

Figure 2.5: Two possible types of t-corridors in a van Kampen Diagram. The diagram on the left has no thin parts, while the one on the left has exactly one thin part whose label is t.

2.8 Automorphisms of groups

A basic way to construct groups is via free and direct products of groups. It is then natural to study if a group G can be decomposed as a free or as a direct product of groups. This perspective might allow to reduce the study of the group G to that of the groups realizing it as a free or direct product. This strategy turns out to be relevant when we study the Dehn functions of mapping tori of RAAGs (see Sections 4.2 and 4.3).

In this section we recall standard and well-known facts about automorphisms of direct products and free products of groups which we use in Chapter 4. We start with automorphisms of free products of groups.

2.8.1 Automorphisms of free products of groups

Given a group G and a subgroup $H \leq G$, we say H is a *free factor* of G provided there exists another subgroup $K \leq G$ such that $G \cong H * K$ where the isomorphism is the one naturally induced by the inclusions of H and K into G. We then say that $H * K$ is a *free (product) decomposition* of G and call the groups H and K *free factors of* G. By abuse of notation we write $G = H * K$. We say that G is *freely indecomposable* if there does not exist a free decomposition of G via two non-trivial free factors.

Presentations for the automorphism group of a free product of groups were found by Fouxe-Rabinovitch [FR40, FR41] and later on by Gilbert [Gil87]. Moreover, provided that $G = H * K$ is a free product of two freely indecomposable and not infinite cyclic groups, then $\mathrm{Aut}(G)$ is generated by the following automorphisms (see [Gil87, p. 116]):

- *Factor automorphisms.* An automorphism $\phi \in \mathrm{Aut}(G)$ is a *factor automorphism* if $\phi|_K \in \mathrm{Aut}(K)$ and $\phi|_H \in \mathrm{Aut}(H)$.
- *Permutation automorphisms.* Suppose that there exists an isomorphism $f\colon K \to H$, then we say that $\phi \in \mathrm{Aut}(G)$ is a *permutation automorphism induced by* f if ϕ permutes the factors according to f, namely if ϕ is defined as

$$\begin{aligned} \phi\colon H * K &\to K * H \\ h_1 k_1 \dots h_s k_s &\mapsto f^{-1}(h_1) f(k_1) \dots f^{-1}(h_s) f(k_s) \end{aligned}$$

  where $K * H \cong G$ via f.
- *Partial conjugations.* An automorphism $\phi \in \mathrm{Aut}(G)$ is a *partial conjugation of* K if there exists $h \in H$ such that $\phi|_K\colon k \mapsto h^{-1}kh$ for every $k \in K$ and $\phi|_H = id_H$, in such case we say that ϕ is a *partial conjugation of* G *by* $h \in H$.

Note that the permutation automorphisms have order two, i.e. $\phi^2 = id_G$.

Given $G = H * K$ under reasonable assumptions on the factors we have a nice description of the automorphisms of G. Namely, we have the following result.

Proposition 2.46 ([Kar23, Lemma 3.3])**.** *If $G = H * K$ is a free product decomposition of G such that the free factors are freely indecomposable and not infinite cyclic groups, then the group* $\mathrm{Out}(G)$ *is generated by factor automorphisms and a single permutation automorphism.*

Proof. For the readers' convenience we include a sketch of proof following [Kar23, Lemma 3.3]. Let ϕ be a partial conjugation of G by $h \in H$. Consider the inner automorphism $c_{h^{-1}} \in \mathrm{Aut}(G)$ defined as $g \mapsto hgh^{-1}$. Then, we have

$$(c_{h^{-1}} \circ \phi)(g) = \begin{cases} hgh^{-1}, & \text{if } g \in H, \\ g, & \text{if } g \in K. \end{cases}$$

Therefore, for the factor automorphism $\psi \in \mathrm{Aut}(G)$ defined as $\psi|_H\colon g \mapsto hgh^{-1}$ and $\psi|_K = id_K$, we have that $[\phi] = [\psi] \in \mathrm{Out}(G)$. Analogously, for a partial conjugation of G by $k \in K$, we see that up to an inner automorphism it is a factor automorphism.

Given two different permutation automorphisms ϕ_1 and ϕ_2, then they differ by a product of two factor automorphisms. Indeed, if for $i \in \{1, 2\}$ we have that ϕ_i is induced by the isomorphism $f_i\colon H \to K$, then we define the factor automorphisms as follows. Let $\psi_H \in \mathrm{Aut}(H)$ and $\psi_K \in \mathrm{Aut}(K)$ such that the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{f_1} & K \\ {\scriptstyle \psi_H}\downarrow & & \downarrow{\scriptstyle \psi_K} \\ H & \xrightarrow[f_2]{} & K \end{array}$$

Let ψ_1 (respectively ψ_2) be the factor automorphism of G induced by ψ_H (respectively ψ_K), that is $\psi_1|_H = \psi_H$ and $\psi_1|_K = id_K$ (respectively $\psi_2|_K = \psi_K$ and $\psi_2|_H = id_H$). Thus, we have $\psi_2 \circ \psi_1 \circ \phi_1 = \phi_2$. □

We would like to mention that an alternative proof of Proposition 2.46, without using the generating set of the automorphism group of a free product decomposition, can be found in Nicolás thesis [Nic21, p. 35]. We opted to include this version of the proof since the use of a generating set aligned with the spirit of this work. Moreover, we would like to mention, that the generating set for the automorphisms of a free product decomposition bears some similarities with the generating set of the automorphism group of a RAAG found independently by Laurence and Servatius, for the definition we refer the reader to [Ser89, Lau95].

We have the following consequence of Proposition 2.46.

Corollary 2.47. *If $G = H * K$ is a free product of two freely indecomposable and not infinite cyclic groups, then for every automorphism $\psi \in \mathrm{Aut}(G)$ there exist $p \in \{1, 2\}$ and $\phi \in \mathrm{Aut}(G)$ such that $\phi|_H \in \mathrm{Aut}(H)$, $\phi|_K \in \mathrm{Aut}(K)$, and $[\phi] = [\psi^p] \in \mathrm{Out}(G)$.*

Proof. It suffices to show that the factor automorphisms generate a normal index 2 subgroup of $\mathrm{Out}(G)$. Let N be the subgroup of $\mathrm{Out}(G)$ generated by factor automorphisms and Φ be the outer class of the single permutation automorphism. By Proposition 2.46 we know that N and Φ generate $\mathrm{Out}(G)$.

To see that N is normal in $\mathrm{Out}(G)$, by Proposition 2.46, it suffices to show that $\Phi \cdot N = N \cdot \Phi$. To see this observe that any factor automorphism $\psi \in \mathrm{Aut}(G)$ can be expresses as $\psi = \psi_H * \psi_K$. Thus, $\Phi \cdot [\psi] = \Phi \cdot [\psi_H * \psi_K] = [\psi_K * \psi_H] \cdot \Phi$. This proves normality of N. Finally, since $\Phi^2 = 1$, it follows that N has index 2 in $\mathrm{Out}(G)$. □

We now proceed with the results regarding the automorphism group of a direct product.

2.8.2 Automorphisms of direct products of groups

We say that a group G is *decomposable as a direct product* if there exist non-proper and non-trivial subgroups H_1 and H_2 of G such that $G \cong H_1 \times H_2$, we call the subgroups H_1 and H_2 *direct factors of* G, or simply *factors* if it is clear from the context that it concerns a direct product decomposition. If such a decomposition does not exist we say that G is *indecomposable (as a direct product)*.

We now state a well-known result for the automorphism groups of the direct product of groups. To state the result we require the following notation. Let $\ell \geqslant 2$, and for each $i \in \{1, \dots, \ell\}$ let G_i be an infinite group, and set $G \coloneqq G_1 \times \ldots \times G_\ell$. Up to permutation of the factors we can collect all the isomorphic copies of the G_i's and write $G = G_1^{m_1} \times \cdots \times G_k^{m_k}$ for $k \leqslant \ell$ and $m_i \geqslant 1$.

Lemma 2.48. *Let $\ell \geqslant 2$. For each $i \in \{1, \dots \ell\}$ let G_i be a non-trivial indecomposable group with trivial centre. Let $G \coloneqq G_1 \times \ldots \times G_\ell$, then*

$$\mathrm{Aut}(G) \cong \prod_{i=1}^{k} \Big(\prod_{j=1}^{m_i} \Big(\mathrm{Aut}(G_j) \Big) \rtimes \mathfrak{S}_{m_i} \Big)$$

where $\mathfrak{S}_{m_i}$ is the permutation group on $\{1, \dots, m_i\}$ which acts on $\prod_{j=1}^{m_i} \mathrm{Aut}(G_i)$ by permuting the factors.

Proof. Although this result is well-known to the experts, we could not find a proof in the literature. For the reader's convenience we include the proof for the case of only two factors. This avoids unnecessary technicalities and conveys the main idea.

Let G and H be two non-trivial indecomposable groups with trivial centres. Let $\phi \in \mathrm{Aut}(G \times H)$, $\iota_G \colon G \hookrightarrow G \times H$ and $\iota_H \colon H \hookrightarrow G \times H$ the canonical inclusion, $\pi_G \colon G \times H \twoheadrightarrow G$ and $\pi_H \colon G \times H \twoheadrightarrow H$ the canonical projections, and define the following subgroups

$$A_1 \coloneqq \mathrm{im}(\pi_G \circ \phi \circ \iota_G) \leq G, \quad B_1 \coloneqq \mathrm{im}(\pi_H \circ \phi \circ \iota_G) \leq H,$$
$$A_2 \coloneqq \mathrm{im}(\pi_G \circ \phi \circ \iota_H) \leq G, \quad B_2 \coloneqq \mathrm{im}(\pi_H \circ \phi \circ \iota_H) \leq H.$$

From the definition of the A_i's and B_i's it follows that $[A_1, A_2] = \{1\}$ and also $[B_1, B_2] = \{1\}$. Moreover, since ϕ is an automorphism, we also have that $G = A_1 \cdot A_2$ and $H = B_1 \cdot B_2$.

Since G and H are indecomposable we have that either A_1 or A_2 is trivial, and either B_1 or B_2 is trivial.

If $A_1 = \{1\}$, then, since G is non-trivial we have that $A_2 = G$. This implies, by definition, that $B_1 \neq \{1\}$. Since H is indecomposbale we have that B_2 is trivial. Thus, we have an isomorphism $G \cong H$ induced by $\pi_G \circ \phi \circ \iota_H$ with inverse $\pi_H \circ \phi^{-1} \circ \iota_G$. Therefore, $\phi = \sigma \circ (\phi_G \times \phi_H)$ for $\phi_G \in \mathrm{Aut}(G)$, $\phi_H \in \mathrm{Aut}(H)$, and permutation $\sigma \in \mathfrak{S}_2$.

If $A_2 = \{1\}$, then, arguing as before, we get $A_1 = G$, $B_1 = \{1\}$, and $B_2 = H$. Therefore we have $\phi = (\phi_G \times \phi_H)$ for $\phi_G \in \mathrm{Aut}(G)$, $\phi_H \in \mathrm{Aut}(H)$. □

A direct consequence of Lemma 2.48 is the following.

Corollary 2.49. *Let G and H be two non-trivial indecomposable groups. Then, the subgroup $\mathrm{Aut}(G) \times \mathrm{Aut}(H) \leq \mathrm{Aut}(G \times H)$ has index at most two. In particular, for every $\phi \in \mathrm{Aut}(G \times H)$ there exist $k \in \{1, 2\}$ such that $\phi^k := \phi_G \times \phi_H$, with $\phi_G \in \mathrm{Aut}(G)$ and $\phi_H \in \mathrm{Aut}(H)$.*

2.9 The growth of automorphisms of groups

As mentioned before, in this work we study the Dehn functions of mapping tori of RAAGs, namely groups of the form $A_L \rtimes_\phi \mathbb{Z}$ where $\phi \in \mathrm{Aut}(A_L)$. For this purpose we study the so-called growth function of the automorphism ϕ. For this, we restrict to the study of automorphisms of *finitely generated groups* which is an important subclass of compactly generated groups.

We start with the definition of growth for a general automorphism of a finitely generated group.

Definition 2.50 (Growth)**.** Given a finitely generated group G, a finite generating set S of G, and an automorphism $\phi \in \mathrm{Aut}(G)$, the *growth function of ϕ with respect to S* is the function $gr_{\phi,S} \colon \mathbb{N} \to \mathbb{N}$ defined via

$$gr_{\phi,S} \colon n \mapsto \max\{ \, |\phi^n(s)|_S \ \mid s \in S\}.$$

Similarly, as with Dehn functions, the asymptotic behaviour of the growth function of an automorphism is well-defined up to an equivalence of functions that captures the asymptotic behaviour, in this case $\sim$-equivalence which we defined in Section 2.1.2.

Proposition 2.51 ([BS19, Proposition 2.4 and Remark 2.5])**.** *Let G be a finitely generated group with generating set S and let $\phi \in \mathrm{Aut}(G)$. If S' is another finite generating set of G, then $gr_{\phi,S} \sim gr_{\phi,S'}$.*

Notation 2.52. In view of Proposition 2.51, we define the *growth of an automorphism* ϕ as the $\sim$-class of $gr_{\phi,S}$, and we denote it by gr_ϕ. By an abuse of notation we do not distinguish the growth of an automorphism ϕ and the growth of an automorphism ϕ with respect to a generating set.

We now mention some of the possible asymptotic behaviours that the growth of an automorphism can exhibit. As we discuss below, in the cases where the finitely generated group G is either a free group or an abelian group, these behaviours are the only possibilities, see Theorems 2.58 and 2.60.

Definition 2.53 (Types of growth)**.** Let G be a finitely generated group and $\phi \in \mathrm{Aut}(G)$. We say that ϕ has

- *polynomial growth* if there exists $d \geqslant 0$ such that $gr_\phi(n) \sim n^d$.
- *exponential growth* if for some $\lambda > 1$ we have that $gr_\phi(n) \sim \lambda^n$.

In the particular case that ϕ has polynomial growth with $d = 0$ we say that ϕ has *constant growth* or *trivial growth*. Note that in this case $gr_\phi \sim gr_{id} \sim K$ for some constant $K > 0$. It is clear that automorphisms of finite order have trivial growth.

Example 2.54. Let $F_{\{a,b\}}$ be the free group of rank 2 with generating set $\{a, b\}$ and $\phi \in \mathrm{Aut}(F_{\{a,b\}})$ defined by $a \mapsto ab, b \mapsto a$. A short computation shows that the number of a's appearing in $\phi^n(b)$ grows like the Fibonacci sequence. This implies, that ϕ has exponential growth.

Remark 2.55. The linear term used in the definition of $\asymp$-equivalence, see (2.2), is essential to prove that the Dehn function is a quasi-isometry invariant. For instance, consider the group $\mathbb{Z}$ with the following two finite presentations $\mathcal{P}_1 = \langle a \mid \varnothing \rangle$ and $\mathcal{P}_2 = \langle a, b \mid b = 1 \rangle$. Then, the Dehn function of $\mathcal{P}_1$ satisfies $\delta_{\mathcal{P}_1}(n) = 0$ for all n, whereas the Dehn function of $\mathcal{P}_2$ satisfies $\delta_{\mathcal{P}_2}(n) = n$ for all n. Nevertheless, when studying automorphisms, we require the finer $\sim$-equivalence, since we actually want to distinguish between linear growth and constant growth.

We restrict the study of automorphisms of finitely generated groups to the study of automorphisms of RAAGs. The main motivation for this, as mentioned in the introduction, is that for certain RAAGs A_L, the growth of their automorphisms helps us to determine the precise asymptotics of the Dehn functions of the mapping tori $A_L \rtimes_\phi \mathbb{Z}$ (see for instance [BP94, PR19] and Theorems E and F).

Regarding the class of RAAGs, the growth of automorphisms of free and free abelian groups has been extensively studied, see for instance [BH92, BFH00, BFH05, Lev09, Bri00] for the free case and [BG96] for the free abelian case. Below we recall some facts about the growth of automorphisms of free and free abelian groups, but first we would like to close this section with an example and a remark.

Example 2.56 ([PR19])**.** The automorphism $\phi \in \mathrm{Aut}(F_{\{a,b\}})$ defined by $a \mapsto b^{-1}ab$ and $b \mapsto b$ has linear growth. By conjugating it by b^{-1} the automorphisms is just the identity which has constant growth.

Remark 2.57. When studying automorphisms of free groups, there exists a different notion of the growth of automorphisms, the so-called *cyclic growth*, which happens to be invariant under the composition with inner automorphisms. This notion was introduced by Pueschel and Riley in [PR19, Definition 7.5] and plays a significant role in their computations for Dehn functions. For details, we refer the reader to [PR19, Section 7.2].

2.9.1 Growth of automorphisms of free abelian groups

After choosing a generating set for $\mathbb{Z}^m$ we can identify $\mathrm{Aut}(\mathbb{Z}^m) \cong \mathrm{GL}_m(\mathbb{Z})$. Therefore an automorphism ϕ corresponds to an invertible $m \times m$-matrix A_ϕ or simply A with integer entries. Moreover, choosing a basis for $\mathbb{Z}^m$ allows us to embed it into $\mathbb{C}^m$ which we can then think of as vector space with the chosen basis. This allows us to think of the invertible matrix $A := (a_{ij})_{i,j=1}^m$ as an element in $\mathrm{Aut}(\mathbb{C}^m)$ where we can define the *supremum-norm* as

$$||A||_{sup} := \max\{\ |a_{ij}|\ \ |\ 1 \leqslant i, j \leqslant m\}.$$

In this setting, it is natural to study the asymptotic behaviour of the function $gr_A \colon \mathbb{N} \to \mathbb{C}$ defined via

$$gr_A \colon n \mapsto ||A^n||_{sup},$$

Observe that by definition we have $gr_A \sim gr_\phi$.

The asymptotic behaviour of gr_A is fully understood [BG96]. First, the $\sim$-equivalence class of gr_A depends only on the conjugacy class of A in $\mathrm{GL}_m(\mathbb{C})$; in particular, it is the same as the growth of the associated Jordan normal form. Second, we have the following *growth dichotomy*: gr_A is $\sim$-equivalent to either a polynomial or an exponential function:

Theorem 2.58 ([BG96, Theorem 2.1])**.** *If $A \in GL_m(\mathbb{Z})$ then the gr_A is $\sim$-equivalent to a polynomial or exponential function. Moreover, it is exponential if and only if there exists an eigenvalue of A whose absolute value is strictly greater* *than 1; else it is polynomial of degree $d+1$, where $d \times d$ is the size of a largest Jordan block associated to A.*

Since by definition $gr_A \sim gr_\phi$ the above growth dichotomy holds for gr_ϕ as well. We finish the discussion about the free abelian case with the following direct consequence of Theorem 2.58.

Corollary 2.59 ([BG96, Corollary 2.3])**.** *For all $A \in GL_m(\mathbb{Z})$ we have $gr_A \sim gr_{A^{-1}}$. In particular, for all $\phi \in \mathrm{Aut}(\mathbb{Z}^m)$ we have $gr_\phi \sim gr_{\phi^{-1}}$.*

2.9.2 Growth of automorphisms of free groups

Recall that in the case of a free group of finite rank every *group element* is represented by a class of words, all of which are freely equivalent to each other. Therefore, in the case of automorphisms of free groups the growth is really the growth of words as we iteratively apply the automorphism. This does not necessarily mean that studying their growth is easy.

In the case of free groups, we observe a similar growth "dichotomy" as in the free abelian case. Nevertheless, achieving this took a non-trivial amount of work and the machinery of train tracks to show the "dichotomy" on the possible growth rates of automorphisms:

Theorem 2.60 ([BH92, Lev09])**.** *Let F_m be a free group of rank m and $\phi \in \mathrm{Aut}(F_m)$. For every $g \in F_m$, precisely one of the following holds:*

- *there exists $d \in \mathbb{N}$ (possibly zero) such that $gr_\phi(n) \sim n^d$;*
- *there exists $\lambda > 1$ such that $gr_\phi(n) \sim \lambda^n$.*

We now record an analogue to Corollary 2.59 for automorphisms of free groups with polynomial growth, which we use in Section 4.3.

Theorem 2.61 ([Pig04, Theorem 0.4])**.** *Let F_m be a free group of finite rank $m > 0$ and $\phi \in \mathrm{Aut}(F_m)$. Then $gr_\phi \sim gr_{\phi^{-1}}$.*

2.10 General bounds on Dehn functions

In this section we start by stating a few known basic results for general upper bounds on Dehn functions for finitely generated groups. We then present a general upper bound on the Dehn functions of mapping tori of finitely presented groups.

We start with an observation which follows from the fact that a finite index subgroup H of a group G is quasi-isometric to G and the fact that the Dehn function of a group is a quasi-isometry invariant.

Lemma 2.62 ([Alo90]). *Let G be a finitely presented group and $H \leq G$ a finite index subgroup. Then, H is also finitely presented and $\delta_H \asymp \delta_G$.*

Proof. For a proof see [Alo90] or [BH99, 8.24 Proposition], a sketch can be found in [Ril17, Theorem 4.3]. □

Recall that a group H is called a *retract* of a group G if there exist a morphism $\iota\colon H \to G$ and an epimorphism $r\colon G \to H$ such that $r \circ \iota = id_H$. The epimorphism is called a *retraction*. In the special case that $H \leqslant G$ we sometimes call it *subgroup retract*.

Lemma 2.63 ([BMS93, Lemma 1]). *Let G be a finitely presented group and H a subgroup of G. If H is a subgroup retract of G, then H is also finitely presented and $\delta_H \preccurlyeq \delta_G$.*

Proof. A proof can be found in [BMS93, Lemma 1], [Alo90, Theorem 2.2], or [BGSS92, Theorem G]. □

The next result is a really rough estimate on the Dehn function of mapping tori of finitely presented groups.

Lemma 2.64. *Let G be a finitely presented group and $\phi \in \mathrm{Aut}(G)$, then the Dehn function of $M_\phi \coloneqq G \rtimes_\phi \mathbb{Z}$ satisfies*

$$\delta_{M_\phi}(n) \preccurlyeq \delta_G \circ \exp(n).$$

In particular, if $\delta_G(n) \preccurlyeq n^d$ for some $d \in \mathbb{N}$, then $\delta_{M_\phi}(n) \preccurlyeq 2^n$

Proof. The proof of this is an adaptation of the proof of [PR19, Lemma 3.4]. Recall from Definition 2.2, that if $\langle S|R\rangle$ is a finite presentation for G, then

$$\mathcal{P} \coloneqq \langle S, t \mid R,\ t^{-1}st = \phi(s) \text{ for all } s \in S\rangle$$

is a finite presentation for M_Φ. Thus, a word $w \coloneqq w(S,t)$ in M_ϕ can be written as

$$w =_{\mathrm{def}} t^{k_0}a_0 \ldots t^{k_m}a_m$$

for some $a_i \in S$ and some $k_i \in \mathbb{Z}$ $(0 \leqslant i \leqslant k)$. Now, suppose w is a null-homotopic word in M_ϕ. We reduce w as follows: we shuffle all the $t^{\pm 1}$ to the right end of the word w. Whenever a $t^{\pm 1}$ passes a letter s from S we replace it by the freely reduced word representing $\phi^{\pm 1}(s)$. Therefore, by shuffling all the $t^{\pm 1}$'s we end up with an identity of the form

$$w =_{\mathcal{P}} u(X)t^{k_0+\ldots+k_m} \tag{2.6}$$

for $u \coloneqq u(X)$ a null-homotopic word in G and $k_0 + \ldots + k_m = 0$. Note that applying $\phi^{\pm 1}$ to a letter in S increases its length by at most the constant $C \coloneqq \max\{|\phi^{\pm 1}(s)| \mid s \in S\}$. Therefore, the length of u is bounded by

$$|u| \leqslant mC^{|k_0|+\ldots+|k_m|} \leqslant |w|C^{|w|}.$$

Moreover, the area of the identity (2.6) is $\lesssim_{\mathcal{P}} |w|C^{|w|}$.

Finally, since we need at most $\delta_G(|u|)$ many relations to reduce u, this implies that

$$\mathrm{Area}_{\mathcal{P}}(w) \lesssim \delta_G(|w|C^{|w|}) + |w|C^{|w|}$$

which proves the desired upper bound.

For the particular case where $\delta_G(n) \preccurlyeq n^d$, observe that from the above estimate we get

$$\mathrm{Area}_{\mathcal{P}}(w) \lesssim_{\mathcal{P}} \delta_G(|w|C^{|w|}) \lesssim_{\mathcal{P}} (|w|C^{|w|})^d.$$

Finally observe that since $n(C^d)^n \preceq (C^{2d})^n$ and $\alpha^n \asymp \beta^n$ for all $\alpha, \beta > 1$ we obtained $\delta_{M_\phi}(n) \preccurlyeq 2^n$. □

It was mentioned in Section 2.6.1 that Dehn functions can grow faster than an exponential function. Nevertheless, in the case of mapping tori of RAAGs this is not the case as the following direct consequence of Lemma 2.64 states.

Corollary 2.65 ([PR19, Lemma 3.4])**.** *Let A_L be a non-free RAAG and let $\phi \in \mathrm{Aut}(A_L)$. Then, the Dehn function of M_ϕ satisfies*

$$n^2 \preccurlyeq \delta_{M_\phi}(n) \preccurlyeq 2^n.$$

Proof. Note that the only word hyperbolic RAAGs are the free ones, all the others contain a $\mathbb{Z}^2$ as a subgroup. Thus, the lower bound follows from A_L being a non-word hyperbolic group. For the upper bound first observe that since A_L is a CAT(0) group (see for instance [Cha07, Theorem 2.6]), it has at most quadratic Dehn function [Bri02, Theorem 6.2.1]. Thus the conclusion follows from the upper bound on δ_{M_ϕ} given by Lemma 2.64. □

As we see in Chapter 4 the upper bound offered by Corollary 2.65 is not always sharp. Finally, we have the following result bounding from below the Dehn function of a group for which there is a quasi-isometrically embedded abelian group.

Lemma 2.66 ([PR19, Lemma 3.5] [BG96, Theorem 4.1])**.** *Suppose that K is a free abelian group quasi-isometrically embedded in a finitely presented group G. If $\Phi \in \mathrm{Aut}(G)$ and $\Phi|_K = \phi \in \mathrm{Aut}(K)$, then the Dehn function of $M_\Phi := G \rtimes_\Phi \mathbb{Z}$ satisfies*

$$n^2 \cdot \max\{ \, |\Phi^{\pm n}(k_i)| \, \mid 1 \leqslant i \leqslant m\} \preccurlyeq \delta_{M_\Phi}(n).$$

Equivalently, if $\Phi|_K = \phi$ has associated matrix A, then the following holds:

1. *If A has an eigenvalue λ such that $|\lambda| \neq 1$, then M_Φ has exponential Dehn function.*
2. *If every eigenvalue of A has norm one, then $n^{c+1} \preccurlyeq \delta_{M_\Phi}(n)$, where $c \times c$ is the size of the largest Jordan block matrix for A.*

Proof. For a proof of this result we refer the reader to [BG96, Theorem 4.1]. □

We conclude this section with the following result. It allows us to find a suitable representative of the outer class of an automorphism $\psi \in \mathrm{Aut}(A_L)$ when estimating the Dehn functions of mapping tori of RAAGs in Chapter 4. The underlying idea is that the Dehn function of the corresponding mapping torus M_ψ depends on the outer class of ψ as the following result shows.

Lemma 2.67 ([PR19, Lemma 3.6], [BMV07, Lemma 2.1])**.** *Let G be a finitely presented group and $\psi \in \mathrm{Aut}(G)$. The following mapping tori have equivalent Dehn functions*

1. *M_ψ and M_{ψ^m} for every non-zero $m \in \mathbb{N}$.*
2. *M_ψ and $M_{\psi^{-1}}$.*
3. *M_ψ and M_ϕ for every $\phi \in \mathrm{Aut}(G)$ such that $[\phi] = [\psi] \in \mathrm{Out}(G)$.*

Proof. For a concise proof we refer the reader to [PR19, Lemma 3.6]. □

Chapter 3

Dehn functions of nilpotent groups

In this chapter we study the Dehn functions of nilpotent groups. In particular, we make an extensive analysis of upper bounds on Dehn functions of the central product of nilpotent groups.

We start by giving a brief background in Section 3.1 to put our results into context.

In Section 3.2 we provide compact presentations for the nilpotent Lie groups we study. The results concerning these presentations are standard and since their proofs are not enlightening for the work we omit them and refer the reader to where they can be found in detail. We then present some basic identities between words.

In Section 3.3 we recall Proposition 3.18 a basic result about compactly presented nilpotent Lie groups that allows us to establish area estimates as if we were working with finitely presented nilpotent Lie groups (see Proposition 3.18).

In Section 3.4 is where we do all the work to compute the upper bounds on Dehn functions of central products of the form $G_{p,3} := L_p \times_Z H$ and $G^{\lrcorner}_{p,3} := L^{\lrcorner}_p \times_Z H$ with H a simply connected nilpotent Lie group of nilpotency class $q-1$. The main steps of the proof are similar to the ones in [LIPT23, Section 6], but the actual proofs of some of the steps require significant changes that involve new ideas. We thus follow the overall structure and notation used in [LIPT23] and refer the reader to that work in places where our proofs are completely analogous, while explaining in detail the steps where our proofs are different. We mostly focus on $G^{\lrcorner}_{p,3}$. The proofs for $G_{p,3}$ are similar, but slightly easier, since one of the relations simplifies. An important takeaway from this part is that the approach developed in [LIPT23, Section 6] holds with much higher generality. We achieve this by significantly simplifying the first part of the proof given there. Further refining this approach seems to be a promising route towards extending the upper bound in Conjecture 3 to very general classes of groups in the factor of higher nilpotency class. The fact that we simultaneously cover the groups L_p and $L_p^{\lrcorner}$ showcases this idea, which is one of our motivations for proving this conjecture for both of these cases.

Finally, the results obtained in Section 3.4 are then used to conclude the proofs of Theorems A, B and D and Corollary C which we present at the end of this chapter in Sections 3.5, 3.7 and 3.8, respectively.

3.1 Background

Our results in this part can perhaps be better understood in the context of the general upper bound on the Dehn functions of nilpotent groups given by [GHR03].

Theorem 3.1 ([GHR03]). *Let G be a finitely generated nilpotent group of class c. Then, the Dehn function of G satisfies*

$$\delta_G(n) \preccurlyeq n^{c+1}. \tag{3.1}$$

This upper bound is sharp for some nilpotent groups. For instance, the free nilpotent group of class c has Dehn function $\asymp$-equivalent to n^{c+1} [BMS93, Ger93]. Moreover, the 3-Heisenberg group, which has nilpotency class 2, has cubic Dehn function [ECH+92]. In contrast, the $(2k+1)$-dimensional Heisenberg groups for $k > 1$ all have quadratic Dehn function, see [All98], [Gro93, $5.A_4'$], and [OS99]. Notably, there exist nilpotent groups of arbitrarily large nilpotency class but with quadratic Dehn functions [You13a].

In general, the isoperimetric spectrum of nilpotent is far from being understood. We know that nilpotent groups can have Dehn functions $\asymp$-equivalent to a polynomial of arbitrarily large integer degree [BP94, BG96]; and that there exists a nilpotent group N whose Dehn function satisfies $n^2 \prec \delta_N(n) \preccurlyeq n^2 \log n$ [Wen11].

For central products of nilpotent groups, there is at present growing evidence that the Dehn function is often smaller than the Gersten–Holt–Riley upper bound [All98, OS99, You13b, LIPT23]. The first result in this direction was due to Allcock [All98] and Olshanskii–Sapir [OS99] who proved that the 5-Heisenberg group, which is the central product of two 3-Heisenberg groups (which have cubic Dehn function) has quadratic Dehn function. In Olshanskii–Sapir's proof the decomposition as a central product played a key role and indeed they showed more generally.

Theorem 3.2 (Allcock [All98], Olshanskii–Sapir [OS99], Young [You13b][1]). [2] *The Dehn function of the* $(2m+1)$*-Heisenberg group is* n^2 *if* $m \geqslant 2$*. If, more generally,* $G = K \times_Z K$ *is a central product of a non-trivial finitely generated torsion-free 2-nilpotent group* K *with itself, then* $\delta_G(n) \preccurlyeq n^2 \log(n)$*.*

Moreover, in [LIPT23] the authors computed the Dehn functions of central products of the form $L_p \times_Z L_{p-1}$ and $L_p \times_Z L_p$ which then led them to formulate the following conjecture:

Conjecture 4 ([LIPT23, Conjecture 11.3]). Let $2 \leqslant \ell \leqslant k$ be two integers. Consider a central product $G = K \times_\theta L$ where K and L are simply connected nilpotent Lie groups with one-dimensional centres, and class k and ℓ respectively. Then, $n^k \preccurlyeq \delta_G(n) \prec n^{k+1}$.

In this chapter we present our results that provide supportive evidence for this conjecture by establishing the proofs of Theorems A and B. Said differently, our results show that the Dehn function of the central product of a filiform group with a nilpotent Lie group has Dehn function strictly smaller than that of the maximal filiform subgroup.

3.2 Useful explicit compact presentations

At various points of our proofs, we work with standard explicit compact presentations of the nilpotent groups under consideration. For the reader's convenience we provide them here. For this purpose we adopt the following notation. For $g \in G$ an element of a simply connected nilpotent Lie group G and $a \in \mathbb{R}$, we denote $g^a := \exp(a \cdot \log(g))$.

Recall, from Section 2.3.3, that a nilpotent Lie group of dimension p is called *filiform* if it has maximal possible (finite) nilpotency class, namely $p-1$, among all Lie groups of dimension p. We consider the following two infinite families of filiform Lie groups $\{L_p \mid p \geqslant 3\}$ and $\{L_p^\lrcorner \mid p \geqslant 5\}$ that we defined in Section 2.3.3.

Proposition 3.3. *For every* $p \geqslant 5$*, there exists a Lie algebra that we denote by* $\mathfrak{l}_p^\lrcorner$ *with basis* $X_1, \ldots, X_p$ *such that the only non-zero brackets are*

$$[X_1, X_i] = X_{i+1} \text{ for } 2 \leqslant i \leqslant n-1, \; [X_2, X_3] = X_p. \tag{3.2}$$

[2]The statement about Heisenberg groups is due to Allcock and Olshanskii–Sapir. The second part was stated without proof by Olshanskii–Sapir in [OS99] and a proof was given by Young [You13b].

This Lie algebra is not isomorphic to $\mathfrak{l}_p$, and the associated simply connected Lie group contains a lattice, with the following presentation:

$$\left\langle x_1, \ldots, x_p \;\middle|\; \begin{array}{ll} [x_1, x_i] = x_{i+1}, & \text{for } 2 \leqslant i \leqslant p-1, \\ [x_2, x_{j+1}] = [x_i, x_j] = 1 & \text{for } 3 \leqslant i, j \leqslant p, \\ [x_2, x_3] = x_p, & \end{array} \right\rangle$$

Proof. For a proof we refer the reader to [GMLIP23, Proposition 2.5]. □

Remark 3.4. Although we are not aware of publications considering $\mathfrak{l}_p^{\lrcorner}$ for all p simultaneously, this Lie algebra can be found in the classifications of nilpotent Lie algebras (up to dimension 7) and of filiform nilpotent Lie algebras (up to dimension 11). The correspondences are as follows. In [dG07], $\mathfrak{l}_5^{\lrcorner}$ and $\mathfrak{l}_6^{\lrcorner}$ are $L_{5,6}$ and $L_{6,17}$ respectively. In [Mag08], $\mathfrak{l}_5^{\lrcorner}, \mathfrak{l}_6^{\lrcorner}$ and $\mathfrak{l}_7^{\lrcorner}$ are the unique real forms of $\mathcal{G}_{5,3}$, $\mathcal{G}_{6,17}$ and $\mathcal{G}_{7,1.6}$ respectively. Finally, in [GJMK98] where filiform Lie algebras of dimension less or equal 11 are classified, $\mathfrak{l}_5^{\lrcorner}, \ldots, \mathfrak{l}_{10}^{\lrcorner}$ and $\mathfrak{l}_{11}^{\lrcorner}$ are $\mu_5^2, \mu_6^3, \mu_7^3, \mu_8^{18}, \mu_9^{37}, \mu_{10}^{50}$ and μ_{11}^{105} respectively.

Proposition 3.5. *The group L_p for $p \geqslant 3$, respectively $L_p^{\lrcorner}$ for $p \geqslant 5$, admits the compact presentation $\mathcal{P}_p = \langle \widehat{S} | R \rangle$, respectively $\mathcal{P}_p^{\lrcorner} = \langle \widehat{S} | R^{\lrcorner} \rangle$, over the generating set $\widehat{S} = \{x_i^a \mid i \in \{1, \ldots, p\},\ a \in [-1,1]\}$, and set of relations*

$$R = \left\{ x_i^a x_i^b x_i^{-(a+b)}, \quad [x_1^a, x_i^b] = x_{i+1}^{ab} x_{i+2}^{-\binom{a}{2}b} \cdots x_p^{(-1)^{p+i+1}\binom{a}{p-i}b}, \quad i \in \{2, \ldots, p-1\}, \right.$$
$$\left. [x_i^a, x_j^b] = 1, \quad i, j \in \{2, \ldots, p\}, \quad a, b \in [-1, 1] \right\}$$

and

$$R^{\lrcorner} = \left\{ x_i^a x_i^b x_i^{-(a+b)}, \quad [x_1^a, x_i^b] = x_{i+1}^{ab} x_{i+2}^{-\binom{a}{2}b} \cdots x_p^{(-1)^{p+i+1}\binom{a}{p-i}b}, \quad i \in \{3, \ldots, p-1\}, \right.$$
$$[x_2^a, x_3^b] = x_p^{ab}, \quad [x_1^a, x_2^b] = x_3^{ab} x_4^{-\binom{a}{2}b} \cdots x_p^{(-1)^{p+1}\binom{a}{p-2}b - a\binom{b}{2}},$$
$$\left. [x_i^a, x_{j+1}^b] = 1, \quad i, j \in \{2, \ldots, p\}, \quad a, b \in [-1, 1] \right\}$$

respectively.

Proof. For a proof of this result we refer the reader to [GMLIP23, Proposition 2.8]. □

When considering central products we adapt these presentations via the following result.

Lemma 3.6. *Let K and L be compactly presented nilpotent groups of class k and ℓ respectively, such that $Z(K) = \gamma^k(K)$ and $Z(L) = \gamma^\ell(L)$. Let θ be a continuous isomorphism between $Z(K)$ and $Z(L)$. Then the group $G = K \times_\theta L$ is compactly presented. Moreover, given two compact presentations of K and L, there is a compact presentation of $G = K \times_Z L$, whose generating system and set of relations contain those of K and L, when viewed as closed subgroups of G.*

Proof. Let $\mathcal{P}_K = \langle \mathcal{S}_K \mid \mathcal{R}_K \rangle$ and $\mathcal{P}_L = \langle \mathcal{S}_L \mid \mathcal{R}_L \rangle$ be compact presentations of K and L respectively. $Z(K)$ is a closed subgroup of K, and K is nilpotent, so $Z(K)$ is compactly generated by [CdlH16a, Proposition 5.A.7]. Let $\mathcal{S}_{Z(K)}$ be a compact generating set of Z. By the assumption that $Z(K) = \gamma_k(K)$, there is $N < +\infty$ such that $\mathcal{S}_{Z(K)} \subset \mathcal{S}_K^N$ [CdlH16a, Lemma 5.A.5]. Similarly, if $\mathcal{S}_{Z(L)}$ is a compact generating set of the centre $Z(L)$ of L, then there is $M < +\infty$ such that $\mathcal{S}_{Z(L)} \subset \mathcal{S}_L^M$. Now set

$$\mathcal{S} = \mathcal{S}_K \sqcup \mathcal{S}_L, \quad \mathcal{R} = \mathcal{R}_K \sqcup \mathcal{R}_L \sqcup \mathcal{R}_+ \sqcup \{[a, b] : a \in \mathcal{S}_K, b \in \mathcal{S}_L\}$$

where $\mathcal{R}_+$ is constructed as follows: for every $z \in \mathcal{S}_{Z(K)}$, write z as a product of at most N elements of $\mathcal{S}_K$, write $\theta(z) \in Z(L)$ as a product of at most M elements of $\mathcal{S}_L$; let v and w be the resulting words, and add vw^{-1} in $\mathcal{R}_+$. Clearly these additional relations are of length bounded by $N + M$. Hence, $\mathcal{S}$ is a compact subset of G, $\mathcal{R}$ is a bounded subset of F_S and all the relation that hold true in G are products of conjugates of relations in $\mathcal{R}$. This shows that $\langle S \mid R \rangle$ is a compact presentation for G. ☐

Definition 3.7. We call the presentation of the central product $K \times_Z L$ constructed in the proof of Lemma 3.6 *adapted* to the central product decomposition.

From now on, whenever a compact presentation has been provided for the factors, we equip their central product with an adapted presentation as produced by Lemma 3.6. We denote by y_i the generators of the right-hand side factor in a central product. As an example, the group $L_p^{\lrcorner} \times_Z L_3$ has as generating set

$$\left\{ x_i^a,\ y_j^b \mid\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant 3,\ a, b \in [-1, 1] \right\},$$

and a set of relators $R^{\lrcorner}$ together with the relations $[y_1^a, y_2^b] = y_3^{ab}$, $x_p^a = y_3^a$, and $[x_i^a, y_j^b] = 1$ for all $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant 3$, and $a, b \in [-1, 1]$.

We recall the following easy consequence of Proposition 3.5 from [LIPT23, Lemma 5.14].

Lemma 3.8. *For $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbb{R}$ let $w = x_2^{m_1} x_1^{n_1} \ldots x_2^{m_k} x_1^{n_k}$ and let $\ell = \sum_{i=1}^{k}(|m_1| + |n_i|)$. There exist $b_1, \ldots, b_p \in \mathbb{R}$ with $|b_1| = O_p(\ell) + O_p(1)$ and $|b_i| = O_p(\ell^{i-1}) + O_p(1)$ for $2 \leqslant i \leqslant p$ such that the identity*

$$w =_{\mathcal{P}} x_1^{b_1} \ldots x_p^{b_p}.$$

holds in $L_p \times_Z L_3$ for $p \geqslant 3$ (respectively $L_p^{\lrcorner} \times_Z L_3$ for $p \geqslant 5$).

Proof. The idea is to move all the x_1's in w to the left starting with the right most one using Proposition 3.5. The bound on the b_i's follow from the fact that $\binom{x}{i}$ is a polynomial of degree i on x. For the details we refer the reader to [LIPT23, Lemma 5.14]. □

We finish this section by providing a general fact that provides an upper bound on the Dehn function of a central product in terms of a subgroup which is itself a central product.

Lemma 3.9. *Let K and L be compactly presented nilpotent groups with isomorphic centres and let G be their central product. Assume that K has a closed subgroup K_0 containing the centre which is isomorphic to L, and that $\delta_{G_0}(n) \preccurlyeq n^p$ where $G_0 = L \times_Z L$ for some $p \geqslant 2$. If every word of length n over the generators of $K < G$ representing the trivial element has area at most n^p in G, then $\delta_G(n) \preccurlyeq n^p$.*

Proof. For a proof we refer the reader to [GMLIP23, Lemma 2.4]. □

3.2.1 Ω-words

We use the following notation for words representing elements in the groups $L_p \times_Z L_3$ and $L_p^{\lrcorner} \times_Z L_3$.

$$[x_1,\ x_2, \ldots, x_{k-1},\ x_k] := [x_1, [x_2, \ldots, [x_{k-1}, x_k]]]$$

to denote a simple k-fold commutator of elements $x_1, \ldots, x_k$ in a group or Lie algebra. Moreover, for $j \geqslant 2$, $k \geqslant 1$, and $\underline{n} := (n_1, \ldots, n_k) \in \mathbb{R}^k$ we denote by

$$\Omega_k^j(\underline{n}) := [x_1^{n_1}, \ldots, x_1^{n_{k-1}}, x_j^{n_k}].$$

For $j = 2$ we only denote it by $\Omega_k(\underline{n})$. We refer to them as Ω_k^j*-words*, and when we do not need to specify the parameters (k, j) we shall refer to them simply as

Ω-*words*. Moreover, we extend the notation of Ω-words to the generators y_i of the second factor as follows:

$$\widetilde{\Omega}^j_k(\underline{n}) := [y_1^{n_1}, \dots, y_1^{n_{k-1}}, y_j^{n_k}].$$

3.2.2 Preliminary identities

The following consequences of Proposition 3.5 are used in Section 3.4 when we give upper bounds on the Dehn functions of $L_p \times_Z L_3$ and $L_p^{\lrcorner} \times_Z L_3$.

Remark 3.10. The results are stated for $L_p^{\lrcorner} \times_Z L_3$ for all $p \geqslant 5$, but the reader should note that the same results hold for $L_p \times_Z L_3$ for all $p \geqslant 3$ with the same area estimates, the only minor change is that there is no error term of the form $[y_1, y_{p-1}]^{\pm 1}$.

Corollary 3.11. *Let $p \geqslant 5$.*

1. *For all $b \in \mathbb{R}$ the identities*

$$[x_1, x_j^b] =_{\mathcal{P}} \begin{cases} x_{j+1}^b, & \text{if } j > 2, \\ x_3^b \cdot [y_1^{\tilde{b}}, y_{p-1}^{\tilde{b}}]^{\pm 1}, & \text{if } j = 2. \end{cases}$$

   hold in $L_p^{\lrcorner} \times_Z L_3$ with area $\lesssim_p |b|^2$, where $\tilde{b} \in \mathbb{R}$ with $|\tilde{b}| \lesssim_p |b|$.

2. *For all $b \in \mathbb{R}$ the identities*

$$[x_1^{-1}, x_j^b] =_{\mathcal{P}} \begin{cases} x_{j+1}^{-b} \dots x_p^{-b}, & \text{if } j > 2, \\ x_3^{-b} \dots x_p^{-b} \cdot [y_1^{\tilde{b}}, y_{p-1}^{\tilde{b}}]^{\pm 1}, & \text{if } j = 2. \end{cases}$$

   hold in $L_p^{\lrcorner} \times_Z L_3$ with area $\lesssim_p |b|^2$ where $\tilde{b} \in \mathbb{R}$ with $|\tilde{b}| \lesssim_p |b|$.

Proof. In case (1) the identity for $j > 2$ is a direct consequence of Proposition 3.5 since $\binom{1}{m} = 0$ for all $m \in \mathbf{Z}_{\geqslant 2}$. Now suppose $j = 2$. For $\beta := \lfloor b \rfloor - b$ we have that the identities

$$x_2^{-b} x_1 \overset{\text{free}}{=} x_2^{\beta} x_2^{-\lfloor b \rfloor} x_1 =_{\mathcal{P}} x_2^{\beta} x_1 (x_2 x_3^{-1})^{-\lfloor b \rfloor}$$

hold in $L_p^{\lrcorner} \times_Z L_3$ with area $\lesssim_p |b|$. By Proposition 3.5 it follows that

$$x_2^{\beta} x_1 (x_2 x_3^{-1})^{-\lfloor b \rfloor} =_{\mathcal{P}} x_1 x_3^{-\beta} x_p^{\binom{\beta}{2}} x_2^{\beta} (x_2 x_3^{-1})^{-\lfloor b \rfloor}$$

holds in $L_p^{\lrcorner} \times_Z L_3$ with area $\lesssim_p |b|$. Finally, since $\langle x_2, x_3, y_1, y_{p-1} \rangle$ generate a subgroup isomorphic to the integral 5-Heisenberg group, it follows that

$$x_1 x_3^{-\beta} x_p^{\binom{\beta}{2}} x_2^{\beta} (x_2 x_3^{-1})^{-\lfloor b \rfloor} \overset{\text{free}}{=} x_1 x_3^{-\beta} x_2^{\beta} (x_2 x_3^{-1})^{-\lfloor b \rfloor} x_p^{\binom{\beta}{2}} =_{\mathcal{P}} x_1 x_3^{b} \cdot [y_1^{\tilde{b}}, y_{p-1}^{\tilde{b}}] \cdot x_2^{-b}$$

holds in $L_p^{\lrcorner} \times_Z L_3$ with area $\lesssim_p |b|^2$ and $|\tilde{b}| \lesssim_p |b|$.

The proof of Case (2) is similar, only that now $\binom{-1}{m} = \pm 1$ leading to the slightly modified formulas. We briefly explain the case $j = 2$, the case $j > 2$ being similar but without the error term $[y_{p-1}^{\tilde{b}}, y_1^{\tilde{b}}]$. It follows from Proposition 3.5 that

$$x_2^b x_1^{-1} = x_1^{-1} x_2^b x_3^b \dots x_p^{b-\binom{b}{2}}$$

holds in $L_p^{\lrcorner} \times_Z L_3$. Thus, applying the identity $x_p^{\binom{b}{2}} =_{\mathcal{P}} [y_{p-1}^{\tilde{b}}, y_1^{\tilde{b}}]$, which has area $\lesssim_p |b|^2$ for suitable $|\tilde{b}| \lesssim |b|$, we obtain the desired statement. □

Corollary 3.12. *Let $p \geqslant 5$. Let $a, b \in \mathbb{R}$.*

The identities

$$[x_1^a, x_j^b] =_{\mathcal{P}} \begin{cases} x_{j+1}^{-b} \cdot [x_{j+1}^{-b}, x_1^{a-1}] \cdot [x_1^{a-1}, x_j^b], \\ \left(\prod_{i=j+1}^{p} x_i^{-b} \cdot [x_i^{-b}, x_1^{a+1}] \right) \cdot [x_1^{a+1}, x_j^b]. \end{cases} \tag{3.3}$$

hold in $L_p^{\lrcorner} \times_Z L_3$ for $2 < j \leqslant p-1$ with area $\lesssim_p |b|^2$.

Moreover, there exists $\tilde{b} \in \mathbb{R}$ with $|\tilde{b}| \lesssim_p |b|$, such that the identities

$$[x_1^a, x_2^b] =_{\mathcal{P}} \begin{cases} x_3^{-b} \cdot [x_3^{-b}, x_1^{a-1}] \cdot [x_1^{a-1}, x_2^b] \cdot [y_1^{\tilde{b}}, y_{p-1}^{\tilde{b}}]^{\pm 1}, \\ \left(\prod_{i=3}^{p} x_i^{-b} [x_i^{-b}, x_1^{a+1}] \right) \cdot [x_1^{a+1}, x_2^b] \cdot [y_1^{\tilde{b}}, y_{p-1}^{\tilde{b}}]^{\pm 1}. \end{cases} \tag{3.4}$$

hold in $L_p^{\lrcorner} \times_Z L_3$ with area $\lesssim_p |b|^2$.

Proof. We prove the second identity in (3.4). The other cases are analogous, but slightly simpler. It follows from Corollary 3.11 (2) that the identity

$$x_1^{-a} x_2^{-b} x_1^a x_2^b =_{\mathcal{P}} x_1^{-(a+1)} x_3^{-b} \dots x_p^{-b} \cdot [y_1^{\tilde{b}}, y_{p-1}^{\tilde{b}}]^{\pm 1} x_2^{-b} x_1^{a+1} x_2^b$$

holds in $L_p^{\lrcorner} \times_Z L_3$ with area $\lesssim_p |b|^2$ where $|\tilde{b}| \lesssim_p |b|$. Therefore, since the free identity

$$x_1^{-(a+1)} x_i^{-b} \overset{\text{free}}{=} x_i^{-b} \cdot [x_i^{-b}, x_1^{a+1}] \cdot x_1^{-(a+1)}$$

holds for all $i \geqslant 3$, we can shuffle $x_1^{-(a+1)}$ to the right in front of the suffix-word $x_2^{-b} x_1^{a+1} x_2^b$ to obtain the desired identity with area $\lesssim_p |b|^2$. □

We record a rewriting of the identities of Corollaries 3.11 and 3.12 that are used in §3.4.

Addendum 3.13. *Let $p \geqslant 5$ and let $3 \leqslant i \leqslant p$. Since for all $i \geqslant j$ and for all $b \in \mathbb{R}$ the identity*

$$x_i^{-b} =_{\mathcal{P}} \Omega_{i-j}^{j+1}(1, \dots, 1, -b)$$

holds in $L_p^{\lrcorner}\times_Z L_3$ with area $\lesssim_p |b|^2$, we can rewrite the identities from Corollary 3.11 (2) as

$$[x_1^{-1}, x_j^b] =_{\mathcal{P}} \begin{cases} \Omega_1^{j+1}(-b)\cdot \ldots \cdot \Omega_{p-j}^{j+1}(1,\ldots,1,-b), & \text{if } j>2, \\ \Omega_1^{3}(-b)\cdot \ldots \cdot \Omega_{p-2}^{3}(1,\ldots,1,-b)\cdot [y_{p-1}^{\tilde{b}}, y_1^{\tilde{b}}]^{\pm 1}, & \text{if } j=2, \end{cases}$$

with area $\lesssim_p |b|^2$ by Corollary 3.11 part 1.

Similarly, we can rewrite the two identities that correspond to increasing the x_1-exponent by one in Corollary 3.12 as

$$[x_1^a, x_j^b] =_{\mathcal{P}} \left(\prod_{i=j+1}^{p} \Omega_{i-j}^{j+1}(1,\ldots,1,-b)\cdot \Omega_{i-j+1}^{j+1}(a+1,1,\ldots,1,-b)^{-1}\right)\cdot [x_1^{a+1}, x_j^b],$$

for $j>2$, and

$$[x_1^a, x_2^b] =_{\mathcal{P}} \left(\prod_{i=3}^{p} \Omega_{i-2}^{3}(1,\ldots,1,-b)\cdot \Omega_{i-1}^{3}(a+1,1,\ldots,1,-b)^{-1}\right)[x_1^{a+1}, x_2^b]\cdot [y_{p-1}^{\tilde{b}}, y_1^{\tilde{b}}]^{\pm 1},$$

each with area $\lesssim_p |b|^2$.

We obtain the following consequence of Lemma 3.8.

Lemma 3.14. *Let $p \geqslant 5$ and $k \geq 1$. For every $\underline{n} \in \mathbb{R}$ there are $t_i \in \mathbb{R}$ with $|t_i| \lesssim_p |\underline{n}|^{i-1}$, $k \leqslant i \leqslant p$ which satisfy the following identity in $L_p^{\lrcorner}\times_Z L_3$*

$$\Omega_k(\underline{n}) =_{\mathcal{P}} x_{k+1}^{t_{k+1}} \ldots x_p^{t_p}.$$

Proof. As consequence of Lemma 3.8 we get that $\Omega_k(\underline{n}) =_{\mathcal{P}} x_1^{t_1}\ldots x_p^{t_p}$. For $i<k+1$ the exponent t_i must be zero, since their images in the quotient of $L_p^{\lrcorner}\times_Z L_3$ by the k-th lower central series term are trivial. Indeed, if this were not the case, then we would obtain a contradiction with the fact that in such quotient the image of the k-folded commutator has to be trivial. □

The following is a well-known result whose proof is straightforward.

Lemma 3.15 ([LIPT23, Lemma 5.7])**.** *Let G be a group, and let u, v, w be words in some generating set of G. The following free identities hold in G*

1. $[u\cdot v, w] \overset{\text{free}}{=} [u,w]^v \cdot [v,w]$.
2. $[u, v\cdot w] \overset{\text{free}}{=} [u,w]\cdot [u,v]^w$.
3. $u^w \overset{\text{free}}{=} u\cdot [u,w]$.

Lemma 3.15 is used in §3.4 to allow ourselves to rewrite and manipulate Ω-words. An example of this is the following result from [LIPT23, Lemma 6.13] which can be seen as a converse to Lemma 3.14.

Lemma 3.16. *For $n \geqslant 1$, $k \geqslant 3$, and $w = x_k^{t_k} \cdots x_{p-1}^{t_{p-1}} x_p^{t_p}$ with $|t_i| \lesssim_p n^{i-1}$ there are $\underline{n}_i \in \mathbb{R}^i$, $k-1 \leqslant i \leqslant p-1$ with $|\underline{n}_i| \lesssim_p n$ such that the identity*

$$w =_{\mathcal{P}} \prod_{i=k-1}^{p-1} \Omega_i(\underline{n}_i)$$

holds in $L_p^{\lrcorner} \times_Z L_3$ for $p \geqslant 5$.

Proof. The proof is done by descending induction on k. For the base of induction it is clear that the identity

$$x_p^{t_k} =_{\mathcal{P}} \Omega_{p-1}(t_p^{\frac{1}{p-1}}, \ldots, t_p^{\frac{1}{p-1}})$$

holds in $L_p^{\lrcorner} \times_Z L_3$. For the induction step, suppose that the result holds for $k = k_0 + 1$. Let $\beta = t_{k_0}^{\frac{1}{k_0-1}}$, so that $|t_{k_0}| \leqslant n$. Arguing, as in the proof of [LIPT23, Lemma 5.13], by iteratively applying Proposition 3.5, Lemma 3.15(2), and the fact that $L_p^{\lrcorner}$ is metabelian to the innermost commutator in $\Omega_{k_0-1}(\beta, \ldots, \beta)$ we get that there are polynomials $q_j(\beta)$ of degree $j-1$ for $k_0 \leqslant j \leqslant p$ such that the identity

$$\Omega_{k_0-1}(\beta, \ldots, \beta) =_{\mathcal{P}} \prod_{j=k_0}^{p} x_j^{-q_j(\beta)}$$

holds in $L_p^{\lrcorner} \times_Z L_3$ for $p \geqslant 5$. Therefore the identity

$$x_{k_0}^{t_{k_0}} =_{\mathcal{P}} \Omega_{k_0-1}(\beta, \ldots, \beta) \cdot \prod_{j=k_0+1}^{p} x_j^{-q_j(\beta)}$$

holds in $L_p^{\lrcorner} \times_Z L_3$ for $p \geqslant 5$. Since $q_j(\beta) \lesssim_p n^{j-1}$ for $k_0 \leqslant j \leqslant p$ we conclude by applying the induction hypothesis to $\prod_{j=k_0+1}^{p} x_j^{-q_j(\beta)}$. ☐

3.3 Reduction to products of efficient words

We start this section by establishing an auxiliary result (see Proposition 3.18 below) which allows us to obtain upper bounds for Dehn functions by bounding the area of particular words belonging to smaller subsets called *efficient sets* (see Definition 3.17). This idea originates from an observation of Gromov [Gro93, $5.A_3''$].

For details we refer to [dCT10] and [LIPT23]. We then construct efficient sets for the groups $L_p \times_Z L_3$ and $L_p^\lrcorner \times_Z L_3$ (Corollary 3.20) which we require in Section 3.4 when we prove the upper bounds on their Dehn functions.

Throughout this section G denotes a compactly presented group, $\mathcal{P} := \langle S \mid R \rangle$ a compact presentation of G, and F_S the free group generated by S. Given a subset $\mathcal{F} \subset F_S$ and an integer $k \geqslant 1$, we denote by $\mathcal{F}[k]$ the set of concatenations of at most k elements of $\mathcal{F}$.

Definition 3.17. Given an integer $r \geqslant 1$, a subset $\mathcal{F} \subset F_S$ is called *r-efficient with respect to $\mathcal{P}$*, if there exists a constant $C \geqslant 1$ such that for every $w \in F_S$ there exists $w' \in \mathcal{F}[r]$ such that $w =_{\mathcal{P}} w'$ and $|w'|_S \leqslant C|w|_S$.

Note that this generalises the definition of efficient sets given in [dCT10]. What they call efficient corresponds to a 1-efficient set in our definition.

Proposition 3.18. *Let G be a compactly presented group and $\mathcal{P} := \langle S \mid R \rangle$ a compact presentation for G. Suppose that for some $r \geqslant 1$ there is an r-efficient set $\mathcal{F}$ with respect to $\mathcal{P}$ and that there exists $d > 1$ such that for all $k \geqslant 1$ and all $n \geqslant 0$, the area of each null-homotopic word in $\mathcal{F}[k]$ of length at most n is $\lesssim_k n^d$. Then, the Dehn function of G satisfies $\delta_G(n) \preccurlyeq n^d$.*

Proof. For a proof we refer the reader to [dCT10, Proposition 4.3] (in this case we apply this result to the 1-efficient set $\mathcal{F}[r]$) and to [LIPT23, Proposition 5.10]. □

We now recall from [LIPT23, Section 2] the existence of $O_p(1)$-efficient sets for the groups $L_p \times_Z L_3$ and $L_p^\lrcorner \times_Z L_3$ with respect to the compact presentations obtained from Proposition 3.5 and Lemma 3.6. For this we first need to recall some notation from [LIPT23]:

$$\Sigma := \{x_1^{a_1}, x_2^{a_2} \mid |a_1|, |a_2| \leqslant 1\}, \quad \mathcal{F} := \{s^n \mid s \in \Sigma, n \in \mathbb{N}\},$$

$$T := \{x_1^{a_1}, x_2^{a_2}, y_1^{a_3}, y_{p-1}^{a_4} \mid |a_1|, |a_2|, |a_3|, |a_4| \leqslant 1\}, \text{ and } \mathcal{G} := \{s^n \mid s \in T, n \in \mathbb{N}\}.$$

Proposition 3.19. *The subset $\mathcal{F}$ is $O_p(1)$-efficient with respect to the compact presentation $\mathcal{P}_p^\lrcorner$ (respectively $\mathcal{P}_p$) of $L_p^\lrcorner$ for all $p \geqslant 5$ (respectively L_p for all $p \geqslant 3$).*

Proof. For a proof we refer the reader to [GMLIP23, 2.18]. □

Corollary 3.20. *For all $p \geqslant 5$ the subset $\mathcal{G}$ is $O_p(1)$-efficient with respect to the compact presentation $\mathcal{P}$ of $L_p^\lrcorner \times_Z L_3$ (respectively of $L_p \times_Z L_3$ for all $p \geqslant 3$).*

Proof. For a proof see [GMLIP23, Corollary 2.19]. □

3.4 Dehn functions of central products of nilpotent groups

In the first part of this section, Section 3.4.1, we compute the upper bounds of Dehn functions of the central products

$$G_{p,3} := L_p \times_Z L_3 \quad \text{and} \quad G^{\lrcorner}_{p,3} := L^{\lrcorner}_p \times_Z L_3$$

introduced in Section 2.3.3. As mentioned before, our proof follows the strategy and techniques from [LIPT23]. Nevertheless, we generalise them and as a consequence we managed to simplify some of the arguments.

In the second part, Section 3.5, we show how to obtain the upper bounds for a bigger class of central products of nilpotent groups using the results obtained for $G_{p,3}$ and $G^{\lrcorner}_{p,3}$, namely central products of the form

$$L_p \times_Z H \quad \text{and} \quad L^{\lrcorner}_p \times_Z H,$$

for H a simply connected nilpotent Lie group of class $\leqslant q-1$ with one-dimensional centre. Then, in Section 3.5 we explain how these results imply Theorems A and B.

3.4.1 Upper bounds on the Dehn functions of $G^{\lrcorner}_{p,3}$ and $G_{p,3}$

We fix once and for all the compact presentation $\mathcal{P}$ for $G^{\lrcorner}_{p,3}$ (respectively for $G_{p,3}$) obtained using Lemma 3.6 and the compact presentation for $L^{\lrcorner}_p$ (respectively for L_p) given in Proposition 3.5. By abuse of notation we denote both presentations by $\mathcal{P}$ and only specify to which groups it refers whenever confusion may arise. We start by stating the main results of this section.

Theorem 3.21. *Let $p \geqslant 3$. The Dehn function of the group $G_{p,3} := L_p \times_Z L_3$ satisfies $\delta_{G_{p,3}}(n) \preccurlyeq n^{p-1}$.*

Theorem 3.22. *Let $p \geqslant 5$. The Dehn function of the group $G^{\lrcorner}_{p,3} := L^{\lrcorner}_p \times_Z L_3$ satisfies $\delta_{G^{\lrcorner}_{p,3}}(n) \preccurlyeq n^{p-1}$.*

Strategy of the proofs of Proposition 3.21 and Proposition 3.22

The proof of Theorems 3.21 and 3.22 are done simultaneously by ascending induction on p, where we emphasise that the *induction step from $p-1$ to p only requires the induction hypothesis for $G_{q,3}$ with $q < p$.*

For the proof of the induction step for $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$) we need to show that the area of null-homotopic words $w := w(x_1, x_2)$ in $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$) is $\lesssim_p n^{p-1}$. By Proposition 3.18 it suffices to show that null-homotopic words $w(x_1, x_2) \in \mathcal{F}[\alpha]$ have area $\lesssim_{p,\alpha} n^{p-1}$ for every $\alpha > 0$. To show this we first prove a result about commuting certain words in $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$), which we call the Main commuting Lemma for $G^{\lrcorner}_{p,3}$, see Lemma 3.26 (respectively Lemma 3.23 for $G_{p,3}$).

We would like to emphasise that it is in the proof of the Main commuting Lemmas for $G_{p,3}$ and $G^{\lrcorner}_{p,3}$ that our proof significantly simplifies the one in [LIPT23]. Each one of the Main commuting Lemmas is a consequence of two corresponding results about commutators involving the Ω^j_k-words – namely the First and Second commuting (k,j)-Lemmas – and the Reduction Lemma 3.29. It should be noted that the First commuting (k,j)-Lemma is a generalisation of [LIPT23, Lemma 6.5 and Lemma 6.7], while the Second commuting (k,j)-Lemma is a generalisation of [LIPT23, Lemma 6.4 and Lemma 6.6]. Once we have proven the First and Second commuting (k,j)-Lemmas, the proof follows the arguments in [LIPT23]: in particular, key steps include the Reduction Lemma 3.29, which allows us to deduce the Main Commuting Lemmas from the First and Second Commuting (k,j)-Lemmas, and the Cancelling k-Lemma 3.31 for $G_{p,3}$ (respectively $G^{\lrcorner}_{p,3}$). We start by stating the main results.

Lemma 3.23 (Main commuting Lemma for $G_{p,3}$)**.** *Let $p \geqslant 3$, $\alpha \geqslant 1$, $n \geqslant 1$, and let $w_1(x_1, x_2)$, $w_2(x_1, x_2)$ be either powers of x_2 or words in $\mathcal{F}[\alpha]$ representing elements in the derived subgroup of $G_{p,3}$. If $|w_1|, |w_2| \leqslant n$, then the identity*

$$[w_1, w_2] =_{\mathcal{P}} 1$$

holds in $G_{p,3}$ with area $\lesssim_{\alpha,p} n^{p-1}$.

Lemma 3.24 (First commuting (k,j)-Lemma for $G_{p,3}$)**.** *Let $p \geqslant 3$, $n \geqslant 1$, $j \geqslant 2$, $k \geqslant 1$, $\underline{n} \in \mathbb{R}^k$ with $|\underline{n}| \leqslant n$, and $m \in \mathbb{R}$. The identity*

$$[\Omega^j_k(\underline{n}), x_2^m] =_{\mathcal{P}} 1$$

holds in $G_{p,3}$ with area $\lesssim_p |m|n^{p-j} + n^{p-j+1}$ if $j < p$ and area $\lesssim_p |m|n + n^2$ if $j = p$.

Lemma 3.25 (Second commuting (k,j)-Lemma for $G_{p,3}$)**.** *Let $p \geqslant 3$, $n \geqslant 1$, $j \geqslant 2$, $k \geqslant 1$, $\underline{n} \in \mathbb{R}^k$ with $|\underline{n}| \leqslant n$, $\alpha \geqslant 1$ and $w := w(x_1, x_2) \in \mathcal{F}[\alpha]$ be a word* *of length at most n representing an element in the derived subgroup of $G_{p,3}$. The identity*

$$[\Omega_k^j(\underline{n}), w] =_{\mathcal{P}} 1$$

holds in $G_{p,3}$ with area $\lesssim_{\alpha,p} n^{p-j+1}$ if $j < p$ and area $\lesssim_{\alpha,p} n^2$ if $j = p$.

Lemma 3.26 (Main commuting Lemma for $G^{\lrcorner}_{p,3}$)**.** *Let $p \geqslant 5$, $\alpha \geqslant 1$, and $n \geqslant 1$. If w_1 and w_2 are either powers of x_2 or words of length at most n in $\mathcal{F}[\alpha]$ representing elements of the derived subgroup of $G^{\lrcorner}_{p,3}$ with $|w_1|, |w_2| \leqslant n$, then the identities*

$$[w_1, w_2] =_{\mathcal{P}} \begin{cases} 1, & w_1, w_2 \text{ are both powers of } x_2, \\ 1, & w_1, w_2 \text{ both represent elements in } [G^{\lrcorner}_{p,3}, G^{\lrcorner}_{p,3}], \\ \prod_{i=1}^{D} \Omega_3^{p-2}(\underline{\eta}_i)^{\pm 1}, & w_1 \text{ represents a word in } [G^{\lrcorner}_{p,3}, G^{\lrcorner}_{p,3}] \text{ and } w_2 := x_2^k. \end{cases}$$

hold in $G^{\lrcorner}_{p,3}$ with area $\lesssim_{\alpha,p} n^{p-1}$ for some $D = O_{\alpha,p}(1)$, and $\underline{\eta}_i \in \mathbb{R}^3$ with $|\underline{\eta}_i| \lesssim_p n$.

Lemma 3.27 (First commuting (k, j)-Lemma for $G^{\lrcorner}_{p,3}$)**.** *Let $p \geqslant 5$, $n \geqslant 1, j \geqslant 2$, $k \geqslant 1$, $\underline{n} \in \mathbb{R}^k$, and $m \in \mathbb{R}$ with $|\underline{n}| \leqslant n$. Then, the identities*

$$[\Omega_k^j(\underline{n}), x_2^m] =_{\mathcal{P}} \begin{cases} 1, & \text{if } k + j \neq 4, \\ \Omega_2^{p-1}(m, -n_1), & \text{if } (k, j) = (1, 3), \\ \Omega_2^{p-1}(m, -n_2)^{n_1}, & \text{if } (k, j) = (2, 2), \end{cases}$$

hold in $G^{\lrcorner}_{p,3}$. with area $\lesssim_p |m| n^{p-j} + n^{p-j+1}$ if $j < p$ and area $\lesssim_p |m|n + n^2$ if $j = p$.

If, moreover, $|m| \leqslant n$ and $(k, j) = (2, 2)$, then an identity of the form

$$[\Omega_2(\underline{n}), x_2^m] =_{\mathcal{P}} \Omega_3^{p-2}(\underline{\tilde{m}})$$

holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^{p-1}$ for some suitable $\underline{\tilde{m}} \in \mathbb{R}^3$ with $|\underline{\tilde{m}}| \lesssim_p n$.

Lemma 3.28 (Second commuting (k, j)-Lemma for $G^{\lrcorner}_{p,3}$)**.** *Let $p \geqslant 5$, $n \geqslant 1, j \geqslant 2$, $k \geqslant 1$, $\underline{n} \in \mathbb{R}^k$, and $\alpha \geqslant 1$ with $|\underline{n}| \leqslant n$. If $w := w(x_1, x_2) \in \mathcal{F}[\alpha]$ is a word of length at most n representing an element in the derived subgroup of $G^{\lrcorner}_{p,3}$, then the identity*

$$[\Omega_k^j(\underline{n}), w] =_{\mathcal{P}} 1,$$

holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_{\alpha,p} n^{p-j+1}$ if $j < p$ and area $\lesssim_{\alpha,p} n^2$ if $j = p$.

A key step in the proof of the Main Commuting Lemmas is the Reduction Lemma; the proof of the latter relies on the First and Second commuting Lemmas.

Lemma 3.29 (Reduction Lemma)**.** *Let $\alpha \geqslant 1$, let $w = w(x_1, x_2)$ be a word of length at most n in $\mathcal{F}[\alpha]$ representing an element in the derived subgroup of $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$). There exists $L = O_{\alpha,p}(1)$ such that the identity*

$$w =_{\mathcal{P}} \prod_{i=1}^{L} \Omega_{l_i}(\underline{\eta}_i)^{\pm 1}$$

holds in $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$) with area $\lesssim_{\alpha,p} n^{p-1}$ for some $2 \leqslant l_i \leqslant p-1$ and some $\underline{\eta}_i \in \mathbb{R}^{l_i}$ with $|\underline{\eta}_i| \lesssim_p n$.

Remark 3.30. As we mentioned before, in this work we simplified the approached in [LIPT23, Section 6]. The most important simplification is that we avoid the use of the highly technical Fractal Form Lemma [LIPT23, Lemma 6.17] by instead using an inductive argument to prove the First and Second commuting Lemmas and then deduce the Main Commuting Lemma.

Finally, we deduce the Cancelling k-Lemma 3.31 from the Main Commuting Lemmas 3.26 and 3.23 and then use it to complete the proofs of Theorems 3.21 and 3.22.

Lemma 3.31 (Cancelling k-Lemma)**.** *Let $n \geqslant 1$, $2 \leqslant k \leqslant p-1$, M_j be a positive integer for all $k \leqslant j \leqslant p-1$, and $M := \max\{M_j \mid k \leqslant j \leqslant p-1\}$. Consider the word*

$$w(x_1, x_2) := \left(\prod_{i=1}^{M_k} \Omega_k(\underline{n}_{k,i})^{\pm 1}\right) \left(\prod_{i=1}^{M_{k+1}} \Omega_{k+1}(\underline{n}_{k+1,i})^{\pm 1}\right) \cdots \left(\prod_{i=1}^{M_{p-1}} \Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1}\right),$$

for some $\underline{n}_{l,i} \in \mathbb{R}^j$ with $|\underline{n}_{l,i}| \leqslant n$. If $w := w(x_1, x_2)$ is null-homotopic in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$), then it has area $\lesssim_{M,p} n^{p-1}$.

We now proceed with the proofs of Theorems 3.21 and 3.22.

3.4.2 Start of the induction on p

As mentioned above, the proofs of Theorems 3.21 and 3.22 are done in parallel by ascending induction on p. The corresponding base of induction is as follows:

Base of induction on p.

By [All98] and [OS99] the Dehn function of $G_{3,3}$ satisfies $\delta_{G_{3,3}}(n) \asymp n^2$ and by [LIPT23] the Dehn function of $G_{4,3}$ satisfies $\delta_{G_{4,3}}(n) \asymp n^3$.

Induction hypothesis for p.

As mentioned above, the induction step from $p-1$ to p only requires the induction hypothesis for $G_{q,3}$ with $q < p$

(IH-p) Let $p-1 \geqslant 3$. Suppose Theorem 3.21 holds for all $G_{q,3}$ with $q \leqslant p-1$. Moreover, suppose the Main commuting Lemma 3.23 for $G_{q,3}$ along with the First commuting (k,j)-Lemma 3.24 and the Second commuting (k,j)-Lemma 3.25 for $G_{q,3}$ hold.

For the induction step from $p-1$ to p for $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$) we first prove the First and Second commuting (k,j)-Lemmas for $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$).

3.4.2.1 Proof of the First and Second Commuting (k,j)-Lemmas

Strategy of the proofs

We prove the First and Second commuting (k,j)-Lemmas for $G^{\lrcorner}_{p,3}$ and $G_{p,3}$ in *parallel by descending induction on* j for all $(k,j) \neq (1,2)$. The case $(1,2)$ follows from the Main Commuting Lemma, whose proof only relies on the cases $(k,j) \neq (1,2)$.

The induction in j for $G^{\lrcorner}_{p,3}$ is done for all pairs (k,j) such that $k+j \geqslant 5$, while for $G_{p,3}$ it is done for *all* pairs (k,j). In the induction step for j, namely to prove the statement for the pairs (k,j_0) assuming the statements for the pairs (k,ℓ) with $j_0+1 \leqslant \ell \leqslant p$, we distinguish between two cases:

- *Case (i): pairs (k,j_0) with $k \neq 1$.* In this case the First (respectively Second) commuting (k,j_0)-Lemma requires the First (respectively Second) commuting (r,j_0+1)-Lemmas for $G^{\lrcorner}_{p,3}$ for all $r \geqslant k-1$.
- *Case (ii): the pair $(1,j_0)$.* The First commuting $(1,j_0)$-Lemma will follow easily from the presentation of $G^{\lrcorner}_{p,3}$, while the Second commuting $(1,j_0)$-Lemma requires the First commuting (k,j_0)-Lemma for $G^{\lrcorner}_{p,3}$ with $k > 1$.

Finally, the statements for the remaining pairs $(1,3)$ and $(2,2)$ for $G^{\lrcorner}_{p,3}$ require extra arguments to address the appearing "error terms" coming from the relation $[x_2,x_3] =_{\mathcal{P}} x_p$. For a pictorial description of the implications between the statements see Figure 3.1.

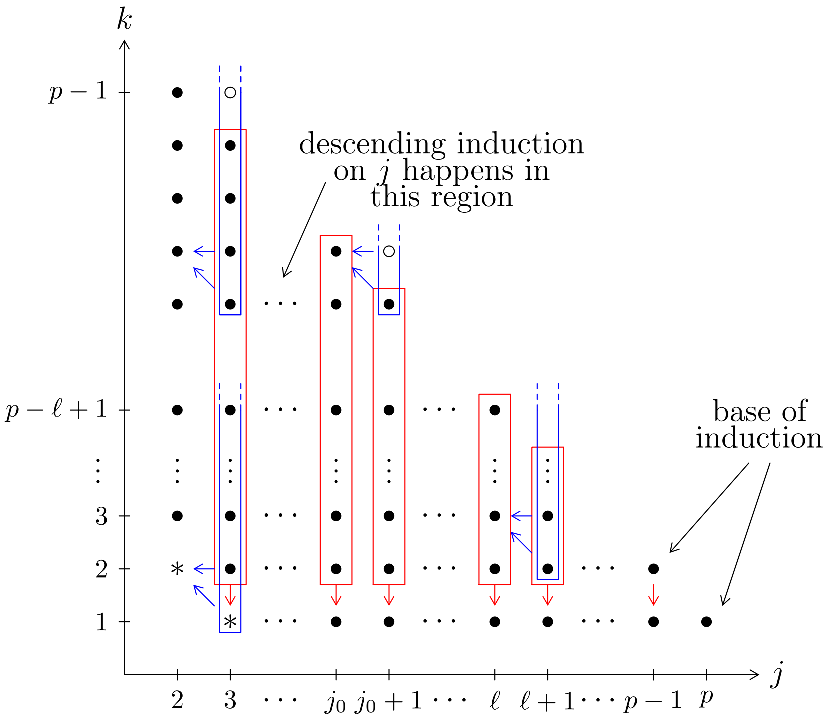

Figure 3.1: Implications of the Commuting (k, j)-Lemmas: the points $\bullet$, $*$, and $\circ$ correspond to both the First and Second commuting (k, j)-Lemmas 3.27 and 3.28 respectively (see also Addendum 3.39). The drawn blue (horizontal and upward diagonally pointing) arrows and blue (unbounded) boxes encode the fact that for the corresponding statement for (k, j) with $k \neq 1$ we need the statements corresponding to the pairs $(r, j + 1)$ for all $r \geqslant k-1$ (see Lemma 3.38). The red (pointing downwards) arrows and the red (bounded) boxes correspond to the fact that for the statement corresponding to $(1, j)$ we need the statements for the pairs (k, j) for all $k \geqslant 2$ (see Lemma 3.42). All the points beside $*$ correspond to the proof by descending induction on j; the points $*$ require special treatment and can only be done once we are done with the proof of the induction step.

Bases of induction on j

Let $W \in \{x_2^m, w\}$ where $w := w(x_1, x_2) \in \mathcal{F}[\alpha]$ is a word in the derived subgroup of $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$). We start the induction with the pairs $(1,p)$ and $(2,p-1)$.

For $(1,p)$ we have that $\Omega_1^p(\underline{n}) := x_p^{n_1}$ is central, and it follows from the presentation $\mathcal{P}$ that the identity $[x_p^{n_1}, W] =_{\mathcal{P}} 1$ has area $\lesssim_{\alpha,p} |W|n$ in $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$).

For $(2, p-1)$, we first use the fact that $\langle x_1, x_{p-1}, y_1, y_{p-1}\rangle$ is a 5-Heisenberg subgroup of $G^{\lrcorner}_{p,3}$ to rewrite $\Omega_2^{p-1}(\underline{n}) =_{\mathcal{P}} \widetilde{\Omega}_2^{p-1}(\underline{n})$ with area $\lesssim_p n^2$ in $G^{\lrcorner}_{p,3}$. Since y_i commutes with x_i, we deduce that the identity $[\Omega_2^{p-1}(\underline{n}), W] =_{\mathcal{P}} 1$ holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_{\alpha,p} n^2$.

Induction hypotheses for j

- **(IH-$G_{p,3}$)** Let $3 \leqslant j_0 + 1 \leqslant p$. Suppose that for all pairs (k,ℓ) such that $j_0 + 1 \leqslant \ell \leqslant p$ the First commuting (k,ℓ)-Lemma 3.24 and the Second commuting (k,ℓ)-Lemma 3.25 for $G_{p,3}$ hold.
- **(IH-$G^{\lrcorner}_{p,3}$)** Let $3 \leqslant j_0 + 1 \leqslant p$. Suppose that for all pairs $(k,\ell) \neq (1,3)$ such that $j_0 + 1 \leqslant \ell \leqslant p$ the First commuting (k,ℓ)-Lemma 3.27 and the Second commuting (k,ℓ)-Lemmas 3.28 for $G^{\lrcorner}_{p,3}$ hold.

Auxiliary results for the induction steps for j

Lemma 3.32. *Let $\ell \geqslant j_0$, and $\beta, n, m \in \mathbb{R}$ with $|\beta| \leqslant 1$. The identity*

$$[x_1^{\beta}, x_\ell^n] =_{\mathcal{P}} x_{\ell+1}^{\beta \cdot n} x_{\ell+2}^{s_{\ell+2}} \dots x_{p-1}^{s_{p-1}} x_p^{s_p}$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) for suitable $|s_i| \lesssim_p n$ with area $\lesssim_p n^2$.

Proof. The identity is a direct consequence of Proposition 3.5. The area estimate follows from a light computation, for the details we refer the reader to [LIPT23, Lemma 6.12(3)]. □

Lemma 3.33. *Let $k \geqslant 2$, $\ell \geqslant j_0 + 1$, $n \geqslant 1$, and $|n_i|, |s_r| \leqslant n$ for $1 \leqslant i \leqslant k-1$, $\ell \leqslant r \leqslant p$. The identity*

$$\left[x_1^{n_1}, \dots, x_1^{n_{k-1}}, x_\ell^{s_\ell} \dots x_p^{s_p}\right] =_{\mathcal{P}} \prod_{r=\ell}^{p-1} \Omega_k^r(n_1, \dots, n_{k-1}, s_r)$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p n^{p-\ell+1}$.

Proof. Since by assumption $\ell > 2$, the statement is a direct consequence of the induction hypothesis (IH-p) using the fact that the Dehn function of $G_{p-\ell+2,3} \hookrightarrow G^{\lrcorner}_{p,3}$ is $\preccurlyeq n^{p-\ell+1}$. □

Lemma 3.34. *Let (k, ℓ) with $k \neq 1$ and $\ell \geqslant j_0 + 1$, and $\underline{n} := (n_1, \ldots, n_k) \in \mathbb{R}^k$. The identity*

$$[x_1^{n_1}, \Omega^{\ell}_{k-1}(n_2, \ldots, n_k)^{-1}] =_{\mathcal{P}} \Omega^{\ell}_k(\underline{n})^{-1}$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area

$$\lesssim_p \max\{|\underline{n}|^{p-\ell+1}, |\underline{n}|^2\}.$$

Proof. For $\ell = p$ this is an easy consequence of x_p being central, so assume $\ell < p$. Since by assumptions $\ell > 2$, the statement is then a direct consequence of (i) the induction hypothesis (IH-p) that the Dehn function of $G_{p-\ell+2,3} \hookrightarrow G^{\lrcorner}_{p,3}$ is $\preccurlyeq n^{p-\ell+1}$ and (ii) the following free identity in $G_{p-\ell+2,3}$:

$$\left[x_1^{n_1}, \Omega^{\ell}_{k-1}(n_2, \ldots, n_k)^{-1}\right] \overset{\text{free}}{=} \left[\Omega^{\ell}_{k-1}(n_2, \ldots, n_k), x_1^{n_1}\right]^{\Omega^{\ell}_{k-1}(n_2,\ldots,n_k)^{-1}}.$$

□

Corollary 3.35. *Let $k \geqslant 2$, $\ell \geqslant j_0 + 1$, $\underline{n} \in \mathbb{R}^k$, and $|l| \leqslant |\underline{n}|$. The identity*

$$[\Omega^{\ell}_k(\underline{n})^{\pm 1}, x_1^l] =_{\mathcal{P}} \Omega^{\ell}_{k+1}(l, \underline{n})^{\mp 1}$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$(respectively $G_{p,3}$ for all $p \geqslant 3$) with area

$$\lesssim_p \max\{|\underline{n}|^{p-\ell+1}, |\underline{n}|^2\}.$$

Proof. Lemma 3.34 implies that the identity

$$[\Omega^{j}_k(\underline{n})^{\pm 1}, x_1^l] \overset{\text{free}}{=} [x_1^l, \Omega^{j}_k(\underline{n})^{\pm 1}]^{-1} =_{\mathcal{P}} \Omega^{j}_{k+1}(l, \underline{n})^{\mp 1}$$

holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p \max\{|\underline{n}|^{p-\ell+1}, |\underline{n}|^2\}$. □

Lemma 3.36. *Let $j_0 \leqslant \ell \leqslant p-1$, $n \geqslant 1$, $\nu \geqslant 1$, and $\eta \in \mathbb{R}$ with $|\eta| \lesssim_p n$. For words $v_i := \Omega^{\ell+1}_{k_i}(\underline{n_i})^{\pm 1}$, with $1 \leqslant k_i \leqslant p - \ell$ and $|\underline{n_i}| \lesssim_p n$ $(1 \leqslant i \leqslant \nu)$, and $w := \Omega^{\ell}_k(\underline{m})^{\pm 1}$, with $2 \leqslant k \leqslant p - \ell + 1$, the identity*

$$[x_1^{\eta}, v_1 \cdots v_{\nu} \cdot w] =_{\mathcal{P}} [x_1^{\eta}, w] \cdot [x_1^{\eta}, v_{\nu}] \cdots [x_1^{\eta}, v_1]$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$). Moreover, if $\ell \leqslant p-2$, then it has area $\lesssim_{\nu,p} n^{p-\ell}$; and if $\ell = p-1$, then it has area $\lesssim_{\nu,p} n^2$.

Proof. Assume $\ell = 2$. (If $p-1 > \ell > 2$ one can either apply the same reasoning in $G_{p-\ell+2,3} \hookrightarrow G^{\lrcorner}_{p,3}$ or observe that this assertion was already proved in the induction step from $G_{p-\ell+1,3}$ to $G_{p-\ell+2,3}$. In the special case $\ell = p-1$ we use that the power of x_p appearing in $v_1, \cdots, v_\nu$ is central.) We assume that $v_i = \Omega^{\ell+1}_{k_i}(\underline{n_i})^{-1}$ for $1 \leqslant i \leqslant \nu$; the case where some exponents are $+1$ is analogous, but slightly easier. Since $\ell + 1 > 2$, the words $v_1, \cdots, v_\nu$ represent elements in $G_{p-\ell+1,3} \hookrightarrow G^{\lrcorner}_{p,3}$.

In particular, it follows from the induction hypothesis (IH-p) and Lemma 3.34 that the identities

$$[x_1^\eta, v_i] \overset{\text{free}}{=} (\Omega^{\ell+1}_{k_i+1}(\eta, \underline{n_i})^{-1})^{v_i^{-1}} =_{\mathcal{P}} \Omega^{\ell+1}_{k_i+1}(\eta, \underline{n_i})^{-1} \tag{3.5}$$

hold in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^{p-\ell}$ for $1 \leqslant i \leqslant \nu$.

Finally, the identities

$$\begin{aligned}[x_1^\eta, v_1 \cdots v_\nu \cdot w] &\overset{\text{free}}{=} [x_1^\eta, w] \cdot \prod_{i=0}^{\nu-1} [x_1^\eta, v_{\nu-i}]^{v_{\nu-i+1}\cdots v_\nu w} \\ &=_{\mathcal{P}} [x_1^\eta, w] \cdot \prod_{i=0}^{\nu-1} [x_1^\eta, v_{\nu-i}]^{w} \\ &=_{\mathcal{P}} [x_1^\eta, w] \cdot [x_1^\eta, v_\nu] \cdots [x_1^\eta, v_1]\end{aligned}$$

have area $\lesssim_{\nu,p} n^{p-\ell}$ in $G^{\lrcorner}_{p,3}$. The first identity is free, so it has no area, the second holds in $G_{p-\ell+1,3} \hookrightarrow G^{\lrcorner}_{p,3}$, and for the last identity we apply the identity (3.5) and the Second commuting $(k_i+1, \ell+1)$-Lemmas in $G^{\lrcorner}_{p,3}$ at most ν times. □

Remark 3.37. In our applications of Lemma 3.36 later we will always have $k_i \geqslant k-1$ meaning that in those case the above proof only relies on the Second commuting $(r, \ell+1)$-Lemmas for $r \geqslant k$.

The following statement is a key ingredient of the induction step that allows us to decompose Ω^ℓ_k-words into a product of $\Omega^{\ell+1}$-words (see blue arrows/boxes in Figure 3.1).

Lemma 3.38. *Let $(k, \ell) \neq (2,2)$ with $k \neq 1$ and $j_0 \leqslant \ell \leqslant p-2$, $\underline{n} := (n_1, \dots, n_k) \in \mathbb{R}^k$, and $\beta := n_{k-1} - \lfloor n_{k-1} \rfloor$.*
If $n_{k-1} \geqslant 0$, then the identity

$$\Omega^\ell_k(\underline{n}) =_{\mathcal{P}} \Omega^\ell_k(n_1, \dots, \beta, n_k) \cdot \Big(\prod_{r=0}^{\lfloor n_{k-1} \rfloor - 1} \Omega^{\ell+1}_k(n_1, \dots, r+\beta, n_k)^{-1} \cdot \Omega^{\ell+1}_{k-1}(n_1, \dots, n_{k-2}, n_k) \Big) \tag{3.6}$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p |\underline{n}|^{p-\ell+1}$.

If $n_{k-1} < 0$, then the identity

$$\Omega_k^\ell(\underline{n}) =_{\mathcal{P}} \Omega_k^\ell(n_1,\dots,\beta,n_k) \prod_{r=0}^{-\lfloor n_{k-1}\rfloor -1} \prod_{i=\ell+1}^{p} \Omega^{\ell+1}_{k+i-(\ell+1)}(n_1,\dots,n_{k-2},\beta-r,1,\dots,1,-n_k)^{-1} \cdot \Omega^{\ell+1}_{k+i-(\ell+1)-1}(n_1,\dots,n_{k-2},1,\dots,1,-n_k) \quad (3.7)$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p |\underline{n}|^{p-\ell+1}$.

Addendum 3.39. *For $3 \leqslant j_0+1 \leqslant \ell \leqslant p$ and $k \geqslant p-\ell+2$ the words $\Omega_k^\ell(\underline{n})$ are null-homotopic in $G_{p-\ell+2,3} \hookrightarrow G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$). By the induction hypothesis (IH-p) their area is $\lesssim_p \delta_{G_{p-\ell+2,3}}(n) \lesssim_p n^{p-\ell+1}$.*

This observation has the following consequences in $G^{\lrcorner}_{p,3}$ and $G_{p,3}$:

1. *For $j_0+1 \leqslant \ell \leqslant p$ and $k \geqslant p-\ell+2$, the First and Second commuting (k,ℓ)-Lemmas may be interpreted as the process of removing and creating the Ω_k^ℓ-words with area $\lesssim_p |m|n^{p-\ell} + n^{p-\ell+1}$ and $\lesssim_p n^{p-\ell+1}$ respectively;*
2. *For $j_0+1 \leqslant \ell \leqslant p$ and the pairs $(p-\ell+1,\ell)$, in the identity (3.6) (respectively (3.7)) the $|\lfloor n_{k-1}\rfloor|$ appearances of the $\Omega^{\ell+1}_{p-\ell+1}$-words (respectively the $\Omega^{\ell+1}_{p-\ell+1}$-$,\dots,\Omega^{\ell+1}_{2p-1}$-words) can be removed with area $\lesssim_p |\lfloor n_{k-1}\rfloor| \cdot \delta_{G_{p-\ell+1,3}}(n) \lesssim_p n^{p-\ell+1}$. For notational convenience we do not distinguish cases and leave these terms in the identities;*
3. *For all $k \geqslant p$ by Lemma 3.38 and using the induction hypothesis (IH-p) at most n times we get that $\Omega_k(\underline{n})$ is a null-homotopic word with area $\lesssim_p n^{p-1}$.*

Proof of Lemma 3.38. We assume $n_{k-1} \geqslant 0$. The proof for $n_{k-1} < 0$ is analogous, but produces slightly longer terms when we use the corresponding formula from Addendum 3.13. The proof is by induction in $\lfloor n_{k-1}\rfloor$. The case $\lfloor n_{k-1}\rfloor = 0$ is trivial, and we now assume that we have proven the result for $\lfloor n_{k-1}\rfloor - 1$.

We deduce from $k \neq 1$ and Corollary 3.12 that

$$\begin{aligned}\Omega_k^\ell(\underline{n}) &=_{\mathcal{P}} \begin{cases} \left[x_1^{n_1},\dots,x_1^{n_{k-2}}, x_{\ell+1}^{n_k}\cdot[x_{\ell+1}^{n_k}, x_1^{n_{k-1}-1}]\cdot[x_1^{n_{k-1}-1}, x_\ell^{n_k}]\cdot[y_1^{\tilde{n}}, y_{p-1}^{\tilde{n}}]\right], & \ell = 2 \\ \left[x_1^{n_1},\dots,x_1^{n_{k-2}}, x_{\ell+1}^{n_k}\cdot[x_{\ell+1}^{n_k}, x_1^{n_{k-1}-1}]\cdot[x_1^{n_{k-1}-1}, x_\ell^{n_k}]\right], & \ell \neq 2 \end{cases} \\ &=_{\mathcal{P}} \left[x_1^{n_1},\dots,x_1^{n_{k-2}}, x_{\ell+1}^{n_k}\cdot[x_{\ell+1}^{n_k}, x_1^{n_{k-1}-1}]\cdot[x_1^{n_{k-1}-1}, x_\ell^{n_k}]\right],\end{aligned}$$

where the last identity holds in $G^{\lrcorner}_{p,3}$ at cost $\lesssim_p |\underline{n}|^2$, since the y_i commute with the x_i. Therefore, by applying Lemma 3.36 to the above equality for $\nu = 2$ and for the words $v_1 := x_{\ell+1}^{n_k}$, $v_2 := [x_1^{n_{k-1}-1}, x_{\ell+1}^{n_k}]^{-1}$, and $w := [x_1^{n_{k-1}-1}, x_\ell^{n_k}]$,[3] we get that the identity

[3]Note that in the case $n_{k-1} < 0$, $\nu \leqslant p$.

$$\left[x_1^{n_1},\dots,x_1^{n_{k-2}},x_{\ell+1}^{n_k}\cdot[x_{\ell+1}^{n_k},x_1^{n_{k-1}-1}]\cdot[x_1^{n_{k-1}-1},x_\ell^{n_k}]\right]=_{\mathcal{P}}$$
$$\left[x_1^{n_1},\dots,x_1^{n_{k-3}},[x_1^{n_{k-2}},x_{\ell+1}^{n_k}]\cdot[x_1^{n_{k-2}},[x_1^{n_{k-1}-1},x_{\ell+1}^{n_k}]^{-1}]\cdot[x_1^{n_{k-2}},[x_1^{n_{k-1}-1},x_\ell^{n_k}]]\right].$$

holds in $G_{p,3}^{\lrcorner}$ with area $\lesssim_p |\underline{n}|^{p-\ell}$. By Lemma 3.34 the identity

$$\left[x_1^{n_{k-2}},[x_1^{n_{k-1}-1},x_{\ell+1}^{n_k}]^{-1}\right]=_{\mathcal{P}}\left[x_1^{n_{k-2}},[x_1^{n_{k-1}-1},x_{\ell+1}^{n_k}]\right]^{-1}$$

holds with area $\lesssim_p |\underline{n}|^{p-\ell}$.

Overall, on the first iteration of applying Lemma 3.36 we get that the identity

$$\Omega_k^\ell(\underline{n})=_{\mathcal{P}}\left[x_1^{n_1},\dots,x_1^{n_{k-3}},[x_1^{n_{k-2}},x_{\ell+1}^{n_k}]\left[x_1^{n_{k-2}},[x_1^{n_{k-1}-1},x_{\ell+1}^{n_k}]\right]^{-1}[x_1^{n_{k-2}},[x_1^{n_{k-1}-1},x_\ell^{n_k}]]\right]$$

holds in $G_{p,3}^{\lrcorner}$ with area $\lesssim_p |\underline{n}|^{p-\ell}$. Iterating this argument $O_p(1)$-times we get that the identity

$$\begin{aligned}\Omega_k^\ell(\underline{n})=_{\mathcal{P}}\ &\Omega_k^\ell(n_1,\dots,n_{k-1}-1,n_k)\cdot\Omega_k^{\ell+1}(n_1,\dots,n_{k-1}-1,n_k)^{-1}\\&\cdot\Omega_{k-1}^{\ell+1}(n_1,\dots,n_{k-2},n_k)\end{aligned}$$

holds in $G_{p,3}^{\lrcorner}$ with area $\lesssim_p |\underline{n}|^{p-\ell}$. Applying our induction hypothesis for $\lfloor n_{k-1}\rfloor-1$ to the word $\Omega_k^\ell(n_1,\dots,n_{k-1}-1,n_k)$ shows that identity (3.6) holds in $G_{p,3}^{\lrcorner}$ with area $\lesssim_p |\underline{n}|^{p-\ell+1}$. □

If $(k,j)=(2,2)$ the same arguments as in the proof of Lemma 3.38 yield a similar identity with the only difference that now the terms of the form $[y_1^{\tilde{n}},y_{p-1}^{\tilde{n}}]$ do not cancel. We record it below.

Addendum 3.40. *Let $p\geqslant 5$, $(k,j)=(2,2)$, $\underline{n}=(n_1,n_2)\in\mathbb{R}^k$ and $\beta=n_1-\lfloor n_1\rfloor$.*

If $n_1\geqslant 0$, then the identity

$$\Omega_2(\underline{n})=_{\mathcal{P}}[x_1^\beta,x_2^{n_2}]\cdot\left(\prod_{r=0}^{\lfloor n_1\rfloor-1}\Omega_1^3(n_2)\cdot\Omega_2^3(r+\beta,n_2)^{-1}\cdot[y_1^{\tilde{n}},y_{p-1}^{\tilde{n}}]^{\pm 1}\right), \tag{3.8}$$

holds in $G_{p,3}^{\lrcorner}$ for all $p\geqslant 5$ with area $\lesssim_p |\underline{n}|^{p-1}$ and $|\tilde{n}|\lesssim_p |n_2|$.

Analogously, if $n_1<0$, then the identity

$$\Omega_2(\underline{n})=_{\mathcal{P}}[x_1^\beta,x_2^{n_2}]\cdot\Big(\prod_{r=0}^{-\lfloor n_1\rfloor-1}\Big(\prod_{i=3}^{p}\Omega_{i-1}^3(\beta-r,1,\dots,1,-n_2)^{-1} \tag{3.9}$$

$$\cdot\, \Omega^3_{i-2}(1,\dots,1,-n_2)\Big)\cdot [y_1^{\tilde n}, y_{p-1}^{\tilde n}]^{\pm 1}\Big),$$

holds in $G^{\lrcorner}_{p,3}$ *for all* $p \geqslant 5$ *with area* $\lesssim_p |\underline{n}|^{p-1}$ *and* $|\tilde n| \lesssim_p |n_2|$.

Remark 3.41. For all $p \geqslant 3$ the identity (3.8) (respectively (3.9)) without the error terms $[y_1^{\tilde n}, y_{p-1}^{\tilde n}]$ holds in $G_{p,3}$ with area $\lesssim_p |\underline{n}|^{p-1}$.

The following result for $(k,\ell) = (1, j_0)$ is a key ingredient in our derivation of the case $(1, j_0)$ of the Commuting Lemmas from the cases (k, j_0) for $k \geqslant 2$ (see the red arrows/boxes in Figure 3.1.) In particular, in its proof we can assume that the Commuting (k, j_0)-Lemmas with $k \geqslant 2$ have already been proved. We prove a more general version for arbitrary (k,ℓ), as this is required in other parts of this work (after the proofs of the Commuting (k,ℓ)-Lemmas).

Lemma 3.42. *Let* $p \geqslant 5$, $n \geqslant 1$, $\alpha \geqslant 0$, $j_0 \leqslant \ell \leqslant p-1$, $k \geqslant 1$, *and* $\underline{n} \in \mathbb{R}^k$ *with* $|\underline{n}| \lesssim_p n$. *For* $u := u(x_1, x_2) \in \mathcal{F}[2\alpha]$ *with* $|u| \leqslant n$, *the identities*

$$[\Omega^\ell_k(\underline{n})^{\pm 1}, u] =_{\mathcal{P}} \begin{cases} \prod_{i=1}^{\nu} \Omega^\ell_{l_i}(\underline{\eta}_i)^{\pm 1}, & \text{if } k+\ell \geqslant 5,\\ \left(\prod_{i=1}^{\nu} \Omega^\ell_{l_i}(\underline{\eta}_i)^{\pm 1}\right)\left(\prod_{i=1}^{\mu} \Omega^{p-1}_2(\underline{\hat n}_i)^{\pm 1}\right), & \text{if } (k,\ell) = (1,3),\\ \left(\prod_{i=1}^{\nu} \Omega_{l_i}(\underline{\eta}_i)^{\pm 1}\right)\left(\prod_{i=1}^{\mu} \Omega^{p-2}_3(\underline{\tilde n}_i)^{\pm 1}\right), & \text{if } (k,\ell) = (2,2).\end{cases}$$

hold in $G^{\lrcorner}_{p,3}$ *with area* $\lesssim_{\alpha,p} n^{p-\ell+1}$, *for suitable* $\nu = O_{p,\alpha}(1)$, $\mu \leqslant 2\alpha$, $l_i \geqslant k+1$, $\underline{\eta}_i \in \mathbb{R}^{l_i}$ *with* $|\underline{\eta}_i| \lesssim_p n$ *for* $1 \leqslant i \leqslant \nu$, $\hat n_i \in \mathbb{R}^2$, *and* $\tilde n_i \in \mathbb{R}^3$ *with* $|\hat n_i|, |\tilde n_i| \lesssim_p n$ *for* $1 \leqslant i \leqslant \mu$.

Proof. We start by proving the identities for $k+\ell \geqslant 5$ for $\Omega^\ell_k(\underline{n})$, the proof for $\Omega^\ell_k(\underline{n})^{-1}$ being similar. The proof for $k+\ell \geqslant 5$ relies only on the First commuting $(k+r,\ell)$-Lemmas for $G^{\lrcorner}_{p,3}$ where $r \geqslant 1$.

Since $u \in \mathcal{F}[2\alpha]$, there exists $\mu \leqslant 2\alpha$ such that

$$u := \prod_{i=1}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}.$$

Without loss of generality we can assume $\mu = 2\alpha$. The proof is done by induction on 2α. For the base of induction, $\alpha = 0$, the statement is trivially true with $\nu = \mu = 0$, with the convention that in this case the product is empty. Suppose that the statement is true for $2\alpha \geqslant 0$ and let $u \in \mathcal{F}[2(\alpha+1)]$. The following identities hold in $G^{\lrcorner}_{p,3}$. The area estimates are explained below.

$$\Omega_k^\ell(\underline{n}) \cdot \prod_{i=1}^{\mu} x_1^{\beta_i} x_2^{\gamma_i} \overset{(1)}{=}_{\mathcal{P}} x_1^{\beta_1} \cdot \Omega_k^\ell(\underline{n}) \cdot \Omega_{k+1}^\ell(\beta_1, \underline{n})^{-1} \cdot x_2^{\gamma_1} \cdot \prod_{i=2}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}$$

$$\overset{(2)}{=}_{\mathcal{P}} x_1^{\beta_1} x_2^{\gamma_1} \cdot \Omega_k^\ell(\underline{n}) \cdot \Omega_{k+1}^\ell(\beta_1, \underline{n})^{-1} \cdot \prod_{i=2}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}$$

$$\overset{(3)}{=}_{\mathcal{P}} \left(\prod_{i=1}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}\right) \cdot \Omega_k^\ell(\underline{n}) \cdot \left(\prod_{i=1}^{\nu} \Omega_{l_i}^\ell(\underline{\eta}_i)^{\pm 1}\right)$$

(1) is a consequence of Corollary 3.35, in fact this is a free identity for $\Omega_k^\ell(\underline{n})$, but Corollary 3.35 is required for $\Omega_k^\ell(\underline{n})^{-1}$. It then follows that the identity has area $\lesssim_p \max\{n^{p-\ell+1}, n^2\} \leqslant n^{p-\ell+1}$;

(2) is obtained as follows: note that $k+\ell \geqslant 5$, if $k=1$, then this follows because $[x_2^{\gamma_2}, x_\ell^{n_1}]$ has area $\lesssim_p n^2$ for all $\ell \geqslant 4$. Otherwise, by the induction hypothesis (IH-$G_{p,3}^{\lrcorner}$) we can apply the First commuting (k,ℓ)-and $(k+1,\ell)$-Lemmas for $G_{p,3}^{\lrcorner}$. Therefore, it has area $\lesssim_p n^{p-\ell+1}$. In either case it has area $\lesssim_p n^{p-\ell+1}$;

(3) is a consequence of the induction hypothesis for 2α. More specifically, first we apply the induction hypothesis to the Ω_{k+1}^ℓ-word to obtain the error term

$$\prod_{i=1}^{\nu_2} \Omega_{l_{2,i}}^\ell(\underline{\eta}_{2,i})^{\pm 1},$$

we then apply the induction hypothesis on 2α to the Ω_k^ℓ-word to obtain the error term $\prod_{i=1}^{\nu_1} \Omega_{l_{1,i}}^\ell(\underline{\eta}_{1,i})^{\pm 1}$, where $\nu_1, \nu_2 = O_{\alpha,p}(1)$, $l_{1,i} \geqslant k+1$, $l_{2,i} \geqslant k+2$, $\underline{\eta}_{1,i} \in \mathbb{R}^{l_{i,1}}$, and $\underline{\eta}_{2,i} \in \mathbb{R}^{l_{2,1}}$ with $|\underline{\eta}_{1,i}|, |\underline{\eta}_{2,i}| \lesssim_p n$. Finally, we merge these terms into the single product $\prod_{i=1}^{\nu} \Omega_{l_i}^\ell(\underline{\eta}_i)^{\pm 1}$. In total, this has area $\lesssim_{p,\alpha} n^{p-\ell+1}$.

This concludes the proof for the pairs (k,ℓ) with $k+\ell \geqslant 5$. The proofs for the pairs $(1,3)$ and $(2,2)$ are analogous. However, some additional error terms are being produced. Rather than repeating the line of argument several times, we just give a brief explanation of the main differences and leave the details to the reader.

Case (1,3): In this case we also have to commute terms of the form $x_3^{n_1}$ with terms of the form $x_2^{\gamma_i}$, which produces additional error terms of the form $[x_3^{n_1}, x_2^{\gamma_i}]$. However, since $\langle x_3, x_2, y_1, y_{p-1}\rangle$ and $\langle x_1, x_{p-1}, y_1, y_{p-1}\rangle$ generate subgroups of $G_{p,3}^{\lrcorner}$ isomorphic to the 5-Heisenberg group, each such commutation can be performed by producing an error term of the form $\Omega_2^{p-1}(\hat{\underline{n}}_i)$ on the very right with area $\lesssim_p n^2$. In particular, the total area of the transformations is still $\lesssim_p n^{p-2}$.

Case (2,2): This is similar to the case (1,3) only that now the error terms $\Omega_3^{p-2}(\underline{\tilde{n}}_i)$ arise from commutators of the form $[\Omega_2(\underline{n}), x_2^{\gamma_i}]$. Which by the First commuting $(2,2)$-Lemma 3.27 for $G^{\lrcorner}_{p,3}$ has area $\lesssim_p n^{p-1}$. It should be noted that at the point where we use the identity for (2,2), we have already proven the First commuting $(2,2)$-Lemma for $G^{\lrcorner}_{p,3}$.

□

Addendum 3.43. *The above proof for the pairs (k,ℓ) with $k+\ell \geqslant 5$ translates without modification to $G_{p,3}$ with $p \geqslant 3$ for all pairs (k,ℓ). So we get that the identity*

$$[\Omega_k^\ell(\underline{n})^{\pm 1}, u] =_{\mathcal{P}} \prod_{i=1}^{\nu} \Omega_{l_i}^\ell(\underline{\eta}_i)^{\pm 1}$$

holds in $G_{p,3}$ for all $p \geqslant 3$ with area $\lesssim_{\alpha,p} n^{p-\ell+1}$ for suitable $\nu = O_{p,\alpha}(1)$, $l_i \geqslant k+1$, and $\underline{\eta}_i \in \mathbb{R}^{l_i}$ with $|\underline{\eta}_i| \lesssim_p n$ for all $1 \leqslant i \leqslant \nu$.

With the required auxiliary results we can now complete the induction step for the First commuting (k,j_0)-Lemma 3.27 and the Second commuting (k,j_0)-Lemma 3.28 for $G^{\lrcorner}_{p,3}$.

Proof of the induction step for j

Recall that the induction in j for $G^{\lrcorner}_{p,3}$ is done for all pairs (k,j) such that $k+j \geqslant 5$. So assume $k+j_0 \geqslant 5$. As mentioned before we distinguish the cases (i) $k \neq 1$ and (ii) $k=1$.

Case (i) $k \neq 1$. Let $W \in \{x_2^m, w(x_1,x_2)\}$ with $w := w(x_1,x_2)$ a word in the derived subgroup of $G^{\lrcorner}_{p,3}$. We need to show that the area of the null-homotopic word $[\Omega_k^{j_0}(\underline{n}), W]$ is $\lesssim_{\alpha,p} |W| \cdot n^{p-j_0} + n^{p-j_0+1}$. We assume that $n_k \geqslant 0$, the proof for $n_k < 0$ is analogous, but involves slightly more terms. Let $\beta := n_{k-1} - \lfloor n_{k-1} \rfloor$. We have the following identities in $G^{\lrcorner}_{p,3}$ whose area estimates are explained below.

$$\begin{aligned}\Omega_k^{j_0}(\underline{n}) \cdot W &\overset{(1)}{=}_{\mathcal{P}} \Big(\Omega_k^{j_0}(n_1,\dots,\beta,n_k) \cdot \\ &\quad \prod_{r=0}^{\lfloor n_{k-1}\rfloor -1} \Big(\Omega_k^{j_0+1}(n_1,\dots,r+\beta,n_k)^{-1} \cdot \Omega_{k-1}^{j_0+1}(n_1,\dots,n_{k-2},n_k) \Big) \Big) \cdot W \\ &\overset{(2)}{=}_{\mathcal{P}} \Omega_k^{j_0}(n_1,\dots,\beta,n_k) \cdot W \\ &\quad \cdot \Big(\prod_{r=0}^{\lfloor n_{k-1}\rfloor -1} \Omega_k^{j_0+1}(n_1,\dots,r+\beta,n_k)^{-1} \cdot \Omega_{k-1}^{j_0+1}(n_1,\dots,n_{k-2},n_k) \Big)\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&\overset{(3)}{=}_{\mathcal{P}} \left(\prod_{r=j_0+1}^{p-1} \Omega_{k-1}^{r}(n_1,\dots,n_{k-2},s_r) \right) \cdot W \\
&\quad \cdot \Big(\prod_{r=0}^{\lfloor n_{k-1} \rfloor -1} \Omega_k^{j_0+1}(n_1,\dots,r+\beta,n_k)^{-1} \cdot \Omega_{k-1}^{j_0+1}(n_1,\dots,n_{k-2},n_k) \Big) \\
&\overset{(4)}{=}_{\mathcal{P}} W \cdot \left(\prod_{r=j_0+1}^{p-1} \Omega_{k-1}^{r}(n_1,\dots,n_{k-2},s_r) \right) \\
&\quad \cdot \Big(\prod_{r=0}^{\lfloor n_{k-1} \rfloor -1} \Big(\Omega_k^{j_0+1}(n_1,\dots,r+\beta,n_k)^{-1} \cdot \Omega_{k-1}^{j_0+1}(n_1,\dots,n_{k-2},n_k) \Big) \Big) \\
&\overset{(5)}{=}_{\mathcal{P}} W \cdot \Omega_k^{j_0}(\underline{n}),
\end{aligned}
$$

(1) follows from Lemma 3.38, so the area is $\lesssim_p |\underline{n}|^{p-j_0+1}$;

(2) is a consequence of the induction hypothesis (IH-$G^{\lrcorner}_{p,3}$) applied $2 \cdot \lfloor n_k \rfloor$ many times. [4] Therefore, it has area $\lesssim_{\alpha,p} |W| n^{p-j_0} + n^{p-j_0+1}$;

(3) follows from applying Lemma 3.32, since $|\beta| \leqslant 1$, and then Lemma 3.33. Therefore, it has area $\lesssim_p n^{p-j_0}$;

(4) follows from the induction hypothesis (IH-$G^{\lrcorner}_{p,3}$) applied $O_p(1)$ times, thus has area $\lesssim_{\alpha,p} |W| n^{p-j_0-1} + n^{p-j_0}$;

(5) follows from Lemmas 3.32, 3.33 and 3.38, so it has area $\lesssim_p |\underline{n}|^{p-j_0+1}$.

Overall, the null-homotopic word $[\Omega_k^{j_0}(\underline{n}), W]$ has area $\lesssim_{\alpha,p} |W| n^{p-j_0} + n^{p-j_0+1}$. This finishes the proof for all pairs (k, j_0) with $k \neq 1$ and $k + j_0 \geqslant 5$.

Case (ii) $k = 1$. In this case $\Omega_1^{j_0}(\underline{n}) := x_{j_0}^{n_1}$. Observe that since $1 + j_0 \geqslant 5$ the First commuting $(1, j_0)$-Lemma for $G^{\lrcorner}_{p,3}$ follows from the relation $[x_{j_0}, x_2] =_{\mathcal{P}} 1$. For the Second commuting $(1, j_0)$-Lemma for $G^{\lrcorner}_{p,3}$, we have that $W =_{\text{def}} w(x_1, x_2) \in \mathcal{F}[\alpha]$ represents an element in the derived subgroup of $G^{\lrcorner}_{p,3}$. So from Lemma 3.42 for $1 + j_0 \geqslant 5$ we obtain the identity

$$
[x_{j_0}^{n_1}, w] =_{\mathcal{P}} \prod_{i=1}^{\nu} \Omega_{l_i}^{j_0}(\underline{\eta}_i)^{\pm 1}
$$

in $G^{\lrcorner}_{p,3}$ with area $\lesssim_{\alpha,p} n^{p-j_0+1}$, where $\nu = O_{\alpha,p}(1)$, $l_i \geqslant 2$, and $\underline{\eta}_i \in \mathbb{R}^{l_i}$ with $|\underline{\eta}_i| \lesssim_p n$.

Since $G^{\lrcorner}_{p,3}$ is metabelian, it follows that the identity $[x_{j_0}^{n_1}, w] =_{\mathcal{P}} 1$ holds. Thus, the word $\prod_{i=1}^{\nu} \Omega_{l_i}^{j_0}(\underline{\eta}_i)^{\pm 1}$ is null-homotopic in $G_{p-j_0+2,3} \hookrightarrow G^{\lrcorner}_{p,3}$. It follows from the induction hypothesis (IH-p) that it has area $\lesssim_{\alpha,p} n^{p-j_0+1}$. Thus, the identity $[x_{j_0}^{n_1}, w] =_{\mathcal{P}} 1$ has area $\lesssim_{\alpha,p} n^{p-j_0+1}$ in $G^{\lrcorner}_{p,3}$.

[4] In the case $n_{k-1} < 0$ there are $2(p - j_0)|\lfloor n_{k-1} \rfloor|$ terms appearing.

This concludes the proof of the induction step of the First and Second commuting (k, j_0)-Lemmas for $G^{\lrcorner}_{p,3}$ with $k + j_0 \geqslant 5$. $\square$

Addendum 3.44. *The above proof translates verbatim to the group* $G_{p,3}$ for all pairs (k, j_0), *completing the proof of the First and Second commuting Lemmas for* $G_{p,3}$*; there is no need to treat the cases* $(1, 3)$ *and* $(2, 2)$ *separately for* $G_{p,3}$.

We now address the cases $(1, 3)$ and $(2, 2)$ of the Commuting Lemmas for $G^{\lrcorner}_{p,3}$. Since the proof of these cases shares some arguments with the above proof, rather than repeating all arguments, we explain the parts where the proofs differ and provide details whenever a new argument is needed.

Proof of case $(1, 3)$**.**

The First commuting $(1, 3)$-Lemma Lemma 3.27 for $G^{\lrcorner}_{p,3}$ follows from the fact that $\langle x_3, x_2, y_1, y_{p-1}\rangle$ and $\langle x_1, x_{p-1}, y_1, y_{p-1}\rangle$ generate subgroups of $G^{\lrcorner}_{p-1,3}$ isomorphic to the 5-Heisenberg group, so that the identities $[x_3^{n_1}, x_2^{m}] =_{\mathcal{P}} [y_1^{m}, y_{p-1}^{-n_1}] =_{\mathcal{P}} \Omega_2^{p-1}(m, -n_1)$ hold in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p |m|n$.

For the proof of the Second commuting $(1, 3)$-Lemma for $G^{\lrcorner}_{p,3}$ we proceed as in *Case (ii)*. We first use Lemma 3.42 for $(1, 3)$ to obtain

$$[x_3^{n_1}, w] =_{\mathcal{P}} \left(\prod_{i=1}^{\nu} \Omega^3_{l_i}(\underline{\eta}_i)^{\pm 1}\right)\left(\prod_{i=1}^{\mu} \Omega_2^{p-1}(\underline{\hat{n}_i})^{\pm 1}\right) \tag{3.11}$$

in $G^{\lrcorner}_{p,3}$ with area $\lesssim_{\alpha,p} n^{p-2}$. We then observe that the left side of (3.11) is null-homotopic in $G^{\lrcorner}_{p,3}$, since $G^{\lrcorner}_{p,3}$ is metabelian, and thus the same is true for the right side. The induction hypothesis (IH-p) for $G_{p-1,3} \hookrightarrow G^{\lrcorner}_{p,3}$ then implies that the right side has area $\lesssim_{\alpha,p} n^{p-2}$. Thus, the word $[x_3^{n_1}, w]$ is null-homotopic with area $\lesssim_{\alpha,p} n^{p-2}$ in $G^{\lrcorner}_{p,3}$. $\square$

Proof of case $(2, 2)$**.**

We start with the proof of the First commuting $(2, 2)$-Lemma. Arguing as in *Case (i)*, we first prove that the identities

$$[\Omega_2(\underline{n}), x_2^{m}] =_{\mathcal{P}} [y_1^{m}, y_{p-1}^{-n_2}]^{n_1} =_{\mathcal{P}} \Omega_2^{p-1}(m, -n_2)^{n_1} \tag{3.12}$$

hold in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p |m|n^{p-2} + n^{p-1}$. We first use the decomposition of $\Omega_2(\underline{n})$ given by Addendum 3.40 (as before we assume that $n_1 \geqslant 0$ with the case $n_1 < 0$ being analogous, but involving slightly more terms). We then follow the chain of identities in (3.10) applying the First commuting $(1, 3)$- and $(2, 3)$-Lemmas at most

$\lfloor n_1 \rfloor$ times. Each application of the First commuting $(1,3)$-Lemma produces an error term of the form $[y_1^m, y_{p-1}^{-n_2}]$. The first identity in (3.12) then follows by shifting each of these error terms to the right, with total area $\lesssim_p n \cdot (|m| n^{p-3} + n^{p-2})$. The second identity in (3.12) follows with area $\lesssim_p |m| n^2$ by arguing with 5-Heisenberg subgroups, similar as in Case $(1,3)$.

For the moreover-part of the First commuting $(2,2)$-Lemma it now suffices to show that the identity

$$\Omega_2^{p-1}(m, -n_2)^{n_1} =_{\mathcal{P}} \Omega_3^{p-2}(m, n_1, -n_2)$$

holds in $G_{p,3}^{\lrcorner}$ with area $\lesssim_p n^{p-1}$. For this observe that one can first argue as in the proof of Lemma 3.38, expanding the commutator $[x_1^{n_1}, x_{p-2}^{-n_2}]$, to obtain that the identity

$$\left[x_1^m, [x_1^{n_1}, x_{p-2}^{-n_2}]\right] =_{\mathcal{P}} \left[x_1^m, x_{p-1}^{-n_2}\right]^{n_1}$$

holds in $G_{4,3} \hookrightarrow G_{p,3}^{\lrcorner}$ with area $\lesssim_p n^4$, since $|m| \leqslant n$. Combining this with (3.12) and using again that $|m| \leqslant n$, the assertion follows.

The Second commuting $(2,2)$-Lemma is done as follows. For simplicity assume that $n_1 \geqslant 0$, the proof for $n_1 < 0$ is analogous but involves slightly more terms and thus uses more instances of the corresponding First and Second commuting Lemmas. First, from Addendum 3.40 we obtain the identity

$$\Omega_2(\underline{n}) =_{\mathcal{P}} [x_1^{\beta}, x_2^{n_2}] \cdot \left(\prod_{r=0}^{\lfloor n_1 \rfloor - 1} \Omega_1^3(n_2) \cdot \Omega_2^3(r, n_2)^{-1} \cdot [y_1^{\tilde{n}}, y_{p-1}^{\tilde{n}}] \right)$$

with area $\lesssim_p n^{p-1}$ in $G_{p,3}^{\lrcorner}$. Then, using the Second commuting $(1,3)$- and $(2,3)$-Lemmas for $G_{p,3}^{\lrcorner}$, and the fact that the y_i commute with the x_i we now proceed using the chain of identities in (3.10) to commute w with $\Omega_2(\underline{n})$ with area $\lesssim_{\alpha,p} n^{p-1}$ in $G_{p,3}^{\lrcorner}$. □

This completes the proof of the First and Second commuting (k,j)-Lemmas for $G_{p,3}^{\lrcorner}$. To close this section we record a consequence of Lemmas 3.33 and 3.38 that will only be used in the proof of the Cancelling k-Lemma 3.31 and in Section §3.5.

Corollary 3.45. *For all $\underline{n} \in \mathbb{R}^{p-1}$, there exists $m \in \mathbb{R}$ with $|m| \lesssim_p |\underline{n}|$ such that the identity*

$$\Omega_{p-1}(\underline{n})^{\pm 1} =_{\mathcal{P}} (\Omega_{p-2}^3(|\underline{n}|, \dots, |\underline{n}|))^m$$

holds in $G_{p,3}^{\lrcorner}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p |\underline{n}|^{p-1}$.

Proof. For the proof we refer the reader to [LIPT23, Corollary 6.24] where it uses Lemmas 3.33 and 3.38, and [LIPT23, Lemma 6.14] which is stated for $G_{p-1,p-1}$ in [LIPT23] but can be proved in precisely the same way for $G_{p-1,3}$. □

3.4.2.2 Proofs of the Main commuting Lemmas and of the Cancelling k-Lemmas

To deduce the Main commuting Lemmas from the First and Second commuting (k,j)-Lemmas for $G_{p,3}^{\lrcorner}$ (respectively for $G_{p,3}$) we require the Reduction Lemma 3.29. Its proof follows the one of [LIPT23], but requires some modifications; we thus include it here. We start with two auxiliary results.

Lemma 3.46. *Let $\underline{n} := (n_1, n_2, n_3) \in \mathbb{R}^3$. The identity*

$$\Omega_3^{p-2}(n_1, n_2, n_3) =_{\mathcal{P}} \Omega_{p-1}(n_1, n_2, 1, \ldots, 1, n_3)$$

holds in $G_{p,3}^{\lrcorner}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) with area $\lesssim_p |\underline{n}|^2$.

Proof. In $G_{p,3}^{\lrcorner}$ this is a straight-forward consequence of iteratively applying the identities $x_j^{n_3} =_{\mathcal{P}} [x_1, x_{j-1}^{n_3}]$ for $3 \leqslant j \leqslant p-2$ and the identity $\Omega_1^3(n_3) =_{\mathcal{P}} [y_1^{\tilde{n}_3}, y_3^{\tilde{n}_3}]^{\pm 1}\Omega_2(1, n_3)$, which hold with area $\lesssim_p |\underline{n}|^2$. The proof for $G_{p,3}$ is analogous, but simpler. □

Using Lemma 3.46 we can rewrite the identity in Lemma 3.42 for $(2,2)$ as follows.

Corollary 3.47. *Let $p \geqslant 5$, $n \geqslant 1$, $\alpha \geqslant 0$, and $\underline{n} \in \mathbb{R}^2$. If $u := u(x_1, x_2) \in \mathcal{F}[\alpha]$ with $|u| \leqslant n$, then there exists $M = O_{\alpha,p}(1)$ such that the identity*

$$\Omega_2(\underline{n})^{\pm 1} \cdot u =_{\mathcal{P}} u \cdot \prod_{i=1}^{M} \Omega_{\ell_i}(\underline{\tilde{n}}_i)^{\pm 1}$$

holds in $G_{p,3}^{\lrcorner}$ with area $\lesssim_{\alpha,p} n^{p-1}$ for suitable $2 \leqslant \ell_i \leqslant p-1$, and $\underline{\tilde{n}}_i \in \mathbb{R}^{\ell_i}$ with $|\underline{\tilde{n}}_i| \lesssim_{\alpha,p} n$.

Proof. Apply the identity from Lemma 3.46 at most $O_{\alpha,p}(1)$-times to the identity in Lemma 3.42 for $(2,2)$ to obtain the desired identity and the area estimate. Finally, for all $l_i \geqslant p$ we remove the Ω_{l_i}-words as explained in Addendum 3.39(3) with area $\lesssim_{\alpha,p} n^{p-1}$. □

Note that the above corollary is not needed for $G_{p,3}$ since Addendum 3.43 already gives us the required identity.

Proof of the Reduction Lemma 3.29. The proof is done by induction on 2α. The base of induction, $\alpha = 1$, is trivial with $L = 0$ and the product being empty, since if $w = x_1^{\beta_1} x_2^{\gamma_1}$ represents an element in the derived subgroup of $G^{\lrcorner}_{p,3}$, then $\beta_1 = \gamma_1 = 0$.

Suppose that the statement is true for some $2\alpha \geqslant 1$ and let $w := \prod_{i=1}^{\mu} x_1^{\beta_i} x_2^{\gamma_i} \in \mathcal{F}[2(\alpha+1)]$. Observe that we can assume $\mu = 2(\alpha + 1)$. The following identities hold in $G^{\lrcorner}_{p,3}$, the area estimates of which are explained below.

$$
\begin{aligned}
\prod_{i=1}^{\mu} x_1^{\beta_i} x_2^{\gamma_i} &\overset{(1)}{=}_F x_1^{\beta_1+\beta_2} x_2^{\gamma_1} \cdot [x_2^{\gamma_1}, x_1^{\beta_2}] \cdot x_2^{\gamma_2} \cdot \prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i} \\
&\overset{(2)}{=}_F x_1^{\beta_1+\beta_2} x_2^{\gamma_1+\gamma_2} \cdot [x_2^{\gamma_1}, x_1^{\beta_2}] \cdot \Big[[x_2^{\gamma_1}, x_1^{\beta_2}], x_2^{\gamma_2}\Big] \cdot \prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i} \\
&\overset{(3)}{=}_{\mathcal{P}} x_1^{\beta_1+\beta_2} x_2^{\gamma_1+\gamma_2} \cdot \Omega_2(\beta_2, \gamma_1)^{-1} \cdot \Omega_3^{p-2}(\underline{\hat{\gamma}}_1) \cdot \prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i} \\
&\overset{(4)}{=}_{\mathcal{P}} x_1^{\beta_1+\beta_2} x_2^{\gamma_1+\gamma_2} \cdot \Omega_2(\beta_2, \gamma_1)^{-1} \cdot \Omega_{p-1}(\underline{\tilde{\gamma}}_1) \cdot \prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i} \\
&\overset{(5)}{=}_{\mathcal{P}} x_1^{\beta_1+\beta_2} x_2^{\gamma_1+\gamma_2} \cdot \Omega_2(\beta_2, \gamma_1)^{-1} \cdot \left(\prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}\right) \cdot \prod_{i=1}^{\nu} \Omega_{s_i}(\underline{\eta}_i)^{\pm 1} \\
&\overset{(6)}{=}_{\mathcal{P}} x_1^{\beta_1+\beta_2} x_2^{\gamma_1+\gamma_2} \cdot \left(\prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}\right) \cdot \left(\prod_{i=1}^{N} \Omega_{l_i}(\underline{\tilde{n}}_i)^{\pm 1}\right) \cdot \left(\prod_{i=1}^{\nu} \Omega_{s_i}(\underline{\eta}_i)^{\pm 1}\right) \\
&\overset{(7)}{=}_{\mathcal{P}} x_1^{\beta_1+\beta_2} x_2^{\gamma_1+\gamma_2} \cdot \left(\prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}\right) \cdot \prod_{i=1}^{L} \Omega_{l_i}(\underline{\hat{\eta}}_i)^{\pm 1}
\end{aligned}
$$

(1) is a free identity so it has no area;

(2) is also a free identity;

(3) is a consequences of the First commuting $(2, 2)$-Lemma 3.27 for $G^{\lrcorner}_{p,3}$ so it has area $\lesssim_p n^{p-1}$ where $|\underline{\hat{\gamma}}_1| \lesssim_p n$;

(4) follows from applying Lemma 3.46, thus it has area $\lesssim_p n^2$ where $|\underline{\tilde{\gamma}}_1| \lesssim_p n$;

(5) is a consequence of Lemma 3.42, so it has area $\lesssim_p n^2$, where $|\underline{\eta}_i| \lesssim_{\alpha,p} n$, $\nu = O_{\alpha,p}(1)$, and $s_i \geqslant p$;

(6) follows from Corollary 3.47, so it has area $\lesssim_p n^{p-1}$, where $l_i \geqslant 2$, $|\underline{\tilde{n}}_i| \lesssim_{\alpha,p} n$, and $N = O_{\alpha,p}(1)$;

(7) follows by first removing all the Ω_{l_i}- and Ω_{s_i}-words with $l_i, s_i \geqslant p$ using Addendum 3.39(3). Since there are at most $O_{\alpha,p}(1)$ of them, this has total area $\lesssim_{\alpha,p} n^{p-1}$. We then merge the remaining Ω-words into a single product without any area.

The word $v(x_1, x_2) := x_1^{\beta_1+\beta_2} x_2^{\gamma_1+\gamma_2} \cdot \left(\prod_{i=3}^{\mu} x_1^{\beta_i} x_2^{\gamma_i}\right)$ has the same exponent sum for x_1 and x_2 as that of the word w, therefore it represents an element in the derived subgroup of $G^{\lrcorner}_{p,3}$ of length at most n and $v(x_1, x_2) \in \mathcal{F}[2\alpha]$. Hence, applying the induction hypothesis for 2α to the word $v(x_1, x_2)$ we obtain the desired result. □

Proof of the Main commuting Lemmas 3.23 and 3.26 If both w_1 and w_2 are powers of x_2 the statement is clear. Otherwise, we apply the Reduction Lemma 3.29 to w_1 to obtain the identity $w_1 =_{\mathcal{P}} \prod_{i=1}^{L} \Omega_{l_i}(\underline{\eta}_i)^{\pm 1}$ in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^{p-1}$ where $2 \leqslant l_i \leqslant p-1$. Thus, if w_2 represents an element of the derived subgroup of $G^{\lrcorner}_{p,3}$, then by applying the Second commuting $(l_i, 2)$-Lemma 3.28 for $G^{\lrcorner}_{p,3}$ we get the desired identity; if $w_2 := x_2^k$, then by applying the First commuting $(l_i, 2)$-Lemma 3.27 for $G^{\lrcorner}_{p,3}$ we get that the identity

$$[w_1, w_2] =_{\mathcal{P}} \prod_{i=1}^{D} \Omega_3^{p-2}(\underline{\eta}_i)^{\pm 1}$$

holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^{p-1}$. Moreover, D equals the number of l_i that are equal to 2, in particular $D \leqslant L = O_{\alpha,p}(1)$.

The same arguments apply verbatim to $G_{p,3}$, with the only observation that we use the corresponding First and Second commuting Lemmas for this group. □

Proof of the Cancelling k-Lemma 3.31 The proof of the Cancelling k-Lemma 3.31 follows the same strategy as in [LIPT23, Section 6.7]. Although in this work we consider the group $G^{\lrcorner}_{p,3}$, the main arguments of [LIPT23, Section 6.7] carry over without major modifications.

For the reader's convenience we present the general arguments of the proof and offer full details whenever our proof differs or requires specific arguments. Prioritizing readability and aiming to convey the main idea, we reduce technical details to a minimum. Specifically, we keep explanations short in the parts of the argument that involve standard manipulations of Ω-words that do not involve area considerations.

The proof of the Cancelling k-Lemma 3.31 is done by **descending induction on** k. For the base case $k = p-1$ suppose that for some positive integer M_{p-1} and $\underline{n}_{p-1,i} \in \mathbb{R}^{p-1}$ with $|\underline{n}_{p-1,i}| \leqslant n$, the word

$$w(x_1, x_2) := \prod_{i=1}^{M_{p-1}} \Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1}$$

is null-homotopic in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$). It follows from Corollary 3.45 that for all $1 \leqslant i \leqslant p-1$ there exist some $m_i \in \mathbb{R}$ with $|m_i| \lesssim_p |\underline{n}_{p-1,i}|$ such that the identity

$$\Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1} =_{\mathcal{P}} \left(\Omega^3_{p-2}(|\underline{n}_{p-1,i}|, \dots, |\underline{n}_{p-1,i}|)\right)^{m_i}$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p n^{p-1}$.

For all $1 \leqslant i \leqslant p-1$, the word $\Omega^3_{p-2}(|\underline{n}_{p-1,i}|, \dots, |\underline{n}_{p-1,i}|)$ represents an element in $G_{p-1,3} \hookrightarrow G_{p,3}$ (respectively $G^{\lrcorner}_{p,3}$). In particular, the word

$$\prod_{i=1}^{M_{p-1}} \left(\Omega^3_{p-2}(|\underline{n}_{p-1,i}|, \dots, |\underline{n}_{p-1,i}|)\right)^{m_i}$$

is null-homotopic in $G_{p-1,3}$. Thus, by merging suitable pairs of Ω^3_{p-2}-words into new shorter Ω^3_{p-2}-words, in $\lesssim M_{p-1}n$ steps we conclude from the induction hypothesis (IH-p) for $G_{p-1,3}$ that it has area $\lesssim_{M_{p-1},p} n^{p-1}$. It now follows that the null-homotopic word $w(x_1, x_2)$ has area $\lesssim_{M_{p-1},p} n^{p-1}$ in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$). This proves the base case.

For the **induction hypothesis**, we now assume that the statement of the Cancelling $(k+1)$-Lemma 3.31 holds. To prove the induction step we need some auxiliary results that we now present. We start with the Cutting in half k-Lemma Lemma 3.48, which is our version of [LIPT23, Lemma 6.10]; its proof requires a modification for $k = 2$ with respect to the proof in [LIPT23] which we explain in detail. Aside from this, the same arguments as in [LIPT23] can be applied without modification.

Lemma 3.48 (Cutting in half k-Lemma, cf. [LIPT23, Lemma 6.10])**.** *Let $2 \leqslant k \leqslant p-2$ and $\underline{n} \in \mathbb{R}^k$. The following identities hold in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$):*

$$\Omega_k(2\underline{n}) =_{\mathcal{P}} \Omega_k(\underline{n})^{2^k} \cdot w_k(\underline{n}) \quad \textit{and} \quad \Omega_k(2\underline{n}) =_{\mathcal{P}} w_k(\underline{n}) \cdot \Omega_k(\underline{n})^{2^k},$$

where $w_k(\underline{n}) =_{\mathcal{P}} \prod_{i=1}^{L} \Omega_{l_i}(\underline{\eta}_i)^{\pm 1}$ with $L = O_p(1)$, $l_i \geqslant k+1$, and $|\underline{\eta}_i| \lesssim_p |\underline{n}|$ for $1 \leqslant i \leqslant L$. Moreover, each of the identities has area $\lesssim_p |\underline{n}|^{p-1}$.

Proof. Like for [LIPT23, Lemma 6.10] the proof is by ascending induction in k. The case $G^{\lrcorner}_{p,3}$ requires a minor adjustment for $k = 2$, which we explain below. The remainder of the induction step is completely analogous to the one in [LIPT23], and we omit it. So assume $k = 2$. Then the following identities hold in $G^{\lrcorner}_{p,3}$

$$\Omega_2(2\underline{n}) := [x_1^{2n_1}, x_2^{2n_2}] \overset{(1)}{=}_F [x_1^{2n_1}, x_2^{n_2}] \cdot \left[x_1^{2n_1}, x_2^{n_2}\right]^{x_2^{n_2}}$$

$$\overset{(2)}{=}_{\mathcal{P}} \left[x_1^{2n_1}, x_2^{n_2}\right]^2 \cdot \prod_{i=1}^{D} \Omega_3^{p-2}(\underline{\eta}_i)^{\pm 1}$$

$$\overset{(3)}{=}_F \left([x_1^{n_1}, x_2^{n_2}]^{x_1^{n_1}} \cdot [x_1^{n_1}, x_2^{n_2}]\right)^2 \cdot \prod_{i=1}^{D} \Omega_3^{p-2}(\underline{\eta}_i)^{\pm 1}$$

$$\overset{(4)}{=}_F \left(\Omega_2(\underline{n}) \cdot [\Omega_2(\underline{n}), x_1^{n_1}] \cdot [x_1^{n_1}, x_2^{n_2}]\right)^2 \cdot \prod_{i=1}^{D} \Omega_3^{p-2}(\underline{\eta}_i)^{\pm 1}$$

$$\overset{(5)}{=}_{\mathcal{P}} \Omega_2(\underline{n})^4 \cdot \Omega_3(n_1, n_1, n_2)^{-2} \cdot \prod_{i=1}^{D} \Omega_3^{p-2}(\underline{\eta}_i)^{\pm 1}$$

$$\overset{(6)}{=}_{\mathcal{P}} \Omega_2(\underline{n})^4 \cdot \Omega_3(n_1, n_1, n_2)^{-2} \cdot \prod_{i=1}^{D} \Omega_{p-1}(\underline{\tilde{n}})^{\pm 1}.$$

(1) is just the free identity $[u, v \cdot w] =_F [u, w] \cdot [u, v]^w$, so it has no area;
(2) is a consequence of the Main commuting Lemma 3.26 for $G_{p,3}^{\lrcorner}$, where $|\underline{\eta}_i| \lesssim_p |\underline{n}|$ and $D = O_{\alpha,p}(1)$, so it has area $\lesssim_p n^{p-1}$;
(3) is an application of the free identity $[u \cdot v, w] =_F [u, w]^v \cdot [v, w]$, so it has no area;
(4) is a free identity, so it has no area;
(5) is a consequence of the Main commuting Lemma 3.26, hence it has area $\lesssim_p n^{p-1}$;
(6) follows from Lemma 3.46 applied $O_{\alpha,p}(1)$ times, thus it has area $\lesssim_p n^2$.

Setting $w_2 := \Omega_3(n_1, n_1, n_2)^{-2} \cdot \prod_{i=1}^{D} \Omega_{p-1}(\underline{\tilde{n}})^{\pm 1}$ completes the proof for $k = 2$.

After dealing with the case $k = 2$, detailed above, the induction step requires applying the Main commuting Lemma 3.26 for $G_{p,3}^{\lrcorner}$ (respectively 3.23 for $G_{p,3}$), but since the Ω-words involved are of the form $\Omega_l(\cdot)$ for $l \geqslant 3$ no error terms will be produced. Therefore, the remainder of the induction step is completely analogous to the one in [LIPT23]. We refer the reader to [LIPT23, Section 6.6] for details. □

We now present another result necessary for the induction step of the Cancelling k-Lemma Lemma 3.31.

Lemma 3.49. *Let $n \geqslant 1$. For $2 \leqslant k \leqslant p-2$ and $\underline{n} \in \mathbb{R}^k$ with $|\underline{n}| \leqslant n$ an identity of the form*

$$\Omega_k(\underline{n})^{\pm 1} =_{\mathcal{P}} x_{k+1}^{\beta} \cdot E_{p,k}(\underline{n})$$

holds in $G_{p,3}^{\lrcorner}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p n^{p-1}$, where $E_{p,k}(\underline{n})$ is equal to $\prod_{i=k+1}^{p-1} \Omega_i(\underline{m}_i)^{\pm 1}$ with $|\underline{m}_i| \lesssim_p n$. Moreover, $|\beta| \lesssim_p n^k$.

Proof. We treat the case for $\Omega_k(\underline{n})^{+1}$ the other case $\Omega_k(\underline{n})^{-1}$ is analogous. The proof is done by induction in $m := \lceil \log_2(\underline{n}) \rceil$. The base of induction, $m = 1$, is basically a direct consequence of the choice of relations (see Proposition 3.5). In particular, if $m = 1$ we have that $|\underline{n}| \leqslant 1$. It then follows that we can transform the word $\Omega_k(\underline{n})$ into the word $x_{k+1}^{\beta} E_{n,k}(\underline{n})$ as follows: we start with the innermost commutator in $\Omega_k(\underline{n})$ and repeatedly apply the relation involving $[x_1^a, x_i^b]$, for $a, b \in [-1, 1]$ (see Proposition 3.5), the free identity $[u, v \cdot w] =_F [u, w] \cdot [u, v]^w$ (see Lemma 3.15) and use the fact that $L_p^{\lrcorner}$ for all $p \geqslant 5$ (respectively L_p for all $p \geqslant 3$) is metabelian.

The induction step uses the following Claim 3.50, which in turn requires the Cutting in half k-Lemma 3.48.

Claim 3.50. There exists a constant $C = C(p)$ such that if the statement of Lemma 3.49 holds for $\underline{n}/2 \in \mathbb{R}^k$ satisfying $\lceil \log_2(|\underline{n}/2|) \rceil = m$ with $\beta = \beta_m$ and area $\lesssim_p \delta_m$, then Lemma 3.49 also holds for $\underline{n} \in \mathbb{R}^k$ satisfying $\lceil \log_2(|\underline{n}|) \rceil = m + 1$ with $\beta = \beta_{m+1} = 2^k \beta_m$ and area $\lesssim_p \delta_{m+1} \leqslant 2^k \delta_m + C 2^{m(p-1)}$.

We explain the proof of Claim 3.50 after explaining how this implies the induction step of Lemma 3.49. A simple calculation shows that

$$\beta_m \leqslant 2^{km} \beta_1 = O_p(|\underline{n}|^k).$$

Moreover, setting $v_i := 2^{-ki} \delta_i$ for $i \geqslant 1$ we get

$$\begin{aligned} v_{m+1} &= 2^{-k(m+1)} \delta_{m+1} \\ &\leqslant 2^{-km} \delta_m + C 2^{-k(m+1)+m(p-1)} \\ &\leqslant v_m + C 2^{m(p-1-k)}. \end{aligned}$$

Since $k \leqslant p - 2 < p - 1$ from these inequalities we get

$$v_m \leqslant v_1 + C \sum_{i=1}^{m-1} 2^{i(p-1-k)} = O_p(2^{m(p-1-k)}).$$

Therefore, from the definitions of v_m and m it follows that

$$\delta_m = O_p(2^{m(p-1)}) = O_p(|\underline{n}|^{p-1}).$$

All together we get

$$\delta_{m+1} = O_p(|\underline{n}|^{p-1}) \quad \text{and} \quad \beta_{m+1} = O_p(|\underline{n}|^k)$$

Therefore, Claim 3.50 implies that Lemma 3.49 holds with the desired area estimate and bound on $|\beta|$. This concludes the induction step of Lemma 3.49. It remains to prove Claim 3.50. ☐

Proof of Claim 3.50. Observe that for the proof of this claim we are assuming the induction hypothesis for m in the above proof of Lemma 3.49.

Let $\underline{n} \in \mathbb{R}^k$ and $m+1 = \lceil \log_2(|\underline{n}|) \rceil$. By the Cutting in half k-lemma 3.48 we get the identity

$$\Omega_k(\underline{n}) =_{\mathcal{P}} \Omega_k(\underline{n}/2)^{2^k} \cdot w_k(\underline{n}/2) \tag{3.13}$$

in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) for $w_k(\underline{n}/2) =_{\mathcal{P}} \prod_{i=1}^{L} \Omega_{l_i}(\underline{\eta}_i)^{\pm 1}$ where $L = O_p(1)$, $l_i \geqslant k+1$, and $|\underline{\eta}_i| \lesssim_p \underline{n}/2$ for each $1 \leqslant i \leqslant L$. Moreover, the area of the above identity is $\lesssim_p |\underline{n}|^{p-1}$.

The proof of Claim 3.50 has two main steps:

- For *Step 1* the idea is to apply the induction hypothesis for m, which is assumed in the proof of Lemma 3.49 where Claim 3.50 is needed, to successively extract the powers of the letter x_{k+1} from the word $\Omega_k(\underline{n}/2)^{2^k} \cdot w_k(\underline{n}/2)$ appearing on the right-hand side of (3.13). We do this in an ordered manner. First we apply the induction hypothesis for m to extract the power of x_{k+1} from the left-most Ω-word $\Omega_k(\underline{n}/2)$. Then, we apply the Main commuting Lemma 3.26 to send the created error term $E_{p,k}(\underline{n}/2)$ to the right. We repeat this 2^k times.
- *Step 2* consists of rearranging the error terms $E_{p,k}(\underline{n}/2)$, created in the previous step, and the word $w_k(\underline{n}/2)$. Note that both words consist of a product of Ω-words. To arrange these products we use the induction hypothesis for $k+1$ of the Cancelling k-Lemma 3.31, namely we apply the Cancelling $(k+1)$-Lemma.

An important point here is that, whenever we apply the Main commuting Lemma for $G^{\lrcorner}_{p,3}$ 3.26 we do *not* create new error terms since we are only commuting words representing elements in the derived subgroup of $G^{\lrcorner}_{p,3}$. We now proceed to explain *Step 1* and *Step 2* in further detail.

In *Step 1*, after the ith application of the induction hypothesis for m and the Main commuting k-Lemma 3.26 for $G^{\lrcorner}_{p,3}$ (respectively 3.23 for $G_{p,3}$), we obtain an identity of the form

$$\Omega_k(\underline{n}) =_{\mathcal{P}} x_{k+1}^{i\beta_m} \cdot \left(\Omega_k(\underline{n}/2)\right)^{2^k - i} \cdot \left(E_{p,k}(\underline{n}/2)\right)^{i} \cdot w_k(\underline{n}/2)$$

in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$). In the next application, we first obtain the identity

$$\Omega_k(\underline{n}) =_{\mathcal{P}} x_{k+1}^{(i+1)\beta_m} \cdot E_{p,k}(\underline{n}/2) \cdot \big(\Omega_k(\underline{n}/2)\big)^{2^k-(i+1)} \cdot \big(E_{p,k}(\underline{n}/2)\big)^{i} \cdot w_k(\underline{n}/2).$$

in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$). Recall, by the induction hypothesis for m in the proof of Lemma 3.49 we have that for $m = \lceil \log_2(\underline{n}/2) \rceil$ the statement of Lemma 3.49 holds for $\beta = \beta_m$ and area $\lesssim_p \delta_m$. Therefore, the above identity has area $\lesssim_p \delta_m$.

We now apply the Main commuting Lemma 3.26 for $G^{\lrcorner}_{p,3}$ (respectively 3.23 for $G_{p,3}$) to shuffle to the right the error term $E_{p,k}(\underline{n}/2)$ past the word

$$(\Omega_k(\underline{n}/2))^{2^k-(i+1)}.$$

We apply it at most $p{\cdot}2^k$ times. As mentioned above, both of these words represent elements in the derived subgroup of $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$) so this process does not create any new error terms. This gives us the identity

$$\Omega_k(\underline{n}) =_{\mathcal{P}} x_{k+1}^{(i+1)\beta_m} \cdot \big(\Omega_k(\underline{n}/2)\big)^{2^k-(i+1)} \cdot \big(E_{p,k}(\underline{n}/2)\big)^{i+1} \cdot w_k(\underline{n}/2)$$

in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) with area $\lesssim_p |\underline{n}|^{p-1}$.

All together, after having done the 2^k steps, we get that the identity

$$\Omega_k(\underline{n}) =_{\mathcal{P}} x_{k+1}^{2^k\beta_m} \cdot \big(E_{p,k}(\underline{n}/2)\big)^{2^k} \cdot w_k(\underline{n}/2) \tag{3.14}$$

holds in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) with area $\lesssim_p 2^k{\cdot}\delta_m + O_p(|\underline{n}|^{p-1})$. This finishes *Step 1*.

Now, for *Step 2*, we have to rearrange the Ω-words on the right-hand side of the above identity (3.14). More concretely, it remains to transform the word $\big(E_{p,k}(\underline{n}/2)\big)^{2^k} \cdot w_k(\underline{n}/2)$ into a single error term of the form $E_{p,k}(\underline{n})$.

We proceed as follows. Without area considerations, from Lemma 3.14 we have that the identity

$$\Omega_k(\underline{n}) =_{\mathcal{P}} x_{k+1}^{t_{k+1}} \dots x_p^{t_p} \tag{3.15}$$

holds in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) for $|t_i| \lesssim_p |\underline{n}|^{i-1}$ for each $k+1 \leqslant i \leqslant p$. Now, by modding out by the $(k+1)$-term of the lower central series of $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$), we see from (3.15) and (3.14) that $2^k\beta_m = t_{k+1}$ (the terms involving x_{k+1} are the only ones that survive in the quotient; since the word $\big(E_{p,k}(\underline{n}/2)\big)^{2^k} \cdot w_k(\underline{n}/2)$ contains only Ω-words of the form $\Omega_i(\cdot)$ for $i \geqslant k+1$).

Therefore, we obtain the identity

$$x_{k+2}^{t_{k+2}} \dots x_p^{t_p} =_{\mathcal{P}} \big(E_{p,k}(\underline{n}/2)\big)^{2^k} \cdot w_k(\underline{n}/2)$$

in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) without any area considerations.

Therefore, it follows from Lemma 3.16 that there exists $\underline{m}_i \in \mathbb{R}^i$ with $|\underline{m}_i| \lesssim_p |\underline{n}|$ such that the identity

$$\prod_{i=k+1}^{p-1} \Omega_i(\underline{m}_i)^{\pm 1} =_{\mathcal{P}} \big(E_{p,k}(\underline{n}/2)\big)^{2^k} \cdot w_k(\underline{n}/2)$$

holds in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$). Note that the word on the right-hand side of this last identity has $O_p(1)$ many Ω-words. The same is true for $\big(E_{p,k}(\underline{n}/2)\big)^{2^k}$ and $w_k(\underline{n}/2)$, they consist of $O_p(1)$ many terms of the form $\Omega_\ell(\underline{m})^{\pm 1}$ for $|\underline{m}| \lesssim_p |\underline{n}|$ and $\ell \geqslant k+1$. The Cancelling $(k+1)$-Lemma 3.31 for $G^{\lrcorner}_{p,3}$ (respectively 3.23 for $G_{p,3}$) applied to the null-homotopic word

$$w(x_1, x_2) := \big(E_{p,k}(\underline{n}/2)\big)^{2^k} \cdot w_k(\underline{n}/2) \cdot \Big(\prod_{i=k+1}^{p-1} \Omega_i(\underline{m}_i)^{\pm 1}\Big)^{-1}.$$

in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) tell us that $w(x_1, x_2)$ has area $\lesssim_p n^{p-1}$. Namely, the area involved in transforming the word $\big(E_{p,k}(\underline{n}/2)\big)^{2^k} \cdot w_k(\underline{n}/2)$ into a single error term $\prod_{i=k+1}^{p-1} \Omega_i(\underline{m}_i)^{\pm 1}$ is $\lesssim_p n^{p-1}$.

Finally, from *Step 1* and *Step 2*, by setting $\beta_{m+1} := 2^k \beta_m$, we obtain that the identity

$$\Omega_k(\underline{n})^{\pm 1} =_{\mathcal{P}} x_{k+1}^{\beta_{m+1}} \cdot E_{p,k}(\underline{n}).$$

holds in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) with area $\lesssim_p 2^k \delta_m + O_p(|\underline{n}|^{p-1})$ Since by definition we have $|\underline{n}| \leqslant 2^m$. This implies that the area is $\lesssim_p \delta_{m+1} = 2^k \delta_m + O_p(2^{m(p-1)})$. This finishes the proof of Claim 3.50. □

We can now conclude the proof of the **induction step** of the Cancelling k-Lemma for $G^{\lrcorner}_{p,3}$ 3.31 (respectively $G_{p,3}$ 3.23). Recall we are assuming that the Cancelling $(k+1)$-Lemma holds.

Let $M \geqslant 1$ and let

$$w(x_1, x_2) := \Big(\prod_{i=1}^{M_k} \Omega_k(\underline{n}_{k,i})^{\pm 1}\Big) \cdot \Big(\prod_{i=1}^{M_{k+1}} \Omega_{k+1}(\underline{n}_{k+1,i})^{\pm 1}\Big) \cdot \dots \cdot \Big(\prod_{i=1}^{M_{p-1}} \Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1}\Big) \tag{3.16}$$

be a null-homotopic word in $G^{\lrcorner}_{p,3}$ for $p \geqslant 5$ (respectively $G_{p,3}$ for $p \geqslant 3$) with $\underline{n}_{j,i} \in \mathbb{R}^j$, $|\underline{n}_{j,l}| \leqslant n$, for $1 \leqslant l \leqslant M_j$, $k \leqslant j \leqslant p-1$, and $M_j \leqslant M$.

The idea is to reduce to the case of the Cancelling $(k+1)$-Lemma 3.31 for $G^{\lrcorner}_{p,3}$ (respectively 3.23 $G_{p,3}$) by manipulating the first product $\prod_{i=1}^{M_k} \Omega_k(\underline{n}_{k,i})^{\pm 1}$ in (3.16) by means of iteratively applying Lemma 3.49. This will create error terms that we then need to rearrange using the Main commuting Lemma 3.26 for $G^{\lrcorner}_{p,3}$ (respectively Lemma 3.23 for $G_{p,3}$). By rearranging the terms we end up with a "product of products" of Ω-words which is ordered in an increasing manner as in (3.16) starting with Ω-words of the form $\Omega_{k+1}(\cdot)$.

Note that since $k \geqslant 2$, whenever we use the Main commuting Lemma 3.26 for $G^{\lrcorner}_{p,3}$ we will not be creating extra terms. This means that the proof presented in [LIPT23, p.51] carries over without needing to do extra modifications. For the reader's convenience we present the main arguments and refer the reader to [LIPT23, p.51] for the extra details.

As mentioned above the first step is to transform the first product appearing in (3.16), namely the word $\prod_{i=1}^{M_k} \Omega_k(\underline{n}_{k,i})^{\pm 1}$, by applying Lemma 3.49. We do this by starting with the left most $\Omega_k(\underline{n}_{k,1})$ and then shuffle to the right the error term $E_{p,k}(\underline{n}_{1,k})$ created by means of the Main commuting Lemma 3.26 for $G^{\lrcorner}_{p,3}$ (respectively 3.23 for $G_{p,3}$). We then proceed similarly with $\Omega_k(\underline{n}_{k,2})$ and move the error term $E_{p,k}(\underline{n}_{2,k})$. This process has $\lesssim_p M_k$ steps and finishes once we have extracted all the x_{k+1}'s and moved all the created error terms.

Thus, by applying Lemma 3.49 and the Main commuting Lemma 3.26 for $G^{\lrcorner}_{p,3}$ (respectively 3.23 for $G_{p,3}$) a total number of $\lesssim_p M_k$ times we obtain the identity

$$w(x_1, x_2) =_{\mathcal{P}} \Big(\prod_{i=1}^{M_k} x_{k+1}^{\beta_i}\Big) \cdot \Big(\prod_{i=1}^{M_k} E_{p,k}(\underline{n}_{i,k})\Big) \cdot \Big(\prod_{i=1}^{M_{k+1}} \Omega_{k+1}(\underline{n}_{k+1,i})^{\pm 1}\Big) \cdot \dots \cdot \Big(\prod_{i=1}^{M_{p-1}} \Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1}\Big).$$

in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p M_k n^{p-1}$.

Recall that for each $i \in \{1, \dots, M_k\}$ the error terms $E_{p,k}(\underline{n}_{i,k})$ are a product of at most p many Ω-words of the form $\Omega_l(\underline{m})^{\pm 1}$ for $|\underline{m}| \lesssim_p |\underline{n}_{i,k}| \lesssim_p n$ and $l \geqslant k+1 > 2$.

We now rearrange the error terms appearing in the above identity, namely the product $\prod_{i=1}^{M_k} E_{p,k}(\underline{n}_{i,k})$, to obtain a product of Ω-words arranged in increasing order (with respect to k). We do this by applying the Main commuting Lemma

3.26 for $G^{\lrcorner}_{p,3}$ (respectively 3.23 for $G_{p,3}$) a total number of $\lesssim_p p \cdot M_k \cdot M(p-1)$ times. Thus, we obtain the identity

$$w(x_1,x_2) =_{\mathcal{P}} \Big(\prod_{i=1}^{M_k} x_{k+1}^{\beta_i}\Big) \cdot \Big(\prod_{i=1}^{\widetilde{\mathbf{M}}_{k+1}} \Omega_{k+1}(\underline{n}_{k+1,i})^{\pm 1}\Big) \cdot \ldots \cdot \Big(\prod_{i=1}^{\widetilde{\mathbf{M}}_{p-1}} \Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1}\Big)$$

in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p Mn^{p-1}$, for suitable $\underline{n}_{l,i}$, where $\widetilde{\mathbf{M}}_l \lesssim_p M_l + M_k$ for $l \geqslant k+1$. Recall, we are assuming that the word $w(x_1,x_2)$ is null-homotopic. Thus, by modding out by the $(k+1)$-term of the lower central series of $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$), we get

$$\sum_{i=1}^{M_k} \beta_i = 0$$

Therefore, the identity

$$w(x_1,x_2) =_{\mathcal{P}} \Big(\prod_{i=1}^{\widetilde{\mathbf{M}}_{k+1}} \Omega_{k+1}(\underline{n}_{k+1,i})^{\pm 1}\Big) \cdot \ldots \cdot \Big(\prod_{i=1}^{\widetilde{\mathbf{M}}_{p-1}} \Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1}\Big)$$

holds in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$) with area $\lesssim_p Mn^{p-1}$. Finally, by the Cancelling $(k+1)$-Lemma 3.31, which we are assuming by hypothesis of induction for k, we get that the null-homotopic word on the right-hand side of the identity above has area $\lesssim_{p,2M} n^{p-1}$ in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$). Therefore, the null-homotopic word $w(x_1,x_2)$ has area $\lesssim_{p,2M} n^{p-1}$ in $G^{\lrcorner}_{p,3}$ for all $p \geqslant 5$ (respectively $G_{p,3}$ for all $p \geqslant 3$).

This finishes the proof of the induction step for the Cancelling k-Lemma 3.31.

Induction step from $p-1$ to p

We now have all the results required to finish the induction step from $p-1$ to p, therefore concluding the proof of Theorems 3.21 and 3.22.

Let us reiterate that for the proof of the induction step for $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$) we need to show that the area of null-homotopic words $w := w(x_1,x_2)$ in $G^{\lrcorner}_{p,3}$ (respectively $G_{p,3}$) is $\lesssim_p n^{p-1}$. By Proposition 3.18 it suffices to show that null-homotopic words $w(x_1,x_2) \in \mathcal{F}[\alpha]$ have area $\lesssim_{p,\alpha} n^{p-1}$ for every $\alpha > 0$. To show we need the Main commuting Lemma 3.26 for $G^{\lrcorner}_{p,3}$ (respectively the Main commuting Lemma 3.23 for $G_{p,3}$).

Proof of Theorems 3.21 and 3.22. The proof is analogous to the one presented in [LIPT23, Theorem 6.1, p.756]. We include it here for the readers convenience, since it is short and explains how the various technical results proved above fit together to yield the upper bound on the Dehn function.

One has to show that for all $\alpha \geqslant 1$ every null-homotopic word in $\mathcal{G}[\alpha]$ of length at most n has area $\lesssim_p n^{p-1}$.

Consider a null-homotopic word $w := w(x_1, x_2, y_1, y_{p-1}) \in \mathcal{G}[\alpha]$ of length $|w| \leqslant n$. Then, there are $u := u(x_1, x_2)$ and $v := v(y_1, y_{p-1})$ with $|u|, |v| \leqslant n$ such that the equality $w =_{\mathcal{P}} u \cdot v$ holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^2$. Since $\langle x_1, x_{p-1}, y_1, y_{p-1}\rangle$ generates a 5-Heisenberg subgroup of $G^{\lrcorner}_{p,3}$, the identity $v(y_1, y_{p-1}) =_{\mathcal{P}} v(x_1, x_{p-1})$ holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^2$.

Since the word $v(x_1, x_{p-1})$ represents a central element, there exists $\underline{\tilde{n}} \in \mathbb{R}^{p-1}$ with $|\underline{\tilde{n}}| \lesssim_p n$ such that $v(x_1, x_{p-1}) =_{\mathcal{P}} \Omega_{p-1}(\underline{\tilde{n}})$ holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^{p-2}$, and for α sufficiently large we have that $\Omega_{p-1}(\underline{\tilde{n}}) \in \mathcal{F}[\alpha]$.

Therefore, after possibly increasing α, the word $u \cdot \Omega_{p-1}(\underline{\widetilde{n}})$ is null-homotopic in $\mathcal{F}[\alpha]$ of length at most n, which allows us to apply the results of the previous sections as follows: first, the Reduction Lemma 3.29 implies that the identity

$$w =_{\mathcal{P}} u \cdot \Omega_{p-1}(\underline{\widetilde{n}}) =_{\mathcal{P}} \prod_{i=1}^{L} \Omega_{l_i}(\underline{\eta}_i)^{\pm 1} \tag{3.17}$$

holds in $G^{\lrcorner}_{p,3}$ with area $\lesssim_p n^{p-1}$, where $|\underline{\eta}_i| \lesssim_p n$ and $2 \leqslant l_i \leqslant p-1$.

We can then apply the Main commuting Lemma 3.26 to rearrange the Ω-words on the left-hand side of (3.17); note that all the Ω-words represent elements in the derived subgroup of $G^{\lrcorner}_{p,3}$, so they commute with area $\lesssim_{\alpha,p} n^{p-1}$. We thus get

$$w =_{\mathcal{P}} \left(\prod_{i=1}^{M_k} \Omega_k(\underline{n}_{k,i})^{\pm 1}\right) \left(\prod_{i=1}^{M_{k+1}} \Omega_{k+1}(\underline{n}_{k+1,i})^{\pm 1}\right) \cdots \left(\prod_{i=1}^{M_{p-1}} \Omega_{p-1}(\underline{n}_{p-1,i})^{\pm 1}\right) \tag{3.18}$$

Since w is a null-homotopic word, the word on the right hand side of (3.18) is also null-homotopic, therefore by the Cancelling 2-Lemma 3.31 we get that w has area $\lesssim_p n^{p-1}$ in $G^{\lrcorner}_{p,3}$. This finishes the proof of the induction step (IH-p) for p. □

3.5 From $G_{p,3}$ and $G^{\lrcorner}_{p,3}$ to the general case

We can now obtain upper bounds on Dehn functions for a larger class of central products using the results obtained for $G_{p,3}$ and $G^{\lrcorner}_{p,3}$. In particular, we prove Theorem A and Theorem B.

Theorem A. *Let $k > \ell \geqslant 2$ be integers. Let K be either the group L_{k+1} or $L^{\lrcorner}_{k+1}$. Let L be a simply connected nilpotent Lie group with one-dimensional centre of nilpotency class ℓ. Let $G = K \times_Z L$. Then $\delta_G(n) \preccurlyeq n^k$.*

Proof. We prove the statement for $K = L_{k+1}$ the argument for $K = L^{\lrcorner}_{k+1}$ is exactly the same. Let $\mathcal{P}$ be the presentation for G given by Proposition 3.5 (setting $p = k+1$) and Lemma 3.6. Let w be a word of length at most n in G. Then, the identity $w =_{\mathcal{P}} u(x_1, x_2)v_L$ holds in G with area $\lesssim_{k+1} n^2$ where v_L is a word in the generating set of L and $|u|, |v_L| \leqslant n$.

Since the centre Z is distorted in L with polynomial distortion of degree $\ell - 1$ and v_L represents a central element of G, there exists b with $|b| \lesssim_{k+1} n^{\ell-1}$ such that $v_L =_{\mathcal{P}} z^b$. Since $\ell - 1 \leqslant (k+1) - 2 = k-1$ there exists $\underline{\tilde{n}}$ with $|\underline{\tilde{n}}| \leqslant n$ such that we can rewrite v_L as $\Omega^3_{k-1}(\underline{\tilde{n}})$ in $L_k \times_Z L \hookrightarrow L_{k+1} \times_Z L$. It follows from [GHR03], that the identity $v_L =_{\mathcal{P}} \Omega^3_{k-1}(\underline{\tilde{n}})$ has area $\lesssim_{k+1} n^k$. Therefore, since $w =_{\mathcal{P}} u(x_1, x_2) \cdot \Omega^3_{k-1}(\underline{\tilde{n}})$ is a null-homotopic word in $L_{k+1} \times_Z L_3$ of length $\lesssim_{k+1} n$, it follows from Theorem 3.21 (respectively Theorem 3.22) that it has area $\lesssim_{k+1} n^k$. □

An immediate consequence of Theorem A is the following result that establishes Theorem B for $k > \ell$.

Corollary 3.52. *Let L be either L_q with $q \geqslant 3$ or $L^{\lrcorner}_q$ with $q \geqslant 5$, and let K be either L_p with $p > q$ or $L^{\lrcorner}_p$ with $p > \max\{q, 4\}$. Then the group $G := K \times_Z L$ has Dehn function $\delta_G(n) \preccurlyeq n^{p-1}$.*

We close this section with a proof of Theorem B for $k = \ell$.

Proposition 3.53. *Let each of K and L be either L_p with $p \geqslant 3$ or $L^{\lrcorner}_p$ with $p \geqslant 5$. Then the Dehn function of $G := K \times_Z L$ satisfies $\delta_G(n) \preccurlyeq n^{p-1}$.*

Proof. Let $\mathcal{P}$ be the presentation for G given by Lemma 3.6 and Proposition 3.5. The proof is done by ascending induction on p [5]. For the base case we have that by [All98] and [OS99] the Dehn function of $L_3 \times_Z L_3$ is quadratic and by [LIPT23] the Dehn function of $L_p \times_Z L_p$ is cubic if $p = 4$ and quartic if $p = 5$. Assume now that the statements hold for $p - 1 \geqslant 4$.

Let $n \geqslant 1$ be an integer. Let $w := w(x_1, x_2, y_1, y_2)$ be a null-homotopic word in G of length at most n. Using the fact that the y_i's commute with the x_i's we can rewrite w as $w_1(x_1, x_2)w_2(y_1, y_2)$ in G with area $\lesssim_p n^2$. Since $w_1 w_2$ is also

[5] Note that the proofs for $L^{\lrcorner}_p \times_Z L^{\lrcorner}_p$ and $L^{\lrcorner}_p \times_Z L_p$ only use the induction hypothesis for $G_{q,q}$ with $q < p$.

a null-homotopic word in G and $\langle x_1, x_2\rangle \cap \langle y_1, y_2\rangle = \langle z\rangle$, where $x_p = z = y_p$ is the generator of the centre of G, we get that w_1 and w_2 represent elements in the centre. Thus, there exists $d \in \mathbb{R}$ such that the identities $w_1(x_1, x_2) =_{\mathcal{P}} z^d$ and $w_2(y_1, y_2) =_{\mathcal{P}} z^{-d}$ hold in G.

Since the distortion of $\langle z\rangle$ in G is $\sim n^{p-1}$, we get $|d| \lesssim_p n^{p-1}$. It follows from Proposition 3.5 that the identities

$$z^d =_{\mathcal{P}} \Omega_{p-1}(\hat{n}) \quad \text{and} \quad z^d =_{\mathcal{P}} \widetilde{\Omega}_{p-1}(\hat{n})$$

hold in G for some $\hat{n} \in \mathbb{R}^{p-1}$ with $|\hat{n}| \lesssim_p n$. Therefore, the words $w_1(x_1, x_2) \cdot (\Omega_{p-1}(\hat{n}))^{-1}$ and $\widetilde{\Omega}_{p-1}(\hat{n}) \cdot w_2(y_1, y_2)$ are null-homotopic words in $G^{\lrcorner}_{p,3} \hookrightarrow G$ with area $\lesssim_p n^{p-1}$.

By Corollary 3.45 we can rewrite the Ω_{p-1}- and $\widetilde{\Omega}_{p-1}$-words as products of Ω^3_{p-2}- and $\widetilde{\Omega}^3_{p-2}$-words in $G_{p-1,p-1} \hookrightarrow G$. It then follows by applying the induction hypothesis on $p \lesssim_p n$ times to pairs of Ω^3_{p-2}- and $\widetilde{\Omega}^3_{p-2}$-words, that the identity $\Omega_{p-1}(\hat{n}) =_{\mathcal{P}} \widetilde{\Omega}_{p-1}(\hat{n})$ holds in G with area $\lesssim_p n^{p-1}$. Overall, the identities

$$\begin{aligned} w &=_{\mathcal{P}} w_1(x_1, x_2) w_2(y_1, z_2) \\ &\overset{\text{free}}{=} w_1(x_1, x_2) \cdot \Omega_{p-1}(\hat{n})^{-1} \widetilde{\Omega}_{p-1}(\hat{n}) \cdot w_2(y_1, y_2) \\ &=_{\mathcal{P}} 1 \end{aligned}$$

hold in G with area $\lesssim_p n^{p-1}$. □

Proof of Theorem B. This now follows directly from Corollary 3.52 and Proposition 3.53. □

3.6 Some words on the lower bounds

In this section, we briefly outline the results from [GMLIP23] on lower bounds for the Dehn function of Theorems A and B to give the reader an idea of the underlying arguments. For details, we refer the reader to [GMLIP23, Section 3].

A general strategy to obtain lower bounds on Dehn functions is to find a suitable family of null-homotopic words with area the desired lower bound, in some sense these words are thought of as “hard to fill” words, meaning that the number of relations needed is as high as possible, see for instance [BG96, Section 3]. In the realm of nilpotent groups this can sometimes be achieved by means of central extensions.

3.6.1 Lower bounds and central extensions

A common technique to obtain lower bounds for Dehn functions of a finitely generated nilpotent group Γ is to look at distorted central extensions of Γ. Recall that elements of $H^2(\Gamma, \mathbb{Z})$ are in one-to-one correspondence with the central extensions of Γ up to isomorphism of group extensions. More concretely, the idea lies in the following principle.

Recall that if H is a subgroup of a group G, where H and G are generated by the finite sets S and T, respectively, then there are two possible metrics we can define on H, namely d_S and d_T which are induced by the word lengths $|\cdot|_S$ and $|\cdot|_T$, respectively. The relation between these two metrics can be expressed numerically by means of the *distortion function* of H in G defined as

$$\Delta_H^G(n) = \max\{|h|_S \mid h \in H, |h|_T \leqslant n\}.$$

We then say that H is *undistorted* in G if $\Delta_H^G(n) \asymp n$. Else, we say that H is *distorted* in G and its distortion is the $\asymp$-class of Δ_H^G.

Lemma 3.54 ([You08, Proposition 4]). *Let Γ be a finitely generated torsion-free nilpotent group and let*

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

be a central extension of Γ. Let $d \geqslant 1$. If $\iota(\mathbb{Z}) \leq \widetilde{\Gamma}$ is n^d-distorted, then $n^d \preccurlyeq \delta_\Gamma(n)$.

Proof. For a proof we refer the reader to [You08, Proposition 4]. □

In fact, Lemma 3.54 computes the precise asymptotics of the so-called *centralized Dehn function* δ_G^{cent} as defined in [BMS93], where they also prove that $\delta_G^{\mathrm{cent}}(n) \preccurlyeq \delta_G(n)$. For a precise definition of the centralized Dehn function we refer the reader to [BMS93] or [You08, Section 3].

Example 3.55. Let $\Gamma \coloneqq \mathbb{Z}^2$ generated by $\{a, b\}$ and $\widetilde{\Gamma} \coloneqq H_3(\mathbb{Z})$ be the 3-dimensional Heisenberg group with generating set $\{x_1, x_2, z\}$. Let $\pi\colon H_3(\mathbb{Z}) \to \mathbb{Z}^2$ be the homomorphism defined by $\pi : (x_1, x_2, z) \mapsto (a, b)$. Then we have a central extension:

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\iota} H_3(\mathbb{Z}) \xrightarrow{\pi} \mathbb{Z}^2 \longrightarrow 1$$

where $\mathbb{Z} = \langle z \rangle$ is quadratically distorted in $H_3(\mathbb{Z})$. Therefore, by Lemma 3.54 we get $n^2 \preccurlyeq \delta_{\mathbb{Z}^2}(n)$.

The underlying idea is as follows: observe that for $\Omega_2(n,n) = [x_1^n, x_2^n]$ the identity $\Omega_2(n,n) = z^{n^2}$ holds in $H_3(\mathbb{Z})$. In particular, the word z^{n^2} has word length n^2 in $\mathbb{Z} = \langle z \rangle$. Observe that, if $\overline{\pi}\colon F_{\{x_1,x_2,z\}} \to F_{a,b}$ is a lift of π, then $\overline{pi}(\Omega_2(n,n))$ is equal to $[a^n, b^n]$ in $\mathbb{Z}^2$, so its area must be at least n^2.

Example 3.56. Let $\Gamma \coloneqq H_3(\mathbb{Z})$ be the 3-dimensional Heisenberg with generating set $\{x, y, z\}$ and let $\widetilde{\Gamma}$ be the mapping torus $\mathbb{Z}^3 \rtimes_\phi \mathbb{Z}$ for $\phi(x_1) = x_1x_2, \phi(x_2) = x_2x_3$ and $\phi(x_3) = x_3$. In particular, $\widetilde{\Gamma}$ admits the following finite presentation

$$\langle x_1, x_2, x_3, t \mid [x_i, x_j] = 1 \text{ for all } 1 \leqslant i, j \leqslant 3, \text{ and } x_i^t = x_{i+1} \text{ for all } 1 \leqslant i \leqslant 2 \rangle$$

and embeds as a lattice in the simply connected nilpotent Lie group L_4.

Consider the homomorphism defined by $\pi : t \mapsto x, x_1 \mapsto y$ and $x_2 \mapsto z$. We have the following central extension:

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^3 \rtimes_\phi \mathbb{Z} \xrightarrow{\pi} H_3(\mathbb{Z}) \longrightarrow 1$$

where $\mathbb{Z} = \langle z \rangle$ is n^3-distorted in $\widetilde{\Gamma}$ [Osi01]. Thus, by Lemma 3.54 we get $n^3 \preccurlyeq \delta_\gamma(n)$.

In [GMLIP23] we established a Lie group theoretic analogue of Lemma 3.54. We would like to emphasise that these ideas were already used in [LIPT23, §8]. In [GMLIP23] we mainly reframed their techniques in a more general framework.

Proposition 3.57 ([GMLIP23, Proposition 3.7])**.** *Let $k \geqslant 2$. Let G be a simply connected nilpotent Lie group of nilpotency class $k-1$. Assume that for some $\omega \in Z^2(G, \mathbf{R})$ the central extension*

$$1 \to \mathbf{R} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} G \to 1$$

has nilpotency class k. Then the Dehn function of G satisfies $n^k \preccurlyeq \delta_G(n)$.

We would like to emphasise that the ideas from Lemma 3.54 were already used in [LIPT23, §8]. In [GMLIP23] we reframed their techniques in a more flexible framework.

Remark 3.58. A novelty in our work in [GMLIP23] is that Proposition 3.57 applies to G possibly without a lattice. Moreover, Proposition 3.57 is of independent interest since together with [GMLIP23, Theorem B.1] it allows us to evaluate new Dehn functions, see [GMLIP23, Remark 6.1].

3.6.2 Lower bounds in central products

In [GMLIP23] we computed lower bounds for the Dehn functions of central products:

$$G \coloneqq K \times_Z H$$

where K is either L_p for $p \geqslant 3$ or $L_p^{\lrcorner}$ for $p \geqslant 5$, and H is a simply connected nilpotent group with a one-dimensional centre. For these groups we obtained the following result:

Proposition 3.59 ([GMLIP23, Proposition 3.2]). *Let G be as above. Then*

$$n^{p-1} \preccurlyeq \delta_G(n).$$

The idea in [GMLIP23, Section 3.B] to obtain a lower bound on the Dehn functions of the central product G is as follows:

- We find a family of null-homotopic words in K that have length n and area n^{p-1}. A first candidate for such a family consists of the image under the inclusion $K \hookrightarrow G$ of the words $\Omega_p(n, \dots, n)$. In fact, we consider the family of null-homotopic words

  $$w \coloneqq x_1^L \Omega_{p-1}(n, \dots, n) x_1^{-2L} \Omega_{p-1}(n, \dots, n)^{-1} x_1^L$$

  in G, where $\Omega_{p-1}(n, \dots, n)$ represents a central element in K.
- To estimate the area of such words we first need to consider them as loops. Second, integrate a well-chosen 1-form along them, by means of a version of Stoke's Theorem, see Lemma 3.60 below. Here the choice of L is relevant, see [LIPT23, p. 20]. The underlying idea for choosing this one-form is using central extensions of G. Moreover, the derivative of this one-form, restricted to certain domains, coincide with two-forms defining highly distorted central extensions, see [GMLIP23, p. 19]. These are the main steps involved in this second point:

  1. We first consider the cocycle ω associated to the central extension

     $$1 \to Z \to K \to K/Z \to 1,$$

     where $Z \cong \mathbb{R}$ is the centre, and consider a representative β_1 in $H^2(G, \mathbb{R})$ which is a 2-form, see [GMLIP23, p.17].

2. Second, we construct a loop Λ which is induced by the closed path $\overline{w}$ in G. The major part of the work is done to establish [GMLIP23, Lemma 3.5] which guarantees us that:

$$\int_{\Lambda} \beta_1 = 2n^{p-1}.$$

3. Finally, Step 2 allows us to apply a version of Stokes theorem [LIPT23, Proposition 8.2] to conclude

$$\mathrm{Area}_{\mathcal{P}}(w) \geqslant \frac{1}{C} n^{p-1},$$

where C is a positive constant which is made explicit by our version of Stoke's Theorem Lemma 3.60, see also [GMLIP23, p.20].

- There is a choice of $L > 0$, with L of order n, that allows us to conclude that the words from the first bullet point above have the required length and area.

We now recall the version of Stoke's theorem from [LIPT23, Proposition 8.2], which up to a reformulation (see [GMLIP23, Lemma 2.1]) can be stated as follows.

Lemma 3.60 ([LIPT23, Proposition 8.2])**.** *Let G be a simply connected Lie group and let $\mathcal{P} := \langle S \mid R \rangle$ be a compact presentation for G. Let $\alpha \in \Omega^1(TG)$ be a continuous 1-form on G. Assume that there exists a constant C such that $\left| \int_{g_* \overline{r}} \alpha \right| \leqslant C$ for all $r \in R$. Then for any null-homotopic word w over the presentation*

$$\mathrm{Area}_{\mathcal{P}}(w) \geqslant \frac{1}{C} \left| \int_{\overline{w}} \alpha \right|.$$

Remark 3.61. We finish this summary by recording here that Proposition 3.59 is the result that allows us to conclude that the upper bounds for the Dehn functions of the central products presented in this work are indeed sharp. Concretely, the Dehn functions of the groups in Theorems A and B are bounded below by n^k.

3.7 Proof of Theorem D

We start by recalling Theorem D.

Theorem D. *Let $k \geqslant 2$. Let $\underline{e}$ be the infimum of the exponents e such that there exists a $O(r^e)$-sublinear bilipschitz equivalence between $G = L_{k+1} \times_Z L_3$ and its asymptotic cone. Then*

$$\frac{1}{2k+2} \leqslant \underline{e} \leqslant \frac{2}{k} = e_G.$$

Also, recall from Section 2.3.2 and Section 2.4 that Carnot groups of central products of simply connected nilpotent groups arise as asymptotic cones.

Proof of Theorem D. Let $p \geqslant 3$. As in [LIPT23], it readily follows from our proof that $L_p \times_Z L_3$ admits (n^{p-1}, n) as a filling pair[6]. For this one checks that all prefix words of the transformations we apply represent group elements of distance $\lesssim n$ to the identity element in the Cayley graph for our chosen generating set. Thus, one argues as in [LIPT23, Section 9] that $L_p \times_Z L_3$ and its associated Carnot graded group are not $O(r^e)$-equivalent for $e \in [0, 1/(2p))$. On the other hand Cornulier shows in [Cor17, Proposition 6.13] that the two groups are $O(r^{2/(p-1)})$-bilipschitz equivalent. This shows that Cornulier's bound is optimal in the limit as the nilpotency class $p-1$ tends to $+\infty$. □

3.8 Proof of Corollary C

We begin by recalling Corollary C.

Corollary C. *Let K and L be simply connected nilpotent Lie groups of classes k and ℓ with $k > \ell$ satisfying the assumption in Theorem A or Theorem B. Then, the group $G = K \times_Z L$ is not quasi-isometric to its asymptotic cone.*

Let us point out that Corollary C applies to groups G obtained as central products of K and L, where the pair (K, L) should satisfy the assumption in one of Theorem A or Theorem B and have different nilpotency classes k and ℓ. In particular, K is always either the model filiform group L_{k+1} or the filiform group $L^{\lrcorner}_{k+1}$, while L can be either $L_{\ell+1}$, $L^{\lrcorner}_{\ell+1}$, or a general simply connected nilpotent Lie group of class ℓ which does not necessarily admit a lattice.

Proposition 3.62. *Let G, K and L be as above. Then*

$$\delta_G(n) \asymp n^k \quad \textit{whereas} \quad \delta_{\mathrm{gr}(G)}(n) \asymp n^{k+1}.$$

Proof. For the lower bounds see Remark 3.61 and [GMLIP23]. Regarding the upper bounds, the statement concerning G is precisely the conclusions of Theorems A and B. As for the statement concerning $\mathrm{gr}(G)$, we start by observing that since K and L have different nilpotency classes k and ℓ, and $k > \ell$,

$$\mathrm{gr}(G) \cong \mathrm{gr}(K) \times \mathrm{gr}(L/Z(L)).$$

[6] See [LIPT23, §3.3] for the definition of a filling pair we use here.

Now K is always either L_{k+1}, $L_{k+1}^{\lrcorner}$, in which case $\mathrm{gr}(K)$ is always L_{k+1}, or $L_{5,5}$ in which case $\mathrm{gr}(K)$ is $L_4 \times \mathbb{R}$. In every case, $\mathrm{gr}(K)$ has Dehn function n^{k+1} (for instance, by [LIPT23, Example 7.8] and the Gersten–Holt–Riley upper bound). □

Corollary C then directly follows from Proposition 3.62 and the quasi-isometry invariance of the Dehn function.

Chapter 4

Dehn functions of mapping tori of right-angled Artin groups

This chapter treats the Dehn functions of mapping tori of RAAGs, namely of groups of the form $A_L \rtimes_\phi \mathbb{Z}$ with A_L a RAAG. We start this chapter by providing a brief background to put into context the results obtained here.

Our main contribution in this chapter is then presented as follows. In Section 4.2 we present the proof of Theorem E. In Section 4.3 we give a precise statement of Theorem F and present a proof, see Theorem 4.7. Finally, in Section 4.4 we offer an alternative proof of Pueschel and Riley's result regarding the mapping tori of $\mathbb{Z}^2 * \mathbb{Z}$ in the case where the automorphism acts on $\mathbb{Z}^2$ trivially, see Theorem 4.10.

4.1 Background

Let us recall from the introduction what is known about the Dehn functions of mapping tori of RAAGs. Namely, groups of the form

$$A_L \rtimes_\phi \mathbb{Z}$$

where A_L is a RAAG and $\phi \in \mathrm{Aut}(A_L)$. Recall that the class of RAAGs extrapolates between free abelian and free groups. The Dehn functions of the mapping tori of the two "extremal cases" in the family of RAAGs were the first to be characterised. For $A_L = \mathbb{Z}^m$ Bridson and Pittet [BP94] obtained the upper bounds for the Dehn functions and later on Bridson and Gersten [BG96] obtained a matching lower bound. Their results can be summarized as follows:

Theorem 4.1 ([BG96, Main Theorem], [BP94, Theorem 5.1]). *Let $\phi \in \mathrm{Aut}(\mathbb{Z}^m) \cong GL(m, \mathbb{Z})$. The Dehn function of the mapping torus $M_\phi := \mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ satisfies*

$$\delta_{M_\phi}(n) \asymp n^2 gr_\phi(n).$$

Subsequently, for the RAAG $A_L = F_m$ Bridson and Groves [BG10] using the machinery of train-tracks obtained:

Theorem 4.2 ([BG10, Main Theorem]). *Let $\psi \in \mathrm{Aut}(F_m)$. The Dehn function of the mapping torus $M_\psi := F_m \rtimes_\psi \mathbb{Z}$ satisfies*

$$\delta_{M_\psi}(n) \preccurlyeq n^2.$$

An alternative proof of Theorem 4.2 can be found in [Gho23].

Since RAAGs extrapolate between free abelian groups and free groups, it is natural to pose the problem of studying Dehn functions of mapping tori of arbitrary RAAGs. Vogtmann posed this question (see [PR19, p.2]) motivating the work of Pueschel and Riley which we now summarise.

Besides the cases mentioned above, Pueschel and Riley [PR19] computed the Dehn functions for the cases where A_L is $F_2 \times \mathbb{Z}$, $F_k \times F_l$ for $k, l \geqslant 2$, and $\mathbb{Z}^2 * \mathbb{Z}$. Moreover, Pueschel in her thesis [Pue16, Chapter 5] presented some partial results to obtain the Dehn function of certain mapping tori of $\mathbb{Z}^m * \mathbb{Z}$ with $m \geqslant 2$. In particular, Pueschel classified the Dehn functions of the mapping tori of $\mathbb{Z}^3 * \mathbb{Z}$, and presented some arguments to obtain the Dehn function of certain mapping tori of $\mathbb{Z}^m * \mathbb{Z}$, namely those for which the automorphism restricted to $\mathbb{Z}^m$ consists of a matrix with a single Jordan block [Pue16, p.75].

The results in [Pue16, PR19], in contrast with Theorems 4.1 and 4.2, have a less compact formulation. This is because the range of possible Dehn functions for the mapping tori of a single RAAG they considered exhibits different behaviour related to the growth of the corresponding automorphism.

We finish this section by recalling that Lemma 2.67 allows us to choose suitable representatives of the outer class of an automorphism $\phi \in \mathrm{Aut}(A_L)$ which facilitates the task of obtaining the Dehn functions of the mapping tori treated in this part of the work.

4.2 Dehn functions of $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Phi \mathbb{Z}$

The main result of this section is the full characterisation of Dehn functions of the mapping tori of $\mathbb{Z}^2 * \mathbb{Z}^2$. Let us recall Theorem E from the introduction.

Theorem E. *For every* $\Psi \in \mathrm{Aut}(\mathbb{Z}^2 * \mathbb{Z}^2)$ *there exists* $\Phi \in \mathrm{Aut}(\mathbb{Z}^2 * \mathbb{Z}^2)$ *such that* $[\Phi] = [\Psi^2] \in \mathrm{Out}(\mathbb{Z}^2 * \mathbb{Z}^2)$ *and* Φ *restricted to the free factors* $\mathbb{Z}^2$ *induces* $\phi_1, \phi_2 \in \mathrm{Aut}(\mathbb{Z}^2)$. *In particular, the Dehn functions of the associated mapping tori* M_Ψ *and* M_Φ *are equivalent and* δ_{M_Φ} *satisfies the following. If there exists* $i \in \{1, 2\}$ *such that the Dehn function of* $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ *is exponential, then the Dehn function of* M_Φ *is exponential. Else, precisely one of the following holds:*

1. *There exists* $i \in \{1, 2\}$ *such that the Dehn function of* $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ *is cubic, in which case the Dehn function of* M_Φ *is cubic.*
2. *For every* $i \in \{1, 2\}$ *the Dehn function of* $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ *is quadratic, in which case the Dehn function* M_Φ *is quadratic.*

We start by observing that the statement regarding the automorphism is a direct consequence of Corollary 2.47. The fact that the corresponding mapping tori have $\asymp$-equivalent Dehn function follows from Lemma 2.67. From this combined with Theorem 4.1 it follows that the cases in Theorem E are mutually exclusive and exhaustive.

Thus, it remains to prove the precise Dehn functions. Their proof relies on a refined upper bound on the Dehn function of the mapping tori of $\mathbb{Z}^2$. Recall, that an upper bound for it was first obtained by [BP94]. In particular, their result states that if $\phi \in \mathrm{Aut}(\mathbb{Z}^m)$ has only eigenvalues of norm 1 and d is the size of the largest Jordan block, then the Dehn function of $M_\phi := \mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ satisfies

$$\delta_{M_\phi}(n) \asymp n^{d+1}.$$

We provide an alternative proof of this upper bound for the case $m = 2$, which has the advantage that it keeps track of the number of letters of the generating set of $\mathbb{Z}^2$, see Lemma 4.3.

4.2.1 Key Lemma

In this section we provide an alternative proof of the upper bound obtained by Bridson and Pittet [BP94] for the mapping tori of $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ for the case $m = 2$.

Lemma 4.3. *Suppose that* $\mathbb{Z}^2 = \langle X \rangle$. *Let* $\phi \in \mathrm{Aut}(\mathbb{Z}^2)$ *such that all its eigenvalues have norm* 1, $M_\phi = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, *and let* $\mathcal{P}$ *be the standard finite presentation for* M_ϕ *as HNN-extension (with generating set* $\{X, t\}$*). Suppose* d *is the size of the largest Jordan block of* ϕ. *If* $w := w(X, t)$ *is a null-homotopic word in* M_ϕ *of length* n, *then*

$$\mathrm{Area}_{\mathcal{P}}(w) \preccurlyeq \ell_X(w) \cdot n^d.$$

Proof. Let $X := \{x_1, x_2\}$ be a generating set for $\mathbb{Z}^2$. Let d be the size of the largest Jordan block of $\phi \in \mathrm{Aut}(\mathbb{Z}^2)$. When $d = 2$ (respectively $d = 1$) we use the following presentation of M_ϕ which presents it as a lattice in the model filiform group $L_3 = H_3(\mathbb{R})$ (respectively $\mathbb{R}^3$):

$$\mathcal{P}_d := \langle x_1, x_2, t \mid [x_1, x_2] = [x_2, t] = 1,\ t^{-1}x_1 t = x_1 x_2^{d-1} \rangle$$

Note that x_2 represents a central element in M_ϕ. For simplicity write $\mathcal{P} := \mathcal{P}_d$.

Let $w := w(x_1, x_1, t)$ be a null-homotopic word in M_ϕ of length n. To estimate the area of w we reduce it in an ordered manner that we now describe.

First, we shuffle all the x_2's appearing in w to the right end of w to obtain the identity

$$w(x_1, x_2, t) =_{\mathcal{P}} u(x_1, 1, t) x_2^{\epsilon_{x_2}(w)}$$

in M_ϕ. Since there are at most $\ell_X(w)$ many x_2's in w and we shuffled them past at most n other letters, the above identity has area $\lesssim_{\mathcal{P}} \ell_X(w) \cdot n$ in M_ϕ. Moreover, since x_2 represents a central element, this shuffling does not create any letter from X, therefore $|u| \leqslant |w|$.

Second, we reduce the word $u(x_1, 1, t)x_2^{\epsilon_{x_2}(w)}$. The idea is fairly simple: we shuffle the t's appearing in $u(x_1, 1, t)$ to the right end of it. We do this one by one. Step 1 consists of starting with the right-most $t^{\pm 1}$ appearing in $u(x_1, 1, t)$. This process might create x_2's which we then shuffle to the right end and gathered right of the $t^{\pm 1}$ we just shuffled. Again, this does not create any x_1's. After this shuffling we obtain a new word, of smaller length, for which we proceed as in Step 1. We proceed in this manner, each time applying Step 1 to the new word until all the t's (as well as the possible x_2's created) have been collected on the right to obtain the identity

$$u(x_1, 1, t)x_2^{\epsilon_{x_2}(w)} =_{\mathcal{P}} x_1^{\epsilon_{x_1}(w)} t^{\epsilon_t(w)} x_2^{\epsilon_{x_2}(w) + \beta_1 + \ldots + \beta_{\ell_t(w)}}.$$

Since the word w is finite, in each step we manipulate a word of smaller length. The total number of steps needed is $\ell_t(w)$.

We now explain in detail how this process can be formally carried out. We claim that for every $1 \leqslant k \leqslant \ell_t(w)$ there exist $\sigma_k \in \{\pm 1\}$, $\beta_k \in \mathbb{Z}$ with $|\beta_k| \leqslant \ell_X(w)^{d-1}$, and subwords $u_k(x_1, 1, t)$ and $v_k(x_1, 1, 1)$ of w with

$$|u_k(x_1, 1, t)|, |v_k(x_1, 1, 1)| \leqslant |w|$$

such that the identity

$$u(x_1,1,t)x_2^{\epsilon_{x_2}(w)} =_{\mathcal{P}} u_k(x_1,1,t)v_k(x_1,1,1)t^{\sigma_1+\ldots+\sigma_i}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_i}$$

holds in M_ϕ with area

$$\lesssim_{\mathcal{P}} \sum_{j=1}^{k}\left(|v_j(x_1,1,1)| + |\beta_j||v_j(x_1,1,1)| + |\beta_j|\cdot\left(\sum_{\nu=1}^{j}|\sigma_1+\ldots+\sigma_\nu|\right)\right).$$

For $k = \ell_t(w)$ we show that, in fact, $u_{\ell_t(w)}(x_1,1,t)$ contains no t's.

We prove this by induction on $\ell_t(w)$. For the base of induction, **Step 1**, if $\ell_t(w) = 1$, then there exist $\sigma_1 \in \{\pm 1\}$, and subwords $u_1(x_1,1,1)$ and $v_1(x_1,1,1)$ such that we can write

$$u(x_1,1,t) =_{\text{def}} u_1(x_1,1,1)t^{\sigma_1}v_1(x_1,1,1).$$

for subwords $u_1(x_1,1,1)$ and $v_1(x_1,1,1)$ of w whose length is bounded above by $|w|$. Therefore,

$$u(x_1,1,t)x_2^{\epsilon_{x_2}(w)} =_{\text{def}} u_1(x_1,1,1)t^{\sigma_1}v_1(x_1,1,1)x_2^{\epsilon_{x_2}(w)}.$$

We now shuffle t^{σ_1} to the right past the word $v_1(x_1,1,1)$ and collect all the x_2's that this might have produced, say β_1, with $x_2^{\epsilon_{x_2}(w)}$, using the fact that x_2 commutes with t. This corresponds to the following identities in M_ϕ:

$$\begin{aligned} u_1(x_1,1,1)t^{\sigma_1}v_1(x_1,1,1)x_2^{\epsilon_{x_2}(w)} &=_{\mathcal{P}} u_1(x_1,1,1)v_1(x_1,1,1)x_2^{\beta_1}t^{\sigma_1}x_2^{\epsilon_{x_2}(w)} \\ &=_{\mathcal{P}} u_1(x_1,1,1)v_1(x_1,1,1)t^{\sigma_1}x_2^{\epsilon_{x_2}(w)+\beta_1} \end{aligned}$$

where $\beta_1 \in \mathbb{Z}$ is such that $|\beta_1| \lesssim_{\mathcal{P}} |v_1(x_1,1,1)|^{d-1}$. Thus we get that the total area of the above identities is

$$\lesssim_{\mathcal{P}} |v_1(x_1,1,1)| + |\beta_1||v_1(x_1,1,1)| + |\beta_1||\sigma_1|.$$

Observe that if ϕ is the trivial automorphism, that is if $d = 1$, then $\beta_1 = 0$.

For the induction hypothesis assume that the claim holds for $k-1$. That is, assume that the identity

$$u(x_1,1,t)x_2^{\epsilon_{x_2}(w)} =_{\mathcal{P}} u_{k-1}(x_1,1,t)v_{k-1}(x_1,1,1)t^{\sigma_1+\ldots+\sigma_{k-1}}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_{k-1}} \tag{4.1}$$

holds in M_ϕ, where for every $j \in \{1, \ldots, k-1\}$ we have $\sigma_j \in \{\pm 1\}$ and

$$|\beta_j| \lesssim_{\mathcal{P}} |v_j(x_1,1,1)|^{d-1} \leqslant \ell_X(w)^{d-1};$$

so if ϕ is trivial, we have that $\beta_j = 0$ for every $j \in \{1, \ldots, k-1\}$. Moreover, we assume that this identity has total area

$$\lesssim_{\mathcal{P}} \sum_{j=1}^{k-1} \left(|v_j(x_1,1,1)| + |\beta_j||v_j(x_1,1,1)| + |\beta_j| \cdot \left(\sum_{\nu=1}^{j} |\sigma_1 + \ldots + \sigma_\nu| \right) \right)$$

For the induction step we proceed as follows. **From Step $\mathbf{k-1}$ to Step $\mathbf{k}$**: we now want to shuffle to the right the right-most $t^{\pm 1}$ in $u_k(x_1,1,t)v_k(x_1,1,1)$. For this write the word on the right-hand side of (4.1) as

$$u_k(x_1,1,t)t^{\sigma_k}v_k(x_1,1,t)t^{\sigma_1+\ldots+\sigma_{k-1}}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_{k-1}},$$

where $\sigma_k \in \{\pm 1\}$, and proceed as we did in the base of induction, Step 1, manipulating this word. Thus, we have the following identities in M_ϕ:

$$\begin{aligned} &u_k(x_1,1,t)t^{\sigma_k}v_k(x_1,1,t)t^{\sigma_1+\ldots+\sigma_{k-1}}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_{k-1}} \\ &\qquad \overset{(1)}{=}_{\mathcal{P}} u_k(x_1,1,t)v_k(x_1,1,t)x_2^{\beta_k}t^{\sigma_1+\ldots+\sigma_k}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_{k-1}} \\ &\qquad \overset{(2)}{=}_{\mathcal{P}} u_k(x_1,1,t)v_k(x_1,1,t)t^{\sigma_1+\ldots+\sigma_k}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_k} \end{aligned}$$

where $|\beta_k| \lesssim_{\mathcal{P}} |v_k(x_1,1,1)|^{d-1}$. The respective area estimates are as follows. The identity (1) has area $\lesssim_{\mathcal{P}} |v_{k+1}(x_1,1,1)| + |\beta_{k+1}||v_{k+1}(x_1,1,1)|$; while the identity (2) has area $\lesssim_{\mathcal{P}} |\beta_{k+1}| \cdot |\sigma_1 + \ldots + \sigma_{k+1}|$.

Therefore, all together we have that the identity

$$u(x_1,1,t)x_2^{\epsilon_{x_2}(w)} =_{\mathcal{P}} u_k(x_1,1,t)v_k(x_1,1,1)t^{\sigma_1+\ldots+\sigma_k}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_k}$$

holds in M_ϕ with total area

$$\lesssim_{\mathcal{P}} \sum_{j=1}^{k} \left(|v_j(x_1,1,1)| + |\beta_j||v_j(x_1,1,1)| + |\beta_j| \cdot \left(\sum_{\nu=1}^{j} |\sigma_1 + \ldots + \sigma_\nu| \right) \right).$$

This concludes the induction step and therefore the proof of the claim.

In particular, it follows that for $k = \ell_t(w)$ we have that the identities

$$\begin{aligned} u(x_1,1,t)x_2^{\epsilon_{x_2}(w)} &=_{\mathcal{P}} u_{\ell_t(w)}(x_1,1,1)v_{\ell_t(w)}(x_1,1,1)t^{\sigma_1+\ldots+\sigma_{\ell_t(w)}}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_{\ell_t(w)}} \\ &=_{\mathrm{def}} x_1^{\epsilon_{x_1}(w)}t^{\epsilon_t(w)}x_2^{\epsilon_{x_2}(w)+\beta_1\ldots+\beta_{\ell_t(w)}} \end{aligned}$$

hold in M_ϕ with area

$$\begin{aligned} &\lesssim_{\mathcal{P}} \sum_{j=1}^{\ell_t(w)} \left(|v_j(x_1,1,1)| + |\beta_j||v_j(x_1,1,1)| + |\beta_j| \cdot \left(\sum_{\nu=1}^{j} |\sigma_1 + \ldots + \sigma_\nu| \right) \right) \\ &\lesssim_{\mathcal{P}} \sum_{j=1}^{\ell_t(w)} \left(|v_j(x_1,1,1)| + |\beta_j||v_j(x_1,1,1)| + |\beta_j| \cdot \ell_t(w) \right). \end{aligned} \tag{4.2}$$

On one hand, if ϕ is trivial, that is if $d = 1$, we have that $\beta_j = 0$ for every $j \in \{1, \ldots, \ell_t(w)\}$. Therefore, the area estimate in (4.2) reduces to

$$\lesssim_{\mathcal{P}} \sum_{j=1}^{\ell_t(w)} |v_j(x_1,1,1)| \leqslant \ell_X(w)\ell_t(w) \leqslant \ell_X(w)n.$$

On the other hand, if ϕ is not trivial, that is if $d = 2$. Since $|\beta_j| \leqslant \ell_X(w)^{d-1}$, it follows that the area estimate from (4.2) is

$$\begin{aligned} &\lesssim_{\mathcal{P}} \sum_{j=1}^{\ell_t(w)} \left(\ell_X(w) + \ell_X(w)^d + \ell_X(w)^{d-1}\ell_t(w) \right) \\ &\lesssim_{\mathcal{P}} \ell_t(w)\ell_X(w) + \ell_t(w)\ell_X(w)^d + \ell_t(w)^2\ell_X(w)^{d-1} \\ &\lesssim_{\mathcal{P}} \ell_X(w) \cdot n^d. \end{aligned}$$

In both cases, we get that the identity

$$u(x_1,1,t)x_2^{\epsilon_{x_2}(w)} =_{\mathcal{P}} x_1^{\epsilon_{x_1}(w)} t^{\sigma_1+\ldots+\sigma_{\ell_t(w)}} x_2^{\epsilon_{x_2}(w)+\beta_1+\ldots+\beta_{\ell_t(w)}}$$

holds in M_ϕ with total area $\lesssim_{\mathcal{P}} \ell_X(w) \cdot n^d$. In consequence, the identity

$$w =_{\mathcal{P}} x_1^{\epsilon_{x_1}(w)} t^{\sigma_1+\ldots+\sigma_{\ell_t(w)}} x_2^{\epsilon_{x_2}(w)+\beta_1+\ldots+\beta_{\ell_t(w)}}$$

holds in M_ϕ with area $\lesssim_{\mathcal{P}} \ell_X(w) \cdot n^d + \ell_X(w) \cdot n$.

Finally, observe that the fact that w is a null-homotopic word in M_ϕ, implies that the word

$$x_1^{\epsilon_{x_1}(w)} t^{\sigma_1+\ldots+\sigma_{\ell_t(w)}} x_2^{\epsilon_{x_2}(w)+\beta_1+\ldots+\beta_{\ell_t(w)}}$$

is null-homotopic in M_ϕ. We claim that this null-homotopic word is freely equal to the trivial word in M_ϕ. Indeed, the corresponding word in the abelianization M_ϕ^{ab} has to be null-homotopic, therefore $\epsilon_{x_1}(w) = \sigma_1 + \ldots + \sigma_{\ell_t(w)} = 0$. From which the following free identity follows:

$$x_1^{\epsilon_{x_1}(w)} t^{\sigma_1+\ldots+\sigma_{\ell_t(w)}} x_2^{\epsilon_{x_2}(w)+\beta_1+\ldots+\beta_{\ell_t(w)}} \stackrel{\text{free}}{=} x_2^{\epsilon_{x_2}(w)+\beta_1+\ldots+\beta_{\ell_t(w)}}.$$

In particular, the word $x_2^{\epsilon_{x_2}(w)+\beta_1+\ldots+\beta_{\ell_t(w)}}$ is null-homotopic in M_ϕ. Thus, we also have $\epsilon_{x_2}(w)+\beta_1+\ldots+\beta_{\ell_t(w)}=0$ proving our claim.

Therefore, we have $\mathrm{Area}_{\mathcal{P}}(w) \lesssim_{\mathcal{P}} \ell_X(w)\cdot n^d + \ell_X(w)\cdot n$. This finishes the proof. □

Comments on the general case. It seems plausible to obtain Lemma 4.3 for general mapping tori $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$. We conjecture that the result holds in this generality.

Conjecture 5. Let $m \geqslant 2$. Suppose that $\langle X\rangle = \mathbb{Z}^m$. Let $\phi \in \mathrm{Aut}(\mathbb{Z}^m)$ such that all its eigenvalues λ are such that $|\lambda| = 1$. Suppose d is the size of the largest Jordan block of ϕ. If $w = w(X,t)$ is a null-homotopic word in $M_\phi := \mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ of length n, then

$$\mathrm{Area}_{\mathcal{P}}(w) \preccurlyeq n^d \cdot \ell_X(w).$$

With Lemma 4.3 at hand we now proceed with the proof of Theorem E.

4.2.2 Proof of Theorem E

With the aid of Lemma 4.3 we prove Theorem E, obtaining in this way a complete characterisation of the Dehn functions of mapping tori of the RAAG $\mathbb{Z}^2 * \mathbb{Z}^2$. For this we first prove the a preliminary result which holds for mapping tori of $\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$ in general.

Recall, that by Corollary 2.47, given an automorphism $\Psi \in \mathrm{Aut}(\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2})$ up to taking a power of it we can assume that it preserves the factors. Namely, there exists $\Phi \in \mathrm{Aut}(\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2})$ such that $[\Phi^2] = [\Psi] \in \mathrm{Out}(\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2})$ and $\Phi = \phi_1 * \phi_2$ with $\phi_i = \Phi|_{\mathbb{Z}^{m_i}} \in \mathrm{Aut}(\mathbb{Z}^{m_i})$. For the purpose of computing the Dehn function, working with Φ instead is not a loss of generality as guaranteed by Lemma 2.67.

Proposition 4.4. *Let $m_1, m_2 \geq 2$. Let $\Phi = \phi_1 * \phi_2 \in \mathrm{Aut}(\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2})$ and $M_\Phi := (\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}) \rtimes_\Phi \mathbb{Z}$. Suppose that the eigenvalues of $\phi_i \in \mathrm{Aut}(\mathbb{Z}^{m_i})$ have norm one. Let d_i be the size of the largest Jordan block of ϕ_i. Then, the Dehn function of M_Φ satisfies*

$$\delta_{M_\Phi}(n) \asymp \max\{n^{d_1+1}, n^{d_2+1}\}.$$

Proof. We start by proving the lower bound on the Dehn function. The lower bound follows from Lemma 2.66 by observing that the groups $\mathbb{Z}^{m_1}$ and $\mathbb{Z}^{m_2}$ quasi-isometrically embed in $\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$ and that the automorphism Φ preserves them, namely $\Phi|_{\mathbb{Z}^{m_1}} = \phi_1 \in \mathrm{Aut}(\mathbb{Z}^{m_1})$ and $\Phi|_{\mathbb{Z}^{m_2}} = \phi_2 \in \mathrm{Aut}(\mathbb{Z}^{m_2})$.

We now proceed with the proof of the upper bound. For this we first set up the presentation(s) that we use throughout the arguments. Let A (respectively B) be a finite generating set for $\mathbb{Z}^{m_1}$ (respectively m_2) and let $\langle t\rangle = \mathbb{Z}$. Denote by $G_1 \coloneqq \mathbb{Z}^{m_1} \rtimes_{\phi_1} \langle t\rangle$ and $G_2 \coloneqq \mathbb{Z}^{m_2} \rtimes_{\phi_2} \langle t\rangle$ equipped with the standard finite presentation $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, as HNN-extensions (see Definition 2.2). Let

$$\mathcal{P} \coloneqq \langle A, B, t \mid [a, a'] = [b, b'] = 1,\ a^t = \phi_1(a),\ b^t = \phi_2(b)\ \forall a, a' \in A, b, b' \in B\rangle.$$

Note that $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$.

Let $w \coloneqq w(A, B, t)$ be a reduced null-homotopic word in M_Φ of length n. Moreover, assume without loss of generality that w is not freely trivial. Observe that, for some integer $s \geq 1$, we can write w in the following, not necessarily unique, way

$$w \coloneqq \prod_{i=1}^{s} u_i(A, 1, t)v_i(1, B, t). \tag{4.3}$$

Note that the words $u_1(A, 1, t)$ and $v_1(1, B, t)$ may be trivial, and that for every $i \in \{2, \ldots, s\}$ (respectively $i \in \{1, \ldots, s-1\}$) the word $u_i(A, 1, t)$ (respectively $v_i(1, B, t)$) contains at least one letter from A (respectively from B) and may not contain t's.

By **induction on s** we show that

$$\operatorname{Area}_{\mathcal{P}}(w) \lesssim_{\mathcal{P}} \left(\sum_{j=1}^{s} \ell_A(u_i(A, 1, t))\right) \cdot n^{d_1} + \left(\sum_{j=1}^{s} \ell_B(v_i(1, B, t))\right) \cdot n^{d_2}.$$

For the **base of induction**, $s = 1$, we have

$$w =_{\text{def}} u_1(A, 1, t)v_1(1, B, t).$$

Since w is not freely trivial, we have that at least one of the following holds:

(i) $u_1 \coloneqq u_1(A, 1, t)$ is not freely trivial.

(ii) $v_1 \coloneqq v_1(1, B, t)$ is not freely trivial.

Suppose without loss of generality that (i) holds. Then we proceed (without area considerations) as follows:

- Step 1: we shuffle all the t's to the left to get

$$u_1 =_{\mathcal{P}} t^{\epsilon_t(u_1)}\overline{u}_1(A, 1, 1)$$
$$v_1 =_{\mathcal{P}} t^{\epsilon_t(v_1)}\overline{v}_1(1, B, 1).$$

- Step 2: we shuffle $t^{\epsilon_t(v_1)}$ to the left. Note that the effect of shuffling one $t^{\pm 1}$ past a letter of A corresponds to apply $\phi_1^{\pm 1}$. After this we to obtain the word

$$u_1'(A,1,1) := \phi_1^{\epsilon_t(v_1)}(\overline{u}_1(A,1,1)),$$

Let us denote $v_1'(1,B,1) := \overline{v}_1(1,B,1)$.

After performing the above two steps we obtain the word

$$w =_{\mathcal{P}} t^{\epsilon_t(u_1)+\epsilon_t(v_1)} u_1'(A,1,1) v_1'(1,B,1).$$

Since w is null-homotopic, it follows that $\epsilon_t(u_1)+\epsilon_t(v_1)=0$ and that the word

$$u_1'(A,1,1)v_1'(1,B,1)$$

represents the neutral element in $\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$. Let g_1 (respectively h_1) be the element in $\mathbb{Z}^{m_1}$ (respectively $\mathbb{Z}^{m_2}$) represented by $u_1'(A,1,1)$ (respectively $v_1'(1,B,1)$). The element $g_1 h_1 \in \mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$ is the neutral element. Therefore, from the normal form for free products of groups (see for instance [LS01, p.175]) we have that the following holds:

(1) g_1 is the neutral element in $\mathbb{Z}^{m_1}$.
(2) h_1 is the neutral element in $\mathbb{Z}^{m_2}$.

Assume without loss of generality that (1) holds. This implies that the word $u_1'(A,1,1)$ is null-homotopic in $G_1 \hookrightarrow M_\Phi$. Since $\phi_1 \in \mathrm{Aut}(\mathbb{Z}^{m_1})$ it follows, together with the second step above, that $\overline{u}_1(A,1,1)$ is null-homotopic in $G_1 \leqslant M_\Phi$. From this, together with the first step above, it then follows that

$$u_1 =_{\mathcal{P}} t^{\epsilon_t(u_1)}$$

holds in $G_1 \hookrightarrow M_\phi$. It then follows from Lemma 4.3 (applied to G_1) that this last identity has area

$$\lesssim_{\mathcal{P}} \ell_A(u_1) \cdot |u_1|^{d_1} \leqslant \ell_A(u_1) \cdot n^{d_1},$$

where we use that the presentaiton for G_1 is embedded in the one for M_Φ.

Therefore, the identity

$$w =_{\mathcal{P}} t^{\epsilon_t(u_1)} v_1(1,B,t)$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} \ell_A(u_1) \cdot n^{d_1}$. Since w is a null-homotopic word, it follows that $t^{\epsilon_t(u_1)} v_1(1,B,t)$ is a null-homotopic word in $G_2 \leqslant M_\Phi$. Thus, by Lemma 4.3

(applied to G_2), we get that it has area $\lesssim_{\mathcal{P}} \ell_B(v_1) \cdot n^{d_2}$. All together we get

$$\mathrm{Area}_{\mathcal{P}}(w) \lesssim_{\mathcal{P}} \ell_A(u_1) \cdot n^{d_1} + \ell_B(v_1) \cdot n^{d_2}.$$

Now, as **induction hypothesis**, suppose that the statement is true for $s-1$ with $s > 1$. We now prove that the statement holds for s. Write, as in (4.3),

$$w \coloneqq \prod_{i=1}^{s} u_i(A,1,t) v_i(1,B,t).$$

We claim the following:

Claim 4.5 (Pinching word)**.** There exists $i \in \{1, \ldots, s\}$ such that at least one of the following holds:

(i) $u_i \coloneqq u_i(A,1,t) =_{\mathcal{P}} t^{\epsilon_t(u_i)}$ in $G_1 \leqslant M_\Phi$.
(ii) $v_i \coloneqq v_i(1,B,t) =_{\mathcal{P}} t^{\epsilon_t(v_i)}$ in $G_2 \leqslant M_\Phi$.

The reader should think of the word whose existence is guaranteed by Claim 4.5 as a word in w where reduction should start, that is, as a "pinching word" for w. Before proving this claim let us explain how the induction step, and consequently the statement of Proposition 4.4, follows from it.

Observe that if Condition (i) (respectively Condition (ii)) from Claim 4.5 holds, then it follows from Lemma 4.3 for G_1 (respectively G_2) that the identity

$$u_i(A,1,t) =_{\mathcal{P}} t^{\epsilon_t(u_i)} \quad \text{(respectively } v_i(1,B,t) =_{\mathcal{P}} t^{\epsilon_t(v_i)}\text{)}$$

has area $\lesssim_{\mathcal{P}} \ell_A(u_i(A,1,t)) \cdot n^{d_1}$ (respectively $\lesssim_{\mathcal{P}} \ell_B(v_i(1,B,t) \cdot n^{d_2})$. Moreover, from Claim 4.5 we deduce that the identity

$$w =_{\mathcal{P}} \begin{cases} u_1(A,1,t) \ldots v_{i-1}(1,B,t) t^{\epsilon_t(u_i)} v_i(1,B,t) \ldots u_s(A,1,t), & \text{in Case (i)}, \\ u_1(A,1,t) \ldots u_i(A,1,t) t^{\epsilon_t(v_i)} u_{i+1}(A,1,t) \ldots u_s(A,1,t), & \text{in Case (ii)}, \end{cases}$$

holds in $G_1 \leqslant M_\Phi$ (respectively $G_2 \leqslant M_\Phi$) with area

$$\lesssim_{\mathcal{P}} \begin{cases} \ell_A(u_i(A,1,t)) \cdot n^{d_1}, & \text{in Case (i)}, \\ \ell_B(v_i(1,B,t) \cdot n^{d_2}, & \text{in Case (ii)}. \end{cases} \tag{4.4}$$

We can assume, without loss of generality, that condition (i) holds and write:

$$\widetilde{u}_j(A,1,t) \coloneqq \begin{cases} u_j(A,1,t), & \text{for } j \in \{1, \ldots, i-1\} \\ u_{j+1}(A,1,t), & \text{for } j \in \{i, \ldots, s-1\} \end{cases}$$

and

$$\widetilde{v}_j(1,B,t) \coloneqq \begin{cases} v_j(1,B,t), & \text{for } j \in \{1,\dots,i-2\}, \\ v_{i-1}(1,B,t)t^{\epsilon_t(u_i)}v_i(1,B,1), & \text{for } j = i-1, \\ v_{j+1}(1,B,t), & \text{for } j \in \{i,\dots,s-1\} \end{cases}$$

Thus, we have

$$w =_{\mathcal{P}} w^{(1)} \coloneqq \prod_{j=1}^{s-1} \widetilde{u}_j(A,1,t)\widetilde{v}_j(1,B,t)$$

Note that since $u_i(A,1,t)$ has at least one letter from A, we have that $|w^{(1)}| \leq |w|$.

By the induction hypothesis we get that

$$\mathrm{Area}_{\mathcal{P}}(w^{(1)}) \lesssim_{\mathcal{P}} \left(\sum_{j=1}^{s-1} \ell_A(\widetilde{u}_j)\right) \cdot n^{d_1} + \left(\sum_{j=1}^{s-1} \ell_B(\widetilde{v}_j)\right) \cdot n^{d_2}. \tag{4.5}$$

By definition of the $\widetilde{v}_j$'s and $\widetilde{u}_j$'s we have that the area estimate in (4.5) is

$$\lesssim_{\mathcal{P}} \left(\sum_{j=1}^{i-1} \ell_A(u_j) + \sum_{j=i+1}^{s} \ell_A(u_j)\right) \cdot n^{d_1}$$
$$+ \left(\sum_{j=1}^{i-2} \ell_B(v_j) + \ell_B(v_{i-1}) + \ell_B(v_i) + \sum_{j=i+1}^{s} \ell_B(v_j)\right) \cdot n^{d_2}.$$

We would like to stress out that in this step Lemma 4.3 is crucial, since the "pinching point" might happen in the subword that absorbed the t's but the nice upper bound from Lemma 4.3 allows us to deal with this without counting the length of the previous subword.

From, (4.4), the identity $w =_{\mathcal{P}} w^{(1)}$ has area $\lesssim_{\mathcal{P}} \ell_A(u_i) \cdot n^{d_1}$. Therefore, we have that

$$\begin{aligned} \mathrm{Area}_{\mathcal{P}}(w) &\lesssim_{\mathcal{P}} \left(\sum_{j=1}^{s} \ell_A(u_i)\right) \cdot n^{d_1} + \left(\sum_{j=1}^{s} \ell_B(v_j)\right) \cdot n^{d_2} \\ &\lesssim_{\mathcal{P}} \ell_{A\cup B}(w) \cdot \left(n^{d_1} + n^{d_2}\right) \\ &\lesssim_{\mathcal{P}} n^{d_1+} + n^{d_2+1} \end{aligned}$$

Finally, observe that $n^{d_1+1}+n^{d_2+1} \asymp \max\{n^{d_1+1}, n^{d_2+1}\}$.This concludes the proof of the induction step and therefore of the desired upper bound on the Dehn function of M_Φ.

From the upper bound and the lower bounds obtained for δ_{M_Φ}, we have

$$\delta_{M_\Phi}(n) \asymp \max\{n^{d_1+1}, n^{d_2+1}\}. \tag{4.6}$$

It remains to prove Claim 4.5, which we do now. □

Proof of Claim 4.5. To see this we do the following manipulations without area considerations:

- Step 1: For all $i \in \{1, \ldots, s\}$ we shuffle to the left all the t's in $u_i := u_i(A, 1, t)$ (respectively $v_i := v_i(1, B, t)$) to obtain:

$$u_i(A, 1, t) =_{\mathcal{P}} t^{\epsilon_t(u_i)} \overline{u}_i(A, 1, t)$$
$$v_i(1, B, t) =_{\mathcal{P}} t^{\epsilon_t(v_i)} \overline{v}_i(1, B, t).$$

- Step 2: For all $i \in \{1, \ldots, s\}$ we shuffle to the left all the $t^{\epsilon_t(u_i)}$ and $t^{\epsilon_t(v_i)}$ from Step 1 to obtain the following words for every $i \in \{1, \ldots, s\}$:

$$u_i'(A, 1, 1) := \phi_1^{\sum_{j=i}^{s} \epsilon_t(v_j) + \sum_{j=i+1}^{s} \epsilon_t(u_j)} \big(\overline{u}_i(A, 1, 1)\big)$$
$$v_i'(1, B, 1) := \phi_2^{\sum_{j=i+1}^{s} \epsilon_t(v_j) + \epsilon_t(u_j)} \big(\overline{v}_i(1, B, 1)\big)$$

After performing both steps above we obtain the identity

$$w =_{\mathcal{P}} t^{\sum_{i=1}^{s} \epsilon_t(u_i) + \epsilon_t(v_i)} \cdot \prod_{i=1}^{s} u_i'(A, 1, 1) v_i'(1, B, 1).$$

Since the word w is null-homotopic it follows that

$$\sum_{i=1}^{s} \epsilon_t(u_i) + \epsilon_t(v_i) = 0$$

and that the word

$$\prod_{i=1}^{s} u_i'(A, 1, 1) v_i'(1, B, 1)$$

is null-homotopic in $\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$.

Now for every $i \in \{1, \ldots, s\}$ let $g_i \in \mathbb{Z}^{m_1}$ (respectively $h_i \in \mathbb{Z}^{m_2}$) be the element that is represented by $u_i'(A, 1, 1)$ (respectively $v_i'(1, B, 1)$). Therefore the element $g_1 h_1 \ldots g_s h_s \in \mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$ is equal to the neutral element. It then follows from the normal forms in free products that there exists $i \in \{1, \ldots, s\}$ such that at least one of the following conditions hold:

(1) g_i is the neutral element in $\mathbb{Z}^{m_1}$.
(2) h_i is the neutral element in $\mathbb{Z}^{m_2}$.

We can assume without loss of generality that condition (1) holds. This implies that $u_i'(A,1,1)$ is a null-homotopic word in $G_1 \leqslant M_\Phi$. By definition of u_i', see Step 2 above, it then follows that the identity

$$\phi_1^{\sum_{j=i}^{s}\epsilon_t(v_j)+\sum_{j=i+1}^{s}\epsilon_t(u_j)}\big(\overline{u}_i(A,1,1)\big) =_{\mathcal{P}} 1$$

holds in $G_1 \leqslant M_\Phi$. Since $\phi_1 \in \mathrm{Aut}(\mathbb{Z}^{m_1})$, this implies that $\overline{u}_i(A,1,1)$ is a null-homotopic word in $G_1 \leqslant M_\Phi$. Therefore, from step 1 above we get

$$u_i(A,1,t) =_{\mathcal{P}} t^{\epsilon_t(u_i)}$$

in $G_1 \leq M_\Phi$. Note that if Condition (2) above holds, then by reasoning in a similar way, we obtain the identity $v_i(1,B,t) =_{\mathcal{P}} t^{\epsilon_t(v_i)}$. This proofs the claim. □

Proof of Theorem E. To conclude the proof of Theorem E first observe that under the assumptions of Theorem E, we have the following two mutually exclusive possibilities:

(A) There exists $i \in \{1,2\}$ such that ϕ_i has an eigenvalue λ_i such that $|\lambda_i| \neq 1$.
(B) All the eigenvalues of ϕ_1 and ϕ_2 have norm one.

Therefore, if (A) holds, namely if ϕ_i has an eigenvalue λ_i such that $|\lambda_i| \neq 1$, then the Dehn function of $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ is exponential. It then follows from Lemma 2.66 and Corollary 2.65 that the Dehn function of the mapping torus $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Phi \mathbb{Z}$ is also exponential.

Finally, if (B) holds, then Proposition 4.4 together with the above characterisation give us the Dehn functions of the mapping tori of $(\mathbb{Z}^2 * \mathbb{Z}^2)$: For every $i \in \{1,2\}$ exactly one of the following holds:

1. $d_1 = d_2 = 1$. In which case the Dehn functions of $\mathbb{Z}^2 \rtimes_{\phi_1} \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_{\phi_2} \mathbb{Z}$ are both quadratic and from Proposition 4.4 it follows that the Dehn function of $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Phi \mathbb{Z}$ is quadratic.
2. $d_1 = 2$ and $d_2 = 1$. In which case, the Dehn function of $\mathbb{Z}^2 \rtimes_{\phi_1} \mathbb{Z}$ is cubic and by Proposition 4.4 the Dehn function of $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Phi \mathbb{Z}$ is cubic. Similarly, for $d_1 = 1$ and $d_2 = 2$.
3. $d_1 = d_2 = 2$. In which case the Dehn function of $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ is cubic and by Proposition 4.4 the Dehn function of $(\mathbb{Z}^2 * \mathbb{Z}^2) \rtimes_\Phi \mathbb{Z}$ is also cubic.

□

Remark 4.6. Since Proposition 4.4 holds for general mapping tori of RAAGs of the form $\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$ with $m_1, m_2 \geqslant 2$, *modulo* having Conjecture 5 for general mapping tori $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$, the strategy to prove Proposition 4.4 applies in general. Thus, this is a promising strategy to obtain the Dehn functions of the mapping tori of $\mathbb{Z}^{m_1} * \mathbb{Z}^{m_2}$ for $m_1, m_2 \geq 2$.

4.3 Dehn functions of the mapping tori of the direct product of free groups

In this section we give a proof of Theorem F, thus obtaining a full characterisation of the Dehn functions of mapping tori of a direct product of finitely many free groups of finite rank. We first give a more detailed statement of Theorem F.

Theorem 4.7 (cf. Theorem F)**.** *Let $k \geqslant 2$ and for each $i \in \{1, \ldots, k\}$ let $m_i \geqslant 2$ be an integer. Let $G \coloneqq F_{m_1} \times \ldots \times F_{m_k}$. For every $\Psi \in \mathrm{Aut}(G)$, there exists $\Phi \in \mathrm{Aut}(G)$ such that $[\Phi] = [\Psi^p] \in \mathrm{Out}(G)$ and $\Phi = \phi_1 \times \ldots \times \phi_k$ where for every $i \in \{1, \ldots, k\}$ we have that $\phi \in \mathrm{Aut}(F_{m_i})$. In particular, the Dehn functions of the associated mapping tori M_Ψ and M_Φ are $\asymp$-equivalent and their asymptotic can be read off from $\phi_1, \ldots, \phi_k$ in that:*

1. *If for all $i \in \{1, \ldots, k\}$ we have that $[\phi_i^p] = [id] \in \mathrm{Out}(F_{m_i})$ for some $p \in \mathbb{N}$, then $\delta_{M_\Phi}(n) \asymp n^2$.*
2. *Let $d > 1$ be an integer. If there exists $\sigma \in \mathfrak{S}_k$ a permutation of the set $\{1, \ldots, k\}$ such that: $gr_{\phi_{\sigma(k-1)}}(n) \sim n^d$ and for all $j \in \{1, \ldots, k-2\}$ we have that $gr_{\phi_{\sigma(j)}} \preceq gr_{\phi_{\sigma(k-1)}} \preceq gr_{\phi_{\sigma(k)}}$, then $\delta_{M_\Phi}(n) \asymp n^{d+2}$.*
3. *If for some distinct $i, j \in \{1, \ldots, k\}$ we have that $gr_{\phi_i}(n) \sim gr_{\phi_j}(n) \sim 2^n$, then $\delta_{M_\Phi}(n) \asymp 2^n$.*

We prove Theorem 4.7 in two parts. First we prove the statement regarding the automorphism and second we prove the precise estimates of the Dehn functions. For the second the lower bounds follow from [PR19, Theorem 1.3] by noting that the group $G \rtimes_\Phi \mathbb{Z}$ retracts into $F_{\ell_1} \times F_{\ell_2} \rtimes_{\phi_{\ell_1} \times \phi_{\ell_2}} \mathbb{Z}$ for $\ell_1, \ell_2 \in \{m_1, \ldots, m_k\}$; see Remark 4.9.

4.3.1 Proof of Theorem F

Before proceeding with the proof let us point out that the three cases in Theorem 4.7 are mutually exclusive and exhaustive. Indeed, it follows from Theorem 2.60 that the growth of an automorphism of a free group of finite rank is either polynomial (with integer degree) or exponential.

Lemma 4.8 (cf [PR19, Lemma 7.1]). *Let $G \coloneqq F_{m_1} \times \ldots \times F_{m_k}$, with $k \geqslant 3, m_i \geqslant 2$, and $\Psi \in \mathrm{Aut}(G)$. There exist $p \in \mathbb{N}$ and for every $1 \leqslant i \leqslant k$ there exists $\phi_i \in \mathrm{Aut}(F_{m_i})$ such that $\Phi \coloneqq \phi_1 \times \ldots \times \phi_k$ satisfies $[\Phi] = [\Psi^p] \in \mathrm{Out}(G)$. Moreover, the Dehn function of the corresponding mapping tori satisfy $\delta_{M_\Psi} \asymp \delta_{M_\Phi}$.*

Proof. This is a direct consequence of Lemma 2.48 in the case that all the RAAGs are free groups. The moreover part follows from the first part together with Lemma 2.67. □

Given an automorphism Ψ of the RAAG $A_L = F_{m_1} \times \ldots \times F_{m_k}$, where each F_{m_i} is a free group on m_i generators, Lemma 4.8 together with Lemma 2.67 allow us to work with a preferred choice of representative of the outer class of Ψ to compute the Dehn function of the mapping torus of A_L. With this in mind we define the following finite presentation for M_Φ.

Let $G \coloneqq F_{m_1} \times \ldots \times F_{m_k}$ and $\Phi = \phi_1 \times \ldots \times \phi_k \in \mathrm{Aut}(G)$ with $\phi_i \in \mathrm{Aut}(F_{m_i})$ for each $i \in \{1, \ldots, k\}$. For each $i \in \{1, \ldots, k\}$ let X_i be a free generating set for F_{m_i}, $X_G \coloneqq \sqcup_{i=1}^k X_i$ a generating set for G, and $X \coloneqq X_G \cup \{t\}$ a generating set for $M_\Phi \coloneqq G \rtimes_\Phi \mathbb{Z}$. This gives us the finite presentation

$$\mathcal{P} \coloneqq \langle X \mid [x, y] = 1 \text{ for } x \in X_i, y \in X_j, i \neq j \text{ and } a_i^t = \phi_i(a_i) \text{ for all } a_i \in X_i \rangle \quad (4.7)$$

Remark 4.9. Note that the presentation for G given by X_G and the set of relations R_G defined by the relations $[x, y] = 1$ for all $x, y \in X_G$ is a *subrepresentation* of $\mathcal{P}$ in the sense that $X_G \subset X$ and the set of relators for the given presentation for G is contained in $R_\mathcal{P}$ the set of relators of $\mathcal{P}$. Similarly, for any $\ell_1, \ell_2 \in \{m_1, \ldots, m_k\}$, we have that the obvious finite presentation for $(F_{\ell_1} \times F_{\ell_2}) \rtimes_{\phi_{\ell_1} \times \phi_{\ell_2}} \mathbb{Z}$ with generating set $X_{\ell_1} \sqcup X_{\ell_2}$ is a subrepresentation of $\mathcal{P}$. Furthermore, the natural surjective homomorphism $G \rtimes_\Phi \mathbb{Z} \to (F_{\ell_1} \times F_{\ell_2}) \rtimes_{\phi_{\ell_1} \times \phi_{\ell_2}} \mathbb{Z}$ is a retraction.

We now proceed with the proof of Theorem 4.7 working with the automorphism Φ given by Lemma 4.8. We make use of Remark 4.9 at several point throughout the proof.

Proof of Theorem 4.7. Observe that the case $k = 2$ was done by Pueschel and Riley [PR19, Theorem C], so we know explain the general case $k \geqslant 3$. We work with the presentation $\mathcal{P}$ defined above, see (4.7).

We start by proving Case 1. In this case, by Lemma 2.67 we can assume that the mapping torus $G \rtimes_{\Phi} \mathbb{Z}$ is actually the group $G \times \mathbb{Z}$ which is a RAAG, therefore a CAT(0) group [Cha07, Theorem 2.6]) and thus has at most quadratic Dehn function [Bri02, Theorem 6.2.1]. Moreover, since the group $G \times \mathbb{Z}$ contains a subgroup isomorphic to $\mathbb{Z}^2$, it is not hyperbolic. Therefore, its Dehn function must be quadratic.

For Case 3 assume that there exist $i, j \in \{1, \dots, k\}$ such that ϕ_i, ϕ_j have exponential growth. Recall that Remark 4.9 implies that the mapping torus $H \coloneqq (F_i \times F_j) \rtimes_{\phi_i \times \phi_j} \mathbb{Z}$ is a retract of M_{Φ}. Therefore, by Lemma 2.63 it follows that $\delta_H(n) \preccurlyeq \delta_{M_{\Phi}}(n)$. Second, since ϕ_i and ϕ_j have exponential growth, then by [Pue16, Proposition 5.2.4] the Dehn function of H satisfies $\delta_H(n) \asymp 2^n$. Therefore, we have that $2^n \preccurlyeq \delta_{M_{\Phi}}(n)$. Finally, the upper bound for Case 3 follows from Corollary 2.65.

It remains to prove Case 2, which is the most challenging one. For simplicity, assume that $\sigma \in \mathfrak{S}_k$ is the trivial permutation. Moreover, suppose that $gr_{\phi_{k-1}}(n) \sim n^d$ and for every $j \in \{1, \dots, k-2\}$ we have that $gr_{\phi_j} \preceq gr_{\phi_{k-1}} \preceq gr_{\phi_k}$.

The lower bound on the Dehn function follows from a retraction argument, using the fact that the group epimorphism $G \rtimes_{\Phi} \mathbb{Z} \to (F_{k-1} \times F_k) \rtimes_{\phi_{k-1} \times \phi_k} \mathbb{Z}$ defines a retraction. Thus, if $H \coloneqq (F_{k-1} \times F_k) \rtimes_{\phi_{k-1} \times \phi_k} \mathbb{Z}$, then $\delta_H(n) \preccurlyeq \delta_{M_{\Phi}}(n)$ and from [PR19, Proposition 5.2.4] it follows that $\delta_H(n) \asymp n^{d+2}$.

We now proceed with the proof of the upper bound on the Dehn function for Case 2. Consider a null-homotopic word $W = W(X)$ in M_{Φ} of length n. We can write W as follows

$$W = W_1 t^{\epsilon_1} \dots W_r t^{\epsilon_r}$$

for some $r \geqslant 1$, where for each $i \in \{1, \dots, r\}$ we have that W_i is a word in X and $\epsilon_i \in \mathbb{Z}$. Now, for each $i \in \{1, \dots, r\}$ we have an identity

$$W_i =_{\mathcal{P}} A_{1,i} \dots A_{k,i}$$

in $G \hookrightarrow M_{\Phi}$ where for every $j \in \{1, \dots, k\}$ we have that $A_{j,i}$ is a word in X_i. Moreover, from Remark 4.9, it follows that this identity has area $\lesssim_{\mathcal{P}} |W_i|^2$ in M_{Φ} and $|W_i|_X = |A_{1,i} \dots A_{k,i}|_X$. From this it follows that the identity

$$W =_{\mathcal{P}} \prod_{i=1}^{r} (A_{1,i} \dots A_{k,i}) t^{\epsilon_i}$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} n^2$.

To reduce the word on the right-hand side of the last identity we proceed as follows. We first shuffle to the right end of the word the words $A_{j,i}$ for each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, k-1\}$. We do so in a controlled manner: first we shuffle all the words $A_{k-1,i}$ to the right end starting from $A_{k-1,1}$, then we shuffle $A_{k-1,2}$ to the right end, and so on, to produce a word $\overline{\mathbf{A}}_{k-1}$ in X_{k-1}. Then, we shuffle all the $A_{k-2,i}$'s to the right all the way before the newly created $\overline{\mathbf{A}}_{k-1}$, we start with $A_{k-2,1}$, continuing with $A_{k-2,2}$, and so on. This will create a word $\overline{\mathbf{A}}_{k-2}$ in X_{k-2} to the left of the word $\overline{\mathbf{A}}_{k-1}$. We proceed similarly with the remaining $A_{j,i}$'s with $j \in \{1, \dots, k-3\}$ and $i \in \{1, \dots, r\}$. Note that the only words we do not move are the words $A_{k,1}, \dots, A_{k,r}$. After this procedure we obtain the identity

$$\prod_{i=1}^{r} \left(A_{1,i} \dots A_{k,i}\right) t^{\epsilon_i} =_{\mathcal{P}} \left(A_{k,1} t^{\epsilon_1} \dots A_{k,r} t^{\epsilon_k}\right) \cdot \overline{\mathbf{A}}_1 \cdot \dots \cdot \overline{\mathbf{A}}_{k-1} \tag{4.8}$$

in M_Φ where for each $j \in \{1, \dots, k-1\}$ we have $\overline{\mathbf{A}}_j$ is a word in X_j.

To estimate the area of (4.8) first observe that shuffling a letter x from X_{k-1} past a t (respectively t^{-1}) corresponds to applying ϕ_{k-1} (respectively ϕ_{k-1}^{-1}). Since, by Theorem 2.61 $gr_{\phi_{k-1}} \sim gr_{\phi_{k-1}^{-1}}$, we have that after shuffling the first word $A_{k-1,1}$ to the right end, the resulting word will have length

$$\lesssim_{\mathcal{P}} |A_{k-1,1}| gr_{\phi_{k-1}}\left(|\epsilon_1 + \dots + \epsilon_k|\right) \preccurlyeq |A_{k-1,1}| gr_{\phi_{k-1}}(n).$$

Now, each letter of $A_{k-1,1}$ has to pass all the other letters of W from which there are at most n. Therefore, the area involved in the shuffling of the word $A_{k-1,1}$ to the right end is

$$\lesssim_{\mathcal{P}} n \cdot |A_{k-1,1}| gr_{\phi_{k-1}}(n) \lesssim_{\mathcal{P}} |A_{k-1,1}| n^{d+1}.$$

where for the last inequality we used the hypothesis $gr_{\phi_{k-1}}(n) \sim n^d$. In general, after shuffling all the words $A_{k-1,i}$ with $i \in \{1, \dots, r\}$ to the right end of the word to create the word $\overline{\mathbf{A}}_{k-1}$ the area involved is

$$\lesssim_{\mathcal{P}} \left(\sum_{i=1}^{r} |A_{k-1,i}|\right) \cdot n^{d+1}.$$

Second, recall that, by hypothesis, for each $j \in \{1, \dots, k-2\}$ we have that $gr_{\phi_j} \preceq gr_{\phi_{k-1}}$. Moreover, by Theorem 2.61 we have $gr_{\phi_j} \sim gr_{\phi_j^{-1}}$. Thus, proceeding similarly with each $A_{j,1}, \dots, A_{j,r}$ for $j \in \{1, \dots, k-2\}$ we can use the same estimates as for the words $A_{k-1,1}, \dots, A_{k-1,r}$, to get that the area involved in shuffling all the $A_{j,1},\dots,A_{j,r}$ for $j\in\{1,\dots,k-2\}$ to create the word $\overline{\mathbf{A}}_j$ is

$$\lesssim_{\mathcal{P}} \left(\sum_{i=1}^{r}|A_{j,i}|\right)\cdot n^{d+1}.$$

All together we get that the identity

$$\prod_{i=1}^{r}\left(A_{1,i}\dots A_{k,i}\right)t^{\epsilon_i} =_{\mathcal{P}} \left(A_{k,1}t^{\epsilon_1}\dots A_{k,r}t^{\epsilon_k}\right)\cdot\overline{\mathbf{A}}_1\cdot\ldots\cdot\overline{\mathbf{A}}_{k-1}$$

has area

$$\lesssim_{\mathcal{P}} \left(\sum_{i=1}^{k-1}\sum_{j=1}^{r}|A_{j,i}|\right)\cdot n^{d+1}\lesssim_{\mathcal{P}} n^{d+2}.$$

Now, we claim that the word $A_{k,1}t^{\epsilon_1}\dots A_{k,r}t^{\epsilon_k}$ represents the identity in $F_{m_k}\rtimes_{\phi_k}\mathbb{Z}\hookrightarrow M_\Phi$ and the word $\overline{\mathbf{A}}_1\cdot\ldots\cdot\overline{\mathbf{A}}_{k-1}$ represents the identity in $F_{m_1}\times\dots,\times F_{m_{k-1}}\hookrightarrow M_\Phi$. Indeed, first observe that by modding out by the derived subgroup it follows that the exponent sum in t's of the word $A_{k,1}t^{\epsilon_1}\dots A_{k,r}t^{\epsilon_k}$ is zero. Second, by shuffling all the t's to the left, we will end up with the null-homotopic word

$$\overline{\mathbf{A}}_k\cdot\left(\overline{\mathbf{A}}_1\cdot\ldots\cdot\overline{\mathbf{A}}_{k-1}\right)$$

in $G\hookrightarrow M_\Phi$, see Remark 4.9, which can be then freely reduced. This implies, in particular, that the word $A_{k,1}t^{\epsilon_1}\dots A_{k,r}t^{\epsilon_k}$ represents the identity in $F_{m_k}\rtimes_{\phi_k}\mathbb{Z}\hookrightarrow M_\Phi$ as we claimed. Moreover, for each $j\in\{1,\dots,k_1\}$ the word $\overline{\mathbf{A}}_j$ is null-homotopic in $F_{m_i}\hookrightarrow M_\Phi$, see Remark 4.9, so it can be freely reduced to the trivial word. Therefore, we obtain the identity

$$\left(A_{k,1}t^{\epsilon_1}\dots A_{k,r}t^{\epsilon_k}\right)\cdot\overline{\mathbf{A}}_1\cdot\ldots\cdot\overline{\mathbf{A}}_{k-1} =_{\mathcal{P}} A_{k,1}t^{\epsilon_1}\dots A_{k,r}t^{\epsilon_k}$$

in M_Φ with no area involved.

Finally, by [BG10] (see Theorem 4.2), the null-homotopic word $A_{k,1}t^{\epsilon_1}\dots A_{k,r}t^{\epsilon_k}$ in $F_{m_k}\rtimes_{\phi_k}\mathbb{Z}$ has area $\lesssim_{\mathcal{P}} n^2$. Overall, we get that $\mathrm{Area}_{\mathcal{P}}(W)\lesssim_{\mathcal{P}} n^{d+2}$. □

4.4 The Dehn function of $(\mathbb{Z}^2*\mathbb{Z})\rtimes_\Phi\mathbb{Z}$ with Φ having trivial growth

In this section we treat a special case of the mapping tori $(\mathbb{Z}^2*\mathbb{Z})\rtimes_\Phi\mathbb{Z}$. Namely we recover the following result from [PR19, Section 6].

Theorem 4.10. *For every $\Psi \in \operatorname{Aut}(\mathbb{Z}^2 * \mathbb{Z})$ there exists Φ such that $\Phi|_{\mathbb{Z}^2} = \phi \in \operatorname{Aut}(\mathbb{Z}^2)$ and $[\Phi] = [\Psi] \in \operatorname{Out}(\mathbb{Z}^2 * \mathbb{Z})$. In particular, the Dehn functions of the associated mapping tori M_Ψ and M_Φ have $\asymp$-equivalent Dehn functions. Moreover, if $\Phi|_{\mathbb{Z}^2} = \phi$ has trivial growth, then the Dehn function of M_Φ satisfies*

$$\delta_{M_\Phi}(n) \asymp n^2.$$

Before proceeding with the proof we would like to point out that although this case is treated in [PR19, Section 6] our strategy to proof the upper bound significantly differs from the one presented there. Moreover, the strategy presented in Section 4.2 leaves open the problem of classifying the Dehn functions of mapping tori of $\mathbb{Z}^m * \mathbb{Z}$ with $m \geqslant 2$. The goal of this section is to present a strategy for this case.

We start by recalling Pueschel and Riley's arguments to prove the first assertion and the lower bound on the Dehn function. We then proceed to present our new arguments for the upper bound.

The first assertion in Theorem 4.10 follows from the same idea as in Section 4.3: find the right representative of $[\Psi] \in \operatorname{Out}(\mathbb{Z}^2 * \mathbb{Z})$ (see Lemma 4.8). The analogous result for $\mathbb{Z}^2 * \mathbb{Z}$ follows from two results in [PR19] that also provide us with a finite presentation for M_Φ. Before stating the results we set the following notation.

Notation 4.11. Let $X \coloneqq \{x_1, x_2\}$ be a generating set for $\mathbb{Z}^2$ and let $\langle x_1, x_2, c \mid [x_1, x_2] = 1\rangle$ be a finite presentation for $\mathbb{Z}^2 * \mathbb{Z}$.

Lemma 4.12 ([PR19, Lemmas 6.1 and 6.2])**.** *For every $\Psi \in \operatorname{Aut}(\mathbb{Z}^2 * \mathbb{Z})$ there exist $\Phi \in \operatorname{Aut}(\mathbb{Z}^2 * \mathbb{Z})$ and a word $z \coloneqq z(X)$ such that $[\Phi] = [\Psi] \in \operatorname{Out}(\mathbb{Z}^m * \mathbb{Z})$, $\Phi|_{\mathbb{Z}^m} = \phi \in \operatorname{Aut}(\mathbb{Z}^m)$, and $\Phi(c) = cz$.*

Since by assumptions the automorphism ϕ has trivial growth, we may assume without loss of generality, see Lemma 2.67, that ϕ is in fact the trivial automorphism of $\mathbb{Z}^2$. From Lemma 4.12 we obtain the following finite presentation given by Pueschel and Riley [PR19, Section 6]:

$$\mathcal{P} \coloneqq \langle X, c, t \mid [x_i, x_j] = [x_i, t] = 1, \text{ for all } i \in \{1, 2\}, \text{ and } c^t = cz\rangle. \tag{4.9}$$

Remark 4.13. We would like to point out that we might assume that $z \coloneqq z(X)$ is a non-trivial word. Else, we would be looking at the RAAG $(\mathbb{Z}^2 * \mathbb{Z}) \times \mathbb{Z}$. Since RAAGs are CAT(0) groups (see for instance [Cha07, Theorem 2.6]), they have at most quadratic Dehn function [Bri02, Theorem 6.2.1].

We fix the above presentation once and for all with z non-trivial. Observe that Lemma 2.67 ensures us that working with this presentation $\mathcal{P}$ is enough to establish the Dehn function of the mapping torus of $\mathbb{Z}^m * \mathbb{Z}$. That is, we obtain the first assertion of Theorem 4.10, namely that $\delta_\Psi(n) \asymp \delta_\Phi(n)$.

The lower bound is a direct consequence of Lemma 2.66. Indeed, $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^2 * \mathbb{Z}$ is a quasi-isometric embedding and $\Phi|_{\mathbb{Z}^2} = \phi \in \mathrm{Aut}(\mathbb{Z}^2)$, so we get

$$n^2 \preccurlyeq \delta_{M_\Phi}(n) \asymp \delta_{M_\Psi}(n).$$

This conlcudes the proof of the first assertion and the lower bounds in Theorem 4.10 as presented in [PR19, Section 6]. We now proceed to explain our new arguments for the upper bound on the Dehn function.

4.4.1 Quadratic upper bound

In this section we give the quadratic upper bound on the Dehn function of Theorem 4.10. To achieve this we need a series of preliminary results that involve analysing the geometric structure of van Kampen diagrams and manipulating identities in the group M_Φ. Before proceeding let us briefly explain the strategy of the proof to provide a better intuition for the combinatorial arguments presented afterwards.

4.4.1.1 Strategy of the proof

The idea to prove the quadratic upper bound of Theorem 4.10 is as follows. We start with a null-homotopic word $w \coloneqq w(X, c, t)$, which we write as

$$w(X, c, t) \coloneqq c^{\sigma_0} w_0(X, 1, t) \dots c^{\sigma_r} w_r(X, 1, t), \tag{4.10}$$

for some $r \in \mathbb{N}$ where for every $j \in \{0, \dots, r\}$ we have that $\sigma_j \in \{\pm 1\}$ and the words $w_j(X, 1, t)$ might be empty. We then locate the first word where reductions might be possible, this would be one of the words among $w_0(X, 1, t), \dots, w_r(X, 1, t)$; we call this word a *pinching word* for w (see Definition 4.29). This pinching word corresponds to a matching pair of c's in the word w. In a van Kampen diagram for w, this matching pair of c's, corresponds to a c-corridor. We reduce the pinching word to obtain a new word $w^{(1)}$, which will have length at most $|w|$.

Geometrically, we can think about this as "cutting off" a van Kampen diagram for w along the c-corridor, so that the only remaining part is either the bottom or top of the c-corridor, depending on the exponent of c. At the level of words, this means, that the reduction of the pinching word will create a "residual word" whose word length is bounded above by the length of the pinching word (see Lemma 4.28 and Notation 4.31). An important observation is that we can use the (quadratic) Dehn function of $M_\phi \hookrightarrow M_\Phi$ to estimate the area it takes to reduce this pinching word. A naive approach would then be to proceed in the same manner as for w now with $w^{(1)}$. Since w has finite length, this process ends after all the c's have been cancelled. In particular, the number of steps required is $\lesssim_{\mathcal{P}} |w|$. Figure 4.1 illustrates this procedure.

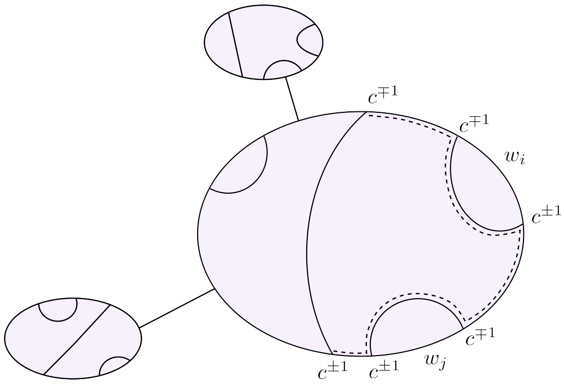

Figure 4.1: A van Kampen diagram D for a null-homotopic word w. The arcs represent c-corridors. The words w_i and w_j are pinching words for w. The dotted line has boundary label a "mixed pinching word" for the word $w^{(k)}$ obtained after k-steps. The parts of the dotted line that border a c-corridor would be labelled by "residual words". Observe that "cutting along" these two residual words we get three subdiagrams of D.

The crux involved in controlling the area estimates in this reduction process lies in the following situation: suppose that after step k of the reduction, we obtain a word in which the next pinching word includes several "residual words" from the previous steps, see Figure 4.1. This next pinching word can be thought of as a "mixed pinching word". This mixed pinching word will contain both residual words and words $w_i(X, 1, t)$ originally appearing in the writing (4.10) of w that were not part of the reduction process until this step.

Roughly speaking, by cutting a van Kampen diagram along the paths that are labelled by each of these residual words we would get L disjoint subdiagrams of the van Kampen diagram, where $L - 1$ would be the number of distinct residual words, see the dotted line in Figure 4.1. This means, that if we try to naively use the Dehn function of $M_\phi \hookrightarrow M_\Phi$ to estimate the area of this mixed pinching word, we might end up exceeding the quadratic bound that we require (e.g. by having to use an area $\lesssim_{\mathcal{P}} |w|^2$ at every step and having $\lesssim_{\mathcal{P}} |w|$ steps resulting in a cubic estimate).

To overcome this difficulty we require a more sophisticated and tailored approach. For this, we make use of the geometry behind this algorithm. We think about this reduction process using the geometric structure of van Kampen diagrams and their so-called *dual tree*, see Definition 4.20. This view point allows us to control the area in this reduction process. In particular, this allows us to obtain Lemma 4.34 to treat the case of a mixed pinching word.

We think of this reduction process as contracting the dual tree (to a van Kampen diagram), see Section 4.4.1.5. In the dual tree, the pinching words of w correspond to leaves in the dual tree; the mixed pinching words of the process correspond to vertices in the dual tree that have degree greater or equal than 3. Using this view point, we show that contracting to the root a whole subtree, rooted at a vertex at distance k from the base point, can be done by an area estimate that only involves the squared lengths of some words $w_i(X, 1, t)$ in the writing (4.10) for w and possibly products of residual words with them and other residual words (in case this step constitutes a mixed pinching word). This is done in Proposition 4.37 which requires Lemma 4.33. The key points here are the following:

1. The words $w_i(X, 1, t)$ contributing to the area estimates of step k belong to the original word w (namely, they are a subset of the words $w_0(X, 1, t)$, ..., $w_r(X, 1, t)$ from w). Moreover, these words do *not* contribute to any of the jth steps for $j \neq k$.
2. At step k, the products involving the residual words are *only* considered at this step k and do not contribute to any of the jth steps for $j \neq k$ (see Remark 4.35).

We make a detailed description of these two points in Section 4.4.1.5. We first proceed to explain the geometry in van Kampen diagrams over the given presentation for M_Φ.

4.4.1.2 Structure of van Kampen diagrams

As mentioned above, we consider van Kampen diagrams with respect to the given presentation

$$\mathcal{P} := \langle X, c, t \mid [x_i, x_j] = [x_i, t] = 1, \text{ for } 1 \leqslant i \leqslant 2, \text{ and } c^t = cz \rangle.$$

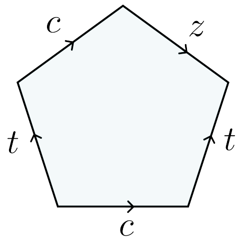

Figure 4.2: The 2-cell corresponding to the relation $t^{-1}ct = cz$.

For this section fix a null-homotopic word w in M_Φ and let D be a van Kampen diagram for w in $\mathcal{P}$. We use the notions defined in Section 2.7, where we defined van Kampen diagrams and corridors.

The reduction process and the area estimates are motivated by the structure of van Kampen diagrams over $\mathcal{P}$ as well as c- and t-corridors. In contrast to t-corridors, c-corridors are formed by just one type of 2-cell, namely the 2-cells corresponding to the relation $c^t = cz$, see Figure 4.2. We sometimes call the top part of a c-corridor *mixed* wall. We may assume that all the c-corridors are reduced in the sense that their boundary labels are words that cannot be freely reduced. We call the number of 2-cells in a c-corridor the *length of the corridor*. See Figure 4.3 for a generic picture of a c-corridor.

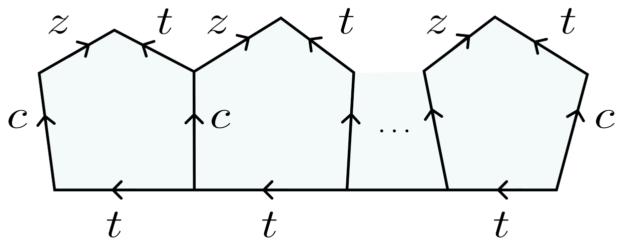

Figure 4.3: A c-corridor. When reading from the bottom left corner in a counter-clockwise direction, the bottom part is the word t^{-N} for some $N \in \mathbb{N}$ and the top part is the word $(tz^{-1})^N$.

Lemma 4.14. *Let $w(X, c, t) =_{\mathcal{P}} 1$. Let D be a van-Kampen diagram for $w(X, c, t)$ in $\mathcal{P}$ and K_1, K_2 two c-corridors in D. If $K_1 \neq K_2$, then $K_1 \cap K_2$ is at most 1-dimensional.*

Proof. Since the only relations involving c are of the form $c^t = cz$, the 2-cells forming a c-corridor K are labelled by the word $t^{-1}ct(cz)^{-1}$. Thus, the bottom of K is the word in t^k for some $k \in \mathbb{Z}$, and the top is a word in $\{X, t\}$. In particular, the bottom and top of K do not have instances of $c^{\pm 1}$'s. Let K_1, K_2 be two distinct c-corridors. If they are disjoint, then $K_1 \cap K_2$ is 0-dimensional.

Suppose $K_1 \cap K_2 \neq \emptyset$, then by the above paragraph they can only intersect on the boundary parts corresponding to the bottom/top parts, which are at most 1-dimensional. □

In view of Lemma 4.14 it makes sense to say that c-corridors do not *cross* each other.

Remark 4.15. Observe that we may assume, without loss of generality, that there are no c-annuli, that is, all the c-corridors start and end in the boundary of D. Indeed, if we were to have such an annulus, by the van Kampen Theorem 2.43, in particular, it would define a van Kampen diagram for the bottom word of the c-annulus. But this word contains only t's and all the edges have the same orientation, so the word would have to be freely-reduced.

Note that Remark 4.15 implies that the number of c-corridors in a van Kampen diagram for w is $\lesssim_{\mathcal{P}} |w|$. Given a van Kampen diagram with n many c-corridors, since c-corridors do not cross each other, we have that these c-corridors partitions the 2-cells of D into $n+1$ many connected components. We call these regions *c-complementary regions of D* or if there is no risk of confusion simply *complementary regions of D*. Given a K-corridor we refer to the two c-complementary regions of D whose boundary intersect the boundary of K as *c-complementary regions attached to K*; we denote by $R_B(K)$ (respectively $R_T(K)$) the c-complementary region of D whose boundary intersects the bottom (respectively top) part of K. We say that two c-complementary regions in D are separated by a c-corridor if there exists a c-corridor in K whose boundary intersects both c-free regions in D.

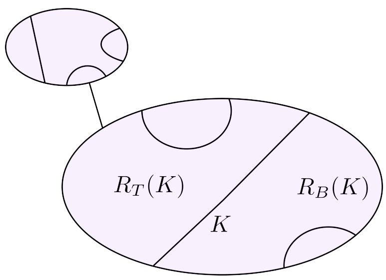

Figure 4.4: A c-corridor K and the c-complementary regions $R_B(K)$ and $R_T(K)$ of D attached to K.

Definition 4.16 (Consecutive c-corridors)**.** Given two c-corridors K and K' in a van Kampen diagram D, we say that the pair (K, K') is a *pair of consecutive c-corridors* if one of the following is non-empty $R_T(K) \cap R_B(K')$, $R_T(K) \cap R_T(K)$, $R_B(K) \cap R_T(K')$, or $R_B(K) \cap R_B(K')$.

Definition 4.17 (Facing and opposing c-corridors)**.** Let K and K' be two c-corridors in a van Kampen diagram D. We say that the pair (K, K') is *facing* if the top parts of K and K' belong to the boundary of the same complementary region. We say that the pair (K, K') is *opposing* if the bottom parts of K and K' belong to the boundary of the same complementary region.

Note that a non-facing pair is not necessarily opposing. To make this clear we define the following.

Definition 4.18 (Positive and negative pairs of c-corridors)**.** Let K and K' be two c-corridors in a van Kampen diagram D. We say that the pair (K, K') is *positively aligned* (respectively *negatively aligned*) if the top (respectively bottom) of K and the bottom (respectively top) of K' belong to the boundary of the same c-complementary region.

Since the null-homotopic word w is freely reduced we have the following observation.

Remark 4.19. Let K_1, K_2, and K_3 be three distinct c-corridors such that (K_1, K_2) and (K_2, K_3) are consecutive pairs and (K_1, K_3) is not a consecutive pair. If the pair (K_1, K_2) is facing, then the pair (K_2, K_3) is opposing.

In the reduction process we make use of the so-called dual tree to a van Kampen diagram in $\mathcal{P}$ which we define below. To define it we need to reduce to van Kampen diagrams that are topological discs. In Section 4.4.1.5 we use this (rooted) tree to estimate the area reduction of a null-homotopic word. The geometric picture of the tree and its branches, is what allows us to carry out the reduction process for a given null-homotopic word while keeping the area estimates low, see Remark 4.35 and Proposition 4.37.

In particular, in Section 4.4.1.5 we use the dual tree when computing area estimates. There are two kinds of c-corridors: thin and thick, see Definition 2.45. The thin ones separate D into a finite disjoint union of topological discs, each of them constitutes a van Kampen diagram for a null-homotopic word $u := u(X, c, t)$ in w, whose length is smaller than $|w|$, so that

$$w =_{\mathrm{def}} w_1 c^{\pm 1} u c^{\mp 1} w_2$$

for some words $w_1 := w_1(X, c, t)$ and $w_2 := w_2(X, c, t)$. Reducing u and cancelling the c's corresponding to the thin corridor, we are left with the null-homotopic word

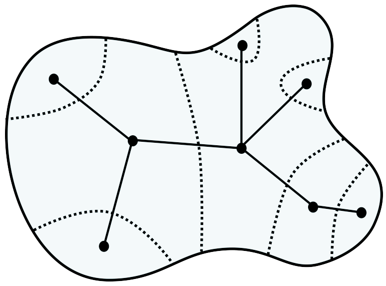

Figure 4.5: The dual tree of a van Kampen diagram, the dotted lines represent c-corridors.

w_1w_2 whose length is less than that of w. In particular, this means that

$$\mathrm{Area}_{\mathcal{P}}(w) \lesssim_{\mathcal{P}} \mathrm{Area}_{\mathcal{P}}(u) + \mathrm{Area}_{\mathcal{P}}(w_1w_2).$$

This allows us to reduce to the problem of studying words whose van Kampen diagrams do not have thin c-corridors. The same is true for t-corridors. Thus, we can assume without loss of generality that D is a single topological disk.

Definition 4.20 (Dual tree)**.** The *dual tree* T is the graph defined as follows: vertices correspond to c-complementary regions in D and an edge between two vertices corresponds to a c-corridor whose boundary intersects the two c-complementary regions attached to the c-corridor.

As mentioned above we may assume without loss of generality that D is a topological disc. Moreover, since c-corridors do not cross each other and there are finitely many c-corridors in a van Kampen diagram, the dual tree is indeed a tree. Figure 4.5 illustrates this fact.

4.4.1.3 Auxiliary identities

For the readers convenience we start this section by recalling the finite presentation for M_Φ:

$$\mathcal{P} := \langle X, c, t \mid [x_i, x_j] = [x_i, t] = 1, \text{ for all } i \in \{1, 2\}, \text{ and } c^t = cz\rangle. \tag{4.11}$$

Recall we are using the convention $c^t = t^{-1}ct$ and that by Remark 4.13 we might assume that $z \neq 1$. The below results are used in the reduction process of a null-homotopic word w in M_Φ. In particular, they describe the process (as well the area estimates) of shuffling t's past a c. The reader might find it useful to have in mind the 2-cell corresponding to the relation $t^{-1}ct = cz$ while reading through the following statements. For convenience we record it here.

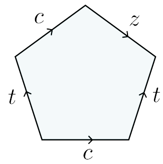

Figure 4.6

Lemma 4.21. *The following identities hold in M_Φ with area 1.*

(i) $ct =_{\mathcal{P}} tcz$,
(ii) $c^{-1}t =_{\mathcal{P}} tz^{-1}c^{-1}$,
(iii) $c^{-1}t^{-1} =_{\mathcal{P}} zt^{-1}c^{-1}$,
(iv) $czt^{-1} =_{\mathcal{P}} t^{-1}c$,
(v) $ct^{-1} =_{\mathcal{P}} t^{-1}c(tz^{-1}t^{-1})$.
(vi) $ctz^{-1} =_{\mathcal{P}} tc$.

Proof. (v): from the relation $t^{-1}ct = cz$ we get $t^{-1}c =_{\mathcal{P}} czt^{-1} \overset{\text{free}}{=} ct^{-1}(tzt^{-1})$. Thus, $t^{-1}c =_{\mathcal{P}} ct^{-1}(tzt^{-1})$. Multiplying from the right by $(tz^{-1}t^{-1})$ we get the desired identity and area estimate. The other identities are easily deduced from the relation involving c, we leave the details to the reader. □

Lemma 4.22. *Let $N \in \mathbb{Z}$. The identity*

$$c^{-1}t^N =_{\mathcal{P}} (tz^{-1})^N c^{-1}, \tag{4.12}$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} |N|$.

Proof. The proof is done by induction on N. We start with the proof of the case $N \geqslant 0$. For the base of induction, Lemma 4.21 (ii) implies that the equality $c^{-1}t =_{\mathcal{P}} tz^{-1}c^{-1}$ holds in M_Φ. Suppose the result holds for $N-1$ with $N > 1$. For the induction step we have by the induction hypothesis that the identity

$$c^{-1}t^N =_{\mathcal{P}} (tz^{-1})^{N-1}c^{-1}t$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} N-1$. It then follows from the base of induction that by applying one relation we get $(tz^{-1})^{N-1}c^{-1}t =_{\mathcal{P}} (tz^{-1})^N c^{-1}$. Overall, we obtain that the identity $c^{-1}t^N =_{\mathcal{P}} (tz^{-1})^N c^{-1}$ holds in M_Φ with area $\lesssim_{\mathcal{P}} N$.

Now, for $N < 0$ we have for the base of induction that from Lemma 4.21(iii) the identity

$$c^{-1}t^{-1} =_{\mathcal{P}} zt^{-1}c^{-1} =_{\text{def}} (tz^{-1})^{-1}c$$

holds in M_Φ with area one. Suppose the result holds for $N+1$ with $-N > 1$. For the induction step we have by the induction hypothesis that the identity

$$c^{-1}t^N =_{\mathcal{P}} (tz^{-1})^{N+1}c^{-1}t^{-1}$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} |N|-1$. From the base of induction it follows that by applying one relation we get $(tz^{-1})^{N+1}c^{-1}t^{-1} =_{\mathcal{P}} (tz^{-1})^N c$. Overall we obtain that the identity $c^{-1}t^N =_{\mathcal{P}} (tz^{-1})^N c^{-1}$ holds in M_Φ with area $\lesssim_{\mathcal{P}} |N|$. □

Lemma 4.23. *Let $N \in \mathbb{Z}$. The identity*

$$c(tz^{-1})^N =_{\mathcal{P}} t^N c$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} |N|$.

Proof. This is just a direct consequence of Lemma 4.22 where we multiply (4.12) by c from the left and from the right. □

The following results help us track the area needed to reduce positively and negatively aligned pairs of c-corridors. The first one concerns the length of the c-corridors, while the second one the area to reduce the corresponding matching c's. Before stating the results let us recall from Chapter 2 the following notation. Given a word $w := w(X, c, t)$ and a letter $a \in X \cup \{t\}$ we denote by $\ell_a(w)$ the number of $a^{\pm 1}$'s in w and by $\varepsilon_a(w)$ the exponent sum in $a^{\pm 1}$'s in w. Moreover, we denote by $\ell_X(w)$ the number of letters from the alphabet X appearing in w.

Lemma 4.24 (Bounding t's between positive and negative pairs)**.** *Let $N \in \mathbb{Z}$ and for $i \in \{1, 2\}$ let $w_i := w_i(X, 1, t)$ and $N_i := \epsilon_t(w_i)$.*

1. *If the identity*

   $$t^{-(N_1+N+N_2)}w_1(tz^{-1})^N w_2 =_{\mathcal{P}} 1$$

   holds in M_Φ, then

   $$|N| \leqslant |w_1| + |w_2|.$$

2. *If the identity*

   $$(tz^{-1})^{-(N_1+N+N_2)}w_1 t^N w_1 =_{\mathcal{P}} 1$$

   holds in M_ϕ, then

   $$|N| \leqslant 2(|w_1| + |w_2|).$$

Proof. In the manipulations of words and identities in this proof we do *not* care about the area estimates.

We start with 1. Assume that the identity

$$t^{-(N_1+N+N_2)}w_1(tz^{-1})^N w_2 =_{\mathcal{P}} 1 \tag{4.13}$$

holds in M_Φ. Then, it follows that the exponent sum in x_1's (respectively x_2's) has to be zero. In particular, we have

$$\varepsilon_{x_1}(w_1) + \varepsilon_{x_1}((tz^{-1})^N) + \varepsilon_{x_1}(w_2) = 0.$$

A similar inequality holds for $\epsilon_{x_2}(\cdot)$. Recall we are assuming that z is non-trivial. Therefore for some $j \in \{1, 2\}$ we have $\varepsilon_{x_j}((tz^{-1})^N) = -N\varepsilon_{x_j}(z) \neq 0$. Since $\varepsilon_{x_j}(w_i) \leqslant |w_i|$ it then follows from the above equality that

$$|N| \leqslant |w_1| + |w_2|.$$

This proves 1.

Proof of case 2: Since the word $(tz^{-1})^{-(N_1+N+N_2)}w_1t^Nw_2$ is null-homotopic, we have

$$\varepsilon_{x_1}(z^{-1})(-(N_1 + N + N_2)) + N_1 + N_2 = 0.$$

Analogously for $\varepsilon_{x_2}(\cdot)$. Since for $j \in \{1, 2\}$ we have $\varepsilon_{x_j}(z^{-1}) = -\varepsilon_{x_j}(z)$ we get

$$\epsilon_{x_j}(z)N + (1 + \varepsilon_{x_j}(z))(N_1 + N_2) = 0.$$

Thus, we get

$$|N| \leqslant \Big(1 + \frac{1}{|\epsilon_{x_j}(z)|}\Big)(|w_1| + |w_2|),$$

where x_j is such that $\epsilon_{x_j}(z) \neq 0$. □

Lemma 4.25 (Area in a positive and negative pair)**.** *For every $i \in \{1, 2\}$ let $w_i := w_i(X, t, c)$, $N_i := \varepsilon_t(w_i)$, and $N \in \mathbb{Z}$. Then the following holds:*

1. *If the identity*
$$t^{-(N_1+N+N_2)}w_1(tz^{-1})^N w_2 =_{\mathcal{P}} 1$$
holds in M_Φ, then it has area $\lesssim_{\mathcal{P}} (|w_1| + |w_2|)^2$.
2. *If the identity*
$$(tz^{-1})^{-(N_1+N+N_2)}w_1t^N w_2 =_{\mathcal{P}} 1$$
holds in M_Φ, then it has area $\lesssim_{\mathcal{P}} (|w_1| + |w_2|)^2$.

Proof. By Lemma 4.24 the null-homotopic words in each case have length $\lesssim_{\mathcal{P}} |w_1|+|w_2|$. Thus, the area estimates follow directly from the fact that This follows directly from the fact that $\langle X, t\rangle \cong \mathbb{Z}^3$ inside M_Φ. □

Corollary 4.26. *For every $i \in \{1,2\}$ let $w_i := w_i(X,t,c)$, $N_i := \varepsilon_t(w_i)$, and $N \in \mathbb{Z}$. Then the following holds:*

1. *If the identity*
$$c^{-1}w_1(tz^{-1})^N w_2 c =_{\mathcal{P}} (tz^{-1})^{N_1+N+N_2}$$
holds in M_Φ, then it has area $\lesssim_{\mathcal{P}} (|w_1|+|w_2|)^2$.
2. *If the identity*
$$cw_1 t^N w_2 c^{-1} =_{\mathcal{P}} t^{N_1+N+N_2}$$
holds in M_Φ, then it has area $\lesssim_{\mathcal{P}} (|w_1|+|w_2|)^2$.

Proof. This follows from Lemma 4.25 as we now explain. First, suppose the identity
$$c^{-1}w_1(tz^{-1})^N w_2 c =_{\mathcal{P}} (tz^{-1})^{N_1+N+N_2} \tag{4.14}$$
holds in M_Φ. For brevity set $M := N_1 + N + N_2$. The word on the left side of this identity satisfies
$$\begin{aligned} c^{-1}w_1(tz^{-1})^N w_2 c &\overset{\text{free}}{=} c^{-1}t^M t^{-M} w_1 (tz^{-1})^N w_2 c \\ &=_{\mathcal{P}} (tz^{-1})^M c^{-1} t^{-M} w_1 (tz^{-1})^N w_2 c, \end{aligned} \tag{4.15}$$
where the first identity is a free identity, so it has no area contribution; while the second is a consequence of Lemma 4.22, so it has area $\lesssim_{\mathcal{P}} |M|$. Since $|M| \leqslant |N_1| + |N| + |N_2|$ and Lemma 4.24 tells us that $|N| \leqslant 2(|w_1|+|w_2|)$, we then get that $|M| \lesssim_{\mathcal{P}} |w_1|+|w_2|$. It then follows from (4.14) and (4.15) that the identity
$$c^{-1}t^{-M}w_1(tz^{-1})^N w_2 c =_{\mathcal{P}} 1.$$
holds in M_Φ with area $\lesssim_{\mathcal{P}} |w_1|+|w_2|$. In particular, this last identity implies that the word $t^{-M}w_1(tz^{-1})^N w_2$ is a null-homotopic word in $M_\phi \hookrightarrow M_\Phi$. Thus, Lemma 4.25 (1) implies that it has area $\lesssim_{\mathcal{P}} (|w_1|+|w_2|)^2$. Overall, the identity
$$c^{-1}w_1(tz^{-1})^N w_2 c =_{\mathcal{P}} (tz^{-1})^{N_1+N+N_2}$$
has area $\lesssim_{\mathcal{P}} (|w_1|+|w_2|)^2$ in M_Φ.

The proof for 2 is analogous. Alternatively, it follows from 1 similarly as explained in the proof of Lemma 4.25. For convenience of the reader we briefly sketch the "combinatorial argument". Suppose that the identity
$$cw_1 t^N w_2 c^{-1} =_{\mathcal{P}} t^{N_1+N+N_2}$$

holds in M_Φ. The word on the right-hand side of the above equality is freely equal to the word

$$c(tz^{-1})^M(tz^{-1})^{-M}w_1t^Nw_2c^{-1}.$$

Applying Lemma 4.23 to this last word, we can shuffle the word $(tz^{-1})^M$ to the left past c to obtain the word $t^Mc(tz^{-1})^{-M}w_1t^Nw_2c^{-1}$ with area $\lesssim_{\mathcal{P}} |M|$ in M_Φ.

It then follows from our assumptions, that the word $c(tz^{-1})^{-M}w_1t^Nw_2c^{-1}$, and consequently the word $(tz^{-1})^{-M}w_1t^Nw_2$, is a null-homotopic word in M_Φ. We conclude from Lemma 4.25 (2) the desired area estimates. □

4.4.1.4 Reducing words

In this section we consider a null-homotopic word $w = w(X, c, t)$ in M_Φ admitting a van Kampen diagram which is a topological disc, and we explain the algorithm that we use to reduce it as well as the area estimates for each step. Recall that a word $w(X, c, t)$ can be written as

$$w(X, c, t) := c^{\sigma_0}w_0(X, 1, t)\dots c^{\sigma_r}w_r(X, 1, t),$$

for some $r \in \mathbb{N}$ and for every $j \in \{0, \dots, r\}$, $\sigma_j \in \{\pm 1\}$. Note that in this case the words $w_j := w_j(X, 1, t)$ can be empty. For each $j \in \{1, \dots, r\}$ let $N_j := \epsilon_t(w_j)$.

Lemma 4.27. *If a word $w = w(X, c, t)$ is null-homotopic in M_Φ, then $\epsilon_t(w) = \epsilon_c(w) = 0$.*

Proof. Let $w := c^{\sigma_0}w_0 \cdots c^{\sigma_r}w_r$ be a null-homotopic word in M_Φ. By shuffling all the $t^{\pm 1}$'s appearing in w to the beginning of w we get the identity

$$w =_{\mathcal{P}} t^Nu(X, c, 1),$$

where $N = \epsilon_t(w)$. Since $t^Nu(X, c, 1) =_{\mathcal{P}} 1$ we have that $N = 0$. Indeed, the homomorphism $(\mathbb{Z}^2 * \mathbb{Z}) \rtimes_\Phi \langle t\rangle \twoheadrightarrow (\mathbb{Z}, +)$ defined as $w \mapsto \epsilon_t(w)$, sends the word $t^Nu(X, c, 1)$ to N, but since the word w is null-homotopic the N has to be 0.

Now, for $\epsilon_c(w)$ note that the exponent sum in c's of $u(X, c, 1)$ remains the same as the one in w. Since the word $u(X, c, 1)$ represents an element in $\mathbb{Z}^m * \mathbb{Z} \hookrightarrow M_\Phi$ it can be written as

$$u(X, c, 1) =_{\mathcal{P}} c^{\sigma_0}u_0(X, 1, 1)\cdots c^{\sigma_r}u_r(X, 1, 1).$$

Finally, since $u(X, c, 1)$ is a null-homotopic word in $\mathbb{Z}^m * \mathbb{Z}$ we must have that $\sigma_0 + \cdots + \sigma_r = 0$. □

Lemma 4.28 (Pinching Lemma). *If*

$$w(X,c,t) := c^{\sigma_0}w_0(X,1,t)\dots c^{\sigma_j}w_j(X,1,t)c^{\sigma_{j+1}}\dots c^{\sigma_r}w_r(X,1,t)$$

is a freely reduced null-homotopic word in M_Φ, then there exists $i \in \{0,\dots,r-1\}$ such that the following holds:

1. $\sigma_i + \sigma_{i+1} = 0$ *and*
2. $w_i(X,1,t) \neq_F 1$.

Moreover, if $\sigma_i = 1$, then the equality

$$(tz^{-1})^{-N_i}w_i =_{\mathcal{P}} 1$$

holds in M_Φ with area $\lesssim |w_i|^2$, where the sign in the exponent of the word $(tz^{-1})^{\pm N_i}$ depends on the sign of $N_i = \epsilon_t(w_i)$. If $\sigma_i = -1$, then the equality

$$t^{-N_i}w_i =_{\mathcal{P}} 1$$

holds in M_Φ with area $\lesssim |w_i|^2$.

Proof. Let D be a minimal van-Kampen diagram for w in $\mathcal{P}$. Recall that a c-corridor in D induces a decomposition of D into two disjoint subdiagrams. Since there are finitely many c-corridors, there exists an "outer most" c-corridor such that one of the components do not have a $c^{\pm 1}$ on its boundary label. In particular, the shared boundary of this component and D has boundary label one of the words

$$w_i := w_i(X,1,t)$$

in w. This is the word that we would like to first remove from w to match the cancelling c^{σ_i} and $c^{\sigma_{i+1}}$ that correspond to this "outer most" c-corridor. Denote this c-corridor by $K(\sigma_i,\sigma_{i+1})$. In particular, we have that $\sigma_i + \sigma_{i+1} = 0$. Since the word w is freely reduced we get the following in $F_{\{X,c,t\}}$

$$c^{\sigma_i}w_i c^{\sigma_{i+1}} \neq_F 1,$$

and therefore $w_i(X,1,t) \neq_F 1$. This proves points 1 and 2. For the moreover part we consider two cases for the matching c's corresponding to the "outer most" c-corridor, i.e. for the pair (σ_i,σ_{i+1}).

Case 1: $(\sigma_i, \sigma_{i+1}) = (1, -1)$. Assume that $N_i > 0$, the case $N_i < 0$ is completely analogous. We have the following identities in M_Φ:

$$\begin{aligned} cw_i(X,1,t)c^{-1} &\overset{\text{free}}{=} c(tz^{-1})^{N_i}(tz^{-1})^{-N_i}w_ic^{-1} && \text{(free insertion, no area)} \\ &=_{\mathcal{P}} t^{N_i}c(tz^{-1})^{-N_i}w_ic^{-1} && \text{(Lemma 4.23: area } \lesssim_{\mathcal{P}} |N_i|) \end{aligned}$$

Observe that one of the c-complementary regions attached to $K(\sigma_i, \sigma_{i+1})$ has boundary label

$$(tz^{-1})^{-N_i}w_i.$$

In particular, the word $(tz^{-1})^{-N_i}w_i$ is null-homotopic in $M_\phi \hookrightarrow M_\Phi$. Thus, by the van Kampen Theorem 2.43, for this null-homotopic word we can construct a van Kampen diagram in the presentation for $M_\phi \hookrightarrow M_\Phi$ having area

$$\lesssim_{\mathcal{P}} (|z||N_i| + |w_i|)^2 \lesssim_{\mathcal{P}} |w_i|^2.$$

This proves the case for the pair $(1, -1)$.

Case 2: $(\sigma_i, \sigma_{i+1}) = (-1, 1)$. In this case we have the following identities

$$\begin{aligned} c^{-1}w_ic &\overset{\text{free}}{=} c^{-1}t^{N_i}t^{-N_i}w_ic && \text{(free insertion, no area)} \\ &=_{\mathcal{P}} (tz^{-1})^{N_i}c^{-1}t^{-N_i}w_ic && \text{(Lemma 4.22: area } \lesssim_{\mathcal{P}} |N_i|) \end{aligned}$$

Just as in the previous case, we have that the word $t^{-N_i}w_i$ is null-homotopic in $M_\phi \hookrightarrow M_\Phi$ and it thus has area $\lesssim_{\mathcal{P}} (|w_i| + |N_i|)^2 \lesssim_{\mathcal{P}} |w_i|^2$. □

Definition 4.29 (Pinching word)**.** Given a null-homotopic word w in M_Φ, we will refer to the word w_i from Lemma 4.28 as a *pinching word for* w.

We have the following consequence of Lemma 4.28.

Corollary 4.30. *Given a null-homotopic word w in M_Φ and a pinching word w_i for w. Then, the following holds*

1. *If $\sigma_i = 1$, then the identity*

$$cw_ic^{-1} =_{\mathcal{P}} t^{N_i}$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} |w_i|^2$.

2. *If $\sigma_i = -1$, then the identity*

$$c^{-1}w_ic =_{\mathcal{P}} (tz^{-1})^{N_i}$$

holds in M_Φ with area $\lesssim_{\mathcal{P}} |w_i|^2$.

As seen in the proof of *Lemma* 4.28, for a van Kampen diagram for w, the pinching word w_i belongs to the boundary label of the c-corridor starting/ending at σ_{i+1} and ending/starting at σ_i, which we denoted by $K(\sigma_i, \sigma_{i+1})$. By *reducing* the c-corridor we mean that we transform the word

$$w \coloneqq c^{\sigma_0} w_0 \dots w_{i-1} c^{\sigma_i} w_i c^{\sigma_{i+1}} w_{i+1} \dots c^{\sigma_r} w_r$$

to the word

$$c^{\sigma_0} w_0 \dots c^{\sigma_{i-1}} w_{i-1} (t\alpha^{-1})^{N_i} w_{i+1} c^{\sigma_{i+2}} \dots c^{\sigma_r} w_r,$$

using relations from $\mathcal{P}$, where $\alpha \in \{z, 1\}$. Note, that by Corollary 4.30 the identity between the words above has area $\lesssim_{\mathcal{P}} |w_i|^2$ in M_Φ.

We now establish some notation that we use through the further reduction process.

Notation 4.31. Suppose $w(X, c, t) =_{\mathcal{P}} 1$ and let $s(i) \in \mathbb{N}$. Suppose we have a family of nested c-corridors $\{K_j \coloneqq K(\sigma_{i-j}, \sigma_{i+j+1}) \mid 1 \leqslant j \leqslant s(i)\}$. For every $0 \leqslant j \leqslant s(i)$ let $\alpha_j \in \{1, z\}$. Moreover, let $N_{i,0} \coloneqq N_i$ and for every $0 < j < s(i)$ define

$$N_{i,j} \coloneqq N_{i-j} + \dots + N_i + \dots N_{i+j}, \tag{4.16}$$

and denote

$$v_{i,j+1} \coloneqq v_{i,j+1}(X, 1, t) \coloneqq (t\alpha_j^{-1})^{N_{i,j}}. \tag{4.17}$$

Observe that these words are what we call "residual words" in Section 4.4.1.1. Note that by definition $N_{i,j} = N_{i-1} + N_{i,j-1} + N_{i+1}$.

Definition 4.32 (Nested c-corridors)**.** Let

$$w(X, c, t) \coloneqq c^{\sigma_0} w_0(X, 1, t) \dots c^{\sigma_r} w_r(X, 1, t)$$

be a null-homotopic word in M_Φ. We say that a collection $\{K_j \mid 0 \leqslant j \leqslant s(i)\}$ is a *family of $s(i)$-nested c-corridors around w_i* if the following two conditions are satisfied:

(P1) The word w_i is a pinching word for w.

(P2) For every $1 \leqslant j \leqslant s(i)$ the word $w_{i-j} v_{i,j} w_{i+j}$ is a pinching word for the word resulting from reducing the c-corridors $K_0, \dots, K_{(j-1)}$ in w.

The results from Lemmas 4.24 and 4.25 help us to keep track of the area needed to reduce a family of nested c-corridors. The first one concerns the length of the c-corridors, while the second one the area to reduce them. In particular, Lemma 4.25 helps us to keep track of the area estimates to reduce a pinching word in w.

We would like to mention that the above results, namely Lemmas 4.24 and 4.25, only consider positively and negatively aligned pairs. For the opposing and facing pairs we are not able to control the length of the c-corridors, so we do not have an equivalent result for such pairs. Instead, we have to proceed by carefully bookkeeping the area estimates involved as we now explain. Ultimately, this careful book keeping allows us to control the area contribution of reducing "mixed pinching words", see Lemma 4.34, and thus to obtain the desired upper bound, see Proposition 4.37.

We now state the result that bounds the area of a family of nested c-corridors.

Lemma 4.33 (Area in a nest). *Let $w(X,c,t)$ be a null-homotopic word in M_Φ. Suppose that $w_i(X,1,t)$ is a pinching word for w, and that there exists $s(i) \in \mathbb{N}_{>0}$ such that there is a family of $s(i)$-nested c-corridors around w_i. Then, the identity*

$$c^{\sigma_{s(i)}} w_{i-s(i)} \dots c^{\sigma_i} w_i c^{\sigma_{i+1}} \dots w_{i+s(i)} c^{\sigma_{i+s(i)+1}} =_{\mathcal{P}} (tz^{-\beta_{i,s(i)}})^{N_{i,s(i)}}$$

holds in M_Φ with area

$$\lesssim_{\mathcal{P}} |w_i|^2 + \sum_{j=1}^{s} \Big(|w_{i-j}| + |w_{i+j}|\Big)^2 + \sum_{j=1}^{s} |N_{i,j}|\big(|w_{i-j}| + |w_{i+j}|\big)$$

Proof. The identity holds by definition of a nested family. For the area we proceed by induction on $j \in \{0, \dots, s(i)\}$. The base of induction follows directly from Lemma 4.28 since w_i is a pinching word for w. So suppose for the induction hypothesis that the statement is true for $j > 0$, namely that the identity

$$c^{\sigma_{i-j}} w_{i-j} \dots c^{\sigma_i} w_i c^{\sigma_{i+1}} \dots w_{i+j} c^{\sigma_{i+(j+1)}} =_{\mathcal{P}} (t\alpha_j^{-1})^{N_{i,j}} \tag{4.18}$$

holds in M_Φ, where $\alpha_j \in \{1, z\}$ with area

$$\lesssim_{\mathcal{P}} |w_i|^2 + \sum_{k=1}^{j} \big(|w_{i-k}| + |w_{i+k}|\big)^2 + \sum_{k=1}^{j} |N_{i,k}|\big(|w_{i-k}| + |w_{i+k}|\big). \tag{4.19}$$

Therefore, it follows that the word

$$c^{\sigma_{i-(j+1)}} w_{i-(j+1)} \dots c^{\sigma_i} w_i c^{\sigma_{i+1}} \dots w_{i+(j+1)} c^{\sigma_{i+(j+1)+1}}$$

is equivalent to the word

$$W_j = \begin{cases} c w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} c^{-1}, & \text{if} \quad \sigma_{i-(j+1)} = 1, \\ c^{-1} w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} c, & \text{if} \quad \sigma_{i-(j+1)} = -1, \end{cases}$$

in M_Φ and the area estimate to transform one to the other is the one given by (4.19).

By free insertions we obtain

$$W_j \overset{\text{free}}{=} \begin{cases} c(tz^{-1})^{N_{i,j+1}}(tz^{-1})^{-N_{i,j+1}} w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} c^{-1}, & \text{if} \quad \sigma_{i-(j+1)} = 1, \\ c^{-1} t^{N_{i,j+1}} t^{-N_{i,j+1}} w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} c, & \text{if} \quad \sigma_{i-(j+1)} = -1. \end{cases}$$

It follows from Lemmas 4.22 and 4.23 that the identities

$$W_j =_{\mathcal{P}} \begin{cases} t^{N_{i,j+1}} c (tz^{-1})^{-N_{i,j+1}} w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} c^{-1}, & \text{if} \quad \sigma_{i-(j+1)} = 1, \\ (tz^{-1})^{N_{i,j+1}} c^{-1} t^{-N_{i,j+1}} w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} c, & \text{if} \quad \sigma_{i-(j+1)} = -1, \end{cases}$$

hold in M_Φ each with area $\lesssim_{\mathcal{P}} |N_{i,j+1}|$. Note that by hypothesis, we have

$$(tz^{-1})^{-N_{i,j+1}} w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} =_{\mathcal{P}} 1 \quad \text{if } \sigma_{i-(j+1)} = 1, \tag{4.20}$$

$$t^{-N_{i,j+1}} w_{i-(j+1)} (t\alpha_j^{-1})^{N_{i,j}} w_{i+(j+1)} =_{\mathcal{P}} \quad \text{if } \sigma_{i-(j+1)} = -1. \tag{4.21}$$

Indeed, the word on the left hand side of each identity is a pinching word for the respective word resulting from reducing the first c-corridors $K_0, \ldots, K_j$.

To estimate the areas of these two identities we proceed as follows. First observe that if $\alpha_j = 1$, then Lemma 4.25(2) implies that the identity

$$(tz^{-1})^{-N_{i,j+1}} w_{i-(j+1)} t^{N_{i,j}} w_{i+(j+1)} =_{\mathcal{P}} 1 \quad \text{if } \sigma_{i-(j+1)} = 1$$

has area $\lesssim_{\mathcal{P}} (|w_{i-(j+1)}| + |w_{i+(j+1)}|)^2$ in M_Φ. While if $\alpha_j = z$, then Lemma 4.25(1) implies that the identity

$$t^{-N_{i,j+1}} w_{i-(j+1)} (tz^{-1})^{N_{i,j}} w_{i+(j+1)} \qquad \sigma_{i-(j+1)} = -1$$

has area $\lesssim_{\mathcal{P}} (|w_{i-(j+1)}| + |w_{i+(j+1)}|)^2$ in M_Φ.

Thus, it remains to estimate the areas of the identities

$$(tz^{-1})^{-N_{i,j+1}} w_{i-(j+1)} (tz^{-1})^{N_{i,j}} w_{i+(j+1)} =_{\mathcal{P}} 1 \quad \text{if} \quad \sigma_{i-(j+1)} = 1, \tag{4.22}$$

$$t^{-N_{i,j+1}} w_{i-(j+1)} t^{N_{i,j}} w_{i+(j+1)} =_{\mathcal{P}} 1 \quad \text{if} \quad \sigma_{i-(j+1)} = -1. \tag{4.23}$$

For this we reduce the words as follows:

- For (4.22) we shuffle to the left the word $(tz^{-1})^{N_{i,j}}$ past $w_{i-(j+1)}$ and place it to the right of the word $(tz^{-1})^{-N_{i,j+1}}$.
- For (4.23) we shuffle to the left the word $t^{N_{i,j}}$ past $w_{i-(j+1)}$ and place it to the right of the word $t^{-N_{i,j+1}}$.

Each one of these manipulations has a total area $\lesssim_{\mathcal{P}} |N_{i,j}||w_{i-(j+1)}|$. Thus, after performing the manipulations described above and the corresponding free cancellations we obtain the identities

$$\begin{aligned}(tz^{-1})^{-(N_{i-(j+1)}+N_{i+(j+1)})}w_{i-(j+1)}w_{i+(j+1)} =_{\mathcal{P}} 1 &\quad \text{if} \quad \sigma_{i-(j+1)} = 1,\\ t^{-(N_{i-(j+1)}+N_{i+(j+1)})}w_{i-(j+1)}w_{i+(j+1)}1 =_{\mathcal{P}} &\quad \text{if} \quad \sigma_{i-(j+1)} = -1\end{aligned}$$

in M_Φ with area $\lesssim_{\mathcal{P}} |N_{i,j}||w_{i-(j+1)}|$. Since $|N_{i-(j+1)} + N_{i+(j+1)}| \lesssim_{\mathcal{P}} |w_{i-(j+1)}| + |w_{i+(j+1)}|$, it follows that each one of these last identities has area $\lesssim_{\mathcal{P}} (|w_{i-(j+1)}| + |w_{i+(j+1)}|)^2$.

Thus, we conclude that, overall each of the identities (see (4.20) and (4.21))

$$\begin{aligned}(tz^{-1})^{-N_{i,j+1}}w_{i-(j+1)}(t\alpha_j^{-1})^{N_{i,j}}w_{i+(j+1)} =_{\mathcal{P}} 1 &\quad \text{if } \sigma_{i-(j+1)} = 1,\\ t^{-N_{i,j+1}}w_{i-(j+1)}(t\alpha_j^{-1})^{N_{i,j}}w_{i+(j+1)} =_{\mathcal{P}} 1 &\quad \text{if } \sigma_{i-(j+1)} = -1\end{aligned}$$

has area

$$\lesssim_{\mathcal{P}} (|w_{i-(j+1)}| + |w_{i+(j+1)}|)^2 + |N_{i,j}|\big(|w_{i-(j+1)}| + |w_{i+(j+1)}|\big) \tag{4.24}$$

in M_Φ.

Finally, summing the area estimates from the induction hypothesis (4.19) and from (4.24), we get that the identity

$$c^{\sigma_{i-(j+1)}}w_{i-(j+1)}\dots c^{\sigma_i}w_ic^{\sigma_{i+1}}\dots w_{i+(j+1)}c^{\sigma_{i+(j+1)+1}} =_{\mathcal{P}} (t\alpha_j)^{N_{i,j+1}}$$

has total area

$$\lesssim_{\mathcal{P}} |w_i|^2 + \sum_{k=1}^{j+1}\big(|w_{i-k}| + |w_{i+k}|\big)^2 + \sum_{k=1}^{j+1}|N_{i,k}|\big(|w_{i-k}| + |w_{i+k}|\big).$$

This finishes the proof of the induction step. □

Lemma 4.33 deals with the situation where the pinching words are of the form $w_1t^Nw_2$ or $w_1(tz^{-1})^Nw_2$. Namely, it deals with estimating the area of reducing parts of the dual tree that are isometric to a line segment, i.e. with contracting branches of the dual tree where every vertex has degree $\leqslant 2$. In general, the vertices of the dual tree can have degree $\geqslant 2$. The following results estimate the area in such situations, which we refer to as mixed pinching word.

Lemma 4.34 (Area to reduce a mixed pinching word)**.** *Let $s \geqslant 1$. For every $i \in \{1,\dots,s+1\}$ let $w_i = w_i(X,t,1)$, $N_i := \varepsilon_t(w_i)$, and for every $i \in \{1,\dots,s\}$*

let $M_i \in \mathbb{Z}$ *and* $v_i \in \{t^{M_i}, (tz^{-1})^{M_i}\}$. *Let* $M := \sum_{i=1}^{s+1} N_i + \sum_{i=1}^{s} M_i$. *If one of the two identities*

(1) $c^{-1}w_1v_1 \dots w_sv_sw_{s+1}c =_{\mathcal{P}} (tz^{-1})^M$
(2) $cw_1v_1 \dots w_sv_sw_{s+1}c^{-1} =_{\mathcal{P}} t^M$

holds in M_Φ*, then it has area*

$$\lesssim_{\mathcal{P}} \Big(\sum_{i=1}^{s+1}|w_i|\Big)^2 + \Big(\sum_{i=1}^{s+1}|w_i|\Big)\Big(\sum_{i=1}^{s}|v_i|\Big) + \sum_{\substack{i,j=1\\ i\neq j}}^{s}|v_i||v_j| + \sum_{i=1}^{s}|v_i|.$$

Proof. Define $\mathcal{I}_1, \mathcal{I}_z \subseteq \{1,\dots,s\}$ such that $i \in \mathcal{I}_1$ if and only if $v_i := t^{M_i}$, and $i \in \mathcal{I}_z$ if and only if $v_i := (tz^{-1})^{M_i}$. We only prove the area for the identity *(1)*, the area estimates for *(2)* are analogous.

Suppose that the identity

$$c^{-1}w_1v_1 \dots w_sv_sw_{s+1}c =_{\mathcal{P}} (tz^{-1})^M$$

holds in M_ϕ. This last identity with the aid of Lemma 4.22 can be transformed with area $\lesssim_{\mathcal{P}} |M|$ to the identity

$$t^{-M}w_1v_1 \dots w_sv_sw_{s+1} =_{\mathcal{P}} 1. \tag{4.25}$$

To estimate the area of the identity (4.25) we proceed as follows.

First shuffle to the left, starting with the first from left to right, all the v_i such that $i \in \mathcal{I}_1$ appearing in the above null-homotopic word. We gather them to the right of t^{-M}. Second we shuffle to the left all the v_j such that $j \in \mathcal{I}_z$ and gather them to the right of the v_i's shuffled in the previous step. After this shuffling we obtain the word

$$t^{-M}t^{\sum_{i\in\mathcal{I}_1} M_i}(tz^{-1})^{\sum_{i\in\mathcal{I}_z} M_i}w_1 \dots w_{s+1} \tag{4.26}$$

Recall $M = \sum_{i=1}^{s+1} N_i + \sum_{i=1}^{s} M_i$, which by definition of the partitioning sets $\mathcal{I}_1$ and $\mathcal{I}_z$ is equal to $\sum_{i=1}^{s+1} N_i + \sum_{i\mathcal{I}_1} M_i + \sum_{i\in\mathcal{I}_z} M_i$. Thus, by setting $R = \big(\sum_{j\in\mathcal{I}_Z} M_j + \sum_{i=1}^{s+1} N_i\big)$, the word in (4.26) can be freely reduced to the word

$$t^{-R}(tz^{-1})^{\sum_{i\in\mathcal{I}_z} M_i}w_1 \dots w_{s+1}.$$

This shuffling process has area

$$\lesssim_{\mathcal{P}} \Big(\sum_{i=1}^{s+1}|w_i|\Big)\Big(\sum_{i=1}^{s}|v_i|\Big) + \Big(\sum_{i\in\mathcal{I}_1}|v_i|\Big)\Big(\sum_{j\in\mathcal{I}_z}|v_j|\Big)$$

$$\lesssim_{\mathcal{P}} \Big(\sum_{i=1}^{s+1}|w_i|\Big)\Big(\sum_{i=1}^{s}|v_i|\Big) + \sum_{\substack{i,j=1\\ i\neq j}}^{s}|v_i||v_j|.$$

We point out that in the case of *(2)*, we should first shuffle all the v_i with $i \in \mathcal{J}_z$, and then the corresponding one with indexes in $\mathcal{J}_1$.

Second, observe that since the word (4.26) is null-homotopic in $M_\phi \hookrightarrow M_\Phi$, the exponent sum in x_1's (respectively x_2's) has to be zero. In particular, for $j \in \{1,2\}$ we have that

$$\varepsilon_{x_j}(z^{-1}) \cdot \big(\sum_{i\in\mathcal{I}_z} M_i\big) + \sum_{i=1}^{s+1}\varepsilon_{x_j}(w_i) = 0.$$

In particular, since at least for one $j \in \{1,2\}$ we have that $\varepsilon_{x_j}(z) \neq 0$, it follows that

$$\Big|\sum_{i\in\mathcal{I}_z} M_i\Big| \leqslant \sum_{i=1}^{s+1}|w_i|.$$

Thus, also $|R| \leqslant 2\sum_{i=1}^{s+1}|w_i|$. Therefore, the identity

$$t^{-R}(tz^{-1})^{\sum_{i\in\mathcal{I}_z} M_i} w_1 \ldots w_{s+1} =_{\mathcal{P}} 1$$

has area $\lesssim_{\mathcal{P}} \Big(\sum\limits_{i=1}^{s+1}|w_i|\Big)^2$.

Overall, summing the areas, we have that the identity

$$c^{-1}w_1v_1\ldots w_sv_sw_{s+1}c =_{\mathcal{P}} (tz^{-1})^M$$

has area

$$\begin{aligned}
&\lesssim_{\mathcal{P}} |M| + \Big(\sum_{i=1}^{s+1}|w_i|\Big)\Big(\sum_{i=1}^{s}|v_i|\Big) + \sum_{\substack{i,j=1\\ i\neq j}}^{s}|v_i||v_j| + \Big(\sum_{i=1}^{s+1}|w_i|\Big)^2\\
&\lesssim_{\mathcal{P}} \sum_{i=1}^{s+1}|w_i| + \sum_{i=1}^{s}|v_i| + \Big(\sum_{i=1}^{s+1}|w_i|\Big)\Big(\sum_{i=1}^{s}|v_i|\Big) + \sum_{\substack{i,j=1\\ i\neq j}}^{s}|v_i||v_j| + \Big(\sum_{i=1}^{s+1}|w_i|\Big)^2\\
&\lesssim_{\mathcal{P}} \sum_{i=1}^{s}|v_i| + \Big(\sum_{i=1}^{s+1}|w_i|\Big)\Big(\sum_{i=1}^{s}|v_i|\Big) + \sum_{\substack{i,j=1\\ i\neq j}}^{s}|v_i||v_j| + \Big(\sum_{i=1}^{s+1}|w_i|\Big)^2.
\end{aligned}$$

in M_Φ which proves the statement. □

We now have the all the necessary tools to finish the proof of the upper bound on the Dehn function in Theorem 4.10 which we now treat.

4.4.1.5 Final step: contracting the dual tree

As mentioned before, the idea to reduce the null-homotopic word w in M_Φ is fairly simple: geometrically, the reduction amounts to contract the dual tree of a van Kampen diagram D for w to a base point. We do this in a controlled way by keeping track of the area of each step. For this we require Lemma 4.34.

We work without distinction with the geometric realisation of the dual tree such that each edge is isometric to the unit interval $[0,1]$, denote it by (T, d_T). We pick a base point in T to be any leaf and denote it by $\star$. By construction of T, see Definition 4.20, there is a bijection

$$V(T) \setminus \{\star\} \longrightarrow \{c\text{-corridors in } D\}$$

which assigns to every vertex $V \neq \star$ the c-corridor K_V dual to the first edge (starting from V) of the unique geodesic connecting V to $\star$, see Figure 4.7.

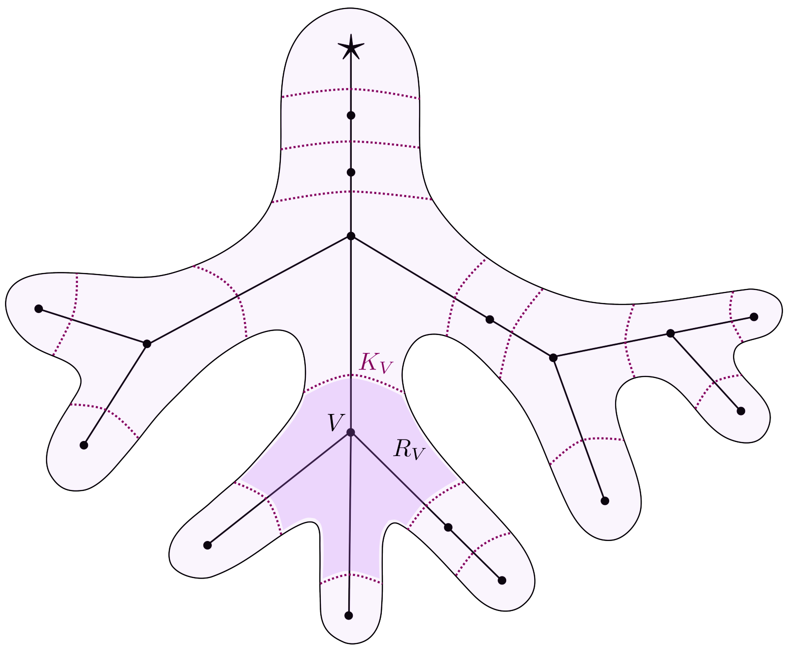

Figure 4.7: Picture of a van Kampen diagram and its dual rooted tree. The dashed lines represent c-corridors. The complementary region R_V dual to V is coloured by a darker tone.

Given a vertex $V \in V(T)$ we denote by R_V the complementary region of D dual to V. By the *area of* T we mean the area of the van Kampen diagram for which the tree is the dual of, and we write it as $\mathrm{Area}_{\mathcal{P}}(T)$.

Every vertex disconnects T into two disjoint trees, one that contains $\star$ and the other that does not, we denote the latter as T_V. We consider T_V as a rooted tree with root V. For a vertex V at distance $j \in \mathbb{N}$ from $\star$, every vertex $v \in T_V$ satisfies $d_T(v, \star) \geq j$. Since every c-corridor divides the van Kampen diagram into two disjoint subdiagrams, it is clear that T_{V_j} induces a subdiagram of D.

Recall that a word w in M_Φ can be written as

$$w(X, c, t) \coloneqq c^{\sigma_0} w_0(X, 1, t) \dots c^{\sigma_r} w_r(X, 1, t).$$

For each pinching word in w the reduction process starts by reducing all the c-corridors nested around it. After this we end up with a new word, different from w, for which the pinching words look like words of the form

$$w_{i_1} v_{i_1} \dots w_{i_s} v_{i_s} w_{i_{s+1}}$$

for some $s < r$ and $i_j \in \{0, \dots, r\}$. Corresponding to these pinching words there are residual words of the form

$$v \in \{(tz^{-1})^{\mathcal{N}}, t^{\mathcal{N}}\};$$

see Lemma 4.34 where we estimate the area to reduce such pinching words.

The following remark guarantees that in our area estimates the number of times that we take into account a subword $w_i(X, 1, t)$ of w (equivalently, a word in the boundary of D) is only once.

Remark 4.35. Geometrically, for every residual word $v \in \{t^{\mathcal{N}}, (tz^{-1})^{\mathcal{N}}\}$ there is a rooted subtree $T(v)$ of T such that $\star \notin T(v)$, see Figure 4.8. Observe that a residual word v is a word belonging to the boundary label of a c-corridor. At every step of the reduction process, the residual words define subsets of the index set $\{0, \dots, r\}$ of subwords of w as follows. Given a residual word

$$v_\ell \in \{t^{\mathcal{N}_\ell}, (tz^{-1})^{\mathcal{N}_\ell}\}$$

there is a subset

$$\mathcal{J}_\ell \subseteq \{0, \dots, r\}$$

such that a subword $w_i = w_i(X, 1, t)$ of w is a boundary word of the subdiagram of D dual to $T(v_\ell)$ if and only if $i \in \mathcal{J}_\ell$, see Figure 4.8. Since T is a tree, given two residual words v_k and v_ℓ with $k \neq \ell$, that belong to the *same* pinching word for a certain step of the reduction process, we have that

$$\mathcal{J}_k \cap \mathcal{J}_\ell = \emptyset.$$

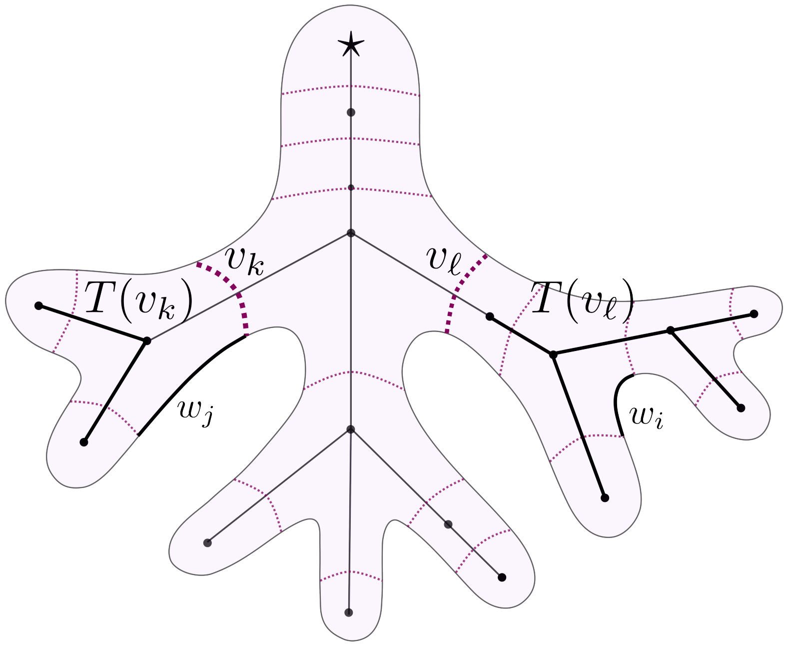

Figure 4.8: Two residual words v_k and v_ℓ with $k \neq \ell$ and their corresponding rooted subtrees $T(v_k)$ and $T(v_\ell)$; drawn with a thicker line. In the boundary of the diagrams dual to these trees there are words w_j with $j \in \mathcal{J}_k$ and w_i with $i \in \mathcal{J}_\ell$ belonging to the boundary of D: the parts of the boundary for which they are labels of is depicted with a thicker curve.

By definition of the set $\mathcal{J}_\ell$, since during the reduction process we do not create any $t^{\pm 1}$'s, it directly follows that

$$|\mathcal{N}_\ell| \leqslant \sum_{i \in \mathcal{J}_\ell} |\epsilon_t(w_i)|,$$

where the words w_i with $i \in \mathcal{J}_\ell$ took part in *previous* reduction steps.

Remark 4.36. The boundary word corresponding to the diagram consisting of the dual of $T(v_\ell)$ and the c-corridor that separates $T(v_\ell)$ from T, is a word of the form

$$v_\ell^{-1} \cdot \left(\prod_{i \in \mathcal{J}_\ell} c^{\sigma_i} w_i \right) c^{\pm 1}$$

for $\alpha \in \{1, z\}$ and $c^{\pm 1}(t\alpha^{-1})^{\mathcal{N}_\ell} c^{\mp 1} v_\ell$ being the boundary word of the c-corridor that separates $T(v_\ell)$ from T, see Figure 4.9.

The proof of the upper bound in Theorem 4.10 follows from the following result which we prove by descending induction on the distance to the root $\star$ in the dual tree T.

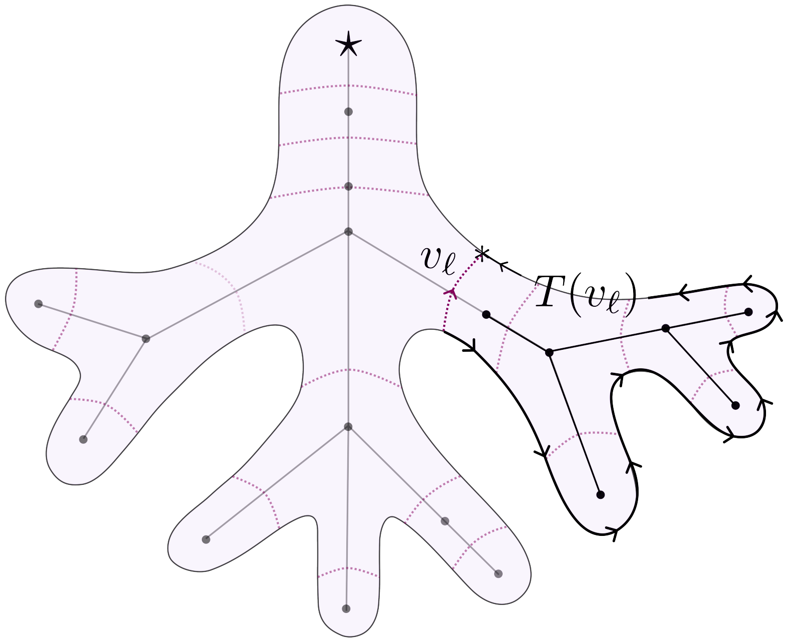

Figure 4.9: The word that labels the boundary of the subdiagram dual to T_ℓ union the c-corridor with boundary word $c^{\pm 1}(t\alpha^{-1})^{\mathcal{N}_\ell} c^{\mp 1} v_\ell$. Reading in the counter-clock wise direction starting from $*$ we read the word $v_\ell^{-1} \cdot \left(\prod_{i \in \mathcal{J}_\ell} c^{\sigma_i} w_i\right) c^{\pm 1}$.

Proposition 4.37 (Tree retraction). *Let $w \coloneqq w(X, c, t)$ be a freely reduced null-homotopic word in M_Φ of length n, written as*

$$w(X, c, t) \coloneqq c^{\sigma_0} w_0(X, 1, t) \dots c^{\sigma_j} w_j(X, 1, t) c^{\sigma_{j+1}} \dots c^{\sigma_r} w_r(X, 1, t), \tag{4.27}$$

such that for every $i \in \{0, \dots, r\}$ we have that $\sigma_i \in \{1, -1\}$ and denote $w_i \coloneqq w_i(X, 1, t)$. Then,

$$\mathrm{Area}_{\mathcal{P}}(w) \lesssim_{\mathcal{P}} \left(\sum_{i=1}^{r} |w_i|\right)^2 \lesssim_{\mathcal{P}} n^2$$

Proof. Let $w \coloneqq w(X, t, c)$ be a null-homotopic word in M_Φ of length n. Let D be a van-Kampen diagram for w in $\mathcal{P}$ and let T be the dual tree of D. As mentioned above, the proof is by descending induction on $d_T(\cdot, \star)$.

The base of induction corresponds to a vertex at maximal distance m from $\star$, which must be a leaf. In this, case, the complementary region dual to this vertex has as boundary label a pinching word for w, say $w_i \coloneqq w_i(X, 1, t)$. Contracting the leaf, say V, corresponds to not only consider the area of the complementary region R_V but also the c-corridor that is dual to the unique edge adjacent to this leaf, this together amounts to the area of the null-homotopic word

$$(t\alpha^{-1})^{\varepsilon_t(w_i)} c^{\pm 1} w_i c^{\mp 1} =_{\mathcal{P}} 1,$$

with $\alpha \in \{1, z\}$. By Corollary 4.30 this word has area $\lesssim_{\mathcal{P}} |w_i|^2 + |w_i|$. This proves the base of induction.

We now suppose that the statement is true for every $j+1$. To precisely state what this means we need some notation. Let V_{j+1} be a vertex of degree $d(j+1)+1$ whose distance to $\star$ is $j+1$. Let $\{V^{(1,j+1)}, \ldots, V^{(d(j+1),j+1)}\}$ be the set of vertices adjacent to V_{j+1} that are at distance $j+2$ from $\star$. Let

$$\mathcal{B} = \{0, \ldots, r\}$$

be the set of indices of words that belong to w, see (4.27). We define the following sets (see Figure 4.10):

(1) For every $\ell \in \{1, \ldots, d(j+1)\}$ define $\mathcal{J}_\ell^{(j+1)}$ as the set of elements $i \in \mathcal{B}$ such that w_i is a boundary word of the dual diagram of $R_{V^{(\ell,j+1)}} \cup T_{V^{(\ell,j+1)}}$.
(2) $\mathcal{J}_T^{(j+1)} := \bigcup_{\ell=1}^{d(j+1)} \mathcal{J}_\ell^{(j+1)}$.
(3) $\mathcal{J}_R^{(j+1)}$ is the set of $i \in \mathcal{B}$ such that w_i is a boundary word of the dual diagram of $R_{V_{j+1}}$.
(4) $\mathcal{J}^{(j+1)} := \mathcal{J}_T^{(j+1)} \cup \mathcal{J}_R^{(j+1)}$.

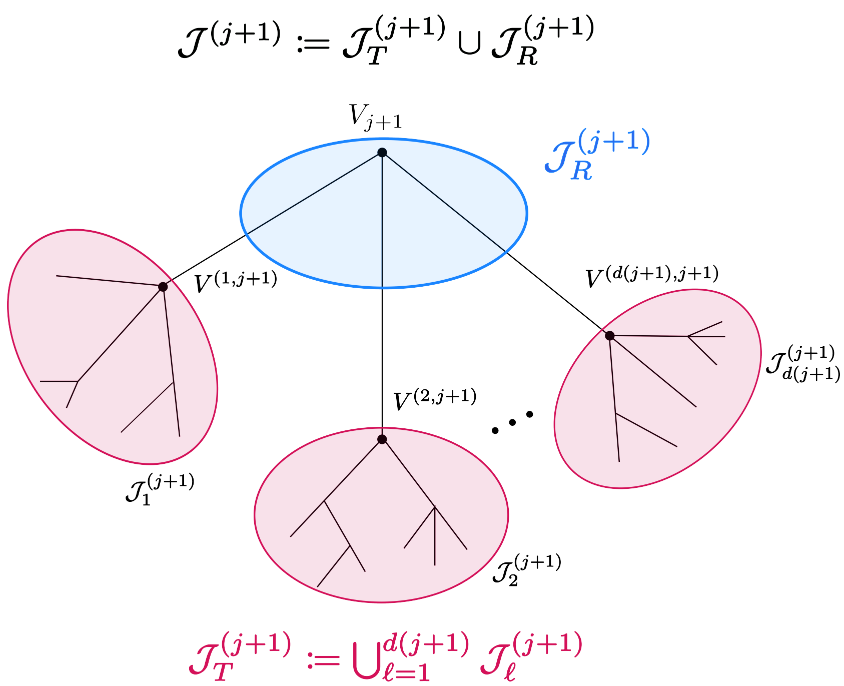

Figure 4.10: A picture of the rooted tree $T_{V_{j+1}}$ where the distinct index sets corresponding to the vertex V_{j+1} are depicted with ovals. The blue top oval represents the index set $\mathcal{J}_R^{(j+1)}$. The three bottom red ovals represent subsets of the index set $\mathcal{J}_T^{(j+1)}$

For the induction hypothesis assume that

$$\mathrm{Area}_{\mathcal{P}}(T_{V_{j+1}} \cup R_{V_{j+1}}) \lesssim_{\mathcal{P}} \sum_{k,\ell \in \mathcal{J}^{(j+1)}} |w_k||w_\ell|.$$

We now prove the induction step. Let V_j be a vertex of degree $d(j)+1$ whose distance to $\star$ is j. Let $\{V^{(1,j)}, \dots, V^{(d(j),j)}\}$ be set of vertices adjacent to V_j which are at distance $j+1$ from $\star$. Following the notation for the induction hypothesis we define the sets:

(1) For every $i \in \{1, \dots, d(j)\}$ set

$$\mathcal{J}^{(i,j)} \coloneqq \{k \in \mathcal{B} \mid w_k \text{ is a boundary word of } T_{V^{(i,j)}} \cup R_{V^{(i,j)}}\}.$$

(2) $\mathcal{J}_T^{(j)} \coloneqq \bigcup_{i=1}^{d(j)} \mathcal{J}^{(i,j)}$.

(3) $\mathcal{J}_R^{(j)} \coloneqq \{i \in \mathcal{B} \mid w_i \text{ is a boundary word of } R_{V_j}\}$.

(4) $\mathcal{J}^{(j)} \coloneqq \mathcal{J}_R^{(j)} \cup \mathcal{J}_T^{(j)}$.

We want to estimate $\mathrm{Area}_{\mathcal{P}}(T_{V_j} \cup R_{V_j})$, From the definition, it follows that

$$\mathrm{Area}_{\mathcal{P}}(T_{V_j} \cup R_{V_j}) = \mathrm{Area}_{\mathcal{P}}(R_{V_j}) + \sum_{i=1}^{d(j)} \mathrm{Area}_{\mathcal{P}}(T_{V^{(i,j)}} \cup R_{V^{(i,j)}}).$$

We estimate the terms $\mathrm{Area}_{\mathcal{P}}(R_{V_j})$ and $\mathrm{Area}_{\mathcal{P}}(T_{V^{(i,j)}} \cup R_{V^{(i,j)}})$ separately.

We start with $\mathrm{Area}_{\mathcal{P}}(T_{V^{(i,j)}} \cup R_{V^{(i,j)}})$. For every $i \in \{1, \dots, d(j)\}$, since the vertex $V^{(i,j)}$ is at distance $j+1$ from $\star$, it follows from the induction hypothesis that

$$\mathrm{Area}_{\mathcal{P}}(T_{V^{(i,j)}} \cup R_{V^{(i,j)}}) \lesssim_{\mathcal{P}} \sum_{k,\ell \in \mathcal{J}^{(i,j)}} |w_k||w_\ell|. \tag{4.28}$$

Now, for $\mathrm{Area}_{\mathcal{P}}(R_{V_j})$ we have two possibilities to consider. Namely when the degree of V_j is 2 and the case where it is greater or equal than 3.

On one hand, if the degree is 2, i.e. if $d(j)+1 = 2$, then there is a single vertex $V^{(1,j)}$ adjacent to V_j which is at distance $j+1$ from the root $\star$. Observe that in this case $\mathcal{J}_R^{(j)}$ contains only two words. Thus, it follows from Corollary 4.26 that

$$\mathrm{Area}_{\mathcal{P}}(R_{V_j}) \lesssim_{\mathcal{P}} \Big(\sum_{i \in \mathcal{J}_R^{(j)}} |w_i| \Big)^2 \tag{4.29}$$

Therefore, from the area estimates (4.28) and (4.29) we get

$$\begin{aligned}
\mathrm{Area}_{\mathcal{P}}(T_{V_j} \cup R_{V_j}) &\lesssim_{\mathcal{P}} \Big(\sum_{i \in \mathcal{J}_R^{(j)}} |w_i| \Big)^2 + \sum_{i=1}^{d(j)} \sum_{k,\ell \in \mathcal{J}^{(i,j)}} |w_k||w_\ell| \\
&\lesssim_{\mathcal{P}} \Big(\sum_{i \in \mathcal{J}_R^{(j)}} |w_i| \Big)^2 + \sum_{k,\ell \in \mathcal{J}_T^{(j)}} |w_k||w_\ell|.
\end{aligned}$$

Since by definition the sets $\mathcal{J}_R^{(j)}$ and $\mathcal{J}_T^{(j)}$ are disjoint we can estimate the above area estimate as

$$\mathrm{Area}_{\mathcal{P}}(T_{V_j} \cup R_{V_j}) \lesssim_{\mathcal{P}} \sum_{k,\ell \in \mathcal{J}^j} |w_k||w_\ell|.$$

This proves the induction step in the case where the degree of V_j is 2.

On the other hand, if $d(j)+1 > 2$, then from Lemma 4.34 and the fact that for every $i \in \{1, \dots, d(j)\}$ we have

$$|v_i| \lesssim_{\mathcal{P}} \sum_{k \in \mathcal{J}^{(i,j)}} |w_i|,$$

it follows that

$$\begin{aligned}
\mathrm{Area}_{\mathcal{P}}(R_{V_j}) \lesssim_{\mathcal{P}} &\Big(\sum_{i \in \mathcal{J}_R^{(j)}} |w_i| \Big)^2 + \Big(\sum_{i \in \mathcal{J}_R^{(j)}} |w_i| \Big) \Big(\sum_{i=1}^{d(j)} \sum_{k \in \mathcal{J}^{(i,j)}} |w_i| \Big) \\
&+ \sum_{\substack{\nu,\mu=1 \\ \nu \neq \mu}}^{d(j)} \sum_{k \in \mathcal{J}^{(\nu,j)}} \sum_{\ell \in \mathcal{J}^{(\mu,j)}} |w_k||w_\ell| + \sum_{i \in \mathcal{J}_R^{(j)}} |w_i|. \qquad (4.30)
\end{aligned}$$

Adding the two area estimates in (4.28) and (4.30) we get

$$\begin{aligned}
\mathrm{Area}_{\mathcal{P}}(T_{V_j} \cup R_{V_j}) \lesssim_{\mathcal{P}} &\Big(\sum_{i \in \mathcal{J}_R^{(j)}} |w_i| \Big)^2 + \Big(\sum_{i \in \mathcal{J}_R^{(j)}} |w_i| \Big) \Big(\sum_{i=1}^{d(j)} \sum_{k \in \mathcal{J}^{(i,j)}} |w_i| \Big) \\
&+ \sum_{\substack{\nu,\mu=1 \\ \nu \neq \mu}}^{d(j)} \sum_{k \in \mathcal{J}^{(\nu,j)}} \sum_{\ell \in \mathcal{J}^{(\mu,j)}} |w_k||w_\ell| + \sum_{i=1}^{d(j)} \sum_{k,\ell \in \mathcal{J}^{(i,j)}} |w_k||w_\ell| \\
&+ \sum_{i \in \mathcal{J}_R^{(j)}} |w_i|. \qquad (4.31)
\end{aligned}$$

Finally, we have the following two observations

1. For $\nu, \mu \in \{1, \dots, d(j)\}$ if $\nu \neq \mu$, then $\mathcal{J}^{(\nu,j)} \cap \mathcal{J}^{(\mu,j)} = \emptyset$.
2. For every $i \in \{1, \dots, d(j)\}$ the sets $\mathcal{J}_R^{(j)}$ and $\mathcal{J}^{(i,j)}$ are disjoint.

By the first of the above observations the terms

$$\sum_{\substack{\nu,\mu=1\\ \nu\neq\mu}}^{d(j)} \sum_{k\in\mathcal{J}^{(\nu,j)}} \sum_{\ell\in\mathcal{J}^{(\mu,j)}} |w_k||w_\ell| \quad \text{and} \quad \sum_{i=1}^{d(j)} \sum_{k,\ell\in\mathcal{J}^{(i,j)}} |w_k||w_\ell|$$

in (4.31) are disjoint. Therefore, together the two above observations imply that

$$\mathrm{Area}_{\mathcal{P}}(T_{V_j} \cup R_{V_j}) \lesssim_{\mathcal{P}} \sum_{k,\ell\in\mathcal{J}^{(j)}} |w_k||w_\ell|.$$

which proves the induction step in the case where $d(j) + 1 > 2$. This concludes the induction step and therefore the proof of Proposition 4.37. This concludes the proof of Theorem 4.10. □

Bibliography

[Abe87] H. Abels. *Finite presentability of S-arithmetic groups. Compact presentability of solvable groups*, volume 1261 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[Ago13] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Groves, and Manning.

[All98] D. Allcock. An isoperimetric inequality for the Heisenberg groups. *Geom. Funct. Anal.*, 8(2):219–233, 1998.

[Alo90] J. M. Alonso. Inégalités isopérimétriques et quasi-isométries. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(12):761–764, 1990.

[Bas72] H. Bass. The degree of polynomial growth of finitely generated nilpotent groups. *Proc. London Math. Soc. (3)*, 25:603–614, 1972.

[BB97] M. Bestvina and N. Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[BB00] N. Brady and M. R. Bridson. There is only one gap in the isoperimetric spectrum. *Geom. Funct. Anal.*, 10(5):1053–1070, 2000.

[BF92] M. Bestvina and M. Feighn. A combination theorem for negatively curved groups. *J. Differential Geom.*, 35(1):85–101, 1992.

[BFH00] M. Bestvina, M. Feighn, and M. Handel. The Tits alternative for $\mathrm{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms. *Ann. of Math. (2)*, 151(2):517–623, 2000.

[BFH05] M. Bestvina, M. Feighn, and M. Handel. The Tits alternative for $\mathrm{Out}(F_n)$. II. A Kolchin type theorem. *Ann. of Math. (2)*, 161(1):1–59, 2005.

[BG84] K. S. Brown and R. Geoghegan. An infinite-dimensional torsion-free FP_∞ group. *Invent. Math.*, 77(2):367–381, 1984.

[BG96] M. R. Bridson and S. M. Gersten. The optimal isoperimetric inequality for torus bundles over the circle. *Quart. J. Math. Oxford Ser. (2)*, 47(185):1–23, 1996.

[BG10] Martin R. Bridson and Daniel Groves. The quadratic isoperimetric inequality for mapping tori of free group automorphisms. *Mem. Amer. Math. Soc.*, 203(955):xii+152, 2010.

[BGSS92] G. Baumslag, S. M. Gersten, M. Shapiro, and H. Short. Automatic groups and amalgams—a survey. In *Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)*, volume 23 of *Math. Sci. Res. Inst. Publ.*, pages 179–194. Springer, New York, 1992.

[BH92] M. Bestvina and M. Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.

[BH99] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[Bie81] R. Bieri. *Homological dimension of discrete groups*. Queen Mary College Mathematics Notes. Queen Mary College, Department of Pure Mathematics, London, second edition, 1981.

[BKMM12] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher. Geometry and rigidity of mapping class groups. *Geom. Topol.*, 16(2):781–888, 2012.

[BKS20] N. Brady, R. Kropholler, and I. Soroko. Homological Dehn functions of groups of type FP_2. *arXiv:2012.00730*, 2020.

[BMS93] G. Baumslag, C. F. Miller, III, and H. Short. Isoperimetric inequalities and the homology of groups. *Invent. Math.*, 113(3):531–560, 1993.

[BMV07] O. Bogopolski, A. Martino, and E. Ventura. The automorphism group of a free-by-cyclic group in rank 2. *Communications in Algebra*, 35(5):1675–1690, 2007.

[Bow95] B. H. Bowditch. A short proof that a subquadratic isoperimetric inequality implies a linear one. *Michigan Math. J.*, 42(1):103–107, 1995.

[BP94] M. R. Bridson and Ch. Pittet. Isoperimetric inequalities for the fundamental groups of torus bundles over the circle. *Geom. Dedicata*, 49(2):203–219, 1994.

[Bri99] M. R. Bridson. Fractional isoperimetric inequalities and subgroup distortion. *J. Amer. Math. Soc.*, 12(4):1103–1118, 1999.

[Bri00] P. Brinkmann. Hyperbolic automorphisms of free groups. *Geom. Funct. Anal.*, 10(5):1071–1089, 2000.

[Bri02] M. R. Bridson. The geometry of the word problem. In *Invitations to geometry and topology*, volume 7 of *Oxf. Grad. Texts Math.*, pages 29–91. Oxford Univ. Press, Oxford, 2002.

[Bro73] I. D. Brown. Representation of finitely generated nilpotent groups. *Pacific J. Math.*, 45:13–26, 1973.

[BRS07] N. Brady, T. Riley, and H. Short. *The geometry of the word problem for finitely generated groups.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2007. Papers from the Advanced Course held in Barcelona, July 5–15, 2005.

[BS19] N. Brady and I. Soroko. Dehn functions of subgroups of right-angled Artin groups. *Geometriae Dedicata*, 200(1):197–239, 2019.

[CdlH16a] Y. Cornulier and P. de la Harpe. *Metric geometry of locally compact groups*, volume 25 of *EMS Tracts in Mathematics.* European Mathematical Society (EMS), Zürich, 2016.

[CdlH16b] Y. Cornulier and P. de la Harpe. *Metric geometry of locally compact groups*, volume 25 of *EMS Tracts in Mathematics.* European Mathematical Society (EMS), Zürich, 2016. Winner of the 2016 EMS Monograph Award.

[CF17] W. Carter and M. Forester. The Dehn functions of Stallings–Bieri groups. *Mathematische Annalen*, 368(1):671–683, 2017.

[CFP96] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[CGMM25] Y.-H. Chang, J. García-Mejía, and M. Migliorini. Complete classification of the Dehn functions of Bestvina–Brady groups, July 2025. arXiv:2507.07566.

[Cha07] R. Charney. An introduction to right-angled Artin groups. *Geom. Dedicata*, 125:141–158, 2007.

[Cha21] Y.C. Chang. Identifying Dehn functions of Bestvina-Brady groups from their defining graphs. *Geom. Dedicata*, 214:211–239, 2021.

[CM18] P. E. Caprace and N. Monod, editors. *New directions in locally compact groups*, volume 447 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2018.

[Cor16] Y. Cornulier. Gradings on Lie algebras, systolic growth, and cohopfian properties of nilpotent groups. *Bull. Soc. Math. France*, 144(4):693–744, 2016.

[Cor17] Y. Cornulier. On sublinear bilipschitz equivalence of groups. arXiv preprint arXiv:1702.06618, 2017.

[Cor18] Y. Cornulier. On the quasi-isometric classification of locally compact groups. In *New directions in locally compact groups*, volume 447 of *London Math. Soc. Lecture Note Ser.*, pages 275–342. Cambridge Univ. Press, Cambridge, 2018.

[CSD21] T. Ceccherini-Silberstein and M. D'Adderio. *Topics in groups and geometry—growth, amenability, and random walks*. Springer Monographs in Mathematics. Springer, Cham, 2021.

[CT17] Y. Cornulier and R. Tessera. Geometric presentations of Lie groups and their Dehn functions. *Publ. Math. Inst. Hautes Études Sci.*, 125:79–219, 2017.

[dCT10] Y. de Cornulier and R. Tessera. Metabelian groups with quadratic Dehn function and Baumslag–Solitar groups. *Confluentes Math.*, 2(4):431–443, 2010.

[Deh11] M. Dehn. Über unendliche diskontinuierliche Gruppen. *Mathematische Annalen*, 71:116–144, 1911.

[DERY09] W. Dison, M. Elder, T. R. Riley, and R. Young. The Dehn function of Stallings' group. *Geom. Funct. Anal.*, 19(2):406–422, 2009.

[dG07] W. A. de Graaf. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. *J. Algebra*, 309(2):640–653, 2007.

[Dis08] W. Dison. An isoperimetric function for Bestvina–Brady groups. *Bull. Lond. Math. Soc.*, 40(3):384–394, 2008.

[DK18] C. Druţu and M. Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[Dro87] C. Droms. Isomorphisms of graph groups. *Proc. Amer. Math. Soc.*, 100(3):407–408, 1987.

[Dt02] Cornelia Dru¸tu. Quasi–isometry invariants and asymptotic cones. *Internat. J. Algebra Comput.*, 12(1-2):99–135, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[ECH+92] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.

[FH91] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[FM00] B. Farb and L. Mosher. Problems on the geometry of finitely generated solvable groups. In *Crystallographic groups and their generalizations (Kortrijk, 1999)*, volume 262 of *Contemp. Math.*, pages 121–134. Amer. Math. Soc., Providence, RI, 2000.

[FR40] D. I. Fouxe-Rabinovitch. Über die Automorphismengruppen der freien Produkte. I. *Rec. Math. [Mat. Sbornik] N.S.*, 8/50:265–276, 1940.

[FR41] D. I. Fouxe-Rabinovitch. Über die Automorphismengruppen der freien Produkte. II. *Rec. Math. [Mat. Sbornik] N.S.*, 9/51:183–220, 1941.

[Ger93] S. M. Gersten. Isoperimetric and isodiametric functions of finite presentations. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 79–96. Cambridge Univ. Press, Cambridge, 1993.

[Gho23] P. Ghosh. Relative hyperbolicity of free-by-cyclic extensions. *Compos. Math.*, 159(1):153–183, 2023.

[GHR03] S. M. Gersten, D. F. Holt, and T. R. Riley. Isoperimetric inequalities for nilpotent groups. *Geom. Funct. Anal.*, 13(4):795–814, 2003.

[Gil87] N. D. Gilbert. Presentations of the automorphism group of a free product. *Proc. London Math. Soc. (3)*, 54(1):115–140, 1987.

[GJMK98] J. R. Gómez, A. Jimenéz-Merchán, and Y. Khakimdjanov. Low-dimensional filiform Lie algebras. *J. Pure Appl. Algebra*, 130(2):133–158, 1998.

[GJMNnV98] J. R. Gómez, A. Jiménez-Merchán, and J. Núñez Valdés. An algorithm to obtain laws of families of filiform Lie algebras. *Linear Algebra Appl.*, 279(1-3):1–12, 1998.

[GK21] T. Gotfredsen and D. Kyed. Cohomological induction and uniform measure equivalence. *Fund. Math.*, 255(1):19–49, 2021.

[GMLIP23] J. García-Mejía, C. Llosa Isenrich, and G. Pallier. On the Dehn functions of central products of nilpotent groups, October 2023. [Submitted]. arXiv:2310.11144.

[Gro81] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, 53:53–73, 1981.

[Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Gub06] V. S. Guba. The Dehn function of Richard Thompson's group F is quadratic. *Invent. Math.*, 163(2):313–342, 2006.

[Gui73] Y. Guivarc'h. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France*, 101:333–379, 1973.

[Hal15] B. Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.

[Ham06] U. Hamenstädt. Geometric properties of the mapping class group. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 215–232. Amer. Math. Soc., Providence, RI, 2006.

[Hel78] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[Kam33] R. V. Kampen, E. On Some Lemmas in the Theory of Groups. *Amer. J. Math.*, 55(1-4):268–273, 1933.

[Kar23] B. Karlhofer. Aut-invariant quasimorphisms on free products. *Ann. Math. Qué.*, 47(2):475–493, 2023.

[KL97] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Publications Mathématiques de l'IHÉS*, 86:115–197, 1997.

[Kne64] M. Kneser. Erzeugende und Relationen verallgemeinerter Einheitengruppen. *J. Reine Angew. Math.*, 214/215:345–349, 1964.

[Lau95] M. R. Laurence. A generating set for the automorphism group of a graph group. *J. London Math. Soc. (2)*, 52(2):318–334, 1995.

[LB17] A. Le Boudec. Compact presentability of tree almost automorphism groups. *Ann. Inst. Fourier (Grenoble)*, 67(1):329–365, 2017.

[LD17] E. Le Donne. A primer on Carnot groups: homogenous groups, Carnot-Carathéodory spaces, and regularity of their isometries. *Anal. Geom. Metr. Spaces*, 5(1):116–137, 2017.

[Lev09] G. Levitt. Counting growth types of automorphisms of free groups. *Geom. Funct. Anal.*, 19(4):1119–1146, 2009.

[LIPT23] C. Llosa Isenrich, G. Pallier, and R. Tessera. Cone-equivalent nilpotent groups with different Dehn functions. *Proc. Lond. Math. Soc. (3)*, 126(2):704–789, 2023.

[LS01] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[Lüc08] W. Lück. Survey on geometric group theory. *Münster J. Math.*, 1:73–108, 2008.

[LY21] E. Leuzinger and R. Young. Filling functions of arithmetic groups. *Annals of Mathematics*, 193(3):733–792, 2021.

[Lyn66] R. C. Lyndon. On Dehn's algorithm. *Math. Ann.*, 166:208–228, 1966.

[Mag08] L. Magnin. Adjoint and trivial cohomologies of nilpotent complex Lie algebras of dimension ≤ 7. *Int. J. Math. Math. Sci.*, pages Art. ID 805305, 12, 2008.

[Mag10] L. Magnin. Determination of 7-dimensional indecomposable nilpotent complex Lie algebras by adjoining a derivation to 6-dimensional Lie algebras. *Algebr. Represent. Theory*, 13(6):723–753, 2010.

[Mal51] A. I. Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Translation*, 1951(39):33, 1951.

[Mon02] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2002.

[Nic21] F. Nicolás. *Finitely generated normal subgroups of Kähler groups.* Theses, Université de Strasbourg, June 2021.

[Ols91] A. Y. Olshanskii. Hyperbolicity of groups with subquadratic isoperimetric inequality. *Internat. J. Algebra Comput.*, 1(3):281–289, 1991.

[OS99] A. Y. Olshanskii and M. V. Sapir. Quadratic isometric functions of the heisenberg groups. a combinatorial proof. *Journal of Mathematical Sciences*, 93(6):921–927, 1999.

[Osi01] D. V. Osin. Subgroup distortions in nilpotent groups. *Comm. Algebra*, 29(12):5439–5463, 2001.

[Pan83] P. Pansu. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems*, 3(3):415–445, 1983.

[Pan89] P. Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[Pap96] P. Papasoglu. On the asymptotic cone of groups satisfying a quadratic isoperimetric inequality. *J. Differential Geom.*, 44(4):789–806, 1996.

[Pig04] A. Piggott. Detecting the growth of free group automorphisms by their action on the homology of subgroups of finite index, 2004.

[Pla04] A. N. Platonov. An isoparametric function of the Baumslag-Gersten group. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, 3:12–17, 70, 2004.

[PR19] K. Pueschel and T. Riley. Dehn functions of mapping tori of right-angled Artin groups. *arXiv preprint arXiv:1906.09368*, 2019.

[Pue16] K. Pueschel. *On Residual Properties Of Groups And Dehn Functions For Mapping Tori Of Right Angled Artin Groups.* PhD thesis, Cornell University, 2016.

[Rag72] M. S. Raghunathan. *Discrete subgroups of Lie groups.* Springer-Verlag, New York-Heidelberg, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[Ril17] T. Riley. Dehn functions. In *Office hours with a geometric group theorist*, pages 146–175. Princeton Univ. Press, Princeton, NJ, 2017.

[Sap11] M. Sapir. Asymptotic invariants, complexity of groups and related problems. *Bull. Math. Sci.*, 1(2):277–364, 2011.

[Sau06] R. Sauer. Homological invariants and quasi-isometry. *Geom. Funct. Anal.*, 16(2):476–515, 2006.

[SBR02] M. V. Sapir, J. C. Birget, and E. Rips. Isoperimetric and isodiametric functions of groups. *Ann. of Math. (2)*, 156(2):345–466, 2002.

[Seg83] D. Segal. *Polycyclic groups*, volume 82 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1983.

[Ser89] H. Servatius. Automorphisms of graph groups. *J. Algebra*, 126(1):34–60, 1989.

[Sha04] Y. Shalom. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.*, 192(2):119–185, 2004.

[Tes18] R. Tessera. Locally compact groups as metric spaces. In *New directions in locally compact groups*, volume 447 of *London Math. Soc. Lecture Note Ser.*, pages 9–16. Cambridge Univ. Press, Cambridge, 2018.

[vdDW84] L. van den Dries and A. J. Wilkie. Gromov's theorem on groups of polynomial growth and elementary logic. *J. Algebra*, 89(2):349–374, 1984.

[Ver70] M. Vergne. Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes. *Bull. Soc. Math. France*, 98:81–116, 1970.

[War83] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[Wen11] S. Wenger. Nilpotent groups without exactly polynomial Dehn function. *J. Topol.*, 4(1):141–160, 2011.

[Wol68] J. A. Wolf. Growth of finitely generated solvable groups and curvature of Riemannian manifolds. *J. Differential Geometry*, 2:421–446, 1968.

[You08] R. Young. Averaged Dehn functions for nilpotent groups. *Topology*, 47(5):351–367, 2008.

[You13a] R Young. The Dehn function of SL(n;$\mathbb{Z}$). *Annals of Mathematics*, pages 969–1027, 2013.

[You13b] R. Young. Filling inequalities for nilpotent groups through approximations. *Groups Geom. Dyn.*, 7(4):977–1011, 2013.

[Zar24] M. C. B. Zaremsky. Finitely presented simple groups with at least exponential Dehn function, July 2024. arXiv:2305.15176v2.