Finance and Stochastics (2025) 29:707–755
https://doi.org/10.1007/s00780-025-00564-6

Measuring risk contagion in financial networks with CoVaR

Bikramjit Das[1] · **Vicky Fasen-Hartmann**[2]

Received: 27 September 2023 / Accepted: 23 September 2024 / Published online: 27 May 2025

Abstract
The stability of a complex financial system may be assessed by measuring risk contagion between various financial institutions with relatively high exposure. We consider a financial network model using a bipartite graph of financial institutions (e.g. banks, investment companies, insurance firms) on one side and financial assets on the other. Following empirical evidence, returns from such risky assets are modelled by heavy-tailed distributions, whereas their joint dependence is characterised by copula models exhibiting a variety of tail-dependence behaviour. We consider CoVaR, a popular measure of risk contagion, and study its asymptotic behaviour under broad model assumptions. We further propose the *extreme CoVaR index* (ECI) for capturing the strength of risk contagion between risk entities in such networks, which is particularly useful for models exhibiting asymptotic independence. The results are illustrated by providing precise expressions of CoVaR and ECI when the dependence of the assets is modelled using two well-known multivariate dependence structures: the Gaussian copula and the Marshall–Olkin copula.

Keywords Bipartite graph · Copula models · CoVaR · Financial network · Heavy tails · Gaussian copula

Mathematics Subject Classification 62G32 · 62H05 · 91G45 · 91G70

JEL Classification C13 · D81 · G32

✉ V. Fasen-Hartmann
vicky.fasen@kit.edu

B. Das
bikram@sutd.edu.sg

1 Engineering Systems and Design, Singapore University of Technology and Design, 8 Somapah Road, Singapore 487372, Singapore

2 Institute for Stochastics, Karlsruhe Institute of Technology, Englerstrasse 2, 76131 Karlsruhe, Germany

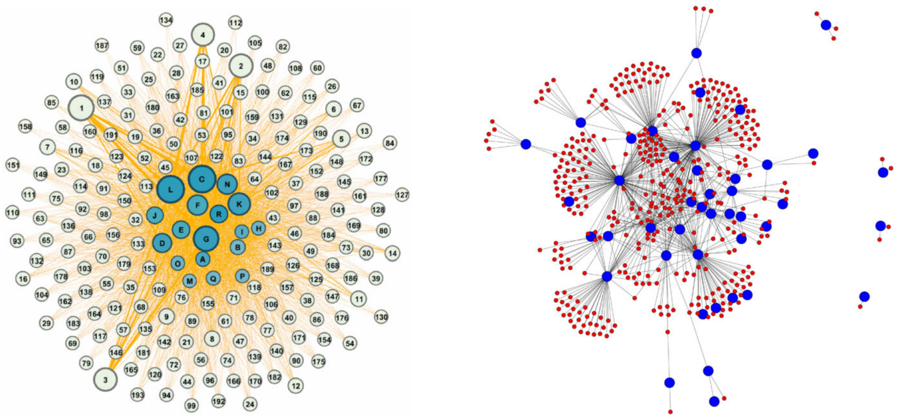

Fig. 1 (left) Bank–asset bipartite network of the Mexican financial system on a particular day. Nodes in the network represent banks (blue) and assets (red). Links between an asset and a bank exist if the bank holds the asset in its portfolio. (Courtesy: Poledna et al. [52]); (right) Bipartite network structure of common assets held by Chinese banks in 2021. The numbers and letters in the circles respectively represent identifiers for banks and industries (Courtesy: Fan and Hu [28])

1 Introduction

The global financial crisis of 2007–2009 brought into immediacy the need to understand risk contagion in order to assess the stability of a financial system. The high level of inter-connectivity between various financial institutions has been argued to be one of few key contributors to such systemic financial instability; see Allen and Gale [5], Eisenberg and Noe [26], Acemoglu et al. [1], Gai and Kapadia [31], Feinstein et al. [29], Benoit et al. [8] for some compelling arguments on such network effects. Among the various frameworks proposed to capture risk contagion in financial systems, a popular one has been to model the financial system as a bipartite graph of financial institutions (e.g. banks, investment companies, insurance firms) on one side and overlapping financial assets where the banks invest on the other. We use this bipartite network as our model which has been observed in many financial markets; see the Chinese and Mexican financial system in Fig. 1. This framework has been used not only for modelling bank–asset risk sharing (Caccioli et al. [14], Fan and Hu [28], Poledna et al. [52]), but also bank–firm credit networks (Marotta et al. [47]), claims in insurance markets (Kley et al. [40, 41]), etc.

To fix notations, consider a vertex set $\mathcal{A} = \mathbb{I}_q = \{1, \dots, q\}$ of banks/agents and a vertex set $\mathcal{O} = \mathbb{I}_d = \{1, \dots, d\}$ of assets/objects. Denote by Z_j the risk attributed to the jth object; then $\mathbf{Z} = (Z_1, \dots, Z_d)^\top$ forms the risk vector. Each bank/agent $k \in \mathcal{A}$ is connected to a number of assets/objects $j \in \mathcal{O}$, and these connections may be modelled in a stochastic manner following some probability distribution; see Fig. 2 for a representative example of such a bipartite network. A basic model assumes that k and j connect (denoted $k \sim j$) with probability

$$\mathbb{P}[k \sim j] = p_{kj} \in [0, 1], \qquad k \in \mathcal{A}, j \in \mathcal{O}.$$

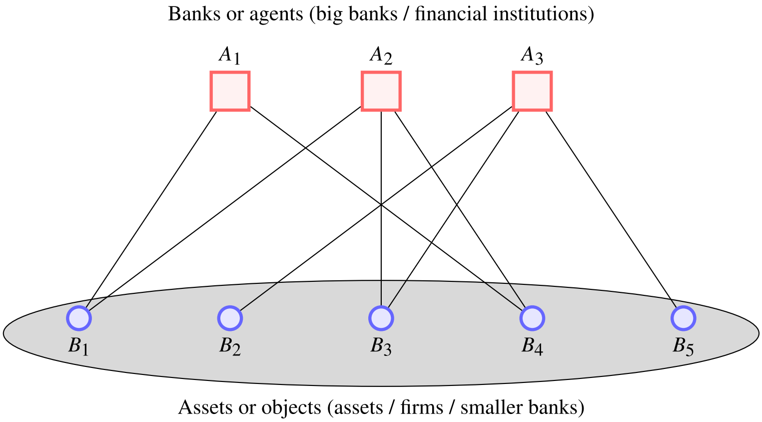

Fig. 2 A bipartite network with $q = 3$ banks (or agents) and $d = 5$ assets (or objects). We model interdependencies via a multivariate model for the assets and the bipartite network (which may be assumed to be random or fixed) connecting assets to banks

The proportion of loss of asset/object j affecting bank/agent k is denoted by

$$f_k(Z_j) = \mathbb{1}_{\{k\sim j\}} W_{kj} Z_j,$$

where $\mathbb{1}$ represents the indicator function and $W_{kj} > 0$ denotes the proportional effect of the jth object on the kth agent. Defining the $q \times d$ adjacency matrix $\mathbf{A}$ by

$$A_{kj} = \mathbb{1}_{\{k\sim j\}} W_{kj},$$

the total exposure of the banks/agents (think of negative log-returns on market equity value) is given by $\mathbf{X} = (X_1, \ldots, X_q)^\top$, where $X_k = \sum_{j=1}^{d} f_k(Z_j)$ can be represented as

$$\mathbf{X} = \mathbf{AZ}. \tag{1.1}$$

We may assume that the graph creation process is independent of $\mathbf{Z}$, i.e., $\mathbf{A}$ and $\mathbf{Z}$ are independent. Now the behaviour of $\mathbf{X}$ will be governed by both the network represented by $\mathbf{A}$ and the underlying distribution of $\mathbf{Z}$, the risk related to the assets/objects. The goal of this paper is to study risk contagion in terms of the extremal behaviour of $\mathbf{X}$ under reasonable assumptions on $\mathbf{Z}$ and $\mathbf{A}$.

In order to model $\mathbf{Z}$, the returns from the asset, we note that financial returns have often been empirically observed to be heavy-tailed; see Adler et al. [3, Chap. 1], Resnick [57, Chap. 1.3], Embrechts et al. [27, Chap. 6.5]. For this paper, we model the distribution of $\mathbf{Z}$ to be heavy-tailed using the paradigm of multivariate regular variation (Bingham et al. [10, Chap. 8], Resnick [57, Chap. 6]); further details are given in Sect. 2. To model the dependence between the components of $\mathbf{Z}$, we resort to a few popular multivariate dependence structures, namely (i) the asymptotically strongly dependent case, (ii) the Gaussian dependence, an erstwhile popular model for dependence in many domains including finance (Fischer et al. [30], Malevergne and Sornette [46]) and (iii) the Marshall–Olkin dependence, which is often used in reliability/failure modelling in large systems (Marshall and Olkin [48], Yuge et al. [58]). Our key results are more generally applicable, and we only provide explicit expressions for the above examples. Interestingly, the latter dependence structures, i.e., the Gaussian copula and the Marshall–Olkin copula, possess the property of (pairwise) asymptotic independence in the tails, i.e., extreme values are less likely to occur simultaneously; see Das and Fasen-Hartmann [19] for details. This property of asymptotic independence has also been empirically observed in international equity markets (Bradley and Taqqu [11], Poon et al. [54]).

We note as well that a linear dependence structure similar to (1.1) is also found in certain factor models. For example, in the popular single-index factor model, namely the capital asset pricing model (CAPM), the rate of return of an asset is divided into a market return and an idiosyncratic risk of the return, which are assumed to be independent, and the weights are the β's introduced in the CAPM (see Zhou [59], de Vries and Hyung [24], Huschens and Kim [37] for details). If the idiosyncratic risks have heavier tails than the market returns of the assets, then the assets turn out to be asymptotically independent, which is again a modelling component of the present paper.

Finally, our key goal is to assess risk contagion in financial systems. Regulatory bodies like the Basel Committee on Banking Supervision, Solvency II and the Swiss Solvency Test for insurance regulations have regularly recommended monitoring of individual measures of risk exposure like value-at-risk (VaR) and expected shortfall (ES) for financial and insurance institutions so that adequate capital is reserved to avoid catastrophic losses; see Artzner et al. [6], Jorion [39, Chap. 1.3]. It is apparent that for capturing *risk contagion*, we need to assess the potential loss for entities/institutions of interest conditioning on an extreme loss event pertaining to one or more other financial institution(s). In this regard, conditional risk measures like CoVaR (Adrian and Brunnermeier [4], Girardi and Ergün [32]), marginal expected shortfall (MES) (Acharya et al. [2]), marginal mean excess (MME) (Das and Fasen-Hartmann [16, 17]), systemic risk (SRISK) (Brownlees and Engle [13]) have become extremely popular in the years following the 2007–2009 crisis. In this paper, we focus on one particular measure of risk contagion, namely CoVaR. Naturally, computing such risks requires an appropriate modelling of joint tail risks which captures the marginal risk behaviour as well as inter-dependencies between financial entities.

We properly define this particular measure of risk contagion, CoVaR, in the following. For a random variable Y, the *value-at-risk* or VaR at level $1-\gamma \in (0,1)$ is defined as

$$\mathrm{VaR}_\gamma(Y) := \inf\{y \in \mathbb{R} : \mathbb{P}[Y > y] \le \gamma\} = \inf\{y \in \mathbb{R} : \mathbb{P}[Y \le y] \ge 1-\gamma\}.$$

Note that we chose γ as the subscript instead of the usual $1-\gamma$ for brevity and notational convenience. Now, given two random variables Y_1, Y_2 and constant levels $1-\gamma_1, 1-\gamma_2 \in (0,1)$, we define the *Co*nditional-VaR or *Co*ntagion-VaR (CoVaR) of Y_1 (at level $1-\gamma_1$) given Y_2 (at level $1-\gamma_2$) as

$$\mathrm{CoVaR}_{\gamma_1|\gamma_2}(Y_1|Y_2) = \inf\{y \in \mathbb{R} : \mathbb{P}[Y_1 > y | Y_2 > \mathrm{VaR}_{\gamma_2}(Y_2)] \le \gamma_1\}. \tag{1.2}$$

Here, $\mathrm{CoVaR}_{\gamma_1|\gamma_2}(Y_1|Y_2)$ represents the VaR at level $1-\gamma_1 \in (0,1)$ of Y_1 given that Y_2 is above its own VaR at level $1-\gamma_2 \in (0,1)$. Although we concentrate on CoVaR throughout the paper, it is clear that other conditional risk measures like MES/MME (Acharya et al. [2], Das and Fasen-Hartmann [16]) and SRISK (Brownlees and Engle [13]) are viable options. The risk measure CoVaR was introduced to capture risk contagion as well as systemic risk by Adrian and Brunnermeier [4] who used the conditioning event to be $\{Y_2 = \mathrm{VaR}_{\gamma_2}(Y_2)\}$. This was later modified by Girardi and Ergün [32] to $\{Y_2 > \mathrm{VaR}_{\gamma_2}(Y_2)\}$ with the restriction that $\gamma_1 = \gamma_2$; this latter definition has been shown to have nicer properties for dependence modelling and is used widely for modelling and estimation (Nolde et al. [51], Mainik and Schaanning [45], Härdle et al. [33]). We use the more general definition given in (1.2) following Girardi and Ergün [32], but allowing γ_1 and γ_2 to be not necessarily equal; see Bianchi et al. [9], Reboredo and Ugolini [55], Kley et al. [41] for related work using this definition.

The computation of CoVaR under a bipartite network framework has been addressed in Kley et al. [41], where the authors have particularly concentrated on distributions of risk exposures that are either asymptotically co-monotone or asymptotically independent. In addition, some of their proofs in the asymptotically independent case are based on an i.i.d. assumption and, more importantly, may often lead to null estimates for the relevant measures.

In this paper, we provide an enhanced characterisation of risk contagion using CoVaR, not only addressing the asymptotically strongly dependent case (which includes the co-monotone case), but especially emphasising models with asymptotic independence which are natural and popular and have not been particularly addressed before; here we use the characterisation for tail asymptotics of random linear functions of regularly varying vectors developed in Das et al. [21]. Our results show that the asymptotic behaviour of CoVaR is affected by both the structure of the bipartite network as well as the strength of dependence of the underlying distribution of risk factors.

The paper is structured as follows. In Sect. 2, we characterise multivariate heavy-tailed models and in particular investigate Gaussian and Marshall–Olkin copula models with Pareto-type marginals. In Sect. 3, we characterise the asymptotic behaviour of $\mathrm{CoVaR}_{g(\gamma)|\gamma}(Y_1|Y_2)$ as $\gamma \downarrow 0$ for some function $g(\gamma)$, where Y_1 is heavy-tailed (note that both marginals need not be heavy-tailed). We also introduce here the *extreme CoVaR index* (ECI), which is a measure of the strength of risk contagion from Y_1 to Y_2 and is particularly useful when they are asymptotically independent. The bipartite network with asymptotically strongly dependent underlying assets is addressed in an example in this section as well. For the remainder of the paper, we investigate the more challenging case of asymptotically independent assets. In Sect. 4, we compute joint and conditional probabilities for extreme events pertaining to the agent's/bank's total exposure modelled by $\mathbf{X} = \mathbf{AZ}$ and the asymptotic behaviour of CoVaR as well as the ECI. This requires an appropriate understanding of transformations of various kinds of sets in the presence of the bipartite network model. We conclude in Sect. 5 with indications for future work. All proofs of relevant results are relegated to the Appendix.

1.1 Notations

The following notations are used throughout the paper. We denote by $\mathbb{I}_d = \{1, \ldots, d\}$ an index set with d elements; the subscript is dropped when evident from the context. For a given vector $\mathbf{z} \in \mathbb{R}^d$ and $S \subseteq \mathbb{I}_d$, we denote by $\mathbf{z}^\top$ the transpose of $\mathbf{z}$ and by $\mathbf{z}_S \in \mathbb{R}^{|S|}$ the vector obtained by deleting the components of $\mathbf{z}$ in $\mathbb{I}_d \backslash S$. Similarly, for non-empty $I, J \subseteq \mathbb{I}_d$, Σ_{IJ} denotes the appropriate submatrix of a given matrix Σ in $\mathbb{R}^{d\times d}$, and we write Σ_I for Σ_{II}. Vector operations are understood componentwise, e.g. for vectors $\mathbf{v} = (v_1, \ldots, v_d)^\top$ and $\mathbf{z} = (z_1, \ldots, z_d)^\top$, $\mathbf{v} \leq \mathbf{z}$ means $v_j \leq z_j$, $\forall j$. We also have the following notations for vectors in $\mathbb{R}^d$: $\mathbf{0} = (0, \ldots, 0)^\top$, $\mathbf{1} = (1, \ldots, 1)^\top$, $\boldsymbol{\infty} = (\infty, \ldots, \infty)^\top$ and $e_j = (0, \ldots, 1, \ldots, 0)^\top$, $j \in \mathbb{I}_d$, where e_j has only one non-zero entry 1 at the jth coordinate.

For a random vector $\mathbf{Z} = (Z_1, \ldots, Z_d)^\top$, we write $\mathbf{Z} \sim F$ if $\mathbf{Z}$ has distribution function F; moreover, we understand that marginally $Z_j \sim F_j$ for $j \in \mathbb{I}_d$. We call the random vector $\mathbf{Z} \sim F$ to be *tail-equivalent* if for all $j, \ell \in \mathbb{I}_d$, $j \neq \ell$, we have $\lim_{t\to\infty}(1 - F_j(t))/(1 - F_\ell(t)) = c_{j\ell}$ for some $c_{j\ell} > 0$; moreover, if $c_{j\ell} = 1$ for all $j, \ell \in \mathbb{I}_d$, we say that $\mathbf{Z}$ is *completely tail-equivalent*. Analogously, we call the associated random variables Z_j, Z_ℓ tail-equivalent or completely tail-equivalent. For functions f, g, we write $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = 1$. The cardinality of a set $S \subseteq \mathbb{I}_d$ is denoted by $|S|$. The indicator function of an event A is denoted by $\mathbb{1}_A$, and for a constant $t > 0$ and a set $A \subseteq \mathbb{R}^d_+$, we define $tA := \{t\mathbf{z} : \mathbf{z} \in A\}$.

2 Multivariate heavy tails

In this paper, the framework that we use for modelling heavy tails is that of multivariate regular variation. We start with a brief primer on multivariate regular variation and then explicitly derive necessary model parameters, limit measures and their supports for our two primary model examples, the Gaussian and Marshall–Olkin copula models.

2.1 Preliminaries: multivariate regular variation

Regular variation is a popular theoretical framework for modelling heavy-tailed distributions; for the models in this paper, we assume this property for all our marginal risk variables.

A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying* (at $+\infty$) with some fixed $\beta \in \mathbb{R}$ if $\lim_{t\to\infty} f(tz)/f(t) = z^\beta, \forall z > 0$. We write $f \in \mathrm{RV}_\beta$, and if $\beta = 0$, we call f a *slowly varying* function. A real-valued random variable $Z \sim F$ is regularly varying (at $+\infty$) if the tail $\overline{F} := 1 - F \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$. Alternatively, if $Z \sim F$, the property $\overline{F} \in \mathrm{RV}_{-\alpha}$ is equivalent to the existence of a measurable function $b : \mathbb{R}_+ \to \mathbb{R}_+$ with $b(t) \to \infty$ as $t \to \infty$ such that

$$t\mathbb{P}[Z > b(t)z] = t\overline{F}\big(b(t)z\big) \longrightarrow z^{-\alpha} \qquad \text{as } t \to \infty, \forall z > 0.$$

Consequently, we have $b \in \mathrm{RV}_{1/\alpha}$, and a canonical choice for b is

$$b(t) = F^{\leftarrow}(1 - 1/t) = \overline{F}^{\leftarrow}(1/t)$$

where $F^{\leftarrow}(z) = \inf\{y \in \mathbb{R} : F(y) \geq z\}$ is the generalised inverse of F. Well-known distributions like Pareto, Lévy, Fréchét, Student-t which are used to model heavy-tailed data are all in fact regularly varying; see Embrechts et al. [27, Chap. 1.3], Resnick [57, Chap. 2] for further details.

For multivariate risks, our interest is in high (positive) risk events, and hence we concentrate on characterising tail behaviour on the positive quadrant $\mathbb{R}^d_+ := [0, \infty)^d$; risk events in any other quadrant can be treated similarly. Multivariate regular variation on such *cones* and their subsets are discussed in detail in Das and Fasen-Hartmann [18], Hult and Lindskog [36], Das et al. [22], Lindskog et al. [44]. We briefly introduce the ideas, notations and tools here. We focus particularly on subcones of $\mathbb{R}^d_+$ of the form

$$\mathbb{O}^{(i)}_d := \mathbb{R}^d_+ \setminus \{\mathbf{z} \in \mathbb{R}^d_+ : z_{(i)} = 0\} = \{\mathbf{z} \in \mathbb{R}^d_+ : z_{(i)} > 0\}, \qquad i \in \mathbb{I}_d,$$

where $z_{(1)} \geq z_{(2)} \geq \cdots \geq z_{(d)}$ is the decreasing order statistic of $z_1, \dots, z_d$. Here $\mathbb{O}^{(i)}_d$ represents the subspace of $\mathbb{R}^d_+$ with all $(i-1)$-dimensional co-ordinate hyperplanes removed; the 0-dimensional hyperplane is the point $\{\mathbf{0}\}$. Clearly,

$$\mathbb{O}^{(1)}_d \supseteq \mathbb{O}^{(2)}_d \supseteq \cdots \supseteq \mathbb{O}^{(d)}_d. \tag{2.1}$$

The type of convergence we use here is called $\mathbb{M}$-convergence of measures [36, 22], of which multivariate regular variation on $\mathbb{O}^{(i)}_d$ is only a special example; see [44, 18]. For a subspace $\mathbb{O}^{(i)}_d$, let $\mathcal{B}(\mathbb{O}^{(i)}_d)$ denote the collection of Borel sets in $\mathbb{O}^{(i)}_d$. Note that two sets $A, B \subseteq \mathbb{R}^d_+$ are *bounded away from each other* if $\overline{A} \cap \overline{B} = \emptyset$, where $\overline{A}, \overline{B}$ are the closures of A, B.

Definition 2.1 Let $i \in \mathbb{I}_d$. A random vector $\mathbf{Z} \in \mathbb{R}^d$ is *multivariate regularly varying (MRV)* on $\mathbb{O}^{(i)}_d$ if there exist a regularly varying function $b_i \in \mathrm{RV}_{1/\alpha_i}$, $\alpha_i > 0$, and a non-null (Borel) measure μ_i which is finite on Borel sets bounded away from the set $\{\mathbf{z} \in \mathbb{R}^d_+ : z_{(i)} = 0\}$ such that

$$\lim_{t\to\infty} t\,\mathbb{P}\left[\frac{\mathbf{Z}}{b_i(t)} \in B\right] = \mu_i(B) \tag{2.2}$$

for all sets $B \in \mathcal{B}(\mathbb{O}^{(i)}_d)$ which are bounded away from $\{\mathbf{z} \in \mathbb{R}^d_+ : z_{(i)} = 0\}$ with $\mu_i(\partial B) = 0$. We write $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}^{(i)}_d)$, where some parameters may be dropped for convenience.

The limit measure μ_i defined in (2.2) turns out to be homogeneous of order $-\alpha_i$, i.e., $\mu_i(\lambda B) = \lambda^{-\alpha_i}\mu_i(B)$, $\forall\,\lambda > 0$. Moreover, if $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}^{(i)}_d)$, $\forall\, i \in \mathbb{I}_d$, then a direct conclusion from (2.1) is that

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d,$$

implying that as $i \in \mathbb{I}_d$ increases, the rate of decay of probabilities of tail sets in $\mathbb{O}_d^{(i)}$ either remains the same or becomes faster.

2.2 Regular variation under a Gaussian copula

Probabilities of tail subsets of $\mathbb{R}_+^d$ for heavy-tailed models using the popular Gaussian dependence structure have been discussed in detail in Das and Fasen-Hartmann [20]. Here, we concentrate on such models where the tails are asymptotically power-law (Pareto-like); this allows explicit computations and insights into this model. To fix notations, if Φ_Σ denotes the distribution function of a d-variate normal distribution with all marginal means zero, variances being one and positive semi-definite correlation matrix $\Sigma \in \mathbb{R}^{d\times d}$, and Φ denotes a standard normal distribution function, then let

$$C_\Sigma(u_1,\ldots,u_d) = \Phi_\Sigma\big(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_d)\big), \qquad 0 < u_1,\ldots,u_d < 1,$$

denote the Gaussian copula with correlation matrix Σ. Next, we define the following model analogous to the RVGC model defined in [20] and analyse its MRV behaviour.

Definition 2.2 An $\mathbb{R}^d$-valued random vector $\mathbf{Z} = (Z_1,\ldots,Z_d)^\top \sim F$ follows a *Pareto-tailed distribution with Gaussian copula* with index $\alpha > 0$, scaling parameter $\theta > 0$ and positive definite correlation matrix Σ if the following hold:

(i) The marginal distributions F_j of Z_j are continuous and strictly increasing with tail $\overline{F}_j(t) := 1 - F_j(t) \sim \theta t^{-\alpha}$, $\forall\, j \in \mathbb{I}_d$, for some $\theta, \alpha > 0$.

(ii) The joint distribution function F of $\mathbf{Z}$ is given by

$$F(\mathbf{z}) = C_\Sigma\big(F_1(z_1),\ldots,F_d(z_d)\big), \qquad \mathbf{z} = (z_1,\ldots,z_d)^\top \in \mathbb{R}^d,$$

where C_Σ is the Gaussian copula with correlation matrix $\Sigma \in \mathbb{R}^{d\times d}$. We write $\mathbf{Z} \in \text{P-GC}(\alpha,\theta,\Sigma)$, where some parameters may be dropped for convenience.

Remark 2.3 The above structure admits a wide variety of distributional behaviour.

(a) The Pareto, Lévy, Student-t and Fréchét distributions have a tail satisfying $\overline{F}_j(t) \sim \theta t^{-\alpha}$ (Embrechts et al. [27, Example 3.3.10], Nair et al. [50, Part I]), thus allowing a few popular heavy-tailed marginals to be chosen from.

(b) A special case here is when the correlation matrix $\Sigma = I_d$ and all marginals $Z_1,\ldots,Z_d$ are identically Pareto-distributed, thus covering the case with i.i.d. marginals as well.

(c) If the Gaussian dependence is defined by an *equicorrelation* matrix given by

$$\Sigma_\rho := \begin{pmatrix} 1 & \rho & \ldots & \rho \\ \rho & 1 & \ldots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \ldots & \ldots & 1 \end{pmatrix}$$

with $-\frac{1}{d-1} < \rho < 1$ (making Σ_ρ positive definite), we write $\mathbf{Z} \in \text{P-GC}(\alpha,\theta,\Sigma_\rho)$. This correlation matrix is the only choice if $d = 2$.

(d) If $\mathbf{Z} \in \text{P-GC}(\alpha, \theta, \Sigma)$, then any subvector $\mathbf{Z}_S$, $S \subseteq \mathbb{I}_d$, satisfies the property that $\mathbf{Z}_S \in \text{P-GC}(\alpha, \theta, \Sigma_S)$.

For the P-GC model, although we have only assumed a Pareto-like tail for the marginal distributions, it turns out that along with the Gaussian dependence, this suffices for the random vectors to admit multivariate regular variation (MRV) on the various subcones $\mathbb{O}_d^{(i)} \subset \mathbb{R}_+^d$; this was derived in further generality in Das and Fasen-Hartmann [20]. We recall and reformulate some of the related results. The following result is from Hashorva and Hüsler [34, Proposition 2.5 and Corollary 2.7] and is used to state and prove Proposition 2.5 below.

Lemma 2.4 *Suppose $\Sigma \in \mathbb{R}^{d\times d}$ is a positive definite correlation matrix. Then the quadratic programming problem*

$$\mathcal{P}_{\Sigma^{-1}} : \min_{\mathbf{z}\geq\mathbf{1}} \mathbf{z}^\top \Sigma^{-1}\mathbf{z}$$

has a unique solution $\mathbf{e}^* = \mathbf{e}^*(\Sigma) \in \mathbb{R}^d$ such that

$$\gamma := \gamma(\Sigma) := \min_{\mathbf{z}\geq\mathbf{1}} \mathbf{z}^\top \Sigma^{-1}\mathbf{z} = \mathbf{e}^{*\top}\Sigma^{-1}\mathbf{e}^* > 1.$$

Moreover, there exists a unique non-empty index set $I := I(\Sigma) \subseteq \{1, \ldots, d\} = \mathbb{I}_d$ with $J := J(\Sigma) := \mathbb{I}_d \setminus I$ such that the unique solution $\mathbf{e}^*$ is given by

$$\mathbf{e}_I^* = \mathbf{1}_I \qquad \textit{and} \qquad \mathbf{e}_J^* = -\big((\Sigma^{-1})_{JJ}\big)^{-1}(\Sigma^{-1})_{JI}\mathbf{1}_I = \Sigma_{JI}(\Sigma_I)^{-1}\mathbf{1}_I \geq \mathbf{1}_J,$$

and $\mathbf{1}_I\Sigma_I^{-1}\mathbf{1}_I = \mathbf{e}^{*\top}\Sigma^{-1}\mathbf{e}^* = \gamma > 1$ as well as $\mathbf{z}^\top\Sigma^{-1}\mathbf{e}^* = \mathbf{z}_I^\top\Sigma_I^{-1}\mathbf{1}_I$ for any $\mathbf{z} \in \mathbb{R}^d$. Also, defining $h_i := h_i(\Sigma) := e_i^\top \Sigma_I^{-1}\mathbf{1}_I > 0$ for $i \in \mathbb{I}_d$, we have $h_i > 0$ for $i \in I$. If $\Sigma^{-1}\mathbf{1} \geq \mathbf{0}$, then $2 \leq |I| \leq d$ and $\mathbf{e}^* = \mathbf{1}$.

With the notations and definitions of Lemma 2.4, we can state the main result on MRV for P-GC models which summarises Das and Fasen-Hartmann [20, Theorems 3.1 and 3.4].

Proposition 2.5 *Let $\mathbf{Z} \in \text{P-GC}(\alpha, \theta, \Sigma)$, where Σ is positive definite. Fix a non-empty set $S \subseteq \mathbb{I}_d$ with $|S| \geq 2$.*

(i) *Let $\gamma_S := \gamma(\Sigma_S)$, $I_S := I(\Sigma_S)$, $\mathbf{e}_S^* := \mathbf{e}^*(\Sigma_S)$ and $h_s^S := h_s(\Sigma_S)$, $s \in I_S$, be defined as in Lemma 2.4.*

(ii) *Let $J_S := \mathbb{I}_d \setminus I_S$. Now define $\mathbf{Y}_{J_S} \sim \mathcal{N}(\mathbf{0}_{J_S}, \Sigma_{J_S} - \Sigma_{J_S I_S}(\Sigma_{I_S})^{-1}\Sigma_{I_S J_S})$ if $J_S \neq \emptyset$, and $\mathbf{Y}_{J_S} = \mathbf{0}_{J_S}$ if $J_S = \emptyset$.*

(iii) *If $J_S \neq \emptyset$, define $\mathbf{l}_S := \lim_{t\to\infty} t(\mathbf{1}_{J_S} - \mathbf{e}_{J_S}^*)$, a vector in $\mathbb{R}^{|J_S|}$ with components either 0 or $-\infty$, and if $J_S = \emptyset$, define $l_S := \mathbf{0}_S$.*

Let $\Gamma_{\mathbf{z}_S} = \{\mathbf{v} \in \mathbb{R}_+^d : v_s > z_s, \forall s \in S\}$ for $\mathbf{z}_S = (z_s)_{s\in S}$ with $z_s > 0, \forall s \in S$. Then as $t \to \infty$,

$$\mathbb{P}[\mathbf{Z} \in t\Gamma_{\mathbf{z}_S}] = \big(1 + o(1)\big)\Upsilon_S(2\pi)^{\frac{\gamma_S}{2}}\theta^{\gamma_S}t^{-\alpha\gamma_S}(2\alpha\log t)^{\frac{\gamma_S - |I_S|}{2}} \prod_{s\in I_S} z_s^{-\alpha h_s^S}$$

with $\Upsilon_S = (2\pi)^{-|I_S|/2}|\Sigma_{I_S}|^{-1/2}\prod_{s\in S}(h_s^S)^{-1}\mathbb{P}[\mathbf{Y}_{J_S} \geq \mathbf{1}_S]$. *Moreover, the following hold*:

(a) *We have* $\mathbf{Z} \in \mathrm{MRV}(\alpha_1, b_1, \mu_1, \mathbb{O}_d^{(1)})$ *with* $\alpha_1 = \alpha$, $b_1(t) = (\theta t)^{1/\alpha}$, $t > 0$, *and* $\mu_1([\mathbf{0}, \mathbf{z}]^c) = \sum_{j=1}^d z_j^{-\alpha}$, $\forall\, \mathbf{z} \in \mathbb{R}_+^d$.

(b) *Let* $2 \leq i \leq d$. *Define*

$$\mathcal{S}_i := \left\{S \subseteq \mathbb{I}_d : |S| \geq i, \mathbf{1}_{I_S}(\Sigma_{I_S})^{-1}\mathbf{1}_{I_S} = \min_{\widetilde{S}\subseteq \mathbb{I}_d, |\widetilde{S}|\geq i} \mathbf{1}_{I_{\widetilde{S}}}(\Sigma_{I_{\widetilde{S}}})^{-1}\mathbf{1}_{I_{\widetilde{S}}}\right\},$$

$$I_i := \arg\min_{S\in\mathcal{S}_i} |I_S|,$$

where I_i *is not necessarily unique. Then* $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$, *where*

$$\gamma_i = \gamma(\Sigma_{I_i}) = \mathbf{1}_{I_i}^\top(\Sigma_{I_i})^{-1}\mathbf{1}_{I_i} = \min_{S\subseteq\mathbb{I}_d, |S|\geq i}\ \min_{\mathbf{z}_S\geq \mathbf{1}_S} \mathbf{z}_S^\top(\Sigma_S)^{-1}\mathbf{z}_S,$$

$$\alpha_i = \alpha\gamma_i, \qquad b_i^{\leftarrow}(t) = (2\pi)^{-\frac{\gamma_i}{2}}\theta^{-\gamma_i}t^{\alpha\gamma_i}(2\alpha\log t)^{\frac{|I_i|-\gamma_i}{2}},$$

$$\mu_i(\Gamma_{\mathbf{z}_S}) = \begin{cases}\Upsilon_S\prod_{s\in I_S} z_s^{-\alpha h_s^S}, & \text{if } S \in \mathcal{S}_i \text{ and } |I_S| = |I_i|,\\ 0, & \text{otherwise.}\end{cases} \tag{2.3}$$

First, we note that in this model, the parameters or indices of regular variation in each subspace $\mathbb{O}_d^{(i)}$ are indeed all different. The proof of this result and the following results of this section are given in Appendix A.

Proposition 2.6 *Let* $\mathbf{Z} \in \text{P-GC}(\alpha, \theta, \Sigma)$, *where* Σ *is positive definite and the assumptions and notations of Proposition 2.5 hold. Then with* $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$, $\forall\, i \in \mathbb{I}_d$, *we have*

$$\alpha_1 < \alpha_2 < \cdots < \alpha_d,$$

and in particular $b_i(t)/b_{i+1}(t) \to \infty$ *as* $t \to \infty$ *for* $i = 1, \ldots, d-1$.

Thus the rate of convergence of tail sets in $\mathbb{O}_d^{(i)}$ for different $i \in \mathbb{I}_d$ are different. Next, we investigate the support of the limit measure μ_i.

Example 2.7 For any positive definite correlation matrix $\Sigma = (\rho_{j\ell})$, we have

$$\mathbf{1}_{\{j,\ell\}}^\top(\Sigma_{\{j,\ell\}})^{-1}\mathbf{1}_{\{j,\ell\}} = \frac{2}{1+\rho_{j\ell}}, \qquad j \neq \ell \in \mathbb{I}_d,$$

and the right-hand side above does not have the same value for all $j \neq \ell \in \mathbb{I}_d$ unless Σ is an equicorrelation matrix. Hence if Σ is not an equicorrelation matrix, using the notation from Proposition 2.5, there exists a set $S = \{j^*, \ell^*\} \notin \mathcal{S}_2$, and

$$\mu_2(\{\mathbf{z} \in \mathbb{R}_+^d : z_j > 0, \forall\, j \in S, z_\ell = 0, \forall\, \ell \in \mathbb{I}_d\backslash S\}) = 0.$$

In the following, we present a general result characterising the support of the limit measure on each of the Euclidean subcones for the P-GC model.

Proposition 2.8 *Let the assumptions and notations of Proposition 2.5 hold and fix some $i \in \mathbb{I}_d$.*

(a) *Suppose that for all $S \subseteq \mathbb{I}_d$ with $|S| = i$, we have $\Sigma_S^{-1}\mathbf{1}_S > \mathbf{0}_S$ and $\mathbf{1}_S^\top \Sigma_S^{-1}\mathbf{1}_S = \gamma_i$. Then the support of the limit measure μ_i on $\mathbb{O}_d^{(i)}$ defined in (2.3) is*

$$\bigcup_{S\subseteq\mathbb{I}_d, |S|=i} \{\mathbf{z} \in \mathbb{R}_+^d : z_j > 0, \forall\, j \in S, z_\ell = 0, \forall\, \ell \in \mathbb{I}_d\backslash S\}.$$

(b) *Suppose that for some $S \subseteq \mathbb{I}_d$ with $|S| = i$, $\Sigma_S^{-1}\mathbf{1}_S > \mathbf{0}_S$ and $\gamma_i \neq \mathbf{1}_S^\top \Sigma_S^{-1}\mathbf{1}_S$. Then*

$$\mu_i(\{\mathbf{z} \in \mathbb{R}_+^d : z_j > 0, \forall\, j \in S, z_\ell = 0, \forall\, \ell \in \mathbb{I}_d\backslash S\}) = 0.$$

2.3 Regular variation under a Marshall-Olkin copula

Another important example of multivariate tail risk modelling is the Marshall–Olkin copula. The Marshall–Olkin distribution is often used in reliability theory to capture the dependence between the failure of subsystems in an entire system and hence is a candidate model for measuring systemic risk. We consider a particular type of Marshall–Olkin survival copula; cf. Lin and Li [43] and Das and Fasen-Hartmann [18, Example 2.14]. Assume that for every non-empty set $S \subseteq \mathbb{I}_d$, there exists a parameter $\lambda_S > 0$ and $\Lambda := \{\lambda_S : \emptyset \neq S \subseteq \mathbb{I}_d\}$. Then the generalised Marshall–Olkin survival copula with rate parameter set Λ is given by

$$\widehat{C}_\Lambda^{\mathrm{MO}}(u_1, \ldots, u_d) = \prod_{i=1}^{d} \prod_{|S|=i} \bigwedge_{j\in S} u_j^{\eta_j^S}, \qquad 0 < u_j < 1,$$

where

$$\eta_j^S = \frac{\lambda_S}{\sum_{J\supseteq\{j\}} \lambda_J}, \qquad j \in S \subseteq \mathbb{I}_d.$$

Similarly to the P-GC model, we define a Pareto–Marshall–Olkin copula (P-MOC) model next.

Definition 2.9 An $\mathbb{R}^d$-valued random vector $\mathbf{Z} = (Z_1, \ldots, Z_d)^\top \sim F$ follows a *Pareto-tailed distribution with Marshall–Olkin copula* with index $\alpha > 0$, scaling parameter $\theta > 0$ and rate parameters $\Lambda = \{\lambda_S : \emptyset \neq S \subseteq \mathbb{I}_d\}$ if the following hold:

(i) The marginal distributions F_j of Z_j are continuous and strictly increasing with tail $\overline{F}_j(t) := 1 - F_j(t) \sim \theta t^{-\alpha}, \forall\, j \in \mathbb{I}_d$, for some $\theta, \alpha > 0$.

(ii) The joint survival distribution function $\overline{F}$ of $\mathbf{Z}$ is given by

$$\overline{F}(\mathbf{z}) = \mathbb{P}[Z_1 > z_1 \ldots, Z_d > z_d] = \widehat{C}_\Lambda^{\mathrm{MO}}\big(\overline{F}_1(z_1), \ldots, \overline{F}_d(z_d)\big)$$

for $\mathbf{z} = (z_1, \dots, z_d)^\top \in \mathbb{R}^d$, where $\widehat{C}_\Lambda^{\mathrm{MO}}$ denotes the Marshall–Olkin survival copula with rate parameter set Λ. We write $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \Lambda)$, where some parameters may be dropped for convenience.

It is also possible to show that $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \Lambda)$ with any feasible rate parameter set Λ is multivariate regularly varying on the cones $\mathbb{O}_d^{(i)}$, but finding the exact parameters and limit measures requires some involved combinatorial computations; hence we concentrate on two specific choices of Λ, cf. Das and Fasen-Hartmann [18, Example 2.14]:

(a) Equal parameter for all sets. Here, $\lambda_S = \lambda$ for all non-empty sets $S \subseteq \mathbb{I}_d$, where $\lambda > 0$. We denote this model by P-MOC$(\alpha, \theta, \lambda^=)$.

(b) Parameters proportional to the cardinality of the sets. Here, $\lambda_S = |S|\lambda$ for all non-empty sets $S \subseteq \mathbb{I}_d$, where $\lambda > 0$. We denote this model by P-MOC$(\alpha, \theta, \lambda^\propto)$.

Note that in both cases, the Marshall–Olkin copula and hence the P-MOC model do not depend on the value of λ.

Remark 2.10 If $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^=)$, then any subvector $\mathbf{Z}_S$ with $S \subseteq \mathbb{I}_d$ also satisfies $\mathbf{Z}_S \in$ P-MOC$(\alpha, \theta, \lambda^=)$, implying a nested structure across dimensions. However, in the case of a Marshall–Olkin copula with proportional parameters where $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^\propto)$, such a nested property does not hold anymore.

In each of these cases, we can explicitly compute all the relevant parameters of the multivariate regular variation, and in fact, the limit measures have positive mass on all feasible support regions in these cases. The result given next is adapted from Das and Fasen-Hartmann [18, Example 2.14] and characterises multivariate regular variation for the P-MOC models for the choices of *equal rate parameters* and *proportional rate parameters*. These are also known as Caudras–Augé copulas [15] and have been used in Lévy frailty models for survival analysis.

Proposition 2.11 *The following statements hold*:

(i) *Let* $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^=)$. *Then* $\mathbf{Z} \in$ MRV$(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$ *for* $i \in \mathbb{I}_d$, *where*

$$\alpha_i = (2 - 2^{-(i-1)})\alpha, \qquad b_i(t) = \theta^{\frac{1}{\alpha}} t^{\frac{1}{\alpha_i}},$$

and for sets $\Gamma_{\mathbf{z}_S}^{(d)} = \{\mathbf{v} \in \mathbb{R}_+^d : v_s > z_s, \forall s \in S\}$ *with* $z_s > 0, \forall s \in S \subseteq \mathbb{I}_d$, $|S| \geq i$, *we have*

$$\mu_i(\Gamma_{\mathbf{z}_S}^{(d)}) = \begin{cases} \prod_{j=1}^{i} z_{(j)}^{-\alpha 2^{-(j-1)}}, & \text{if } |S| = i, \\ 0, & \text{otherwise}, \end{cases}$$

where $z_{(1)} \geq \cdots \geq z_{(i)}$ *denote the decreasing order statistic of* $(z_j)_{j \in S}$.

(ii) *Let* $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^\propto)$. *Then* $\mathbf{Z} \in$ MRV$(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$ *for* $i \in \mathbb{I}_d$, *where*

$$\alpha_i = \frac{\alpha}{d+1}\left(2d - \frac{d-i}{2^{i-1}}\right), \qquad b_i(t) = \theta^{\frac{1}{\alpha}} t^{\frac{1}{\alpha_i}},$$

and for sets $\Gamma^{(d)}_{\mathbf{z}_S} = \{\mathbf{v} \in \mathbb{R}^d_+ : v_s > z_s, \forall s \in S\}$ *with* $z_s > 0, \forall s \in S \subseteq \mathbb{I}_d,\ |S| \geq i$, *we have*

$$\mu_i(\Gamma^{(d)}_{\mathbf{z}_S}) = \begin{cases} \prod_{j=1}^{i} z_{(j)}^{-\alpha(1-\frac{j-1}{d+1})2^{-(j-1)}}, & \text{if } |S| = i, \\ 0, & \text{otherwise}. \end{cases}$$

In both cases, we have $\alpha_1 < \alpha_2 < \cdots < \alpha_d$ *and in particular* $b_i(t)/b_{i+1}(t) \to \infty$ *as* $t \to \infty$ *for* $i = 1, \ldots, d-1$.

It is easy to check that if $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \Lambda)$ with the rate parameters either *all equal* or *proportional* (to the size of the sets) as in Proposition 2.11, and if the marginals are identically distributed, then $\mathbf{Z}$ is an exchangeable random vector (Durrett [25, Example 4.7.8]). This implies that in these two cases, μ_i actually puts positive mass on

$$\{\mathbf{z} \in \mathbb{R}^d_+ : z_j > 0, \forall\, j \in S, z_\ell = 0, \forall\, \ell \in \mathbb{I}_d \backslash S\}$$

for all $S \subseteq \mathbb{I}_d$ with $|S| = i$ for any fixed $i \in \mathbb{I}_d$.

3 Measuring CoVaR

Risk contagion is often assessed using conditional measures of risk, and in this regard, CoVaR has turned out to be both reasonable and popular; cf. Sect. 1 and also see Kley et al. [41], Bianchi et al. [9] for computations of CoVaR in various setups. By its definition, CoVaR measures the effect of severe stress of one risk factor, say Y_2, on the risk behaviour of another factor, say Y_1. To facilitate the computation of CoVaR in the bipartite network setup described previously, we first provide a result on its asymptotic behaviour for a bivariate random vector assuming appropriate multivariate tail behaviour, i.e., multivariate regular variation. Proofs of the results in this section are given in Appendix B.

Theorem 3.1 *Let* $\mathbf{Y} = (Y_1, Y_2)^\top$ *be a bivariate random vector with marginals* F_1 *and* F_2, *respectively. Suppose* $\mathbf{Y}' := (Y_1, F_1^{\leftarrow} \circ F_2(Y_2))^\top \in \mathrm{MRV}(\alpha_i, b_i, \mu'_i, \mathbb{O}_2^{(i)})$ *for* $i = 1, 2$. *Define the functions* $h, h_\gamma : (0, \infty) \to (0, \infty)$ *as*

$$h(y) := \mu'_2\big((y, \infty) \times (1, \infty)\big),$$
$$h_\gamma(y) := \gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big)\mathbb{P}[Y_1 > y\,\mathrm{VaR}_\gamma(Y_1) | Y_2 > \mathrm{VaR}_\gamma(Y_1)]$$

for fixed $0 < \gamma < 1$. *By the MRV assumptions, we have* $h_\gamma(y) \to h(y)$ *as* $\gamma \downarrow 0$ *for* $y \in (0, \infty)$ *and* h *is decreasing with* $\lim_{y \uparrow \infty} h(y) = 0$. *Further, assume the following*:

(i) *Let* $\lim_{y \downarrow 0} h(y) = r \in (0, \infty]$ *and for some* $\ell \geq 0$, $h : (\ell, \infty) \to (0, r)$ *is strictly decreasing and continuous with inverse* h^{-1}.

(ii) *Let* $g : (0, 1) \to (0, \infty)$ *be a measurable function with*

$$g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) \in (0, r).$$

Now let one of the following three conditions hold:

(a)
$$0 < \liminf_{\gamma\downarrow 0} g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) \\ \leq \limsup_{\gamma\downarrow 0} g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) < r; \tag{3.1}$$

(b)
$$\lim_{\gamma\downarrow 0} g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) = 0$$

and $h_\gamma^{\leftarrow}(\upsilon)/h^{-1}(\upsilon) \to 1$ *uniformly on* $(0, R]$ *as* $\gamma \downarrow 0$, *for some* $0 < R < r$;

(c)
$$\lim_{\gamma\downarrow 0} g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) = r$$

and $h_\gamma^{\leftarrow}(\upsilon)/h^{-1}(\upsilon) \to 1$ *uniformly on* $[L, r)$ *as* $\gamma \downarrow 0$, *for some* $0 < L < r$.

Then for any $0 < \upsilon < 1$,

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2) \sim \mathrm{VaR}_\gamma(Y_1)h^{-1}\Big(\upsilon g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big)\Big), \qquad \gamma \downarrow 0.$$

Remark 3.2 A few remarks may aid in understanding the assumptions and consequences of the above result.

(a) If F_2 is continuous, then $F_2(Y_2)$ is uniform on $(0, 1)$. Hence the tail behaviour of Y_2 has no influence on the asymptotic behaviour of $\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2)$, as is expected.

(b) If $\overline{F}_1 \in \mathrm{RV}_{-\alpha}$, the condition $(Y_1, F_1^{\leftarrow} \circ F_2(Y_2))^\top \in \mathrm{MRV}(\alpha_i, b_i, \mu_i', \mathbb{O}_2^{(i)})$, $i = 1, 2$, can be formulated as a condition on the survival copula $\widehat{C}$ of $(Y_1, Y_2)^\top$. Necessary and sufficient conditions on $\widehat{C}$ are given in Das and Fasen-Hartmann [17, Theorems 3.11 and 3.12].

(c) Suppose $(Y_1, Y_2)^\top \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_2^{(i)})$ for $i = 1, 2$, and that for some $K > 0$, we have $\mathbb{P}[Y_2 > t] \sim K\mathbb{P}[Y_1 > t]$ as $t \to \infty$. Then $\mathbf{Y}' := (Y_1, F_1^{\leftarrow} \circ F_2(Y_2))^\top$ is in $\mathrm{MRV}(\alpha_i, b_i, \mu_i', \mathbb{O}_2^{(i)})$ for $i = 1, 2$, and

$$h(y) = \mu_2'\big((y, \infty) \times (1, \infty)\big) = \mu_2\big((y, \infty) \times (K^{1/\alpha}, \infty)\big). \tag{3.2}$$

(d) The assumption $0 < \upsilon < 1$ is only sufficient and not necessary. Indeed, if $r = \infty$ and $g(\gamma) \to 0$ as $\gamma \downarrow 0$ (which is the standard case), then for any $\upsilon \in (0, \infty)$, we have $\upsilon g(\gamma) \in (0, 1)$ for small γ, and thus $\upsilon \in (0, \infty)$ is allowed as well.

(e) In general, we cannot guarantee that as $\gamma \downarrow 0$, we have $h_\gamma^{\leftarrow}(\upsilon)/h^{-1}(\upsilon)$ converging uniformly to 1 on bounded but non-compact intervals. However, such a uniform convergence does hold for compact intervals; cf. Lemma 3.3. The need for assuming uniform convergence on non-compact intervals becomes evident from the proof of Proposition 3.12(a)(i), thus providing a justification for the additional assumptions in Theorem 3.1(b,c).

Lemma 3.3 *Let the assumptions of Theorem 3.1 hold. Then for any closed interval* $[a_1, a_2] \subseteq (0, r)$, *we have*

$$\sup_{\upsilon\in[a_1,a_2]} \left| \frac{h_\gamma^{\leftarrow}(\upsilon)}{h^{-1}(\upsilon)} - 1 \right| \longrightarrow 0, \qquad \gamma \downarrow 0.$$

Example 3.4 Prior to discussing further implications of Theorem 3.1, it is instructive to note the behaviour of CoVaR when $Y_1 = Y_2$ a.s. and when Y_1 and Y_2 are independent. For convenience, assume that Y_1, Y_2 are identically Pareto(α_1)-distributed and $g : (0, 1) \to (0, 1)$ is a measurable function. Using the notations in Theorem 3.1, we have the following:

(a) If $Y_1 = Y_2$ a.s., then $(\alpha_1, b_1, \mu_1) = (\alpha_2, b_2, \mu_2)$ and

$$h_\gamma(y) = h(y) = \max(y, 1)^{-\alpha_1}.$$

For any $0 < \upsilon < 1$, we have

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2) = \big(\upsilon g(\gamma)\big)^{-\frac{1}{\alpha_1}} \mathrm{VaR}_\gamma(Y_1) = \big(\upsilon\gamma g(\gamma)\big)^{-\frac{1}{\alpha_1}}, \qquad 0 < \gamma < 1.$$

In particular, if $g(\gamma) = \gamma$, then

$$\mathrm{CoVaR}_{\upsilon\gamma|\gamma}(Y_1|Y_2) = \mathrm{VaR}_{\upsilon\gamma^2}(Y_1), \qquad 0 < \gamma < 1,$$

and if $g(\gamma) \equiv 1$, then

$$\mathrm{CoVaR}_{\upsilon|\gamma}(Y_1|Y_2) = \mathrm{VaR}_{\upsilon\gamma}(Y_1) = \upsilon^{-1/\alpha_1} \mathrm{VaR}_\gamma(Y_1), \qquad 0 < \gamma < 1. \tag{3.3}$$

(b) If Y_1, Y_2 are independent, then $\alpha_2 = 2\alpha_1$ and the canonical choice of taking $b_2^{\leftarrow}(t) = (b_1^{\leftarrow}(t))^2$ results in $h_\gamma(y) = h(y) = y^{-\alpha_1}$. Hence for $0 < \upsilon < 1$,

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2) = \mathrm{VaR}_{\upsilon g(\gamma)}(Y_1), \qquad 0 < \gamma < 1.$$

In particular, if $g(\gamma) = \gamma$, then

$$\mathrm{CoVaR}_{\upsilon\gamma|\gamma}(Y_1|Y_2) = \mathrm{VaR}_{\upsilon\gamma}(Y_1) = \upsilon^{-1/\alpha_1} \mathrm{VaR}_\gamma(Y_1), \qquad 0 < \gamma < 1. \tag{3.4}$$

Comparing (3.3) and (3.4), we observe that for different levels of strength of dependence between Y_1 and Y_2, there exist different choices of $g(\gamma)$ allowing the asymptotic behaviour

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2) \sim \upsilon^{-1/\alpha_1} \mathrm{VaR}_\gamma(Y_1), \qquad \gamma \downarrow 0.$$

Such a characterisation may be obtained for a variety of dependence behaviours, and hence we define the following quantity.

Definition 3.5 Let $\mathbf{Y} = (Y_1, Y_2)^\top$ be a bivariate random vector. Suppose some measurable function $g : (0, 1) \to (0, 1)$ with $g(t^{-1}) \in \mathrm{RV}_{-\beta}$, $\beta \geq 0$, satisfies

$$\lim_{\gamma\downarrow 0} \frac{\mathrm{CoVaR}_{g(\gamma)|\gamma}(Y_1|Y_2)}{\mathrm{VaR}_\gamma(Y_1)} = c > 0 \tag{3.5}$$

for some constant c. If the index β is unique for all regularly varying functions g satisfying (3.5), then we call β^{-1} the *extreme CoVaR index* of Y_1 given Y_2, or *ECI*$(Y_1|Y_2)$ in short.

Remark 3.6 The index $\mathrm{ECI}(Y_1|Y_2)$ provides a value to assess the strength of risk contagion from Y_2 to Y_1 and may take any value between 0 and ∞.

(a) In Example 3.4, we observed that when Y_1 and Y_2 are strongly dependent and $g(\gamma) = \upsilon\gamma^0 \equiv \upsilon \in (0, 1)$, that $\mathrm{CoVaR}_{g(\gamma)|\gamma}(Y_1|Y_2) \sim c\,\mathrm{VaR}_\gamma(Y_1)$ for some $c > 0$ as $\gamma \downarrow 0$, and hence the ECI is $1/0 = \infty$. On the other hand, if Y_1 and Y_2 are independent and $g(\gamma) = \gamma$, we have $\mathrm{CoVaR}_{g(\gamma)|\gamma}(Y_1|Y_2) = \mathrm{VaR}_\gamma(Y_1)$ and hence $\mathrm{ECI}(Y_1|Y_2) = 1$.

(b) The ECI of Y_1 given Y_2 provides a measure of risk contagion between Y_1 and Y_2, in particular for systemic risks. Given Y_2 has a value larger than its VaR at level γ, ECI allows us to compute the level $g(\gamma) = \ell(\gamma)\gamma^{1/\mathrm{ECI}}$ for Y_1 which makes $\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2)$ to be of the same order as $\mathrm{VaR}_\gamma(Y_1)$, where ℓ is some slowly varying function. Lower values of ECI reflect lower values of $\mathrm{CoVaR}_{\gamma|\gamma}(Y_1|Y_2)$.

If $\mathbf{Y}' := (Y_1, F_1^{\leftarrow} \circ F_2(Y_2))^\top \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_2^{(i)})$ for $i = 1, 2$ satisfies the assumptions of Theorem 3.1, the choice of a function g satisfying (3.1) must also satisfy $g(t^{-1}) \in \mathrm{RV}_{-\beta}$ with $\beta = \alpha_2/\alpha_1 - 1$, and hence (3.5) is satisfied. Let us summarise this result.

Proposition 3.7 *Let* $\mathbf{Y} = (Y_1, Y_2)^\top$ *be a bivariate random vector with marginals* F_1 *and* F_2*, respectively. Suppose* $\mathbf{Y}' := (Y_1, F_1^{\leftarrow} \circ F_2(Y_2))^\top \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_2^{(i)})$, $i = 1, 2$ *satisfies the assumptions of Theorem* 3.1*. Then*

$$\mathrm{ECI}(Y_1|Y_2) = \frac{\alpha_1}{\alpha_2 - \alpha_1},$$

where $\alpha_1/0 := \infty$.

Remark 3.8 Note that for $(Y_1, Y_2)^\top \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_2^{(i)})$, $i = 1, 2$, the index α_2 is often called the parameter of hidden regular variation in bivariate regularly varying models (Resnick [56], Maulik and Resnick [49]) and is closely related to the coefficient η of tail-dependence, which is defined in Ledford and Tawn [42] under the assumption $\alpha_1 = 1$. If $\alpha_1 = 1$ and Y_1, Y_2 are tail-equivalent, then we have the relation $\eta = 1/\alpha_2 = \mathrm{ECI}(Y_1|Y_2)/(1 + \mathrm{ECI}(Y_1|Y_2))$. An important difference is as well that we only require the MRV of $(Y_1, F_1^{\leftarrow} \circ F_2(Y_2))^\top$ on $\mathbb{O}_2^{(i)}$ for $i = 1, 2$ and not of $(Y_1, Y_2)^\top$.

Remark 3.9 Suppose $(Y_1^\perp, Y_2^\perp)^\top$ is a bivariate random vector with independent components, but with the same marginal distribution as $(Y_1, Y_2)^\top$, which satisfies the assumptions of Theorem 3.1. Then by the definition of the ECI, the function g in Definition 3.5 and Remark 3.6 (a), we have

$$\mathrm{CoVaR}_{g(\gamma)|\gamma}(Y_1|Y_2) \sim c\ \mathrm{VaR}_\gamma(Y_1) = c\ \mathrm{CoVaR}_{\gamma|\gamma}(Y_1^\perp|Y_2^\perp), \qquad \gamma \downarrow 0,$$

for some $c > 0$. Thus the value of $g(\gamma)$ helps in assessing the probability or confidence with which $\mathrm{CoVaR}(Y_1|Y_2)$ will have the same asymptotic behaviour as the CoVaR of the independent model, i.e., $\mathrm{CoVaR}(Y_1^{\perp}|Y_2^{\perp})$. In other words, higher values of ECI (and hence, $g(\gamma)$) imply a higher risk contagion of Y_2 on Y_1; by definition, this also means that with probability $1-g(\gamma)$, the value of $\mathrm{CoVaR}_{g(\gamma)|\gamma}(Y_1|Y_2)$ is sufficient to cover the losses of Y_1 if the value of Y_2 is already above $\mathrm{VaR}_\gamma(Y_2)$. Clearly, larger values of α_2 result in weaker tail-dependence between Y_1 and Y_2 and in a smaller ECI.

A similar phenomenon may also be observed for the popular systemic risk measure marginal-mean-excess (MME) defined as

$$\mathrm{MME}_\gamma(Y_1|Y_2) := \mathbb{E}\big[\big(Y_1 - \mathrm{VaR}_\gamma(Y_2)\big)^+\big|Y_2 > \mathrm{VaR}_\gamma(Y_2)\big].$$

We know from Das and Fasen-Hartmann [16, 17] that under some mild assumptions, for some $c_1, c_2 > 0$,

$$g(\gamma)^{-1}\,\mathrm{MME}_\gamma(Y_1|Y_2) \sim c_1\,\mathrm{VaR}_\gamma(Y_1) \sim c_2\gamma^{-1}\,\mathrm{MME}_\gamma(Y_1^{\perp}|Y_2^{\perp}), \qquad \gamma \downarrow 0.$$

We can check that for example the P-GC and the P-MOC model satisfy these assumptions. To summarise, under some regularity conditions, for increasing values of ECI and hence g, the MME increases as well to attain the same confidence level $1-\gamma$ as in the independent model, implying higher risk contagion of Y_2 on Y_1. In particular, a conclusion of the asymptotic behaviour of CoVaR and MME is that although the underlying models exhibit asymptotic independence, there is still a certain amount of dependence on the tails which will have an influence on risk contagion, and this strength of tail-dependence is measured using the function g and finally reflected in the ECI.

3.1 Measuring CoVaR under different model assumptions

In this section, we show the direct consequences of Theorem 3.1 for various underlying distributions discussed in this paper, in particular asymptotically dependent, Gaussian copula and Marshall–Olkin copula models. These models provide a flavour of expected results, although computations for complex networks are more complicated, as we shall see in Sect. 4.

First, we consider the asymptotically dependent case, where $\alpha_1 = \alpha_2$ and the result is a direct consequence of Theorem 3.1 and the definition of the function h in (3.2).

Proposition 3.10 *Let $\mathbf{Y} = (Y_1, Y_2)^\top \in \mathbb{R}^2$ be a bivariate tail-equivalent random vector with*

$$\mathbf{Y} \in \mathrm{MRV}(\alpha, b, \mu, \mathbb{O}_2^{(1)}) \cap \mathrm{MRV}(\alpha, b, \mu, \mathbb{O}_2^{(2)})$$

and $h(y) = \mu((y,\infty) \times (K^{1/\alpha}, \infty))$ with

$$K = \frac{\mu(\mathbb{R}_+ \times (1,\infty))}{\mu((1,\infty)\times\mathbb{R}_+)} \qquad \textit{and} \qquad c = \mu\big((1,\infty)\times\mathbb{R}_+\big).$$

Suppose $h : (\ell, \infty) \to (0, r)$ *is strictly decreasing and continuous for some* $r, \ell > 0$. *Then for* $0 < \upsilon < \min\{r/c, 1\}$, *we have*

$$\mathrm{CoVaR}_{\upsilon|\gamma}(Y_1|Y_2) \sim h^{-1}(\upsilon c)\,\mathrm{VaR}_\gamma(Y_1), \qquad \gamma \downarrow 0.$$

Moreover, $\mathrm{ECI}(Y_1|Y_2) = \infty$.

For the bipartite network model of Sect. 1 where $\mathbf{Z}$ has asymptotically dependent pairs, we obtain directly the following result, some restricted versions of which have been shown in Kley et al. [40, 41].

Example 3.11 Here we consider a bipartite network with asymptotically dependent objects. Let $\mathbf{Z} \in \mathbb{R}_+^d$ be a random vector, $\mathbf{A} \in \mathbb{R}_+^{2\times d}$ a random matrix and $\mathbf{X} = \mathbf{AZ}$. Now also assume $\mathbf{Z} \in \mathrm{MRV}(\alpha, b, \mu, \mathbb{O}_d^{(1)}) \cap \mathrm{MRV}(\alpha, b, \mu, \mathbb{O}_d^{(2)})$ and $\mathbf{Z}$ has completely tail-equivalent marginals. Then due to Basrak et al. [7, Proposition A.1], which generalises Breiman's theorem to the multivariate setup, we have

$$\mathbf{X} = \mathbf{AZ} \in \mathrm{MRV}(\alpha, \overline{\mu}, \mathbb{O}_2^{(1)}) \cap \mathrm{MRV}(\alpha, \overline{\mu}, \mathbb{O}_2^{(2)})$$

with $\overline{\mu}(\cdot) = \mathbb{E}[\mu(\mathbf{A}^{-1}(\cdot))]$. Define $h(y) = \overline{\mu}((y, \infty) \times (K^{1/\alpha}, \infty))$, where

$$K = \frac{\overline{\mu}(\mathbb{R}_+ \times (1, \infty))}{\overline{\mu}((1, \infty) \times \mathbb{R}_+)} \qquad \text{and} \qquad c = \overline{\mu}\big((1, \infty) \times \mathbb{R}_+\big).$$

Suppose $h : (\ell, \infty) \to (0, r)$ is strictly decreasing and continuous for some $r, \ell > 0$. Then a conclusion of Proposition 3.10 is that for $0 < \upsilon < \min\{r/c, 1\}$,

$$\mathrm{CoVaR}_{\upsilon|\gamma}(X_1|X_2) \sim h^{-1}(\upsilon c)\,\mathrm{VaR}_\gamma(X_1), \qquad \gamma \downarrow 0.$$

Moreover, $\mathrm{ECI}(X_1|X_2) = \infty$.

In Kley et al. [40, 41], the authors investigate the asymptotically co-monotone case $\mu([0, \mathbf{x}]^c) = \max_{j=1,\ldots,d}(K_j x_j^{-\alpha})$ for $\mathbf{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}_+^d$ with positive constants $K_1, \ldots, K_d > 0$. Our result turns out to be more general.

In the next two results, we investigate the asymptotically independent case with $\alpha_1 < \alpha_2$, where the dependence is modelled by a Marshall–Olkin copula (Proposition 3.12) and a Gaussian copula (Proposition 3.14), respectively.

Proposition 3.12 *Let* $\mathbf{Y} = (Y_1, Y_2)^\top \in \mathbb{R}^2$ *be a random vector and* $0 < \upsilon < 1$.

(a) *Let* $\mathbf{Y} \in \text{P-MOC}(\alpha, \theta, \lambda^{=})$ *for* $\lambda > 0$. *Then* $\mathrm{ECI}(Y_1|Y_2) = 2$, *and the following statements hold*:

(i) *Suppose either*

$$\beta = 1/2$$

or

$$\beta > 1/2 \quad \text{and} \quad \frac{\upsilon^{1/\alpha}\overline{F}_1^{\leftarrow}(\upsilon\gamma)}{\overline{F}_1^{\leftarrow}(\gamma)} \text{ converges uniformly on } (0, 1] \text{ to } 1 \text{ as } \gamma \downarrow 0.$$

Then

$$\mathrm{CoVaR}_{\upsilon\gamma^{\beta}|\gamma}(Y_1|Y_2) \sim (\upsilon\gamma^{\beta})^{-\frac{1}{\alpha}}\gamma^{\frac{1}{2\alpha}}\,\mathrm{VaR}_{\gamma}(Y_1), \qquad \gamma \downarrow 0.$$

(ii) *Suppose* $0 \leq \beta < 1/2$ *and* $\frac{v^{1/\alpha}\overline{F}_1^{\leftarrow}(v\gamma)}{\overline{F}_1^{\leftarrow}(\gamma)}$ *converges uniformly on* $[1, \infty)$ *to* 1 *as* $\gamma \downarrow 0$. *Then*

$$\mathrm{CoVaR}_{\upsilon\gamma^{\beta}|\gamma}(Y_1|Y_2) \sim (\upsilon\gamma^{\beta})^{-\frac{2}{\alpha}}\gamma^{\frac{1}{\alpha}}\,\mathrm{VaR}_{\gamma}(Y_1), \qquad \gamma \downarrow 0.$$

(b) *Let* $\mathbf{Y} \in$ P-MOC$(\alpha, \theta, \lambda^{\propto})$ *for* $\lambda > 0$. *Then* $\mathrm{ECI}(Y_1|Y_2) = 3$, *and the following statements hold*:

(i) *Suppose either*

$$\beta = 1/3$$

or

$$\beta > 1/3 \quad \textit{and} \quad \frac{v^{1/\alpha}\overline{F}_1^{\leftarrow}(v\gamma)}{\overline{F}_1^{\leftarrow}(\gamma)} \text{ converges uniformly on } (0, 1] \text{ to } 1 \text{ as } \gamma \downarrow 0.$$

Then

$$\mathrm{CoVaR}_{\upsilon\gamma^{\beta}|\gamma}(Y_1|Y_2) \sim (\upsilon\gamma^{\beta})^{-\frac{1}{\alpha}}\gamma^{\frac{1}{3\alpha}}\,\mathrm{VaR}_{\gamma}(Y_1), \qquad \gamma \downarrow 0.$$

(ii) *Suppose* $0 \leq \beta < 1/3$ *and* $\frac{v^{1/\alpha}\overline{F}_1^{\leftarrow}(v\gamma)}{\overline{F}_1^{\leftarrow}(\gamma)}$ *converges uniformly on* $[1, \infty)$ *to* 1 *as* $\gamma \downarrow 0$. *Then*

$$\mathrm{CoVaR}_{\upsilon\gamma^{\beta}|\gamma}(Y_1|Y_2) \sim (\upsilon\gamma^{\beta})^{-\frac{3}{\alpha}}\gamma^{\frac{1}{\alpha}}\,\mathrm{VaR}_{\gamma}(Y_1), \qquad \gamma \downarrow 0.$$

Remark 3.13 (a) Let $\mathbf{Y} \in$ P-MOC as in Proposition 3.12. Then the uniform convergence of the quantity $v^{\alpha}\overline{F}_j^{\leftarrow}(v\gamma)/\overline{F}_j^{\leftarrow}(\gamma)$ on some set $I \subseteq (0, \infty)$ is necessary and sufficient for the uniform convergence of $h_{\gamma}^{\leftarrow}(v)/h^{-1}(v)$ on I. However, we can check that even if $\overline{F}_j^{\leftarrow}(v\gamma)/\overline{F}_j^{\leftarrow}(\gamma)$ converges uniformly to $v^{-\alpha}$ on $[1, \infty)$, it need not necessarily converge uniformly on (0, 1] (counterexamples exist). Nevertheless, exactly Pareto(α)-distributed marginals satisfy $v^{\alpha}\overline{F}_j^{\leftarrow}(v\gamma)/\overline{F}_j^{\leftarrow}(\gamma) = 1$ for any $v > 0$ and γ small, and hence the asymptotic behaviour of CoVaR holds for Pareto marginals and any $\beta \geq 0$ without any additional assumption.

(b) Suppose $\mathbf{Y}^{=} \in$ P-MOC$(\alpha, \theta, \lambda^{=})$ and $\mathbf{Y}^{\propto} \in$ P-MOC$(\alpha, \theta, \lambda^{\propto})$ are bivariate random vectors. Then Proposition 3.12 implies that $\mathrm{ECI}(Y_1^{\propto}|Y_2^{\propto}) > \mathrm{ECI}(Y_1^{=}|Y_2^{=})$, and thus the risk contagion in the P-MOC$(\alpha, \theta, \lambda^{=})$-model is higher than in the P-MOC$(\alpha, \theta, \lambda^{\propto})$-model if CoVaR is used as risk measure. Even though both models exhibit asymptotic independence, the result shows that there is still some dependence on the tails influencing CoVaR.

Proposition 3.14 *Let* $\mathbf{Y} = (Y_1, Y_2)^\top \in \text{P-GC}(\alpha, \theta, \Sigma_\rho)$ *with* $\rho \in (-1, 1)$ *and suppose* $g(\gamma) = \gamma^{\frac{1-\rho}{1+\rho}}(-\log\gamma)^{-\frac{\rho}{1+\rho}}$. *Then* $\mathrm{ECI}(Y_1|Y_2) = \frac{1+\rho}{1-\rho}$, *and for* $0 < \upsilon < 1$,

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2) \sim B^* \upsilon^{-\frac{1+\rho}{\alpha}}\, \mathrm{VaR}_\gamma(Y_1), \qquad \gamma \downarrow 0,$$

where $B^* = B^*(\rho, \alpha) = (4\pi)^{-\frac{\rho}{\alpha}}(1+\rho)^{\frac{3(1+\rho)}{2\alpha}}(1-\rho)^{-\frac{1+\rho}{2\alpha}}$.

Remark 3.15 (a) Although the logarithm is a slowly varying function, $\log(vt)/\log t$ converges uniformly only on compact intervals. Hence for the Gaussian dependence case, it seems that $h_\gamma^{\leftarrow}(v)/h^{-1}(v)$ need not converge uniformly on intervals of the form $(0, R]$ or $[L, \infty)$ for any $L, R > 0$. Thus we do not attempt to verify conditions (b) or (c) in Theorem 3.1.

(b) The measure $\mathrm{ECI}(Y_1|Y_2) = 2/(1-\rho) - 1$ is increasing in ρ, suggesting, not quite surprisingly, that as the Gaussian correlation ρ increases, the risk contagion measured by CoVaR increases as well; in fact, as ρ increases from -1 to 1, ECI increases from 0 to ∞.

4 Risk contagion in a bipartite network with asymptotically independent objects

Recall the bipartite network structure defined in Sect. 1, where the risk exposure of q entities of a financial system given by $\mathbf{X} \in \mathbb{R}_+^q$ is captured by using the risk exposure of the underlying assets $\mathbf{Z} \in \mathbb{R}_+^d$ and the bipartite network is defined via the matrix $\mathbf{A} \in \mathbb{R}_+^{q\times d}$. In this section, we derive asymptotic tail probabilities of the risk exposure $\mathbf{X} = \mathbf{AZ}$, where the objects in $\mathbf{Z}$ are asymptotically independent; the case of pairwise asymptotically dependent objects was already covered in Example 3.11. We are particularly interested in tail probabilities of *rectangular sets*

$$\Gamma_{\mathbf{x}_S}^{(q)} = \{\mathbf{v} \in \mathbb{R}_+^q : v_s > x_s, \forall s \in S\},$$

where $x_s > 0$ for all $s \in S \subseteq \mathbb{I}_q$ and $\mathbf{x}_S = (x_s)_{s\in S}$, as these help us first in computing conditional probabilities and eventually conditional risk measures like CoVaR. In the bivariate setup, the rectangular sets are of the form $[\mathbf{0}, \mathbf{x}]^c$ and $(\mathbf{x}, \infty)$, $\mathbf{x} \in \mathbb{R}_+^2$. The proofs for the results in this section are given in Appendix C.

Before stating the results, we need some definitions and notations following Das et al. [21]. Recall that we denote by $\mathcal{A} = \mathbb{I}_q$ the set of agents/entities and by $\mathcal{O} = \mathbb{I}_d$ the set of assets/objects.

Definition 4.1 For $k \in \mathbb{I}_q$ and $i \in \mathbb{I}_d$, the functions $\tau_{(k,i)} : \mathbb{R}_+^{q\times d} \to \mathbb{R}_+$ are defined as

$$\tau_{(k,i)}(\mathbf{A}) = \sup_{\mathbf{z}\in\mathbb{O}_d^{(i)}} \frac{(\mathbf{Az})_{(k)}}{z_{(i)}}.$$

The functions $\tau_{(k,i)}$ are meant to be like norms for the matrices $\mathbf{A} \in \mathbb{R}_+^{q\times d}$. Although the $\tau_{(k,i)}$ are not necessarily norms (or even semi-norms) on the induced vector space (see Horn and Johnson [35, Sect. 5.1]), they do admit some useful properties; cf. Das et al. [21, Lemma 3.4].

Definition 4.2 Let $\mathbf{A} \in \mathbb{R}_+^{q\times d}$ be a random matrix. For $k \in \mathbb{I}_q$ and $\omega \in \Omega$, define $\mathbf{A}_\omega := \mathbf{A}(\omega)$ and

$$i_k(\mathbf{A}_\omega) := \max\{j \in \mathbb{I}_d : \tau_{(k,j)}(\mathbf{A}_\omega) < \infty\},$$

which creates a partition $\Omega^{(k)}(\mathbf{A}) = (\Omega_i^{(k)}(\mathbf{A}))_{i=1,\dots,d}$ of Ω given by

$$\Omega_i^{(k)} := \Omega_i^{(k)}(\mathbf{A}) := \{\omega \in \Omega : i_k(\mathbf{A}_\omega) = i\}, \quad i \in \mathbb{I}_d.$$

We write $\mathbb{P}_i^{(k)}[\,\cdot\,] := \mathbb{P}[\,\cdot \cap \Omega_i^{(k)}]$ and $\mathbb{E}_i^{(k)}[\,\cdot\,] := \mathbb{E}[\,\cdot\,\mathbf{1}_{\Omega_i^{(k)}}]$.

Now we are ready to characterise the asymptotic probabilities of $\mathbf{X} = \mathbf{AZ}$ for various tail sets C of $\mathbb{R}_+^q$; cf. [21, Theorem 3.4 and Proposition 3.2] and the details in Appendix C.

Theorem 4.3 *Let* $\mathbf{Z} \in \mathbb{R}_+^d$ *be a random vector,* $\mathbf{A} \in \mathbb{R}_+^{q\times d}$ *a random matrix and* $\mathbf{X} = \mathbf{AZ}$. *Moreover, for fixed* $k \in \mathbb{I}_q$, *let* $C \subseteq \mathbb{O}_q^{(k)}$ *be a Borel set bounded away from* $\{\mathbf{x} \in \mathbb{R}_+^q : x_{(k)} = 0\}$. *Now also assume the following*:

(i) $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$ *for all* $i \in \mathbb{I}_d$ *and* $\lim_{t\to\infty} b_i(t)/b_{i+1}(t) = \infty$ *for* $i = 1,\dots,d-1$.

(ii) $\mathbf{A}$ *has almost surely no trivial rows and is independent of* $\mathbf{Z}$.

(iii) $\mathbb{E}_i^{(k)}[\mu_i(\partial\mathbf{A}^{-1}(C))] = 0$ *for all* $i \in \mathbb{I}_d$.

(iv) *For all* $i \in \mathbb{I}_d$, *we have* $\mathbb{E}_i^{(k)}[(\tau_{(k,i)}(\mathbf{A}))^{\alpha_i+\delta}] < \infty$ *for some* $\delta = \delta(i,k) > 0$.

Then the following hold:

(a) *Define* $i_k^* := \arg\min\{i \in \mathbb{I}_d : \mathbb{P}[\Omega_i^{(k)}] > 0\}$. *Then we have*

$$b_{i_k^*}^{\leftarrow}(t)\mathbb{P}[\mathbf{X} \in tC] \longrightarrow \mathbb{E}_{i_k^*}^{(k)}\big[\mu_{i_k^*}\big(\mathbf{A}^{-1}(C)\big)\big] = \overline{\mu}_{i_k^*,k}(C) =: \overline{\mu}_k(C), \qquad t \to \infty.$$

Moreover, if $\overline{\mu}_k$ *is a non-null measure, then* $\mathbf{AZ} \in \mathrm{MRV}(\alpha_{i_k^*}, \overline{\mu}_k, \mathbb{O}_q^{(k)})$.

(b) *Define*

$$\bar{\imath} := \bar{\imath}_C := \min\Big\{d, \inf\big\{i \in \{i_k^*,\dots,d\} : \mathbb{E}_i^{(k)}\big[\mu_i\big(\mathbf{A}^{-1}(C)\big)\big] > 0\big\}\Big\}.$$

Suppose for all $i = i_k^*,\dots,\bar{\imath}-1$ *and* $\omega \in \Omega_i^{(k)}$ *that* $\mathbf{A}_\omega^{-1}(C) = \emptyset$. *Then we have*

$$\mathbb{P}[\mathbf{X} \in tC] = \big(b_{\bar{\imath}}^{\leftarrow}(t)\big)^{-1}\mathbb{E}_{\bar{\imath}}^{(k)}\big[\mu_{\bar{\imath}}\big(\mathbf{A}^{-1}(C)\big)\big] + o\Big(\big(b_{\bar{\imath}}^{\leftarrow}(t)\big)^{-1}\Big), \qquad t \to \infty.$$

Remark 4.4 Assumption (i) of Theorem 4.3 is satisfied by many popular models:

(a) For both $\mathbf{Z} \in$ P-GC(α, θ, Σ) with Σ positive definite (cf. Proposition 2.6) and $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^{=})$ or $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^{\propto})$, respectively (cf. Proposition 2.11), assumption (i) of Theorem 4.3 is satisfied.

(b) Assumption (i) in Theorem 4.3 excludes the asymptotically dependent case, where $\mathbf{Z} \in \text{MRV}(\alpha, b, \mu, \mathbb{O}_d^{(i)})$ for $i = 1, 2$ with $\mu(\mathbb{O}_d^{(2)}) > 0$. But in this case, we can use the well-known Breiman theorem generalised to the multivariate setup in Basrak et al. [7, Proposition A.1]; see Example 3.11.

Remark 4.5 If $\mathbf{A} = \mathbf{A}_0$ is a deterministic matrix, we have a few direct consequences:

(a) First, $i_k^* = i_k(\mathbf{A}_0)$ and also $\Omega_j^{(k)} = \emptyset$ for all $j \neq i_k^*$.

(b) If $C = \Gamma_{\mathbf{x}_S}^{(q)} \subseteq \mathbb{O}_q^{(k)}$ is a rectangular set, then the intersection of $\mathbf{A}_0^{-1}(\Gamma_{\mathbf{x}_S}^{(q)})$ with the support of $\mu_{i_k^*}$ is a finite union of rectangular sets; however, it is clearly not necessarily a rectangular set itself (Das et al. [21, Example 3.2]). When d is large, the number of sets in this union might be quite large, and hence it may be computationally expensive not only to get an explicit expression for $\overline{\mu}_k(\Gamma_{\mathbf{x}_S}^{(q)})$, but even to assess whether $\overline{\mu}_k(\Gamma_{\mathbf{x}_S}^{(q)}) > 0$. Nevertheless, it turns out that there are reasonable sufficient conditions under which we can guarantee $\overline{\mu}_k(\Gamma_{\mathbf{x}_S}^{(q)}) > 0$, where $|S| = k$; see the next result which is a consequence of Das et al. [21, Proposition 3.1].

Lemma 4.6 *Suppose the assumptions of Theorem 4.3 hold. For any rectangular set $\Gamma_{\mathbf{x}_S}^{(q)}$ with $|S| = k$, we have $\overline{\mu}_k(\Gamma_{\mathbf{x}_S}^{(q)}) > 0$ with $|S| = k$ if $\mu_{i_k^*}$ has mass on all $\binom{d}{i_k^*}$ coordinate hyperplanes comprising $\mathbb{O}_d^{(i_k^*)}$.*

Remark 4.7 With regard to Lemma 4.6, the following examples satisfy $\overline{\mu}_k(\Gamma_{\mathbf{x}_S}^{(q)}) > 0$ for any rectangular set $\Gamma_{\mathbf{x}_S}^{(q)}$ with $|S| = k \geq 2$ and any random matrix $\mathbf{A}$ with non-trivial rows:

(i) $\mathbf{Z}$ is exchangeable (including independence).
(ii) $\mathbf{Z} \in$ P-GC$(\alpha, \theta, \Sigma_\rho)$, where $\rho \in (-1/(d-1), 1)$.
(iii) $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^{=})$.
(iv) $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^{\propto})$.

Note that the distributions of $\mathbf{Z}$ in (ii)–(iv) are close to exchangeability (they become exchangeable if we assume that the marginals are identically distributed instead of completely tail-equivalent).

For a P-GC(α, θ, Σ)-model, Proposition 2.8 gives sufficient criteria on the correlation matrix Σ such that μ_i has mass on all $\binom{d}{i}$ coordinate hyperplanes comprising $\mathbb{O}_d^{(i)}$, naturally indicating that there exist PG-C models which may not satisfy this criterion; see Example 2.7. However, if $\mathbf{Z} \in$ P-GC(α, θ, Σ), we can often use a technique of dimension reduction as in the proof of Proposition 4.15 to obtain the correct tail probability rates.

4.1 Risk contagion between two portfolios

In the context of risk contagion, an important task is to understand the probability of an extremely large loss for a single asset or a linear combination of assets, given an extremely large loss for some other asset or a linear combination of assets. This allows us to concentrate on $\mathbf{A} \in \mathbb{R}_+^{q\times d}$ with $q = 2$. For a particular type of event, we should also concentrate on risk exposures of pairs of financial entities that may invest in disjoint sets of assets; but because of the dependence of the underlying variables, the joint probability is not necessarily a product measure. First, we obtain a general result for such joint tail probabilities characterising limit measures $\overline{\mu}_1$ and $\overline{\mu}_2$ as obtained in Theorem 4.3.

Proposition 4.8 *Let* $\mathbf{Z} \in \mathbb{R}_+^d$ *be a random vector with* $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$ *for* $i = 1, 2$ *and* $b_1(t)/b_2(t) \to \infty$ *as* $t \to \infty$. *Let* $\mathbf{A} \in \mathbb{R}_+^{2\times d}$ *be a random matrix which has almost surely no trivial rows and is independent of* $\mathbf{Z}$. *With the notations from Theorem* 4.3, *the following hold for* $\overline{\mu}_1$ *and* $\overline{\mu}_2$:

(a) *Suppose* $\mathbb{E}[\|\mathbf{A}\|^{\alpha_1+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then* $i_1^* = 1$ *and for* $\mathbf{x} \in \mathbb{R}_+^2$,

$$\overline{\mu}_1([\mathbf{0}, \mathbf{x}]^c) = \sum_{\ell=1}^{d} \mathbb{E}\Big[\mu_1\Big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \max\{a_{1\ell}z_\ell/x_1, a_{2\ell}z_\ell/x_2\} > 1\big\}\Big)\Big].$$

(b) *Suppose* $\max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] > 0$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\alpha_1+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then* $i_2^* = 1$ *and for* $\mathbf{x} \in (\mathbf{0}, \infty)$,

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \sum_{\ell=1}^{d} \mathbb{E}\Big[\mu_1\Big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \min\{a_{1\ell}z_\ell/x_1, a_{2\ell}z_\ell/x_2\} > 1\big\}\Big)\Big].$$

(c) *Suppose* $\max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] = 0$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\alpha_2+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then* $i_2^* = 2$ *and for* $\mathbf{x} \in (\mathbf{0}, \infty)$,

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \sum_{\ell,j=1}^{d} \mathbb{E}\Big[\mu_2\Big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \min\{a_{1\ell}z_\ell/x_1, a_{2j}z_j/x_2\} > 1\big\}\Big)\Big].$$

In particular, if $\mathbf{Z}$ *is exchangeable, each measure in* (a)–(c) *is non-zero.*

Note that in Proposition 4.8, it is sufficient to assume $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$ for $i = 1, 2$ instead of $i = 1, \ldots, d$ as in Theorem 4.3 because $\mathbf{X}$ is a bivariate random vector.

Next, we provide sufficient conditions for the limit measures in Proposition 4.8 to be positive so that our probability approximations for $\mathbf{X} = \mathbf{AZ}$ belonging to some extreme rectangular set are non-trivial. These approximations turn out to be sufficient for obtaining the asymptotic behaviour of CoVaR under the given assumptions as well.

Proposition 4.9 *Let the assumptions of Proposition 4.8 hold. Let* $\alpha := \alpha_1$. *Suppose further that* $\mathbf{Z}$ *is completely tail-equivalent and* $\mathbb{E}[\|\mathbf{A}\|^{\alpha+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then the following hold*:

(a) *We have* $\mathbf{X} = (X_1, X_2)^\top \in \mathrm{MRV}(\alpha, b_1, \overline{\mu}_1, \mathbb{O}_2^{(1)})$, *where*

$$\overline{\mu}_1([\mathbf{0}, \mathbf{x}]^c) = \frac{\mu_1([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^{d} \mathbb{E}[\max\{a_{1\ell}/x_1, a_{2\ell}/x_2\}^\alpha], \qquad \mathbf{x} \in \mathbb{R}_+^2.$$

(b) *If* $\max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] > 0$, *then* $\mathbf{X} \in \mathrm{MRV}(\alpha, b_1, \overline{\mu}_2, \mathbb{O}_2^{(2)})$, *where*

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \frac{\mu_1([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^{d} \mathbb{E}[\min\{a_{1\ell}/x_1, a_{2\ell}/x_2\}^\alpha], \qquad \mathbf{x} \in (\mathbf{0}, \infty).$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[X_1 > tx_1 | X_2 > tx_2] \sim x_2^\alpha \frac{\sum_{\ell=1}^d \mathbb{E}[\min\{a_{1\ell}/x_1, a_{2\ell}/x_2\}^\alpha]}{\sum_{\ell=1}^d \mathbb{E}[a_{2\ell}^\alpha]}. \tag{4.1}$$

With

$$h(y) = \frac{\mu([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^{d} \mathbb{E}[\min\{a_{1\ell}/y, a_{2\ell}/K^{1/\alpha}\}^\alpha],$$

$$K = \frac{\sum_{\ell=1}^d a_{2\ell}^\alpha}{\sum_{\ell=1}^d a_{1\ell}^\alpha} \qquad \textit{and} \qquad c = \frac{\mu([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^{d} a_{1\ell}^\alpha,$$

we have for $0 < \upsilon < \min\{h(0)/c, 1\}$ *that*

$$\mathrm{CoVaR}_{\upsilon|\gamma}(X_1|X_2) \sim h^{-1}(\upsilon c)\,\mathrm{VaR}_\gamma(X_1), \qquad \gamma \downarrow 0.$$

Additionally, if the non-zero components of $\mathbf{A}$ *have bounded support, bounded away from zero, there exists* $\upsilon^* \in (0, 1)$ *such that for all* $0 < \upsilon < \upsilon^*$, *we have*

$$\mathrm{CoVaR}_{\upsilon|\gamma}(X_1|X_2) \sim \upsilon^{-\frac{1}{\alpha}}\,\mathrm{VaR}_\gamma(X_1), \qquad \gamma \downarrow 0. \tag{4.2}$$

Finally, $\mathrm{ECI}(X_1|X_2) = \infty$.

Remark 4.10 (a) The multivariate regular variation of $\mathbf{X}$ on $\mathbb{O}_2^{(1)}$ is a direct consequence of the multivariate version of Breiman's theorem from Breiman [12] in Basrak et al. [7, Proposition A.1]; see as well Kley et al. [40, Proposition 3.1].

(b) If $Z_1, \ldots, Z_d$ are completely tail-equivalent with marginals being exactly Pareto distributions and $\mathbf{Z} = (Z_1, \ldots, Z_d)^\top \in \mathrm{MRV}(\alpha_1 = \alpha, b_1, \mu_1, \mathbb{O}_2^{(1)})$ with μ_1 having only mass on the axes (which is satisfied if $\alpha_1 < \alpha_2$), statement (4.1) is a special case of Kley et al. [41, Corollary 2.6] and (4.2) of [41, Theorem 3.4]. The CoVaR results obtained in [41] address the specific case where $\mathrm{ECI}(X_1|X_2) = \infty$.

(c) If bank/agent 1 (with risk variable X_1) is connected to object j (with risk variable Z_j), then by defining $X_2 = Z_j$, we can directly apply Proposition 4.9 to obtain the tail behaviour of (X_1, Z_j) and (Z_j, X_1), respectively. Hence we can get the asymptotic behaviour of $\mathrm{CoVaR}_{\upsilon|\gamma}(X_1|Z_j)$ and $\mathrm{CoVaR}_{\upsilon|\gamma}(Z_j|X_1)$; in other words, we are able to measure the risk contagion between agents and their connected objects. If additional new objects are introduced into the market (i.e., d increases), this has no impact on the asymptotic behaviour of $\mathrm{CoVaR}_{\upsilon|\gamma}(X_1|Z_j)$ and $\mathrm{CoVaR}_{\upsilon|\gamma}(Z_j|X_1)$ even if the agent, in this case agent 1, connects to these new objects.

Incidentally, if $\max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] = 0$, reflecting that the two banks or agents are not connected to the same asset or object at the same time, the right-hand side of (4.1) turns out to be 0, $\mathrm{ECI}(X_1|X_2)$ becomes finite leading to the asymptotically independent model, and the CoVaR approximation of Proposition 4.9 is not valid anymore. In the next few results, we concentrate on the case where

$$\max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] = 0,$$

which relates to a scenario where the aggregate returns of financial entities are represented by disjoint sets of assets. For example, this covers as well the case where we want to understand the influence of an asset/object on a bank/agent with which it is not connected. Note that although the two banks/agents may be represented by almost surely disjoint assets, they are nevertheless related by the dependence assumption on the underlying set of assets whose risk is given by $\mathbf{Z}$. We provide explicit tail rates and eventually also CoVaR computations under such a setup, assuming different dependence structures for the underlying random vector $\mathbf{Z}$. The following result assumes that $\mathbf{Z}$ has i.i.d. components.

Proposition 4.11 *Let $\mathbf{Z} \in \mathbb{R}_+^d$ be a random vector with i.i.d. components $Z_1, \dots, Z_d$ having distribution function F_α, where $\overline{F}_\alpha \in \mathrm{RV}_{-\alpha}$, $\alpha > 0$, $b_1(t) = F_\alpha^{\leftarrow}(1 - 1/t)$ and $b_i^{\leftarrow}(t) = (b_1^{\leftarrow}(t))^i$. Further, let $\mathbf{A} \in \mathbb{R}_+^{2\times d}$ be a random matrix which has almost surely no trivial rows and is independent of $\mathbf{Z}$ and suppose that*

$$\max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] = 0$$

and $\mathbb{E}[\|\mathbf{A}\|^{2\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$. Then $(X_1, X_2)^\top \in \mathrm{MRV}(2\alpha, b_2, \overline{\mu}_2, \mathbb{O}_2^{(2)})$, where

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = (x_1x_2)^{-\alpha} \sum_{\ell,j=1}^{d} \mathbb{E}[a_{1\ell}^\alpha a_{2j}^\alpha], \qquad \mathbf{x} \in (\mathbf{0}, \infty).$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[X_1 > tx_1|X_2 > tx_2] \sim \big(b_1^{\leftarrow}(t)\big)^{-1} x_1^{-\alpha} \frac{\sum_{\ell,j=1}^d \mathbb{E}[a_{1\ell}^\alpha a_{2j}^\alpha]}{\sum_{j=1}^d \mathbb{E}[a_{2j}^\alpha]}.$$

Additionally, for $0 < \upsilon < 1$, *we have*

$$\mathrm{CoVaR}_{\upsilon\gamma|\gamma}(X_1|X_2) \sim \upsilon^{-\frac{1}{\alpha}} \frac{(\sum_{\ell,j=1}^{d} \mathbb{E}[a_{1\ell}^{\alpha} a_{2j}^{\alpha}])^{\frac{1}{\alpha}}}{(\sum_{\ell=1}^{d} \mathbb{E}[a_{1\ell}^{\alpha}] \sum_{j=1}^{d} \mathbb{E}[a_{2j}^{\alpha}])^{\frac{1}{\alpha}}} \mathrm{VaR}_{\gamma}(X_1), \qquad \gamma \downarrow 0.$$

Finally, $\mathrm{ECI}(X_1|X_2) = 1$.

Remark 4.12 The arrival of new independent assets/objects to the financial market has no influence on the ECI or the asymptotic behavior (rate) of CoVaR. Regardless of whether the new asset/object connects to one of the banks/agents, we still have $\mathrm{CoVaR}_{\upsilon\gamma,\gamma}(X_1|X_2) = O(\mathrm{VaR}_{\gamma}(X_1))$ as $\gamma \downarrow 0$. In case at least one of agents 1 and 2 happens to connect to a new object, the only change may appear in the value of the finite positive limit of $\mathrm{CoVaR}_{\upsilon\gamma|\gamma}(X_1|X_2)/\,\mathrm{VaR}_{\gamma}(X_1)$ as $\gamma \downarrow 0$.

The next result assumes a Marshall–Olkin dependence.

Proposition 4.13 *Let* $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \Lambda)$ *and* $\mathbf{A} \in \mathbb{R}_+^{2\times d}$ *be a random matrix which has almost surely no trivial rows, is independent of* $\mathbf{Z}$ *and satisfies*

$$\max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] = 0.$$

(a) *Suppose* $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^{=})$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\frac{3\alpha}{2}+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then*

$$\mathbf{X} = \mathbf{A}\mathbf{Z} \in \mathrm{MRV}(\alpha_2, b_2, \overline{\mu}_2, \mathbb{O}_2^{(2)}),$$

where $\alpha_2 = \frac{3\alpha}{2}$, $b_2(t) = \theta^{\frac{1}{\alpha}} t^{\frac{2}{3\alpha}}$ *and for* $\mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty)$,

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \sum_{\ell,j=1}^{d} \mathbb{E}[\min\{a_{1\ell}/x_1, a_{2j}/x_2\}^{\alpha} \max\{a_{1\ell}/x_1, a_{2j}/x_2\}^{\alpha/2}].$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[X_1 > tx_1 | X_2 > tx_2] \sim (\theta t^{-\alpha})^{\frac{1}{2}} x_2^{\alpha}\, \overline{\mu}_2\big((\mathbf{x}, \infty)\big) \Bigg(\sum_{j=1}^{d} \mathbb{E}[a_{2j}^{\alpha}] \Bigg)^{-1}.$$

Additionally, if the non-zero components of $\mathbf{A}$ *have bounded support, bounded away from zero, then there exist* $0 < \upsilon_1^* < \upsilon_2^* < \infty$ *such that for all* $0 < \upsilon < \upsilon_1^*$, *we have*

$$\mathrm{CoVaR}_{\upsilon\gamma^{\frac{1}{2}}|\gamma}(X_1|X_2)$$
$$\sim \upsilon^{-\frac{1}{\alpha}} \frac{(\sum_{\ell,j=1}^{d} \mathbb{E}[a_{1\ell}^{\alpha} a_{2j}^{\alpha/2}])^{\frac{1}{\alpha}}}{(\sum_{\ell=1}^{d} \mathbb{E}[a_{1\ell}^{\alpha}])^{\frac{1}{\alpha}} (\sum_{j=1}^{d} \mathbb{E}[a_{2j}^{\alpha}])^{\frac{1}{2\alpha}}} \mathrm{VaR}_{\gamma}(X_1), \qquad \gamma \downarrow 0,$$

and for all $\upsilon_2^* < \upsilon < \infty$, *we have*

$$\mathrm{CoVaR}_{\upsilon\gamma^{\frac{1}{2}}|\gamma}(X_1|X_2) \sim \upsilon^{-\frac{2}{\alpha}} \frac{(\sum_{\ell,j=1}^d \mathbb{E}[a_{1\ell}^{\alpha/2} a_{2j}^{\alpha}])^{\frac{2}{\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{1\ell}^{\alpha}])^{\frac{1}{\alpha}} (\sum_{j=1}^d \mathbb{E}[a_{2j}^{\alpha}])^{\frac{2}{\alpha}}} \mathrm{VaR}_{\gamma}(X_1), \qquad \gamma \downarrow 0.$$

Finally, $\mathrm{ECI}(X_1|X_2) = 2$.

(b) *Suppose* $\mathbf{Z} \in \text{P-MOC}(\alpha, \theta, \lambda^{\propto})$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\alpha\frac{3d+2}{2(d+1)}+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then*

$$\mathbf{X} = \mathbf{AZ} \in \mathrm{MRV}(\alpha_2, b_2, \overline{\mu}_2, \mathbb{O}_2^{(2)}),$$

where $\alpha_2 = \alpha\frac{3d+2}{2(d+1)}$, $b_2(t) = \theta^{\frac{1}{\alpha}} t^{\frac{2(d+1)}{(3d+2)\alpha}}$ *and for* $\mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty)$,

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \sum_{\ell,j=1}^{d} \mathbb{E}[\min\{a_{1\ell}/x_1, a_{2j}/x_2\}^{\alpha} \max\{a_{1\ell}/x_1, a_{2j}/x_2\}^{\frac{d\alpha}{2(d+1)}}].$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[X_1 > tx_1|X_2 > tx_2] \sim (\theta t^{-\alpha})^{\frac{d}{2(d+1)}} x_2^{\alpha}\, \overline{\mu}_2\big((\mathbf{x}, \infty)\big) \left(\sum_{j=1}^{d} \mathbb{E}[a_{2j}^{\alpha}]\right)^{-1}.$$

Additionally, if the non-zero components of $\mathbf{A}$ *have bounded support, bounded away from zero, then there exist* $0 < \upsilon_1^* < \upsilon_2^* < \infty$ *such that for all* $0 < \upsilon < \upsilon_1^*$, *we have*

$$\mathrm{CoVaR}_{\upsilon\gamma^{\frac{d}{2(d+1)}}|\gamma}(X_1|X_2) \sim \upsilon^{-\frac{1}{\alpha}} \frac{(\sum_{\ell,j=1}^d \mathbb{E}[a_{1\ell}^{\alpha} a_{2j}^{\frac{d\alpha}{2(d+1)}}])^{\frac{1}{\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{1\ell}^{\alpha}])^{\frac{1}{\alpha}} (\sum_{j=1}^d \mathbb{E}[a_{2j}^{\alpha}])^{\frac{d}{2(d+1)\alpha}}} \mathrm{VaR}_{\gamma}(X_1), \qquad \gamma \downarrow 0,$$

and for all $\upsilon_2^* < \upsilon < \infty$, *we have*

$$\mathrm{CoVaR}_{\upsilon\gamma^{\frac{d}{2(d+1)}}|\gamma}(X_1|X_2) \sim \upsilon^{-\frac{2(d+1)}{d\alpha}} \frac{(\sum_{\ell,j=1}^d \mathbb{E}[a_{1\ell}^{\frac{d\alpha}{2(d+1)}} a_{2j}^{\alpha}])^{\frac{2(d+1)}{d\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{1\ell}^{\alpha}])^{\frac{1}{\alpha}} (\sum_{j=1}^d \mathbb{E}[a_{2j}^{\alpha}])^{\frac{2(d+1)}{d\alpha}}} \mathrm{VaR}_{\gamma}(X_1), \qquad \gamma \downarrow 0.$$

Finally, $\mathrm{ECI}(X_1|X_2) = 2 + \frac{2}{d}$.

Remark 4.14 (a) If $\mathbf{Z} \in \text{P-MOC}(\alpha, \theta, \lambda^{\propto})$, then as the number of assets/objects increases, i.e., if d increases, $\mathrm{ECI}(X_1|X_2) = 2 + 2/d$ decreases and hence the rate of convergence of $\mathrm{CoVaR}_{\gamma,\gamma}(X_1|X_2)$ to ∞ as $\gamma \downarrow 0$ becomes slower. In this case, the dependence of X_1 and X_2 in the tails gets weaker as the network becomes more diversified. It is important to note here that the P-MOC$(\alpha, \theta, \lambda^{\propto})$-model does not have a nested structure, in contrast to P-MOC$(\alpha, \theta, \lambda^{=})$ which is nested. In the P-MOC$(\alpha, \theta, \lambda^{=})$-model, an increase in the value of d has no influence on the ECI value.

(b) Under the assumption of independent objects, we have $\mathrm{ECI}(X_1|X_2) = 1$, which is less than $\mathrm{ECI}(X_1|X_2) = 2$ when the objects have a Marshall–Olkin dependence structure with equal parameters (Proposition 4.13(a)), which is again less than $\mathrm{ECI}(X_1|X_2) = 2 + 2/d$ when the objects have a Marshall–Olkin copula with proportional parameters (Proposition 4.13(b)), reflecting that the dependence in the tails becomes progressively stronger.

(c) Note that the degree of tail-dependence plays a role only if $\mathbf{A}$ contains a column with all entries zero; otherwise the entries of $\mathbf{A}$ do not influence ECI. In other words, although it is important to know if agents are connected to the same object, the weights or magnitudes of the connections are not essential for the value of ECI. However, this is not always the case as we can see in Proposition 4.15 under a Gaussian copula model, where the location (index) of the zero column is also important. Note that if $\mathbf{Z} \in$ P-GC(α, θ, Σ) and Σ is not an equicorrelation matrix, then $\mathbf{Z}$ is not exchangeable, in contrast to the other examples above (assuming identical marginals).

Finally, we provide a result where the components of $\mathbf{Z}$ have Gaussian dependence.

Proposition 4.15 *Let* $\mathbf{Z} \in$ P-GC(α, θ, Σ) *with* $\Sigma = (\rho_{\ell j})_{1\le \ell, j \le d}$ *positive definite. Let* $\mathbf{A} \in \mathbb{R}_+^{2\times d}$ *be a random matrix which has almost surely no trivial rows, is independent of* $\mathbf{Z}$ *and satisfies* $\max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{1\ell}, a_{2\ell}\} > 0] = 0$. *Also, define*

$$\rho^{\vee} = \max\{\rho_{\ell j} : \ell, j \in \mathbb{I}_d, \ell \neq j\},$$

$$\rho^{*} = \max\{\rho_{\ell j} : \ell, j \in \mathbb{I}_d, \ell \neq j \text{ and } \mathbb{P}[\min\{a_{1\ell}, a_{2j}\} > 0] > 0\}.$$

(a) *Suppose* $\rho^* = \rho^{\vee}$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\frac{2\alpha}{1+\rho^{\vee}}+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then we have*

$$\mathbf{X} = \mathbf{AZ} \in \mathrm{MRV}(\alpha_2, b_2, \overline{\mu}_2, \mathbb{O}_2^{(2)}),$$

with

$$\alpha_2 = \frac{2\alpha}{1+\rho^{\vee}}, \qquad b_2^{\leftarrow}(t) = C(\rho^{\vee}, \alpha)(\theta t^{-\alpha})^{-\frac{2}{1+\rho^{\vee}}} (\log t)^{\frac{\rho^{\vee}}{1+\rho^{\vee}}},$$

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = D(\rho^{\vee}, \alpha, \mathbf{A})(x_1 x_2)^{-\frac{\alpha}{1+\rho^{\vee}}}, \qquad \mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty),$$

where for $\rho \in (-1, 1), \alpha > 0, \theta > 0$ *and* $\mathbf{A} \in \mathbb{R}_+^{2\times d}$, *we define*

$$C(\rho, \alpha) = (2\pi)^{-\frac{1}{1+\rho}} (2\alpha)^{\frac{\rho}{1+\rho}},$$

$$D(\rho, \alpha, \mathbf{A}) = \frac{1}{2\pi} \frac{(1+\rho)^{3/2}}{(1-\rho)^{1/2}} \sum_{(\ell, j):\, \rho_{\ell j} = \rho} \mathbb{E}\big[a_{1\ell}^{\frac{\alpha}{1+\rho}} a_{2j}^{\frac{\alpha}{1+\rho}}\big]. \tag{4.3}$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[X_1 > x_1 t | X_2 > x_2 t] \sim (\theta t^{-\alpha})^{\frac{1-\rho^\vee}{1+\rho^\vee}} (\log t)^{-\frac{\rho^\vee}{1+\rho^\vee}} x_1^{-\frac{\alpha}{1+\rho^\vee}} x_2^{\frac{\alpha\rho^\vee}{1+\rho^\vee}} \frac{C(\rho^\vee,\alpha)^{-1} D(\rho^\vee,\alpha,\mathbf{A})}{\sum_{j=1}^d \mathbb{E}[a_{2j}^\alpha]}.$$

Additionally, with $g(\gamma) = \gamma^{\frac{1-\rho^\vee}{1+\rho^\vee}} (-\alpha^{-1}\log\gamma)^{-\frac{\rho^\vee}{1+\rho^\vee}}$ *and* $0 < \upsilon < \infty$, *we have*

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(X_1|X_2) \sim \upsilon^{-\frac{1+\rho^\vee}{\alpha}} \frac{(C(\rho^\vee,\alpha)^{-1} D(\rho^\vee,\alpha,\mathbf{A}))^{\frac{1+\rho^\vee}{\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{1\ell}^\alpha] \sum_{j=1}^d \mathbb{E}[a_{2j}^\alpha])^{\frac{1}{\alpha}}} \mathrm{VaR}_\gamma(X_1), \qquad \gamma \downarrow 0.$$

Finally, $g(t^{-1}) \in \mathrm{RV}_{-\frac{1-\rho^\vee}{1+\rho^\vee}}$ *and hence* $\mathrm{ECI}(X_1|X_2) = \frac{1+\rho^\vee}{1-\rho^\vee}$.

(b) *Suppose* $\rho^* < \rho^\vee$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\frac{2\alpha}{1+\rho^*}+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then* $\overline{\mu}_2$ *as defined in Theorem 4.3 is identically zero. But* $\mathbf{X} = \mathbf{AZ} \in \mathrm{MRV}(\alpha_2^*, b_2^*, \overline{\mu}_2^*, \mathbb{O}_2^{(2)})$, *with*

$$\alpha_2^* = \frac{2\alpha}{1+\rho^*},$$

$$b_2^{*\leftarrow}(t) = C(\rho^*,\alpha)(\theta t^{-\alpha})^{-\frac{2}{1+\rho^*}} (\log t)^{\frac{\rho^*}{1+\rho^*}},$$

$$\overline{\mu}_2^*\big((\mathbf{x},\infty)\big) = D(\rho^*,\alpha,\mathbf{A})(x_1 x_2)^{-\frac{\alpha}{1+\rho^*}}, \qquad \mathbf{x} = (x_1, x_2) \in (\mathbf{0},\infty),$$

and as $t \to \infty$,

$$\mathbb{P}[X_1 > x_1 t | X_2 > x_2 t] \sim (\theta t^{-\alpha})^{\frac{1-\rho^*}{1+\rho^*}} (\log t)^{-\frac{\rho^*}{1+\rho^*}} x_1^{-\frac{\alpha}{1+\rho^*}} x_2^{\frac{\alpha\rho^*}{1+\rho^*}} \frac{C(\rho^*,\alpha)^{-1} D(\rho^*,\alpha,\mathbf{A})}{\sum_{j=1}^d \mathbb{E}[a_{2j}^\alpha]},$$

where $C(\cdot)$ *and* $D(\cdot)$ *are as defined in (4.3). Additionally, with the function* g *given by* $g(\gamma) = \gamma^{\frac{1-\rho^*}{1+\rho^*}} (-\alpha^{-1}\log\gamma)^{-\frac{\rho^*}{1+\rho^*}}$ *and* $0 < \upsilon < \infty$, *we have*

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(X_1|X_2) \sim \upsilon^{-\frac{1+\rho^*}{\alpha}} \frac{(C(\rho^*,\alpha)^{-1} D(\rho^*,\alpha,\mathbf{A}))^{\frac{1+\rho^*}{\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{1\ell}^\alpha] \sum_{j=1}^d \mathbb{E}[a_{2j}^\alpha])^{\frac{1}{\alpha}}} \mathrm{VaR}_\gamma(X_1), \qquad \gamma \downarrow 0.$$

Finally, $g(t^{-1}) \in \mathrm{RV}_{-\frac{1-\rho^*}{1+\rho^*}}$ *and hence* $\mathrm{ECI}(X_1|X_2) = \frac{1+\rho^*}{1-\rho^*}$.

Remark 4.16 (a) Suppose $\rho^* < \rho^\vee$. Then there exist $\ell^*, j^* \in \mathbb{I}$ with $\ell^* \neq j^*$ and

$$\rho^* < \rho_{\ell^* m^*} = \rho_{m^* \ell^*} \leq \rho^\vee.$$

By the definition of ρ^*, we have

$$\mathbb{P}[\min\{a_{1\ell^*}, a_{2j^*}\} > 0] = 0 = \mathbb{P}[\min\{a_{1j^*}, a_{2\ell^*}\} > 0],$$

resulting in $a_{1\ell^*} = a_{1j^*} = a_{2\ell^*} = a_{2j^*} = 0$ a.s. Hence the ℓth and jth columns of $\mathbf{A} \in \mathbb{R}_+^{2\times d}$ are a.s. zero columns.

(b) Both values ρ^* and $\mathrm{ECI}(X_1|X_2) = \frac{1+\rho^*}{1-\rho^*}$ depend, on the one hand, on the location of the non-zero entries of $\mathbf{A}$ and, on the other hand, on the dependence structure of the underlying object $\mathbf{Z}$ modelled by the correlation matrix Σ. Naturally, the dependence in the network becomes stronger if either the off-diagonal entries of Σ increase or if the zero components of $\mathbf{A}$ are replaced by non-zero components, i.e., we have more connections between the agents and the objects. Both effects might increase as well the tail-dependence of the agents and the ECI.

(c) If an additional object, which is not connected to any of the two agents, is introduced into the market, this has no influence on the CoVaR behaviour. However, if the new object does connect with one of the agents, say agent 1, then the change in behaviour of CoVaR depends on the correlation of this new object with the other existing objects which are connected to the other agent, namely agent 2, in the Gaussian copula model.

Remark 4.17 Note that the limit measures $\overline{\mu}_1$ and $\overline{\mu}_2$ found in Propositions 4.11–4.15 are all non-zero measures, and hence the computed conditional probabilities are asymptotically non-trivial as well.

4.2 Risk contagion between various aggregates

In Sect. 4.1, we obtained asymptotic conditional tail probabilities with two portfolios in a bipartite structure. For a financial institution with more than two portfolios, it is also interesting to assess systemic risk between the entire system and a part of the system in terms of *aggregate risks*. Therefore, suppose that $T, S \subseteq \mathbb{I}_q$ are groups of agents and $g : (0, 1) \to (0, 1)$ is a measurable function. Given that the aggregate risk $\sum_{m\in T} X_m$ of the agents in T is above its VaR at level γ, we may be interested in finding the VaR of the aggregate risk $\sum_{k\in S} X_k$ of the agents in S at level $\upsilon g(\gamma)$, i.e.,

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}\left(\sum_{k\in S} X_k \,\middle|\, \sum_{m\in T} X_m\right). \tag{4.4}$$

Of course, the following special cases are of particular interest:

– CoVaR of the kth entity at level $\upsilon g(\gamma)$ given the aggregate of the entire system at level γ,

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}\left(X_k \,\middle|\, \sum_{m=1}^{q} X_m\right),$$

– CoVaR of the aggregate of the entire system at level $\upsilon g(\gamma)$ given a particular entity k at level γ,

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}\left(\sum_{m=1}^{q} X_m \middle| X_k\right).$$

Incidentally, the results obtained in the previous sections suffice for computations of the CoVaR asymptotics of (4.4), as well as computing the corresponding ECI. Indeed, by defining $\mathbf{e}^S \in \mathbb{R}^q$ as $\mathbf{e}^S_S := \mathbf{1}_S$ and $\mathbf{e}^S_{S^c} := \mathbf{0}_{S^c}$, which is a vector containing only 1's and 0's (similarly $\mathbf{e}^T \in \mathbb{R}^q$), and finally, by defining $\mathbf{A}^* := (\mathbf{e}^S, \mathbf{e}^T)^\top \mathbf{A}$, we obtain

$$(Y_1, Y_2)^\top := \left(\sum_{k\in S} X_k, \sum_{m\in T} X_m\right)^\top = (\mathbf{e}^S, \mathbf{e}^T)^\top \mathbf{A}\mathbf{Z} = \mathbf{A}^*\mathbf{Z}.$$

Thus if $\mathbf{A}^*$ and $\mathbf{Z}$ satisfy the assumptions of Sect. 4.1, we can directly apply the associated tail probability results and CoVaR asymptotics. Since

$$\max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a^*_{1\ell}, a^*_{2\ell}\} > 0] = \max_{\ell\in\mathbb{I}_d} \mathbb{P}\left[\min\left\{\sum_{k\in S} a^*_{k\ell}, \sum_{m\in T} a^*_{m\ell}\right\} > 0\right],$$

the essential condition $\max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a^*_{1\ell}, a^*_{2\ell}\} > 0] = 0$ (or > 0) for determining the MRV on $\mathbb{O}_2^{(2)}$ and hence the CoVaR rate and ECI is equivalent to

$$\max_{\ell\in\mathbb{I}_d} \max_{k\in S} \max_{m\in T} \mathbb{P}\big[\min\{a_{k\ell}, a_{m\ell}\} > 0] = 0 \qquad (\text{or} > 0). \tag{4.5}$$

Thus subject to determining (4.5), the asymptotic behaviour of CoVaR in (4.4) and the value of its associated ECI are direct consequences of the results in Sect. 4.1.

4.3 Risk contagion in more than two portfolios

In this section, for a financial system with two or more portfolios, we assess the risk of one financial entity performing poorly given that at least one other entity in the system is under stress. Alternatively, given that an individual entity has high negative returns, what is the effect of this to the entire system? We use the ideas from the results in the bivariate structure developed previously in Sect. 4.1 to obtain the asymptotics here. Clearly, the sparsity of the matrix defining the bipartite network given by $\mathbf{A}$ has a role to play. For this section, we concentrate on $\mathbf{Z} \in \mathbb{R}^d_+$ which are completely tail-equivalent.

Proposition 4.18 *Let $\mathbf{Z} \in \mathbb{R}^d_+$ be a completely tail-equivalent random vector and $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$, $\forall i \in \mathbb{I}_d$, with $b_i(t)/b_{i+1}(t) \to \infty$ as $t \to \infty$ for $i = 1, \ldots, d-1$. Let $\mathbf{A} \in \mathbb{R}^{q\times d}_+$ be a random matrix which has almost surely no trivial rows and is independent of $\mathbf{Z}$. Moreover, let $\mathbb{E}[\|\mathbf{A}\|^{\alpha+\epsilon}] < \infty$, where $\alpha := \alpha_1$. Define $\mathbf{X} = \mathbf{A}\mathbf{Z}$. Then for a fixed $k \in \mathbb{I}_q$ and $\mathbf{Y} := (Y_1, Y_2)^\top := (X_k, \max_{m\in\mathbb{I}_q\setminus\{k\}} X_m)$ the following hold*:

(a) *We have* $\mathbf{Y} \in \mathrm{MRV}(\alpha, b_1, \mu_1^*, \mathbb{O}_2^{(1)})$, *where for* $\mathbf{x} \in \mathbb{R}_+^2$,

$$\mu_1^*([\mathbf{0}, \mathbf{x}]^c) = \frac{\mu_1([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^{d} \mathbb{E}\Big[\max\Big\{a_{k\ell}/x_1, \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}/x_2\Big\}^{\alpha}\Big].$$

(b) *If* $\max_{m \in \mathbb{I}_q \setminus \{k\}} \max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{k\ell}, a_{m\ell}\} > 0] > 0$, *then*

$$\mathbf{Y} \in \mathrm{MRV}(\alpha, b_1, \mu_2^*, \mathbb{O}_2^{(2)}),$$

where for $\mathbf{x} \in (\mathbf{0}, \infty)$,

$$\mu_2^*\big((\mathbf{x}, \infty)\big) = \frac{\mu_1([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^{d} \mathbb{E}\Big[\max_{m \in \mathbb{I}_q \setminus \{k\}} \big\{\min\{a_{k\ell}/x_1, a_{m\ell}/x_2\}\big\}^{\alpha}\Big].$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[Y_1 > tx_1 | Y_2 > tx_2] \sim x_2^{\alpha} \frac{\sum_{\ell=1}^{d} \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} \{\min\{a_{k\ell}/x_1, a_{m\ell}/x_2\}\}^{\alpha}]}{\sum_{\ell=1}^{d} \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^{\alpha}]},$$

$$\mathbb{P}[Y_2 > tx_2 | Y_1 > tx_1] \sim x_1^{\alpha} \frac{\sum_{\ell=1}^{d} \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} \{\min\{a_{k\ell}/x_1, a_{m\ell}/x_2\}\}^{\alpha}]}{\sum_{\ell=1}^{d} \mathbb{E}[a_{k\ell}^{\alpha}]}.$$

Additionally, if the non-zero components of $\mathbf{A}$ *have a bounded support, bounded away from zero, then there exist* $0 < \upsilon_1^*, \upsilon_2^* < 1$ *such that for all* $0 < \upsilon < \upsilon_1^*$, *we have*

$$\mathrm{CoVaR}_{\upsilon|\gamma}(Y_1|Y_2) \sim \upsilon^{-\frac{1}{\alpha}}\, \mathrm{VaR}_{\gamma}(Y_1), \qquad \gamma \downarrow 0,$$

and for all $0 < \upsilon < \upsilon_2^*$, *we have*

$$\mathrm{CoVaR}_{\upsilon|\gamma}(Y_2|Y_1) \sim \upsilon^{-\frac{1}{\alpha}}\, \mathrm{VaR}_{\gamma}(Y_2), \qquad \gamma \downarrow 0.$$

Finally, $\mathrm{ECI}(Y_1|Y_2) = \mathrm{ECI}(Y_2|Y_1) = \infty$.

The first part of the proof is a direct application of Theorem 4.3 and the mapping theorem in Lindskog et al. [44, Theorem 2.3]. The second part of the proof then follows from Theorem 3.1. Hence the detailed proof of this proposition is omitted.

Remark 4.19 Note that if for fixed $k \in \mathbb{I}_q$, we have

$$\max_{m \in \mathbb{I}_q \setminus \{k\}} \max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{k\ell}, a_{m\ell}\} > 0] > 0,$$

then the measure μ_2^* in Proposition 4.18 is defined via μ_1, the limit measure of $\mathbf{Z}$ on $\mathbb{O}_d^{(1)}$. On the other hand, if

$$\max_{m \in \mathbb{I}_q \setminus \{k\}} \max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{k\ell}, a_{m\ell}\} > 0] = 0,$$

then μ_2^* will as well involve μ_2, the limit measure of $\mathbf{Z}$ on $\mathbb{O}_d^{(2)}$; thus the dependence structure of $\mathbf{Z}$ plays a role here. The CoVaR and ECI results for $\mathbf{Z}$ with an i.i.d., P-GC and P-MOC dependence structure are provided in Appendix D and follow a similar pattern to the results obtained in Sect. 4.1.

5 Conclusion

The goal of this paper has been to understand the tail risk behaviour in complex financial network models and to use this knowledge to provide asymptotic approximations for conditional risk measures like CoVaR. We have used the framework of a bipartite network to assess risk contagion in this regard. In our study, we have managed to accomplish a few things:

(i) For modelling via bipartite networks, it is natural to assume that various banks will invest in different assets, possibly non-overlapping; however, the exposure of these assets to the market may still make them dependent (regardless of whether they are asymptotically dependent or independent). We have shown that modelling via such distributions still allows us to compute conditional tail probabilities, which earlier computations have shown to be negligible. Moreover, the computed CoVaR measures can be drastically different depending on the behaviour of the joint distribution of the assets.

(ii) We have proposed the *extreme CoVaR index* which measures the strength of risk contagion between entities in a system; this is especially useful when the risks of underlying objects are asymptotically independent.

We have restricted to CoVaR measures, but we surmise that other conditional risk measures like mean expected shortfall (MES), mean marginal excess (MME) and systemic risk (SRISK) can also be computed under such models; we have briefly discussed some connections with MME. We have also particularly focused on Pareto or Pareto-like tails in this paper for convenience; naturally, some of the results can be extended to general regularly varying marginal tails as well. For Gaussian copulas with regularly varying tails, we still need certain restrictions on the marginal tail behaviour (see Das and Fasen-Hartmann [20] for details); on the other hand, for Marshall–Olkin copulas, assuming tail-equivalent regularly varying marginals will lead to similar results. In this paper, we focused on these particular copula models because of their flexibility and inherent connections with risk, reliability and specifically systemic risk. Clearly, our results need not be restricted to these copulas only. Although bivariate copula families are usually popular, Joe [38, Chap. 4] lists multiple extensions of bivariate copulas to general high dimensions. We believe that many such copulas can be explored for creating models with particular levels of asymptotic independence as necessitated by the context. Finally, we have left model calibration and statistical estimation to be pursued in future work.

Appendix A: Proofs of Sect. 2

For the proof of Proposition 2.6, we use the following auxiliary result given in Das and Fasen-Hartmann [19, Lemma A.2].

Lemma A.1 *Let $\Sigma \in \mathbb{R}^{d\times d}$ be positive definite and $\gamma(\Sigma)$, $I(\Sigma)$ be defined as in Lemma 2.4.*

(a) *Suppose $\Sigma^{-1}\mathbf{1} > \mathbf{0}$. Then for any $S \subseteq \mathbb{I}_d$ with $S \neq \mathbb{I}_d$, we have the inequality*

$$\gamma(\Sigma) > \gamma(\Sigma_S).$$

(b) *Suppose $\Sigma^{-1}\mathbf{1} \not> \mathbf{0}$. Then $I(\Sigma) \neq \mathbb{I}_d$ and for any $S \neq \mathbb{I}_d$ with $I(\Sigma) \subseteq S \subseteq \mathbb{I}_d$, we have*

$$\gamma(\Sigma) = \gamma(\Sigma_S).$$

For $S \subseteq \mathbb{I}_d$ with $S^c \cap I(\Sigma) \neq \emptyset$, we have $I(\Sigma) = I(\Sigma_S)$ and

$$\gamma(\Sigma) > \gamma(\Sigma_S).$$

Proof of Proposition 2.6 Note that

$$\gamma_i = \min_{S\subseteq\mathbb{I}_d, |S|\geq i} \min_{\mathbf{z}_S\geq\mathbf{1}_S} \mathbf{z}_S^\top \Sigma_S^{-1}\mathbf{z}_S$$

by definition (see Proposition 2.5). Hence there exists a set $S \subseteq \mathbb{I}_d$ with $|S| \geq i$ and $\gamma_i = \min_{\mathbf{z}_S\geq\mathbf{1}_S} \mathbf{z}_S^\top \Sigma_S^{-1}\mathbf{z}_S$. However, due to Lemma A.1, there exists as well a set $M \subseteq S$ with $|M| = |S| - 1 \geq i - 1$ and

$$\gamma_i = \gamma(\Sigma_S) > \gamma(\Sigma_M) \geq \gamma_{i-1},$$

and finally, $\alpha_i = \alpha\gamma_i > \alpha\gamma_{i-1} = \alpha_{i-1}$ and $b_i(t)/b_{i-1}(t) \to \infty$ as $t \to \infty$ for $i = 2, \ldots, d$. □

Proof of Proposition 2.8 (a) Suppose that for all $S \subseteq \mathbb{I}_d$ with $|S| = i$, we have $\Sigma_S^{-1}\mathbf{1}_S > \mathbf{0}_S$ and $\mathbf{1}_S^\top\Sigma_S^{-1}\mathbf{1}_S = \gamma_i$. Then $I(\Sigma_S) = I_S = S$ and $\gamma(\Sigma_S) = \mathbf{1}_S^\top\Sigma_S^{-1}\mathbf{1}_S = \gamma_i$. Thus $\mathcal{S}_i = \{S \subseteq \mathbb{I}_d : |S| = i\}$ and the statement follows directly from Proposition 2.5.

(b) Suppose that for some $S \subseteq \mathbb{I}_d$ with $|S| = i$, we have $\Sigma_S^{-1}\mathbf{1}_S > \mathbf{0}_S$ and $\gamma_i \neq \mathbf{1}_S^\top\Sigma_S^{-1}\mathbf{1}_S$. Then $I_S = S$, but $\gamma_i \neq \mathbf{1}_{I_S}^\top\Sigma_{I_S}^{-1}\mathbf{1}_{I_S}$, giving $S \notin \mathcal{S}_i$, and finally, the statement follows again from Proposition 2.5. □

Appendix B: Proofs of Sect. 3

Proof of Theorem 3.1 (a) **Case 1.** Suppose $F_1 = F_2$. Then we obviously obtain $\mathrm{VaR}_\gamma(Y_1) = \mathrm{VaR}_\gamma(Y_2)$ and $\mathbf{Y} = \mathbf{Y}' \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_2^{(i)})$ for $i = 1, 2$. For

$0 < \gamma < 1$ and $0 < \upsilon < 1$, we get

$$\begin{aligned}
&\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2)\\
&= \inf\{y \in \mathbb{R} : \mathbb{P}[Y_1 > y|Y_2 > \mathrm{VaR}_\gamma(Y_2)] \le \upsilon g(\gamma)\}\\
&= \mathrm{VaR}_\gamma(Y_1) \inf\{y \in \mathbb{R} : \mathbb{P}[Y_1 > y\,\mathrm{VaR}_\gamma(Y_1)|Y_2 > \mathrm{VaR}_\gamma(Y_1)] \le \upsilon g(\gamma)\}\\
&= \mathrm{VaR}_\gamma(Y_1) \inf\left\{y \in \mathbb{R} : h_\gamma(y) \le \upsilon g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big)\right\}\\
&= \mathrm{VaR}_\gamma(Y_1) h_\gamma^{\leftarrow}\Big(\upsilon g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big)\Big).
\end{aligned}$$

Due to our assumption, there exist constants $0 < a_1 < a_2 < r$ such that

$$a_1 < \liminf_{\gamma\downarrow 0} g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) \le \liminf_{\gamma\downarrow 0} g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) < a_2,$$

and hence there exists $\gamma_0 \in (0, 1)$ such that

$$g(\gamma)\gamma b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big) \in [a_1/2, (a_2 + \min\{r, a_2\})/2] \qquad \text{for } 0 < \gamma < \gamma_0.$$

Due to Lemma 3.3, as $\gamma \downarrow 0$, $h_\gamma^{\leftarrow}(y)/h^{-1}(y) \to 1$ uniformly on the compact interval $[a_1/2, (a_2 + \min\{r, a_2\})/2]$, and thus

$$\lim_{\gamma\downarrow 0} \frac{h_\gamma^{\leftarrow}(\upsilon g(\gamma)\gamma b_2^{\leftarrow}(\mathrm{VaR}_\gamma(Y_1))}{h^{-1}(\upsilon g(\gamma)\gamma b_2^{\leftarrow}(\mathrm{VaR}_\gamma(Y_1))} = 1.$$

Therefore we have

$$\begin{aligned}
\lim_{\gamma\downarrow 0} \frac{\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2)}{\mathrm{VaR}_\gamma(Y_1)h^{-1}(\upsilon g(\gamma)\gamma b_2^{\leftarrow}(\mathrm{VaR}_\gamma(Y_1)))} &= \lim_{\gamma\downarrow 0} \frac{h_\gamma^{\leftarrow}(\upsilon g(\gamma)\gamma b_2^{\leftarrow}(\mathrm{VaR}_\gamma(Y_1))}{h^{-1}(\upsilon g(\gamma)\gamma b_2^{\leftarrow}(\mathrm{VaR}_\gamma(Y_1)))}\\
&= 1.
\end{aligned}$$

Case 2. Suppose F_2 is arbitrary. Define $\mathbf{Y}' := (Y_1', Y_2') := (Y_1, F_1^{\leftarrow} \circ F_2(Y_2))$; by assumption, we have $\mathbf{Y}' \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_2^{(i)})$, $i = 1, 2$, and $Y_1' \stackrel{\mathrm{d}}{=} Y_2'$. Since $F_1^{\leftarrow} \circ F_2$ is an increasing function,

$$\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2) = \mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1'|Y_2')$$

and we can directly apply Case 1 to $\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1'|Y_2')$ and obtain the statement.

(b), (c) By the assumption that $h_\gamma^{\leftarrow}(v)/h^{-1}(v)$ converges uniformly on appropriate intervals, the proof follows analogously as the proof of (a). □

Proof of Lemma 3.3 From the fact that

$$\big(Y_1, F_1^{\leftarrow} \circ F_2(Y_2)\big) \in \mathrm{MRV}(\alpha_1, b_1, \mu_1, \mathbb{O}_2^{(1)}) \cap \mathrm{MRV}(\alpha_2, b_2, \mu_2, \mathbb{O}_2^{(2)})$$

and the continuity of h, it follows that $\mathrm{VaR}_\gamma(Y_1) \to \infty$ as $\gamma \downarrow 0$ and for any $y > 0$,

$$h_\gamma(y) \to h(y), \qquad \gamma \downarrow 0.$$

Since h is continuous and decreasing, we know from de Haan and Ferreira [23, Lemma 1.1.1] (taking $-h$, which is increasing) that for any $v \in (0, r)$,

$$h_\gamma^{\leftarrow}(v) \to h^{-1}(v), \qquad \gamma \downarrow 0. \tag{B.1}$$

Note that h^{-1} is a strictly decreasing, continuous function due to (i) such that

$$0 < h^{-1}(a_2) = \inf_{v\in[a_1,a_2]} h^{-1}(v) \le \sup_{v\in[a_1,a_2]} h^{-1}(v) = h^{-1}(a_1) < \infty.$$

Therefore it is sufficient to prove that

$$\sup_{v\in[a_1,a_2]} |h_\gamma^{\leftarrow}(v) - h^{-1}(v)| \longrightarrow 0, \qquad \gamma \downarrow 0,$$

or equivalently with $v = w^{-1}$,

$$\sup_{w\in[a_2^{-1},a_1^{-1}]} |h_\gamma^{\leftarrow}(w^{-1}) - h^{-1}(w^{-1})| \longrightarrow 0, \qquad \gamma \downarrow 0.$$

Define

$$F_\gamma(w) = \frac{h_\gamma^{\leftarrow}(w^{-1})}{h_\gamma^{\leftarrow}(a_1)} \quad \text{and} \quad F(w) = \frac{h^{-1}(w^{-1})}{h^{-1}(a_1)} \qquad \text{for } w \in [a_2^{-1}, a_1^{-1}],$$

with $F_\gamma(w) = F(w) = 1$ for $w > a_1^{-1}$ and $F_\gamma(w) = F(w) = 0$ for $w < a_2^{-1}$. Then F_γ and F are distribution functions (in particular, F_γ is right-continuous with left limits since $h_\gamma^{\leftarrow}$ is left-continuous with right limits), and due to (B.1), we have as well $\lim_{\gamma\downarrow 0} F_\gamma(w) = F(w)$ for every $w \in \mathbb{R}$. Since F is continuous, Polya's theorem [53] actually gives the uniform convergence

$$\lim_{\gamma\downarrow 0} \sup_{v\in[a_1,a_2]} \left| \frac{h_\gamma^{\leftarrow}(v)}{h_\gamma^{\leftarrow}(a_1)} - \frac{h^{-1}(v)}{h^{-1}(a_1)} \right| = \lim_{\gamma\downarrow 0} \sup_{w\in[a_2^{-1},a_1^{-1}]} |F_\gamma(w) - F(w)| = 0.$$

Finally,

$$\begin{aligned}
&\sup_{v\in[a_1,a_2]} |h_\gamma^{\leftarrow}(v) - h^{-1}(v)| \\
&\le \sup_{v\in[a_1,a_2]} \left| h_\gamma^{\leftarrow}(v) - h^{-1}(v)\frac{h_\gamma^{\leftarrow}(a_1)}{h^{-1}(a_1)} \right| + \sup_{v\in[a_1,a_2]} \left| h^{-1}(v)\frac{h_\gamma^{\leftarrow}(a_1)}{h^{-1}(a_1)} - h^{-1}(v) \right| \\
&= h_\gamma^{\leftarrow}(a_1) \sup_{v\in[a_1,a_2]} \left| \frac{h_\gamma^{\leftarrow}(v)}{h_\gamma^{\leftarrow}(a_1)} - \frac{h^{-1}(v)}{h^{-1}(a_1)} \right| + \left| \frac{h_\gamma^{\leftarrow}(a_1)}{h^{-1}(a_1)} - 1 \right| \sup_{v\in[a_1,a_2]} |h^{-1}(v)| \\
&\to 0 \qquad \text{as } \gamma \downarrow 0,
\end{aligned}$$

which gives the statement. □

Proof of Proposition 3.12 (a)(i) First of all,

$$h_\gamma(y) := b_2^{\leftarrow}\big(\mathrm{VaR}_\gamma(Y_1)\big)\mathbb{P}[Y_1 > y\,\mathrm{VaR}_\gamma(Y_1), Y_2 > \mathrm{VaR}_\gamma(Y_1)],$$

and due to Proposition 2.11, we have $h(y) = y^{-\alpha}$ for $y \geq 1$ and $h^{-1}(v) = v^{-1/\alpha}$ for $v \in (0, 1]$. Define $\eta = \frac{1}{2}$. Since

$$b_2^{\leftarrow}(t) = \theta^{-(\eta+1)}t^{(\eta+1)\alpha} \sim \big(\overline{F}_1(t)\overline{F}_2(t)^\eta\big)^{-1}, \qquad t \to \infty,$$

we take without loss of generality $b_2^{\leftarrow}(t) = (\overline{F}_1(t)\overline{F}_2(t)^\eta)^{-1}$. Then we have for $y \geq 1$ that

$$\begin{aligned} b_2^{\leftarrow}(t)\mathbb{P}[Y_1 > yt, Y_2 > t] &= b_2^{\leftarrow}(t)\widehat{C}_\Lambda^{\mathrm{MO}}\big(\overline{F}_1(yt), \overline{F}_2(t)\big) \\ &= b_2^{\leftarrow}(t)\overline{F}_1(yt)\overline{F}_2(t)^\eta \\ &= \frac{\overline{F}_1(yt)}{\overline{F}_1(t)} =: \widetilde{h}_t(y). \end{aligned}$$

Moreover,

$$\frac{\widetilde{h}_t^{\leftarrow}(v)}{v^{-\frac{1}{\alpha}}} = \frac{\overline{F}_1^{\leftarrow}(v\overline{F}_1(t))}{v^{-\frac{1}{\alpha}}t} = \frac{\overline{F}_1^{\leftarrow}(v\overline{F}_1(t))}{v^{-\frac{1}{\alpha}}\overline{F}_1^{\leftarrow}(\overline{F}_1(t))}\,\frac{\overline{F}_1^{\leftarrow}(\overline{F}_1(t))}{t}.$$

The first factor converges to 1 uniformly on $(0, 1]$ as $t \to \infty$ by our assumption, and the second factor converges to 1 as $t \to \infty$ as well. Hence $\widetilde{h}_t^{\leftarrow}(v)/h^{-1}(v)$ converges uniformly on $(0, 1]$ to 1 as $t \to \infty$. Finally, $h_\gamma^{\leftarrow}(v)/h^{-1}(v)$ converges uniformly on $(0, 1]$ to 1 as $\gamma \downarrow 0$ so that the assumption in Theorem 3.1(b) is satisfied. Since $v\gamma^{\beta+1}b_2^{\leftarrow}(\mathrm{VaR}_\gamma(Y_1)) \sim v\gamma^{\beta-\eta}$, we have $h^{-1}(v\gamma^{\beta+1}b_2^{\leftarrow}(\mathrm{VaR}_\gamma(Y_1)) \sim v^{-\frac{1}{\alpha}}\gamma^{-\frac{\beta-\eta}{\alpha}}$ as $\gamma \downarrow 0$. The final statement is then an application of Theorem 3.1(b).

(a)(ii) For $0 < y \leq 1$, we have $h(y) = y^{-\alpha\eta}$, and for $v \in [1, \infty)$, we have $h^{-1}(v) = v^{-\frac{1}{\alpha\eta}}$. Furthermore,

$$b_2^{\leftarrow}(t)\mathbb{P}[Y_1 > yt, Y_2 > t] = b_2^{\leftarrow}(t)\overline{F}_1(yt)^\eta\overline{F}_2(t).$$

The rest of the proof follows analogously as the proof of (i) by using Theorem 3.1(c).

(b) The proof is analogous to that of part (a) by taking $\eta = 1/3$. □

Proof of Proposition 3.14 In the bivariate case, we have the parameters $I_2 = \{1, 2\}$, $\gamma_2 = \frac{2}{1+\rho}$, $\alpha_2 = \frac{2\alpha}{1+\rho}$ and $h_s^S = \frac{1}{1+\rho}$ for $S = \{1, 2\}$; cf. Das and Fasen-Hartmann [20, Example 3.8]. Define

$$\begin{aligned} b_2^{\leftarrow}(t) &= (\theta t^{-\alpha})^{-\frac{2}{1+\rho}}(2\alpha\log t)^{\frac{\rho}{1+\rho}}(2\pi)^{\frac{\rho}{1+\rho}}\frac{(1-\rho)^{\frac{1}{2}}}{(1+\rho)^{\frac{3}{2}}}, \\ \mu_2\big((\mathbf{z}, \infty)\big) &= (z_1z_2)^{-\frac{\alpha}{1+\rho}}, \qquad \mathbf{z} = (z_1, z_2)^\top \in \mathbb{R}_+^2, \\ h^{-1}(v) &= v^{-\frac{1+\rho}{\alpha}}, \qquad v > 0. \end{aligned}$$

A conclusion of Proposition 2.5 is that $(Y_1, Y_2)^\top \in \mathrm{MRV}(\alpha_2, b_2, \mu_2, \mathbb{O}_2^{(2)})$ (note that the constant $\Upsilon_{\{1,2\}}$ is moved to b_2). The proof follows now by applying the forms of b_2, μ_2 in Theorem 3.1(a). □

Appendix C: Proofs of Sect. 4

Proof of Theorem 4.3 Define $b_i^*(t) = F_{Z_{(i)}}^{\leftarrow}(1 - 1/t)$. Then

$$\lim_{t\to\infty} \frac{b_i^{\leftarrow}(t)}{b_i^{*\leftarrow}(t)} = \lim_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}[Z_{(i)} > t]$$
$$= \mu_i(\{\mathbf{z} \in \mathbb{R}_+^d : z_{(i)} > 1\}) =: c_i \in (0, \infty). \tag{C.1}$$

Thus $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)})$ is equivalent to $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i^*, \mu_i^*, \mathbb{O}_d^{(i)})$ with the measure $\mu_i^* = (c_i)^{-1}\mu_i$. Furthermore, due to (C.1), we also have $b_i^*(t)/b_{i+1}^*(t) \to \infty$ as $t \to \infty$, $i = 1, \ldots, d-1$. Thus (a) is a consequence of Das et al. [21, Theorem 3.4] and (b) of [21, Proposition 3.2] in combination with (C.1) and the fact that $\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i^*, \mu_i^*, \mathbb{O}_d^{(i)})$ for all $i \in \mathbb{I}_d$. □

Proof of Proposition 4.8 (a) First of all, $\tau_{(1,1)}(\mathbf{A}) < \infty$ and $\tau_{(1,2)}(\mathbf{A}) = \infty$ a.s. so that $i_1^* = 1$ and $\mathbb{P}[\Omega_1^{(1)}] = 1$. Since $[\mathbf{0}, \mathbf{x}]^c \in \mathcal{B}(\mathbb{O}_2^{(1)})$, we obtain by the definition of $\overline{\mu}_1$ that

$$\overline{\mu}_1([\mathbf{0}, \mathbf{x}]^c) = \mathbb{E}\big[\mu_1\big(\mathbf{A}^{-1}([\mathbf{0}, \mathbf{x}]^c)\big)\big]. \tag{C.2}$$

Furthermore, from Das et al. [21, Remark 7], we already know that due to

$$\mathbf{Z} \in \mathrm{MRV}(\alpha_i, b_i, \mu_i, \mathbb{O}_d^{(i)}), \quad i = 1, 2,$$

with $b_1(t)/b_2(t) \to \infty$, the support of μ_1 is restricted to

$$\{\mathbf{z} \in \mathbb{R}_+^d : z_{(2)} = 0\}\backslash\{\mathbf{0}\} = \bigcup_{\ell=1}^{d} \big\{\mathbf{z} \in \mathbb{R}_+^d : z_\ell > 0, z_j = 0, \ \forall\, j \in \mathbb{I}_d\backslash\{\ell\}\big\} =: \bigcup_{\ell=1}^{d} \mathbb{T}_\ell,$$

which is a disjoint union. Together with (C.2), this implies that

$$\begin{aligned}
\overline{\mu}_1([0, \mathbf{x}]^c) &= \mathbb{E}\big[\mu_1\big(\mathbf{A}^{-1}([0, \mathbf{x}]^c)\big)\big] \\
&= \sum_{\ell=1}^{d} \mathbb{E}\big[\mu_1\big(\mathbf{A}^{-1}([0, \mathbf{x}]^c) \cap \mathbb{T}_\ell\big)\big] \\
&= \sum_{\ell=1}^{d} \mathbb{E}[\mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : a_{1\ell}z_\ell > x_1 \text{ or } a_{2\ell}z_\ell > x_2\})] \\
&= \sum_{\ell=1}^{d} \mathbb{E}\big[\mu_1\big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \max\{a_{1\ell}z_\ell/x_1, a_{2\ell}z_\ell/x_2\} > 1\big\}\big)\big].
\end{aligned}$$

(b) The set $(\mathbf{x}, \infty)$ is in $\mathcal{B}(\mathbb{O}_2^{(2)})$. When $k = 2$, we have

$$\Omega_1^{(2)} = \Big\{ \min_{\ell \in \mathbb{I}_d} \{a_{1\ell}, a_{2\ell}\} > 0 \Big\} \quad \text{and} \quad \Omega_2^{(2)} = \Big\{ \min_{\ell \in \mathbb{I}_d} \{a_{1\ell}, a_{2\ell}\} = 0 \Big\}.$$

Now $\mathbb{P}[\Omega_1^{(2)}] > 0$ by our assumption and hence $i_2^* = 1$. A consequence of the definition of $\overline{\mu}_2$ and $(\mathbf{x}, \infty) \in \mathcal{B}(\mathbb{O}_2^{(2)})$ is then that

$$\begin{aligned}
\overline{\mu}_2\big((\mathbf{x}, \infty)\big) &= \mathbb{E}\Big[\mu_1\Big(\mathbf{A}^{-1}\big((\mathbf{x}, \infty)\big) \cap \Omega_1^{(2)}\Big)\Big] \\
&= \sum_{\ell=1}^{d} \mathbb{E}[\mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : a_{1\ell} z_\ell > x_1, a_{2\ell} z_\ell > x_2\} \cap \Omega_1^{(2)})] \\
&= \sum_{\ell=1}^{d} \mathbb{E}\big[\mu_1\big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \min\{a_{1\ell} z_\ell / x_1, a_{2\ell} z_\ell / x_2\} > 1\big\} \cap \Omega_1^{(2)}\big)\big] \\
&= \sum_{\ell=1}^{d} \mathbb{E}\big[\mu_1\big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \min\{a_{1\ell} z_\ell / x_1, a_{2\ell} z_\ell / x_2\} > 1\big\}\big)\big].
\end{aligned}$$

(c) By assumption, we have $\mathbb{P}[\Omega_1^{(2)}] = 0$ so that $i_2^* = 2$ and $\mathbb{P}[\Omega_2^{(2)}] = 1$. Hence the definition of $\overline{\mu}_2$ implies that

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \mathbb{E}\Big[\mu_2\Big(\mathbf{A}^{-1}\big((\mathbf{x}, \infty)\big) \cap \Omega_2^{(2)}\Big)\Big] = \mathbb{E}\Big[\mu_2\Big(\mathbf{A}^{-1}\big((\mathbf{x}, \infty)\big)\Big)\Big].$$

Again from Das et al. [21, Remark 7], we already know that the support of μ_2 is restricted to

$$\begin{aligned}
&\{\mathbf{z} \in \mathbb{R}_+^d : z_{(3)} = 0\} \setminus \{\mathbf{z} \in \mathbb{R}_+^d : z_{(2)} = 0\} \\
&= \bigcup_{1 \le \ell < j \le d} \big\{\mathbf{z} \in \mathbb{R}_+^d : z_\ell > 0, z_j > 0, z_m = 0, \ \forall m \in \mathbb{I}_d \setminus \{\ell, j\}\big\} \\
&=: \bigcup_{1 \le \ell < j \le d} \mathbb{T}_{\ell, j}
\end{aligned}$$

so that

$$\begin{aligned}
\overline{\mu}_2\big((\mathbf{x}, \infty)\big) &= \sum_{1 \le \ell < j \le d} \mathbb{E}\Big[\mu_2\Big(\mathbf{A}^{-1}\big((\mathbf{x}, \infty)\big)\Big) \cap \mathbb{T}_{\ell, j}\Big] \\
&= \sum_{\substack{\ell, j = 1 \\ \ell \ne j}}^{d} \mathbb{E}\big[\mu_2\big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \min\{a_{1\ell} z_\ell / x_1, a_{2j} z_j / x_2\} > 1\big\}\big)\big],
\end{aligned}$$

completing the proof. □

Proof of Proposition 4.9 (a) Since $Z_1, \dots, Z_d$ are completely tail-equivalent, we have

$$\mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : z_\ell > 1\}) = \mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : z_1 > 1\}), \qquad \ell \in \mathbb{I}_d,$$

and by the choice of b_1,

$$\begin{aligned}\mu_1([\mathbf{0}, \mathbf{1}]^c) &= \mu_1\left([\mathbf{0}, \mathbf{1}]^c \cap \bigcup_{\ell=1}^d \mathbb{T}_\ell\right) \\ &= \sum_{\ell=1}^d \mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : z_\ell > 1\}) \\ &= d\mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : z_1 > 1\}).\end{aligned}$$

This implies

$$\mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : z_\ell > 1\}) = \frac{\mu_1([\mathbf{0}, \mathbf{1}]^c)}{d}, \qquad \ell \in \mathbb{I}_d. \tag{C.3}$$

Furthermore, μ_1 is homogeneous of order $-\alpha$ so that, together with Proposition 4.8(a), we obtain

$$\begin{aligned}\overline{\mu}_1([\mathbf{0}, \mathbf{x}]^c) &= \sum_{\ell=1}^d \mathbb{E}\big[\mu_1\big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \max\{a_{1\ell}z_\ell/x_1, a_{2\ell}z_\ell/x_2\} > 1\big\}\big)\big] \\ &= \sum_{\ell=1}^d \mathbb{E}[(\max\{a_{1\ell}/x_1, a_{2\ell}/x_2\})^\alpha]\mu_1(\{\mathbf{z} \in \mathbb{R}_+^d : z_\ell > 1\}) \\ &= \frac{\mu_1([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^d \mathbb{E}[(\max\{a_{1\ell}/x_1, a_{2\ell}/x_2\})^\alpha].\end{aligned}$$

Finally, the almost surely non-zero rows of $\mathbf{A}$ imply that the right-hand side above is strictly positive, and hence $\overline{\mu}_1$ is a non-null measure. From $i_1^* = 1$ and Theorem 4.3, we then conclude that $(X_1, X_2)^\top \in \mathrm{MRV}(\alpha_1, b_1, \overline{\mu}_1, \mathbb{O}_2^{(1)})$.

(b) From Proposition 4.8(b), we already know that $i_2^* = 2$ and

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \mathbb{E}\big[\mu_1\big(\big\{\mathbf{z} \in \mathbb{R}_+^d : \min\{a_{1\ell}z_\ell/x_1, a_{2\ell}z_\ell/x_2\} > 1\big\}\big)\big].$$

Again the homogeneity of μ_1 of order $-\alpha$ and (C.3) give

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \frac{\mu_1([\mathbf{0}, \mathbf{1}]^c)}{d} \sum_{\ell=1}^d \mathbb{E}[(\min\{a_{1\ell}/x_1, a_{2\ell}/x_2\})^\alpha],$$

which is strictly positive because of $\max_{\ell\in\mathbb{I}_d}\mathbb{P}[\min\{a_{1\ell},a_{2\ell}\}>0]>0$. Then a consequence of Theorem 4.3 is that $(X_1,X_2)^\top\in\mathrm{MRV}(\alpha_1,b_1,\overline{\mu}_2,\mathbb{O}_2^{(2)})$. Finally,

$$\begin{aligned}\mathbb{P}[X_1>tx_1|X_2>tx_2]&=\frac{\mathbb{P}[X_1>tx_1,X_2>tx_2]}{\mathbb{P}[X_2>tx_2]}\\&\sim\frac{\overline{\mu}_2((\mathbf{x},\infty))}{\overline{\mu}_1([0,\infty)\times(x_2,\infty))}\\&=x_2^\alpha\frac{\mathbb{E}[(\min\{a_{1\ell}/x_1,a_{2\ell}/x_2\})^\alpha]}{\sum_{\ell=1}^d\mathbb{E}[a_{2\ell}^\alpha]}\qquad\text{as } t\to\infty\end{aligned}$$

by the former results. The asymptotic behaviour of CoVaR is then a consequence of Theorem 3.1(a) and (3.2). □

Proof of Proposition 4.11 First note that

$$\mu_2(\{\mathbf{z}\in\mathbb{R}_+^d:z_\ell>v_\ell,z_j>v_j\})=(v_\ell v_j)^{-\alpha},\qquad\mathbf{z}\in(\mathbf{0},\infty).$$

Plugging this into Proposition 4.8(c) results in

$$\begin{aligned}\overline{\mu}_2\big((\mathbf{x},\infty)\big)&=\sum_{\substack{\ell,j=1\\\ell\neq j}}^d\mathbb{E}[\mu_2(\{\mathbf{z}\in\mathbb{R}_+^d:a_{1\ell}z_\ell>x_1,a_{2j}z_j>x_2\})\\&=\sum_{\ell,j=1}^d\mathbb{E}\bigg[\bigg(\frac{a_{1\ell}a_{2j}}{x_1x_2}\bigg)^\alpha\bigg],\end{aligned}$$

where we used in the last step the assumption $\max_{\ell\in\mathbb{I}_d}\mathbb{P}[\min\{a_{1\ell},a_{2\ell}\}>0]=0$. Furthermore, $b_1^{\leftarrow}(t)=1/\overline{F}_\alpha(t)$ and $b_2^{\leftarrow}(t)=1/\overline{F}_\alpha(t)^2$ so that as $t\to\infty$,

$$\begin{aligned}\mathbb{P}[X_1>tx_1|X_2>tx_2]&=\frac{b_1^{\leftarrow}(t)}{b_2^{\leftarrow}(t)}\frac{b_2^{\leftarrow}(t)\mathbb{P}[X_1>tx_1,X_2>tx_2]}{b_1^{\leftarrow}(t)\mathbb{P}[X_2>tx_2]}\\&\sim\overline{F}_\alpha(t)\frac{\overline{\mu}_2((\mathbf{x},\infty))}{\overline{\mu}_1((x_1,\infty)\times\mathbb{R}_+)}\\&=\overline{F}_\alpha(t)x_2^\alpha\frac{\sum_{\ell,j=1}^d\mathbb{E}[(\frac{a_{1\ell}}{x_1}\frac{a_{2j}}{x_2})^\alpha]}{\sum_{j=1}^d\mathbb{E}[a_{2j}^\alpha]}.\end{aligned}$$

Again the asymptotic behaviour of CoVaR follows then from Theorem 3.1(a) and (3.2) with similar arguments as in Proposition 4.9. □

Proof of Proposition 4.13 Similarly as in Proposition 4.11, the proof of the MRV is a combination of Proposition 4.8(c) and Proposition 2.11, and the asymptotic behaviour of CoVaR can be derived from Theorem 3.1(a). □

Proof of Proposition 4.15 For any set $S = \{\ell, j\} \subseteq \mathbb{I}_d$ with $|S| = 2$, we have

$$\Sigma_S^{-1} = \frac{1}{1-\rho_{\ell j}^2}\begin{pmatrix} 1 & -\rho_{j\ell} \\ -\rho_{\ell j} & 1 \end{pmatrix},$$

$\Sigma_S^{-1}\mathbf{1}_S > \mathbf{0}_S$ and $\mathbf{1}_S^\top \Sigma_S^{-1}\mathbf{1}_S = \frac{2}{1+\rho_{\ell j}}$. Thus $\mathcal{S}_2$ defined in Proposition 2.5 is equal to

$$\mathcal{S}_2 = \left\{\{\ell, j\} \subseteq \mathbb{I}_d : \frac{2}{1+\rho_{\ell j}} = \frac{2}{1+\rho^\vee}\right\} = \left\{\{\ell, j\} \subseteq \mathbb{I}_d : \rho_{\ell j} = \rho^\vee\right\},$$

$|I(\Sigma_S)| = |I_S| = |S| = 2$ and $\gamma_S = \frac{2}{1+\rho^\vee}$ for any $S \in \mathcal{S}_2$. Finally, with α_2, b_2 as given above, $z_\ell, z_j > 0$ and

$$\mu_2(\{\mathbf{z} \in \mathbb{R}_+^d : z_\ell > v_\ell, z_j > v_j\}) = \begin{cases} \frac{1}{2\pi}\frac{(1+\rho^\vee)^{3/2}}{(1-\rho^\vee)^{1/2}}(v_\ell v_j)^{-\frac{\alpha}{1+\rho_{\ell j}}}, & \text{if } \rho_{\ell j} = \rho^\vee, \\ 0, & \text{otherwise,} \end{cases}$$

we have due to Proposition 2.5 that $\mathbf{Z} \in \mathrm{MRV}(\alpha_2, b_2, \mu_2, \mathbb{O}_d^{(2)})$. The representation of μ_2 and Proposition 4.8(c) imply that for $\mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty)$, we have

$$\overline{\mu}_2\big((\mathbf{x}, \infty)\big) = \frac{1}{2\pi}\frac{(1+\rho^\vee)^{3/2}}{(1-\rho^\vee)^{1/2}} \sum_{(\ell, j):\, \rho_{\ell j} = \rho^\vee} \mathbb{E}\left[\left(\frac{a_{1\ell}a_{2j}}{x_1 x_2}\right)^{\frac{\alpha}{1+\rho^\vee}}\right]. \tag{C.4}$$

(a) If $\rho^* = \rho^\vee$, then of course $\overline{\mu}_2((\mathbf{x}, \infty)) > 0$ and it follows from Theorem 4.3 that $(X_1, X_2)^\top \in \mathrm{MRV}(\alpha_2, b_2, \overline{\mu}_2, \mathbb{O}_2^{(2)})$. Finally, the asymptotic behaviour of the conditional probability can be calculated as in the previous statements, and the CoVaR asymptotics follow from Theorem 3.1(a).

(b) If $\rho^* < \rho^\vee$, then (C.4) results in $\overline{\mu}_2((\mathbf{x}, \infty)) = 0$. Then define

$$M := \{\ell \in \mathbb{I}_d : \rho_{\ell j} \leq \rho^* \ \forall\ j \in \mathbb{I}_d\}.$$

For any $\ell \in \mathbb{I}_d \backslash M$, there exists $j \in \mathbb{I}_d$, $j \neq \ell$, with $\rho_{\ell j} > \rho^*$, and hence Remark 4.16 gives that the ℓth column of $\mathbf{A}$ is a.s. a zero column. Therefore we define $\mathbf{A}_{\{1,2\},M}$ by deleting the ℓth column in $\mathbf{A}$ for all $\ell \in \mathbb{I}_d \backslash M$ and similarly, we define $\mathbf{Z}_M$. Then

$$\mathbf{X} = \mathbf{A}\mathbf{Z} = \mathbf{A}_{\{1,2\},M}\mathbf{Z}_M.$$

But $\mathbf{Z}_M \in$ P-GC$(\alpha, \theta, \Sigma_M)$ and $\rho^* = \max_{\ell, j \in M, \ell \neq j} \rho_{\ell j}$. Hence $\mathbf{X} = \mathbf{A}_{\{1,2\},M}\mathbf{Z}_M$ satisfies the assumption of (a), and an application of (a) gives the statement. □

Appendix D: Risk contagion with more than two portfolios

In Sect. 4.3, we obtained asymptotic conditional tail probabilities and CoVaR asymptotics comparing the risk of high negative returns for one single entity in the system versus the worst (or at least one other entity in the entire system) having poor returns, for a general regularly varying underlying distribution $\mathbf{Z}$. In this section, we detail results for particular choices of $\mathbf{Z}$. First, we consider the case with i.i.d. components.

Proposition D.1 *Let $\mathbf{Z} \in \mathbb{R}_+^d$ be a random vector with i.i.d. components $Z_1, \ldots, Z_d$ with distribution function F_α, where $\overline{F}_\alpha \in \mathrm{RV}_{-\alpha}$, $\alpha > 0$, $b_1(t) = F_\alpha^{\leftarrow}(1 - 1/t)$ and $b_i^{\leftarrow}(t) = (b_1^{\leftarrow}(t))^i$. Further, let $\mathbf{A} \in \mathbb{R}_+^{q \times d}$ be a random matrix which has almost surely no trivial rows and is independent of $\mathbf{Z}$, and suppose for fixed $k \in \mathbb{I}_q$ that*

$$\max_{m \in \mathbb{I}_q \setminus \{k\}} \max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{k\ell}, a_{m\ell}\} > 0] = 0$$

and $\mathbb{E}[\|\mathbf{A}\|^{2\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$. Define $\mathbf{X} = \mathbf{A}\mathbf{Z}$. Then

$$\mathbf{Y} = (Y_1, Y_2)^\top = \Big(X_k, \max_{m \in \mathbb{I}_q \setminus \{k\}} X_m\Big)^\top \in \mathrm{MRV}(2\alpha, b_2, \mu_2^*, \mathbb{O}_2^{(2)}),$$

where

$$\mu_2^*\big((\mathbf{x}, \infty)\big) = (x_1 x_2)^{-\alpha} \sum_{\ell, j=1}^{d} \mathbb{E}\Big[a_{k\ell}^\alpha \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^\alpha\Big], \qquad \mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty).$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[Y_1 > t x_1 | Y_2 > t x_2] \sim \big(b_1^{\leftarrow}(t)\big)^{-1} x_1^{-\alpha} \frac{\sum_{\ell, j=1}^d \mathbb{E}[a_{kj}^\alpha \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^\alpha]}{\sum_{\ell=1}^d \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^\alpha]},$$

$$\mathbb{P}[Y_2 > t x_2 | Y_1 > t x_1] \sim \big(b_1^{\leftarrow}(t)\big)^{-1} x_2^{-\alpha} \frac{\sum_{\ell, j=1}^d \mathbb{E}[a_{kj}^\alpha \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^\alpha]}{\sum_{j=1}^d \mathbb{E}[a_{kj}^\alpha]}.$$

Additionally, for $0 < \upsilon < 1$, we have as $\gamma \downarrow 0$ that

$$\mathrm{CoVaR}_{\upsilon\gamma|\gamma}(Y_1|Y_2) \sim \upsilon^{-\frac{1}{\alpha}} \frac{(\sum_{\ell, j=1}^d \mathbb{E}[a_{kj}^\alpha \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^\alpha])^{\frac{1}{\alpha}}}{(\sum_{j=1}^d \mathbb{E}[a_{kj}^\alpha] \sum_{\ell=1}^d \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^\alpha])^{\frac{1}{\alpha}}} \mathrm{VaR}_\gamma(Y_1),$$

$$\mathrm{CoVaR}_{\upsilon\gamma|\gamma}(Y_2|Y_1) \sim \upsilon^{-\frac{1}{\alpha}} \frac{(\sum_{\ell, j=1}^d \mathbb{E}[a_{kj}^\alpha \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^\alpha])^{\frac{1}{\alpha}}}{(\sum_{j=1}^d \mathbb{E}[a_{kj}^\alpha] \sum_{\ell=1}^d \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{m\ell}^\alpha])^{\frac{1}{\alpha}}} \mathrm{VaR}_\gamma(Y_2).$$

Finally, $\mathrm{ECI}(Y_1|Y_2) = \mathrm{ECI}(Y_2|Y_1) = 1$.

Next we consider an underlying vector $\mathbf{Z}$ with Marshall–Olkin dependence.

Proposition D.2 *Let $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \Lambda)$ and $\mathbf{A} \in \mathbb{R}_+^{q \times d}$ be a random matrix which has almost surely no trivial rows, is independent of $\mathbf{Z}$ and for fixed $k \in \mathbb{I}_q$, we have*

$$\max_{m \in \mathbb{I}_q \setminus \{k\}} \max_{\ell \in \mathbb{I}_d} \mathbb{P}[\min\{a_{k\ell}, a_{m\ell}\} > 0] = 0.$$

Further, let $\mathbf{X} = \mathbf{A}\mathbf{Z}$ and $\mathbf{Y} = (Y_1, Y_2)^\top = (X_k, \max_{m \in \mathbb{I}_q \setminus \{k\}} X_m)^\top$.

(a) *Suppose* $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^{=})$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\frac{3\alpha}{2}+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then we have* $\mathbf{Y} \in \text{MRV}(\alpha_2, b_2, \mu_2^*, \mathbb{O}_2^{(2)})$, *where* $\alpha_2 = (3\alpha)/2$, $b_2(t) = \theta^{\frac{1}{\alpha}} t^{\frac{2}{3\alpha}}$, *and for any* $\mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty)$, *we have*

$$\mu_2^*\big((\mathbf{x}, \infty)\big) = \sum_{\ell, j=1}^{d} \mathbb{E}\left[\min\left\{\frac{a_{k\ell}}{x_1}, \max_{m \in \mathbb{I}_q \setminus \{k\}} \frac{a_{mj}}{x_2}\right\}^{\alpha} \max\left\{\frac{a_{k\ell}}{x_1}, \max_{m \in \mathbb{I}_q \setminus \{k\}} \frac{a_{mj}}{x_2}\right\}^{\frac{\alpha}{2}}\right].$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[Y_1 > tx_1 | Y_2 > tx_2] \sim (\theta t^{-\alpha})^{\frac{1}{2}} x_2^{\alpha} \mu_2^*\big((\mathbf{x}, \infty)\big)\left(\sum_{j=1}^{d} \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^{\alpha}]\right)^{-1},$$

$$\mathbb{P}[Y_2 > tx_2 | Y_1 > tx_1] \sim (\theta t^{-\alpha})^{\frac{1}{2}} x_1^{\alpha} \mu_2^*\big((\mathbf{x}, \infty)\big)\left(\sum_{\ell=1}^{d} \mathbb{E}[a_{k\ell}^{\alpha}]\right)^{-1}.$$

Additionally, if the non-zero components of $\mathbf{A}$ *have bounded support, bounded away from zero, then there exist* $0 < \upsilon_1^* < \upsilon_2^* < \infty$ *such that for all* $0 < \upsilon < \upsilon_1^*$,

$$\begin{aligned}
&\text{CoVaR}_{\upsilon\gamma^{\frac{1}{2}}|\gamma}(Y_1|Y_2) \\
&\sim \upsilon^{-\frac{1}{\alpha}} \frac{(\sum_{\ell,j=1}^{d} \mathbb{E}[a_{k\ell}^{\alpha} \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^{\alpha/2}])^{\frac{1}{\alpha}}}{(\sum_{\ell=1}^{d} \mathbb{E}[a_{k\ell}^{\alpha}])^{\frac{1}{\alpha}} (\sum_{j=1}^{d} \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^{\alpha}])^{\frac{1}{2\alpha}}} \text{VaR}_{\gamma}(Y_1), \qquad \gamma \downarrow 0,
\end{aligned}$$

and for all $\upsilon_2^* < \upsilon < \infty$,

$$\begin{aligned}
&\text{CoVaR}_{\upsilon\gamma^{\frac{1}{2}}|\gamma}(Y_1|Y_2) \\
&\sim \upsilon^{-\frac{2}{\alpha}} \frac{(\sum_{\ell,j=1}^{d} \mathbb{E}[a_{k\ell}^{\alpha/2} \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^{\alpha}])^{\frac{2}{\alpha}}}{(\sum_{\ell=1}^{d} \mathbb{E}[a_{k\ell}^{\alpha}])^{\frac{1}{\alpha}} (\sum_{j=1}^{d} \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^{\alpha}])^{\frac{2}{\alpha}}} \text{VaR}_{\gamma}(Y_1), \qquad \gamma \downarrow 0.
\end{aligned}$$

Finally, $\text{ECI}(Y_1|Y_2) = \text{ECI}(Y_2|Y_1) = 2$.

(b) *Suppose* $\mathbf{Z} \in$ P-MOC$(\alpha, \theta, \lambda^{\propto})$ *and* $\mathbb{E}[\|\mathbf{A}\|^{\alpha\frac{3d+2}{2(d+1)}+\epsilon}] < \infty$ *for some* $\epsilon > 0$. *Then we have* $\mathbf{Y} \in \text{MRV}(\alpha_2, b_2, \mu_2^*, \mathbb{O}_2^{(2)})$, *where* $\alpha_2 = \alpha\frac{3d+2}{2(d+1)}$, $b_2(t) = \theta^{\frac{1}{\alpha}} t^{\frac{2(d+1)}{(3d+2)\alpha}}$, *and for any* $\mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty)$, *we have*

$$\mu_2^*\big((\mathbf{x}, \infty)\big) = \sum_{\ell, j=1}^{d} \mathbb{E}\left[\min\left\{\frac{a_{k\ell}}{x_1}, \max_{m \in \mathbb{I}_q \setminus \{k\}} \frac{a_{mj}}{x_2}\right\}^{\alpha} \max\left\{\frac{a_{k\ell}}{x_1}, \max_{m \in \mathbb{I}_q \setminus \{k\}} \frac{a_{mj}}{x_2}\right\}^{\alpha\frac{d}{2(d+1)}}\right].$$

Moreover, as $t \to \infty$,

$$\mathbb{P}[Y_1 > tx_1 | Y_2 > tx_2] \sim (\theta t^{-\alpha})^{\frac{d}{2(d+1)}} x_2^\alpha \mu_2^*\big((\mathbf{x}, \infty)\big)\Bigg(\sum_{j=1}^d \mathbb{E}[\max_{m\in\mathbb{I}_q\setminus\{k\}} a_{mj}^\alpha]\Bigg)^{-1},$$

$$\mathbb{P}[Y_2 > tx_2 | Y_1 > tx_1] \sim (\theta t^{-\alpha})^{\frac{d}{2(d+1)}} x_1^\alpha \mu_2^*\big((\mathbf{x}, \infty)\big)\Bigg(\sum_{\ell=1}^d \mathbb{E}[a_{k\ell}^\alpha]\Bigg)^{-1}.$$

Additionally, if the non-zero components of $\mathbf{A}$ *have bounded support, bounded away from zero, then there exist* $0 < \upsilon_1^* < \upsilon_2^* < \infty$ *such that for all* $0 < \upsilon < \upsilon_1^*$,

$$\mathrm{CoVaR}_{\upsilon\gamma^{\frac{d}{2(d+1)}}|\gamma}(Y_1|Y_2)$$

$$\sim \upsilon^{-\frac{1}{\alpha}} \frac{(\sum_{\ell,j=1}^d \mathbb{E}[a_{k\ell}^\alpha \max_{m\in\mathbb{I}_q\setminus\{k\}} a_{mj}^{\frac{\alpha d}{2(d+1)}}])^{\frac{1}{\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{k\ell}^\alpha])^{\frac{1}{\alpha}} (\sum_{j=1}^d \mathbb{E}[\max_{m\in\mathbb{I}_q\setminus\{k\}} a_{mj}^\alpha])^{\frac{d}{2(d+1)\alpha}}} \mathrm{VaR}_\gamma(Y_1), \qquad \gamma \downarrow 0,$$

and for all $\upsilon_2^* < \upsilon < \infty$,

$$\mathrm{CoVaR}_{\upsilon\gamma^{\frac{d}{2(d+1)}}|\gamma}(Y_1|Y_2)$$

$$\sim \upsilon^{-\frac{2(d+1)}{d\alpha}} \frac{(\sum_{\ell,j=1}^d \mathbb{E}[a_{k\ell}^{\frac{\alpha d}{2(d+1)}} \max_{m\in\mathbb{I}_q\setminus\{k\}} a_{mj}^\alpha])^{\frac{2(d+1)}{d\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{k\ell}^\alpha])^{\frac{1}{d\alpha}} (\sum_{j=1}^d \mathbb{E}[\max_{m\in\mathbb{I}_q\setminus\{k\}} a_{mj}^\alpha])^{\frac{2(d+1)}{d\alpha}}} \mathrm{VaR}_\gamma(Y_1), \qquad \gamma \downarrow 0.$$

Finally, $\mathrm{ECI}(Y_1|Y_2) = \mathrm{ECI}(Y_2|Y_1) = 2 + \frac{2}{d}$.

Finally, we consider an underlying vector $\mathbf{Z}$ with Gaussian dependence.

Proposition D.3 *Let* $\mathbf{Z} \in \text{P-GC}(\alpha, \theta, \Sigma)$ *with* $\Sigma = (\rho_{\ell j})_{1\le \ell, j\le d}$ *positive definite. Suppose* $\mathbf{A} \in \mathbb{R}_+^{q\times d}$ *is a random matrix which has almost surely no trivial rows, is independent of* $\mathbf{Z}$ *and for fixed* $k \in \mathbb{I}_q$*, we have*

$$\max_{m\in\mathbb{I}_q\setminus\{k\}} \max_{\ell\in\mathbb{I}_d} \mathbb{P}[\min\{a_{k\ell}, a_{m\ell}\} > 0] = 0.$$

Also, define

$$\rho^* = \max\Big\{\rho_{\ell j} : \ell, j \in \mathbb{I}_d, \ell \neq j \text{ and } \max_{m\in\mathbb{I}_q\setminus\{k\}} \mathbb{P}[\min\{a_{k\ell}, a_{mj}\} > 0] > 0\Big\}.$$

Suppose $\mathbb{E}[\|\mathbf{A}\|^{\frac{2\alpha}{1+\rho^*}+\epsilon}] < \infty$ *for some* $\epsilon > 0$ *and let* $\mathbf{X} = \mathbf{AZ}$*. Then we have*

$$\mathbf{Y} = (Y_1, Y_2)^\top = \Big(X_k, \max_{m\in\mathbb{I}_q\setminus\{k\}} X_m\Big)^\top \in \mathrm{MRV}(\alpha_2^*, b_2^*, \mu_2^*, \mathbb{O}_2^{(2)}),$$

with

$$\alpha_2^* = \frac{2\alpha}{1+\rho^*},$$

$$b_2^{*\leftarrow}(t) = C^*(\rho^*, \alpha)(\theta t^{-\alpha})^{-\frac{2}{1+\rho^*}} (\log t)^{\frac{\rho^*}{1+\rho^*}},$$

$$\mu_2^*\big((\mathbf{x}, \infty)\big) = D^*(\rho^*, \alpha, \mathbf{A})(x_1 x_2)^{-\frac{\alpha}{1+\rho^*}}, \qquad \mathbf{x} = (x_1, x_2) \in (\mathbf{0}, \infty),$$

where for $\rho \in (-1, 1), \alpha > 0, \theta > 0$ *and* $\mathbf{A} \in \mathbb{R}_+^{2\times d}$, *we define*

$$C^*(\rho, \alpha) = (2\pi)^{-\frac{1}{1+\rho}} (2\alpha)^{\frac{\rho}{1+\rho}},$$

$$D^*(\rho, \theta, \mathbf{A}) = \frac{1}{2\pi} \frac{(1+\rho)^{3/2}}{(1-\rho)^{1/2}} \sum_{(\ell, j):\rho_{\ell j} = \rho} \mathbb{E}\Big[a_{k\ell}^{\alpha/(1+\rho)} \max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^{\alpha/(1+\rho)}\Big].$$

Moreover, as $t \to \infty$,

$$\begin{aligned} &\mathbb{P}[Y_1 > x_1 t | Y_2 > x_2 t] \\ &\sim (\theta t^{-\alpha})^{\frac{1-\rho^*}{1+\rho^*}} (\log t)^{-\frac{\rho}{1+\rho}} x_1^{-\frac{\alpha}{1+\rho^*}} x_2^{\frac{\alpha\rho^*}{1+\rho^*}} \frac{C^*(\rho^*, \alpha)^{-1} D^*(\rho^*, \alpha, \mathbf{A})}{\sum_{j=1}^d \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^\alpha]}, \\ &\mathbb{P}[Y_2 > t x_2 | Y_1 > t x_1] \\ &\sim (\theta t^{-\alpha})^{\frac{1-\rho^*}{1+\rho^*}} (\log t)^{-\frac{\rho}{1+\rho}} x_1^{\frac{\alpha\rho^*}{1+\rho^*}} x_2^{-\frac{\alpha}{1+\rho^*}} \frac{C^*(\rho^*, \alpha)^{-1} D^*(\rho^*, \alpha, \mathbf{A})}{\sum_{\ell=1}^d \mathbb{E}[a_{k\ell}^\alpha]}. \end{aligned}$$

Additionally, with $g(\gamma) = \gamma^{\frac{1-\rho^*}{1+\rho^*}} (-\alpha^{-1} \log \gamma)^{-\frac{\rho^*}{1+\rho^*}}$ *and* $0 < \upsilon < 1$, *we have*

$$\begin{aligned} &\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_1|Y_2) \\ &\sim \upsilon^{-\frac{1+\rho^*}{\alpha}} \frac{(C^*(\rho^*, \alpha)^{-1} D^*(\rho^*, \alpha, \mathbf{A}))^{\frac{1+\rho^*}{\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{k\ell}^\alpha] \sum_{j=1}^d \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^\alpha])^{\frac{1}{\alpha}}} \mathrm{VaR}_\gamma(Y_1), \qquad \gamma \downarrow 0, \end{aligned}$$

and

$$\begin{aligned} &\mathrm{CoVaR}_{\upsilon g(\gamma)|\gamma}(Y_2|Y_1) \\ &\sim \upsilon^{-\frac{1+\rho^*}{\alpha}} \frac{(C^*(\rho^*, \alpha)^{-1} D^*(\rho^*, \alpha, \mathbf{A}))^{\frac{1+\rho^*}{\alpha}}}{(\sum_{\ell=1}^d \mathbb{E}[a_{k\ell}^\alpha] \sum_{j=1}^d \mathbb{E}[\max_{m \in \mathbb{I}_q \setminus \{k\}} a_{mj}^\alpha])^{\frac{1}{\alpha}}} \mathrm{VaR}_\gamma(Y_2), \qquad \gamma \downarrow 0. \end{aligned}$$

Finally, $g(t^{-1}) \in \mathrm{RV}_{-\frac{1-\rho^*}{1+\rho^*}}$ *and hence* $\mathrm{ECI}(Y_1|Y_2) = \mathrm{ECI}(Y_2|Y_1) = \frac{1+\rho^*}{1-\rho^*}$.

Acknowledgement We should like to thank the Editor, the Associate Editor and both referees for giving us valuable and informative comments which helped us in improving the quality of the paper.

Funding information Open Access funding enabled and organized by Projekt DEAL.

Declarations

Competing interests The authors declare no competing interests.

References

1. Acemoglu, D., Ozdaglar, A., Tahbaz-Salehi, A.: Systemic risk and stability in financial networks. Am. Econ. Rev. **105**, 564–608 (2015)
2. Acharya, V.V., Pedersen, L.H., Philippon, T., Richardson, M.: Measuring systemic risk. Rev. Financ. Stud. **30**, 2–47 (2017)
3. Adler, R., Feldman, R., Taqqu, M.: A Practical Guide to Heavy Tails: Statistical Techniques for Analysing Heavy Tailed Distributions. Birkhäuser, Boston (1998)
4. Adrian, T., Brunnermeier, M.K.: CoVaR. Am. Econ. Rev. **106**, 1705–1741 (2016)
5. Allen, F., Gale, D.: Financial contagion. J. Polit. Econ. **108**, 1–33 (2000)
6. Artzner, P., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk. Math. Finance **9**, 203–228 (1999)
7. Basrak, B., Davis, R., Mikosch, T.: Regular variation of GARCH processes. Stoch. Process. Appl. **99**, 95–115 (2002)
8. Benoit, S., Colliard, J.-E., Hurlin, C., Pérignon, C.: Where the risks lie: a survey on systemic risk. Rev. Finance **21**, 109–152 (2016)
9. Bianchi, M.L., De Luca, G., Rivieccio, G.: Non-Gaussian models for CoVaR estimation. Int. J. Forecast. **39**, 391–404 (2023)
10. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Cambridge University Press, Cambridge (1989)
11. Bradley, B.O., Taqqu, M.: An extreme value theory approach to the allocation of multiple assets. Int. J. Theor. Appl. Finance **07**, 1031–1068 (2004)
12. Breiman, L.: On some limit theorems similar to the arc-sin law. Theory Probab. Appl. **10**, 323–331 (1965)
13. Brownlees, C., Engle, R.F.: SRISK: A conditional capital shortfall measure of systemic risk. Rev. Financ. Stud. **30**, 48–79 (2016)
14. Caccioli, F., Shrestha, M., Moore, C., Farmer, J.D.: Stability analysis of financial contagion due to overlapping portfolios. J. Bank. Finance **46**, 233–245 (2014)
15. Cuadras, C.M., Augé, J.: A continuous general multivariate distribution and its properties. Commun. Stat., Theory Methods **10**, 339–353 (1981)
16. Das, B., Fasen-Hartmann, V.: Risk contagion under regular variation and asymptotic tail independence. J. Multivar. Anal. **165**, 194–215 (2018)
17. Das, B., Fasen-Hartmann, V.: Conditional excess risk measures and multivariate regular variation. Stat. Risk. Model. **36**, 1–23 (2019)
18. Das, B., Fasen-Hartmann, V.: Aggregating heavy-tailed random vectors: from finite sums to Lévy processes. Preprint, (2023). Available online at https://arxiv.org/abs/2301.10423
19. Das, B., Fasen-Hartmann, V.: On asymptotic independence in higher dimensions and its implications on risk management Preprint, (2024). Available online at https://arxiv.org/abs/2406.19186
20. Das, B., Fasen-Hartmann, V.: On heavy-tailed risks under Gaussian copula: the effects of marginal transformation. J. Multivar. Anal. **202**, 105310 (2024)

21. Das, B., Fasen-Hartmann, V., Klüppelberg, C.: Tail probabilities of random linear functions of regularly varying random vectors. Extremes **25**, 721–758 (2022)
22. Das, B., Mitra, A., Resnick, S.: Living on the multidimensional edge: seeking hidden risks using regular variation. Adv. Appl. Probab. **45**, 139–163 (2013)
23. de Haan, L., Ferreira, A.: Extreme Value Theory: An Introduction. Springer, New York (2006)
24. de Vries, C., Hyung, N.: Portfolio diversification effects and regular variation in financial data. Allg. Stat. Arch. **86**, 69–82 (2002)
25. Durrett, R.: Random Graph Dynamics. Cambridge University Press, Cambridge (2010)
26. Eisenberg, L., Noe, T.: Systemic risk in financial systems. Manag. Sci. **47**, 236–249 (2001)
27. Embrechts, P., Klüppelberg, C., Mikosch, T.: Modelling Extreme Events for Insurance and Finance. Springer, Berlin (1997)
28. Fan, H., Hu, C.: Research on systemic risk of China's bank-asset bipartite network. Heliyon **10**, e26952 (2024)
29. Feinstein, Z., Rudloff, B., Weber, S.: Measures of systemic risk. SIAM J. Financ. Math. **8**, 672–708 (2017)
30. Fischer, M., Koeck, C., Schlueter, S., Weigert, F.: An empirical analysis of multivariate copula models. Quant. Finance **9**, 839–854 (2009)
31. Gai, P., Kapadia, S.: Contagion in financial networks. Proc. Royal Soc. A, Math. Phys. Eng. Sci. **466**, 2401–2423 (2010)
32. Girardi, G., Ergün, A.T.: Systemic risk measurement: multivariate GARCH estimation of CoVaR. J. Bank. Finance **37**, 3169–3180 (2013)
33. Härdle, W.K., Wang, W., Yu, L.: TENET: tail-event driven NETwork risk. J. Econom. **192**, 499–513 (2016)
34. Hashorva, E., Hüsler, J.: On asymptotics of multivariate integrals with applications to records. Stoch. Models **18**, 41–69 (2002)
35. Horn, R.A., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Press, Cambridge (2013)
36. Hult, H., Lindskog, F.: Regular variation for measures on metric spaces. Publ. Inst. Math. **80**, 121–140 (2006)
37. Huschens, S., Kim, J.-R.: A stable CAPM in the presence of heavy-tailed distributions. In: Franke, J., et al. (eds.) Measuring Risk in Complex Stochastic Systems, pp. 175–188. Springer, New York (2000)
38. Joe, H.: Dependence Modeling with Copulas, vol. 134. CRC Press, Boca Raton (2015)
39. Jorion, P.: Value at Risk: The New Benchmark for Measuring Financial Risk, 3rd edn. McGraw-Hill, New York (2006)
40. Kley, O., Klüppelberg, C., Reinert, G.: Risk in a large claims insurance market with bipartite graph structure. Oper. Res. **64**, 1159–1176 (2016)
41. Kley, O., Klüppelberg, C., Reinert, G.: Conditional risk measures in a bipartite market structure. Scand. Actuar. J. **2018**, 328–355 (2018)
42. Ledford, A.W., Tawn, J.A.: Statistics for near independence in multivariate extreme values. Biometrika **83**, 169–187 (1996)
43. Lin, J., Li, X.: Multivariate generalized Marshall-Olkin distributions and copulas. Methodol. Comput. Appl. Probab. **16**, 53–78 (2014)
44. Lindskog, F., Resnick, S., Roy, J.: Regularly varying measures on metric spaces: hidden regular variation and hidden jumps. Probab. Surv. **11**, 270–314 (2014)
45. Mainik, G., Schaanning, E.: On dependence consistency of CoVaR and some other systemic risk measures. Stat. Risk. Model. **31**, 49–77 (2014)
46. Malevergne, Y., Sornette, D.: Testing the Gaussian copula hypothesis for financial assets dependences. Quant. Finance **3**, 231–250 (2003)
47. Marotta, L., Micciche, S., Fujiwara, Y., Iyetomi, H., Aoyama, H., Gallegati, M., Mantegna, R.N.: Bank-firm credit network in Japan: an analysis of a bipartite network. PLoS ONE **10**, e0123079 (2015)
48. Marshall, A.W., Olkin, I.: A multivariate exponential distribution. J. Am. Stat. Assoc. **62**, 30–44 (1967)
49. Maulik, K., Resnick, S.: Characterizations and examples of hidden regular variation. Extremes **7**, 31–67 (2005)
50. Nair, J., Wierman, A., Zwart, B.: The Fundamentals of Heavy Tails: Properties, Emergence, and Estimation. Cambridge University Press, Cambridge (2022)
51. Nolde, N., Zhou, C., Zhou, M.: An extreme value approach to CoVaR estimation Preprint, (2022). Available online at https://arxiv.org/abs/2201.00892

52. Poledna, S., Martínez-Jaramillo, S., Caccioli, F., Thurner, S.: Quantification of systemic risk from overlapping portfolios in the financial system. J. Financ. Stab. **52**, 100808 (2021)
53. Pólya, G.: Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung und das Momentenproblem. Math. Z. **8**, 171–181 (1920)
54. Poon, S.-H., Rockinger, M., Tawn, J.: Modelling extreme-value dependence in international stock markets. Stat. Sin. **13**, 929–953 (2003)
55. Reboredo, J.C., Ugolini, A.: Systemic risk in European sovereign debt markets: a CoVaR-copula approach. J. Int. Money Financ. **51**, 214–244 (2015)
56. Resnick, S.: Hidden regular variation, second order regular variation and asymptotic independence. Extremes **5**, 303–336 (2002)
57. Resnick, S.: Heavy Tail Phenomena: Probabilistic and Statistical Modeling. Springer, New York (2007)
58. Yuge, T., Maruyama, M., Yanagi, S.: Reliability of a k-out-of-n system with common-cause failures using multivariate exponential distribution. In: Howlett, R., et al. (eds.) Knowledge-Based and Intelligent Information and Engineering Systems: Proceedings of the 20th International Conference KES-2016. Procedia Computer Science, vol. 96, pp. 968–976 (2016)
59. Zhou, C.: Dependence structure of risk factors and diversification effects. Insur. Math. Econ. **46**, 531–540 (2010)