Finiteness properties of totally disconnected locally compact groups

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Introduction

In group theory, finiteness properties play a central role in classifying and analyzing algebraic structures. These properties have extensive applications in areas such as algebraic topology, geometric group theory and algebraic geometry.
In the field of geometric group theory, the main idea is to consider groups as geometric objects or more precisely as metric spaces. To do so we can construct the Cayley graph of a given group G with respect to a finite generating set and equip it with the word metric, which induces a metric on G. A main tool in geometric group theory is to characterize groups up to quasi-isometries. This is possible since the choice of the metric of the group is itself only unique up to quasi-isometry.
Another way to study groups with an additional structure is to consider a group as a topological space. Here we consider groups equipped with a topology such that the group operation and forming inverses are continuous maps. Any group can be considered as a topological group by choosing the discrete topology. These groups are called discrete groups. Many classical definitions of topological spaces also hold in the context of topological groups, for example being Hausdorff or connected. We will introduce the main concepts and ideas of this theory in Chapter 1.
If we equip a group with additional structures we are able to study the properties of the group as an algebraic object in contrast to the properties of the new structures. In the following, we focus on topological groups, which we equip with a metric and describe interactions between groups as algebraic objects, metric spaces and topological spaces.

Classical geometric group theory usually focuses on discrete groups, or in particular on finitely generated groups. For discrete groups there is a higher dimensional generalization of being finitely generated or finitely presented, which was introduced by R. Bieri and K. Brown as the topological and homological finiteness properties F_n and FP_n over some commutative ring R (see [4] and [9]). That these properties are a generalization of being finitely generated, respectively finitely presented, is due to the fact that being of type F_1 is equivalent to being finitely generated, and being of type F_2 is equivalent to being finitely presented. In the case of finitely presented groups the topological and homological finiteness properties over $\mathbb{Z}$ are equivalent. Similar to the lower dimensional cases of being finitely generated and finitely presented, these higher dimensional finiteness properties are quasi-isometry invariants.
One important question for quasi-isometry invariants of groups is if there are groups or classes of groups which can be classified by this property. In the case of finiteness properties of discrete groups, there are some known examples of groups or more precisely, of simple groups where this is the case.
One of the first examples of a group which is of type F_2, but not of type FP_3 over $\mathbb{Z}$ was introduced by J. Stalling [29]. This example was generalized by R. Bieri [3] to create

groups that are of type FP_n, but not of type FP_{n+1} over $\mathbb{Z}$ for each $n \in \mathbb{N}$. Other examples of groups that are of type F_n, but not of type FP_{n+1} over $\mathbb{Z}$ were discovered by H. Abels and K. Brown [1]. In 1997 Bestvina and Brady introduced a connection between the classical Morse theory and finiteness properties of right angled Artin groups (see [2]). Later R. Skipper, S. Witzel and M. Zaremsky used Thompson-like groups, called Röver-Nekrachevych groups, to create simple discrete groups which can also be separated by their finiteness properties (see [26] or [34]). A similar construction, which used cloning systems, is used in [27], to create groups of type F_∞.

Finiteness properties of topological groups are also studied in the context of totally disconnected locally compact (tdlc) groups. These groups arise naturally in the area of number theory and Lie theory. Key examples of such groups include profinite groups and automorphism groups of infinite trees. In 2016 I. Castellano and T. Weigel introduced a first generalization of being compactly generated or compactly presented for tdlc groups [13]. These ideas were further advanced by I. Castellano and G. Corob Cook, who also found analogies for most results of the classical finiteness properties of discrete groups in the context of tdlc groups [11]. In particular they showed that these finiteness properties are again a quasi-isometry invariant.

A first example of tdlc groups that have been analyzed on the basis of their finiteness properties are the almost automorphisms of trees. This class of groups includes the Higmann-Thompson group V_q and the Neretin group N_q. In 2015 R. Sauer and W. Thumann showed that all groups in this class are of type F_∞ (see [21]). In the following, we are interested in tdlc groups which can be classified by their finiteness properties, i. e. we are searching for tdlc groups which are of type F_n or FP_n over $\mathbb{Z}$, but not of type F_{n+1}, respectively FP_{n+1} over $\mathbb{Z}$.

In this thesis we show that there are two different constructions of generating tdlc groups from given groups preserving the finiteness properties. We use these constructions in a two step process to generate simple tdlc groups classified by their finiteness properties. In step one, we start with a discrete group and generate a tdlc group. In step two, we choose this tdlc group and a second discrete group to generate a simple tdlc group. Since both constructions preserve the finiteness properties we get a way to create simple tdlc groups which can be classified by their finiteness properties.

The first construction we use is the Schlichting completion, which was introduced by G. Schlichting in 1980 [22]. The Schlichting completion associates a discrete group together with a commensurated subgroup to a tdlc group similar to the construction of the profinite completion. One well known example is the Neretin group, which can be described as a Schlichting completion of Higmann-Thomson group V_q with respect to a locally finite commensurated subgroup.
One aspect we are interested in are the finiteness properties of this construction. For the lower dimensional case A. Le Boudec has shown that the Schlichting completion is compactly presented, if the discrete group is finitely presented and the commensurated subgroup is finitely generated, [19]. In Chapter 3 we first introduce the construction and

to emphasize some of its properties. After that we generalize the theorem of A. Le Boudec such that we can deduce the lower dimensional finiteness properties of the discrete group from those of the Schlichting completion and the commensurated subgroup. The following two theorems, which were introduced in a joint work with R. Sauer [5], extend this to the higher dimensional case.

Theorem A ([5, Theorem 1.1])
Let $G := \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$. Let R be a commutative ring. Then the following holds.

(i) If Λ and G are of type FP_n over R, then Γ is of type FP_n over R.

(ii) If Λ and G are of type F_n, then Γ is of type F_n.

Theorem B ([5, Theorem 1.2])
Let $G := \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$. Let R be a commutative ring. Then the following holds.

(i) If Λ is of type FP_{n-1} over R and Γ is of type FP_n over R, then G is of type FP_n over R.

(ii) If Λ is of type F_{n-1} and Γ is of type F_n, then G is of type F_n.

These theorems allow us to construct new tdlc groups from known discrete groups, which can be classified by their finiteness properties, if the discrete groups are of type F_n or FP_n over R, but not of type F_{n+1} respectively FP_{n+1} over R.

In the case of the Neretin group N_q and Higmann-Thompson group V_q we can not use these results, since the commensurated subgroup is not finitely generated, which is an assumption in these theorems. From R. Sauer and W. Thumann [21] we know that they are of type F_∞ though. In Chapter 4, we look at the fixpoint subgroups of the tree almost automorphisms, generalizing the result of R. Sauer and W. Thumann [21].

Theorem C
Let $r \geq 1$, $q \geq 2$, $D \leq \mathrm{Sym}(q)$, $\mathtt{B}_q = \partial\mathcal{T}_q$ and $x \in r\mathtt{B}_q$. Then the fixpoint subgroup $(\mathcal{A}^D_{qr})_x$ of the tree almost automorphism group $\mathcal{A}^D_{qr}$ is of type F_∞.

For the proof of this theorem we use the same constructions as for the non-fixpoint case in [21]. In the last step we can use that the criterion of Brown also holds in the case of tdlc groups, due to [11].

The second construction are the universal Smith groups introduced by S. Smith (see [28]). These types of groups were constructed for creating examples of pairwise non-isomorphic simple tdlc groups and are a generalization of the universal Burger-Mozes groups introduced by M. Burger and S. Mozes in [10]. In particular, S. Smith has shown cases where these groups are compactly generated. This result can be generalized to statements about the higher dimensional topological and homological finiteness properties.

Theorem D

Let $M \leq \mathrm{Sym}(X)$, $N \leq \mathrm{Sym}(Y)$ be non-empty, closed with compact point stabilizers and $G \coloneqq \mathcal{U}(M, N)$. Assume M and N have only finitely many orbits and one of M and N is transitive. Then

(i) G is of type FP_n over $\mathbb{Z}$ if and only if M and N are of type FP_n over $\mathbb{Z}$,

(ii) for $n \geq 2$, G is of type F_n and not of type F_{n+1} if M and N are of type F_n and one of them is not of type F_{n+1}.

Since we are interested in simple tdlc groups, we can again use a result by S. Smith that provides conditions for when his construction creates simple tdlc groups (see [28]). Considering the results of Chapter 3 and Chapter 5 together with the criterion of simplicity of the universal Smith groups, we obtain a recipe for how we can construct simple tdlc groups classified by their finiteness properties.

Theorem E

Let Γ be a discrete group of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$ together with a commensurated subgroup Λ of type FP_n over $\mathbb{Z}$ and Γ be generated by $\{\gamma\Lambda\gamma^{-1} \mid \gamma \in \Gamma\}$. Let $M = \Gamma/\!/\Lambda$ and $N = \mathrm{Sym}(3)$. Then $\mathcal{U}(M, N)$ is a simple tdlc group of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$.

1 Basics

In this first chapter we introduce some basic definitions and statements of the theory of topological groups in particular of totally disconnected locally compact groups. For more details about these groups and their theory we refer to [18]. In the second part we see the connection between graphs and topological groups, coming from Bass-Serre theory, which was introduced by J.-P. Serre in [24]. Then we introduce one of the classical examples of totally disconnected locally compact groups, tree automorphisms. We will use these groups in Chapter 4 and Chapter 5 to construct new tdlc groups.

1.1 Totally disconnected locally compact groups

Topological groups are mathematical structures that combine the concepts of group theory and topology. This structure allows the group to be studied in the context of algebra and topology, providing a deep connection between these two areas. Among the various types of topological groups, locally compact groups play a particularly significant roll. An important class of locally compact groups are the totally disconnected locally compact (tdlc) groups. Basic examples of tdlc groups are the p-adic numbers $\mathbb{Q}_p$ or discrete groups. The theory of profinite groups, which are inverse limits of finite groups, is closely related to totally disconnected groups and much of their theory builds on the fact that they do not contain non-trivial connected components.

A *topological group* is a group G equipped with a topology such that the group operations, multiplication $\cdot : G \times G \to G$ and inversion $i \colon G \to G$, are continuous functions.
Due to this we can use all the definitions known from topological spaces for topological groups as well. This includes for example connectedness, Hausdorff and so on. Every group G can be turned into a topological group by equipping it with the discrete topology. In the following lemma we list a few facts and properties of topological groups and their subgroups.

Lemma 1.1.1
Let G be a topological group. Then the group fulfills the following conditions.

- *If $H \leq G$ is an open subgroup of G, then H is also closed.*
- *If $H \leq G$ contains an open subset, then H is open.*
- *If G is connected, then the only open subgroup of G is G.*
- *If $H \leq G$ is a normal subgroup of G, then the quotient group ${}^{G}/_{H}$ is discrete if and only if H is open.*
- *G is a discrete group if and only if the one element subgroup is open.*

- *Each compact open subgroup of G is a commensurated subgroup and all compact open subgroups are commensurated.*

On the one hand we have the discrete groups and on the other hand we want to introduce a subset of topological groups which share some properties with discrete groups but give us more flexibility.
For topological spaces there is the definition of connected components and for topological groups we can also look at connected components. In the following we want to look at a special connected component of topological groups.

Definition 1.1.2
Let G be a topological group and 1_G be the identity of the group. Then we call the connected component of G which contains the identity 1_G the *identity component* G_0.

For each $g \in G$ we have that gG_0 is the connected component of G which contains g. The connected components and the connectedness are preserved under translations. Therefore we get another fact about the identity component.

Proposition 1.1.3
Let G be a topological group. Then the identity component G_0 is a closed normal subgroup of G.

Proof:
Since G_0 is a connected component we have that G_0 is connected and therefore a closed subset. The group multiplication is a continuous homomorphism, since G is a topological group. Therefore the multiplication preserves the connectedness and we get that $g^{-1}G_0$ is connected and contains the identity for each $g \in G_0$. Hence G_0 is closed under taking inverses. On the other hand for $g, h \in G_0$, we have $gh \in gG_0 = G_0$, so G_0 is a subgroup of G. Multiplication preserves connectedness and G_0 is the largest subset of G which is connected and contains the identity therefore we get $gG_0 = G_0g$. □

We know that the identity component contains only one element if the group is discrete. The next step is to generalize this property.

Definition 1.1.4
We call a topological space *totally disconnected* if each connected component contains only one element.
For topological groups this means, that they are *totally disconnected* if the identity component only contains the identity.

Therefore we get that the identity component of a totally disconnected group is open if and only if the group is discrete. In the next step we introduce a method for making topological groups totally disconnected.

Proposition 1.1.5
Let G be a topological group and G_0 be the identity component. Then ${}^{G}/_{G_0}$ is a totally disconnected group.

Proof:
Let $\pi\colon G \to {}^{G}\!/_{G_0} =: Q$ be the quotient map and Q_0 be the identity component of Q. Then $1 \in \pi^{-1}(Q_0)$, therefore $G_0 \subseteq \pi^{-1}(Q_0)$. If $\pi^{-1}(Q_0)$ is connected we have $G_0 = \pi^{-1}(Q_0)$, which is the identity in Q and so Q_0 is only the identity and Q is totally disconnected.
Therefore we need to show that $\pi^{-1}(Q_0)$ is connected. If $\pi^{-1}(Q_0)$ is not connected we can find two disjoint non-empty closed subsets L_1 and L_2, such that $\pi^{-1}(Q_0) = L_1 \cup L_2$. Since $G_0 \subseteq \pi^{-1}(Q_0)$ is connected we have for each $g \in \pi^{-1}(Q_0)$ either $gG_0 \subseteq L_1$ or $gG_0 \subseteq L_2$. Then L_1 and L_2 are unions of G_0-cosets and we get that Q_0 is a disjoint union of non empty closed subsets which is a contradiction to the fact that Q_0 is connected. Therefore $\pi^{-1}(Q_0)$ must be connected. □

Remark 1.1.6
Every totally disconnected compact group is a profinite group, that is, an inverse limit of finite groups.

The next concept we want to describe are locally compact groups. Then we collect some facts about totally disconnected locally compact groups and discuss a few of their properties.

Definition 1.1.7
A topological space X is called *locally compact* if every point has a compact neighborhood. A topological group G is called *locally compact* if G is locally compact as a topological space.

Since G is a group, it is sufficient to show that the identity 1_G has a compact neighborhood to see that G is locally compact. If U is a compact neighborhood of 1_G, then for every $g \in G$, gU is a compact neighborhood of g.
In the following we present a few cases in which subgroups of locally compact groups are also locally compact.

Fact 1.1.8

(i) A closed subgroup of a locally compact group is locally compact.

(ii) Every quotient of a locally compact group is locally compact.

(iii) The product of locally compact groups is locally compact.

From now on we will call a group G a *tdlc group* if G is totally disconnected locally compact and Hausdorff. In the following we give the most important theorem about tdlc groups which was introduced by D. van Dantzig in 1936 (see [33]).

Theorem 1.1.9 (van Dantzig's Theorem)
Let G be a tdlc group. Then every neighborhood of the identity contains a compact open subgroup.

Here we see that a tdlc group G always contains small compact open subgroups. Additionally compact open subgroups are closed since each open subgroup of a topological group is also closed. Therefore the subgroup by itself is locally compact and as a sub-

group of a totally disconnected group it is also a totally disconnected group. So each compact open subgroup of a tdlc group is a compact tdlc group, which means that it is a profinite group.
After having introduced van Dantzig's theorem we continue by discussing a few of its consequences.

Lemma 1.1.10
Let G be a locally compact group. Then G_0 coincides with the intersection of all open subgroups.

Lemma 1.1.11
The quotient of a tdlc group by a closed normal subgroup is totally disconnected.

Lemma 1.1.12
Each tdlc group has a neighborhood basis of compact open subgroups.

Lemma 1.1.13
If a topological group G admits a neighborhood basis at the identity consisting of compact open subgroups, then G is a tdlc group.

We have described the concept of tdlc groups and we can directly see that each discrete group is a tdlc group, but not each tdlc group is discrete. The aim later will be to apply some of the definitions and facts we have for discrete groups to the bigger set of groups and generalize them to tdlc groups. Here we give a short example of a tdlc group and after the next section we see another class of examples, the automorphisms of locally finite trees.

Example 1.1.14
Let $\mathbb{Q}_p$ be the field of p-adic numbers, then $\mathbb{Z}_p$, the ring of p-adic integers, is a compact open subgroup and $\{p^n\mathbb{Z}_p\}_{n\in\mathbb{N}}$ is a neighborhood basis of compact open subgroups. Therefore, by the consequences of the theorem of van Dantzig, $\mathbb{Q}_p$ is a tdlc group.

1.2 Graphs, trees and automorphisms

Graphs of groups and Bass-Serre theory are fundamental frameworks in combinatorial and geometric group theory. Based on the work of J.-P. Serre [24], this theory provides a powerful means to understand groups as algebraic and geometric objects. By representing a group as the fundamental group of a graph of groups, Bass-Serre theory gives a way to combine the combinatorial structure of graphs with the algebraic properties of groups. Here we repeat some notation and fundamental arguments which allow us to fix the notation of graphs and graphs of groups in the following.

A *graph* $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a tuple with a set of vertices $V(\mathcal{G})$ and a set of edges $E(\mathcal{G}) \subseteq V(\mathcal{G}) \times V(\mathcal{G})$. For non-oriented graphs we don't allow multiple edges, so we have no element in $E(\mathcal{G})$ occurring more than once. For oriented graphs $E(\mathcal{G})$ be symmetric, which means that for each $e = (v, w) \in E(\mathcal{G})$ also $(w, v) \in E(\mathcal{G})$. We choose an orientation on $E(\mathcal{G})$, $\sigma \subseteq E(\mathcal{G})$ such that $E(\mathcal{G}) = \sigma \dot\cup \{(w, v) \mid (v, w) \in \sigma\}$. There are three maps from the edge set

$$\begin{aligned} o\colon E(\mathcal{G}) &\to V(\mathcal{G}), & e=(v,w) &\mapsto o(e) := v & &\text{the origin map,}\\ t\colon E(\mathcal{G}) &\to V(\mathcal{G}), & e=(v,w) &\mapsto t(e) := w & &\text{the terminal map and}\\ \bar{\ }\colon E(\mathcal{G}) &\to E(\mathcal{G}), & e=(v,w) &\mapsto \overline{e} := (w,v) & &\text{the edge inversion map.} \end{aligned}$$

We have the following conditions for these maps: $t(\overline{e}) = o(e)$, $o(\overline{e}) = t(e)$, $\overline{\overline{e}} = e$ for all $e \in E(\mathcal{G})$ and $\bar{\ }$ is a bijection. This means for oriented graphs that e is the edge in σ and $\overline{e}$ is the inversion of the edge in σ. We also have for each loop of an oriented graph this loop in both directions. For each vertex $v \in V(\mathcal{G})$ we denote the set of all outgoing edges of v by $o(v) = \{e \in E(\mathcal{G}) \mid o(e) = v\}$ and by $t(v) = \{e \in E(\mathcal{G}) \mid t(e) = v\}$ the set of incoming edges of v.

Graph of groups

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a connected oriented graph, then we define a *graph of groups* based on $\mathcal{G}$ is defined by the following data:

(i) for each vertex $v \in V(\mathcal{G})$ we have a group $\mathcal{A}_v$,

(ii) for each edge $e \in E(\mathcal{G})$ we have a group $\mathcal{A}_e$, where $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$,

(iii) for each edge $e \in E(\mathcal{G})$, there is an open embedding $\iota_e\colon \mathcal{A}_e \to \mathcal{A}_{t(e)}$.

We call the groups $\mathcal{A}_v$ the *vertex groups* and the groups $\mathcal{A}_e$ the *edge groups* of the graph. Since each edge exists in both directions $\iota_{\overline{e}}\colon \mathcal{A}_{\overline{e}} \to \mathcal{A}_{t(\overline{e})}$ induces a map $\mathcal{A}_e \to \mathcal{A}_{o(e)}$. So it is possible to embed each edge group in both adjacent vertex groups.
Let T be a spanning tree of G, then we define the *fundamental group* $\pi_1(\mathcal{G}, \mathcal{A})$ *of the graph of groups* $(\mathcal{G}, \mathcal{A})$ as follows

$$\begin{aligned} F(\mathcal{G}, \mathcal{A}) &= \langle \bigcup_{v \in V(\mathcal{G})} \mathcal{A}_v \cup E(\mathcal{G}) \mid \overline{e} = e^{-1},\ e\iota_e(x)\overline{e} = \iota_{\overline{e}}(x),\ \forall x \in \mathcal{A}_e,\ \forall e \in E(\mathcal{G})\rangle,\\ \Pi = \pi_1(\mathcal{G}, \mathcal{A}) &= {F(\mathcal{G}, \mathcal{A})}/{\langle e \mid e \in E(T)\rangle}. \end{aligned}$$

If we have the graph of groups $(\mathcal{G}, \mathcal{A})$ and a spanning tree T, then we define the *Bass-Serre graph* $\hat{T}$ such that

$$\begin{aligned} V(\hat{T}) &:= \left\{(v, x) \mid v \in V(\mathcal{G}),\ x \in {\Pi}/{\mathcal{A}_v}\right\},\\ E(\hat{T}) &:= \left\{(e, x) \mid e \in E(\mathcal{G}),\ x \in {\Pi}/{\iota_e(\mathcal{A}_e)}\right\}, \end{aligned}$$

where $o(e, g\iota_e(\mathcal{A}_e)) = (o(e), g\mathcal{A}_{o(e)})$ and $t(e, g\iota_e(\mathcal{A}_e)) = (t(e), g\overline{e}\mathcal{A}_{t(e)})$.
All these definitions of the fundamental group of a graph of groups and the Bass-Serre graph are independent of the choice of the spanning tree, therefore mostly we do not include T in the notation of these constructions.

Theorem 1.2.1
Let $(\mathcal{G}, \mathcal{A})$ be a graph of groups and T be a spanning tree of $\mathcal{G}$. Then the Bass-Serre graph of $(\mathcal{G}, \mathcal{A})$ is a tree.

Therefore in the following and in most of the literature the Bass-Serre graph is called Bass-Serre tree. In the next lemma we give a criterion in which case the Bass-Serre tree is a locally finite tree.

Lemma 1.2.2
Let $(\mathcal{G}, \mathcal{A})$ be a graph of groups, T be a spanning tree of $\mathcal{G}$, $\Pi = \pi_1(\mathcal{G}, \mathcal{A})$ with respect to T and $\hat{T}$ be the Bass-Serre tree. Then $\hat{T}$ is a locally finite tree if and only if $[\mathcal{A}_{t(e)} \colon \iota_e(\mathcal{A}_e)] < \infty$ and $[\mathcal{A}_{o(e)} \colon \iota_{\overline{e}}(\mathcal{A}_{\overline{e}})] < \infty$ for all $e \in E(\mathcal{G})$.

In the next step we introduce the action of the fundamental group of the graph of groups on the Bass-Serre tree as follows. Let $(\mathcal{G}, \mathcal{A})$ be a graph of groups, $\Pi = \pi_1(\mathcal{G}, \mathcal{A})$ be the fundamental group with respect to the spanning tree T and $\hat{T}$ be the Bass-Serre tree. Then Π acts with the following action on $\hat{T}$: for all $g \in \Pi$, all $(v, h\mathcal{A}_v) \in V(\hat{T})$ and all $(e, h\mathcal{A}_e) \in E(\hat{T})$ we have

$$\begin{aligned} g(v, h\mathcal{A}_v) &= (v, gh\mathcal{A}_v) \text{ and} \\ g(e, h\iota_e(\mathcal{A}_e)) &= (e, gh\iota_e(\mathcal{A}_e)) \,. \end{aligned}$$

With this action we get ${}_{\Pi}\backslash^{\hat{T}} = \mathcal{G}$.

Automorphisms of graphs

The last concept we introduce in this section are automorphisms of graphs and especially automorphisms of trees, since we will use the latter later for other constructions. A graph $\mathcal{G}$ is called a *tree* if $\mathcal{G}$ is connected and has no loops and no closed paths. Sometimes we denote a tree with $\mathcal{T}$. Since we want to introduce a topology we use the metric on $\mathcal{T}$ such that each edge has length 1. In particular we use regular trees. They are split into two types, on the one hand we have rooted regular trees. A *d-regular rooted tree* $\mathcal{T}_d$ is defined by having one vertex, the root, with degree d and every other vertex having degree $d+1$. On the other hand the *d-regular unrooted tree* $\mathcal{T}_d$ has degree d for each vertex.

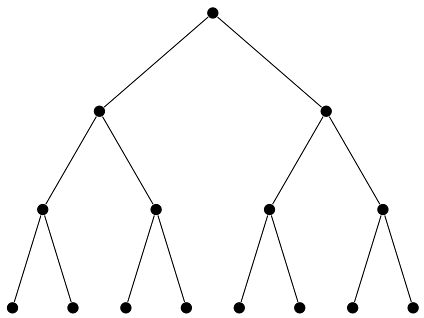

Figure 1.1: Ball of radius 3 of the rooted 2-regular tree

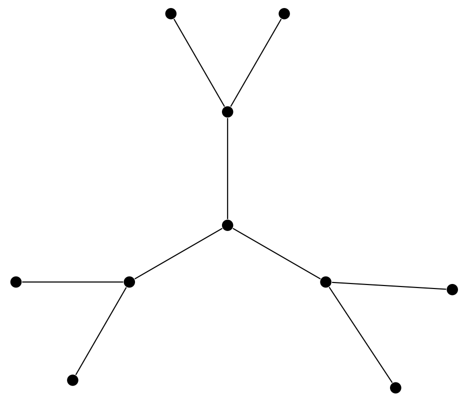

Figure 1.2: Ball of radius 2 of the unrooted 3-regular tree

An automorphism of a graph consists of two bijective maps, one mapping vertices to vertices and the other mapping edges to edges, such that the image of two vertices is connected by an edge if and only if the vertices are connected by an edge. This means that the degree of a vertex does not change under automorphisms. For unrooted trees we can map each vertex to each vertex and for rooted trees we fix the root and can then only map the descendants of each vertex.

As the topology of the automorphism group of a tree $\mathcal{T}$ we choose the topology of pointwise convergence, which coincides with the compact open topology. Now we show that the automorphism group of a locally finite tree $\mathcal{T}$ is a tdlc group with this topology. For this we use the fact that we need a neighborhood basis which consits of compact open subgroups. Let $v \in V(\mathcal{T})$ be a vertex of $\mathcal{T}$ and $\text{stab}_{\text{Aut}(\mathcal{T})}(v) = \{\phi \in \text{Aut}(\mathcal{T}) \mid \phi(v) = v\}$ be the stabilizer of this vertex, then this subset is open by construction of the topology. Let $B_n(v)$ be the n-ball around the vertex v. Then for all $\phi \in \text{stab}_{\text{Aut}(\mathcal{T})}(v)$ we get $\phi(B_n(v)) = B_n(v)$. We define $H_n := \{\psi \in \text{Aut}(B_n(v)) \mid \psi(v) = v\}$. Then each H_n is finite, since each B_n is finite, which follows from the fact that $\mathcal{T}$ is locally finite. Therefore $\text{stab}_{\text{Aut}(\mathcal{T})}(v) = \varprojlim H_n$ is the inverse limit of finite groups, so it is profinite and therefore compact. The stabilizers of finite sets are compact open subgroups of the automorphism group and they form a neighborhood basis of the identity. This shows by the consequence of the theorem of van Dantzig, Lemma 1.1.10, that $\text{Aut}(\mathcal{T})$ is a tdlc group if $\mathcal{T}$ is a locally finite tree.

For any tree $\mathcal{T}$ and any vertex $v \in V(\mathcal{T})$ there exists a natural partition of the vertices into two sets. Let $v \in V(\mathcal{T})$, then $V(\mathcal{T}) = V_1 \cup V_2$, where V_1 are all vertices of $\mathcal{T}$ which have odd distance to v and V_2 are all vertices of $\mathcal{T}$ which have even distance to v. If all vertices of one set of this partition have valence m and all vertices of the other set have valence n, then we say $\mathcal{T}$ is (m, n)-*biregular*.

Let $\mathcal{T}$ be an oriented tree and $e = (v, w) \in E(\mathcal{T})$. Then we can split $\mathcal{T}$ into two connected components if we delete the edge e. This results in the following decomposition $\mathcal{T} \setminus \{(v, w)\} = \mathcal{T}_{(v,w)} \cup \mathcal{T}_{\overline{(v,w)}}$ such that $v \in \mathcal{T}_{(v,w)}$ and $w \in \mathcal{T}_{\overline{(v,w)}}$. Therefore we get $\mathcal{T}_{(v,w)} = \mathcal{T}_{\overline{(w,v)}}$ and $\mathcal{T}_{(w,v)} = \mathcal{T}_{\overline{(v,w)}}$.

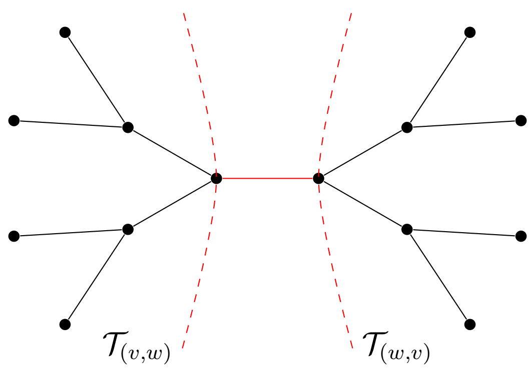

Figure 1.3: Here is an example for the result of deleting the red edge (v, w)

In Figure 1.3 we see an example for the deletion of an edge and the two resulting connected components for the regular tree $\mathcal{T}_3$, then $T_{(v,w)}$ is rooted at v and $\mathcal{T}_{(w,v)}$ is rooted at w. An unrooted oriented q-regular tree splits into two rooted oriented q-regular trees.

2 Finiteness properties of groups

Here we first give a short introduction to the theory of finiteness properties of discrete groups. These properties generalize the definition of being finitely generated and finitely presented. On the one hand there is a topological version and on the other hand we have an algebraic version of finiteness properties of discrete groups. Therefore this theory allows us to use some topological and algebraic techniques to study discrete groups. We explain some properties of these conditions. In the second part of this chapter we give an introduction into the setting of totally disconnected locally compact groups and how we can transfer some finiteness properties from the discrete case to the non-discrete case. This theory for tdlc groups was introduced in 2020 by I. Castellano and G. Corob Cook [11]. In this part we give at first an introduction into being compactly generated and compactly presented, since the finiteness properties form a generalization of these properties. Here we have also a topological and an algebraic version of the conditions but we see that in the algebraic version we need to use some different techniques than in the discrete case.

2.1 Finiteness properties of discrete groups

For more details about the finiteness properties of discrete groups we refer to the books of K. Brown [9] and R. Geoghegan [16]. In the following G is a discrete group. First we define the topological version of the finiteness properties and then we define the homological version and give a few properties of and connections between these definitions.

Before we start with the definitions we need a short recap of what we mean by a classifying space of a group. A CW-complex X is called a *classifying space* of G if X is aspherical and $\pi_1(X)$ is isomorphic to G. Such a space is unique up to homotopy equivalence, therefore we can use these spaces for some properties of the group.

Definition 2.1.1
Let G be a discrete group. Then G is of *type* F_n if there exists a classifying space X such that the n-skeleton of X is finite, G is of *type* F_∞ if G is of type F_n for all $n \in \mathbb{N}$ and G is of *type* F if there exists a finite classifying space X.

Theorem 2.1.2 ([16, Proposition 7.2.1])
Every group G has type F_0. G has type F_1 if and only if G is finitely generated, and G has type F_2 if and only if G is finitely presented. For $n \geq 2$ G has type F_n if and only if there exists a finite pointed n-dimensional $(n-1)$-aspherical CW complex X such that $\pi_1(X) \cong G$.

Since for totally disconnected locally compact groups in general we cannot find a classifying space which is finite in any dimension, we give here an alternative definition of the finiteness properties which is the type of definition which we can generalize later. We first need the definition of a G-CW complex. For this let X be a non-empty CW complex, such that G acts on X. In the following we only consider left actions but all the notation works for right actions in the same way. We call X a G-CW *complex* if G acts cellulary on X. Such a G-CW complex is called *proper* if each cell stabilizer is finite.

Then G is of type F_n if there exists an $(n-1)$-connected free G-CW complex whose n-skeleton is finite mod G. Or if we look only at the n-skeleton of this complex, then we have that G is of type F_n if and only if there exists an $(n-1)$-connected n-dimensional free G-CW complex which is finite mod G.

In the next step we introduce some facts about the finiteness properties of discrete groups.

Proposition 2.1.3 ([16, Proposition 7.2.3 and Corollary 7.2.4])
Let $H \leq G$ be a finite index subgroup of G. For $0 \leq n \leq \infty$ we have that G is of type F_n if and only if H is of type F_n.

Theorem 2.1.4 ([16, Theorem 7.2.20])
Let $n \geq 1$, G be a group of type F_n and X be a classifying space with finite n-skeleton. Then G is of type F_{n+1} if and only if there exists a classifying space Y with finite $(n+1)$-skeleton and $Y^n = X^n$.

We can also give a similar statement for the universal cover of the classifying space.

Theorem 2.1.5 ([16, Theorem 8.2.1])
Let X be an $(n-1)$-connected n-dimensional free G-CW complex which is finite mod G. Then G is of type F_n and G is of type F_{n+1} if and only if it is possible to attach finitely many G-orbits of $(n+1)$-cells to X to get an n-connected $(n+1)$-dimensional free G-CW complex which is finite mod G.

Theorem 2.1.6 ([16, Theorem 7.2.21])
Let $0 \to N \to G \to Q \to 0$ be a short exact sequence of discrete groups.

(i) If G is of type F_n and N is of type F_{n-1}, then Q is of type F_n.

(ii) If Q and N are of type F_n, then G is of type F_n.

Now we have collected the topological part of the finiteness properties of discrete groups. In the next step we want to repeat the homological finiteness properties of discrete groups.

Definition 2.1.7
Let G be a discrete group and R be a commutative ring. Then G is of *type* FP_n *over* R if there exists a projective resolution $P_* \to R$ of the trivial $R[G]$-module R, such that $P_0, \dots, P_n$ are finitely generated as $R[G]$-modules. As before we have that G is of *type* FP_∞ *over* R if G is of type FP_n over R for all $n \in \mathbb{N}$. And G is of *type* FP *over* R if P_i is finitely generated in each dimension and trivial in all but finitely many dimensions.

If G is of type FP_n over $\mathbb{Z}$ then G is of type FP_n over R for all commutative rings R. Some properties we have for the topological setting we also have for the homological setting.

Theorem 2.1.8 ([9, Exercise *8, Section VIII.7])
Let $0 \to N \to G \to Q \to 0$ be a short exact sequence of groups.

(i) If G is of type FP_n over R and N is of type FP_{n-1} over R, then Q is of type FP_n over R.

(ii) If Q and N are of type FP_n over R, then G is of type FP_n over R.

Proposition 2.1.9 ([9, (5.1) Proposition, p.197])
Let $H \leq G$ be a finite index subgroup of G and R be a commutative ring. Then H is of type FP_n over R if and only if G is of type FP_n over R.

The other important fact about the finiteness properties is the connection between the topological and the homological version.

Theorem 2.1.10 ([16, Section 8.2])
Let G be a discrete group and R be a commutative ring. Then we have the following

(i) if G is of type F_n then G is also of type FP_n over R,

(ii) if G is of type FP_n over $\mathbb{Z}$ and finitely presented, then G is also of type F_n.

A really powerful criterion for the proof that a group is of type FP_n over R is Brown's criterion, for more details see [8]. To state it we need some more notions.

Definition 2.1.11
Let X be a G-CW complex and R be a commutative ring. Then we call X *n-good over R* if

(i) X is acyclic in dimension $< n$, i.e. the reduced homology $\tilde{H}_i(X, R) = 0$ for $i < n$,

(ii) for $0 \leq p \leq n$ the cell stabilizer G_σ for any p-cell σ is of type FP_{n-p} over R.

In the next step we need a few definitions of filtrations of the G-CW complex. For this let X be a G-CW complex and $(X_j)_{j\in D}$ be a family of subcomplexes of X. Then $(X_j)_{j\in D}$ is called *G-filtration* if D is countable, each X_j is G-invariant, $X_j \subseteq X_k$ for $j \leq k$ and $\bigcup_{j\in D} X_j = X$. A G-filtration $(X_j)_{j\in D}$ is of *finite n-type* if for each X_j the n-skeleton is finite mod G.
The last definition we need for G-filtrations is based on the level of groups. Therefore we need that a system of groups $(\mathcal{A}_j)_{j\in\mathbb{N}}$ is called *essentially trivial* if for all j there exists a $k \geq j$ such that the map $\mathcal{A}_j \to \mathcal{A}_k$ is trivial. We then call a G-filtration $(X_j)_{j\in\mathbb{N}}$ *H-essentially trivial over R in dimension n* if the system of groups $(H_n(X_j, R))_{j\in\mathbb{N}}$ is essentially trivial. And the G-filtration is called *essentially n-connected* if $(\pi_k(X_j))_{j\in\mathbb{N}}$ is essentially trivial for $k \leq n$.
Now we have all definitions which we need for all the versions of Brown's criterion, which we reformulate later for the tdlc case. In the following let G be a discrete group and R be a commutative ring.

Proposition 2.1.12 ([8, Proposition 1.1])
Suppose that G admits an n-good G-CW complex X, such that the n-skeleton of X is finite mod G. Then G is of type FP_n.

Theorem 2.1.13 ([8, Theorem 2.2])
Let X be an n-good G-CW complex with G-filtration $(X_j)_{j\in\mathbb{N}}$ of finite n-type. Then G is of type FP_n if and only if the G-filtration is H-essentially trivial over R in dimension n.

In the next step we give the formulation of Brown's criterion for the topological finiteness properties.

Theorem 2.1.14 ([8, Theorem 3.2])
Let X be a simply connected G-CW complex such that

(i) all vertex stabilizers are finitely presented,

(ii) all edge stabilizers are finitely generated,

(iii) $(X_j)_{j\in\mathbb{N}}$ be a G-filtration of finite 2-type and

(iv) $v \in \bigcap_{j\in\mathbb{N}} X_j$.

Then if G is finitely generated, G is finitely presented if and only if the direct system $(\pi_1(X_j, v))_{j\in\mathbb{N}}$ is essentially trivial.

Corollary 2.1.15 ([8, Corollary 3.3])
Let X be a contractible G-CW complex such that

(i) all cell stabilizers are finitely presented,

(ii) all cell stabilizers are of type FP_∞ over R,

(iii) $(X_j)_{j\in\mathbb{N}}$ be a G-filtration of X such that each X_j is finite mod G.

Then if the connectivity of the pair (X_{j+1}, X_j) tends to infinity, if j tends to infinity G is finitely presented and of type FP_∞ over R.

Remark 2.1.16
In Corollary 2.1.15 we can replace the condition FP_∞ over $\mathbb{Z}$ and finitely presented for the cell stabilizers by being of type F_∞ and G is then of type F_∞ over $\mathbb{Z}$ since it is finitely presented and of type FP_∞ over $\mathbb{Z}$.

We now can combine all these theorems to get directly a criterion for being of type F_n.

Theorem 2.1.17 (Browns criterion, [16, Theorem 7.4.1])
Let X be an $(n-1)$-connected free G-CW complex with a G-filtration $(X_j)_{j\in\mathbb{N}}$ where each X_j has finite n-skeleton mod G. Then G is of type F_n if and only if the G-filtration $(X_j)_{j\in\mathbb{N}}$ is essentially $(n-1)$-connected.

2.2 Finiteness properties of totally disconnected locally compact groups

As a generalization of discrete groups we look in the following at totally disconnected locally compact groups. All the statements and definitions about the theory of finiteness properties of tdlc groups we use here come from I. Castellano and G. Corob Cook in 2020 [11].
In the case of tdlc groups the topological and homological finiteness properties are a generalization of being compactly generated and compactly presented.

2.2.1 Compactly presented groups

Before we look at the higher dimensional finiteness properties of totally disconnected locally compact groups we repeat two different definitions of compactly generated, respectively presented and give an example of that concept. We start with the definition which comes naturally from the definition of being finitely generated or finitely presented.

Definition 2.2.1
Let G be a topological group. Then we call G *compactly generated*, if there exists a compact subset $S \subseteq G$, such that S generates G. And we call G *compactly presented*, if G is compactly generated by a compact subset $S \subseteq G$, and $G = \langle S \mid R \rangle$, where R is a set of relations with bounded length.

For the second definition of compactly presented, we need a graph of profinite groups and before we can define the condition for compact generation we need to introduce the notion of a generalized presentation.

Definition 2.2.2 ([13, Section 5.7])
Let G be a tdlc group. A *generalized presentation* of G is a graph of profinite groups $(\mathcal{G}, \mathcal{A})$ together with a continuous open surjective homomorphism

$$\Phi \colon \pi_1(\mathcal{G}, \mathcal{A}) \to G,$$

such that $\Phi|_{\mathcal{A}_v}$ is injective for all $v \in V(\mathcal{G})$.

First we show that such a presentation exists for each tdlc group. We give the proof here, since we use the construction in this proof to show the equivalence of both definitions.

Proposition 2.2.3 ([13, Proposition 5.10 (a)])
Let G be a tdlc group. Then G has a generalized presentation $((\mathcal{G}, \mathcal{A}), \Phi)$, where $\mathcal{G}$ is a graph with a single vertex.

Proof:
Since G is a tdlc group there exists a compact open subgroup $\mathcal{O} \subseteq G$, by van Dantzig's theorem. Choose a subset $S \subseteq G$, such that $G = \langle \mathcal{O}, S \rangle$. Without loss of generality we assume that S is symmetric, this means $S = S^{-1}$. So we can choose $S_+ \subseteq S$, such that $S = S_+ \dot{\cup} S_+^{-1}$. Now we can define the graph $\mathcal{G}$, as a graph with a single vertex v and a loop attached for each $s \in S_+$. This looks like the graph depicted in Figure 2.1.

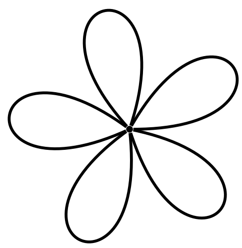

Figure 2.1: Graph $\mathcal{G}$ for the graph of groups

Then we have $V(\mathcal{G}) = \{v\}$ and $E(\mathcal{G}) = \{e_s, \bar{e}_s \mid s \in S_+\}$. For the single vertex v and for each edge we define the profinite groups as follows

$$\mathcal{A}_v = \mathcal{O},$$

$$\mathcal{A}_{e_s} = \mathcal{A}_{\bar{e}_s} = \mathcal{O} \cap s^{-1}\mathcal{O}s \text{ for all } s \in S_+,$$

$$\iota_{e_s} \colon \mathcal{A}_{e_s} \to \mathcal{A}_v,\ a \mapsto sas^{-1},$$

$$\iota_{\bar{e}_s} \colon \mathcal{A}_{\bar{e}_s} \to \mathcal{A}_v,\ a \mapsto a.$$

Then $(\mathcal{G}, \mathcal{A})$ is a graph of profinite groups. The maximal tree in $\mathcal{G}$ is a single vertex, so it is not relevant for the fundamental group of $(\mathcal{A}, \mathcal{G})$. We define the canonical map

$$\Phi_0 \colon E(\mathcal{G}) \cup \mathcal{O} \to G\ ,\ \Phi_0(e_s) = \Phi_0(\bar{e}_s)^{-1} = s\ ,\ \Phi_0(\omega) = \omega\ ,\ \omega \in \mathcal{O}. \tag{2.2.1}$$

The relations of the fundamental group of the graph of profinite groups are fulfilled. Therefore we can find a unique group homomorphism $\Phi \colon \pi_1(\mathcal{G}, \mathcal{A}) \to G$, such that $\Phi|_{E(\mathcal{G}) \cup \mathcal{O}} = \Phi_0$. By construction Φ is surjective and $\Phi|_{\mathcal{O}}$ is injective. Hence $((\mathcal{G}, \mathcal{A}), \Phi)$ is a generalized presentation of G. □

Now we can give the alternative definition of compactly generated and compactly presented.

Definition 2.2.4
Let G be a tdlc group. Then G is called *compactly generated* if and only if there exists a generalized presentation $((\mathcal{G}, \mathcal{A}), \Phi)$, such that $\mathcal{G}$ is a finite connected graph. And G is called *compactly presented* if and only if there exists a generalized presentation $((\mathcal{G}, \mathcal{A}), \Phi)$, such that

(i) $\mathcal{G}$ is a finite connected graph,

(ii) $N = \ker(\Phi)$ is finitely generated as a normal subgroup of $\pi_1(\mathcal{G}, \mathcal{A})$.

In the following we show that both definitions of compactly generated and compactly presented are equivalent, because we want to use both of them later to show some properties of tdlc groups.

Theorem 2.2.5
Let G be a tdlc group.

(i) The two definitions of being compactly generated are equivalent.

(ii) The two definitions of being compactly presented are equivalent.

Proof:
Let G be a tdlc group.

(i) Let G be compactly generated according to Definition 2.2.1, then we have a $\mathcal{C} \subseteq G$, which is compact and generates G. Now we need to show that we have a generalized presentation $((\mathcal{G}, \mathcal{A}), \Phi)$, such that the graph $\mathcal{G}$ is a finite connected graph. By Proposition 2.2.3, we know that a generalized presentation exists. We can choose now the compact open subgroup $\mathcal{O}$, from the proof of Proposition 2.2.3, as a subset of $\mathcal{C}$. This is possible by the theorem of van Dantzig, Theorem 1.1.9. We can choose a finite S, such that $\mathcal{O} \cup S$ generates G. Indeed we can find an open covering of $\mathcal{C}$ with translates of $\mathcal{O}$, we can also find a finite open covering, therefore we can find the finite subset S. With the construction in the proof of Proposition 2.2.3 we get a generalized presentation with a finite connected graph.

On the other hand let G be compactly generated according to Definition 2.2.4, then there exists a generalized presentation $((\mathcal{G}, \mathcal{A}), \Phi)$, such that $\mathcal{G}$ is a finite connected graph. Since $\pi_1(\mathcal{G}, \mathcal{A})$ is generated by $\mathcal{C} = \bigcup_{v \in V(\mathcal{G})} \mathcal{A}_v \cup E(\mathcal{G})$ and we have a surjective map from $\pi_1(\mathcal{G}, \mathcal{A})$ to G, G is also generated by the image of $\mathcal{C}$. We know that $\mathcal{C}$ is compact, since $\mathcal{G}$ is finite, and so $\mathcal{C}$ is a union of finitely many profinite groups and a finite set, so it is compact. The image is also compact, since Φ is continuous.

(ii) Let G be compactly presented by Definition 2.2.1, then we have $G = \langle \mathcal{C} \mid R \rangle$, such that $\mathcal{C} \subseteq G$ is compact and all relations in R are of bounded length. By (i) we can find a generalized presentation $((\mathcal{G}, \mathcal{A}), \Phi)$, such that $\mathcal{G}$ is a finite connected graph. We need to show that $N = \ker(\Phi)$ is finitely generated as a normal subgroup. By [13, Proposition 5.10 (b)] N is a discrete free group. Using a short exact sequence we can show now, that N is compactly generated as a normal subgroup, and by the fact that N is discrete we see that N is finitely generated as a normal subgroup.

$$0 \to N \hookrightarrow \pi_1(\mathcal{G}, \mathcal{A}) \xrightarrow{\Phi} G \to 0$$

The sequence is exact by construction, since N is defined as the kernel of Φ. Since G is compactly presented and $\pi_1(\mathcal{G}, \mathcal{A})$ is compactly generated [14, Proposition 4.15] shows that N must be compactly generated as a normal subgroup. Hence N is compactly generated as a normal subgroup and discrete so N is finitely generated as a normal subgroup. Therefore G fulfills the conditions of being compactly presented by Definition 2.2.4.

Conversely, let G be compactly presented according to Definition 2.2.4. Let $((\mathcal{G}, \mathcal{A}), \Phi)$ be the generalized presentation of G, such that $\mathcal{G}$ is a finite connected graph and $N = \ker(\Phi)$ is finitely generated as a normal subgroup. By (i) we know that G has a compact subset $\mathcal{C}$, such that $\mathcal{C}$ generates G. We need to show that we have a presentation of $G = \langle \mathcal{C} \mid R \rangle$, such that all relations in R are of bounded length.
Since we have a surjective grouphomomorphism $\Phi \colon \pi_1(\mathcal{G}, \mathcal{A}) \to G$, we know that $G \cong \pi_1(\mathcal{G}, \mathcal{A})/N$ since N is finitely generated as a normal subgroup we know that all generators of N have bounded length and all relations in the fundamental group

of the graph of groups are of length ≤ 3. Then we have a presentation of G with a compact generating set $\mathcal{C}$ and a set of relations of bounded length.

□

In the next step we see that it is not relevant which generalized presentation for a finite connected graph we use for a compactly presented group.

Lemma 2.2.6
Let G be a compactly presented tdlc group. Then for each generalized presentation of G $((\mathcal{G},\mathcal{A}),\Phi)$ with finite connected graph the kernel of Φ is finitely generated as a normal subgroup.

Proof:
Let G be compactly presented and $((\mathcal{G},\mathcal{A}),\Phi)$ be an arbitrary generalized presentation with finite connected graph. Then we have the following short exact sequence

$$0 \to \ker(\Phi) \to \pi_1(\mathcal{G},\mathcal{A}) \xrightarrow{\Phi} G \to 0$$

and in this sequence G is compactly presented and $\pi_1(\mathcal{G},\mathcal{A})$ is compactly generated. Therefore by [14, Proposition 4.5] the kernel is compactly generated as a normal subgroup. Since the kernel is a discrete group it is finitely generated as a normal subgroup.

□

Example 2.2.7
Let p be prime and $G = \mathrm{SL}_2(\mathbb{Q}_p)$. Then G is compactly presented and we give an example for both characterizations.

(i) Let $S = \{\mathrm{SL}_2(\mathbb{Z}_p) \cup P\}$ where $P = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$.

Then we have on the one hand the relations which verify that $\mathrm{SL}_2(\mathbb{Z}_p)$ is a group, these are all relations of length 3. On the other hand we have relations which come from the multiplication of P or powers of P with some elements of $\mathrm{SL}_2(\mathbb{Z}_p)$. Since $\mathrm{SL}_2(\mathbb{Z}_p)$ is compact in $\mathrm{SL}_2(\mathbb{Q}_p)$ and P is only one element, S is a compact subset of $\mathrm{SL}_2(\mathbb{Q}_p)$. We need to show that S generates $\mathrm{SL}_2(\mathbb{Q}_p)$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$, with $\det(A) = 1 = ad-bc$. There exists $n, m \in \mathbb{Z}$ such that $ap^{n+m} = a', bp^{(n-m)} = b', cp^{m-n} = c', dp^{n+m} = d' \in \mathbb{Z}_p$, so $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p)$ and $A = P^m A' P^n$. Thus S is a compact generating set and we have only relations of bounded length.

(ii) For the generalized presentation of $\mathrm{SL}_2(\mathbb{Q}_p)$ we follow the construction of the proof of Theorem 2.2.5. Let $H = \mathrm{SL}_2(\mathbb{Z}_p)$ and $P = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ as above.

Let $\mathcal{G}$ be a graph with one vertex v and one loop e. Then we have with $\mathcal{A}_v = H$ and $\mathcal{A}_e = \mathcal{A}_{\overline{e}} = H \cap P^{-1}HP$, together with the embeddings $\iota_e \colon \mathcal{A}_e \to \mathcal{A}_v$, $A \mapsto PAP^{-1}$, $\iota_{\overline{e}} \colon \mathcal{A}_{\overline{e}} \to \mathcal{A}_v$, $A \mapsto A$, a graph of profinite groups and $\Pi = \pi_1(\mathcal{G},\mathcal{A})$ be the fundamental group. The continuous open surjective homomorphism is defined as follows:

$$\begin{aligned}\Phi\colon \Pi &\to G\\ A &\mapsto A, \text{ if } A \in \mathcal{A}_v\\ e &\mapsto P, \text{ for } e \in E(\mathcal{G}).\end{aligned}$$

Then Φ is surjective and $\Phi|_{\mathcal{A}_v}$ is injective by construction. So we have a generalized presentation of G with a finite connected graph $\mathcal{G}$, the only thing we need to show is that the kernel $N = \ker(\Phi)$ is normally finitely generated, but this is given by Theorem 2.2.5.

Another concept we introduce for compactly generated tdlc groups is their Cayley-Abels graphs. In the discrete case if we have a finitely generated group G together with a finite generating set S we can define the Cayley graph of G with respect to S. We can define a similar construction for compactly presented tdlc groups. The construction in the following follows [20, Construction 2.1].

Construction 2.2.8
Let G be a compactly generated tdlc group, K be a compact generating set and $H \subseteq G$ be a compact open subgroup. Then let $\hat{\mathcal{G}}$ be the Cayley graph of G with respect to K, this is a connected but not locally finite graph. In the next step we define a equivalence relation $\sim$, by $x \sim y \Leftrightarrow xH = yH$. This relation is a G-congruence on $\hat{\mathcal{G}}$. Then we define a *Cayley-Abels graph* by $\mathrm{CA}(G, K, H) := \hat{\mathcal{G}}/\sim$. The action of G on CA is induced by the action of G on $\hat{\mathcal{G}}$.

2.2.2 Higher dimensional generalization

Here we introduce the generalization of the higher dimensional finiteness properties for tdlc groups. All the definitions and statements follow [11] and [13].
Before we start with the higher dimensional finiteness properties of tdlc groups we need a few definitions about G-CW complexes.
In the case of discrete groups we have seen that we can define the property of being of type F_n by a free action on a G-CW complex. We define in Section 2.1 proper G-CW complexes for discrete groups. We define a similar construction for tdlc groups in the next step.
Since the identity is not an open subgroup in a tdlc group we need another type of subgroups for the fixpoint sets. In Theorem 1.1.9 we have seen that each tdlc group has small compact open subgroups, therefore we want to use these groups in the following definitions.

In the following let G be a tdlc group. Further let $\mathcal{F}$ be a non-empty set of open subgroups of G with the following properties

$(F1)$ for all $A \in \mathcal{F}$ and $g \in G$ we want that $A^g = gAg^{-1} \in \mathcal{F}$,

$(F2)$ for all $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$.

This means that $\mathcal{F}$ is closed under conjugation and intersection. Let X be a non-trivial topological space and G acts by a continuous left action on X, then X is a *left G-space*. We sometimes only call it a G-space because we only look at left spaces, but all these definitions work for right spaces in the same way.
Let X be a G-space, then we call it *$\mathcal{F}$-discrete* if $\mathrm{stab}_G(x) = \{g \in G \mid g \cdot x = x\} \in \mathcal{F}$ for all $x \in X$. We say X is a discrete G-space if $\mathcal{F}$ is clear from context. If all elements in $\mathcal{F}$ are compact open then we call the $\mathcal{F}$-discrete G-space *proper*. This compares to proper G-CW complexes in the discrete case, there we have finite stabilizers and here we have compact stabilizers.
Let X and Y be two G-spaces then $f\colon X \to Y$ is called a *map of G-spaces* if it is continuous and commutes with the G-action on the spaces.

Definition 2.2.9
Let G be a tdlc group, $\mathcal{F}$ a family of open subgroups of G as before and X be a non-empty CW complex. Then X is called *$\mathcal{F}$-discrete* G-CW *complex* if G acts by cellulary on X and X^0 is an $\mathcal{F}$-discrete subspace of X.

If the family of $\mathcal{F}$ is clear we say discrete G-CW complex instead of $\mathcal{F}$-discrete G-CW complex.

Remark 2.2.10 ([11, Fact 2.3])
Let G be a tdlc group and X be a discrete G-CW complex. Then

(i) the action of G on X is continuous,

(ii) an element in G fixes a cell σ of X pointwise if it fixes it setwise,

(iii) if $\mathcal{F}$ only contains compact open subgroups X is called *proper*.

Now we can define the analogues of a classifying space for the tdlc case.

Definition 2.2.11
We call a topological space X a *G-model* for a tdlc group G if X is a contractible proper discrete G-CW complex.

In the next step we give a short example of how we can construct a G-model for a compactly generated tdlc group.

Example 2.2.12
Let G be a compactly generated tdlc group. Then we have seen in Construction 2.2.8 that G has a Cayley-Abels graph. The geometric realization of this graph is a one skeleton of a G-model for the group, since the stabilizers of a vertex is a conjugate of the compact open subgroup which we use for the construction of the Cayley-Abels graph.

Definition 2.2.13
We say G is *of type* F_n if there exists a G-model X such that the n-skeleton of X has finitely many G-orbits, G is *of type* F_∞ if G is of type F_n for all $n \in \mathbb{N}$ and *of type* F if there exists a finite dimensional G-model, which means that $X^k = X$ for some k.

Remark 2.2.14
If G is a discrete group then a G-model corresponds to a universal cover of a classifying

space. Therefore the two definitions of being of type F_n are equivalent in the case of discrete groups.

In the next step we prove a similar statement as for the discrete case, which shows that this definition is a generalization of compactly generated respectively presented.

Lemma 2.2.15 ([11, Proposition 3.4])

(i) G is of type F_1 if and only if G is compactly generated.

(ii) G is of type F_2 if and only if G is compactly presented.

Proof:
Let X be a G-model. We show how we can construct a generalized presentation of G from X. Then we see that this presentation is finite if the 1-skeleton is finite mod G and the kernel is normally finitely generated if the 2-skeleton is finite mod G. In the following let $\mathcal{G} = {}_{G}\backslash^{X^1}$, which is a connected graph, since X is contractible. Then $V(\mathcal{G})$ is a representative set of the 0-cells of ${}_{G}\backslash^{X}$ and $E(\mathcal{G})$ is a representative set of the 1-cells. In [7, Section 1] Brown shows that we can construct a graph of groups in the following way, and that this graph coincides with a generalized presentation of a tdlc group. For each $v \in V(\mathcal{G})$ we choose $\mathcal{A}_v$ as the fixpoint group of the corresponding 0-cell in X. Since X is a proper discrete G-CW complex we know that all fixpoint groups are compact open subgroups of G, so all fixpoint groups are compact totally disconnected groups, therefore they are profinite groups. We use the same way for the edge groups, so for each $e \in E(\mathcal{G})$ let $\mathcal{A}_e$ be the fixpoint group of the corresponding 1-cell in X. Since if an element fixes a cell setwise it fixes it pointwise we know that the fixpoint group of a 1-cell is the intersection of the fixpoint groups of the 0-cells which are inside this 1-cell. Therefore we can embed the edge groups naturally into the corresponding vertex groups. Then we get a graph of groups $(\mathcal{G}, \mathcal{A})$ and we only need to find the right map from the fundamental group to G. Let $\Pi = \pi_1(\mathcal{G}, \mathcal{A})$ and $\Phi\colon \Pi \to G$, then $\Phi|_{\mathcal{A}_v}$ is the identity for each $v \in V(\mathcal{G})$. The map from the edge set to G comes from the embeddings of the edge groups inside the vertex groups and the relations $e\iota_e(\mathcal{A}_e)\overline{e} = \iota_{\overline{e}}(\mathcal{A}_{\overline{e}})$ inside the fundamental group. On the one hand, if $e(v, w) \in E(\mathcal{G})$ is an edge between two different vertices, then $\mathcal{A}_e \subseteq \mathcal{A}_v \cap \mathcal{A}_w$ and $\mathcal{A}_w = g\mathcal{A}_v g^{-1}$ for some $g \in G$, since $\mathcal{A}_v$ and $\mathcal{A}_w$ are both fixpoint groups of a 0-cell of X which are connected by a 1-cell. So we can map e to this g. On the other hand, if $e = (v, v) \in E(\mathcal{G})$ is a loop in $\mathcal{G}$, then $\mathcal{A}_e \subseteq \mathcal{A}_v \cap g\mathcal{A}_v g^{-1}$ for some $g \in G$, since $\mathcal{A}_e$ is a fixpoint group of a 1-cell in X which is attached to two 0-cells which can be mapped onto each other by some $g \in G$. Then we have a generalized presentation of G by $((\mathcal{G}, \mathcal{A}), \Phi)$, where $\mathcal{G} = {}_{G}\backslash^{X^1}$ and this graph is finite if and only if the one skeleton of X is finite mod G, and this is only possible if G is of type F_1. The kernel of the map Φ corresponds to the 2-cells of X, so the normal generators of the kernel are exactly the G orbits of the 2-cells of X. So we get that G is compactly presented if G is of type F_2.
For the other direction of the proof we follow the proof of I. Castellano and G. Corob Cook in [11]. Let G be a compactly generated (respectively presented) group and $((\mathcal{G}, \mathcal{A}), \Phi)$ be a generalized presentation with finite connected graph. Let $\hat{T}$ be the Bass-Serre tree of $(\mathcal{G}, \mathcal{A})$, $\Pi = \pi_1(\mathcal{G}, \mathcal{A})$ be the fundamental group of the generalized presentation and $N = \ker(\Phi)$. Then $\mathrm{CA} = {}^{\hat{T}}/_{N}$ is a Cayley-Abels graph of G which has only finitely

many G-orbits and the geometric realization can be a 1-skeleton of a G-model. Since we have that $\hat{T}$ is the universal cover of CA we know that $\pi_1(\mathrm{CA}) = N$. We have an action of G on the Cayley-Abels graph, this induces a conjugation on N. Then we have a normal generating set $\{w_i\}_{i\in I}$ of N in G and we can identify each normal generator with a corresponding loop in CA. For each normal generator w_i we need to add one G-orbit of 2-cells to CA to make it simply connected. If G is compactly presented we have that N is finitely normally generated which means that we have only finitely many w_i, so we need to attach only finitely many G-orbits of 2-cells. Then we can add higher cells to kill higher homotopy classes and by Whitehead we get a G-model with finite 1- (resp 2-) skeleton mod G. □

For the homological finiteness properties of tdlc groups we first need a few definitions and results about modules. In the following let G be a tdlc group and R be a commutative ring with discrete topology. We always use left modules in the description but we can also use right modules and find analogous definitions. The following explanations follow the paper of I. Castellano and G. Corob Cook [11].
Let ${}_{R[G]}\mathbf{mod}$ be the category of all left $R[G]$-modules and we call a module $M \in {}_{R[G]}\mathbf{mod}$ *discrete* if the action $G \times M \to M$ is continuous with the discrete topology on M. This means that all point stabilizers $G_m = \{g \in G \mid gm = m\}$ are open subgroups of G. By ${}_{R[G]}\mathbf{dis}$ we denote the full subcategory of discrete $R[G]$-modules. This category contains enough injectives but in general not enough projectives. If G acts continuously on Ω, we call Ω a *discrete left G-space* if all stabilizers $G_\omega = \{g \in G \mid g\omega = \omega\}$ are open for all $\omega \in \Omega$. Then we can define a *discrete permutation* $R[G]$*-module* as follows

$$R[\Omega] = \bigoplus_{\omega\in\mathcal{R}} R\left[{}^{G}/{}_{G_\omega}\right]$$

where $\mathcal{R}$ is a set of representatives of the G-orbits of Ω. These permutation modules are called *proper* if all stabilizers are compact. So we have a proper discrete permutation $R[G]$-module if G acts on a G-space Ω with compact open stabilizers.
Then we call a left resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to R \to 0$$

of R in ${}_{R[G]}\mathbf{dis}$ *proper discrete permutation resolution* of R if each P_i is a proper discrete permutation $R[G]$-module. In general such a resolution is not projective but if $R = \mathbb{Q}$ then each proper discrete permutation module is projective, therefore this resolution is a projective resolution. So we can use some statements about projective resolutions only if we have $R = \mathbb{Q}$, or more precisely if $\mathbb{Q} \subseteq R$. A discrete $R[G]$-module M is called *finitely generated* if M contains a finite set S such that S is not contained in any proper submodule of M.

Definition 2.2.16
Let M be a discrete $R[G]$-module. Then M is called of *type* FP_n if there exists a proper discrete resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0$$

such that P_i is finitely generated for $0 \leq i \leq n$. Then M is of *type* FP_∞ if M is of type FP_n for all $n \in \mathbb{N}$ and M is of *type* FP if the resolution can be finite, i.e. all P_i are finitely generated and all but finitely many modules in the resolution are trivial.

Corollary 2.2.17 ([11, Corollary 3.12])
Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of discrete $R[G]$-modules. Then the following holds:

(i) If M' is of type FP_{n-1} and M is of type FP_n, then M'' is of type FP_n;

(ii) If M is of type FP_{n-1} and M'' is of type FP_n, then M' is of type FP_{n-1};

(iii) If M' and M'' are of type FP_n, then M is of type FP_n.

Now we know in which case a discrete $R[G]$-module is of type FP_n, respectively FP_∞ and FP. We can define the same properties for tdlc groups.

Definition 2.2.18
Let G be a tdlc group. Then G is of *type* FP_n *over* R, respectively FP_∞ *over* R and FP *over* R, if the trivial $R[G]$-module R is of type FP_n, respectively FP_∞ and FP.

Since $\mathbb{Q}$ is flat over $\mathbb{Z}$ we have that if a group G is of type FP_n over $\mathbb{Z}$ the group is also of type FP_n over $\mathbb{Q}$.
Then we can formulate a similar result as in the discrete case for the connection between both finiteness conditions, the topological and the homological. If G is of type F_n then G is of type FP_n over any ring R, since the cellular chain complex of a G-model is a proper discrete permutation resolution of R. On the other hand, in the discrete case we can conclude type F_n from type FP_n over $\mathbb{Z}$ only if the group is finitely presented. In the non-discrete case we can formulate a similar statement.

Proposition 2.2.19 ([11, Proposition 3.13])
Let G be a compactly presented tdlc group. Then G is of type FP_n over $\mathbb{Z}$ if and only if G is of type F_n.

By this proposition we see that we mostly want to work over $\mathbb{Z}$ and not over $\mathbb{Q}$, but only over $\mathbb{Q}$ we always have a projective resolution. Therefore we cannot use some statements about projective resolutions for some proofs, so we need some other techniques than in the discrete case. In the following we have two statements about subgroups which explain in which cases we can transport some finiteness conditions from subgroups or to subgroups.

Proposition 2.2.20 ([11, Corollary 3.18])
Let $H \leq G$ be a open subgroup of the tdlc group G and M be a discrete $R[H]$-module. Then M is of type FP_n over R if and only if $\mathrm{ind}_H^G(M) = R[G] \otimes_H M$ is of type FP_n over R.

Theorem 2.2.21 ([11, Theorem 3.20])
Let $0 \to N \to G \to Q \to 0$ be a short exact sequence of tdlc groups such that N is a closed normal subgroup of G. Then we have the following

(i) if N is of type F_m and G is of type F_n, then Q is of type $\mathrm{F}_{\min\{m+1,n\}}$,

(ii) if N is of type FP_m *over R and G is of type* FP_n *over R, then Q is of type* $\mathrm{FP}_{\min\{m+1,n\}}$ *over R,*

(iii) if N is of type F_m *and Q is of type F_n, then G is of type* $\mathrm{F}_{\min\{m,n\}}$*,*

(iv) if N is of type FP_m *over R and Q is of type* FP_n *over R, then G is of type* $\mathrm{FP}_{\min\{m,n\}}$ *over R.*

In the last part of this section we introduce the criterion of Brown in the non-discrete case. For this but rather we use the same definitions as in the discrete case for being an n-good complex. But we use a discrete G-CW complex instead of an G-CW complex. The fact that such an n-good complex always exists is described in [13, §6.2].

Proposition 2.2.22 ([11, Proposition 4.5])
*Let G be a tdlc group and X be an n-good G-*CW *complex for G over R. Then G is of type* FP_n *over R if the n-skeleton of X is finite mod G.*

We also use the same definitions for the G-filtration and the properties of this filtration, but we use discrete G-CW complexes instead of G-CW complexes.

Theorem 2.2.23 ([12], Browns criterion)
*Let G be a tdlc group, X be an n-good discrete G-*CW *complex for G over R and $(X_j)_{j\in\mathbb{N}}$ be a filtration of finite n-type. Then G is of type* FP_n *over R if and only if the filtration $(X_j)_{j\in\mathbb{N}}$ is H-essentially trivial in each dimension $k < n$.*

This theorem gives us a criterion to show that a group is of type FP_n over R. If we want to use it for type F_n we need something about the compact presentability of a group. In the following theorem we use the same notation as in [7] for the construction of the graph of groups and the generators of the kernel. Let X be a simply connected G-complex and $\mathcal{G}$ be the graph which is based on the 1-skeleton of X. Then G acts by graph automorphisms on $\mathcal{G}$ and we don't know if G acts with edge inversions. Construct a graph $\mathcal{G}'$ such that G acts similar as on $\mathcal{G}$ but without edge inversions. For this let $\mathcal{G}^+$ be the subgraph of $\mathcal{G}$, which contains all vertices and all edges which can not be invert by an element of G. Let $\mathcal{G}'$ be the partial barycentric subdivision of $\mathcal{G}$, this means that we subdivided only the edges which are not in $\mathcal{G}^+$. Then G acts on $\mathcal{G}'$ without inversions. Define $\mathcal{L}$ as the graph which is given by ${}_G\backslash^{\mathcal{G}'}$ and $(\mathcal{L},\mathcal{A})$ be the graph of groups which is given by this graph and the corresponding fixpoint groups of the action of G on X. Moreover, let $\tilde{G} = \pi_1(\mathcal{L},\mathcal{A})$, then $\tilde{G}$ can be generated by G and some relations r_τ. For more details about these construction we refer to [7].

Theorem 2.2.24 ([11, Theorem 4.8])
*Let G be an abstract group and X be a simply connected discrete G-*CW *complex. Then the canonical map $\Phi\colon \tilde{G} \to G$ is surjective and its kernel is the normal subgroup of $\tilde{G}$ generated by r_τ.*

The following are the same constructions as in the proof of Theorem 2.1.2, but here we don't have directly profinite groups since we don't have a proper discrete G-CW complex.

Theorem 2.2.25 ([11, Theorem 4.9])
Let G be a tdlc group which admits a simply connected G-CW complex X satisfying:

a) All vertex isotropy groups are compactly presented,

b) all edge isotropy groups are compactly generated,

c) X has finite 2-skeleton mod G.

Then G is compactly presented.

Now we can formulate the criterion of Brown for the compact presentability of a group.

Theorem 2.2.26 ([11, Theorem 4.10])
Let X be a simply connected discrete G-CW complex with compactly presented vertex stabilizers and compactly generated edge stabilizers. Let $(X_j)_{j\in\mathbb{N}}$ be a filtration of X, such that each X_j has finite 2-skeleton mod G and $v \in \bigcap_{j\in\mathbb{N}} X_j$ be a basepoint. Then if G is compactly generated, G is compactly presented if and only if the system $(\pi_1(X_j, v))_{j\in\mathbb{N}}$ is essentially trivial.

Corollary 2.2.27 ([11, Corollary 3.3])
Let X be a contractible discrete G-CW complex such that

(i) each cell stabilizer is of type F_∞,

(ii) $(X_j)_{j\in\mathbb{N}}$ be a G-filtration of X such that each X_j is finite mod G.

Then if the connectivity of the pair (X_{j+1}, X_j) tends to infinity, if j tends to infinity G is of type F_∞.

In this chapter we have collected the definitions and statements about the higher dimensional finiteness properties of discrete and tdlc groups. For the case of discrete groups there are some known examples of classes of groups which are of type F_n or FP_n over R but not of type F_{n+1} or FP_{n+1} over R. In the next chapters we use all the statements for the tdlc case to show how we can construct examples of classes of tdlc groups which can be classified by their finiteness properties.

3 Schlichting completion

The following descriptions and statements follow a joint work with R. Sauer [5]. Some fundamental statements from the paper are described in the previous chapter, hence we do not repeat them here.
In 1980 G. Schlichting formulated a statement about subgroups of permutation groups over a quotient group (see [22]). In 1988 he reformulated this theorem in [23]. For this he used a group together with a subgroup and defined a new group, as a subgroup of the permutation group over the quotient group, depending on the group. In 2003 the construction was introduced for commensurated subgroups [32], and this is the construction we use here. We refer to this construction as the Schlichting completion. However some sources also refer to it as relative profinite completion. This name comes from the fact that this completion is also near to a profinite group and not so far away from the profinite completion of a group. Y. Shalom and G. Willis studied in [25] properties of this completion. In particular they showed that the result of the construction is always a tdlc group. Therefore the Schlichting completion introduced a concept which creates a new class of tdlc groups from a discrete group together with a commensurated subgroup.
In 2017 A. Le Boudec has shown in [19] how to transfer the lower dimensional finiteness properties of the discrete groups to the tdlc groups which come from the Schlichting completion.
Here we introduce first the construction of the Schlichting completion together with some fundamental statements and properties. Then we look at the compactly generated Schlichting completion, thereby reformulating the statement of Le Boudec, such that we can use it in both directions. In the last step of this chapter we generalize this for the higher dimensional topological and homological finiteness properties.

3.1 The construction

First we introduce the construction defined in [32] and repeat some important properties of the construction. For more background we refer to the paper of Y. Shalom and G. Willis [25], who call the Schlichting completion, the relative profinite completion.

In the following let Γ be a discrete group together with a commensurated subgroup $\Lambda \leq \Gamma$. Such a subgroup is sometimes called almost normal, since by definition $\Lambda \leq \Gamma$ is a commensurated subgroup if and only if for all $\gamma \in \Gamma$ the index $[\Lambda\colon \Lambda \cap \gamma\Lambda\gamma^{-1}]$ is finite. Then Γ acts by left multiplication on Γ/Λ. This group action induces a group homomorphism

$$\alpha\colon \Gamma \to \mathrm{Sym}\left(\Gamma/\Lambda\right).$$

Since Γ is a discrete group we have on the left side of α a topological group on the other side we equip $\mathrm{Sym}(\Gamma/\Lambda)$ with the topology of pointwise convergence. Then the closure

$$G = \Gamma//\Lambda := \overline{\alpha(\Gamma)}$$

is called the *Schlichting completion of* Γ *relative to the commensurated subgroup* Λ.

In the following we collect some fundamental properties of this completion.

Proposition 3.1.1 ([25, Section 3])
Let $G = \Gamma//\Lambda$ be the Schlichting completion of Γ relative to Λ.

(i) If Λ is a normal subgroup of Γ, then $G \cong \Gamma/\Lambda$.

(ii) The map $\alpha\colon \Gamma \to G$ has a dense image.

(iii) The closure of the image of Λ, $\overline{\alpha(\Lambda)}$, is a compact open subgroup of G. In particular it is a commensurated subgroup of G.

(iv) The collection $\{\gamma\overline{\alpha(\Lambda)}\gamma^{-1} \mid \gamma \in \alpha(\Gamma)\}$ forms a neighborhood basis of the identity of G.

(v) G is a tdlc group.

Proof:

(i) If Λ is a normal subgroup of Γ then $\ker(\alpha) = \Lambda$. Therefore the pointwise topology on $\mathrm{Sym}(\Gamma/\Lambda)$ is discrete, which implies $G \cong \Gamma/\Lambda$.

(ii) Since G is the closure of the image of α this follow directly by the construction.

(iii) The Λ-orbits of the Γ action on Γ/Λ are finite, which implies that $\overline{\alpha(\Lambda)}$ is compact. On the other hand $\gamma\Lambda \neq \Lambda$ for all $\gamma \in \Gamma \setminus \Lambda$, therefore $\overline{\alpha(\Lambda)}$ is open.

(iv) Since $\overline{\alpha(\Lambda)}$ is compact open and we chose the topology of pointwise convergence on $\mathrm{Sym}(\Gamma/\Lambda)$ it follows directly.

(v) We have seen before that G has a neighborhood basis which contains only compact open subgroups. Therefore by the Theorem of van Dantzig we get G is a tdlc group.

□

In the following lemma we show a useful connection between the Schlichting completion and the compact open subgroup.

Lemma 3.1.2 ([19, Lemma 6.3 and 6.4])
Let $G = \Gamma//\Lambda$ be the Schlichting completion of Γ relative to Λ. Then

(i) $G = \overline{\alpha(\Lambda)}\alpha(\Gamma)$,

(ii) $\overline{\alpha(\Lambda)} \cap \alpha(\Gamma) = \alpha(\Lambda)$.

Proof:

(i) Since $\overline{\alpha(\Lambda)}$ is open in G we get for all $g \in G$ that $\overline{\alpha(\Lambda)}g$ is open, and thus an open neighborhood of g. Since $\alpha(\Gamma)$ is dense in G we get $\overline{\alpha(\Lambda)} \cdot g \cap \alpha(\Gamma) \neq \emptyset$. Then there exists $\gamma \in \alpha(\Gamma)$ and $\lambda \in \overline{\alpha(\Lambda)}$ such that $\gamma = \lambda g$ which implies $g = \lambda^{-1}\gamma$.

(ii) The inclusion $\alpha(\Lambda) \subseteq \overline{\alpha(\Lambda)} \cap \alpha(\Gamma)$ is clear. For the other case we know that $\alpha(\Lambda)$ stabilizes the coset of Λ in $\mathrm{Sym}(\Gamma/\Lambda)$. Since we use the topology of pointwise convergence $\overline{\alpha(\Lambda)}$ also fixes the coset of Λ. Therefore $\overline{\alpha(\Lambda)} \cap \alpha(\Gamma) \subseteq \alpha(\Lambda)$.
□

In the following we analyze under which conditions a given group G is a Schlichting completion. Therefore we use in the following that we can define the Schlichting completion not only for discrete groups, but also for all topological groups.

Lemma 3.1.3 ([25, Lemma 3.5])
Let G be a topological group with compact open subgroup H. The map $\alpha\colon G \to \mathrm{Sym}(G/H)$ is continuous, open and has closed image. This implies $G//H \cong G/\ker(\alpha)$, where $\ker(\alpha)$ is the largest normal subgroup of G contained in H.

Proof:
Since gHg^{-1} is open for all $g \in G$ we get that the map is continuous. In the next step we show that $\alpha(G)$ is closed. Let $(g_m)_{m\in\mathbb{N}}$ be a sequence in G such that the image $(\alpha(g_m))_{m\in\mathbb{N}}$ converges to x in $G//H$. Then $g_mH = xH$ if m is large enough, which implies that $g_m \in x(H)$ for large m. Therefore $(g_m)_{m\geq N}$ is a sequence in G which is contained in a compact set xH. Let g be an accumulation point of g_m in xH, then $\alpha(g_m) = x \in \alpha(G)$. Therefore $\alpha(G)$ is closed. Since the map is closed we get $G//H = \alpha(G) = G/\ker(\alpha)$ and $\overline{\alpha(H)} = \alpha(H)$ which is a compact open subgroup, hence α is an open map. □

Corollary 3.1.4
Let G be a topological group and $H \leq G$ be a compact open subgroup. Then $G \cong G//H$ if H does not contain any non-trivial normal subgroup of G.

Lemma 3.1.5 ([25, Lemma 3.6])
Let Γ be a group with subgroup $\Lambda \leq \Gamma$ and G be a topological group with compact open subgroup $H \leq G$. Assume that there exists a group homomorphism $\varphi\colon \Gamma \to G$ such that

(i) $\varphi(\Gamma)$ is dense in G and

(ii) $\varphi^{-1}(H) = \Lambda$.

Then Λ is a commensurated subgroup of Γ and $\Gamma//\Lambda \cong G//H$.

Proof:
The assumptions (i) and (ii) imply that $\varphi(\Lambda)$ is dense in H. Then we define for each $\gamma \in \Gamma$ the map

$$\begin{aligned}\psi_\gamma\colon \Lambda/\left(\Lambda \cap \gamma\Lambda\gamma^{-1}\right) &\to H/\left(H \cap \varphi(\gamma)H\varphi(\gamma)^{-1}\right)\\ \lambda\left(\Lambda \cap \gamma\Lambda\gamma^{-1}\right) &\mapsto \varphi(\lambda)\left(H \cap \varphi(\gamma)H\varphi(\gamma)^{-1}\right).\end{aligned}$$

Since $\varphi(\Lambda)$ is dense in H we get ψ_γ is surjective for all $\gamma \in \Gamma$. On the other hand we have for each $\lambda \in \Lambda$, if $\varphi(\lambda) \in H \cap \varphi(\gamma)H\varphi(\gamma)^{-1}$ then $\lambda \in \Lambda \cap \gamma\Lambda\gamma^{-1}$, which shows that ψ_γ is injective, thus bijective. This implies $[\Lambda : \Lambda \cap \gamma\Lambda\gamma^{-1}] = [H : H \cap \varphi(\gamma)H\varphi(\gamma)^{-1}]$, where the right side is finite, since each compact open subgroup of a topological group is commensurated. Therefore Λ is a commensurated subgroup of Γ.

For the second part we define the function

$$\hat{\varphi}\colon {}^{\Gamma}/_{\Lambda} \to {}^{G}/_{H}, \quad \gamma\Lambda \mapsto \varphi(\gamma)H.$$

Since H is open and φ has a dense image $\hat{\varphi}$ is surjective. The second assumption implies the injectivity of $\hat{\varphi}$. Γ acts on ${}^{\Gamma}/_{\Lambda}$ and $\hat{\varphi}$ transfers this action into an action on ${}^{G}/_{H}$. Let $\alpha\colon \Gamma \to \mathrm{Sym}({}^{\Gamma}/_{\Lambda})$ and $\tilde{\alpha}\colon G \to \mathrm{Sym}({}^{G}/_{H})$ be the corresponding action homomorphism. Then we get $\overline{\alpha(\Gamma)} \cong \overline{\hat{\varphi}(\alpha(\Gamma))}$, where the right side is the same as $\overline{\tilde{\alpha}(G)}$, since $\alpha(\Gamma)$ is dense in G. Therefore $\Gamma/\!/\Lambda = \overline{\alpha(\Gamma)} \cong \overline{\tilde{\alpha}(G)} = G/\!/H$. □

Let Γ be a discrete group and $\Lambda \leq \Gamma$ be a commensurated subgroup. The previous statements provide the following universal property of the Schlichting completion of Γ relative to Λ. Let G be an arbitrary tdlc group together with a compact open subgroup $H \leq G$ such that there exists $\varphi\colon \Gamma \to G$ with dense image and $\varphi^{-1}(H) = \Lambda$. Then there exists a continuous and injective ψ such that

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \varphi\ } & G \\ {\scriptstyle\alpha}\big\downarrow & \swarrow{\scriptstyle \exists!\psi} & \\ \Gamma/\!/\Lambda & & \end{array} \tag{3.1.1}$$

is a commutative diagram.

3.2 Compact presentability

A. Le Boudec has shown in [19] how it is possible to generate a compact presentation of the Schlichting completion of Γ relative to the commensurated subgroup Λ if Γ is finitely presented and Λ is finitely generated. In the following we look at generalized presentations of Schlichting completions and how they can be built up from Γ and the commensurated subgroup Λ. These presentations allow us to give an alternative proof of the theorem of A. Le Boudec. Moreover the same construction also provides a proof of the reverse statement.

In Proposition 2.2.3 we have seen that for each tdlc group there exists a generalized presentation with a graph of profinite groups based on a single vertex graph. For Schlichting completions we specify this graph of profinite groups in the following way.

Lemma 3.2.1
Let $G = \Gamma/\!/\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup Λ and S be a generating set of $\alpha(\Gamma)$. Then there exist a generalized presentation $((\mathcal{G}, \mathcal{A}), \Phi)$ of G such that

- *$\mathcal{G}$ is a single vertex graph,*
- *$E(\mathcal{G}) = \{e_s, \overline{e_s} \mid s \in S\}$,*
- *$\mathcal{A}_v = \overline{\alpha(\Lambda)}$,*
- *$\mathcal{A}_{e_s} = \overline{\alpha(\Lambda)} \cap s^{-1}\overline{\alpha(\Lambda)}s$ for each $s \in S$,*

- $\iota_{e_s}\colon \mathcal{A}_{e_s} \to \mathcal{A}_v,\ \lambda \mapsto s\lambda s^{-1}$,
- $\iota_{\overline{e_s}}\colon \mathcal{A}_{\overline{e_s}} \to \mathcal{A}_v,\ \lambda \mapsto \lambda$.

Proof:
Let $(\mathcal{G}, \mathcal{A})$ be the graph of groups with the above assumptions. By Lemma 3.1.2 we have seen that $G = \overline{\alpha(\Lambda)}\alpha(\Gamma)$ which implies that G can be generated by $\overline{\alpha(\Lambda)} \cup S$. Since $\overline{\alpha(\Lambda)}$ is a compact open subgroup of G it is also a profinite group, therefore $(\mathcal{G}, \mathcal{A})$ is a graph of profinite groups. Similar as in the proof of Proposition 2.2.3 we can define the map from $\Pi = \pi_1(\mathcal{G}, \mathcal{A})$ to G.

$$\begin{aligned}\Phi\colon \Pi &\to G\\ \lambda &\mapsto \lambda, \ \text{ if } \lambda \in \mathcal{A}_v\\ e_s &\mapsto s, \ \text{ for all } e_s \in E(\mathcal{G})\end{aligned}$$

Then $((\mathcal{G}, \mathcal{A}), \Phi)$ is generalized presentation of G. □

In the following let $G := \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$, $((\mathcal{G}, \mathcal{A}), \Phi)$ be the generalized presentation of G from Lemma 3.2.1 and $\Pi = \pi_1(\mathcal{G}, \mathcal{A})$ be the fundamental group of the graph of profinite groups. We want to find a subgroup $H \subseteq \Pi$, such that $\Phi(H) = \alpha(\Gamma)$. By Lemma 3.1.2 we know that $\overline{\alpha(\Lambda)} \cap \alpha(\Gamma) = \alpha(\Lambda)$, so we can choose $H = \langle \alpha(\Lambda) \cup E(\mathcal{G})\rangle_{\Pi}$. Therefore we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N = \ker(\Phi) & \longrightarrow & \Pi & \xrightarrow{\Phi} & G & \longrightarrow & 0\\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M = \ker(\Phi|_H) & \longrightarrow & H & \xrightarrow{\Phi|_H} & \alpha(\Gamma) & \longrightarrow & 0\end{array} \tag{3.2.1}$$

Lemma 3.2.2
Let $G = \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$. Let Π and H be as above. Then Π is the Schlichting completion of H relative to the commensurated subgroup $\alpha(\Lambda) \leq H$.

Proof:
By construction, H is a dense subgroup of Π and $\overline{\alpha(\Lambda)}$ is a compact open subgroup of Π. By Lemma 3.1.5 we get that $\alpha(\Lambda) \leq H$ is a commensurate subgroup. Moreover it follows from Lemma 3.1.3 and Lemma 3.1.5 that $\overline{\alpha(\Lambda)}$ contains no non-trivial normal subgroup of $G = \Gamma//\Lambda = G//\overline{\alpha(\Lambda)}$. Since Φ restricted to $\overline{\alpha(\Lambda)}$ is injective, $\overline{\alpha(\Lambda)}$ also contains no non-trivial normal subgroup of Π. Therefore we get $H//\alpha(\Lambda) \cong \Pi//\overline{\alpha(\Gamma)} \cong \Pi$ by Lemma 3.1.3. □

Since $\Pi = H//\alpha(\Lambda)$ is the Schlichting completion of H relative to the commensurated subgroup $\alpha(\Lambda) \leq H$, we obtain with Lemma 3.1.2 that $\Pi = \overline{\alpha(\Lambda)}H$.
This is used in the proof of the following lemma.

Lemma 3.2.3
Let $G, \Pi, H, \alpha(\Gamma), N$ and M as in the diagram (3.2.1). Then N lies in H. In particular, $N = M$.

Proof:
Let $x \in N \subseteq \Pi$. There exists $\lambda \in \overline{\alpha(\Lambda)}$ and $h \in H$ such that $x = \lambda h$. Also, we have by construction of Φ that $\Phi|_{\overline{\alpha(\Lambda)}}$ is the identity on this subgroup. Therefore we get

$$1 = \Phi(x) = \Phi(\lambda h) = \Phi(\lambda)\Phi(h) = \lambda\Phi(h).$$

Thus $\lambda \in \overline{\alpha(\Lambda)} \cap \Phi(H) = \overline{\alpha(\Lambda)} \cap \alpha(\Gamma) = \alpha(\Lambda)$. Hence $x = \lambda h \in H$, so $N \leq H$ and then $N = M$. $\square$

For the proof of Theorem 3.2.6 we need a connection between the compact generation respectively presentation of Γ and Λ and their images under α. Thus we start with the finiteness properties of the kernel of α.

Lemma 3.2.4
Let Γ be a discrete group, $\Lambda \leq \Gamma$ be a commensurated subgroup and $\alpha\colon \Gamma \to \mathrm{Sym}(\Gamma/\Lambda)$. If Γ is finitely generated the kernel of α has the same finiteness properties as Λ.

Proof:
The kernel of α is given by $\ker(\alpha) = \bigcap_{\gamma\in\Gamma} \gamma\Lambda\gamma^{-1}$. Let S be a finite generating set of Γ. Then we get

$$\begin{aligned}
[\Lambda : \ker(\alpha)] &= [\Lambda : \bigcap_{\gamma\in\Gamma} \gamma\Lambda\gamma^{-1}] \\
&= [\Lambda : \bigcap_{\gamma\in S\cup S^{-1}} \gamma\Lambda\gamma^{-1}] \\
&= [\Lambda : \bigcap_{\gamma\in S\cup S^{-1}} (\gamma\Lambda\gamma^{-1} \cap \Lambda)] \\
&\leq \prod_{\gamma\in S\cup S^{-1}} [\Lambda : \gamma\Lambda\gamma^{-1} \cap \Lambda].
\end{aligned}$$

Each Element in the product is finite, since Λ is a commensurated subgroup. Then if S is finite we get that $\ker(\alpha)$ is a finite index subgroup of Λ. Thus $\ker(\alpha)$ and Λ have the same finiteness properties. $\square$

We obtain the following corollary directly from the following short exact sequences and Lemma 3.2.4

$$\begin{gathered}
0 \to \ker(\alpha) \to \Lambda \to \alpha(\Lambda) \to 0 \\
0 \to \ker(\alpha) \to \Gamma \to \alpha(\Gamma) \to 0.
\end{gathered}$$

Corollary 3.2.5

(i) If Λ and Γ are finitely generated, then $\alpha(\Lambda)$ is finitely generated.

(ii) If Λ is finitely presented and Γ is finitely generated, then $\alpha(\Lambda)$ is finitely presented.

(iii) If Γ is finitely generated, then $\alpha(\Gamma)$ is finitely generated.

(iv) If Γ is finitely presented and Λ finitely generated, then $\alpha(\Gamma)$ is finitely presented.

(v) If $\alpha(\Gamma)$ and $\ker(\alpha)$ are finitely generated, then Γ is finitely generated.

(vi) If $\alpha(\Gamma)$ and $\ker(\alpha)$ are finitely presented, then Γ is finitely presented.

For the proof of the following theorem we use the definition of being compactly generated and being compactly presented of Definition 2.2.4 by graphs of profinite groups.

Theorem 3.2.6 ([5, Theorem 2.3])
Let $G = \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$.

(i) If G is compactly generated and Λ is finitely generated, then Γ is finitely generated.

(ii) If G is compactly presented and Λ is finitely presented, then Γ is finitely presented.

(iii) If Γ is finitely generated, then G is compactly generated.

(iv) If Γ is finitely presented and Λ is finitely generated, then G is compactly presented.

For the general case of this theorem we refer to [5] here we only consider the case that we can repace Γ by $\alpha(\Gamma)$ and Λ by $\alpha(\Lambda)$.
Proof:
In case (iii) and (iv) Γ is finitely generated so by Corollary 3.2.5 we can replace Γ by $\alpha(\Gamma)$ and Λ by $\alpha(\Lambda)$. In the case (i) and (ii) we only consider the case that $\ker(\alpha)$ is finitely generated, respectively finitely presented, so we can replace Γ by $\alpha(\Gamma)$ and Λ by $\alpha(\Lambda)$ also.

(i) Let G be compactly generated and $((\mathcal{G}, \mathcal{A}), \Phi)$ be a generalized presentation of G with a finite connected graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ that has a single vertex. Let $\alpha(\Lambda)$ be generated by the finite subset $S' \subseteq \alpha(\Lambda)$. The group G is generated by the compact subset $\mathcal{A}_v \cup E(\mathcal{G})$. Furthermore, we have

$$G = \bigcup_{s \in \alpha(\Gamma)} \overline{\alpha(\Lambda)} s.$$

By compactness, there is a finite subset $S \subseteq \alpha(\Gamma)$ such that

$$\mathcal{A}_v \cup E(\mathcal{G}) \subseteq \bigcup_{s \in S} \overline{\alpha(\Lambda)} s.$$

Therefore, we can define a generalized presentation of G by $((\mathcal{G}', \mathcal{A}'), \Phi')$ as in Lemma 3.2.1 with a single vertex graph $\mathcal{G}'$ and finite edge set S. As in Equation (3.2.1) we get $H = \langle \alpha(\Lambda) \cup S \rangle_\Pi = \langle S' \cup S \rangle_\Pi$ showing that H and $\alpha(\Gamma)$ are finitely generated.

(ii) Let G be compactly presented, $((\mathcal{G}, \mathcal{A}), \Phi)$ be the generalized presentation of Lemma 3.2.1. Then by Lemma 2.2.6 $N = \ker(\Phi)$ is finitely generated as a normal subgroup and $\alpha(\Lambda) = \langle S' | R' \rangle$ be finitely presented. Then H is a finitely presented group and $M = \ker(\Phi|_H) = N$ is finitely generated as a normal subgroup. Therefore, $\Phi(H) = \alpha(\Gamma)$ is finitely presented.

(iii) Let $\alpha(\Gamma)$ be finitely generated and $S \subseteq \alpha(\Gamma)$ be a finite generating set of $\alpha(\Gamma)$. We can choose S to be the finite set from which the edge set of the generalized presentation $((\mathcal{G},\mathcal{A}),\Phi)$ of G from Lemma 3.2.1 comes. Therefore G is compactly generated since $\mathcal{G}$ is finite.

(iv) Let $\alpha(\Lambda)$ be finitely generated by $S' \subseteq \alpha(\Lambda)$ and $\alpha(\Gamma)$ be finitely presented by $\langle S|R\rangle$ where $S' \subseteq S$. For the generalized presentation $((\mathcal{G},\mathcal{A}),\Phi)$ of G we use the same finite graph $\mathcal{G}$ as in (3). Then we need to show that $N = \ker(\Phi)$ is finitely generated as a normal subgroup of Π. By Lemma 3.2.3 we have seen that it is enough to show that $M = \ker(\Phi|_H)$ is finitely generated as a normal subgroup of H. We choose for H the set $\alpha(\Lambda) \cup E(\mathcal{G})$ as generating set in Π, then we get that all relations of $\alpha(\Lambda)$ are hold in H and since S is inside the generating set we get that M is normally generated by all the relations in R, which do not hold in H. But since R is finite, M is normally finitely generated. Therefore, N is too so G is compactly presented. □

In the following we give an example for this construction and then we use Lemma 2.2.15 which gives a way to construct the complexes for the group from the generalized presentation. In this example we show how we can construct the G-model for the Schlichting completion of the classifying spaces of Γ and the commensurated subgroup $\Lambda \leq \Gamma$.

Example 3.2.7
Let $\Gamma := \langle a, b, c \mid bab^{-1} = a^2, cac^{-1} = a^3, bc = cb\rangle$ with $S = \{a, b, c\}$ be the finite generating set and R be the finite set of relations. Let $\Lambda := \langle a\rangle \leq \Gamma$ be the subgroup generated by $S' = \{a\}$. Since $[\Lambda : s\Lambda s^{-1} \cap \Lambda] < \infty$ for all $s \in S \cup S^{-1}$ we get that Λ is a commensurated subgroup of Γ. Equip Γ with the discrete topology and define $G := \Gamma//\Lambda$. Since Γa and Λ are finitely presented we can use the construction of Theorem 3.2.6 to define a G-model. Then we can define the generalized presentation for G as in Lemma 3.2.1.

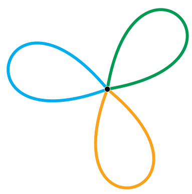

Figure 3.1: Graph of groups for the generalized presentation of G

Let $\mathcal{G}$ be the graph as in Figure 3.1, where the orange edge corresponds to the generator a, the blue edge to the generator b and the green edge to the generator c. We equip this graph with the following profinite groups and embeddings:

$$\mathcal{A}_v = \overline{\alpha(\Lambda)} \quad \text{and} \quad \mathcal{A}_{e_s} = \mathcal{A}_{\overline{e_s}} = \overline{\alpha(\Lambda)} \cap s^{-1}\overline{\alpha(\Lambda)}s \quad \text{for all } s \in S,$$
$$\iota_{e_s} \colon \mathcal{A}_{e_s} \to \mathcal{A}_v, \qquad \lambda \mapsto s\lambda s^{-1},$$
$$\iota_{\overline{e_s}} \colon \mathcal{A}_{\overline{e_s}} \to \mathcal{A}_v, \qquad \lambda \mapsto \lambda.$$

Then $(\mathcal{G}, \mathcal{A})$ is a finite graph of profinite groups and together with

$$\begin{aligned} \Phi\colon \pi_1(\mathcal{G}, \mathcal{A}) &\to G, \\ \lambda &\mapsto \lambda \quad \lambda \in \mathcal{A}_v \\ e_s &\mapsto s \quad e_s \in E(\mathcal{G}) \end{aligned}$$

we have a generalized presentation of G. Let $\Pi = \pi_1(\mathcal{G}, \mathcal{A})$ and $H \coloneqq \langle \alpha(\Lambda) \cup E(\mathcal{G}) \rangle_\Pi$ such that $\Phi(H) = \alpha(\Gamma)$ and we have the same diagram as in 3.2.1.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N = \ker(\Phi) & \longrightarrow & \Pi & \xrightarrow{\Phi} & G & \longrightarrow & 0 \\ & & \updownarrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M = \ker(\Phi|_H) & \longrightarrow & H & \xrightarrow{\Phi|_H} & \alpha(\Gamma) & \longrightarrow & 0 \end{array} \tag{3.2.2}$$

We want to figure out how N look like, therefore we describe the presentation of H and find the normal generators of M. On the one hand we have $\alpha(\Gamma) = \langle \alpha(S) \mid \alpha(R) \rangle$ and on the other hand

$$\begin{aligned} H &= \langle \alpha(\Lambda) \cup E(\mathcal{G}) \rangle_\Pi \\ &\cong \langle a, e_a, e_b, e_c \mid e_a = a,\ e_b a \overline{e_b} = a^2,\ e_c a \overline{e_c} = a^3 \rangle \\ &\cong \langle a, b, c \mid bab^{-1} = a^2,\ cac^{-1} = a^3 \rangle. \end{aligned}$$

Therefore $M \cong \langle\langle bc = cb \rangle\rangle_H$ which has the same normal generators as N in Π.
In the next step we use the proof of Lemma 2.2.15 to show how we can construct the finite 2-skeleton of a G-model from a 2-skeleton of a universal cover of a classifying space of Γ. We use the Cayley graph of Γ with respect to S as universal cover and for each relation in R we glue in a Γ-orbit of 2-cells. In Figure 3.2 we see a small part of this Cayley graph, we use the same coloring as for $\mathcal{G}$.

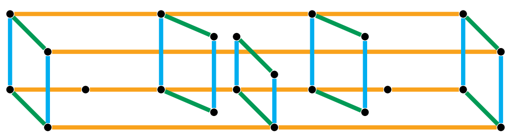

Figure 3.2: Cayley Graph of Γ with respect to $\{a, b, c\}$

In the proof of Lemma 2.2.15 we have seen that we need at first the Bass-Serre tree of $(\mathcal{G}, \mathcal{A})$ to create then the G-model. Since $\sum_{e \in E(\mathcal{G})} [\mathcal{A}_v : \iota_e(\mathcal{A}_e)] + [\mathcal{A}_v : \iota_{\overline{e}}(\mathcal{A}_{\overline{e}}] = 9$, we get the 9-regular tree as in Figure 3.3 as Bass-Serre tree, we use again the same coloring.
If we cut out $\mathcal{A}_v$ from the Bass-Serre tree we get a Cayley-Abels graph for G which looks like Figure 3.4.
Here we see that all closed paths in the Cayley-Abels graph comes from a square (green edge, blue edge, green edge, blue edge), which corresponds to the relation $bc = cb$. Therefore we can fill the Cayley-Abels graph with G-orbits of the generator of N, which is exactly the relation $bc = cb$.

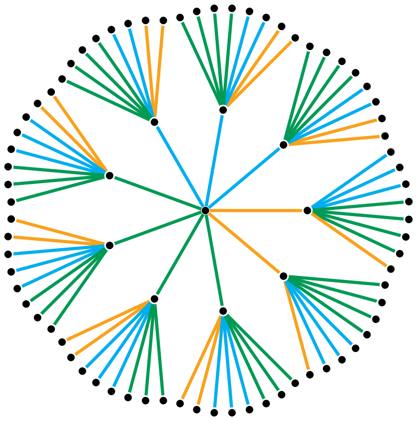

Figure 3.3: Bass-Serre tree of $(\mathcal{G}, \mathcal{A})$

3.3 Finiteness properties

In Theorem 3.2.6 we have seen how we can transfer the lower dimensional finiteness properties between the discrete groups and the Schlichting completion. In the following we prove the generalization of this theorem for the topological and homological finiteness properties.

Theorem A ([5, Theorem 1.1])
Let $G := \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$. Let R be a commutative ring. Then the following holds.

(i) If Λ and G are of type FP_n over R, then Γ is of type FP_n over R.

(ii) If Λ and G are of type F_n, then Γ is of type F_n.

Theorem B ([5, Theorem 1.2])
Let $G := \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$. Let R be a commutative ring. Then the following holds.

(i) If Λ is of type FP_{n-1} over R and Γ is of type FP_n over R, then G is of type FP_n over R.

(ii) If Λ is of type F_{n-1} and Γ is of type F_n, then G is of type F_n.

In the case that Λ is a normal subgroup of Γ, we have seen in Proposition 3.1.1 that the Schlichting completion G is the quotient group. Consequently, this case directly follows from Theorem 2.1.6 and Theorem 2.1.8.
The cases $n = 1$ and $n = 2$ of part (ii) of both theorems are already covered by Theorem 3.2.6. Because the properties FP_n over $\mathbb{Z}$ and F_n are equivalent for groups of type F_2, by Theorem 2.1.2 and Lemma 2.2.15, it remains to show part (i) of both theorems.

Lemma 3.3.1 ([5, Lemma 3.3])
Let $G = \Gamma//\Lambda$ be the Schlichting completion of Γ relative to Λ, R be a commutative ring and $M = R[\Omega]$ be a finitely generated proper discrete permutation G-module over R. Then, if Λ is of type FP_n over R, the restriction $\mathrm{res}^G_\Gamma(M)$ has a projective $R(\Gamma)$-resolution $P_* \twoheadrightarrow M$ such that $P_0, \ldots, P_n$ are finitely generated. If Λ is locally finite and

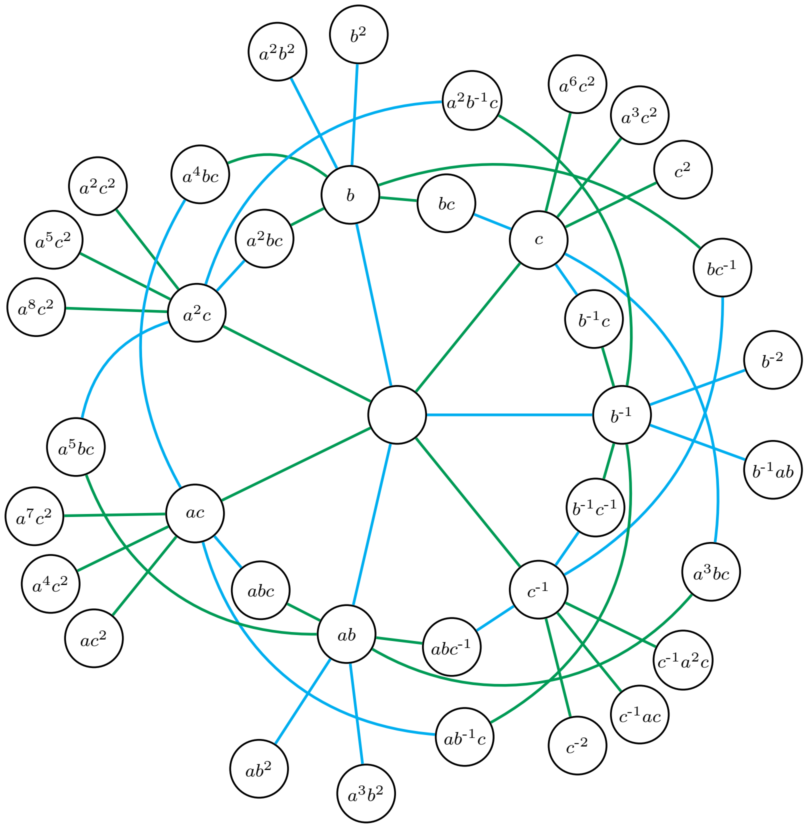

Figure 3.4: Cayley-Abels Graph of G

$\mathbb{Q} \subseteq R$, then $\operatorname{res}^G_\Gamma(M)$ is a flat $R[\Gamma]$-module.

Proof:
By Proposition 2.1.9 finite index subgroups are of type FP_n over R if and only if the group is of type FP_n over R. This implies that two commensurated subgroups are of the same finiteness type. Therefore let $\Lambda' \leq \Gamma$ be a subgroup which is commensurated to Λ. Let $Q_* \to R$ be a projective $R[\Lambda']$-resolution of the trivial module such that $Q_0, \dots, Q_n$ are finitely generated. Then $R[\Gamma] \otimes_{R[\Lambda']} Q_*$ is a projective resolution of $R\left[\Gamma/\Lambda'\right]$ which is finitely generated until degree n. Hence the $R[\Gamma]$-module $R\left[\Gamma/\Lambda'\right]$ is of type FP_n. On the other hand we have that a finitely generated discrete permutation G-module $R[\Omega]$ is a finite sum of modules of the type $R\left[G/U\right]$, where $U \leq G$ is a compact open subgroup. By Lemma 3.1.2 we can conclude that $G/U \cong \Gamma/\alpha^{-1}(U)$ and $\alpha^{-1}(U)$ is commensurated to $\alpha^{-1}(\overline{\alpha(\Lambda)}) = \Lambda$. This implies that $\operatorname{res}^G_\Gamma\left(R[G/U]\right)$ is of type FP_n over R as $R[\Gamma]$-module. For the last step let Λ be locally finite, this implies that Λ' is also locally finite, and let $\mathbb{Q} \subseteq R$. Then R is a flat $R[\Lambda']$-module by [4, Proposition 4.12 on p. 63], therefore $R[\Gamma] \otimes_{R[\Lambda']} R = R\left[\Gamma/\Lambda'\right]$ is a flat $R[\Gamma]$-module. □

For the proof of Theorem A we need one more Lemma by Brown.

Lemma 3.3.2 ([6, Lemma 1.5])
Let C_* be a chain complex over a ring R and $P_*^{(i)}$ be a projective resolution of C_i. Then there exist a chain complex Q_* with $Q_n = \bigoplus_{i+j=n} P_i^{(j)}$ and a weak equivalence $Q_* \to C_*$.

In the next step we use these two lemmas to prove the first theorem.

Proof of Theorem A:
We have seen before that we only need to proof the first part of the theorem, the second part follows immediately by Theorem 3.2.6 and Proposition 2.2.19.
We assume that G and Λ are of type FP_n over R. Then let

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \to R \to 0$$

be a projective resolution of the trivial $R[G]$-module by proper discrete permutation modules, such that they are finitely generated until degree n. By Lemma 3.3.1 there exist for each $R[\Gamma]$-module $\mathrm{res}^G_\Gamma(P_i)$, $i \leq n$, a projective resolution $Q^{(i)}_*$ such that $Q^{(i)}_j$ is finitely generated for $j \in \{1, \dots, n\}$. For $i > n$ let $Q^{(i)}_*$ any projective resolution of the restriction $\mathrm{res}^G_\Gamma(P_i)$. Lemma 3.3.2 shows that there exists a projective resolution Q_* of the trivial $R[\Gamma]$-module R with

$$Q_k = \bigoplus_{i+j=k} Q^{(i)}_j.$$

This implies that Q_k is finitely generated until degree n and therefore Γ is of type FP_n over R. □

For the proof of Theorem B we need the following proposition.

Proposition 3.3.3 ([5, Proposition 3.6])
Let $G = \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda < \Gamma$. Let R be a commutative ring. Let M be a discrete $R[G]$-module. If Λ is of type FP_m over R and $\mathrm{res}^G_\Gamma(M)$ is of type FP_n over R then M is of type $\mathrm{FP}_{\min\{m+1,n\}}$ over R.

Proof:
If $\mathrm{res}^G_\Gamma(M)$ is not finitely generated we are done. If $\mathrm{res}^G_\Gamma(M)$ is finitely generated then M is clearly finitely generated. In particular, there is a short exact sequence

$$0 \to K \to P \to M \to 0 \tag{3.3.1}$$

where P is a finitely generated proper discrete permutation module.
We show the statement by induction over n. The case $n = 1$ just means finite generation, and there is nothing more to do. Suppose the statement holds true for every restriction of an $R[G]$-module of type FP_{n-1} over R. Let $\mathrm{res}^G_\Gamma(M)$ be of type FP_n over R and choose a sequence as in (3.3.1). We apply Corollary 2.2.17 to the short exact sequence (3.3.1) for the tdlc group G and to the short exact sequence

$$0 \to \mathrm{res}^G_\Gamma(K) \to \mathrm{res}^G_\Gamma(P) \to \mathrm{res}^G_\Gamma(M) \to 0 \tag{3.3.2}$$

for the discrete group Γ. By Lemma 3.3.1 the module $\mathrm{res}^G_\Gamma(P)$ is of type FP_m over R. By part (ii) of Corollary 2.2.17 the kernel $\mathrm{res}^G_\Gamma(K)$ is of type $\mathrm{FP}_{\min\{m,n-1\}}$ over R. By induction hypothesis, K is of type $\mathrm{FP}_{\min\{m,n-1\}}$ over R. By part (i) of Corollary 2.2.17, applied to (3.3.1), we obtain that M is of type $\mathrm{FP}_{\min\{m,n-1\}+1} = \mathrm{FP}_{\min\{m+1,n\}}$ over R. This concludes the proof. □

Proof of Theorem B:
The first part of the theorem follows by applying Proposition 3.3.3 to the trivial G-module R. The second part follows immediately by Theorem 3.2.6 and Proposition 2.2.19 from the first part. □

Example 3.3.4 (The Abels-Brown group)
Let R be a commutative ring. Let $\Gamma_n(R)$ denote the subgroup of $\mathrm{GL}_{n+1}(R)$ that consists of upper triangular matrices $(g_{i,j})$ such that $g_{1,1} = g_{n+1,n+1} = 1$. For example, $\Gamma_2(R)$ consists of matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

This group was studied by H. Abels and K. Brown [1]. They showed that $\Gamma_n(\mathbb{Z}[1/p])$ is of type FP_{n-1} over $\mathbb{Z}$ but not of type FP_n over $\mathbb{Z}$. Moreover, for $n \geq 3$ it is finitely presented. The subgroup $\Lambda_n = \Gamma_n(\mathbb{Z})$ has entries ± 1 on the diagonal. Therefore, Λ_n is finitely generated nilpotent, hence of type FP_∞ over $\mathbb{Z}$. Let G_n be the Schlichting completion of $\Gamma_n(\mathbb{Z}[1/p])$ relative to Λ_n. By Theorem B, G_n is of type FP_{n-1} over $\mathbb{Z}$. By Theorem A, G_n is not of type FP_n over $\mathbb{Z}$. Therefore G_n is also of type F_{n-1} but not of type F_n.
By Theorem 3.2.6, G_n is compactly presented for $n \geq 3$.

4 Tree almost automorphism fixpoint groups

The class of tree almost automorphisms is one of the first classes of tdlc groups for which it was proven that they are of type F_∞ (see [21]). These groups generalize the Neretin group N_q, which can be described as the Schlichting completion of the Higmann-Thompson group V_q. But the techniques of Chapter 3 cannot be used for the finiteness properties, since the commensurated subgroup is not compactly generated.
Tree almost automorphism groups can be viewed as certain subgroups of the full automorphism group of a regular tree, consisting of elements that act almost everywhere as an automorphism but may differ on a finite subtree. This perspective makes them a natural generalization of both Thompson group V and the Neretin group, while also providing new examples of tdlc groups.
In this chapter, we focus on the fixpoint subgroups of tree almost automorphisms and investigate their finiteness properties. First we recall the fundamental definitions and structural properties of tree almost automorphisms. In the following we define the fixpoint subgroups of the tree almost automorphisms and prove their finiteness properties. Therefore we repeat at first some definitions which we need for the tree almost automorphisms. Then we define the fixpoint subgroups and show some properties of them, which we need later. After that we recall an introduction into Morse theory for categories (see [30]). We use this theory in various steps of the proof that the fixpoint subgroups are of type F_∞.

4.1 Tree almost automorphism

In this section, we recall some basic notions about tree almost automorphisms following the notation of [21]. In Section 1.2 we introduced the group of automorphisms of trees. The tree almost automorphisms are a generalization of this construction. Let $q \geq 2$ and $\mathcal{T}_q$ be the q-regular infinite rooted tree with root v_0. We have seen that the group $\mathrm{Aut}(\mathcal{T}_q)$ is a tdlc group, with respect to the compact open topology. Moreover, the sets

$$U_n := \left\{\gamma \in \mathrm{Aut}(\mathcal{T}_q) \mid \gamma|_{B_n(v_0)} = \mathrm{id}|_{B_n(v_0)}\right\}$$

form a neighborhood basis of compact open subgroups of the identity.
It is well known that $\mathrm{Aut}(\mathcal{T}_q)$ is topologically isomorphic to the group of isometries $\mathrm{Isom}(\mathtt{B}_q)$ of the boundary $\mathtt{B}_q$ of $\mathcal{T}_q$ equipped with the visual metric d. In the next step we first recall the definition of this metric and take a look at the connection between open balls in $\mathtt{B}_q$ and subtrees of $\mathcal{T}_q$.

Definition 4.1.1
Let $x, y \in \mathtt{B}_q$ and $[v_0, x)$ and $[v_0, y)$ be the infinite rays in $\mathcal{T}_q$ which start at the root v_0 and represent x, respectively y. Then define $c\colon \mathtt{B}_q \times \mathtt{B}_q \to \mathbb{N}$, which maps (x, y) to the length of the common initial segments of $[v_0, x)$ and $[v_0, y)$. Define the *visual metric* d on $\mathtt{B}_q$ as follows

$$d(x, y) = \exp(-c(x, y)).$$

Remark 4.1.2
Let $x \in \mathtt{B}_q$ and $v \in [v_0, x)$ be a vertex of the infinite ray such that $d(v, v_0) = n$. The space of ends of the full subtree of $\mathcal{T}_q$ that hangs at the root v is the ball of radius e^{-n} around x. Therefore we say the full subtree of $\mathcal{T}_q$ that hangs at v represents $B_{e^{-n}}(x)$ in $\mathtt{B}_q$.

With the compact open topology, which is induced by the visual metric, $\mathrm{Isom}(\mathtt{B}_q)$ is topologically isomorphic to $\mathrm{Aut}(\mathcal{T}_q)$. Therefore we write sometimes $\mathrm{Aut}(\mathcal{T}_q) = \mathrm{Isom}(\mathtt{B}_q)$. Then the neighborhood basis U_n of $\mathrm{Aut}(\mathcal{T}_q)$ induces a neighborhood basis of $\mathrm{Isom}(\mathtt{B}_q)$ and $\mathrm{Isom}(\mathtt{B}_q)$ is also a tdlc group. Before we can start with the generalization of the maps we need two definitions of maps of metric spaces.

Definition 4.1.3
Let (X, d_x) and (Y, d_Y) be two metric spaces and $\phi\colon X \to Y$ be a map of metric spaces. Then ϕ is called a *similarity* if there exist $\lambda > 0$ such that for all $x_1, x_2 \in X$ we get $d_Y(\phi(x_1), \phi(x_2)) = \lambda d_X(x_1, x_2)$. On the other hand we call ϕ a *local similarity* if for each $x \in X$ there exist $n > 0$ and $\lambda > 0$ such that ϕ restricted to $B_n(x)$ is a λ-similarity, i.e. for all $y \in B_n(x)$ we get $d_Y(\phi(x), \phi(y)) = \lambda d_X(x, y)$.

Definition 4.1.4
Let $\phi\colon \mathtt{B}_q \to \mathtt{B}_q$ be a map of metric spaces. Then ϕ is called *spheromorphism* if ϕ is a local similarity.

Since we have seen before that balls in the boundary coincide with infinite full subtrees we get the following alternative description for the local similarities of the boundary.

Remark 4.1.5
Let F_1 and F_2 be two finite q-regular rooted subtrees of $\mathcal{T}_q$, with the same number of leaves. Then we can define an isomorphism of forests $f\colon \mathcal{T}_q \setminus F_1 \to \mathcal{T}_q \setminus F_2$ which induces a map on the boundary $\mathtt{B}_q$ of $\mathcal{T}_q$. Each such induced map is a local similarity at $\mathtt{B}_q$ and we call two triples (F_1, F_2, f) and (F_1', F_2', f') of two subtrees and a isomorphism of a forest equivalent, if they induce the same spheromorphism.

Lemma 4.1.6
The set of all spheromorphisms of $\mathtt{B}_q$ *form a subgroup of* $\mathrm{Homeo}(\mathtt{B}_q)$.

Proof:
Let γ_1 and γ_2 be spheromorphisms represented by the triples (F_1, F_2, f), respectively (F_1', F_2', f'). Then there exist a finite q-regular rooted subtree F of $\mathcal{T}_q$ such that F_1' and F_2 are contained in F. We can find $\tilde{F}$ and $\tilde{F}'$ such that $(F_1, F_2, f) \sim (F, \tilde{F}, \tilde{f})$ and $(\tilde{F}', F, \tilde{f}') \sim (F_1', F_2', f')$. Therefore $(\tilde{F}, \tilde{F}', \tilde{f}' \circ \tilde{f})$ represents $\gamma_2 \circ \gamma_1$ and (F_2, F_1, f^{-1}) represents γ_1^{-1}. □

The group of spheromorphisms of $\mathtt{B}_q$ is sometimes called the group of *almost automorphisms* of $\mathcal{T}_q$, denoted by $\mathrm{AAut}(\mathcal{T}_q) \coloneqq \mathcal{A}_q$. Since we are interested in topological groups we need to equip the group of almost automorphisms with a topology. We have a topology on the automorphisms of the tree which is topologically isomorphic to the isometries of the boundary. We equip $\mathcal{A}_q$ with the unique topology, such that the embedding $\mathrm{Isom}(\mathtt{B}_q) \hookrightarrow \mathcal{A}_q$ is continuous and open.

Lemma 4.1.7
With the topology from above $\mathcal{A}_q$ is a tdlc group.

Proof:
The sets $U_n \subseteq \mathrm{Aut}(\mathcal{T}_q) = \mathrm{Isom}(\mathtt{B}_q) \hookrightarrow \mathcal{A}_q$ form a neighborhood basis of the identity in $\mathcal{A}_q$ by the assumption that the embedding is continuous and open. Therefore $\mathcal{A}_q$ is a tdlc group by the theorem of van Dantzig. □

We have seen in Remark 4.1.5 that we can write the elements of $\mathcal{A}_q$ as triples of two finite q-regular subtrees and an isometry of forests. For this description we give here a short lemma which allows us to choose some special finite subtrees for the representation of an element.

Lemma 4.1.8 ([19, Lemma 2.3])
Let $F_1, F_1', F_2, F_2' \subseteq \mathcal{T}_q$ be finite q-regular rooted subtrees with the same number of leaves. Then the triples (F_1, F_2, f) and (F_1', F_2', f') represent the same spheromorphism if and only if there exist finite q-regular rooted subtrees T_1 and T_2 such that T_1 contains F_1 and F_1', T_2 contains F_2 and F_2', and $f, f' \colon \mathcal{T}_q \setminus T_1 \to \mathcal{T}_q \setminus T_2$ agree.

Remark 4.1.9 ([19, Remark 2.4])
For each spheromorphism and given finite q-regular rooted subtrees F_1 and F_2, we can find a representing triple (T_1, T_2, f) such that T_1 contains F_1 and T_2 contains F_2.

In the following we generalize the concept of tree automorphisms to multiple copies of $\mathcal{T}_q$. First we need the terminology about the disjoint union of metric spaces and how we equip them with a metric.
Let (X, d_X) and (Y, d_Y) be two metric spaces. Then $X \sqcup Y$ be the disjoint union of X and Y together with the metric d which is defined by

$$\begin{aligned} d(x_1, x_2) &= d_X(x_1, x_2), && \forall x_1, x_2 \in X, \\ d(y_1, y_2) &= d_Y(y_1, y_2), && \forall y_1, y_2 \in Y, \\ d(x, y) &= \infty, && \forall x \in X, y \in Y. \end{aligned}$$

So we extend the metrics of X and Y with the distance ∞ between the spaces. We can use this concept to form a disjoint union of n copies of one metric space X. We denote by nX the n-fold disjoint union of X with itself. If we have an order on the set X, then we can order the elements in nX by first ordering the summands of nX and then ordering inside each X.

Definition 4.1.10
Let $r \geq 1$. A local similarity of $\phi\colon rB \to rB$ is called a *r-spheromorphism*. Then all isometries of the r-copies of the set of ends $\mathrm{Isom}(\mathtt{B}_q)^r$ are r-spheromorphisms. The subgroup of all r-spheromorphisms of $\mathrm{Homeo}(\mathtt{B}_q)^r$ is denoted by $\mathcal{A}_{qr}$ in the following.

We equip $\mathrm{Isom}(\mathtt{B}_q)^r$ with the product topology and get that the $(U_n)^r$ form a neighborhood basis of the identity. Then we equip $\mathcal{A}_{qr}$ with the topology such that the embedding $\mathrm{Isom}(\mathtt{B}_q)^r \hookrightarrow \mathcal{A}_{qr}$ is continuous and open. With the same argument as before we get that $\mathcal{A}_{qr}$ is a tdlc group.
In the next step we generalize this group by using not the whole automorphism group of the tree inside each connected component of the forests. Therefore we define at first a subgroup of $\mathrm{Aut}(\mathcal{T}_q) = \mathrm{Isom}(\mathtt{B}_q)$, which is induced by a subgroup of $\mathrm{Sym}(q)$.

Let $D \leq \mathrm{Sym}(q)$ be a subgroup of the symmetric group. We identify D with the subgroup of $\mathrm{Aut}(\mathcal{T}_q)$ whose action on the first level of $\mathcal{T}_q$ is permutationally isomorphic to the one of D on the set $\{1,\dots,q\}$. We define D_n recursively by $D_1 = D$ and $D_{n+1} = D_n \wr D$. Note that D_n can also be identified with the subgroup of $\mathrm{Aut}(\mathcal{T}_q)$ in the obvious way. We define D_∞ as the subgroup of $\mathrm{Aut}(T_q)$ generated by the family $(D_n)_{n\geq 1}$, and $\mathrm{Isom}_D(\mathtt{B}_q)$ as the closure of D_∞. We call an element of $\mathrm{Aut}(\mathcal{T}_q)$ which can be identified with an element in $\mathrm{Isom}_D(\mathtt{B}_q)$ *D-admissible*.

Definition 4.1.11
An almost automorphism $\gamma \in \mathcal{A}_q$ is called *D-admissible* if the triple (F_1, F_2, f), which represents γ has only D-admissible elements in the isometry of forests f. The group of all D-admissible almost automorphisms of $\mathcal{T}_q$ is denote by $\mathcal{A}_q^D$.

In the next step we combine both generalizations of $\mathrm{Aut}(\mathcal{T}_q)$. Therefore let $r \geq 1$ and $\mathrm{Isom}_D(\mathtt{B}_q)^r$ be r copies of $\mathrm{Isom}_D(\mathtt{B}_q)$ which acts on $r\mathcal{T}_q$, respectively $r\mathtt{B}_q$. Analogous as before we call the elements in $\mathrm{Isom}_D(\mathtt{B}_q)^r$ *D-admissible*. Together with the product topology we get that $\mathrm{Isom}_D(\mathtt{B}_q)^r$ is a tdlc group.

Definition 4.1.12
Let $q \geq 2$, $r \geq 1$ and $D \leq \mathrm{Sym}(q)$. By $\mathcal{A}_{qr}^D$ we denote the subgroup of $\mathrm{Homeo}(\mathtt{B}_q)^r$ consisting of all D-admissible r-spheromorphisms, called the *almost automorphisms of type D of $\mathcal{T}$*.

We equip $\mathcal{A}_{qr}^D$ with the topology such that the embedding $\mathrm{Isom}_D(\mathtt{B}_q)^r \hookrightarrow \mathcal{A}_{qr}^D$ is continuous and open. This implies with the same arguments as before that $\mathcal{A}_{qr}^D$ is a tdlc group.

Remark 4.1.13
(i) For $D = \mathrm{Sym}(q)$ we get $\mathrm{Isom}_D(\mathtt{B}_q) = \mathrm{Isom}(\mathtt{B}_q)$ and so $\mathcal{A}_{qr}^D = \mathcal{A}_{qr}$.
(ii) For $D = \mathrm{Sym}(q)$ and $r = 1$, the group $\mathcal{A}_{qr}^D$ is the Neretin group N_q.
(iii) For $D = \{\mathrm{id}\}$, the group $\mathcal{A}_{qr}^D$ is the Higmann-Thompson group V_{qr}.
(iv) For all subgroups $D' \leq D \leq \mathrm{Sym}(q)$ we get $\mathcal{A}_{qr}^{D'} \leq \mathcal{A}_{qr}^D$. In particular, for each $D \leq \mathrm{Sym}(q)$ we have the inclusion $V_{qr} \leq \mathcal{A}_{qr}^D$.

In Chapter 3 we have seen how we can transfer finiteness properties along the Schlichting completion and A. Le Boudec has shown in [19, Example 6.7] that $\mathcal{A}_{qr}^D = V_{qr}//D_\infty^r$. Which can not be used to say something about the finiteness properties, since the commensurated subgroup D_∞^r is not finitely generated. This construction shows directly why the Neretin group N_q is sometimes called the tdlc version of the Thompson group V.

4.2 Fixpoint groups

In the following we fix $q \geq 2$, $r \geq 1$ and $D \leq \mathrm{Sym}(q)$. In short, we write $\mathcal{A}$, $\mathtt{B}$ and $\mathcal{T}$ instead of $\mathcal{A}_{qr}^D$, $\mathtt{B}_q$ and $\mathcal{T}_q$, respectively. In this section we introduce the fixpoint subgroup and prove that this subgroup is closed. This property is necessary for the proof of the finiteness properties in Section 4.4.

Definition 4.2.1
For $x \in r\mathtt{B}$ we denote the *fixpoint subgroup* of x in $\mathcal{A}$ as

$$\mathcal{A}_x \coloneqq \{\gamma \in \mathcal{A} \mid \gamma(x) = x\}.$$

In the following, $\mathcal{A}_x$ will be always endowed with the subspace topology from $\mathcal{A}$.

Lemma 4.2.2
For every $x \in r\mathtt{B}$, the subgroup $\mathcal{A}_x$ is closed in $\mathcal{A}$.

Proof:
We choose an arbitrary $\gamma \in \mathcal{A} \setminus \mathcal{A}_x$ and find an open neighborhood U_γ of γ in $\mathcal{A}$ such that U_γ does not intersect with $\mathcal{A}_x$.
Let $\gamma \in \mathcal{A} \setminus \mathcal{A}_x$ and $y \in r\mathtt{B}$, such that $\gamma(x) = y \neq x$. Then γ is a r-spheromorphism which implies that γ can be written as an isometry of forests. Let $F_1, \ldots, F_r, F_1', \ldots, F_r' \subseteq \mathcal{T}$ such that $f\colon (\mathcal{T} \setminus F_1 \times \cdots \times \mathcal{T} \setminus F_r) \to (\mathcal{T} \setminus F_1' \times \cdots \times \mathcal{T} \setminus F_r')$ represents γ such that x and y are not in the same connected component of the domain and the codomain.
Let v be the leaf of F_j such that x is in the ball which coincides with the full subtree rooted at v and w be the leaf of F_i' such that y is in the ball which coincides with the full subtree rooted at w. This implies that $f(v) = w$. Then let $n = d(v_0, v)$ and $U_n = \{\psi \in \mathcal{A} \mid \psi|_{B_n(v_0)} = \mathrm{id}|_{B_n(v_0)}\}$ is an open neighborhood of the identity in $\mathcal{A}$. Therefore γU_n is an open neighborhood of γ in $\mathcal{A}$. By the construction of U_n we get for each $\lambda \in \gamma U_n$ that $\lambda(x) \notin B_{e^{-n}}(x)$ which implies that $\lambda(x) \neq x$ and $U_n \cap \mathcal{A}_x = \emptyset$. □

Since $\mathcal{A}_x$ is a closed subgroup endowed with the subspace topology from $\mathcal{A}$, we can directly deduce the following corollary.

Corollary 4.2.3
The group $(\mathrm{Isom}_D(B)^r)_x$ is compact open in $\mathcal{A}_x$.

We have seen before that the group of almost automorphisms of type D can be described as a Schlichting completion. Since $\mathcal{A}_x$ is a closed subgroup of $\mathcal{A}$ we can choose the groups for the Schlichting completion very similarly. Therefore we can show with the same techniques as in [19, Example 6.7] that $\mathcal{A}_x = (V_{qr})_x//(D_\infty^r)_x$. This implies by [5] that we can deduce some properties for the fixpoint almost automorphisms from the properties of the fixpoint Higmann-Thompson group, but not the finiteness properties, since $(D^r_\infty)_x$ is not finitely generated but locally finite.

4.3 Morse theory

In this section we recall how we can interpret small categories as topological spaces and simplical sets, and collect some properties. After that we give a short introduction into the Morse theory of categories, for more details see [30, Section 2.10] or [21, Section 3]. In the following, we look at categories as topological objects via the nerve construction. The *nerve of a category* is a simplical set and its geometric realization is a CW complex built in the following way: the 0-cells correspond to the objects of the category, and the n-cells correspond one-to-one to chains of n composable non-identity morphisms, i.e.

$$c_0 \xrightarrow{\mu_1} c_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_n} c_n,$$

where μ_i is not the identity for every $1 \leq i \leq n$.

Definition 4.3.1
A small category $\mathcal{C}$ is a *generalized poset* if for each two morphisms $\alpha, \beta\colon A \to B$ we have $\alpha = \beta$.

Let $\mathcal{C}$ be a generalized poset and $\mathcal{D}$ a subgroupoid. Then $\mathcal{D}$ is a subcomplex at the level of spaces, and we can define the quotient $\mathcal{C}/\mathcal{D}$ as follows. The objects are the objects of $\mathcal{C}$ modulo the equivalence $\sim$, where $X \sim Y$ if and only if there exists a morphism $X \to Y$ in $\mathcal{D}$. For all objects $[X]$ and $[Y]$ in $\mathcal{C}/\mathcal{D}$, we define

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{D}}([X],[Y]) := \{\alpha\colon A \to B \mid A \in [X], B \in [Y]\}/_\sim$$

where $\alpha\colon A \to B$ and $\alpha'\colon A' \to B'$ are equivalent if there exist two morphisms g and h in $\mathcal{D}$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[\alpha']{} & B' \end{array}$$

For the following, we recall the corresponding notion of contractible spaces at the level of categories.

Definition 4.3.2
A small category $\mathcal{C}$ is called *cofiltered* if

(i) for all $X, Y \in \mathrm{ob}(\mathcal{C})$ there exist $Z \in \mathrm{ob}(\mathcal{C})$ such that there exists $f\colon Z \to X$ $g\colon Z \to Y$ morphisms in $\mathcal{C}$,

(ii) for all $f_1, f_2\colon X \to Y$ there exists $g\colon Z \to X$ such that $f_1 \circ g = f_2 \circ g$.

Lemma 4.3.3
A small category $\mathcal{C}$ is contractible at the level of spaces if it is cofiltered at the level of categories.

Remark 4.3.4 ([21, Remark 3.3])
Let G be a group acting on a small category $\mathcal{C}$ by invertible functors. Then G has an induced action on the CW complex associated to $\mathcal{C}$. If an element $g \in G$ fixes a cell setwise, then it fixes it also pointwise. In particular, the stabilizer of any cell is the intersection of the stabilizers of all the vertices of the cell.

The way of looking at categories as CW complexes will be used later in the proof that the fixpoint almost automorphism groups are of type F_∞ (cf. Theorem C). We recall some Morse theory for categories, which we will also use in that proof. For more details, see [30, Section 2.10].

Definition 4.3.5 ([21, Definition 3.6])
Let $\mathcal{C}$ be a generalized poset and $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. We call a map $f\colon \mathrm{ob}(\mathcal{C}/\mathcal{D}) \to \mathbb{N}$ a *generalized Morse function* if no two objects with the same f-value are connected by a non-invertible arrow. Such an f is called *well-behaved* if no two objects of different Morse height are joined by an isomorphism.

We recall how to use a generalized Morse function to build up the category $\mathcal{C}$ from the full subcategory $\mathcal{D}$. Before this, we recall some preliminary concepts.
Let $X \in \mathrm{ob}(\mathcal{C}) \setminus \mathrm{ob}(\mathcal{D})$, f be a generalized Morse function with $n = f(X)$, and $(\mathcal{C}(k))_{k\geq 1}$ be the Morse filtration of $\mathcal{C}$ with respect to f. We denote by $\mathcal{C}(n-1) \downarrow X$ the category whose objects are the $\mathcal{C}(n-1)$-morphisms $X \to Y$, and the morphisms are all $\mathcal{C}(n-1)$-morphisms below X (i.e. there exists a morphism between $g\colon X \to Y$ and $h\colon X \to Z$, if there exists a $\mathcal{C}(n-1)$-morphism $f\colon Y \to Z$ such that $h = f \circ g$). Similarly, we denote by $X \downarrow \mathcal{C}(n-1)$ the category whose objects are the $\mathcal{C}(n-1)$-morphisms $Y \to X$, and the morphisms are the $\mathcal{C}(n-1)$-morphisms over X (i.e. there exists a morphism between $g\colon Y \to X$ and $h\colon Z \to X$, if there exists a $\mathcal{C}(n-1)$-morphism $f\colon Y \to Z$ such that $g = h \circ f$). The *descending link* $\mathrm{lk}_\downarrow(X)$ of X is defined by the join of $\mathcal{C}(n-1) \downarrow X$ and $X \downarrow \mathcal{C}(n-1)$. For more details we refer to [30, Section 2.10] and [21, Section 3].

Lemma 4.3.6 ([21, Remark 3.5])
Let $\mathcal{C}$ be a generalized poset, $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory, f be a generalized Morse function of $(\mathcal{C}, \mathcal{D})$, and $(\mathcal{C}(k))_{k\geq 1}$ be the Morse filtration of $\mathcal{C}$ with respect to f. If for any object X of $\mathcal{C}$ with Morse height k the descending ling $\mathrm{lk}_\downarrow(X)$ is n-connected, then the pair $(\mathcal{C}(k), \mathcal{C}(k+1))$ is $(n+1)$-connected.

4.4 Finiteness properties

R. Sauer and W. Thumann [21] showed that all tree almost automorphism groups are of type F_∞. In this section, we use the same constructions to prove the same statement for the fixpoint subgroups.

Theorem C
Let $r \geq 1$, $q \geq 2$, $D \leq \mathrm{Sym}(q)$, $\mathtt{B}_q = \partial\mathcal{T}_q$ and $x \in r\mathtt{B}_q$. Then the fixpoint subgroup $(\mathcal{A}_{qr}^D)_x$ of the tree almost automorphism group $\mathcal{A}_{qr}^D$ is of type F_∞.

For the proof the fixpoint x will always be choose in the first copy of $\mathtt{B}_q$, since if we pick it in another $\mathtt{B}_q$ the groups is conjugate to a fixpoint group of a x in the first copy. As before we fix $r > 0$, $q \geq 2$, $D \subseteq \mathrm{Sym}(q)$ and $x \in r\mathtt{B}_q$. We write $\mathcal{A}$, $\mathtt{B}$ and $\mathcal{T}$ instead of $\mathcal{A}_{qr}^D$, $\mathtt{B}_q$ and $\mathcal{T}_q$, respectively.

Our proof strategy is largely similar to the one in [21], except for the last part, where we apply Brown's criterion for tdlc groups provided by I. Castellano and G. Corob Cook [11]. We also need to adapt some definitions, as exposed below.

Definition 4.4.1

- A map $\phi\colon m\mathtt{B} \to n\mathtt{B}$, with $m, n \geq 1$, is called a *sphero-fixvertex* if ϕ is a D-admissible map with $\phi(x) = x$. We define $\mathrm{lvl}(\phi) = m$.
- A sphero-fixvertex $\mu\colon m\mathtt{B} \to n\mathtt{B}$ is called *merge map* if, for each $\mathtt{B}$ in the domain $m\mathtt{B}$, the restriction is a local D-admissible similarity onto a ball $\mathtt{B}$ in the codomain $n\mathtt{B}$. Note that necessarily $m \geq n$.
- A merge map is called *very elementary* if, for each $\mathtt{B}$ in the codomain $n\mathtt{B}$, the preimage $\mu^{-1}(\mathtt{B})$ consists of at most q summands in $m\mathtt{B}$.
- The inverse of a (very elementary) merge map is called *(very elementary) split map*.
- A D-admissible isometry $\nu\colon n\mathtt{B} \to n\mathtt{B}$ with $n \geq 1$ and $\nu(x) = x$ is called a *transformation*. This means that there exists $\sigma \in \mathrm{Sym}(n)$ with $\sigma(1) = 1$ such that the i-th $\mathtt{B}$ in the domain $n\mathtt{B}$ maps onto the $\sigma(i)$-th $\mathtt{B}$ of the codomain $n\mathtt{B}$ via $\mathrm{Isom}_D(\mathtt{B})$ for $i > 1$ and via $(\mathrm{Isom}_D(\mathtt{B}))_x$ for $i = 1$.
- A transformation is called *strict* if $\sigma = \mathrm{id}$.

Remark 4.4.2 ([21, Remark 4.7])
For all $l \in \mathbb{N}$ and each sphero-fixvertex $\phi\colon m\mathtt{B} \to n\mathtt{B}$ there exists $l' \in \mathbb{N}$ such that $\phi \circ (U_{l'}^D)_x^m \circ \phi^{-1} \subseteq (U_l^D)_x^n$. Indeed, one can argue as in [21, Remark 4.7], recalling that $(U_{l'}^D)_x^m = (U_{l'}^D)^m \cap \mathcal{A}_x$ and $(U_l^D)_x^n = (U_l^D)^n \cap \mathcal{A}_x$.

In Section 4.3, we recalled how we can interpret categories as topological spaces. Our strategy is to define a generalized poset together with an $\mathcal{A}_x$-action by invertible functors. The nerve of this generalized poset provides us the $\mathcal{A}_x$-CW complex of type F_∞, with which we conclude the proof of Theorem C.
Let $\mathcal{R}_x$ be the generalized poset whose objects are all the sphero-fixvertices $n\mathtt{B} \to r\mathtt{B}$, as n varies in $\mathbb{N}$ (note that r is fixed). For given objects $\phi\colon n\mathtt{B} \to r\mathtt{B}$ and $\psi\colon m\mathtt{B} \to r\mathtt{B}$, a morphism $\phi \to \psi$ in $\mathcal{R}_x$ is either a merge map or a transformation $\alpha\colon n\mathtt{B} \to m\mathtt{B}$ such that $\psi \circ \alpha = \phi$. The composition of morphisms is given by the composition of merge maps and transformations, which are again either merge maps or transformations. For an object ϕ in $\mathcal{R}_x$ the identity transformation represents the identity.
Let $\mathcal{T}_x$ be the subgroupoid of $\mathcal{R}_x$ whose objects are all the transformations $r\mathtt{B} \to r\mathtt{B}$, i.e., all isomorphisms in $\mathcal{R}_x$. Moreover, let $\mathcal{S}_x$ be the subgroupoid of $\mathcal{R}_x$ whose objects are all the strict transformations $r\mathtt{B} \to r\mathtt{B}$. Then the quotient categories $\mathcal{P}_x := \mathcal{R}_x/\mathcal{T}_x$ and $\mathcal{Q}_x := \mathcal{R}_x/\mathcal{S}_x$ are a poset and a generalized poset respectively.

Note that the level function introduced in Definition 4.4.1 is a well-behaved generalized Morse function on $\mathcal{R}_x$ and it induces generalized Morse functions on $\mathcal{P}_x$ and $\mathcal{Q}_x$. Let $(\mathcal{R}_x(n))_{n\geq 1}$, $(\mathcal{P}_x(n))_{n\geq 1}$ and $(\mathcal{Q}_x(n))_{n\geq 1}$ denote the associated Morse filtrations. For example, recall that $\mathcal{R}_x(n)$ is the full subcategory of $\mathcal{R}_x$ spanned by all the objects of level at most n.
The generalized poset $\mathcal{R}_x$ carries a free $\mathcal{A}_x$-action defined, for all $\phi \in \mathrm{ob}(\mathcal{R}_x)$ and $\gamma \in \mathcal{A}_x$, by

$$\gamma \cdot \phi \coloneqq \gamma \circ \phi.$$

This action induces an action on $\mathcal{P}_x$ (respectively $\mathcal{Q}_x$) by setting

$$\gamma \cdot [\phi] \coloneqq [\gamma \cdot \phi] = [\gamma \circ \phi].$$

Since all elements in $\mathcal{A}_x$ map from $r\mathtt{B}$ to $r\mathtt{B}$, all the actions before preserve the Morse levels. This implies that all the Morse filtrations above are $\mathcal{A}_x$-invariant.

Remark 4.4.3
By Remark 4.4.2 we get that the action of $\mathcal{A}_x$ is continuous on $\mathcal{Q}_x$ and $\mathcal{P}_x$.

Proposition 4.4.4
The cell stabilizers of the action of $\mathcal{A}_x$ on $\mathcal{Q}_x$ are compact open subgroups.

Proof:
The argument is analogous to the one in [21, Proposition 4.9].
In Remark 4.3.4 we have seen that in this set up the cell-stabilizers are the intersection of the vertex-stabilizers. Therefore it is enough to show that the vertex-stabilizers are compact open. At first we give a description of the vertex-stabilizers in this case. Let $[\phi]$ be a sphero-fixvertex in $\mathcal{Q}_x$ and $\gamma \in \mathcal{A}_x$, which fixes $[\phi]$, i.e. $\gamma \cdot [\phi] = [\gamma \circ \phi] = [\phi]$. Since $[\gamma \circ \phi] = [\phi]$ there exists a strict transformation α such that $\gamma \circ \phi = \phi \circ \alpha$, which implies $\gamma = \phi \circ \alpha \circ \phi^{-1}$. On the other hand if an element of $\mathcal{A}_x$ is of this form it fixes $[\phi]$. Hence we get

$$\mathrm{stab}[\phi] \coloneqq \{\phi \circ \alpha \circ \phi^{-1} \mid \alpha \text{ is a strict transformation}\}.$$

We need to show that the group of strict transformations of $n\mathtt{B}$ is compact open. In the first step it is easy to see that this group is isomorphic to $(\mathrm{Isom}_D(B)^n)_x$, which we equip with the product topology. We have the embedding

$$\iota\colon (\mathrm{Isom}_D(B)^n)_x \to \mathcal{A}_x$$
$$\alpha \mapsto \phi \circ \alpha \circ \phi^{-1}$$

whose image is $\mathrm{stab}[\phi]$. In the proof of [21, Proposition 4.9] we see that this embedding is continuous and open. Since we have only the restriction to $\mathcal{A}_x$ which is by Lemma 4.2.2 closed, we get that $\mathrm{stab}[\phi]$ is compact open. □

Since the cell stabilizers of the action of $\mathcal{A}_x$ on $\mathcal{Q}_x$ are compact open, we get that $\mathcal{Q}_x$ is a proper discrete $\mathcal{A}_x$-CW complex.

Proposition 4.4.5
The nerve of $\mathcal{Q}_x$ is contractible.

Proof:
If the nerve of $\mathcal{R}_x$ is contractible then the nerve $\mathcal{Q}_x$ is also contractible, therefore it suffices to show that the nerve of $\mathcal{R}_x$ is contractible. By Lemma 4.3.3 if the category is cofiltered then the nerve of the category is contractible at the level of spaces. Therefore we show in the following that $\mathcal{R}_x$ is a colfiltered category.
Part two of Definition 4.3.2 is still true since $\mathcal{R}_x$ is a generalized poset. Hence we need to show the first part of the definition. Let ϕ_1, ϕ_2 be two sphero-fixvertices as objects of $\mathcal{R}_x$. Then $\phi_i\colon n_i\mathtt{B} \to r\mathtt{B}$ are D-admissible local similarities, which fix x. There exist partitions D_i of the codomains $r\mathtt{B}$ into disjoint balls, such that each ball is D-similar to a ball in $n_i\mathtt{B}$. Then we find a partition D of $r\mathtt{B}$ which refines D_1 and D_2. Let $n \coloneqq |D|$ be the number of balls in D. There exists $\psi\colon n\mathtt{B} \to r\mathtt{B}$ the unique order preserving merge map, mapping the summands of $n\mathtt{B}$ to the balls in D. Since ψ is order preserving it maps x to x and is therefore a sphero-fixvertex and an object of $\mathcal{R}_x$. Define the merge maps or transformations $\alpha_i = \phi_i \circ \psi$ which connects ψ with ϕ_1 and ϕ_2. Therefore $\mathcal{R}_x$ is cofiltered, also contractible at the level of spaces, which implies that $\mathcal{Q}_x$ is contractible. □

So far we have proven that $\mathcal{Q}_x$ is a proper discrete contractible $\mathcal{A}_x$-CW complex. In the following, we use the Morse filtration and some statements of Section 4.3 to show Theorem C.

Proposition 4.4.6
For each $n \in \mathbb{N}$, the nerve of $\mathcal{Q}_x(n)$ is an $\mathcal{A}_x$-CW complex of finite type.

Proof:
We follow the structure of the proof of [21, Proposition 4.12]. At first we show that two sphero-fixvertices of the same level n are $\mathcal{A}_x$-equivariant. Let ϕ_1, ϕ_2 be two sphero-fixvertices of level n. Then $\phi_2 \circ \phi_1^{-1} = \gamma \in \mathcal{A}_x$, so $\gamma \cdot [\phi_1] = [\phi_2]$ this shows they are $\mathcal{A}_x$-equivariant. Therefore we have only finitely many $\mathcal{A}_x$-classes of objects in $\mathcal{Q}$.
Let $[\phi_1], [\phi_2]$ be objects in $\mathcal{Q}_x$ with $\phi_i\colon n_i\mathtt{B} \to r\mathtt{B}$ be sphero-fixvertices. Then there exists an arrow between $[\phi_1]$ and $[\phi_2]$ if there exists a merge map or transformation α such that $\phi_2 \circ \alpha = \phi_1$. If we choose the right representative for $[\phi_1]$ and for $[\phi_2]$ we can assume that $\alpha|_{\mathtt{B}}$ is order preserving for each $\mathtt{B}$ in the domain $n_1\mathtt{B}$. There exist only finitely many order preserving merge maps and transformations form $n_1\mathtt{B}$ to $n_2\mathtt{B}$. Therefore $\mathcal{Q}_x(n)$ is locally finite, this means that each object is only for finitely many arrows the domain or codomain. So we get that $\mathcal{Q}_x(n)\backslash\mathcal{A}_x$ is locally finite and also finite, since we have only finitely many objects and so $\mathcal{Q}_x(n)$ is of finite type. □

The next step is to show that the connectivity of the descending links tends to infinity as the level tends to infinity.

Proposition 4.4.7
For all $k \in \mathbb{N}$, there exits $n = n_k \in \mathbb{N}$ such that, for all $X \in \mathrm{ob}(\mathcal{Q}_x)$ with $\mathrm{lvl}(X) \leq n$ we have that $\mathrm{lk}_{\downarrow}(X)$ is k-connected.

Proof:
If we show this result for $\mathcal{P}_x$, by [21, Remark 3.7] we get the same result for $\mathcal{Q}_x$ and $\mathcal{R}_x$. The strategy to prove the claim for $\mathcal{P}_x$ is analogous to the one adopted in [21, Section

4.4]. For the reader's convenience, we recall the main steps of the strategy. Recall that the objects of $\mathrm{lk}_{\downarrow}(X)$ are merge maps $n\mathtt{B} \to k\mathtt{B}$, with $k < n$, modulo transformations of the codomain.
First, we establish a 1-to-1 correspondence between the set of elements and the set of split maps $k\mathtt{B} \to n\mathtt{B}$, for $k < n$, modulo transformations of the domain.
Secondly, we establish a homotopy equivalence between $\mathrm{lk}_{\downarrow}(X)$ and its full subcomplex $\mathrm{lk}^{*}_{\downarrow}(X)$ spanned by the very elementary split maps. The proof strategy is analogous to [21, Proposition 4.13].
Therefore, it suffices to prove the statement of Proposition 4.4.7 for each $\mathrm{lk}^{*}_{\downarrow}(X)$. In view of this, we establish an isomorphism of posets between $\mathrm{lk}^{*}_{\downarrow}(X)$ and the poset C_n of partitions of $\{1, \dots, n\}$ into sets of cardinality either 1 or q, and assign each term of the partition of cardinality q a representative of $\mathrm{Sym}(q)/D$ fixing 1. This is basically what was done in [21], apart from the requirement of 1 being fixed.
The geometric realization of the poset C_n is the barycentric subdivision of the following flag complex $\mathcal{C}_n$. The vertices of $\mathcal{C}_n$ are all subsets M of $\{1, \dots, n\}$ of cardinality q equipped with an element $\sigma \in \mathrm{Sym}(q)/D$ and, whenever $1 \in M$, we have $\sigma(1) = 1$. Two vertices (M_1, σ_1) and (M_2, σ_2) are joined by an edge if and only if $M_1 \cap M_2 = \emptyset$. If we prove that $\mathcal{C}_n$ is connected enough, then so is its barycentric subdivision, and therefore $\mathrm{lk}^{*}_{\downarrow}(X)$ and so $\mathrm{lk}_{\downarrow}(X)$ are connected enough. The precise statement on the high connectivity of $\mathcal{C}_n$ is given in Proposition 4.4.8. □

Proposition 4.4.8
A lower bound for the connectivity of $\mathcal{C}_n$ is given by

$$v(n) := \left\lfloor \frac{n-q}{2q-1} \right\rfloor - 1.$$

Proof:
Proof analogous to [21, Prop. 4.14], but we provide the proof for the convenience of the reader. We show the statement by induction on n.
Base case: $n = q$
Then $v(n) = v(q) = -1$. Therefore, we need to show that $\mathcal{C}_n$ is not empty. This is given by the fact that $M = \{1, \dots, q\}$ together with $\sigma = \mathrm{id} \in \mathrm{Sym}(q)/D$ is an element of $\mathcal{C}_n$.
Induction step:
We assume the claimed lower bound on the connectivity holds for all $m < n$. Let $b = (M, \sigma)$ be a vertex of $\mathcal{C}_n$, i.e., $M = \{1, \dots, q\}$ and $\sigma \in \mathrm{Sym}(q)/D$ with $\sigma(1) = 1$ if $1 \in M$. If $M \subseteq \{1, \dots, n\}$ such that $1 \notin M$ we choose $\sigma \in \mathrm{Sym}(q)/D$ without further restrictions and the rest of the proof follows with the same arguments. Then $\mathcal{C}'_n$ be the full subcomplex of $\mathcal{C}_n$ spanned by all vertices whose underlying subset is disjoint from M. If we show that the pair $(\mathcal{C}_n, \mathcal{C}'_n)$ is $v(n)$ connected, then $\mathcal{C}_n$ is at least $v(n)$-connected. To prove the connectivity of the pair above, we use discrete Morse theory for simplicial complexes.
At first we need to define a Morse function on $\mathcal{C}_n \setminus \mathcal{C}'_n$. Let $a = (M', \sigma')$ be a vertex of $\mathcal{C}_n \setminus \mathcal{C}'_n$. Then f maps a to the q digit binary number such that the i-th digit is 0 if $i \notin M'$ and 1 if $i \in M'$. Due to the natural order of binary numbers, f yields a Morse function and we build up $\mathcal{C}_n$ from $\mathcal{C}'_n$.

We now show that the descending link $\mathrm{lk}_\downarrow(a)$ of a is $v(n-2q+1)$-connected. Recall that $\mathrm{lk}_\downarrow(a)$ is the full subcomplex of $\mathcal{C}_n$ spanned by all vertices $x = (\tilde{M}, \tilde{\sigma})$ such that $\tilde{M} \cap M' = \emptyset$ and $\min(M') < \min(\tilde{M})$. This subcomplex is isomorphic to $\mathcal{C}_m$, where

$$m = n - q - (\min(M') - 1) \geq n - q - q + 1 = n - (2q-1).$$

By induction, $\mathrm{lk}_\downarrow(a)$ is $v(m)$-connected and therefore $v(n-(2q-1))$-connected. Note that

$$\begin{aligned} v(n-(2q-1)) &= \left\lfloor \frac{n-(2q-1)-q}{2q-1} \right\rfloor - 1 = \left\lfloor \frac{n-q}{2q-1} - \frac{2q-1}{2q-1} \right\rfloor - 1 \\ &= \left\lfloor \frac{n-q}{2q-1} - 1 \right\rfloor - 1 = \left\lfloor \frac{n-q}{2q-1} \right\rfloor - 2 = v(n) - 1. \end{aligned}$$

By standard arguments of discrete Morse theory, we get that the pair $(\mathcal{C}_n, \mathcal{C}'_n)$ of spaces is $v(n-(2q-1)) + 1 = v(n)$-connected. Lastly, we show that the inclusion $\iota\colon \mathcal{C}'_n \to \mathcal{C}_n$ induces a trivial map $\pi_m(\iota)\colon \pi_m(\mathcal{C}'_n) \to \pi_m(\mathcal{C}_n)$ at the level of the m-th homotopy groups, for $m \leq v(n)$. Let $\varphi\colon S^m \to \mathcal{C}'_n$ be a simplical map. Note that we can always construct such a map from a continuous map $S^m \to \mathcal{C}'_n$ by simplical approximation. Then we get $\mathrm{im}(\iota \circ \varphi) \subseteq \mathrm{star}(b)$ and $\iota \circ \varphi$ is homotopy equivalent to a simplicial map $S^m \to \mathcal{C}_n$ which is constant inside $\mathrm{star}(b)$. This implies that $\pi_m(\iota)$ is trivial for $m \leq v(n)$. By the long exact sequence

$$\cdots \to \pi_k(\mathcal{C}'_n) \to \pi_k(\mathcal{C}_n) \to \pi_k(\mathcal{C}_n, \mathcal{C}'_n) \to \pi_{k-1}(\mathcal{C}'_n) \to \cdots$$

and since the pair $(\mathcal{C}_n, \mathcal{C}'_n)$ is $v(n)$-connected, we conclude that $\mathcal{C}_n$ is $v(n)$-connected.

□

Proof of Theorem C:
By Proposition 4.4.4 and Proposition 4.4.5, $\mathcal{Q}_x$ is a proper discrete contractible $\mathcal{A}_x$-CW complex with a filtration $(\mathcal{Q}_x(m))_{m\geq 1}$ such that each $\mathcal{Q}_x(m)$ is finite modulo $\mathcal{A}_x$ (cf. Proposition 4.4.6). Since $\mathcal{Q}_x$ is proper, each cell stabilizer is compact and therefore of type F_∞. By Proposition 4.4.7 and 4.4.8, the connectivity of the descending links $\mathrm{lk}_\downarrow(X)$ of the objects X of $\mathcal{Q}_x$ tends to infinity as the Morse height of X tends to infinity. By Lemma 4.3.6, the connectivity of the pairs $(\mathcal{Q}_x(m+1), \mathcal{Q}_x(m))$ tends to infinity as m tends to infinity. By the criterion of Brown for tdlc groups (see Corollary 2.2.27), we conclude that $\mathcal{A}_x$ is of type F_∞. □

5 Universal groups

Universal Burger-Mozes groups were first introduced by M. Burger and S. Mozes in 2000 [10] as a result of their research into automorphism groups of regular trees. These groups provide examples of totally disconnected locally compact groups that appear in different areas. The universal Burger-Mozes groups are defined as automorphism groups of regular trees that specify specific local symmetry conditions. These groups are universal in the sense that each subgroup of the automorphism group of the tree can be embedded in one of these groups. There is a generalization of this construction which was introduced by S. Smith in 2017 [28]. For this generalization we use two subgroups of possibly different symmetric groups, which induce an action on a bipartite graph. He uses this construction to show that there exist uncountably many pairwise non-isomorphic simple, compactly generated, non-discrete tdlc groups. We use this construction to find new examples of tdlc groups which can be classified by their finiteness properties.

In this chapter, we first briefly introduce the Tits universal property which we later use to prove the simplicity of the universal groups. Then we look at the Burger-Mozes universal groups which sit in the middle of permutation groups and the automorphism groups of regular trees. We show some properties of these, which we later compare with some ideas of the Smith group.
After the introduction of the Burger-Mozes universal groups, we introduce the construction of the universal Smith groups. In the next step, we repeat some properties and conditions of this construction and find connections through the properties of the Burger-Mozes groups.
In the last part, we see which conditions we need for the permutation groups such that we can say something about the finiteness properties of this construction. In particular which conditions we need for $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ such that we create simple tdlc groups which can be classified by their finiteness properties.

5.1 Tits independence property

At first, we give a brief description of the property introduced by J. Tits in [31]. We will use this property later to show in which cases the universal groups are simple.
Let G be a group which acts on a tree $\mathcal{T}$ and $\mathcal{P}$ be a non-empty finite or (bi-)infinite path in $\mathcal{T}$. Then we can define the nearest point projection to $\mathcal{P}$ as a map of the vertices of $\mathcal{T}$, $\pi_{\mathcal{P}}\colon V(\mathcal{T}) \to V(\mathcal{T})$, $v \mapsto \pi_{\mathcal{P}}(v)$, such that $d(v, \pi_{\mathcal{P}}(v)) = \min\{d(v,w) \mid w \in V(P)\}$. Let $q \in V(\mathcal{P})$ be an arbitrary vertex of the path $\mathcal{P}$. By $\pi_{\mathcal{P}}^{-1}(q)$ we get a connected subtree of $\mathcal{T}$ which is rooted at q. Let $G_{\mathcal{P}}$ be the pointwise stabilizer of the path $\mathcal{P}$ in G. This subgroup of G leaves all subtrees of the form $\pi_{\mathcal{P}}^{-1}(p)$ invariant. The subgroup

$G_{\mathcal{P}}$ induces a subgroup $G_{\mathcal{P}}^q \leq \mathrm{Sym}\left(\pi_{\mathcal{P}}^{-1}(q)\right)$, then we define the group homomorphism

$$\Phi\colon G_{\mathcal{P}} \to G_{\mathcal{P}}^q, \quad \alpha \mapsto \alpha|_{\pi_{\mathcal{P}}^{-1}(q)}.$$

These homomorphisms induce the homomorphism

$$\Phi\colon G_{\mathcal{P}} \to \prod_{q\in V(\mathcal{P})} G_{\mathcal{P}}^q, \quad \alpha \mapsto \left(\alpha|_{\pi_{\mathcal{P}}^{-1}(q)}\right)_{q\in V(\mathcal{P})}.$$

Definition 5.1.1 (Tits independence property)
Let $G, \mathcal{T}, \mathcal{P}, G_{\mathcal{P}}, G_{\mathcal{P}}^q$ and Φ as before. Then we say G has *property (P)* if Φ is an isomorphism for each non-empty path $\mathcal{P}$ of $\mathcal{T}$.

We end this subsection by giving the theorem of Tits without a proof.

Theorem 5.1.2 ([31, Theorem 4.5])
Let $\mathcal{T}$ be a tree. Assume $G \leq \mathrm{Aut}(\mathcal{T})$ satisfies property (P), no proper non-empty subtree of $\mathcal{T}$ is invariant under G and no end of $\mathcal{T}$ is fixed by G. Then the group generated by all edge stabilizers $G^+ := \langle G_{(v,w)} \mid e = (v,w) \in E(\mathcal{T})\rangle$ is simple.

Sometimes the fact of invariance and no fixed ends is called geometrically dense. Later we refer that the universal groups fulfill these properties and we use this fact to show that the group generated by the edge stabilizers is simple and in which cases the full group is simple.

5.2 Burger-Mozes groups

The following introduction to the universal Burger-Mozes groups follows the descriptions of [15]. For more details we refer there. Let $\mathcal{T}_d$ be the unrooted d-regular oriented tree. Then we define a legal labeling as follows. Let $1, \dots, n$ be n different labels and we want to label all edges of the tree. So we define the *labeling* as a map $l\colon E(\mathcal{T}_d) \to \{1, \dots, n\}$. We call such a labeling a *legal labeling* if l is bijective on each ball of radius one around a vertex of the tree and each edge has for both directions e and $\overline{e}$ the same label. Then we need for a legal labeling of the unrooted tree $\mathcal{T}_d$ at least d different labels. In the following we use that a legal labeling of $\mathcal{T}_d$ uses exactly d different labels. For each legal labeling l we get for each vertex $v \in V(\mathcal{T}_d)$ a bijective map from the outgoing edges from v, noted by $o(v)$, to $\{1, \dots, d\}$ of the form

$$l_v\colon o(v) \to \{1, \dots, d\}, \quad e = (v,w) \mapsto l(e).$$

Figure 5.1 shows a legal labeling of a ball of radius 2 for $d = 3$. For visualization purposes the labels are indicated by colors.
In the following we write $\mathcal{T}$ instead of $\mathcal{T}_d$ if the degree is clear from context. Let $v \in V(\mathcal{T})$ and l be a legal labeling. Then each $g \in \mathrm{Aut}(\mathcal{T})$ induces a permutation of the 1 ball around v as follows

$$c\colon \mathrm{Aut}(\mathcal{T}) \times V(\mathcal{T}) \to \mathrm{Sym}(d), \quad (g,v) \mapsto l_{g(v)} \circ g \circ l_v^{-1}.$$

Then we have the set of induced local permutations of $g \in \mathrm{Aut}(\mathcal{T})$ as a subset of the symmetry group $\{c(g,v) \mid v \in V(\mathcal{T})\} \subseteq \mathrm{Sym}(d)$.

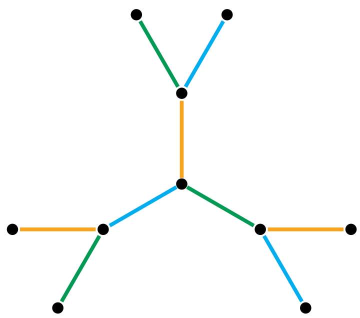

Figure 5.1: A legal coloring of $\mathcal{T}_3$ on a radius 2 ball

Definition 5.2.1
Let $\mathcal{T}$ be the unrooted d-regular oriented tree, $l\colon E(\mathcal{T}) \to \{1, \dots, d\}$ be a legal labeling and $F \subseteq \mathrm{Sym}(d)$. The *universal Burger-Mozes group* is defined as

$$\mathcal{U}^l(F) := \{g \in \mathrm{Aut}(\mathcal{T}_d) \mid c(g, v) \in F \text{ for all } v \in V(\mathcal{T})\}.$$

Lemma 5.2.2
Let $\mathcal{T}$ be the unrooted d-regular oriented tree, $l\colon E(\mathcal{T}) \to \{1, \dots, d\}$ be a legal labeling and $F \subseteq \mathrm{Sym}(d)$. Then $\mathcal{U}^l(F)$ is a subgroup of $\mathrm{Aut}(\mathcal{T})$.

Proof:
Let $g, h \in \mathcal{U}^l(F)$. Then $c(g, v), c(h, v) \in F$ for all $v \in V(\mathcal{T})$. We see directly that $c(gh, v) = c(g, h(v))c(h, v)$ and since F is a subgroup we know that this element is also in F, therefore $gh \in \mathcal{U}^l(F)$. For the inverse we also see that g^{-1} is in $\mathcal{U}(F)$, since $\mathrm{id} = c(\mathrm{id}, v) = c(gg^{-1}, v) = c(g, g^{-1}(v))c(g^{-1}, v) \in F$ so $g^{-1} \in \mathcal{U}(F)$. □

From this we can immediately conclude the following relationship between subgroups of the symmetric group and their universal Burger-Mozes group. If $F, F' \leq \mathrm{Sym}(d)$ are two subgroups such that $F' \leq F$, then we have $\mathcal{U}^l(F') \leq \mathcal{U}^l(F)$. The next thing we want to check is what happens if we change the legal labeling to another legal labeling. For the following statements we fix $d \geq 2$ and $\mathcal{T}$ be the unrooted d-regular oriented tree.

Proposition 5.2.3 ([15, Corollary 6.18])
Let l and l' be two different legal labelings of $\mathcal{T}$ and $F \leq \mathrm{Sym}(d)$. Then $\mathcal{U}^l(F)$ and $U^{l'}(F)$ are conjugated subgroups in $\mathrm{Aut}(\mathcal{T})$.

This proposition follows directly from the following lemma.

Lemma 5.2.4 ([15, Lemma 6.17])
Let l and l' be two legal labelings of $\mathcal{T}$ and $v, v' \in V(\mathcal{T})$. Then there exists a unique $g \in \mathrm{Aut}(\mathcal{T})$ such that $g(v) = v'$ and $l' = l \circ g$.

Proof:
We define $g \in \mathrm{Aut}(\mathcal{T})$ in the following way. Let $g(v) = v'$, then we define g inductively by n-balls around v. So let $B_1(v)$ the one ball around v, then g is defined such that $l'|_{B_1(v)} = l|_{B_1(v')} \circ g|_{B_1(v)}$ and then we do this procedure on each vertex in the one ball and so on. Then we get a unique $g \in \mathrm{Aut}(\mathcal{T})$ which fulfills the conditions of the statement. □

Then $\alpha(l,l',v,v') := g$ defines the map which is described in the proof of Lemma 5.2.4. For $v = v'$ this is the *unique label-changing automorphism*. Therefore the group is independent of the labeling, so we use for $\mathcal{U}^l(F)$ sometimes only $\mathcal{U}(F)$. For $l = l'$, $v = o(e)$ and $v' = t(e)$ this is the unique *label respecting inversion* of the edge e.

Example 5.2.5
For an arbitrary $d \geq 2$ we directly see that $\mathcal{U}(\mathrm{Sym}(d)) \cong \mathrm{Aut}(\mathcal{T}_d)$.
The other extreme is to choose $F = \{\mathrm{id}\} \subseteq \mathrm{Sym}(d)$ the smallest subgroup. At first for $d = 3$ we have $\mathcal{U}(F) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and for an arbitrary d we obtain a free product of d copies of $\mathbb{Z}/2\mathbb{Z}$. To see this let $v \in V(\mathcal{T}_d)$, $i \in \{1,\dots,d\}$ and e_i be an edge such that $o(e_i) = v$ and $l(e_i) = i$. We define $g_i = \alpha(l,l,o(e_i),t(e_i)) \in \mathcal{U}(\{\mathrm{id}\})$ as the inversion map of the edge e_i. Then on the one hand we have $\langle g_1\rangle * \dots * \langle g_d\rangle \leq \mathcal{U}(\{\mathrm{id}\})$. On the other hand let $h = \alpha(l,l,v,h(v)) \in \mathcal{U}(\{\mathrm{id}\})$ and p be the path from v to $h(v)$. The element h can be described as the product of edge inversions of the edges of the path p, therefore $h \in \langle g_1\rangle * \dots * \langle g_d\rangle$ and $\mathcal{U}(\{\mathrm{id}\}) \leq \langle g_1\rangle * \dots * \langle g_d\rangle$.

In the following we want to compare subgroups of the automorphism group of the tree with some subgroups of the symmetric group, for which we may use the following definition.

Definition 5.2.6
Let $H \leq \mathrm{Aut}(\mathcal{T})$ and $F \leq \mathrm{Sym}(d)$. Then H is called *locally permutation isomorphic* to F if for all $v \in V(\mathcal{T})$ the actions $\mathrm{stab}_H(v) \curvearrowright B_1(v)$ and $F \curvearrowright \{1,\dots,d\}$ are isomorphic. In this case we write $H \cong_{\mathrm{per}} F$.

In the next proposition we give a few properties of the universal group, which we use later to show that these groups are compactly generated. This we use finally to show that the universal Burger-Mozes groups are tdlc groups.

Proposition 5.2.7 ([15, Proposition 6.20])
Let $F \subseteq \mathrm{Sym}(d)$ and $\mathcal{U}(F) \leq \mathrm{Aut}(\mathcal{T})$. Then

(i) $\mathcal{U}(F)$ is a closed subgroup of $\mathrm{Aut}(\mathcal{T})$,

(ii) $\mathcal{U}(F) \cong_{\mathrm{per}} F$,

(iii) $\mathcal{U}(F)$ acts vertex transitively on $\mathcal{T}$,

(iv) $\mathcal{U}(F)$ acts edge transitively on edges with the same label on $\mathcal{T}$,

(v) $\mathcal{U}(F)$ is discrete if and only if $F \curvearrowright \{1,\dots,d\}$ is free.

Proof:

(i) On the one hand if $F = \mathrm{Sym}(d)$ we get $\mathcal{U}(F) = \mathrm{Aut}(\mathcal{T}_d)$, which is closed in $\mathrm{Aut}(\mathcal{T}_d)$ by definition. On the other hand let $g \in \mathrm{Aut}(\mathcal{T}) \setminus \mathcal{U}(F)$. Then there exists a $v \in V(\mathcal{T})$ such that $c(g,v) \notin F$. Then $\{h \in \mathrm{Aut}(\mathcal{T}) \mid h|_{B_1(v)} = g|_{B_1(v)}\}$ defines an open neighborhood of g which does not intersect with F. Therefore F must be closed.

(ii) Let $v \in V(\mathcal{T})$ and $\sigma \in F$. Then there exists $g = \alpha(l, \sigma \circ l, v, v)$ such that $c(g,w) = \sigma$ for all $w \in V(\mathcal{T})$. The restriction of g to the 1-ball around v realizes σ. Conversely, for $h \in \mathcal{U}(F)$ we know that $c(h,v) = c \in F$ and c realizes exactly the action on the 1-ball.

(iii) Let $v, v' \in V(\mathcal{T})$. Then $g \coloneqq \alpha(l,l,v,v') \in \mathcal{U}(F)$ is a map which maps v to v' and $c(g,w) = \mathrm{id}$ for all $w \in V(\mathcal{T})$. Since v and v' are chosen arbitrarily $\mathcal{U}(F)$ is vertex transitive.

(iv) Let $e, e' \in E(\mathcal{T})$ be two edges of the tree such that $l(e) = l(e')$, i.e. they have the same label. Then for $v = o(e)$ and $w = o(e')$ we get $g = \alpha(l,l,v,w) \in \mathcal{U}(F)$, therefore $g(e) = e'$.

(v) That $\mathcal{U}(F)$ is discrete means that every one element subset is open, therefore $\mathcal{U}(F)$ is discrete if and only if $\{\mathrm{id}\}$ is an open subgroup. Let $F \subseteq \mathrm{Sym}(d)$ be a subgroup which does not act freely on $\{1,\dots,d\}$. We show that $\{\mathrm{id}\}$ is not open in $\mathcal{U}(F)$. There exists $v \in V(\mathcal{T})$ and $\sigma \in F \setminus \{\mathrm{id}\}$ with $\sigma(i) = i$ for some $i \in \{1,\dots,d\}$. Then we define for each $n \in \mathbb{N}$ an element $g_n \in \mathcal{U}(F)$, such that g_n be the identity on the n-ball around v but not the identity at the $(n+1)$-ball. Therefore choose an edge $e \in E(\mathcal{T})$ such that $o(e) = w \in B_n(v)$, $t(e) \in B_{n-1}(v)$ and $l(e) = i$. Define $(g_n)|_{B_1(w)} = l_w^{-1} \circ \sigma \circ l_w$ and $g_n = \mathrm{id}$ outside of $B_1(w)$. Every open neighborhood of the identity in $\mathcal{U}(F)$ contains some $g_n \neq \mathrm{id}$, therefore $\{\mathrm{id}\}$ is not open and $\mathcal{U}(F)$ is non-discrete.

On the other hand if $\mathcal{U}(F)$ is non-discrete we know that $\{\mathrm{id}\}$ is not open. Hence we get that each neighborhood of the identity contains some g_n defined in the same way as before and for some $v \in V(\mathcal{T})$, $c(g_n, v) \in F$ has a fixpoint, so F acts non-freely on $\{1,\dots,d\}$.

□

Since we have seen before that $\mathrm{Aut}(\mathcal{T})$ is a tdlc group if d is finite and here we have shown that $\mathcal{U}(F)$ is a closed subgroup we directly know that $\mathcal{U}(F)$ is also a tdlc group. Moreover if F acts non-freely on $\{1,\dots,d\}$ we know that $\mathcal{U}(F)$ is non-discrete.

Proposition 5.2.8 ([15, Proposition 6.21])
The universal Burger-Mozes group $\mathcal{U}(F)$ is compactly generated for every $F \leq \mathrm{Sym}(d)$.

Proof:
Let $v \in V(\mathcal{T})$ and for all $w_i \in B_1(v)$ let $g_i = \alpha(l,l,v,w_i)$ be the edge inversion map as defined in Example 5.2.5. Then $\mathcal{U}(F)$ is generated by the compact set $\mathcal{U}(F)_v \cup \{g_1,\dots,g_d\}$, since for $h \in \mathcal{U}(F)$ choose $g \in \mathcal{U}(\{\mathrm{id}\})$, such that $g(h(v)) = v$. This is possible since $\mathcal{U}(\{\mathrm{id}\})$ is vertex transitive. Then $g \circ h \in \mathcal{U}(F)_v$ and therefore $\mathcal{U}(F) = \langle \mathcal{U}(F)_v \cup \{g_1,\dots,g_d\}\rangle$. □

Recall that $\mathcal{U}(F)^+$ is the subgroup of $\mathcal{U}(F)$ generated by all edge stabilizers. To show the simplicity of $\mathcal{U}(F)^+$ we make use of Tits independence property and Theorem 5.1.2.

Theorem 5.2.9 ([15, Theorem 6.22])
Let $F \leq \mathrm{Sym}(d)$. Then $\mathcal{U}(F)^+$ is either trivial or simple.

Proof:
By Theorem 5.1.2 it suffices to show that no proper non-empty subtree of $\mathcal{T}$ is invariant

under $\mathcal{U}(F)$, no end of $\mathcal{T}$ is fixed by $\mathcal{U}(F)$ and that $\mathcal{U}(F)$ has property (P). The first two properties are fulfilled since $\mathcal{U}(F)$ acts vertex transitive and transitive on edges of the same color on $\mathcal{T}$. For the remaining property (P) we refer to Theorem 5.3.15 where we show this property for generalized universal Burger-Mozes groups for which $\mathcal{U}(F)$ is a special case. □

Since some properties of groups and especially the finiteness properties of groups are preserved for finite index subgroups, we are interested in finite index subgroups of the universal group, in particular in finite index simple subgroups. The following theorem shows when the simple subgroup $\mathcal{U}(F)^+$ is of finite index.

Theorem 5.2.10 ([15, Theorem 6.22])
Let $F \leq \mathrm{Sym}(d)$. Then $[\mathcal{U}(F) : \mathcal{U}(F)^+] = 2$ if and only if F is transitive and generated by point stabilizers.

Before we generalize the Burger-Mozes groups we want to see why we call these groups universal.

Theorem 5.2.11 ([15, Proposition 6.23])
Let $H \leq \mathrm{Aut}(\mathcal{T})$ be vertex transitive and locally permutation isomorphic to $F \leq \mathrm{Sym}(d)$. Then if F is transitive there exists a legal labeling l such that $H \leq \mathcal{U}^l(F)$.

Proof:
Let $v \in V(\mathcal{T})$. Since $H \cong_{\text{per}} F$ there exists a bijective map $l_v\colon o(v) \to \{1, \dots, d\}$ such that $H_v|_{o(v)} = l_v^{-1} \circ F \circ l_v$. Then we can define a legal labeling on $\mathcal{T}$ inductively such that $H \leq \mathcal{U}^l(F)$. We start by defining the labeling on the edges in $B_1(v)$, i.e. all edges which start at v. For these we set $l|_{o(v)} = l_v$. In the next step assume that l is defined in all edges in $B_n(v)$. To extend the labeling at $B_{n+1}(v)$ let $w \in B_n(v) \setminus B_{n-1}(v)$ and $e_w \in E(\mathcal{T})$ be an edge such that $o(e_w) = w$ and $d(v, t(e_x)) + 1 = d(v, w) = n$. Since H is vertex-transitive and locally permutation isomorphic to a transitive F there exists a $h_{e_w} \in H$ which inverts the edge e_w. We set $l|_{o(w)} := l \circ h_{e_w}$ for all vertices in $B_n(v) \setminus B_{n-1}(v)$.
We now need to check that $H \leq \mathcal{U}^l(F)$. Let $h \in H$, $w \in V(\mathcal{T})$ and $(v, v_1, \dots, v_n, w)$, $(v, v_1', \dots, v_m', h(w))$ be the reduced paths from v to w, respectively to $h(w)$. We can describe the paths by the edges $e_{v_1}, \dots, e_{v_n}, e_w$, respectively by $e_{v_1'}, \dots, e_{v_n'}, e_{h(w)}$. Define the following map which fixes the vertex v

$$s := \sigma_{e_{v_1'}} \circ \dots \circ \sigma_{e_{v_n'}} \circ \sigma_{e_{h(w)}} \circ h \circ \sigma_{e_w} \circ \sigma_{e_{v_n}} \circ \dots \circ \sigma_{e_{v_1}} \in H_v.$$

We get $c(h, w) = c(s, v) \in F$, since H and F are locally permutation isomorphic. Hence $h \in \mathcal{U}^l(F)$ and $H \leq \mathcal{U}^l(F)$. □

5.3 Smith groups

After we introduced the universal Burger-Mozes groups we describe the universal groups $\mathcal{U}(M, N)$ which act on a biregular tree and built from two permutation subgroups $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$. For the universal Burger-Mozes groups only permutation subgroups of finite sets are considered. In this section we look at what happens if one relaxes this condition and allows arbitrary sets X and Y. We will see that in general this does not result in a tdlc group unless we impose further conditions on $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$. The following explanations follow [28], to which we refer for more details.

Let X and Y be two non-empty, disjoint sets, each containing at least two elements. Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ be two non-empty permutation subgroups. We define the biregular tree $\mathcal{T}$ as follows. Let $V(\mathcal{T}) = V_X \cup V_Y$, where V_X and V_Y are disjoint, V_X are the set of all vertices with degree $|X|$ and V_Y be the set of vertices with degree $|Y|$. Then all edges in $\mathcal{T}$ are between these sets and $\mathcal{T}$ is $(|X|, |Y|)$-biregular. Similar as for the Burger-Mozes groups we need legal labelings, but for a biregular tree. For this we take the tree as an oriented tree, where each edge exists in both directions. The function $l\colon E(T) \to X \cup Y$ is called a *legal labeling* if

(i) for all $v \in V_X$, $l|_{o(v)} \to X$ is a bijection,

(ii) for all $w \in V_Y$, $l|_{o(w)} \to Y$ is a bijection and

(iii) for all $v \in V(\mathcal{T})$, $l|_{t(v)}$ is constant.

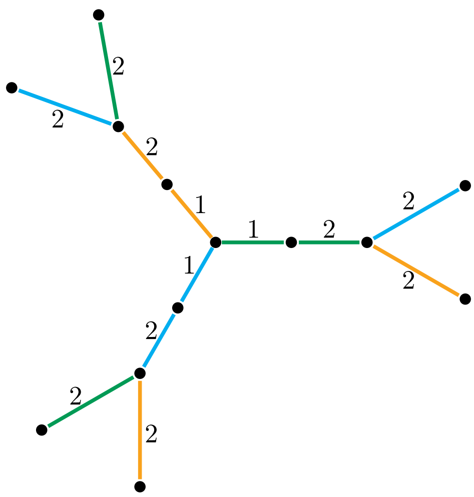

Figure 5.2: A legal coloring of the biregular tree $\mathcal{T}_{2,3}$ on a radius 2 ball, with $X = \{1, 2\}$ and $Y = \{\text{blue, green, orange}\}$.

In Figure 5.2 we see an example of a legal labeling of the tree $\mathcal{T}_{2,3}$. Similar to the case of Burger-Mozes groups we need a few more maps before we can define the universal group. Therefor we need the elements of $\mathrm{Aut}(\mathcal{T})$ which fix V_X setwise, denoted by $\mathrm{Aut}(\mathcal{T})_{\{V_X\}}$. This set also fixes the set V_Y and for this we get $\mathrm{Aut}(\mathcal{T})_{\{V_X\}} = \mathrm{Aut}(\mathcal{T})_{\{V_Y\}}$. Let l be a legal labeling of $\mathcal{T}$ for $v \in V_X$ we define the bijection

$$l_v\colon o(v) \to X, \quad e \mapsto l(e),$$

and for $w \in V_Y$

$$l_w\colon o(w) \to Y, \quad e \mapsto l(e).$$

In the same way as before we get the following maps into the permutation groups,

$$c_X\colon \operatorname{Aut}(\mathcal{T})_{\{V_X\}} \times V_X \to \operatorname{Sym}(X),\ (g,v) \mapsto l_{g(v)} \circ g \circ l_v^{-1},$$
$$c_Y\colon \operatorname{Aut}(\mathcal{T})_{\{V_Y\}} \times V_Y \to \operatorname{Sym}(Y),\ (g,w) \mapsto l_{g(w)} \circ g \circ l_w^{-1}.$$

Definition 5.3.1
Let X, Y be two disjoint sets with at least two elements, $M \leq \operatorname{Sym}(X)$ and $N \leq \operatorname{Sym}(Y)$. Let $\mathcal{T}$ be the $(|X|,|Y|)$- biregular tree and l be a legal labeling of $\mathcal{T}$. Then

$$\mathcal{U}^l(M,N) \coloneqq \{g \in \operatorname{Aut}(\mathcal{T})_{\{V_X\}} \mid \forall v \in V_X\colon c_x(g,v) \in M \text{ and } \forall w \in V_Y\colon c_Y(g,w) \in N\}$$

is called the *universal Smith group* over M and N.

We directly see in this definition that this construction is symmetric, hence we have $\mathcal{U}^l(M,N) \cong \mathcal{U}^l(N,M)$. As before we verify first that this construction is a group.

Proposition 5.3.2
$\mathcal{U}^l(M,N)$ is a subgroup of $\operatorname{Aut}(\mathcal{T})$.

Proof:
Let $g, h \in \mathcal{U}^l(M,N)$. Then $c_X(g,v), c_X(h,v) \in M$ for all $v \in V_X$ and $c_Y(g,w), c_Y(h,w) \in N$ for all $w \in V_Y$. Hence for all $v \in V_X$ and all $w \in V_Y$

$$\begin{aligned} c_X(gh,v) &= l_{g(h(v))} \circ gh \circ l_v^{-1} \\ &= l_{g(h(v))} \circ g \circ l_{h(v)}^{-1} \circ l_{h(v)} \circ h \circ l_v^{-1} \\ &= c_X(g,h(v)) \circ c_X(h,v) \in M, \text{ and} \\ c_Y(gh,w) &= c_Y(g,h(w)) \circ c_Y(h,w) \in N. \end{aligned}$$

Therefore $\mathcal{U}^l(M,N)$ is closed under multiplication. On the other hand we have $\mathrm{id} \in M$, so $\mathrm{id} = c_X(\mathrm{id},v) = c_X(gg^{-1},v) = c_X(g,g^{-1}(v)) \circ c_X(g^{-1},v) \in M \Rightarrow c_X(g^{-1},v) \in M$ for all $v \in V_X$. The same we have for all $w \in V_Y$, so $\mathcal{U}^l(M,N)$ is closed under inverses. □

Before we show the connection between the Burger-Mozes groups and the Smith groups we show that the Smith groups are also independent of the labeling. Therefore we use a similar statement as for the Burger-Mozes groups in Lemma 5.2.4.

Proposition 5.3.3 ([28, Lemma 9])
Let l and l' be two legal labelings of $\mathcal{T}$ and $v, v' \in V_X$ or $v, v' \in V_Y$. Then there exists $\sigma \in \operatorname{Sym}(X \cup Y)_{\{X\}}$ such that $\sigma(l'|_{t(v')}) = l|_{t(v)}$. For all such σ there exit a unique $g \in \operatorname{Aut}(\mathcal{T})_{\{V_X\}}$ such that $g(v) = v'$ and $l = \sigma(l' \circ g)$.

Proof:
Without loss of generality we may assume $v, v' \in V_X$. The case for vertices in V_Y is completely analogous. Then $l|_{t(v)}$ and $l|_{t(v')}$ are constant maps into Y, hence we identify them with the element in Y they map to. Therefore such a $\sigma \in \operatorname{Sym}(X \cup Y)_{\{X\}}$ always exists. For the construction of the map g we construct at first maps which are defined on increasingly larger balls around v mapping to balls around v'. To this end let $B_n = B_n(v)$ and $B'_n = B_n(v')$ and define $g_0\colon B_0 \to B'_0$ such that $g_0(v) = v'$, since $B_0 = \{v\}$ and $B'_0 = \{v'\}$ the map is completely defined. Then construct $g_n\colon B_n \to B'_n$ such that

$\sigma \circ l'|_{B'_n} \circ g_n = l|_{B_n}$ and $g_n|_{B_{n-1}} = g_{n-1}$. We show by induction that the construction of g_n is possible and thereby we show how we construct these maps. Assume that g_n is constructed with the above conditions.

Let $w \in B_n \setminus B_{n-1}$. Then there exist an unique edge $e \in B_n$, such that $o(e) = w$ and $g_n(e) = e'$ with $o(e') = w' = g_n(w)$. Then we get $((l')^{-1}|_{o(w')} \circ \sigma^{-1} \circ l|_{o(w)})(e) = e'$ and this induces a bijection $h_w \colon B_1(w) \setminus V(B_n) \to B_1(w') \setminus V(B'_n)$. Then g_{n+1} is the extension of g_n by all these h_w for $w \in B_n \setminus B_{n-1}$. By construction $g_{n+1}|_{B_n} = g_n$ hence it remains to show that $\sigma \circ l'|_{B'_{n+1}} \circ g_{n+1} = l|_{B_{n+1}}$. Let e be an edge in $B_{n+1} \setminus B_n$. Then either $o(e) \in V(B_n)$ or $t(e) \in V(B_n)$. First we look at the case that $t(e) \in V(B_n)$. Let $\tilde{e}$ be the edge in B_n with $t(e) = t(\tilde{e})$. In this case we get $t(g_{n+1}(e)) = t(g_{n+1}(\tilde{e}))$, hence it follows that $l(e) = l(\tilde{e})$ and $l'(g_{n+1}(e)) = l'(g_{n+1}(\tilde{e}))$. Moreover by the induction hypothesis we have $\sigma(l'(g_{n+1}(\tilde{e}))) = \sigma(l'(\tilde{e})) = l(\tilde{e})$, hence $l(e) = l(\tilde{e}) = \sigma(l'(g_{n+1}(\tilde{e}))) = \sigma(l'(g_{n+1}(e)))$. On the other hand if $w = o(e) \in V(B_n)$, we have $g_{n+1}(w) = g_n(w)$ and by construction $g_{n+1}(t(e)) = h_w(t(e))$ for the other side of the edge. Therefore $\sigma(l'(g_{n+1}(e))) = l(e)$ follows by the definition of h_w.
In the next step we use the maps g_n to construct the map $g \in \mathrm{Aut}(\mathcal{T})_{\{V_X\}}$, such that $g(v) = v'$ and $l = \sigma \circ l' \circ g$. For all $w \in V(\mathcal{T}) \setminus \{v\}$ there exists a unique $n(w) \in \mathbb{N}$ such that $w \in B_{n(w)} \setminus B_{n(w)-1}$. Define g on the vertices of $\mathcal{T}$ as follows:

$$\begin{aligned} g \colon V(\mathcal{T}) \to V(\mathcal{T}),\ & v \mapsto v' \\ & w \mapsto g(w) = g_{n(w)}(w). \end{aligned}$$

Then g fixes V_X and V_Y setwise by construction of the g_n. Let $e = (w, w') \in E(\mathcal{T})$, then there exists $m \in \mathbb{N}$ such that $e \in B_m$ and $m \geq \max\{n(w), n(w')\}$. Define g at the edge set

$$\begin{aligned} g(e) = g(w, w') &= (g(w), g(w')) \\ &= (g_{n(w)}(w), g_{n(w')}(w')) \\ &= (g_m(w), g_m(w')) \\ &= g_m(w, w') = g_m(e). \end{aligned}$$

This directly implies $(\sigma \circ l' \circ g)(e) = l(e)$.
In the last step we need to show that g is unique. Therefore let $h \in \mathrm{Aut}(\mathcal{T})_{\{V_X\}}$ with $h(v) = v'$ and $\sigma \circ l' \circ h = l$. We show that if $gh^{-1}(w) = w$, then gh^{-1} is the identity on $B_1(w)$. Let $w \in V(\mathcal{T})$ and $gh^{-1}(w) = w$. Then for $e \in o(w)$ we get $gh^{-1}(e) \in o(w)$ and

$$\begin{aligned} (\sigma \circ l' \circ g)(h^{-1}(e)) = l(h^{-1}(e)) &= (\sigma \circ l' \circ h)(h^{-1}(e)) \\ &= (\sigma \circ l')(e). \end{aligned}$$

Since $l'|_{o(w)}$ is a bijection we have that gh^{-1} fixes $B_1(w)$ pointwise. As $h(v) = v' = g(v)$, we have that v' is a fixpoint of gh^{-1} and since $\mathcal{T}$ is connected we inductively obtain that gh^{-1} is the identity. Hence $g = h$ and g must be unique. □

Equipped with the above result we can state the analogue of Proposition 5.2.3 for the universal Smith groups.

Theorem 5.3.4 ([28, Proposition 11])
Let l and l' be legal labelings of the biregular tree $\mathcal{T}$. Then $\mathcal{U}^l(M,N)$ and $\mathcal{U}^{l'}(M,N)$ are conjugate in $\operatorname{Aut}(\mathcal{T})$.

Proof:
Choose $v, v' \in V(\mathcal{T})$ such that $l|_{t(v)} = l'_{t(v')}$. Then v, v' are both in V_X or both in V_Y. By Proposition 5.3.3 there exists a unique $g \in \operatorname{Aut}(\mathcal{T})_{\{V_X\}}$ such that $g(v) = v'$ and $l = l' \circ g$. Let $h \in \operatorname{Aut}(\mathcal{T})_{\{V_X\}}$ and define $h' := g \circ h \circ g^{-1} \in \operatorname{Aut}(\mathcal{T})_{\{V_X\}}$. Choose $w \in V(\mathcal{T})$ and define $w' := g(w)$. Then $w, w' \in V_X$ or $w, w' \in V_Y$. Since all the arguments in both cases are the same we may assume w.l.o.g. that $w, w' \in V_X$. Now

$$\begin{aligned} c_X(h', w') &= l'_{h'(w')} \circ h' \circ (l'_{w'})^{-1} \\ &= l'_{g(h(w))} \circ g \circ h \circ g^{-1} \circ (l'_{g(w)})^{-1} \\ &= l_{h(w)} \circ h \circ l_w^{-1} = c_X(h, w), \end{aligned}$$

hence we get $h' = ghg^{-1} \in \mathcal{U}^{l'}(M,N)$ if and only if $h \in \mathcal{U}^l(M,N)$. □

In the following we may suppress the labeling l from the notation if the argument is independent of the given labeling. In this case we write $\mathcal{U}(M,N)$ instead of $\mathcal{U}^l(M,N)$. The next lemma focuses on the connection between the two types of universal groups which explains how the Smith groups are a generalization of the Burger-Mozes groups.

Lemma 5.3.5 ([28, after Remark 4])
Let $F \leq \operatorname{Sym}(d)$ for $d \geq 3$. Then $\mathcal{U}(F)$ is topologically isomorphic to $\mathcal{U}(F, \operatorname{Sym}(2))$.

Proof:
By construction $\mathcal{U}(F)$ acts on the regular tree $\mathcal{T}_d$. This action induces an action on the barycentric subdivision of $\mathcal{T}_d$, which is $\mathcal{T}_{d,2}$. Therefore we may consider $\mathcal{U}(F)$ as a subgroup of $\operatorname{Aut}(\mathcal{T}_{d,2})_{\{1,\dots,d\}}$. On the other hand $\mathcal{U}(F, \operatorname{Sym}(2))$ also acts on $\mathcal{T}_{d,2}$. These actions are isomorphic as topological groups, therefore $\mathcal{U}(F) \cong \mathcal{U}(F, \operatorname{Sym}(2))$. □

In Proposition 5.2.7 we have seen some properties of the universal Burger-Mozes groups and now we collect some of them for the universal Smith groups.

Proposition 5.3.6 ([28, Proposition 10])
The vertices $v, v' \in V_X$ (respectively V_Y) lie in the same orbit of $\mathcal{U}(M,N)$ if and only if $l|_{t(v)}$ and $l|_{t(v')}$ lie in the same orbit of N (respectively M).

Proof:
As before we only consider the case $v, v' \in V_X$. If $l|_{t(v)}$ and $l|_{t(v')}$ lie in the same orbit of N, then there exists $\sigma \in N$ such that $\sigma(l|_{t(v)}) = l|_{t(v')}$. Using this map we define $\hat{\sigma} \in \operatorname{Sym}(X \cup Y)_{\{Y\}}$ by setting $\hat{\sigma}|_X = \mathrm{id}$ and $\hat{\sigma}|_Y = \sigma$. There exists $g \in \operatorname{Aut}(\mathcal{T})_{\{V_X\}}$ such that $g(v) = v'$ and $l = \hat{\sigma} \circ l \circ g$, hence we get

$$\begin{aligned} c_X(g, w) &= l_{g(w)} \circ g \circ l_w^{-1} = l|_{o(g(w))} \circ g|_{o(w)} \circ l^{-1}|_{o(w)} \\ &= (l \circ g)|_{o(w)} \circ l^{-1}|_{o(w)} \\ &= (\hat{\sigma}^{-1} \circ l)|_{o(w)} \circ l^{-1}|_{o(w)} \\ &= \hat{\sigma}^{-1}|_X \in M \end{aligned}$$

for all $w \in V_X$. Analogously we see that for $w \in V_Y$ we get $c_Y(g,w) \in N$. Therefore $g \in \mathcal{U}(M,N)$ and v and v' are in the same orbit of $\mathcal{U}(M,N)$.
On the other hand let $g \in \mathcal{U}(M,N)$ such that $g(v) = v'$. Fix $e = (w,v) \in E(\mathcal{T})$ with $o(e) = w \in V_Y$ and $t(e) = v \in V_X$, so $e \in t(v)$ and $g(e) \in t(v')$. Then $\sigma = c_Y(g,w) \in N$ and $(\sigma \circ l)(e) = l(g(e))$, hence $l|_{t(v)} = l(e)$ and $l|_{t(v')} = l(g(e))$, which are mapped to each other by $\sigma \in N$. In particular lie in the same orbit.

□

Recall the notion of groups being locally permutation isomorphic from Definition 5.2.6. In part (ii) of Proposition 5.2.7 we have seen that $\mathcal{U}(F)$ is locally permutation isomorphic to F. We can formulate a similar statement for the universal Smith groups.

Definition 5.3.7
Let $M \leq \mathrm{Sym}(X)$, $N \leq \mathrm{Sym}(Y)$ and $\mathcal{T}$ be the $(|X|,|Y|)$-biregular tree with edge sets V_X and V_Y. We say $H \leq \mathrm{Aut}(\mathcal{T})$ is *locally* (M,N) if

- $H \leq \mathrm{Aut}(\mathcal{T})_{\{V_X\}}$ and
- for all $v \in V_X$, the fixpoint group H_v is locally permutation isomorphic to M and for $v \in V_Y$ to N.

Lemma 5.3.8 ([28, Lemma 13])
The universal Smith group $\mathcal{U}(M,N)$ is locally (M,N).

Proof:
Let $G \coloneqq \mathcal{U}(M,N)$. Fix $v \in V(\mathcal{T})$ and without loss of generality we may assume $v \in V_X$, since all arguments are similar for the other case. We need to show that $c_X(-,v)\colon G_v|_{B_1(v)} \to M$ is an isomorphism. Injectivity follows directly by the construction of G. For surjectivity let $\sigma \in M$ and $\hat{\sigma} \in \mathrm{Sym}(X \cup Y)_{\{X\}}$ with $\hat{\sigma}|_X = \sigma$ and $\hat{\sigma}|_Y = \mathrm{id}$. Then by Proposition 5.3.3 there exists $g \in G_v$ such that $c_X(g,v) = \sigma$.

□

For the universal Burger-Mozes groups we have seen in Theorem 5.2.11 why they are called universal groups. For the Smith groups a similar statement can be formulated. For the proof we refer to [28].

Theorem 5.3.9 ([28, Proposition 12])
Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ be transitive and $H \leq \mathrm{Aut}(\mathcal{T})$ be locally (M,N). Then there exists a legal labeling l of $\mathcal{T}$ such that $H \leq \mathcal{U}^l(M,N)$.

In Proposition 5.2.7 we have seen that $\mathcal{U}(F)$ is a closed subgroup, which we used to show that it is a tdlc group. For the universal Smith group $\mathcal{U}(M,N)$ we have a similar statement but we can not use it directly to say that the group is a tdlc group, since $\mathcal{T}$ is not locally finite.

Lemma 5.3.10 ([28, Lemma 14])
Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ be closed subgroups. Then $\mathcal{U}(M,N) \leq \mathrm{Aut}(\mathcal{T})$ is closed.

Proof:
Let $A := \mathrm{Aut}(\mathcal{T})_{\{V_X\}} \leq \mathrm{Aut}(\mathcal{T})$. Then A is a closed subgroup. Now define two continuous functions from A to $\mathrm{Sym}(X)$, respectively $\mathrm{Sym}(Y)$, let $v \in V_X$ and $w \in V_Y$

$$\begin{aligned} c_X^v &:= c_X(-,v)\colon\ \mathrm{Aut}(\mathcal{T})_{\{V_X\}} \to \mathrm{Sym}(X),\\ c_Y^w &:= c_Y(-,w)\colon\ \mathrm{Aut}(\mathcal{T})_{\{V_Y\}} \to \mathrm{Sym}(Y). \end{aligned}$$

Then $(c_X^v)^{-1}(M)$ and $(c_Y^w)^{-1}(N)$ are closed, since the functions are continuous, by [28, Lemma 7], and the sets M and N are closed. By the construction of the universal group we get

$$\mathcal{U}(M,N) = \left(\bigcap_{v\in V_X} (c_X^v)^{-1}(M)\right) \cap \left(\bigcap_{w\in V_Y} (c_Y^w)^{-1}(N)\right)$$

and therefore $\mathcal{U}(M,N)$ is, as the intersection of closed subgroups, closed. □

Since it is possible that X or Y or both are not finite the biregular tree $\mathcal{T}$ must not be locally finite. Hence we cannot deduce that $\mathrm{Aut}(\mathcal{T})$ is locally compact. Nevertheless, we know that $\mathrm{Aut}(\mathcal{T})$ is totally disconnected, thus it is not enough to know that $\mathcal{U}(M,N)$ is a closed subgroup to guarantee that we have a tdlc group. We need to show in which case $\mathcal{U}(M,N)$ is locally compact.

Theorem 5.3.11 ([28, Theorem 30])
Let $M \leq Sym(X)$ and $N \leq \mathrm{Sym}(Y)$ be closed and non-trivial. Then the following are equivalent:

(i) $\mathcal{U}(M,N)$ is locally compact,

(ii) all point stabilizers of M and N are compact,

(iii) all edge stabilizers of $\mathcal{U}(M,N)$ are compact.

Proof:
Let $G := \mathcal{U}(M,N)$.

(iii) ⇒ (i): clear

(i) ⇒ (ii): If G is locally compact there exists a finite subset $Z \subseteq V(\mathcal{T})$ such that the pointwise stabilizer G_Z has only finite orbits. We first show that all point stabilizers of M are compact. Let $x \in X$, then $V(\mathcal{T}) \setminus Z$ contains infinitely many v such that $l|_{t(v)} = x$. Let $\mathcal{P}$ be a finite path in $\mathcal{T}$ which contains only the edge (v,w) such that $l|_{t(v)} = x$ and $\pi_{\mathcal{P}}^{-1}(v)$ contains ϕ. In Theorem 5.3.15 we later see that G satisfies Tits independence property (P) which implies that the pointwise stabilizer $G_{\pi_{\mathcal{P}}^{-1}(v)}$ and $G_{(v,w)}$ induce the same permutation group on $B_1(w)$ and this group is permutation isomorphic to the point stabilizer M_x. Since $G_{\pi_{\mathcal{P}}^{-1}(v)} \leq G_\phi$ and all orbits of G_ϕ are finite, so are the orbits of the induced group on $B_1(w)$. Hence all orbits of M_x are finite and therefore M_x is compact. For $y \in Y$ we use the same arguments to show that N_y is compact. Hence all point stabilizers of M and N are compact.

(ii) ⇒ (iii): Since all point stabilizers of M and N are compact all orbits of the point stabilizers are finite. Let $v, w \in V(\mathcal{T})$ be adjacent vertices in $\mathcal{T}$. We show that if all orbits of $G_{(v,w)}$ have finite length, then $G_{(v,w)}$ is compact. Fix $u \in V(\mathcal{T})\backslash\{v, w\}$. Note that either $d(v, u) < d(w, u)$ or $d(w, u) < d(v, u)$. Without loss of generality let u be closer to w. Let $\mathcal{P} = (y_0, y_1, \ldots, y_n) = (v, w, y_2, \ldots, y_{n-1}, u)$ be the geodesic from v to u. Then

$$|G_{(v,w)}u| = |G_{(v,w)} \colon G_{\mathcal{P}}| \leq \prod_{i=0}^{n-2} |G_{(y_i,y_{i+1})}y_{i+2}|,$$

where each orbit of the last product is finite since $y_i, y_{i+2} \in B_1(y_{i+1})$ and $G_{(y_i,y_{i+1})}|_{B_1(y_{i+1})}$ is permutation isomorphic to a point stabilizer of M or N (depending on wether v in V_X or in V_Y).

□

By Lemma 5.3.10 and Theorem 5.3.11 we directly obtain the following corollary.

Corollary 5.3.12
Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ be closed, non-trivial and assume that all point stabilizers of M and N are closed. Then $\mathcal{U}(M, N)$ is a tdlc group.

Now we have seen in which case the universal Smith groups are tdlc groups, but the other case we are interested in was when these groups are non-discrete.

Theorem 5.3.13 ([28, Theorem 32])
Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ be non-trivial. Then the following are equivalent:

(i) $\mathcal{U}(M, N)$ is discrete,

(ii) M acts freely on X and N acts freely on Y,

(iii) $\mathcal{U}(M, N)$ acts freely on $E(\mathcal{T})$.

Proof:
The equivalence between (i) and (iii) follows directly by [24, Section 3.3, Theorem 4], since $\mathcal{T}$ is a tree.
Suppose $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ are subgroups which act freely on X respectively Y. Let $w, w' \in V_Y$ with $d(w, w') = 2$ and let $v \in V_v$ such that (w, v, w') be the geodesic path between w and w'. Then the pointwise stabilizers of $\{w, w'\}$ and $\{w, v, w'\}$ of G are equal. Therefore $G_{(v,w)} \subseteq G_{w,w'}$ and $G_{(v,w')} \subseteq G_{w,w'}$. Since G is locally (M, N) we know that $G_v|_{B_1(v)}$ is isomorphic to M and $G_w|_{B_1(w)}$ is isomorphic to N. This means that the point stabilizer acts freely on the one ball around the fixed point. In particular the action is free on the edges of this ball. Therefore we get that the $G_{(v,w)}$ fixes $B_1(v)$ and $B_1(w)$ pointwise thus they are trivial. Iteratively repeating this procedure on balls with increasing size shows that $G_{(v,w)}$ must be trivial. Since all the arguments are symmetric for V_X and V_Y, G acts freely on $E(\mathcal{T})$.
On the other hand if M or N do not act freely we show that G does not act freely on $E(\mathcal{T})$. Without loss of generality we may assume that the action of M is not free. Then there exist some $x \in X$ such that the fixpoint group M_x is not trivial. There exit infinitely many $w \in V_Y$ such that $l|_{t(w)} = x$. Let $w \in V_Y$ and $v \in V_X$ such that

$l(v,w) = x$ and $\sigma \in M_x$ such that $\sigma \neq \mathrm{id}$. Since G is locally (M,N) there exists an isomorphism between $G_v|_{B_1(v)}$ and M. Let $h \in G_v$ be such that the image of $h|_{b_1(v)}$ is equal to σ. This implies

$$l|_{B_1(v)} \circ h|_{B_1(v)} \circ l^{-1}|_{B_1(v)} = \sigma.$$

Then $h \in G_{(v,w)}$ but $h \neq \mathrm{id}$, hence G acts non-freely on the edge set of $\mathcal{T}$ which implies that G is not discrete. If N acts non-freely on Y we get that there is an edge from V_Y to V_X such that the edge stabilizer is not trivial, by the same arguments. □

The last statement we have shown for the universal Burger-Mozes groups was the simplicity of the edge stabilizer. We show a similar statement here, for this we need to show first that the Tits independence property (P) is satisfied.

Lemma 5.3.14 ([28, Lemma 17])
Let $e \in E(\mathcal{T})$ be an edge of $\mathcal{T}$ and $h \in \mathcal{U}(M,N)_e$. Then there exists a $g \in \mathcal{U}(M,N)$ such that $g|_{\mathcal{T}_e} = \mathrm{id}$ and $g|_{\mathcal{T}_{\overline{e}}} = h|_{\mathcal{T}_{\overline{e}}}$.

Theorem 5.3.15 ([28, Theorem 20])
Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ non-trivial. Then the universal Smith group $\mathcal{U}(M,N)$ satisfies Tits independence property (P).

Proof:
Let $\mathcal{P}$ be a non-empty finite or (bi-)infinite path in $\mathcal{T}$ and $\pi_{\mathcal{P}}$ be the nearest point projection to the path. Denote $G = \mathcal{U}(M,N)$. Choose $g \in G_{\mathcal{P}}$, such that it fixes $\mathcal{P}$ pointwise, and let $p \in V(\mathcal{P})$ be a vertex of the path. If $\mathcal{P}$ is an one element path the statement is clear, so let $\mathcal{P}$ be a path with at least two vertices. Then $|(B_1(p) \setminus \{p\}) \cap V(\mathcal{P})| \in \{1,2\}$, hence let $(B_1(p) \setminus \{p\}) \cap V(\mathcal{P}) = \{q, q'\}$, where possibly $q = q'$. By Lemma 5.3.14 there exists $g' \in G$ such that $g'|_{\mathcal{T}_{(q,p)}} = \mathrm{id}$ and $g'|_{\mathcal{T}_{(p,q)}} = g|_{\mathcal{T}_{(p,q)}}$. Then $g' \in G_{\mathcal{P}}$ fixes $\mathcal{P}$ pointwise, therefore g' fixes the edge (p, q') which is inside the path. Using Lemma 5.3.14 again shows that there exists $g_p \in G$ such that $g_p|_{\mathcal{T}_{(q',p)}} = \mathrm{id}$ and $g_p|_{\mathcal{T}_{(p,q')}} = g|_{\mathcal{T}_{(p,q')}}$. This situation is illustrated in Figure 5.3 Therefore we see that $T_{(p,q')} \cap T_{(p,q)} = \pi_{\mathcal{P}}^{-1}(p)$ and we get for g_p that $g_p|_{\pi_{\mathcal{P}}^{-1}(p)} = g|_{\pi_{\mathcal{P}}^{-1}(p)}$ and $g_p = \mathrm{id}$ otherwise. Therefore $g_p|_{\pi_{\mathcal{P}}^{-1}(p)} \in G_{\mathcal{P}}^p \leq \mathrm{Sym}(\pi_{\mathcal{P}}^{-1}(p))$. Hence $g = \prod_{p \in V(\mathcal{P})} g_p$ and the map

$$\phi_p \colon G_{\mathcal{P}} \to G_{\mathcal{P}}^p, \quad g \mapsto g_p|_{\pi_{\mathcal{P}}^{-1}(p)}$$

is a homomorphism inducing a homomorphism

$$\begin{aligned} \phi \colon G_{\mathcal{P}} &\to \prod_{p \in V(\mathcal{P})} G_{\mathcal{P}}^p \\ g &\mapsto \prod_{p \in V(\mathcal{P})} g_p|_{\pi_{\mathcal{P}}^{-1}(p)}. \end{aligned}$$

That this map is an isomorphism can be easily verified. □

Before we verify in which case the universal Smith groups are simple we first show in which case the subgroup G^+ is simple, so in which case we can guarantee all assumptions of Theorem 5.1.2.

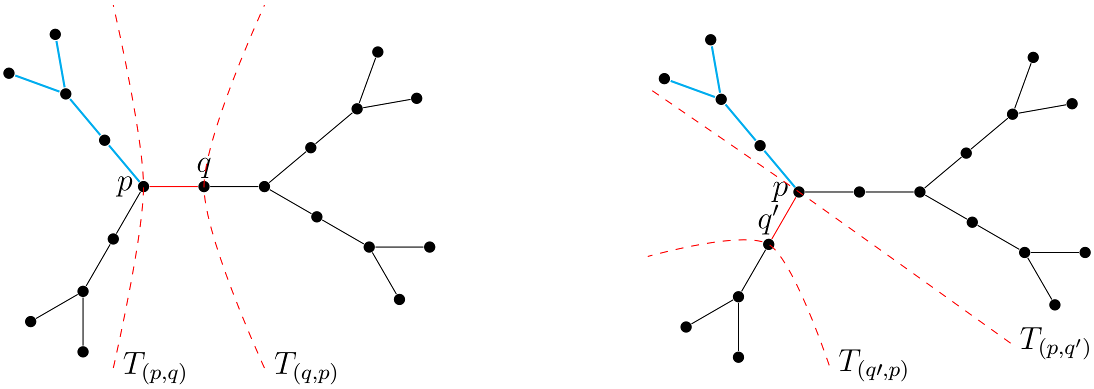

Figure 5.3: Example for the situation in the proof of Theorem 5.3.15, where $\pi_{\mathcal{P}}^{-1}(p)$ is colored blue

Proposition 5.3.16
Let $M \leq \mathrm{Sym}(X)$, $N \leq \mathrm{Sym}(Y)$ and $G = \mathcal{U}(M, N)$. If M or N is transitive then G leaves no proper non-empty subtree of $\mathcal{T}$ invariant and fixes no end of $\mathcal{T}$.

Proof:
Without loss of generality we may assume that N is transitive. As G is locally (M, N) this implies that the group $G_v|_{B_1(v)}$ is transitive for all $v \in V_Y$. Let ε be an end of $\mathcal{T}$ and $\mathcal{T}'$ be a non-empty proper subtree. Then there exists a $v \in V_Y \setminus V(\mathcal{T}')$, and for this vertex we have G_v does not fix ε and G_v also does not leave $\mathcal{T}'$ invariant. □

Corollary 5.3.17
If either M or N is transitive and $G = \mathcal{U}(M, N)$, then G^+ is simple.

In the next step we find the case in which we have that $G = G^+$. We make use of the following result due to Serre.

Lemma 5.3.18 ([24, Corollary 1, Section 5.4])
Let G be a group which acts on a tree $\mathcal{T}$ without inversions and $R = \langle G_v \mid v \in V(\mathcal{T})\rangle$ be the subgroup generated by all vertex stabilizers. Then R is normal and ${}^{G}\!/_{R}$ can be identified with the fundamental group of the graph ${}_{G}\backslash^{\mathcal{T}}$.

In the following, our focus will be on the case where $G = R$, which is precisely the case when ${}_{G}\backslash^{\mathcal{T}}$ is a tree. This condition is satisfied if and only if G is generated by the vertex groups of ${}_{G}\backslash^{\mathcal{T}}$. This assertion follows by the results outlined in Exercise 2 in Section 5.4 of Serres book [24].
By the statement about the orbits of the vertices in Proposition 5.3.6 and a similar statement about the orbits of the edges (see [28, Lemma 16]) we obtain the following.

Lemma 5.3.19 ([28, Lemma 22])
Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$, m be the number of orbits of M and n be the number of orbits of N. Let $G = \mathcal{U}(M, N)$. Then ${}_{G}\backslash^{\mathcal{T}}$ is the complete bipartite graph $K_{m,n}$.

Theorem 5.3.20 ([28, Theorem 23])
Let $M \leq \mathrm{Sym}(X)$, $N \leq \mathrm{Sym}(Y)$ be generated by their point stabilizers and non-trivial. Then $\mathcal{U}(M, N)$ is simple if and only if M or N is transitive.

Proof:
Let $G = \mathcal{U}(M, N)$, $G^+ = \langle G_{(v,w)} \mid (v, w) \in E(\mathcal{T})\rangle$ and G_v^+ be the point stabilizer of $v \in V(\mathcal{T})$ in G^+.
Then if M or N is transitive we have seen that G^+ is simple. Therefore we show that $G = G^+$ if and only if M or N is transitive. Let $v \in V(\mathcal{T})$. Then $G_v|_{B_1(v)}$ is generated by point stabilizers, since G is locally (M, N) and M and N are generated by their point stabilizers. That $G_v|_{B_1(v)}$ is generated by point stabilizers means that it is generated by the edge stabilizers $G_{(v,w)}|_{B_1(v)}$ for all $w \in B_1(v)$.
At first we show that $G_v = G_v^+$. On the one hand $G^+ \leq G$, hence $G_v^+ \leq G_v$. On the other hand $G_{(v,w)} \leq G_v^+$ for all $w \in B_1(v)$. Since $G_{(v,w)}$ fixes the edge (v, w) pointwise all elements also fix v, i.e. $G_{(v,w)} \leq G_v$. Then $G_{(v,w)} \leq G_v^+$ implies $G_{(v,w)}|_{B_1(v)} \leq G_v^+|_{B_1(v)}$. Since $G_v|_{B_1(v)}$ is generated by point stabilizers we get $G_v|_{B_1(v)} \leq G_v^+|_{B_1(v)}$. Then $G_v^+|_{B_1(v)} = G_v|_{B_1(v)}$ and $G_{(v,w)} \leq G_v^+$ for all edges $w \in B_1(v)$ implies $G_v = G_v^+$ for all $v \in V(\mathcal{T})$.
Let $R = \langle G_v \mid v \in V(\mathcal{T})\rangle \leq G^+$. Since M and N are non-trivial R is non-trivial. By Lemma 5.3.18 R is a normal subgroup of G and $G = R$ if and only if ${}_G\backslash^{\mathcal{T}}$ is a tree. In Lemma 5.3.19 we see that ${}_G\backslash^{\mathcal{T}}$ is a tree if and only if M or N is transitive. Therefore $G = G^+$ if and only if M or N is transitive. □

5.4 Finiteness properties

The previous section shows some properties of the universal Smith groups, now we look at the finiteness properties of these groups. In the following let X and Y be two non-empty disjoint sets with at least two elements. Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$, be non-trivial subgroups. Since we only defined the finiteness properties for tdlc groups we need in the following that the universal Smith group over M and N is a tdlc group, hence by Corollary 5.3.12 all point stabilizers of M and N must be compact and M and N must be closed. Let $\mathcal{T}$ be the $(|X|, |Y|)$-biregular tree and l be a legal labeling of $\mathcal{T}$. Then we define the tdlc group $G \coloneqq \mathcal{U}^l(M, N)$ as the universal Smith group over M and N. The goal of this section is the proof of the following theorem.

Theorem D
Let $M \leq \mathrm{Sym}(X)$, $N \leq \mathrm{Sym}(Y)$ be non-empty, closed with compact point stabilizers and $G \coloneqq \mathcal{U}(M, N)$. Assume M and N have only finitely many orbits and one of M and N is transitive. Then

(i) G is of type FP_n over $\mathbb{Z}$ if and only if M and N are of type FP_n over $\mathbb{Z}$,

(ii) for $n \geq 2$, G is of type F_n and not of type F_{n+1} if M and N are of type F_n and one of them is not of type F_{n+1}.

For the case of compactly generated groups we refer to a theorem of Smith, which shows one direction of the second part for the case $n = 1$.

Theorem 5.4.1 ([28, Theorem 31])
Let $M \subseteq \mathrm{Sym}(X)$ and $N \subseteq \mathrm{Sym}(Y)$ with nontrivial degree, both are closed under the permutation topology, and have compact point stabilizers. Then all point stabilizers of $\mathcal{U}(M,N)$ are compactly generated if and only if M and N are both compactly generated. Moreover, if M and N are both compactly generated with only finitely many orbits and M or N is transitive, then $\mathcal{U}(M,N)$ is compactly generated.

For the proof of this statement we refer to the paper of Smith, since we prove a stronger statements.

Theorem 5.4.2
Let $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ be non-trivial, closed under permutation topology and have only compact point stabilizers. Let $G = \mathcal{U}(M,N)$ and R be a commutative ring. Then

(i) M and N are of type FP_n over R if and only if all point stabilizers of G are of type FP_n over R;

(ii) M and N are of type F_n if and only if all point stabilizers of G are of type F_n.

Proof:
Since M and N are closed and all point stabilizers are compact we get by Theorem 5.3.11 and Corollary 5.3.12 that G is a tdlc group and all edge stabilizers of G are compact and therefore of type F_∞, respectively of type FP_∞ over R.
Let $v \in V_X \subseteq V(\mathcal{T})$. Then $G_v|_{B_1(v)}$ is isomorphic to M, hence there exists a surjective group homomorphism $f\colon G_v \to M$ with $\ker(f) = \bigcap_{w \in B_1(v)} G_{(v,w)} = K$. Each edge stabilizer $G_{(v,w)}$ is compact, by Theorem 5.3.11. Therefore we get that K is also compact so of type F_∞, respectively of type FP_∞ over R. The map f induces the following short exact sequence

$$0 \to K \to G_v \xrightarrow{f} M \to 0.$$

By the fact that K is of type F_∞, respectively of type FP_∞ over R and by Theorem 2.2.21 we can deduce that G_v and M have the same topological and homological finiteness type for all $v \in V_X$. As all arguments also work for $v \in V_Y$ we have shown the theorem. □

For the second part of Theorem 5.4.1 we can also find a generalization, we will show this later, for this we use Lemma 5.4.3.
Let m be the number of orbits of M and n be the number of orbits of N. Then we have seen in Lemma 5.3.19 that $G\backslash\mathcal{T} = K_{m,n}$ and Lemma 5.3.18 shows for which $G\backslash\mathcal{T}$ the group can be identified with a fundamental group of a graph of groups. At first we specify these statement.

Lemma 5.4.3
The universal group $G = \mathcal{U}(M,N)$ can be identified as a fundamental group of a finite graph of groups, which underlying graph is a finite tree, if and only if M and N have only finitely many orbits and one of them is transitive.

Proof:
If M or N is transitive then $G\backslash\mathcal{T}$ is a tree and J. P. Serre shows in [24] that G is

isomorphic to the fundamental group of the graph of groups based on the graph $G\backslash\mathcal{T}$. □

The trivial case where M and N are transitive is explained in [24, Section 4.1, Theorem 6]. There it is shown that G has the structure of an amalgamated product

$$G = G_u *_{G(u,v)} G_v$$

where u and v are adjacent vertices of $\mathcal{T}$, therefore $u \in V_X$ and $v \in V_Y$.

Theorem 5.4.4
Let $M \leq \mathrm{Sym}(X)$, $N \leq \mathrm{Sym}(Y)$, be non-trivial, closed, have compact point stabilizers, only finitely many orbits and one of them is transitive. Let $G = \mathcal{U}(M,N)$ and R be a commutative ring. If M and N are of type F_n then G is of type F_n. Moreover, if M and N are of type FP_n over R, then G is of type FP_n over R.

Proof:
Let N be transitive and m be the number of orbits of M. Then $G\backslash\mathcal{T}$ is the finite tree $K_{m,1}$. By Lemma 5.4.3 we get that G is the fundamental group of a finite graph of groups, and by Theorem 5.3.11 we get that all edge stabilizers and therefore the edge groups are compact. Therefore Theorem 5.4.2 gives that the vertex groups have the same homological and topological finiteness properties as N and M. By the criterion of Brown Theorem 2.2.23 and Theorem 2.2.25 we can deduce that G is of type F_n, respectively of type FP_n over R.

□

In the next step we use these fundamental groups of graphs of groups to say something about the finiteness properties of $\mathcal{U}(M,N)$. At first we refer for the discrete case.

Theorem 5.4.5 ([17])
Let G be a discrete group which can be identified as a fundamental group of a finite graph of groups $(\mathcal{G},\mathcal{A})$ with edge groups $\mathcal{A}_e$ which are all of type F_n. Then G is of type F_n if and only if for all vertices $v \in V(\mathcal{G})$ the vertex group $\mathcal{A}_v$ is of type F_n.

One direction of this theorem follows directly by the criterion of Brown and the other direction was shown by Haglund and Wise in [17], for the proof we refer to the paper. Next we generalize this theorem to the case of tdlc groups.

Theorem 5.4.6
Let G be a tdlc group which can be identified as a fundamental group of a finite graph of groups $(\mathcal{G},\mathcal{A})$ with edge groups $\mathcal{A}_e$ all of type FP_n over $\mathbb{Z}$.
Then $G = \pi_1(\mathcal{G},\mathcal{A})$ is of type FP_n over $\mathbb{Z}$ if and only if for all $v \in V(\mathcal{G})$ the vertex group $\mathcal{A}_v$ is of type FP_n over $\mathbb{Z}$.

Proof:
That the finiteness conditions of $\mathcal{A}_v$ implies these for G as well in the tdlc case follows by the criterion of Brown, see Theorem 2.2.23.
Let $\hat{T}$ be the Bass-Serre tree of $(\mathcal{G},\mathcal{A})$. For $e \in E(\hat{T})$ and $v \in V(\hat{T})$ let H_e and H_v denote the edge stabilizers and vertex stabilizers of $\hat{T}$ respectively. By construction of G

and the Bass-Serre tree all stabilizers are open subgroups of G. Note that the edge and vertex stabilizers of the Bass-Serre tree are given by edge and vertex groups of $(\mathcal{G}, \mathcal{A})$ respectively. Therefore by assumption H_e is of type FP_n over $\mathbb{Z}$.
Let G be of type FP_n over $\mathbb{Z}$, then by definition $\mathbb{Z}$ is of type FP_n as a $\mathbb{Z}[G]$-module. H_e is of type FP_n over $\mathbb{Z}$ which implies that $\mathbb{Z}$ is of type FP_n as a $\mathbb{Z}[H_e]$-module. Since H_e is an open subgroup we can use Proposition 2.2.20, which implies that $\mathbb{Z}[G] \otimes_{H_e} \mathbb{Z} = \mathbb{Z}[{}^{G}/_{H_e}]$ is of type FP_n as a $\mathbb{Z}[G]$-module. The cellular chain complex of the Bass-Serre tree $\hat{T}$ is given by the short exact sequence

$$0 \to \bigoplus_{e \in R_e} \mathbb{Z}[{}^{G}/_{H_e}] \to \bigoplus_{v \in R_v} \mathbb{Z}[{}^{G}/_{H_v}] \to \mathbb{Z} \to 0$$

of permutations modules over $\mathbb{Z}$, where R_e and R_v are representative systems of the edges respectively the vertices of the action of $\mathcal{G}$ on $\hat{T}$. By assumption the parts $\bigoplus_{e \in R_e} \mathbb{Z}[{}^{G}/_{H_e}]$ and $\mathbb{Z}$ are both of type FP_n as a $\mathbb{Z}[G]$-modules, therefore it follows by Corollary 2.2.17 that $\bigoplus_{v \in R_v} \mathbb{Z}[{}^{G}/_{H_v}]$ is of type FP_n as a $\mathbb{Z}[G]$-module. By this fact and the short exact sequence

$$0 \to \bigoplus_{v \in R_v \setminus \{v'\}} \mathbb{Z}[{}^{G}/_{H_v}] \to \bigoplus_{v \in R_v} \mathbb{Z}[{}^{G}/_{H_v}] \to \mathbb{Z}[{}^{G}/_{H_{v'}}] \to 0$$

it follows that for all $v \in V(\hat{T})$, ${}^{G}/_{H_v}$ is of type FP_n as $\mathbb{Z}[G]$-module. Since H_v is an open subgroup and $\mathbb{Z}[{}^{G}/_{H_v}] = \mathbb{Z}[G] \otimes_{H_v} \mathbb{Z}$ we can use Proposition 2.2.20 again which implies that $\mathbb{Z}$ is of type FP_n as a $\mathbb{Z}[H_v]$-module. Hence H_v is of type FP_n over $\mathbb{Z}$, so also $\mathcal{A}_v$ is of type FP_n over $\mathbb{Z}$ for all $v \in V(\mathcal{G})$.

□

Proof of Theorem D:
Since M and N are closed and all point stabilizers are compact the universal group $G = \mathcal{U}(M, N)$ is a tdlc group. G can be identified as a fundamental group of a finite graph of groups by Lemma 5.4.3 and the assumption that M and N have only finitely many orbits and one of them is transitive.
The first part follows by Theorem 5.4.2 and Theorem 5.4.6.
For the second part we use the connection between being of type F_n and being of type FP_n over $\mathbb{Z}$ for tdlc groups (see Proposition 2.2.19). Let M and N both be of type F_n, with at least one of them not of type F_{n+1} and $n \geq 2$. By Theorem 5.4.4 G is of type F_n. In particular M, N and G are of type F_2, i.e. compactly presented. This implies that if G is of type F_{n+1}, G is also of type FP_{n+1} over $\mathbb{Z}$. By Theorem 5.4.2 and Theorem 5.4.6 M and N are also of type FP_{n+1} over $\mathbb{Z}$, which implies on the other hand that M and N are of type F_{n+1}, since M and N are compactly presented. This is a contradiction, therefore G must be of type F_n and not of type F_{n+1} if M and N are both of type F_n and one of them is not of type F_{n+1}.

□

5.5 Examples classified by finiteness properties

The previous sections collected some properties of the universal Smith groups, most of them depend on the permutation groups M and N. Here we figure out which conditions we need for M and N such that we can construct new examples of tdlc groups which can be classified by their finiteness properties.
First, we want to ensure that we construct a tdlc group which is not a discrete group. In Lemma 5.3.10 we have seen that $G := \mathcal{U}(M, N)$ is a closed subgroup of the automorphism group of $\mathcal{T}$ if M and N are closed subgroups. Therefore if M and N are closed, G is totally disconnected. In Theorem 5.3.11 we have shown that G is locally compact if and only if all point stabilizers of M and N are compact. Theorem 5.3.13 shows that G is non-discrete if and only if M or N acts non-freely on X, respectively Y. Hence G is a non-discrete tdlc group if M and N are closed subgroups, such that all point stabilizers are compact and one of them acts non-freely on the underlying set.
Section 5.4 shows for which M and N we can transfer the finiteness properties between the permutation subgroups M and N and the universal group $G = \mathcal{U}(M, N)$. In this section we mostly use that G can be identified as a fundamental group of a finite graph of groups. This is only possible if M and N have only finitely many orbits and one of them is transitive (see Lemma 5.4.3). Then Theorem D shows that the homological and topological finiteness properties of M and N coincide with those of G.

Setup 5.5.1
Assume the following situation:

$M \subseteq \mathrm{Sym}(X)$

- non-trivial
- closed
- all point stabilizers are compact
- has finitely many orbits
- of type FP_n over $\mathbb{Z}$
- not of type FP_{n+1} over $\mathbb{Z}$
- acts non-freely on X

$N \subseteq \mathrm{Sym}(Y)$

- non-trivial
- closed
- all point stabilizers are compact
- transitive
- of type FP_n over $\mathbb{Z}$

Then $G = \mathcal{U}(M, N)$ is a non-discrete tdlc group of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$.

In the following we fix N as finite discrete group in the following way. Let $Y = \{1, 2\}$ and $N = \mathrm{Sym}(Y)$, then N is non-trivial, closed, has only finite point stabilizers, therefore the point stabilizers are compact. Since N is finite we get that N is of type F_∞ which implies that N is of type FP_∞ over $\mathbb{Z}$.
Since M must be of type FP_n and not of type FP_{n+1} over $\mathbb{Z}$, M cannot be a finite group and therefore X cannot be finite. Chapter 3 introduced the Schlichting completion which is based on a subgroup of a permutation group. Let Γ be a discrete group and Λ be a commensurated subgroup. Let $\alpha\colon \Gamma \to \mathrm{Sym}(\Gamma/\Lambda)$ be the map induced by the action of Γ on Γ/Λ. By construction the Schlichting completion $\Gamma/\!/\Lambda$ is a closed transitive

subgroup of Sym (Γ/Λ). By Lemma 3.1.2 we have seen that $\Gamma//\Lambda = \alpha(\Gamma)\overline{\alpha(\Lambda)}$ and for the intersection we get $\overline{\alpha(\Lambda)} \cap \alpha(\Gamma) = \alpha(\Lambda)$, which implies that the point stabilizers of $\Gamma//\Lambda$ all look like $\gamma\overline{\alpha(\Lambda)}\gamma^{-1}$ for $\gamma \in \alpha(\Gamma)$. Therefore all point stabilizers are closed and non-trivial, so $\Gamma//\Lambda$ acts non-freely on Γ/Λ if Λ is not a normal subgroup of Γ. In the following let $M = \Gamma//\Lambda \leq \mathrm{Sym}(\Gamma/\Lambda)$. To verify the finiteness conditions we use the two main theorems of Chapter 3.

Theorem A ([5, Theorem 1.1])
Let $G \coloneqq \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$. Let R be a commutative ring. Then the following holds.

(i) If Λ and G are of type FP_n over R, then Γ is of type FP_n over R.

(ii) If Λ and G are of type F_n, then Γ is of type F_n.

Theorem B ([5, Theorem 1.2])
Let $G \coloneqq \Gamma//\Lambda$ be the Schlichting completion of Γ relative to the commensurated subgroup $\Lambda \leq \Gamma$. Let R be a commutative ring. Then the following holds.

(i) If Λ is of type FP_{n-1} over R and Γ is of type FP_n over R, then G is of type FP_n over R.

(ii) If Λ is of type F_{n-1} and Γ is of type F_n, then G is of type F_n.

These theorems imply together with Setup 5.5.1 the following theorem.

Theorem 5.5.2
Let Γ be a discrete group of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$ together with a commensurated subgroup Λ of type FP_n over $\mathbb{Z}$. Then we get with $M = \Gamma//\Lambda$ and $N = \mathrm{Sym}(2)$ a tdlc group $G = \mathcal{U}(M, N)$ such that G is of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$.

In Theorem 5.3.20 we have also seen in which case the universal Smith group over M and N is simple. Therefore we figure out what we need for M and N such that $G = \mathcal{U}(M, N)$ is a simple tdlc group of type FP_n and not of type FP_{n+1} over $\mathbb{Z}$. By Theorem 5.3.20 G is simple if and only if M and N are generated by point stabilizers.

Setup 5.5.3
Assume the following situation:

$M \subseteq \mathrm{Sym}(X)$

- non-trivial
- closed
- all point stabilizers are compact
- generated by their point stabilizers
- has finitely many orbits
- of type FP_n over $\mathbb{Z}$
- not of type FP_{n+1} over $\mathbb{Z}$

$N \subseteq \mathrm{Sym}(Y)$

- non-trivial
- closed
- all point stabilizers are compact
- generated by their point stabilizers
- transitive
- of type FP_n over $\mathbb{Z}$
- acts non-freely on Y

Then $G = \mathcal{U}(M, N)$ is a simple tdlc group of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$.

Since Sym(2) has only trivial point stabilizers it can not be generated by the point stabilizers. Therefore we need another choice for N, but we can choose it in a similar way as finite discrete group. Let $Y = \{1, 2, 3\}$ and $N = \mathrm{Sym}(Y)$, then N is as before non-trivial, closed, transitive with compact point stabilizers and of type F_∞ respectively of type FP_∞ over $\mathbb{Z}$. Moreover N is generated by the point stabilizers and acts non-freely on Y.

For M we choose the same construction as before, $M = \Gamma//\Lambda \subseteq \mathrm{Sym}(\Gamma/\Lambda)$. Then we get again that M is closed, all point stabilizers are compact, it acts transitive and non-freely. For the finiteness properties we get again that M is of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$ if we choose Γ and Λ accordingly. The point stabilizers of M are all of the form $\gamma\overline{\alpha(\Lambda)}\gamma^{-1}$ for $\gamma \in \overline{\alpha(\Gamma)}$ and the question is in which case they generate M.

Lemma 5.5.4
Let Γ be a discrete group with commensurated subgroup Λ. Then the following are equivalent

(i) $\Gamma//\Lambda$ *is generated by the sets* $\{\gamma\overline{\alpha(\Lambda)}\gamma^{-1} \mid \gamma \in \overline{\alpha(\Gamma)}\}$,

(ii) $\alpha(\Gamma)$ *is generated by the sets* $\{\gamma\alpha(\Lambda)\gamma^{-1} \mid \gamma \in \alpha(\Gamma)\}$.

Proof:
On the one hand we have by Lemma 3.1.2

$$\begin{aligned}\alpha(\Gamma) = \overline{\alpha(\Gamma)} \cap \alpha(\Lambda) &= \langle\{\gamma\overline{\alpha(\Lambda)}\gamma^{-1} \mid \gamma \in \overline{\alpha(\Gamma)}\}\rangle \cap \alpha(\Gamma) \\ &= \langle\{\gamma\alpha(\Lambda)\gamma^{-1} \mid \gamma \in \alpha(\Gamma)\}\rangle.\end{aligned}$$

And on the other hand we get

$$\begin{aligned}\overline{\alpha(\Gamma)} &= \overline{\langle\{\gamma\alpha(\Lambda)\gamma^{-1} \mid \gamma \in \alpha(\Gamma)\}\rangle} \\ &= \langle\{\gamma\overline{\alpha(\Lambda)}\gamma^{-1} \mid \gamma \in \overline{\alpha(\Gamma)}\}\rangle.\end{aligned}$$

□

All in all we obtain the following theorem.

Theorem E
Let Γ be a discrete group of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$ together with a commensurated subgroup Λ of type FP_n over $\mathbb{Z}$ and Γ be generated by $\{\gamma\Lambda\gamma^{-1} \mid \gamma \in \Gamma\}$. Let $M = \Gamma//\Lambda$ and $N = \mathrm{Sym}(3)$. Then $\mathcal{U}(M, N)$ is a simple tdlc group of type FP_n over $\mathbb{Z}$ and not of type FP_{n+1} over $\mathbb{Z}$.

We show that Example 3.3.4 does not satisfied this property, therefore we generate with this example a new class of non-simple tdlc groups which can be classified by their finiteness properties. Let

$$\Gamma_n = \Gamma_n\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \text{ and } \Lambda_n = \Gamma_n(\mathbb{Z}) \leq \Gamma_n.$$

Let $A \in \Gamma_n$ and $B \in \Lambda_n$, then all diagonal elements of B are in $\mathbb{Z}$. Then we get for ABA^{-1} that all diagonal elements are also in $\mathbb{Z}$, therefore $\{\gamma\Lambda_n\gamma^{-1} \mid \gamma \in \Gamma_n\}$ cannot generate Γ_n.
The whole constructions and statements also hold in the same way for being of type F_n and not of type F_{n+1} for $n \geq 2$ instead of FP_n and FP_{n+1} over $\mathbb{Z}$.
We constructed a new class of tdlc groups which can be classified by finiteness properties. Further we found a concrete condition for a discrete group Γ together with a commensurated subgroup which creates simple tdlc groups classified by their finiteness properties.

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