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Breather Solutions to Nonlinear Maxwell and Wave Equations

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Contents

CHAPTER 1

Introduction

1.1. Breathers

Overview. We are interested in real-valued, spatially localized and time-periodic solutions to Maxwell's equations, called *breathers*. We consider Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\mathbf{B}_t, \\ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{D}_t, \end{aligned} \qquad \text{for} \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R} \tag{1.1}$$

without charges and currents, and for Kerr-type optical materials. These materials are magnetically inactive and respond in a nonlinear and noninstantaneous way to the presence of electric fields. The material interaction is described by

$$\mathbf{B} = \mu_0 \mathbf{H}, \qquad \mathbf{D} = \epsilon_0(\mathbf{E} + \mathbf{P}(\mathbf{x}, \mathbf{E})) \tag{1.2}$$

where $\mu_0, \epsilon_0 \in (0, \infty)$ denote vacuum permeability and permittivity, and the nonlinear polarization $\mathbf{P}(\mathbf{x}, \mathbf{E})$ is given by

$$\begin{aligned} \mathbf{P}(\mathbf{x}, \mathbf{E}) = & \int_0^\infty \chi^{(1)}(\mathbf{x}, \tau)[\mathbf{E}(\mathbf{x}, t - \tau)] \,\mathrm{d}\tau \\ & + \int_0^\infty \int_0^\infty \int_0^\infty \chi^{(3)}(\mathbf{x}, \tau_1, \tau_2, \tau_3)[\mathbf{E}(\mathbf{x}, t - \tau), \mathbf{E}(\mathbf{x}, t - \tau_2), \mathbf{E}(\mathbf{x}, t - \tau_3)] \,\mathrm{d}\tau_1 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_3. \end{aligned} \tag{1.3}$$

with linear susceptibility tensor $\chi^{(1)}(\mathbf{x}, \tau) \colon \mathbb{R}^3 \to \mathbb{R}^3$ and cubic susceptibility tensor $\chi^{(3)}(\mathbf{x}, \tau_1, \tau_2, \tau_3) \colon \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. We omit a quadratic polarization term since it vanishes for silica glasses due to molecular symmetries (cf. [3]), and higher-order polarization terms are also neglected. Taking the curl of Faraday's law $\nabla \times \mathbf{E} = -\mathbf{B}_t$, from (1.1) and (1.2) we obtain the second-order problem

$$\nabla \times \nabla \times \mathbf{E} + \epsilon_0 \mu_0 \partial_t^2 (\mathbf{E} + \mathbf{P}(\mathbf{x}, \mathbf{E})) = 0, \qquad \nabla \cdot \mathbf{D} = 0. \tag{1.4}$$

Given a function $\mathbf{E}$ solving (1.4), we can recover a solution of (1.1) and (1.2) by setting

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}, 0) - \int_0^t \nabla \times \mathbf{E}(\mathbf{x}, s) \,\mathrm{d}s$$

and defining $\mathbf{D}, \mathbf{H}$ via the constitutive relations (1.2). Then $\nabla \cdot \mathbf{D} = 0$, $\nabla \times \mathbf{E} = -\mathbf{B}_t$ trivially hold and $\mathbf{B}$ is divergence-free if it is divergence-free at time $t = 0$. For the last equation of (1.1), we only obtain $\partial_t(\nabla \times \mathbf{H} - \mathbf{D}_t) = 0$, where we can recover Ampère's law $\nabla \times \mathbf{H} = \mathbf{D}_t$ if the appearing functions have temporal mean 0.

Choice of polarization. We want to use variational methods to find breather solutions to (1.4). For general nonlinearities $\mathbf{P}(\mathbf{x}, \mathbf{E})$ it is not clear whether (1.4) is variational, so we restrict ourselves to three choices in which a variational structure is present. These are given by (1.3) with linear polarization tensor

$$\chi^{(1)}(\mathbf{x}, \tau)[\mathbf{u}] = g(\mathbf{x}, \tau)\mathbf{u} \tag{1.5}$$

and one of the nonlinear tensors

$$\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{u},\mathbf{v},\mathbf{w}] = h(\mathbf{x})\nu(\tau_1)\delta_0(\tau_2-\tau_1)\delta_0(\tau_3-\tau_1)\langle\mathbf{u},\mathbf{v}\rangle\mathbf{w}, \tag{1.5.i}$$

$$\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{u},\mathbf{v},\mathbf{w}] = h(\mathbf{x})\nu(\tau_1)\nu(\tau_2)\nu(\tau_3)\langle\mathbf{u},\mathbf{v}\rangle\mathbf{w}, \tag{1.5.ii}$$

$$\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{u},\mathbf{v},\mathbf{w}] = h(\mathbf{x})\nu(\tau_1)\delta_0(\tau_2-\tau_1)\delta_0(\tau_3)\langle\mathbf{u},\mathbf{v}\rangle\mathbf{w}. \tag{1.5.iii}$$

Here g, h, ν are given real-valued functions and δ_0 denotes the Dirac measure at 0. If we choose $g(\mathbf{x},\tau) = g_0(\mathbf{x})\delta_0(\tau)$ and $\nu(\tau) = \delta_0(\tau)$ above, then for each of the three examples we recover the instantaneous polarization $\mathbf{P}(\mathbf{x},\mathbf{E}) = g_0(\mathbf{x})\mathbf{E} + h(\mathbf{x})|\mathbf{E}|^2\mathbf{E}$. In the following, we refer to the polarization (1.3) with linear kernel (1.5) and nonlinear kernel (1.5.x) as "polarization (1.5.x)".

To further simplify the setting, we consider breather solutions traveling inside a material of either slab or cylindrical geometry.

Slab materials. We consider materials where the nonlinear polarization field $\mathbf{P}(\mathbf{x},\mathbf{E})$ depends only on one direction of $\mathbf{x} = (x, y, z)$, say, the x-direction, and we call these *slab materials*. For $w\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ we make the ansatz

$$\mathbf{E}(x,y,z,t) = w(x, t-\tfrac{1}{c}z)\begin{pmatrix}0\\1\\0\end{pmatrix} \tag{1.6}$$

of a wave traveling with speed c in z-direction that is constant along the y-direction. As the electric field $\mathbf{E}$ points perpendicular to the direction of travel, such a wave is called *transverse electric (TE)* polarized. In contrast, we call a breather *transverse magnetic (TM)* polarized if the magnetic field $\mathbf{B}$ is perpendicular to the direction of travel. We insert the ansatz (1.6) into (1.4), where normalizing the speed of light in vacuum to $c_0 := (\epsilon_0\mu_0)^{-\frac{1}{2}} = 1$ and using t instead of $t - \frac{1}{c}z$ as the "time" variable in the traveling coordinate frame yield

$$-\partial_x^2 w + (1-\tfrac{1}{c^2})\partial_t^2 w + \partial_t^2 P(w) = 0 \qquad \text{for} \quad (x,t)\in\mathbb{R}\times\mathbb{R}, \tag{1.7}$$

where $P(w)$ is the nonlinear scalar polarization. It is given by

$$P(w) = \int_0^\infty w(x,t-\tau)g(x,\tau)\,\mathrm{d}\tau + P_{\mathrm{NL}}(w) \tag{1.8}$$

with one of

$$P_{\mathrm{NL}}(w)(x,t) = h(x)\int_0^\infty w(x,t-\tau)^3\nu(\tau)\,\mathrm{d}\tau, \tag{1.8.i}$$

$$P_{\mathrm{NL}}(w)(x,t) = h(x)\Big(\int_0^\infty w(x,t-\tau)\nu(\tau)\,\mathrm{d}\tau\Big)^3, \tag{1.8.ii}$$

$$P_{\mathrm{NL}}(w)(x,t) = h(x)w(x,t)\int_0^\infty w(x,t-\tau)^2\nu(\tau)\,\mathrm{d}\tau, \tag{1.8.iii}$$

corresponding to the three choices of polarization (1.5.i), (1.5.ii) and (1.5.iii). Since $h(\mathbf{x}) = h(x,y,z)$ is independent of y and z for slab materials, we write $h(x) := h(x,y,z)$ instead, and similarly we use $g(x,\tau) := g(x,y,z,\tau)$.

Cylindrical materials. In addition to slab materials, which reduce Maxwell's equations to (1.7), we also consider material where the parameters g, h depend in space only on the cylinder radius $r := \sqrt{x^2+y^2}$, but not on the z-direction or angular direction. We make the ansatz

$$\mathbf{E}(\mathbf{x},t) = w(r, t - \tfrac{1}{c}z)\begin{pmatrix} -y/r \\ x/r \\ 0 \end{pmatrix}, \qquad r = \sqrt{x^2+y^2} \tag{1.9}$$

of a TE-polarized wave traveling in z-direction. Inserting (1.9) into (1.4), we obtain

$$(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2})w + (1 - \tfrac{1}{c^2})\partial_t^2 w + \partial_t^2 P(w) = 0 \qquad \text{for} \quad (r,t) \in (0,\infty) \times \mathbb{R} \tag{1.10}$$

where the scalar polarization $P(w)$ is the same as in (1.8), (1.8.i)–(1.8.iii), except that x is replaced by r.

Compared with the slab ansatz (1.6), solutions given by the cylindrical ansatz (1.9) are more localized, since we require them to decay in both directions perpendicular to the direction of travel while solutions of the form (1.6) decay in one direction. For both (1.6) and (1.9) the breathers are traveling and time-periodic, so they are also periodic (and hence not localized) in the direction of travel. However, (1.10) is more difficult to treat because of the two additional terms in the spatial operator $-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}$ compared with $-\partial_x^2$ of (1.7). We discuss this in more detail in the overview of Chapter 4.

In our existence results on breathers, we consider the following combinations of material geometry and polarization:

material \ polarization	(1.5.i) and (1.5.ii)	(1.5.iii)
slab with ansatz (1.6)	Chapter 3 and 4	Chapter 5
cylindrical with ansatz (1.9)	Chapter 4	Chapter 5

One distinguishes between *monochromatic* and *polychromatic* breather solutions, where monochromatic breathers have only one temporal frequency whereas polychromatic breathers have multiple supported frequencies. Throughout this thesis, we consider polychromatic breather solutions for materials with bounded material coefficients. However, as well shall see, in some settings we observed numerically that breathers, which are allowed to be polychromatic, seem to be monochromatic instead.

Monochromatic breathers. Monochromatic solutions have only one supported frequency. That is, for (1.7) they are given by

$$w(x,t) = u(x)\mathrm{e}^{\mathrm{i}k\omega t} + \overline{u(x)}\mathrm{e}^{-\mathrm{i}k\omega t} \tag{1.11}$$

with $u\colon \mathbb{R} \to \mathbb{C}$, or more generally by

$$\mathbf{E}(\mathbf{x},t) = \mathcal{E}(\mathbf{x})\mathrm{e}^{\mathrm{i}k\omega t} + \overline{\mathcal{E}(\mathbf{x})}\mathrm{e}^{-\mathrm{i}k\omega t} \tag{1.12}$$

with $\mathcal{E}\colon \mathbb{R}^3 \to \mathbb{C}^3$ for (1.4). Here $\omega > 0$ is the frequency of the breather, and the second term in (1.11) and (1.12) ensures that the breather is real-valued. The ansatz (1.12) is valid if one considers the polarization (1.5.iii) with $\nu = \frac{1}{2T}\mathbb{1}_{[0,T]}$ where $T := \frac{2\pi}{\omega}$ is the (temporal) period of the breather, and leads to the nonlinear curl-curl problem

$$\nabla \times \nabla \times \mathcal{E} - \omega^2 g(\mathbf{x})\mathcal{E} - \omega^2 h(\mathbf{x})|\mathcal{E}|^2\mathcal{E} = 0. \tag{1.13}$$

with $g(\mathbf{x}) = \int_0^\infty \chi^{(1)}(\mathbf{x},\tau)\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau$. Note that (1.13) has a variational structure provided $g(\mathbf{x})$ and $h(\mathbf{x})$ are real-valued. Alternatively to (1.5.iii), if one neglects third harmonics (i.e., terms proportional to $\mathrm{e}^{\pm 3\mathrm{i}\omega t}$) coming from the nonlinear part of the polarization (cf. [82]), then one again obtains (1.13) for all three choices of the polarization $\mathbf{P}(\mathbf{x},\mathbf{E})$ ((1.5.i), (1.5.ii), or (1.5.iii)).

In place of the cubic nonlinearity $h(\mathbf{x})|\mathcal{E}|^2\mathcal{E}$ of (1.13), saturated nonlinearities $h(\mathbf{x},|\mathcal{E}^2|)\mathcal{E}$, i.e., those that grow linearly as $|\mathcal{E}| \to \infty$, were studied early on in a series of papers [62, 87–93] by Stuart et al. They considered divergence-free, TE- or TM-polarized ansatz functions in cylindrically symmetric media to reduce the Maxwell problem to a one-dimensional elliptic problem, which they solved, e.g., using variational methods. More general nonlinearities $h(\mathbf{x},|\mathcal{E}^2|)$, including also power nonlinearities, were considered in [5, 8, 10] for standing monochromatic breathers given by cylindrically or spherically symmetric ansatz functions. The restriction to divergence-free ansatz functions is convenient since it reduces the noninjective curl-curl-operator to $-\Delta$, which is injective on localized functions. Ansatz functions that are not divergence-free were considered by Mederski et al. in [7, 63–67], where the authors used the Helmholtz decomposition of a function into divergence-free and curl-free parts to deal with the kernel of the curl-curl-operator. Mandel et al. considered different approaches using limiting absorption principles [59] or dual variational methods [57, 58, 60]. Dohnal et al. obtained breathers for photonic crystals using bifurcation for frequency parameters near an edge of a spectral band [33], and in [30–32] they considered time-periodic solutions at an interface of metals and dielectrics, as well as time-periodic solutions for $\mathcal{PT}$-symmetric materials. For further information, we refer to the survey paper [7].

Polychromatic breathers We turn to polychromatic breather solutions, i.e., solutions with multiple (usually infinitely many) supported frequencies. These are the subjects that naturally arise when studying higher harmonics or the polarization (1.5.i), (1.5.ii) or the general case of (1.5.iii). As a model problem, let us consider (1.7) with (instantaneous) coefficients $g(x,\tau) = g_0(x)\delta_0(\tau)$, $\nu(\tau) = \delta_0(\tau)$ and $c = 1$, which yields

$$-\partial_x^2 w + g_0(x)\partial_t^2 w + h(x)\partial_t^2(w^3) = 0. \tag{1.14}$$

In settings where g_0 or h contain Dirac measures, existence results on breathers for (1.14) were obtained in [49] using variational methods, and in [19] using bifurcation arguments. In [75] the authors considered a periodic Dirac measure for the linear potential g_0 and embed (1.14) as well as a system of uncoupled nonlinear Schrödinger (NLS) equations into a family of equations. They then sought solutions by local bifurcation from the NLS system, with numerical evidence that some branches of solutions indeed reach (1.14). The authors of [34] obtained approximate breathers on large but finite time scales using amplitude equations for a material consisting of two different Kerr-dielectrics each occupying a halfspace.

Inhomogeneous materials. We consider (1.7) and (1.10) exclusively for inhomogeneous materials, i.e., material parameters $\chi^{(1)}$, $\chi^{(3)}$ that depend nontrivially on space.

Intuitively, waves are partially reflected/deflected when interacting with inhomogeneous materials. So, when set up well, these materials may reflect the wave back towards the waveguide and thereby prevent the electromagnetic energy from radiating to infinity, which keeps the breather localized.

To illustrate the effect of homogeneous vs inhomogeneous materials, consider the (constant coefficient) Sine-Gordon equation

$$-u_{xx} + u_{tt} + \sin(u) = 0,$$

which is known[1] to admit the family of breather solutions

$$u^\star(x,t) = 4\arctan\left(\frac{m}{w}\frac{\sin(\omega t)}{\cosh(mx)}\right) \qquad \text{where} \quad m,\omega > 0,\ m^2 + \omega^2 = 1.$$

The existence of breathers is strongly related to the $\sin(u)$-nonlinearity: Among analytic perturbations $f \approx \sin$ with $f''(0) = 0$, the perturbed Klein-Gordon equation

$$-u_{xx} + u_{tt} + f(u) = 0 \tag{1.15}$$

only admits a family of breather solutions $u \approx u^\star$ when $f(s) = \alpha \sin(\beta s)$, and these solutions are given by a rescaling of the Sine-Gordon breathers (cf. [29]). The authors of [14] conjectured on (1.15) that "breathing [...] takes place only for isolated nonlinearities".

Using inhomogeneous materials allows us to overcome this difficulty and obtain breather solutions to Klein-Gordon equations with nonlinearities $f \neq \sin$: For the inhomogeneous problem

$$-u_{xx} + V(x)u_{tt} + f(x,u) = 0, \tag{1.16}$$

positive results on existence of breathers are known for different nonlinearities f including also sums of power nonlinearities, see [15, 43] for two such results. Both papers strongly use the x-dependency of the potential V.

1.2. MAIN RESULTS

We give a short overview of the topic of each chapter in Figure 1.1. Let us point out:

- In all chapters except Chapter 6 we show existence of breather solutions.
- Among Chapters 3 to 5 where we consider breathers for Maxwell's equations, in Chapter 3 we find breathers only for slab materials. This is because the method used in Chapter 3 (and Chapter 2) depends delicately on the spectral structure of the underlying linear operator, and we currently have a good enough understanding of this structure only for the slab setting.
- Polarizations (1.5.i) and (1.5.ii) have a regularizing effect provided the convolution kernels are sufficiently smooth. This allows us to treat the Maxwell problem with semilinear methods, and also to show that breather solutions are very smooth. For example, they are infinitely differentiable in time.
  On the other hand, the polarization (1.5.iii) has no such regularizing effect and requires different methods. We show that breathers are locally integrable, and numerical simulations suggest that some breathers do not have first-order derivatives.

[1]The Sine-Gordon breathers were originally found by [1].

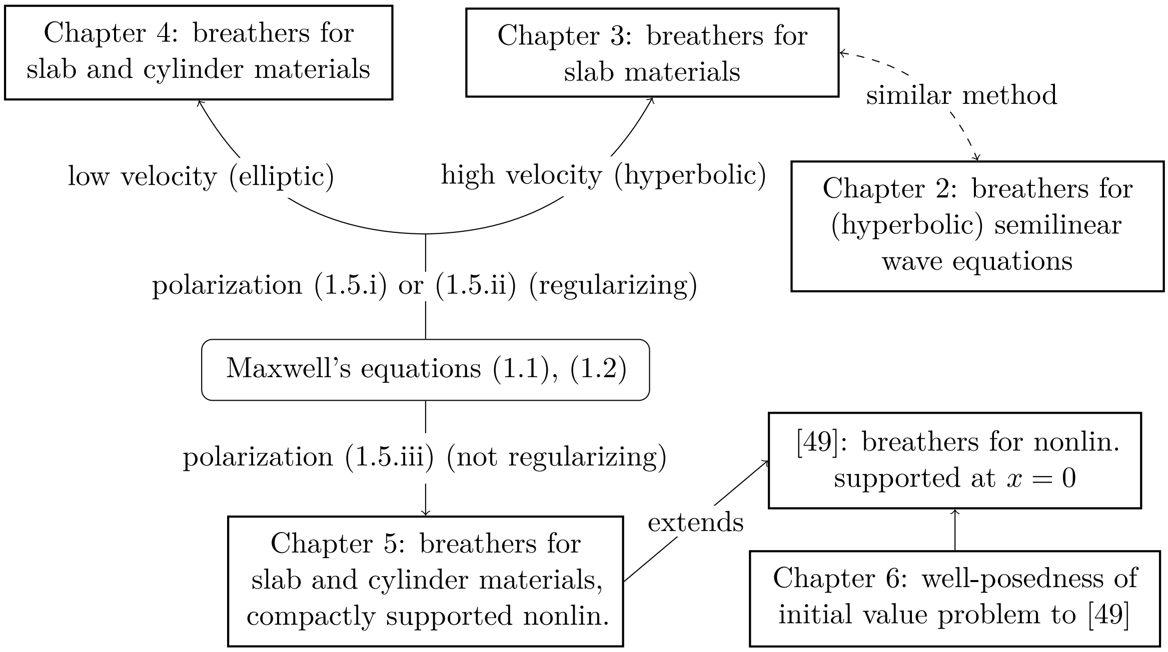

Figure 1.1.: An overview of our results and how they are connected.

Before discussing the individual chapters in detail, we sketch the general setup of the variational methods used to find breather solutions in Chapters 2–5. The process consists of the following four steps.

1. *Identify variational structure:* We verify that our problem of interest, e.g., (1.7) or a variant thereof, formally is the Euler-Lagrange equation of some energy functional J.
2. *Find domain and variational method:* We find a Banach space X on which J is well-defined, C^1-smooth and has a geometry suitable for variational methods. It is important to choose the right domain, which is usually uniquely determined by J, since choosing X too small or large will destroy the geometry or make J ill-defined. Also, the assumptions on breathers will be built into the domain, i.e., each $u \in X$ is real-valued, spatially localized and time-periodic.

   We use three different variational methods throughout the thesis. For functionals J having suitable geometries, each method guarantees the existence of a Palais-Smale sequence (see Definition 1.2.1 below), an approximating sequence for a critical point of J at a particular energy level $\mathfrak{c}$.

   - *Global minimization:* Assuming that the functional J is bounded from below, we search for a minimizer, which necessarily is a critical point at level $\mathfrak{c} := \inf_{u \in X} J(u)$. The corresponding variational method is called Ekeland's variational principle (cf. [86]).
   - *Mountain pass:* As the name suggests, we search for the highest point along the lowest curve connecting two given points $u_0, u_1 \in X$, which should be a saddle point of J. So we set

     $$\mathfrak{c} := \inf_{\substack{\gamma \in C([0,1];X) \\ \gamma(0)=u_0, \gamma(1)=u_1}} \max_{s \in [0,1]} J(\gamma(s))$$

     and we require a "mountain range" between u_0 and u_1, i.e., we assume $J(u) \geq \max\{J(u_0), J(u_1)\} + \varepsilon$ for u in a hypersurface separating u_0 from u_1. In the typical case where $u_0 = 0$ and the hypersurface is given by $\partial B_r(u_0)$ with $r < \|u_1\|$, this is known as the mountain pass method, see Theorem 1.2.2 below.

- *Generalized Nehari manifold:* We consider a semilinear problem of the form $J'(u) = \mathcal{L}u - I'(u)$ with a selfadjoint linear operator $\mathcal{L}$ and a superlinear nonlinearity I'. Here semilinear means that I (and J) are well-defined on the natural domain X of the bilinear form associated to $\mathcal{L}$, called *form domain* of $\mathcal{L}$. Furthermore we require I to be positive and $\mathcal{L}$ to be indefinite with $0 \notin \sigma(\mathcal{L})$.[2] Denoting by X^- the subspace of X corresponding to the negative part of the spectrum of $\mathcal{L}$, we utilize the generalized Nehari manifold
$$\mathcal{M} \coloneqq \{u \in X \setminus \{0\} \colon J'(u)|_{\mathbb{R}u+X^-} = 0\} \tag{1.17}$$
and seek critical points as minimizers of $J|_{\mathcal{M}}$, i.e., $\mathfrak{c} \coloneqq \inf_{u\in\mathcal{M}} J(u)$. This method is known as the method of generalized Nehari manifold, see [94] or Appendix A for details.

3. *Take limit of Palais-Smale sequence:* We show that, up to a subsequence and symmetries of the equation, the Palais-Smale sequence (u_n) from step 2 converges in a suitable topology. In the limit $u = \lim_{n\to\infty} u_n$ we recover a critical point of J with $J(u) = \mathfrak{c}$. The arguments involved depend strongly on the variational method and require some "compactness" properties of the functional J in addition to the geometry assumptions of step 2.
4. *Regularity and justification of formal calculations:* As the calculations of step 1 were done on a formal level, we need to justify them. We do this using the regularity $u \in X$ or, should that be insufficient, we first show further integrability or differentiability properties of u. To obtain these properties, we can exploit $J'(u) = 0$ and/or information coming from the variational method.

The definition of a Palais-Smale sequence and the mountain pass theorem are given next.

Definition 1.2.1 (Palais-Smale sequence). *Let X be a real Banach space and $J \in C^1(X;\mathbb{R})$. A sequence (u_n) in X is called* Palais-Smale sequence for J at level $\mathfrak{c} \in \mathbb{R}$ *if*
$$J(u_n) \to \mathfrak{c} \quad \text{in } \mathbb{R} \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } X'.$$

Theorem 1.2.2 (mountain pass). *Let X be a real Banach space and $J \in C^1(X;\mathbb{R})$ be a functional that has* mountain pass geometry*, i.e.,*

(a) $J(0) = 0$,
(b) *there exist $r, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\| = r$, and*
(c) *there exists $u_0 \in X$ with $\|u_0\| > r$ and $J(u_0) < \alpha$.*

Then there exists a Palais-Smale sequence for J at level
$$\mathfrak{c} \coloneqq \inf_{\substack{\gamma\in C([0;1];X)\\ \gamma(0)=0,\gamma(1)=u_0}} \max_{s\in[0,1]} J(\gamma(s)) \geq \alpha.$$

Proof. See Theorem 6.1, Theorem 3.4, and Remark 3.5 in [86, Chapter II]. □

Since we are interested in time-periodic functions, we fix $T > 0$ as the period of the breather and denote by $\mathbb{T} \coloneqq \mathbb{R}/_{T\mathbb{Z}}$ the torus of length T, which is the natural time-domain. The

[2] With modifications, it is also possible to consider $0 \in \sigma_p(\mathcal{L}) \setminus \sigma_{\text{ess}}(\mathcal{L})$.

associated base frequency is $\omega := \frac{2\pi}{T}$. We equip the torus $\mathbb{T}$ with the Haar measure $\mathrm{d}t$ normalized such that $\int_{\mathbb{T}} 1 \,\mathrm{d}t = 1$. Using this normalization, the Fourier coefficients of a function $f\colon \mathbb{T} \to \mathbb{C}$ and the Fourier series are given by

$$\hat{f}_k := \mathcal{F}_k[f] := \int_{\mathbb{T}} f(t)\mathrm{e}^{-\mathrm{i}k\omega t}\,\mathrm{d}t, \qquad f(t) = \mathcal{F}_t^{-1}[\hat{f}_k] := \sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{\mathrm{i}k\omega t}.$$

In Chapter 2 we consider the semilinear problem (1.16) for power nonlinearities, i.e.,

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u \qquad \text{for} \quad (x,t) \in \mathbb{R}^2 \tag{1.18}$$

with $p \in (1,\infty)$. The chapter is based on the preprint [41] which was written jointly with Julia Henninger and Wolfgang Reichel. We refer to Section 2.1 for a detailed overview of relevant literature on equations like (1.18).

In contrast to the results of [15, 43] discussed earlier, we consider potentials V, Γ that need not be purely periodic: The potential V only has to be periodic in neighborhoods of $\pm\infty$. Furthermore, we work under the assumption that the linear operator of (1.18) is hyperbolic, i.e., V is uniformly positive. These basic assumptions are stated next.

($\mathcal{A}2.1$) $V, \Gamma \in L^\infty(\mathbb{R})$ with $\operatorname{ess\,inf}_{\mathbb{R}} V > 0$ and $\Gamma > 0$ almost everywhere. Moreover, V has locally bounded variation.[3]

($\mathcal{A}2.2$) There exist $X^\pm > 0$, $R^\pm \in \mathbb{R}$ and $X^\pm$-periodic functions $V_{\mathrm{per}}^\pm \in L^\infty(\mathbb{R})$ such that $V(x) = V_{\mathrm{per}}^+(x)$ for $x > R^+$ and $V(x) = V_{\mathrm{per}}^-(x)$ for $x < R^-$.

It is convenient to rescale the linear operator by multiplying with $\frac{1}{V(x)}$ and to thus consider $L + \partial_t^2$ with the weighted Sturm-Liouville operator

$$L := -\frac{1}{V(x)}\partial_x^2.$$

This is easier since L acts only on x and ∂_t^2 only on t. As a result, for the spectrum we have $\sigma(L + \partial_t^2) = \overline{\sigma(L) + \sigma(\partial_t^2)}$.

Since we seek time-periodic functions, we consider the operator ∂_t^2 on $\mathbb{T}$ which has spectrum $\sigma(\partial_t^2) = \{-\omega^2 k^2 \colon k \in \mathbb{Z}\}$. Our analysis is based on the assumption that L has a spectral gap about $\omega^2 k^2$ for each $k \in \mathbb{Z}_{\mathrm{odd}}$, and moreover that the size of the gaps grows linearly in $|k|$. This allows us to invert the operator $L + \partial_t^2$ after restricting to odd frequencies, or equivalently, restricting to functions that are $\frac{T}{2}$-antiperiodic in time. Note that 0 does not lie in a spectral gap of L. The restriction to odd frequencies avoids the $k = 0$ mode, and $\frac{T}{2}$-antisymmetry is compatible with equation (1.18) and the variational structure. As we discuss in Remark 1.2.4, a careful choice of the potential V is required in order to ensure that L has countably many large spectral gaps. In total, we make the following assumptions on the spectrum of L.

($\mathcal{A}2.3$) There exists $\omega > 0$ such that $\inf\{|\sqrt{\lambda} - k\omega| : \lambda \in \sigma(L), k \in \mathbb{N}_{\mathrm{odd}}\} > 0$.

($\mathcal{A}2.4$) The point spectrum $\sigma_p(L)$ satisfies $\sum_{\lambda\in\sigma_p(L)} \lambda^{-r} < \infty$ for all $r > \frac{1}{2}$, which means that $\sigma_p(L)$ grows at least quadratically or is finite.

With these assumptions in place, we formulate our main existence result for breather solutions to equation (2.1).

[3] In this chapter, labels that do not begin with the number 1, such as assumption ($\mathcal{A}2.1$) or Theorem 2.1.1, are references to a statement in another chapter (here Chapter 2) with essentially the same content.

Theorem 2.1.1. *Let $1 < p < \infty$ and assume ($\mathcal{A}$2.1), ($\mathcal{A}$2.2), ($\mathcal{A}$2.3), ($\mathcal{A}$2.4). Then* (1.18) *has infinitely many nontrivial $T = \frac{2\pi}{\omega}$-periodic breathers if additionally one of the following assumptions are satisfied.*

($\mathcal{A}$2.5a) $\lim_{|x|\to\infty} \Gamma(x) = 0$. *(compact case)*

($\mathcal{A}$2.5b) $V = V_{\mathrm{per}}$ *is X-periodic on $\mathbb{R}$ and $\Gamma = \Gamma_{\mathrm{per}} + \Gamma_{\mathrm{loc}}$ where Γ_{per} is X-periodic and $\Gamma_{\mathrm{loc}} \geq 0$, $\lim_{|x|\to\infty} \Gamma_{\mathrm{loc}}(x) = 0$. (asymptotically periodic case)*

The breathers are in $H^2(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ and satisfy (1.18) *pointwise almost everywhere.*

Remark 1.2.3. Let us comment on Theorem 2.1.1 and its assumptions.

- Case ($\mathcal{A}$2.5b) also includes the purely periodic case by choosing $\Gamma_{\mathrm{loc}} = 0$. In the purely periodic setting, compared with results of [43] we improve the range of admissible p from $p \in (1, 3)$ to $p \in (1, \infty)$. Some of the ideas leading to this improvement were previously used in [56], so the main novelty is that our results are applicable also for nonperiodic coefficients.
- If u is a breather solution to (1.18), then the temporal translate $u(\,\cdot\,, \,\cdot\, - \tau)$ is again a breather for any $\tau \in \mathbb{T}$. In the purely periodic case, the same is true for translates by the spatial period. When claiming existence of infinitely many solutions in Theorem 2.1.1, we mean infinitely many solutions that are not translates of one another.
- In case ($\mathcal{A}$2.5b) the linear operator L is periodic. By Floquet-Bloch theory (cf. [36]) the spectrum $\sigma(L)$ consists of essential spectrum and $\sigma_p(L) = \emptyset$, hence ($\mathcal{A}$2.4) trivially holds.

In Appendix 2.B we discuss three examples of potentials V which satisfy the assumptions ($\mathcal{A}$2.1)–($\mathcal{A}$2.4), including both periodic and nonperiodic V. For each example, V is a piecewise constant function with carefully chosen step heights and widths.

Remark 1.2.4. The coefficients in our examples in Appendix 2.B are piecewise constant, thus they have low regularity. Recall that in ($\mathcal{A}$2.3) we require L to have spectral gaps about $\omega^2 k^2$ of size $\sim |k|$, and we expect that rough coefficients are required to obtain such large spectral gaps (cf. [23]).

As an illustration we consider $L = -\frac{1}{V(x)} \partial_x^2$ for even, 1-periodic potentials $V \in H^s(\mathbb{R}/_\mathbb{Z})$ with $s \in [1, \infty)$. By Floquet Bloch theory (cf. [76]), the spectrum of L has a band-gap structure

$$\sigma(L) = \bigcup_{k \in \mathbb{N}_0} [\lambda_{2k}, \lambda_{2k+1}]$$

with an increasing sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \lambda_5 \cdots \to \infty$. The size of the k-th spectral gap is given by $g_k \coloneqq \lambda_{2k} - \lambda_{2k-1} \geq 0$. Due to [23] the map

$$H^s_{\mathrm{even}}(\mathbb{R}/_\mathbb{Z}) \supseteq O \to \mathbb{R} \times h^{s-2}, \; V \mapsto (\omega, (\tilde{g}_k)_{k \in \mathbb{N}}) \quad \text{with} \quad \omega = \frac{\pi}{\int_0^1 \sqrt{V(x)} \,\mathrm{d}x}, \quad \tilde{g}_k \in \{g_k, -g_k\}$$

is an isomorphism from a neighborhood O of $V = 1$ onto its range. Here, the sign of the signed gap size $\tilde{g}_k$ above is chosen depending on whether $(\lambda_{2k-1}, \lambda_{2k})$ is a pair of (Dirichlet, Neumann)- or (Neumann, Dirichlet)-eigenvalues, the spectral gap $(\lambda_{2k-1}, \lambda_{2k})$ is located about $\omega^2 k^2$, and the sequence space h^{s-2} is defined by $\|(\tilde{g}_k)\|^2_{h^{s-2}} \coloneqq \sum_{k \in \mathbb{N}} k^{2(s-2)} |\tilde{g}_k|^2 < \infty$. Thus, assuming L has spectral gaps of size $g_k \sim k^\gamma$, the inequality

$$\|(k^\gamma)\|_{h^{s-2}} < \infty \iff \gamma < \frac{3}{2} - s \tag{1.19}$$

necessarily holds. In particular, for H^1-regular potentials $V \approx 1$ we have $\gamma < \frac{1}{2}$, meaning that for such V the operator L can never have spectral gaps whose size is linear in k. Although the results of [23] are not applicable below $s = 1$, the requirement (1.19) is still consistent with our examples of Appendix 2.B and may suggest that these are chosen optimally: In the examples, the gap sizes grow linearly in k (hence $\gamma = 1$) and we use piecewise constant functions V. These are H^s-regular precisely for $s < \frac{1}{2}$ as discussed in Remark B.2.3.

We now give an overview of the methods used to prove Theorem 2.1.1. We follow the four steps outlined above.

Step 1. Formally, (1.18) is the Euler-Lagrange equation of

$$J(u) := J_0(u) - J_1(u) := \int_{\mathbb{R}\times\mathbb{T}} (L + \partial_t^2)u \cdot u\, V(x)\mathrm{d}(x,t) - \frac{2}{p+1}\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u|^{p+1}\,\mathrm{d}(x,t)$$

which we consider for $\frac{T}{2}$-antiperiodic in time functions u.

Step 2. We use the generalized Nehari method to find critical points of J. Therefore the domain of J is the form domain of $L + \partial_t^2$. Let us explain the construction of the form domain. Using the Fourier series, we transform the linear operator $L + \partial_t^2$ into a sequence of operators

$$L + \partial_t^2 \quad = \quad \mathcal{F}^{-1} \circ \Big(\bigoplus_{k\in\mathbb{Z}_{\mathrm{odd}}} (L - \omega^2 k^2) \Big) \circ \mathcal{F}$$

which are selfadjoint on $L^2_V(\mathbb{R}) := L^2_V(\mathbb{R}; V\mathrm{d}x)$. Since they are semibounded from below, by standard methods (cf. [77]) one can define their form domain $\mathcal{H}_k$. For $L - \omega^2k^2$ one has $H^1(\mathbb{R})$ with the norm

$$\|v\|^2_{\mathcal{H}_k} := \int_{\mathbb{R}} \Big| L - \omega^2 k^2 \Big| v \cdot \overline{v}\, V(x)\mathrm{d}x,$$

where $|L - \omega^2k^2|$ is defined by the functional calculus for L. The form domain of $L + \partial_t^2$ is then given by the Hilbert space

$$\mathcal{H} := \mathcal{F}^{-1}\Big(\bigoplus_{k\in\mathbb{Z}_{\mathrm{odd}}} \mathcal{H}_k \Big).$$

Having defined the space $\mathcal{H}$ according to the linear operator, we show boundedness and local compactness of the embedding $\mathcal{H} \hookrightarrow L^{p+1}(\mathbb{R}\times\mathbb{T})$ for $p \in [1,\infty)$ in Section 2.3 to verify that J_1 is also well-defined on $\mathcal{H}$. This is based on an effective formula and estimates on the spectral measure of L and uses the spectral assumptions ($\mathcal{A}$2.3), ($\mathcal{A}$2.4).

Step 3 and 4. Using the method of generalized Nehari manifold $\mathcal{M}$ introduced in (1.17), we obtain a Palais-Smale sequence (u_n) of J at level $\mathfrak{c} := \inf_{u\in\mathcal{M}} J(u) > 0$. We then use assumptions ($\mathcal{A}$2.5a) or ($\mathcal{A}$2.5b) to pass from (u_n) to a critical point of J:

- In ($\mathcal{A}$2.5a), from decay of Γ we have that J_1' is completely continuous[4]. With this, one can show that J satisfies the Palais-Smale condition.
- For ($\mathcal{A}$2.5b), we first consider the purely periodic case, i.e., $\Gamma^{\mathrm{loc}} = 0$. Using the periodic structure of J and concentration-compactness arguments, we show that suitable spatial translates of the u_n converge up to a subsequence to a nonzero critical point.

[4]This means that $J_1' \colon (\mathcal{H}, \text{weak topology}) \to (\mathcal{H}', \text{norm topology})$ is continuous.

- For the general case of ($\mathcal{A}$2.5b), we compare the energy $\mathfrak{c} = \inf_{u\in\mathcal{M}} J(u)$ to the energy $\mathfrak{c}^{\mathrm{per}}$ of the purely periodic problem. The assumption $\Gamma^{\mathrm{loc}} \geq 0$ ensures that $\mathfrak{c} < \mathfrak{c}^{\mathrm{per}}$ holds.[5] It follows that the u_n are localized near the perturbation Γ^{loc}. Indeed, if u_n has a part $\tilde{u}_n$ that is localized far away from the perturbation, then we have

$$\mathfrak{c}^{\mathrm{per}} > \mathfrak{c} = J(u_n) + \mathrm{o}(1) \geq J(\tilde{u}_n) + \mathrm{o}(1) = J^{\mathrm{per}}(\tilde{u}_n) + \mathrm{o}(1) \geq \mathfrak{c}^{\mathrm{per}} + \mathrm{o}(1),$$

  a contradiction.

These arguments, as well as smoothness and multiplicity of breathers, are carried out in Section 2.2.

Remark 1.2.5. To have the energy comparison $\mathfrak{c} \leq \mathfrak{c}^{\mathrm{per}}$ available for the above argument, we consider perturbations in the nonlinearity Γ but not in the linear potential V, since difficulties appear when perturbing the linear part: Note that by adding local perturbations of the correct sign to both Γ and V, we can always ensure $J \leq \tilde{J}$ (where J is the perturbed and $\tilde{J}$ is the reference functional). This does not imply comparability of ground state energy levels, i.e.,

$$\inf_{\substack{J'(u)=0\\ u\neq 0}} J(u) =: \mathfrak{c} \leq \tilde{\mathfrak{c}} := \inf_{\substack{\tilde{J}'(u)=0\\ u\neq 0}} \tilde{J}(u)$$

is false in general. As an example we consider $J_\lambda(x,y) = 2\lambda x^2 + 2y^2 - x^4 - y^4$ on $\mathbb{R}\times\mathbb{R}$, which is increasing in λ while the ground state energy level

$$\mathfrak{c}_\lambda := \inf_{\substack{J_\lambda'(u)=0\\ u\neq 0}} \tilde{J}_\lambda(u) = \begin{cases} J(\sqrt{\lambda},0) = \lambda^2, & 0<\lambda\leq 1,\\ J_\lambda(0,1) = 1, & \lambda \leq 0 \text{ or } 1\leq\lambda \end{cases}$$

decreases by 1 as λ crosses 0.

In our setting, we can show that energy levels are comparable when we only perturb the nonlinearity. The argument also works when the linear operators in $J, \tilde{J}$ both are elliptic, in which case we can perturb both linear and nonlinear coefficients. In the context of the above example, ellipticity means we consider J_λ only for $\lambda > 0$, where $\lambda \mapsto \mathfrak{c}_\lambda$ indeed is monotone increasing.

In Chapters 3 and 4 we consider the quasilinear problem (1.7) for scalar polarizations (1.8.i) or (1.8.ii). In Chapter 3 we investigate breathers of high velocity c, while in Chapter 4 we seek slow breathers.

For this overview let us consider (1.8.i), i.e.,

$$-\partial_x^2 w + (1 - \tfrac{1}{c^2} + g(x,\cdot\,)*_{\mathbb{R}})\partial_t^2 w + h(x)\partial_t^2(\nu *_{\mathbb{R}} w^3) = 0 \tag{1.20}$$

with $*_{\mathbb{R}}$ denoting convolution in the time variable $t\in\mathbb{R}$.

Step 1. We require that (1.20) is time reversal symmetric, at least after restriction to time-periodic functions w of the fixed period T, so that (1.20) has a variational structure. For ν, this means that its periodization

$$\mathrm{Per}[\nu]\colon \mathbb{T}\to\mathbb{R},\ \mathrm{Per}[\nu](t) := T\sum_{k\in\mathbb{Z}} \nu(t+kT)$$

[5] For simplicity, we assume a strict inequality here. For the equality case, which we cannot exclude in general, we employ a different argument.

is even, and similarly $\mathrm{Per}[g(x;\,\cdot\,)]$ has to be even for all $x \in \mathbb{R}$. Note that $\nu *_{\mathbb{R}} f = \mathrm{Per}[\nu] *_{\mathbb{T}} f$ holds for time-periodic f. To uncover the variational structure, we apply $(\partial_t^2 \nu *_{\mathbb{R}})^{-1}$ to (1.20) and obtain

$$\left(-\partial_t^2 \,\mathrm{Per}[\nu]*_{\mathbb{T}}\right)^{-1}\left(-\partial_x^2 + (1 - \tfrac{1}{c^2} + \mathrm{Per}[g(x,\,\cdot\,)]*_{\mathbb{T}})\partial_t^2\right)w - h(x)w^3 = 0. \tag{1.21}$$

In the following, we abbreviate the linear operator of (1.21) by $\mathcal{L}$, so that this equation reads

$$\mathcal{L}w - h(x)w^3 = 0. \tag{1.22}$$

Note that $\mathcal{L}$ is symmetric since the periodizations are even. Therefore (1.22) formally is the Euler-Lagrange equation of the energy functional

$$J(w) = \int_{\mathbb{R}} \frac{1}{2}\mathcal{L}w \cdot w - \frac{1}{4}h(x)w^4 \,\mathrm{d}(x,t). \tag{1.23}$$

A more general and detailed derivation of the variational problem is given in Sections 3.2 and 4.2. We note that the polarization (1.8.ii) leads to the same variational problem but for the surrogate variable $u = \mathrm{Per}[\nu] *_{\mathbb{T}} w$ instead of w, and thus can be treated almost identically. Furthermore, if $\mathrm{Per}[\nu] *_{\mathbb{T}} (\,\cdot\,)$ is not injective, i.e., if some Fourier coefficients of $\mathrm{Per}[\nu]$ vanish, then this can be built into the variational problem by requiring that the same Fourier coefficients of w (or u) also vanish.

Assuming the velocity c of the wave is large enough, then $1 - \frac{1}{c^2} + g(x,\,\cdot\,)*_{\mathbb{R}}$ is positive and the linear operator of (1.20) is hyperbolic. Existence of breathers in this setting is investigated in Chapter 3. Conversely, for sufficiently small c the linear operator is elliptic, and so is $\mathcal{L}$ provided $\mathrm{Per}[\nu]$ is positive definite. This setting is studied in Chapter 4.

IN CHAPTER 3 we investigate (1.7) for fast-traveling waves and for scalar polarizations (1.8.i) or (1.8.ii). We assume that the linear retardation kernel

$$g(x,\tau) = g_0(x)\delta_0(\tau) + g_1(x,\tau)$$

consists of an instantaneous contribution g_0 plus a (small) noninstantaneous contribution g_1. We abbreviate

$$V(x) := 1 - \tfrac{1}{c^2} + g_0(x)$$

so that equation (1.21) can be written as

$$\left(-\partial_t^2 \,\mathrm{Per}[\nu]*_{\mathbb{T}}\right)^{-1}\left(-\partial_x^2 + (V(x) + \mathrm{Per}[g_1(x,\,\cdot\,)]*_{\mathbb{T}})\partial_t^2\right)w - h(x)w^3 = 0,$$

which is similar to equation (1.18). These similarities motivate us to look at (1.22) through the methods developed in Chapter 2. We now state our main existence result.

Theorem 3.1.5. *Let $T > 0$ be the period of the breather, $\omega := \frac{2\pi}{T}$ be its frequency and $c \in (0,\infty)$ be its speed. Assume that for constants $\alpha > 1$, $\frac{1}{2} \le \beta < 2, \gamma \le 1$, $0 < d < \delta$ we have:*

- ($\mathcal{A}$3.1) $h \in L^\infty(\mathbb{R};(0,\infty))$.
- ($\mathcal{A}$3.2) $\nu \in L^1(\mathbb{R};\mathbb{R})$ *and its periodization $\mathcal{N} := \mathrm{Per}[\nu]$ is even. Denoting its Fourier support restricted to odd frequencies by $\mathfrak{R} := \left\{k \in \mathbb{Z}_{\mathrm{odd}} \colon \hat{\mathcal{N}}_k \neq 0\right\}$, we have $\mathfrak{R} \neq \emptyset$ and $|\hat{\mathcal{N}}_k| \lesssim |k|^{-\alpha}$ for all $k \in \mathfrak{R}$.*

($\mathcal{A}3.3$) $g_0 \in L^\infty(\mathbb{R};\mathbb{R})$ *satisfies* $\operatorname{ess\,inf} g_0 > \frac{1}{c^2} - 1$ *and is locally of bounded variation.*

($\mathcal{A}3.4$) *For the spectrum of the operator* $L = -\frac{1}{V(x)}\partial_x^2 \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ *we have:*

- $(\omega^2 k^2 - \delta|k|^\gamma, \omega^2 k^2 + \delta|k|^\gamma) \subseteq \rho(L)$ *holds for all* $k \in \mathfrak{R}$.
- *The point spectrum* $\sigma_p(L)$ *satisfies* $\sum_{\lambda\in\sigma_p(L)} \lambda^{-\beta-\varepsilon} < \infty$ *for all* $\varepsilon > 0$.

($\mathcal{A}3.5$) $\alpha + \gamma - 2 > \beta$.

($\mathcal{A}3.6$) $g_1 \in L_x^\infty(\mathbb{R}; L_t^1(\mathbb{R};\mathbb{R}))$ *and its periodization* $\mathcal{G}(x) := \operatorname{Per}[g_1(x;\cdot)]$ *is even in* t *and satisfies* $|\hat{\mathcal{G}}_k(x)| \leq \frac{d}{\omega^2|k|^{2-\gamma}} V(x)$ *for all* $x \in \mathbb{R}$, $k \in \mathfrak{R}$.

($\mathcal{A}3.7$) *If the scalar polarization is* (1.8.ii), *assume additionally* $|\hat{\mathcal{N}}_k| \gtrsim |k|^{-s}$ *for* $k \in \mathfrak{R}$ *and some* $s \in \mathbb{R}$, *and that there exist constants* $\tilde{\gamma} \leq 1$, $0 < \tilde{d} < \tilde{\delta}$ *such that*

$$(\omega^2 k^2 - \tilde{\delta}|k|^{\tilde{\gamma}}, \omega^2 k^2 + \tilde{\delta}|k|^{\tilde{\gamma}}) \subseteq \rho(L), \qquad |\hat{\mathcal{G}}_k(x)| \leq \frac{\tilde{d}}{\omega^2|k|^{2-\tilde{\gamma}}} V(x)$$

hold for all $k \in \mathbb{Z}_{\text{odd}} \setminus \mathfrak{R}$.

Further assume g_0, g_1, h *have* <u>*one*</u> *of the two following spatial geometries.*

($\mathcal{A}3.8$a) $h(x) \to 0$ *as* $|x| \to \infty$. *Moreover,* g_0 *is periodic on* $[R^+, \infty)$ *with period* $X^+ > 0$, *and also on* $(-\infty, R^-]$ *with period* X^-.

($\mathcal{A}3.8$b) $h = h^{\text{loc}} + h^{\text{per}}$ *where* $h^{\text{loc}}(x) \geq 0$ *satisfies* $h^{\text{loc}}(x) \to 0$ *as* $|x| \to \infty$, *and* h^{per}, g_0, g_1 *are periodic in* x *with common period.*

Then there exists a nonzero breather solution $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$ *to Maxwell's equations* (1.1), (1.2) *with polarization* (1.5.i) *or* (1.5.ii). *It has period* T *and travels with speed* c. *Each field* $\mathbf{F} \in \{\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}\}$ *has the regularity*

$$\partial_t^n \partial_x^m \mathbf{F} \in L^2(\Omega;\mathbb{R}^3) \cap L^\infty(\Omega;\mathbb{R}^3)$$

for $n \in \mathbb{N}_0$, $m \in \{0, \ldots, \overline{m}(\mathbf{F})\}$ *where* $\overline{m}(\mathbf{E}) = 2, \overline{m}(\mathbf{D}) = 0, \overline{m}(\mathbf{B}) = \overline{m}(\mathbf{H}) = 1$, *and for domains of the form* $\Omega = \mathbb{R} \times [y_1, y_2] \times [z_1, z_2] \times [t_1, t_2]$.

Moreover, if the set $\mathfrak{R}$ *is infinite, there exists infinitely many distinct breathers.*

Remark 1.2.6. In Theorems 3.1.2 and 3.1.3 we give two examples of potentials g_0, g_1, h, ν satisfying the assumptions of Theorem 3.1.5. Let us also point out:

- There are similarities between Theorem 3.1.5 and Theorem 2.1.1. Indeed, the former contains all assumptions of the latter, except that the growth assumptions of ($\mathcal{A}3.4$) and ($\mathcal{A}3.5$) are more general compared with ($\mathcal{A}2.3$) and ($\mathcal{A}2.4$).
- By assumption ($\mathcal{A}3.3$), the potential V is positive. This, together with smallness of g_1 in ($\mathcal{A}3.6$), ensures that the linear operator $\mathcal{L}$ is hyperbolic. Smallness of g_1 also allows us to treat the additional linear term $\partial_t^2 \operatorname{Per}[g_1(x,\cdot)] *_{\mathbb{T}}$ as a perturbation, and so we study the linear operator with methods from Chapter 2.

Now we give some ideas of the proof. We focus on steps 2 and 4 since step 1 was discussed above and step 3 uses the same ideas as in Chapter 2.

Step 2. The main methodical difference to Chapter 2 is the choice of variational method. We use the dual variational method, which compared with the generalized Nehari manifold method is simpler, especially for power nonlinearities. The dual method works as follows: Formally, we invert both the linear operator and the nonlinearity to get

$$\mathcal{L}w - hw^3 = 0 \quad \xLeftrightarrow{u = h^{\frac{3}{4}} w^3} \quad u^{\frac{1}{3}} - h^{\frac{1}{4}} \mathcal{L}^{-1} h^{\frac{1}{4}} u = 0.$$

Note that the dual problem is again variational: The energy functional is given by

$$J(u) = \int_{\mathbb{R}\times\mathbb{T}} \tfrac{3}{4}|u|^{\frac{4}{3}} - \tfrac{1}{2}h^{\frac{1}{4}}\mathcal{L}^{-1}h^{\frac{1}{4}}u \cdot u \,\mathrm{d}(x,t)$$

with domain $L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})$. The simple domain and the simple geometry of J are the main advantages of the dual method. Since the functional J has mountain pass geometry, we solve its Euler-Lagrange equation using the mountain pass method (cf. Theorem 1.2.2).

Step 4. Using boundedness of $|\partial_t|^{\varepsilon}\colon \mathcal{H} \to L^4$ for small $\varepsilon > 0$ in a bootstrapping argument on $\mathcal{L}w - hw^3 = 0$, we obtain C^∞-regularity in time for u by gaining ε derivatives in each step.

In Chapter 4 we again consider (1.7) with scalar polarizations (1.8.i) or (1.8.ii), but for small velocities c so that the linear operator $\mathcal{L}$ is elliptic. In addition to slab materials, we also consider cylindrical materials, i.e., (1.10) in place of (1.7) with the same polarizations. This chapter is based on the preprint [72] which is written together with Wolfgang Reichel.

To simplify the notation of instantaneous and retarded contributions in the linear retardation $g(\mathbf{x},\tau)$, we now assume that $g(\mathbf{x},\,\cdot\,)$ is a real-valued measure in t for each $\mathbf{x}$, which we denote by $G(\mathbf{x})$. We write $\mathcal{M}(\mathbb{R})$ and $\mathcal{M}(\mathbb{T})$ for the sets of $\mathbb{R}$-valued measures on $\mathbb{R}$ and $\mathbb{T}$, respectively. They are equipped with the total variation norm. To unify notation, we also assume that ν is a real-valued measure.[6] Analogous to the periodization of a function, we define the periodization $\operatorname{Per}[\mu] \in \mathcal{M}(\mathbb{T})$ of a measure $\mu \in \mathcal{M}(\mathbb{R})$ by $\operatorname{Per}[\mu](E) = \mu(P_{\mathbb{T}}^{-1}(E))$ for any Borel set $E \subseteq \mathbb{T}$, where $P_{\mathbb{T}}\colon \mathbb{R} \to \mathbb{T}$ is the canonical projection.

We now state the main result of this chapter.

Theorem 4.1.3. *Let $T > 0$ denote the temporal period, $\omega := \frac{2\pi}{T}$ the associated frequency, and $c \in (0,\infty)$ the speed of travel of the breather. We make the following assumptions:*

- ($\mathcal{A}$4.1) *We consider polarization* (1.3) *with linear susceptibility $\chi^{(1)}(\mathbf{x},\tau)\,\mathrm{d}\tau = \mathrm{d}G(\mathbf{x})(\tau)I$, where $G\colon \mathbb{R}^3 \to \mathcal{M}(\mathbb{R})$ is measurable, and nonlinear susceptibility tensor $\chi^{(3)}$ given by* (1.5.i) *or* (1.5.ii)*, where $h \in L^\infty(\mathbb{R}^3;\mathbb{R})$ and $\nu \in \mathcal{M}(\mathbb{R})$.*
- ($\mathcal{A}$4.2) *G and h both have either cylindrical or slab geometry.*
- ($\mathcal{A}$4.3) $\sup_{\mathbf{x}\in\mathbb{R}^3}\|G(\mathbf{x})\|_{\mathcal{M}(\mathbb{R})} < \infty$ *and* $h \not\leq 0$.
- ($\mathcal{A}$4.4) *The periodization $\mathcal{G}(\mathbf{x}) := \operatorname{Per}[G(\mathbf{x})]$ is even in time for all $\mathbf{x} \in \mathbb{R}^3$ and satisfies* $\sup_{\mathbf{x}\in\mathbb{R}^3, k\in\mathbb{Z}} \mathcal{F}_k[\mathcal{G}(\mathbf{x})] < \frac{1}{c^2} - 1$.
- ($\mathcal{A}$4.5) *The periodization $\mathcal{N} := \operatorname{Per}[\nu]$ is even in time, nonconstant, and $|k|^{-\beta} \lesssim \mathcal{F}_k[\mathcal{N}] \lesssim |k|^{-\alpha}$ holds for all $k \in \mathbb{Z}\setminus\{0\}$ with $\mathcal{F}_k[\mathcal{N}] \neq 0$ and some $\beta \geq \alpha > \alpha^\star$ where $\alpha^\star = 1$ in the slab geometry and $\alpha^\star = \frac{3}{2}$ in the cylindrical geometry.*
- ($\mathcal{A}$4.6) *In case of the slab geometry, <u>one</u> of the following holds as well:*
  - ($\mathcal{A}$4.6a) *$h(x) \to 0$ as $x \to \pm\infty$,*
  - ($\mathcal{A}$4.6b) *$\mathcal{G}(x) = \mathcal{G}^{\mathrm{per}}(x) + \mathcal{G}^{\mathrm{loc}}(x)$ and $h(x) = h^{\mathrm{per}}(x) + h^{\mathrm{loc}}(x)$ where $\mathcal{G}^{\mathrm{per}}(x), h(x)$ are periodic with common period, and we have $\left\|\mathcal{G}^{\mathrm{loc}}(x)\right\|_{\mathcal{M}(\mathbb{T})} \to 0$ and $h^{\mathrm{loc}}(x) \to 0$ as $x \to \pm\infty$. Moreover, $\mathcal{G}^{\mathrm{loc}}(x)$ has nonnegative Fourier coefficients for all $x \in \mathbb{R}$ and $h^{\mathrm{loc}} \geq 0, h^{\mathrm{per}} \not\leq 0$ hold.*

[6]The assumptions imposed later ensure that ν is a bounded function.

Under these assumptions, there exists a nontrivial breather solution $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H}$ *of Maxwell's equations* (1.1),(1.2)*. It satisfies*

$$\partial_t^n \mathbf{E} \in W^{2,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{B}, \partial_t^n \mathbf{H} \in W^{1,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{D} \in L^p(\Omega; \mathbb{R}^3)$$

for all $n \in \mathbb{N}_0$*,* $p \in [2, \infty]$ *and all domains* Ω *of the form* $\Omega = \mathbb{R} \times [y, y+1] \times [z, z+1] \times [t, t+1]$ *in the slab case, and* $\Omega = \mathbb{R}^2 \times [z, z+1] \times [t, t+1]$ *in the cylindrical case, with norm bounds independent of* y, z, t*.*

Moreover, if the material parameters are $l \in \mathbb{N}_0$ *times continuously differentiable, i.e., if* $G \in C_b^l(\mathbb{R}^3; \mathcal{M}(\mathbb{R}))$ *and* $h \in C_b^l(\mathbb{R}^3)$*, then the fields* $\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}$ *are* l *additional times differentiable with derivatives in* $L^2(\Omega; \mathbb{R}^3) \cap C_b(\mathbb{R}^4; \mathbb{R}^3)$*.*

Lastly, if $\mathcal{F}_k[\mathcal{N}] \neq 0$ *for infinitely many* $k \in \mathbb{Z} \setminus \{0\}$*, then there exist infinitely many breather solutions with the stated properties.*

In Theorems 4.1.1 and 4.1.2 we give two examples of potentials G, h, ν satisfying our assumptions. Let us point out some important features of Theorem 4.1.3 and differences to previous results.

- In contrast to our results in Chapters 2 and 3, we can consider settings where both the linear and nonlinear potential are perturbed periodic functions. We motivated this in Remark 1.2.5.
- In the cylindrical setting, assumption ($\mathcal{A}$4.6) makes no further requirements on the potentials G, h. This is because the cylindrical setting has inherent compactness properties which are not present in the slab geometry. We discuss this in Section 4.5. As an illustration, consider the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in (2, \infty)$, which is not compact due to translation invariance, but it becomes compact after restriction to radially symmetric functions (cf. [52]).
- Note that in Theorem 4.1.3 we require no information on the spectra of operators. This is due to the fact that our linear operator contains

  $$-\partial_x^2 + (1 - \tfrac{1}{c^2} + \mathcal{G}(x) *_{\mathbb{T}}) \partial_t^2$$

  which is a sum of positive operators. This is different from Chapters 2 or 3, where the linear operator is a difference of positive operators. There we made assumptions to ensure that the spectra of the individual operators do not overlap, so that the entire operator is invertible, which is not needed here.

Let us also comment on selected parts of the proof.

Step 2. For the linear operator

$$\mathcal{L} = \left(-\partial_t^2 \mathcal{N} *_{\mathbb{T}}\right)^{-1} \left(-\partial_x^2 + (1 - \tfrac{1}{c^2} + \mathcal{G}(x) *_{\mathbb{T}}) \partial_t^2\right)$$

appearing in (1.22), notice that $-(1 - \frac{1}{c^2} + \mathcal{G}(x) *_{\mathbb{T}})$ is positive and bounded from above and below by assumptions ($\mathcal{A}$4.3) and ($\mathcal{A}$4.4). Since the Fourier coefficients of $\mathcal{N}$ are all positive by ($\mathcal{A}$4.5), $\mathcal{L}$ is elliptic. Therefore the energy functional (1.23) has mountain pass geometry, and we use the mountain pass method to find critical points.

For the cylindrical geometry, the problem of interest is (1.22) with

$$\mathcal{L} = \left(-\partial_t^2 \mathcal{N} *_{\mathbb{T}}\right)^{-1} \left(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} + (1 - \tfrac{1}{c^2} + \mathcal{G}(x) *_{\mathbb{T}}) \partial_t^2\right) \qquad \text{for} \quad (r, t) \in (0, \infty) \times \mathbb{T}.$$

Motivated by the setting, we treat this as a problem in two space dimensions and identify $w(r,t)$ with the cylindrically symmetric function $w(x,y,t)$, i.e., $r = \sqrt{x^2+y^2}$. The spatial differential operator is the Laplacian $-\partial_r^2 - \frac{1}{r}\partial_r = -\partial_x^2 - \partial_y^2 = -\Delta_{2\mathrm{d}}$, and the $\frac{1}{r^2}$-term is an additional positive term that cannot be controlled by the remaining terms since Hardy's inequality $\|\frac{w}{r}\|_{L^p(\mathbb{R}^2)} \leq \|\nabla w\|_{L^p(\mathbb{R}^2)}$ fails in $\mathbb{R}^2$ (cf. [61]).

For considerations of regularity, we employ a bootstrapping argument similar to that of Chapter 3, which can be lifted from $-\partial_x^2$ to the Laplacian in higher dimensions as the spatial operator. For large r the remaining $\frac{1}{r^2}$-term can be controlled, showing that breather solutions are smooth. However, for small r the perturbation character is lost and we have to understand $\frac{1}{r^2}$ as a part of the differential operator. There we can use $-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} = -r\Delta_{4\mathrm{d}}\frac{1}{r}$ and employ a bootstrapping argument on $\frac{1}{r}w$.[7]

IN CHAPTER 5 we study the slab problem (1.7) and the cylindrical problem (1.10), both with scalar polarization (1.8.iii). This chapter is based on the preprint [71], which is written together with Wolfgang Reichel. Our main assumption is that the nonlinearity is compactly supported in space. We also use spectral (and other) properties of the linear operator.

For brevity, we discuss only our existence result on breathers for slab materials, a similar result for cylindrical materials is given in Theorems 5.2.1 and 5.2.3.

In the slab setting, we make the following assumptions on the material coefficients g_0, h, ν.

($\tilde{\mathcal{A}}$5.1) $g_0, h \in L^\infty(\mathbb{R};\mathbb{R})$ are even, the linear polarization kernel is given by $g(x,\tau) = g_0(x)\delta_0(\tau)$ and $\operatorname{supp}(h) = [-R,R]$ holds for some $R > 0$.

($\tilde{\mathcal{A}}$5.2) The nonlinear polarization $P(w)$ is given by (1.8.iii) where either $\nu = \delta_0$, or $\nu \in L^1(\mathbb{R};\mathbb{R})$ and $\mathcal{N} := \operatorname{Per}[\nu]$ satisfies the following assumptions

$$\begin{cases} \mathcal{N} \in C^\alpha(\mathbb{T}) \text{ for some } \alpha > 0, \\ \mathcal{N}(t) = \mathcal{N}(-t) > 0 \text{ for } t \in \mathbb{T}, \\ L^4(\mathbb{T}) \to \mathbb{R}, v \mapsto \int_{\mathbb{T}} (\mathcal{N} * v^2) v^2 \,\mathrm{d}t \text{ is convex.} \end{cases}$$

($\tilde{\mathcal{A}}$5.3) $\operatorname{ess\,sup}_{[-R,R]} g_0 \leq \frac{1}{c^2} - 1$ and $\operatorname{ess\,sup}_{[-R,R]} h < 0$.

($\tilde{\mathcal{A}}$5.4) For each $k \in \mathbb{N}_{\mathrm{odd}}$, there exists a solution $\widetilde{\phi}_k \in H^2([R,\infty)) \setminus \{0\}$ of

$$(-\partial_x^2 - k^2\omega^2(g_0(x) + 1 - \tfrac{1}{c^2}))\widetilde{\phi}_k = 0.$$

($\tilde{\mathcal{A}}$5.5) The following inequalities are satisfied by the solution $\widetilde{\phi}_k$:

$$\liminf_{k\to\infty} \frac{|\widetilde{\phi}_k(R)|}{\|\widetilde{\phi}_k\|_{L^2([R,\infty))}} > 0, \qquad \sup_k \frac{|\widetilde{\phi}_k'(R)|}{k\|\widetilde{\phi}_k\|_{L^2([R,\infty))}} < \infty.$$

($\tilde{\mathcal{A}}$5.6) There exists $k_0 \in \mathbb{N}_{\mathrm{odd}}$ such that $\widetilde{\phi}_{k_0}(R) \neq 0$ and the following inequality holds:

$$\frac{\widetilde{\phi}_{k_0}'(R)}{\widetilde{\phi}_{k_0}(R)} > \lambda k_0 \tanh(\lambda k_0 R) \quad \text{with} \quad \lambda := \omega\big(\tfrac{1}{c^2} - 1 - \operatorname*{ess\,inf}_{[-R,R]} \tilde{\chi}_1\big)^{1/2}.$$

A sufficient condition for ($\tilde{\mathcal{A}}$5.6) to hold is

[7] This is only helpful for small r because the rescaled equation $\mathcal{L}w - hw^3 = 0 \iff (\frac{1}{r}\mathcal{L}r)\frac{1}{r}w - hr^2(\frac{1}{r}w)^3 = 0$ is singular at $r = \infty$ as an equation for $\frac{1}{r}w$.

($\tilde{\mathcal{A}}$5.6') $\limsup\limits_{k\to\infty} \dfrac{\widetilde{\phi}_k'(R)}{k\widetilde{\phi}_k(R)} > \omega\big(\frac{1}{c^2} - 1 - \operatorname*{ess\,inf}\limits_{[-R,R]} \tilde{\chi}_1\big)^{1/2}$.

Our main result for the slab geometry is given next.

Theorem 5.2.4 and 5.2.5. *Assume ($\tilde{\mathcal{A}}$5.1)–($\tilde{\mathcal{A}}$5.6) for given ν, g_0, h and T. Then there exists a nonzero T-periodic real-valued distributional solution $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H} \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^3)$ of the Maxwell problem* (1.1), (1.2) *with polarization* (1.5.iii), *i.e., a solution in the sense of Definition 5.1.1. The solution is localized in the x-direction to the extent that the electromagnetic energy per unit square in the y, z-directions*

$$\int_{\mathbb{R}\times[y_0,y_0+1]\times[z_0,z_0+1]} (\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H})\,\mathrm{d}(x,y,z)$$

is finite for all $y_0, z_0 \in \mathbb{R}$ and times $t_0 \in \mathbb{R}$, and uniformly bounded w.r.t. t_0, y_0, z_0.

Furthermore, if ($\tilde{\mathcal{A}}$5.6) holds for infinitely many $k_0 \in \mathbb{N}_{\mathrm{odd}}$ (e.g., if ($\tilde{\mathcal{A}}$5.6') is true) then there exist infinitely many distinct breather solutions with the above properties.

In Theorem 5.1.2 we give examples of potentials g_0, h satisfying our assumptions. Examples of admissible ν are discussed in Remark 5.A.2, and a sufficient condition for the convexity part of ($\tilde{\mathcal{A}}$5.2) is given in Lemma 5.A.1. Numerical approximations of some of the breathers from our examples are displayed in Figures 5.1 and 5.2.

Let us sketch the variational method for the slab geometry and comment on the assumptions ($\tilde{\mathcal{A}}$5.1)–($\tilde{\mathcal{A}}$5.6).

Step 1. We illustrate the variational setup for the instantaneous nonlinear polarization $\nu = \delta_0$, where assumptions ($\tilde{\mathcal{A}}$5.1) and ($\tilde{\mathcal{A}}$5.2) let us write (1.7) as

$$-w_{xx} + \big(g_0(x) + 1 - \tfrac{1}{c^2}\big)w_{tt} + h(x)(w^3)_{tt} = 0.$$

To reveal the variational structure, we substitute $w = u_t$ and integrate in t to obtain

$$-u_{xx} + \big(g_0(x) + 1 - \tfrac{1}{c^2}\big)u_{tt} + h(x)(u_t^3)_t = 0, \tag{1.24}$$

which is the Euler-Lagrange equation of

$$J(u) = \int_{\mathbb{R}\times\mathbb{T}} \tfrac{1}{2}u_x^2 + \tfrac{1}{2}V(x)u_t^2 - \tfrac{1}{4}h(x)u_t^4\,\mathrm{d}(x,t),$$

where $V(x) := -(g_0(x) + 1 - \frac{1}{c^2})$.

Step 2. We seek solutions of this variational problem by direct minimization of J on a suitable space of functions

$$X \subseteq \Big\{u\colon \mathbb{R}\times\mathbb{T}\to\mathbb{R} \,\Big|\, u \text{ is even in } x,\ \tfrac{T}{2}\text{-antiperiodic in } t, \text{ solves (1.24) on } [R,\infty)\times\mathbb{T}\Big\}.$$

The evenness condition in the definition of X is convenient as it shortens some formulas, but it is not necessary in general. Evenness is compatible with (1.24) since g_0, h, V are even by assumption ($\tilde{\mathcal{A}}$5.1). Antiperiodicity is also compatible with (1.24) and is used because the assumptions ($\tilde{\mathcal{A}}$5.4) and ($\tilde{\mathcal{A}}$5.5) on the linear operator were imposed only for odd frequencies.

Recall that h is supported on $[-R,R]$ so that (1.24) is linear on $(\mathbb{R}\setminus[-R,R])\times\mathbb{T}$. By symmetry it suffices to consider $[R,\infty)\times\mathbb{T}$. There, by assumption ($\tilde{\mathcal{A}}$5.4) equation (1.24) has the localized solutions

$$u(x,t) = \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \alpha_k \widetilde{\phi}_{|k|}(x)\mathrm{e}^{\mathrm{i}k\omega t} \qquad \text{for} \quad x \geq R, t\in\mathbb{T}. \tag{1.25}$$

This inspires the choice

$$X := \left\{ u\colon \mathbb{R}\times\mathbb{T}\to\mathbb{R} \,\middle|\, \begin{array}{l} u \text{ is even in } x,\ \tfrac{T}{2}\text{-antiperiodic in } t,\ (1.25) \text{ holds with } \alpha_k = \dfrac{\hat{u}_k(R-)}{\tilde{\phi}_k(R)}, \\ u_x \in L^2([0,R]\times\mathbb{T}), u_t \in L^4([0,R]\times\mathbb{T}) \end{array} \right\}$$

where the formula for α_k is obtained from (1.25) by requiring $u(R-,t) = u(R+,t)$ and taking the Fourier series of this identity. Note that we do not assume that $u \in X$ is differentiable in the region $[R,\infty)\times\mathbb{T}$. Despite this, J is well-defined on X. To see this, we formally calculate

$$\begin{aligned} J(u) &= \int_{[0,\infty)\times\mathbb{T}} u_x^2 + V(x)u_t^2 - \tfrac{1}{2}h(x)u_t^4 \,\mathrm{d}(x,t) \\ &= \int_{[0,R]\times\mathbb{T}} u_x^2 + V(x)u_t^2 - \tfrac{1}{2}h(x)u_t^4 \,\mathrm{d}(x,t) + \sum_{k\in\mathbb{Z}_{\text{odd}}} |\alpha_k|^2 \int_R^\infty \Big((\tilde{\phi}_k')^2 + \omega^2 k^2 V(x)\tilde{\phi}_k^2 \Big)\,\mathrm{d}x \\ &= \int_{[0,R]\times\mathbb{T}} u_x^2 + V(x)u_t^2 - \tfrac{1}{2}h(x)u_t^4 \,\mathrm{d}(x,t) + \sum_{k\in\mathbb{Z}_{\text{odd}}} \frac{|\hat{u}_k(R-)|^2}{\tilde{\phi}_k(R)^2}\Big(-\tilde{\phi}_k'(R)\tilde{\phi}_k(R) \Big) \\ &=: J_I(u) + J_B(u), \end{aligned}$$

and we use the last identity as the proper definition of J on X. The boundary term J_B is finite (and weakly continuous) due to the estimates on $\tilde{\phi}_k(R), \tilde{\phi}_k'(R)$ in assumption ($\tilde{\mathcal{A}}$5.5). By ($\tilde{\mathcal{A}}$5.3) both V and $-h$ are strictly positive in $[0,R]$ so that the integral term J_I is convex and coercive.

Step 3. The minimization method relies on two important properties of boundary term J_B and its symbol $-\frac{\tilde{\phi}_k'(R)}{\tilde{\phi}_k(R)}$. First, by ($\tilde{\mathcal{A}}$5.5) the symbol decays fast enough, which combined with a trace inequality shows that J is bounded from below, weakly lower semicontinuous, and coercive. Second, the symbol is large enough to ensure $\inf J < 0$, where assumption ($\tilde{\mathcal{A}}$5.6) is a sufficient condition (and necessary if $V|_{[-R,R]}$ is constant).

Step 4. For the properties of J_B in step 3 it is necessary that we impose no regularity condition on u in $[R,\infty)\times\mathbb{T}$. So we verify differentiability of u in the linear region $[R,\infty)\times\mathbb{T}$ afterwards. For this, using ellipticity of the main part J_I and estimates to control the boundary part J_B, we obtain[8] uniform estimates for higher-order terms such as

$$\Big\| |\partial_t|^{1/2} u_x \Big\|_{L^2([0,R]\times\mathbb{T})} \le C, \quad \Big\| |\partial_t|^{1/2} u_t |u_t| \Big\|_{L^2([0,R]\times\mathbb{T})} \le C, \quad \|u(R-,\cdot)\|_{H^1(\mathbb{T})} \le C.$$

With the third estimate and the formula (1.25) we can verify that u is H^1-regular in $[R,\infty)\times\mathbb{T}$. Since $w = u_t$ is the intensity of the electric field, this is sufficient to prove $w \in L^2$ and hence $\mathbf{E} \in L^2_{\text{loc}}$.

IN CHAPTER 6 we investigate the flow of a particular setup of Maxwell's equations, which is the associated linear wave equation with quasilinear boundary condition

$$\begin{cases} V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, & x\in[0,\infty), t\in[0,\infty), \\ u_x(0,t) = (f(u_t(0,t)))_t, & x = 0, t\in[0,\infty), \\ u(x,t_0) = u_0(x), u_t(x,0) = u_1(x), & x\in[0,\infty), t = 0. \end{cases} \tag{1.26}$$

[8] We give an explicit proof only for analogous estimates in the cylindrical setting, see Proposition 5.4.8.

This chapter is based on the paper [73] written together with Wolfgang Reichel and Roland Schnaubelt. We give an overview of related results from the literature in Section 6.1. To motivate the initial value problem (1.26), we consider (1.7) with the instantaneous polarization

$$P(w)(x,t) = g_0(x)w(x,t) + 2f(w(x,t))\delta_0(x) \tag{1.27}$$

where the nonlinearity is concentrated at one point $x = 0$. If the potential g_0 and the solution w are even in x, we obtain from (1.7) the effective problem

$$\begin{cases} V(x)w_{tt}(x,t) - w_{xx}(x,t) = 0, & x \in [0,\infty), t \in [0,\infty), \\ w_x(0+,t) = \partial_t^2 f(w(0,t)), & t \in [0,\infty) \end{cases} \tag{1.28}$$

on the half-space with $V(x) \coloneqq 1 - \frac{1}{c^2} + g_0(x)$. We substitute $w(x,t) = u_t(x,t)$ and integrate in time to turn (1.28) into

$$\begin{cases} V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in [0,\infty), t \in [0,\infty), \\ u_x(0+,t) = (f(u_t(0,t)))_t, & x = 0, t \in [0,\infty). \end{cases} \tag{1.29}$$

Note that (1.26) is the initial value problem for (1.29). The choice of material law is motivated by [49] where the authors consider (1.27) with cubic nonlinearity $f(s) = \pm s^3$ and show existence of infinitely many breather solutions using variational methods.

We investigate (1.26) for monotone increasing f and positive, piecewise continuously differentiable functions $V \in PC^1([0,\infty);\mathbb{R})$. That is, V is continuously differentiable outside a discrete set $D \subseteq [0,\infty)$ and one-sided limits of V and V' exist on D.

Next we present the two main results of this chapter. The first concerns global well-posedness of (1.26) in $C^1([0,\infty) \times [0,\infty))$, and we explain the notion of a C^1-solution in Definition 6.1.3 below. The second discusses two conserved quantities.

In the following, the space $C^k(X;Y)$ of k-times continuously differentiable functions $\phi\colon X \to Y$ is equipped with the topology of locally uniform convergence (of ϕ and its derivatives), and the subspace $C_b^k(X;Y)$ of bounded functions with bounded derivatives is equipped with the topology of uniform convergence.

Theorem 6.1.1. *Assume on V, f the following.*

- ($\mathcal{A}6.0$) $u_0 \in C^1([0,\infty);\mathbb{R})$, $u_1 \in C([0,\infty);\mathbb{R})$.
- ($\mathcal{A}6.1$) $V \in PC^1([0,\infty);\mathbb{R})$ *with* $V, V' \in L^\infty([0,\infty)), \inf V > 0$.
- ($\mathcal{A}6.2$) $\inf\{|d_1 - d_2|$ *with* $d_1, d_2 \in D(V) \cup \{0\}, d_1 \neq d_2\} > 0$ *where* $D(V)$ *denote the set of discontinuities of* V.
- ($\mathcal{A}6.3$) $f\colon \mathbb{R} \to \mathbb{R}$ *is an increasing homeomorphism.*

Then (1.26) *has a unique solution* $u \in C^1([0,\infty) \times [0,\infty);\mathbb{R})$*. Moreover,* (1.26) *is well-posed in the sense that the solution map*

$$C^1([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R}) \to C([0,\infty) \times [0,\infty);\mathbb{R}), \quad (u_0,u_1) \mapsto u$$

is continuous, and

$$C_b^1([0,\infty);\mathbb{R}) \times C_b([0,\infty);\mathbb{R}) \to C_b([0,\infty) \times [0,T];\mathbb{R}), \quad (u_0,u_1) \mapsto u$$

is continuous for every finite end time $T > 0$.

Theorem 6.1.4. *Assume ($\mathcal{A}$6.0)–($\mathcal{A}$6.3) and that the initial data satisfy $u_0'(x), u_1(x) \to 0$ as $x \to \infty$. Then the momentum*

$$M(u,t) \coloneqq \int_0^\infty V(x) u_t \,\mathrm{d}x + f(u_t(0,t))$$

and the energy

$$E(u,t) \coloneqq \tfrac{1}{2} \int_0^\infty \Big(V(x) u_t(x,t)^2 + u_x(x,t)^2 \Big) \,\mathrm{d}x + F(u_t(0,t)),$$

where $F(s) \coloneqq s f(s) - \int_0^s f(\sigma)\,\mathrm{d}\sigma$, are both constant in time.

While we formulated both results only in forward time $t \geq 0$, they also hold in backward time since (1.26) is time reversal symmetric up to the substitution $f(s) \rightsquigarrow -f(-s)$, which is compatible with the assumptions ($\mathcal{A}$6.0)–($\mathcal{A}$6.3). In Section 6.7 we connect Theorem 6.1.1 to [49] by showing that the breather solutions found in [49] are C^1-smooth and solve (1.26) with their own initial data.

As we discuss below, we cannot expect the solution of (1.26) to be twice differentiable, even for smooth initial data and smooth f, V. This is why we consider solutions in the space $C^1([0,\infty) \times [0,\infty); \mathbb{R})$. Let us clarify the precise sense in which the second derivates in (1.26) exist for C^1-solutions.

Definition 6.1.3. *In the setting of Theorem 6.1.1, a function $u \in C^1([0,\infty) \times [0,\infty); \mathbb{R})$ is called a C^1-solution to* (6.1) *if the following hold:*

(i) *For all $x \geq 0$ on which V and V' are continuous and all $t \geq 0$, the directional derivative $\partial_t - \frac{1}{\sqrt{V(x)}} \partial_x$ applied to $u_t + \frac{1}{\sqrt{V(x)}} u_x$ exists and satisfies*

$$\big(\partial_t - \tfrac{1}{\sqrt{V(x)}} \partial_x\big) \big[u_t + \tfrac{1}{\sqrt{V(x)}} u_x\big] = \tfrac{V'(x)}{2V(x)^2} u_x(x,t).$$

(ii) *$f(u_t(0,\cdot))$ is differentiable with $(f(u_t(0,t)))_t = u_x(0,t)$ for all $t \geq 0$.*

(iii) *$u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$ for all $x \geq 0$.*

Let us discuss the above results by investigating the example $V = 1$, $f(s) = \pm s^3$, i.e.,

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in [0,\infty), t \in [0,\infty), \\ u_x(0,t) = \pm(u_t(0,t)^3)_t, & x = 0, t \in [0,\infty), \\ u(x,t_0) = u_0(x), u_t(x,0) = u_1(x), & x \in [0,\infty), t = 0, \end{cases}$$

which already contains the main features of (1.26). As u solves the homogeneous one-dimensional wave equation, we have

$$\begin{aligned} u_x(x,t) + u_t(x,t) &= \text{const} \qquad \text{along curves} \qquad \{x + t = \text{const}\}, \\ u_x(x,t) - u_t(x,t) &= \text{const} \qquad \text{along curves} \qquad \{x - t = \text{const}\}. \end{aligned}$$

This allows us to rewrite the boundary condition as

$$\begin{aligned} \pm(u_t(0,t)^3)_t &= u_x(0,t) + u_t(0,t) - u_t(0,t) \\ &= u_x(t,0) + u_t(t,0) - u_t(0,t) \\ &= u_0'(t) + u_1(t) - u_t(0,t), \end{aligned}$$

and this differential equation depends only on $u|_{\{x=0\}}$. In terms of the quantity $d(t) = u_t(0,t)^3$, it reads

$$d'(t) = \mp \sqrt[3]{d(t)} \pm (u_0'(t) + u_1(t)) \tag{1.30}$$

with $\sqrt[3]{\cdot}$ denoting the real cube root.

For the "$\pm = -$" case and $u_0 = u_1 = 0$, (1.30) is well-known to be ill-posed, and it admits the solution

$$d(t) = \begin{cases} 0, & t \le t_0, \\ \left(\frac{2}{3}(t - t_0)\right)^{\frac{3}{2}}, & t \ge t_0 \end{cases}$$

for every $t_0 \in [0, \infty]$. From the above boundary data d, we can reconstruct a solution u to (1.26) traveling to the right by

$$\begin{aligned} u(x,t) = u(0, t-x) &= \int_0^{t-x} u_t(0,\tau)\,\mathrm{d}\tau = \int_0^{t-x} \sqrt[3]{d(\tau)}\,\mathrm{d}\tau = d(t-x) \\ &= \begin{cases} 0, & x \ge t - t_0, \\ \left(\frac{2}{3}(t - x - t_0)\right)^{\frac{3}{2}}, & x \le t - t_0, \end{cases} \end{aligned}$$

and we indeed have $u_0 = u|_{t=0} = 0$ and $u_1 = u_t|_{t=0} = 0$. In summary, for the "$\pm = -$" case we have seen that (1.26) is ill-posed since it can have multiple solutions.

Let us now consider the "$\pm = +$" case, where (1.30) is well-posed in forward time since the right-hand side is monotone decreasing in d (cf. [98, §9, Theorem X]). Similar to the above, we consider the solution

$$d(t) = \begin{cases} \left(\frac{2}{3}(t_0 - t)\right)^{\frac{3}{2}}, & t \le t_0 \\ 0, & t \ge t_0, \end{cases}$$

to (1.30) with $t_0 \in (0, \infty)$. It gives rise to the right-traveling wave u given by

$$u(x,t) = \int_0^{t-x} \sqrt[3]{d(\tau)}\,\mathrm{d}\tau = -d(t-x) = \begin{cases} -\left(\frac{2}{3}(t_0 + x - t)\right)^{\frac{3}{2}}, & x \ge t - t_0, \\ 0, & x \le t - t_0, \end{cases}$$

Here we observe a loss of regularity for the solution map, as u is only $C^{\frac{3}{2}}$-smooth while the initial data $u_0 = u|_{t=0}$ and $u_1 = u_t|_{t=0}$ lie in $C^\infty([0,\infty))$.

IN APPENDIX A we discuss the method of the generalized Nehari manifold (cf. [94]), which we use to find critical points in Chapter 2. We state the main result in Theorem A.1.1, which is adapted to fit our use case, and we give a self-contained proof.

IN APPENDIX B we introduce the space $L^2(\mu)$ of vector-valued functions that are square-integrable with respect to an increasing matrix-valued function μ, as well as the fractional Sobolev-Slobodeckij space $W^{s,p}(\mathbb{T})$ on the torus. We also discuss important properties of both spaces.

Publications

Chapter 2 consists of the preprint [41] which was written jointly with Julia Henninger and Wolfgang Reichel. Chapters 4 and 5 contain the preprints [72] and [71], respectively. Both are joint work with Wolfgang Reichel. At the time of submitting this thesis, all three preprints [41, 71, 72] have been submitted, and [72] has been recommended for publication with minor revisions. Chapter 6 comprises the paper [73] which appeared 2023 in Nonlinearity and was written together with Wolfgang Reichel and Roland Schnaubelt. For each of these four co-authored papers and preprints, all the authors made significant contributions.

Appendix B consists of some of the appendices of [41, 71, 73]. Chapter 3 and Appendix A are not co-authored, and at the time of submitting this thesis they have not been published or submitted elsewhere.

1.3. Outlook

Here are several interesting and open questions surrounding breather solutions to Maxwell's equations.

- With regard to Chapter 5, regularity of breather solutions seems not fully explored: We show that the electromagnetic fields are L^2_{loc}-smooth, whereas the numerical approximations are much smoother: the depictions in Figures 5.1 and 5.2 appear to be Hölder-continuous.

  In addition, all approximate ground states appear to be even in time, which would be interesting to verify analytically.
- In the setting of Chapter 3, we currently don't know if it is possible to find breathers for cylindrical materials, because we don't understand the spectral situation well enough. In a different context, we observed in Appendix 5.B that certain eigenfunctions in the slab and cylindrical setting are asymptotically equal (up to a natural rescaling) in the limit when the spectral parameter approaches ∞. One could hope that a similar relation exists between the spectrum of operators in the slab setting and those in the cylindrical setting.
- Additionally, for the problems in Chapters 2 and 3 we want to better understand the spectral assumptions on the linear operator that are needed for breathers to exist. We currently impose strong assumptions that force the entire linear operator to be invertible, which is necessary for our variational methods. This gives rise to the question: Can we find breathers if the linear operators has zero in its essential spectrum, and what effect does such a spectral situation have on breather formation?
- For our existence results on breathers, we rely on the ansatzes (1.6) or (1.9), which are essentially one-dimensional in space. Polychromatic breather solutions in more complicated material geometries, such as geometries that vary in two dimensions, are an interesting topic for future research.
- Perhaps the most important open question concerns the stability of breather solutions, both for the Maxwell system as well as for the effective equations like (1.7), since physically meaningful breathers should possess adequate stability properties, such as asymptotic or orbital stability.

1.4. Preliminaries and Notation

We fix some notation that is used throughout the thesis.

symbol	**definition**
$\mathbb{T}$	$= \mathbb{R}/_{T\mathbb{Z}}$, the one-dimensional torus of a fixed length $T > 0$ and time domain of all our breather problems. It is equipped with the Haar measure $\mathrm{d}t$ normalized such that $\int_{\mathbb{T}} 1 \,\mathrm{d}t = 1$.
ω	$= \frac{2\pi}{T}$, the base frequency on $\mathbb{T}$.
e_k	$: \mathbb{T} \to \mathbb{C}, e_k(t) = \mathrm{e}^{\mathrm{i}\omega kt}$ for $k \in \mathbb{Z}$, the standard orthonormal basis of $L^2(\mathbb{T})$.
$\hat{v}_k$	$= \int_{\mathbb{T}} v\overline{e_k} \,\mathrm{d}t$, the k-th Fourier coefficient of $v\colon \mathbb{T} \to \mathbb{C}$.
$\mathcal{F}$	Fourier transform in time $\mathbb{T}$, $\mathcal{F}[v] = (\mathcal{F}_k[v])_{k\in\mathbb{Z}} = (\hat{v}_k)_{k\in\mathbb{Z}}$, with inverse transform $\mathcal{F}_t^{-1}[\hat{v}_k] = \mathcal{F}_t^{-1}[(\hat{v}_k)_{k\in\mathbb{Z}}] = \sum_{k\in\mathbb{Z}} \hat{v}_k e_k(t)$. *also:* Fourier transform in space $\mathbb{R}^d$, $\mathcal{F}_\xi[v] = \int_{\mathbb{R}^d} v(x)\overline{\mathrm{e}^{\mathrm{i}x\cdot\xi}} \frac{\mathrm{d}x}{(2\pi)^{d/2}}$ with inverse $\mathcal{F}_x^{-1}[v] = \int_{\mathbb{R}^d} v(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi} \frac{\mathrm{d}\xi}{(2\pi)^{d/2}}$ for $v\colon \mathbb{R}^d \to \mathbb{C}$. *also:* Fourier transform in spacetime $\mathbb{R}^d \times \mathbb{T}$, $\mathcal{F}_{\xi,k} = \mathcal{F}_\xi\mathcal{F}_k$, $\mathcal{F}_{x,t}^{-1} = \mathcal{F}_x^{-1}\mathcal{F}_t^{-1}$. Indices of $\mathcal{F}, \mathcal{F}^{-1}$ are omitted when clear from the context.
$X \hookrightarrow Y$	X embeds into Y. That is, the identity map $\mathrm{Id}\colon X \to Y$ is well-defined, injective and continuous.
X'	dual of the Banach space X,
A'	adjoint of the linear operator A.
p'	conjugate Hölder exponent, $\frac{1}{p} + \frac{1}{p'} = 1$.
$\#S$	cardinality of the set S.
$\mathbb{1}_S$	indicator function of the set S, i.e., $\mathbb{1}_S(x) = 1$ if $x \in S$ and $\mathbb{1}_S(x) = 0$ otherwise. *also:* indicator of a statement, $\mathbb{1}_{\{\text{true}\}} = \mathbb{1}_{\text{true}} = 1$ and $\mathbb{1}_{\{\text{false}\}} = \mathbb{1}_{\text{false}} = 0$.
$I_{d\times d}$	the identity matrix in $\mathbb{C}^{d\times d}$.
$\mathbb{N}$	$= \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The subsets of $\mathbb{N}$ and $\mathbb{Z}$ consisting of odd or even numbers are denoted by $\mathbb{N}_{\text{odd}}$, $\mathbb{N}_{\text{even}}$ and $\mathbb{Z}_{\text{odd}}$, $\mathbb{Z}_{\text{even}}$.
$C^k(\Omega; X)$	space of k-times continuously differentiable functions $f\colon \Omega \to X$, equipped with the topology of locally uniform convergence of function and derivatives.
$C_b^k(\Omega; X)$	subspace consisting of k-times bounded continuously differentiable functions, equipped with the topology of uniform convergence of function and derivatives.
$L^p(\Omega; \mu)$	Lebesgue space of p-integrable functions $f\colon \Omega \to \mathbb{C}$ (modulo equality a.e.).
$L^p_{\text{loc}}(\Omega; \mu)$	space of locally p-integrable functions f, i.e., $f\|_K \in L^p(K)$ for compact $K \subseteq \Omega$.
$\ell^p(\Omega)$	$= L^p(\Omega; \#)$.
$W^{k,p}(\Omega)$	Sobolev space of k-times weakly differentiable functions f with $\partial^\alpha f \in L^p(\Omega)$ for $0 \leq \|\alpha\| \leq k$.
$W^{s,p}(\Omega)$	Sobolev-Slobodeckij space for noninteger s, defined in Section B.2 for $\Omega = \mathbb{T}$.
$H^s(\Omega)$	$= W^{s,2}(\Omega)$, or equivalently defined using the Fourier transform on Ω.
$\mathrm{o}(f)$	Landau notation: $\varphi \in \mathrm{o}(f)$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{\varphi(x)}{\|f(x)\|} = 0$.
$\mathcal{O}(f)$	Landau notation: $\varphi \in \mathcal{O}(f)$ as $x \to x_0$ if $\limsup_{x\to x_0} \frac{\\|\varphi(x)\\|}{\|f(x)\|} < \infty$.
$f \lesssim g$	There exists a constant $C > 0$ with $f \leq Cg$.
$f \eqsim g$	$:\Longleftrightarrow f \lesssim g$ and $g \lesssim f$.

We denote special subspaces of function spaces by a subscript, such as:

symbol	**definition**
X_{anti}	subspace of X consisting of $\frac{T}{2}$-antiperiodic (in time) functions. For example, $L^2_{\text{anti}}(\mathbb{R}\times\mathbb{T}) = \left\{f \in L^2(\mathbb{R}\times\mathbb{T}) \mid f(x, t+\frac{T}{2}) = -f(x,t)\right\}$.
X_{rad}	subspace consisting of radially symmetric (in space) functions. For example, $L^2_{\text{rad}}(\mathbb{R}^2\times\mathbb{T}) = \{f \in L^2(\mathbb{R}^2\times\mathbb{T}) \mid \forall O \in \mathrm{SO}(2)\colon f(Ox,t) = f(x,t)\}$.
X_{c}	subspace of compactly supported functions.

Variational methods. We recall some fundamental concepts of variational methods, as these will be used throughout the thesis. On a Banach space X let $J \in C^1(X;\mathbb{R})$ be a functional, which we call the *energy functional*. Fundamentally, we are interested in finding a nonzero *critical point* (CP) of J, i.e., a solution $u \neq 0$ of the *Euler-Lagrange equation*

$$J'(u) = 0 \qquad \text{in} \quad X'.$$

Moreover, a *ground state* of J is a nonzero critical point u with minimal energy among all nonzero critical points. Accordingly, we refer to the energy level

$$J(u) = \inf_{\substack{v\in X\setminus\{0\}\\ J'(v)=0}} J(v)$$

as the *ground state energy level*.

Variational methods such as the mountain pass method (see Theorem 1.2.2), minimization, and the generalized Nehari manifold method (see Appendix A) do not show existence of a critical point directly, but only existence of an approximating sequence.

In this thesis, we use two notions of approximating sequences: We mainly work with Palais-Smale sequences (see Definition 1.2.1), but also use Cerami sequences in Appendix A.

Definition 1.4.1. *Let X be a real Banach space, $J \in C^1(X;\mathbb{R})$, and (u_n) a sequence in X. Then (u_n) is called a* Cerami sequence *for J at level $\mathfrak{c} \in \mathbb{R}$ if*

$$J(u_n) \to \mathfrak{c} \quad \text{in } \mathbb{R} \quad \text{and} \quad (1+\|u_n\|)J'(u_n) \to 0 \quad \text{in } X'.$$

Note that any Cerami sequences is, in particular, a Palais-Smale sequences. However, the reverse is not necessarily true.

To pass from a Palais-Smale sequence to a critical point of J requires further information on J. The simplest case is when J satisfies the *Palais-Smale condition*, i.e., every Palais-Smale sequence has a convergent subsequence. Recall that we also consider settings where the Palais-Smale condition is not satisfied, e.g., due to translational invariances of the problem. These require different arguments, which we sketched in step 3 in the introduction of Chapter 2.

Double symbols. To improve readability, we use double symbols like $\pm$, $\mp$, and statements involving these symbols should be read choosing consistently either the top symbol or the bottom symbol. So "$\mathcal{A}(\pm,\mp)$" means "both $\mathcal{A}(+,-)$ and $\mathcal{A}(-,+)$ are true". For terms involving double symbols, the operator $\sum^{\pm}$ is used to denote summation of top and bottom version of the term, so $\sum^{\pm} a(\pm,\mp) := a(+,-) + a(-,+)$. In contrast, we always use $\{\pm x\}$ as an an abbreviation for the set $\{x,-x\}$.

CHAPTER 2

Breather solutions for semilinear wave equations

This chapter is based on the preprint [41], which was written jointly with Julia Henninger and Wolfgang Reichel. The preprint was adapted to fit the notation and structure of this thesis. In particular, [41, Appendix A] was moved to Section B.1.

Start of Preprint

Abstract. We prove existence of real-valued, time-periodic and spatially localized solutions (breathers) of semilinear wave equations $V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u$ on $\mathbb{R}^2$ for all values of $p \in (1, \infty)$. Using tools from the calculus of variations our main result provides breathers as ground states of an indefinite functional under suitable conditions on V, Γ beyond the limitations of pure x-periodicity. Such an approach requires a detailed analysis of the wave operator acting on time-periodic functions. Hence a generalization of the Floquet-Bloch theory for periodic Sturm-Liouville operators is needed which applies to perturbed periodic operators. For this purpose we develop a suitable functional calculus for the weighted operator $-\frac{1}{V(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with an explicit control of its spectral measure. Based on this we prove embedding theorems from the form domain of the wave operator into L^q-spaces, which is key to controlling nonlinearities. We complement our existence theory with explicit examples of coefficient functions V and temporal periods T which support breathers.

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2.1. Introduction and main result

We are interested in breather solutions for the spatially heterogeneous semilinear wave equation

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u \text{ for } (x,t) \in \mathbb{R}^2 \tag{2.1}$$

for $p > 1$. Breathers are real-valued, periodic in t and localized in x, i.e., $\lim_{|x|\to\infty} u(x,t) = 0$. Our goal is to give sufficient conditions on V and Γ so that non-trivial breathers of (2.1) exist.

Breather solutions to (2.1) are a rare phenomenon in the sense that only few examples of coefficient classes $V(x)$ are known which support breathers – a more detailed account of the literature will be given at the end of the introduction. The results known to us so far [15, 43, 55, 56] and very recently [22] all rely on spatially periodic coefficients V_{per} and on having very good information on the spectrum of the spatial operator $L = -\frac{1}{V_{\mathrm{per}}(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. The main tool to describe periodic differential operators is Floquet-Bloch theory, which provides an explicit description of spectrum and spectral measure in terms of quasiperiodic eigenvalues and eigenfunctions on the periodicity cell. The basic idea in the approaches cited above is then to tailor V_{per} in such a way that L has spectral gaps about $k^2\omega^2$ for $k \in \mathbb{N}_{\mathrm{odd}}$, with ω being the temporal frequency of the breather, e.g., by a careful design of a piecewise constant V_{per}. As a consequence, the linear wave operator $V(x)\partial_t^2 - \partial_x^2$ is invertible in suitable spaces of $T = \frac{2\pi}{\omega}$-periodic functions. In principle, this approach is not limited to spatially periodic coefficients $V_{\mathrm{per}}(x)$, but as soon as one leaves this class, several difficulties arise:

(a) finding a way to describe the spectrum of L and to keep $k^2\omega^2$ out of it,

(b) finding a replacement for the Bloch-transform which diagonalizes the operator L.

In this paper, we overcome the limitations of spatially periodic coefficients to some extent and show existence of breathers to (2.1) for different kinds of perturbed periodic V. A prototypical case where our results apply is a perturbation $V = V_{\text{per}} + V_{\text{loc}}$ of a positive periodic potential V_{per} where V_{loc} has compact support. Concrete examples beyond the purely periodic setting are also provided.

As an essential tool to find breathers in the nonperiodic setting and in extension of [25], we show an explicit formula for the spectral measure of the perturbed periodic operator $L = -\frac{1}{V(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, cf. Theorem 2.3.23. The spectral description uses generalized eigenvalues and eigenfunctions replacing the quasiperiodic Bloch-eigenfunctions, and their interaction in the perturbed region. Once this is available we use the calculus of variations to find breathers of (2.1) as critical points of a functional J. We show that J is well defined in the Hilbert space $\mathcal{H} = \{u \in L^2(\mathbb{R}\times\mathbb{T}) : \int_{\mathbb{R}\times\mathbb{T}} u|\Box|uV(x)\,\mathrm{d}(x,t) < \infty\}$ where $\Box := \partial_t^2 - \frac{1}{V(x)}\partial_x^2$ is the weighted wave operator and $|\Box|$ its spectrally defined absolute value. To achieve this, we show embedding properties of $\mathcal{H}$ into $L^q(\mathbb{R}\times\mathbb{T})$-spaces using the functional calculus for L. Then, saddle-point tools from the calculus of variations [94] provide breathers for (2.1) as ground states of J.

With this brief introduction we can now describe our main result. We consider potentials V which coincide outside a finite interval with a periodic function V_{per}^+ in a neighborhood of $+\infty$ and V_{per}^- in a neighborhood of $-\infty$. This also includes the case where V is a purely periodic function. We use the following basic assumptions.

($\mathcal{A}$2.1) $V, \Gamma \in L^\infty(\mathbb{R})$ with $\operatorname{ess\,inf}_{\mathbb{R}} V > 0$ and $\Gamma > 0$ almost everywhere. Moreover, V has locally bounded variation.

($\mathcal{A}$2.2) There exist $X^\pm > 0$, $R^\pm \in \mathbb{R}$, and $X^\pm$-periodic functions $V_{\text{per}}^\pm \in L^\infty(\mathbb{R})$ such that $V(x) = V_{\text{per}}^+(x)$ for $x > R^+$ and $V(x) = V_{\text{per}}^-(x)$ for $x < R^-$.

The next two assumptions concern the spectrum of the weighted Sturm-Liouville operator L. If we consider the $\frac{1}{V(x)}$-weighted wave operator $\Box = \partial_t^2 - \frac{1}{V(x)}\partial_x^2$ applied to a time periodic function u, then a Fourier-decomposition of $u = \sum_{k\in\mathbb{Z}} \hat{u}_k(x)\mathrm{e}^{\mathrm{i}k\omega t}$ results into a Fourier-decomposition of $\Box$ into a family $(L_k)_{k\in\mathbb{Z}}$ of Sturm-Liouville operators

$$L_k := -\frac{1}{V(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2\omega^2, \tag{2.2}$$

where $\omega = \frac{2\pi}{T}$ is the frequency and T is the time period of the breather. As we shall see later, it is advantageous to consider only $k \in \mathbb{Z}_{\text{odd}}$ which amounts to a restriction to $\frac{T}{2}$-antiperiodic functions u. Since $L_k = L_{-k}$, the main assumption is now to keep 0 out of the spectrum of the family $(L_k)_{k\in\mathbb{N}_{\text{odd}}}$ or, in other words, to keep $k^2\omega^2$ out of the spectrum $\sigma(L)$ of the weighted Sturm-Liouville operator $L = -\frac{1}{V(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. In fact, a stronger assumption is needed as follows.

($\mathcal{A}$2.3) There exists $\omega > 0$ such that $\inf\{|\sqrt{\lambda} - k\omega| : \lambda \in \sigma(L), k \in \mathbb{N}_{\text{odd}}\} > 0$.

($\mathcal{A}$2.4) The point spectrum $\sigma_p(L)$ satisfies $\sum_{\lambda\in\sigma_p(L)} \lambda^{-r} < \infty$ for all $r > \frac{1}{2}$, which means that $\sigma_p(L)$ grows at least quadratically or is finite.

The restriction to $k \in \mathbb{N}_{\text{odd}}$ in ($\mathcal{A}$2.3), or in other words the restriction to $\frac{T}{2}$-antiperiodic functions, avoids the $k = 0$ mode in the Fourier decomposition which would otherwise be incompatible with ($\mathcal{A}$2.3). With these assumptions we can now formulate our main theorem. We use the notation $\mathbb{T} := \mathbb{R}/_{T\mathbb{Z}}$ for the one-dimensional torus of length T.

Theorem 2.1.1. *Let $1 < p < \infty$ and assume ($\mathcal{A}$2.1), ($\mathcal{A}$2.2), ($\mathcal{A}$2.3), ($\mathcal{A}$2.4). Then* (2.1) *has infinitely many nontrivial $T = \frac{2\pi}{\omega}$-periodic breathers if additionally one of the following assumptions are satisfied.*

($\mathcal{A}$2.5a) $\lim_{|x|\to\infty} \Gamma(x) = 0$. *(compact case)*

($\mathcal{A}$2.5b) $V = V_{\text{per}}$ *is X-periodic on $\mathbb{R}$ and $\Gamma = \Gamma_{\text{per}} + \Gamma_{\text{loc}}$ where Γ_{per} is X-periodic and* $\Gamma_{\text{loc}} \geq 0$, $\lim_{|x|\to\infty} \Gamma_{\text{loc}}(x) = 0$. *(asymptotically periodic case)*

The solutions are strong $H^2(\mathbb{R} \times \mathbb{T})$-solutions and satisfy (2.1) *pointwise almost everywhere. The theorem also holds if we replace Γ by $-\Gamma$ in* (2.1).

Remark 2.1.2. Note that in case ($\mathcal{A}$2.5b) we have $V_{\text{per}}^+ = V_{\text{per}}^- = V_{\text{per}}$ and $X^+ = X^- = X$. Case ($\mathcal{A}$2.5b) also includes the purely periodic case where $\Gamma_{\text{loc}} = 0$. Assumption ($\mathcal{A}$2.5a) provides a convenient way to overcome compactness problems for Palais-Smale sequences. It remains open how to generalize ($\mathcal{A}$2.5a), ($\mathcal{A}$2.5b) to cases where $\inf_{\mathbb{R}} \Gamma > 0$ and still V is a perturbation of a periodic potential.

Examples of coefficients V which satisfy ($\mathcal{A}$2.1), ($\mathcal{A}$2.2), ($\mathcal{A}$2.3), ($\mathcal{A}$2.4) are given in Appendix 2.B. The structure of the paper will be given at the end of the introduction after the account of the relevant literature which follows next.

The study of breather solutions to nonlinear wave equations may have its origins in completely integrable cases such as the Sine-Gordon equation $u_{tt} - u_{xx} + \sin u = 0$ on $\mathbb{R} \times \mathbb{R}$ where explicit breather families can be constructed through the inverse scattering method [1]. Although next to Sine-Gordon, a number of completely integrable systems with breather solutions are known, e.g., Korteweg-de Vries, nonlinear Schrödinger, Toda lattice etc., these systems are still rare, and therefore the search for methods to prove existence of breathers beyond inverse scattering continues. The first attempts of studying generalizations of the Sine-Gordon equation turned out to be fruitless since in [14, 29, 51, 81] it was shown that if the $\sin u$ nonlinearity in Sine-Gordon is generalized to an analytic perturbation $f(u)$ with $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$ then only the $f(u) = \sin u$ nonlinearity can support a breather solution. On the other hand, nonlinear lattice equations can support breather solutions [54], e.g., the 2-atomic Fermi-Pasta-Ulam-Tsingou chains [45, 46]. A similar result on the PDE-side of nonlinear wave equations was obtained by Blank, Chirilus-Bruckner, Lescarret and Schneider [15], where breathers for spatially-periodic nonlinear wave equations of the type (2.1) were shown to exist. As we have explained above, the idea was to use the band-structure of the spectrum of the spatially periodic operator $L = -\frac{1}{V(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and design the coefficient function $V(x)$ in such a way that the temporal frequencies $k^2\omega^2$ with $\omega = \frac{2\pi}{T}$ being the breather frequency, fall into the gaps of the spectrum of L, cf. [23] for a systematic approach to such coefficient functions based on inverse spectral theory. By adding a suitable linear potential with a bifurcation parameter to L, breather solutions bifurcating from the edge of the essential spectrum were constructed by the center-manifold theorem and spatial dynamics. The same approach was also used for finding breathers for constant-coefficient nonlinear wave equations on metric graphs [55], where the spatial heterogeneity of the graph generates a banded spectrum of the spatial operator.

A methodically different approach using calculus of variation tools rather than spatial dynamics came up in [43] and was also used in [56]. Variational methods for time-periodic solutions of nonlinear wave equations on spatially bounded intervals have been used before [17, 18, 44]. But its applicability to breathers on unbounded spatial domains like the real line has been substantially obstructed due to essential spectra rather than point spectra of the underlying spatial differential operators. The variational techniques constructed breathers as saddle points of an energy functional on a suitable Hilbert space. It was successful for (2.1) with a periodic potential $V(x)$ [43] or on a periodic metric graph [56], and it required essentially the same spectral situation as in the spatial dynamics approaches. As an additional feature the variational method does not rely on a bifurcation parameter and can therefore generate "large" breathers instead of "small" breathers locally bifurcating from zero as for the spatial dynamics method.

From the results mentioned so far one might get the impression that (except for the completely integrable cases) spatial heterogeneity is a necessary prerequisite for the existence of breathers. It is surprising that in space dimensions 2 or higher this is not the case, cf. [60, 80] for the construction of weakly localized breathers.

Our paper is structured as follows: in Section 2.2 we set up the problem in a way that it can be treated by tools from variational calculus, i.e., we give a proper definition of the energy functional J and its domain $\mathcal{H}$. Then we prove that ground states exists and are strong solutions of (2.1). In Section 2.3 the function calculus for L is introduced and used to prove embedding theorems $\mathcal{H} \hookrightarrow L^q(\mathbb{R}\times\mathbb{T})$. Two appendices complete our paper: In Appendix 2.A we show how L^2 and L^∞-bounds for solutions of $-u'' = \lambda V(x)u$ can be mutually estimated uniformly in λ. Finally, in Section 2.B we give several classes of potential V for which the assumptions ($\mathcal{A}$2.1), ($\mathcal{A}$2.2), ($\mathcal{A}$2.3), ($\mathcal{A}$2.4) our our main result of Theorem 2.1.1 hold.

2.2. Proof of the main result

We begin with the functional analytic framework. The one-dimensional torus $\mathbb{T} = \mathbb{R}/_{T\mathbb{Z}}$ is equipped with the measure $\mathrm{d}t = \frac{1}{T}\mathrm{d}\lambda$ where $\mathrm{d}\lambda$ is the Lebesgue measure on $[0, T]$. We use the notation $L^2_V(\mathbb{R}) := L^2(\mathbb{R}, V(x)\mathrm{d}x)$ and $L^2_V(\mathbb{R}\times\mathbb{T}) := L^2(\mathbb{R}\times\mathbb{T}, V(x)\,\mathrm{d}(x,t))$. Our goal is to obtain breathers as critical points of a suitable functional. Purely formally, this can be achieved via the functional

$$J(u) = \int_{\mathbb{R}\times\mathbb{T}} -V(x)u_t^2 + u_x^2 - \frac{2}{p+1}\Gamma(x)|u|^{p+1}\,\mathrm{d}(x,t)$$

where $u\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a temporally $\frac{T}{2}$-antiperiodic function of time. Note that we have not yet specified the domain of the functional J. This will be our next task. First, we decompose u into its temporal Fourier modes by writing

$$u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e_k(t)$$

where $e_k(t) = \mathrm{e}^{\mathrm{i}\omega kt}$, $k\in\mathbb{Z}$ is an orthonormal basis of $L^2(\mathbb{T})$. Then the functional J takes the form

$$J(u) = J_0(u) - J_1(u)$$

where

$$J_0(u) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} |\hat{u}_k'|^2 - k^2\omega^2 V(x)|\hat{u}_k|^2\,\mathrm{d}x, \quad J_1(u) = \tfrac{2}{p+1}\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u|^{p+1}\,\mathrm{d}(x,t).$$

For the operator L_k as defined in (2.2), we know by ($\mathcal{A}$2.3) that 0 does not belong to its spectrum. Therefore the following constructions are possible. Based on the spectral resolution $L = \int_{\mathbb{R}}\lambda\,\mathrm{d}P_\lambda$ of the selfadjoint operator $L : H^2(\mathbb{R}) \subseteq L^2_V(\mathbb{R}) \to L^2_V(\mathbb{R})$ we can define

$$\sqrt{|L_k|} = \int_{\mathbb{R}} \sqrt{|\lambda - k^2\omega^2|}\,\mathrm{d}P_\lambda,$$

which has domain $H^1(\mathbb{R})$. We equip $H^1(\mathbb{R})$ with the norm $\|v\|_{\mathcal{H}_k} := \|\sqrt{|L_k|}v\|_{L^2_V}$ and thus obtain the Hilbert space $\mathcal{H}_k$. The two orthogonal projections $P_k^\pm\colon \mathcal{H}_k \to \mathcal{H}_k$ given by

$$v^+ := P_k^+[v] := \int_{k^2\omega^2}^{\infty} 1\,\mathrm{d}P_\lambda[v], \qquad v^- := P_k^-[v] := \int_{-\infty}^{k^2\omega^2} 1\,\mathrm{d}P_\lambda[v]$$

yield an orthogonal decomposition $\mathcal{H}_k = \mathcal{H}_k^+ \oplus \mathcal{H}_k^-$, $v = v^+ + v^-$ such that

$$\int_{\mathbb{R}} |v'|^2 - k^2\omega^2 V(x)|v|^2\,\mathrm{d}x = \|v^+\|_{\mathcal{H}_k}^2 - \|v^-\|_{\mathcal{H}_k}^2, \quad \|v\|_{\mathcal{H}_k}^2 = \|v^+\|_{\mathcal{H}_k}^2 + \|v^-\|_{\mathcal{H}_k}^2.$$

For later purposes, let us also introduce the bilinear form associated to L_k by

$$b_k(v,w) = \int_{\mathbb{R}} v'w' - k^2\omega^2 V(x)vw\,\mathrm{d}x, \quad v,w \in \mathcal{H}_k.$$

As we shall see in Proposition 2.3.1, the proper domain for J is the Hilbert space

$$\mathcal{H} := \{u \in L^2(\mathbb{R}\times\mathbb{T}) : u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e_k(t),\ \hat{u}_k = \overline{\hat{u}_{-k}}\ \forall k \in \mathbb{Z}_{\text{odd}},\ \|u\|_{\mathcal{H}} < \infty\}$$

equipped with the norm $\|\cdot\|_{\mathcal{H}}$ according to

$$\|u\|_{\mathcal{H}}^2 := \sum_{k\in\mathbb{Z}_{\text{odd}}} \langle\sqrt{|L_k|}\hat{u}_k, \sqrt{|L_k|}\hat{u}_k\rangle_{L^2_V} = 2\sum_{k\in\mathbb{N}_{\text{odd}}} \|\hat{u}_k\|_{\mathcal{H}_k}^2.$$

By ($\mathcal{A}$2.3) we have that $\mathcal{H}$ continuously embeds into $L^2(\mathbb{R}\times\mathbb{T})$. Moreover, $\mathcal{H} = \oplus_{k\in\mathbb{N}_{\text{odd}}}\mathcal{H}_k$ and the orthogonal decomposition of $\mathcal{H}_k = \mathcal{H}_k^+ \oplus \mathcal{H}_k^-$ now readily extends to $\mathcal{H}$ so that $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, $\mathcal{H}^\pm = \oplus_{k\in\mathbb{N}_{\text{odd}}}\mathcal{H}_k^\pm$ and

$$J_0(u) = \|u^+\|_{\mathcal{H}}^2 - \|u^-\|_{\mathcal{H}}^2, \quad \|u\|_{\mathcal{H}}^2 = \|u^+\|_{\mathcal{H}}^2 + \|u^-\|_{\mathcal{H}}^2$$

where $u = u^+ + u^-$ and $u^\pm = P^\pm[u] \in \mathcal{H}^\pm$ according to the two orthogonal projectors $P^\pm : \mathcal{H} \to \mathcal{H}^\pm$.

Remark 2.2.1. It is clear that functions $u = \sum_{|k|\le K} \hat{u}_k e_k(t)$ with finitely many non-zero Fourier-modes and where $\hat{u}_k \in H^1(\mathbb{R})$ has compact support are dense in $\mathcal{H}$.

Since J is neither bounded from above nor from below, a critical point of J is necessarily a saddle point. A general tool for obtaining existence of saddle points is given by the generalized Nehari-Pankov manifold, cf. [74, 94]. It consists of minimizing J on the set

$$\mathcal{M} = \{u \in \mathcal{H}\setminus\mathcal{H}^- : J'(u)[w] = 0 \text{ for all } w \in [u] + \mathcal{H}^-\}.$$

The following properties of J and $\mathcal{M}$ are important prerequisites for applying abstract results from [94].

Lemma 2.2.2. *The following statements hold true:*

(i) J_1 *is weakly lower semicontinuous,*

$$J_1(0) = 0 \quad \textit{and} \quad \frac{1}{2}J_1'(u)[u] > J_1(u) > 0 \textit{ for } u \neq 0.$$

(ii) $\lim_{u\to 0} \frac{J_1'(u)}{\|u\|_{\mathcal{H}}} = 0$ *and* $\lim_{u\to 0} \frac{J_1(u)}{\|u\|_{\mathcal{H}}^2} = 0$.

(iii) *For a weakly compact set* $U \subseteq \mathcal{H}\setminus\{0\}$ *we have* $\lim_{s\to\infty} \frac{J_1(su)}{s^2} = \infty$ *uniformly w.r.t.* $u \in U$.

(iv) *In case* ($\mathcal{A}$2.5a) *of Theorem 2.1.1 the map* $u \mapsto J_1'(u)$ *is completely continuous from* $\mathcal{H}$ *to* $\mathcal{H}'$.

(v) *For each* $w \in \mathcal{H}\setminus\mathcal{H}^-$ *let* $\mathcal{H}(w) = \mathbb{R}_{\geq 0}w + \mathcal{H}^-$. *Then there exists a unique nontrivial critical point* $m(w)$ *of* $J|_{\mathcal{H}(w)}$. *Moreover,* $m(w) \in \mathcal{M}$ *is the unique global maximizer of* $J|_{\mathcal{H}(w)}$ *as well as* $J(m(w)) > 0$.

(vi) *There exists* $\delta > 0$ *such that* $\|m(w)^+\|_{\mathcal{H}} \geq \delta$ *for all* $w \in \mathcal{H}\setminus\mathcal{H}^-$.

Proof. The proof of (iv) is standard. The proof of the remaining claims is essentially contained in [56]. For (i)–(iii) see the proof of [56, Lemma 5.1] where for Fatou's lemma the positivity of Γ is used, cf. ($\mathcal{A}$2.1). For (v)–(vi) we refer to the proof of [56, Lemma 5.2]. □

Another important property of $\mathcal{M}$ is that it does not generate Lagrange-multipliers for critical points of J when restricted to $\mathcal{M}$. This is the main meaning of the following result whose proof can be found in [8].

Lemma 2.2.3. *The set* $\mathcal{M}$ *is a* C^1*-manifold, Moreover, if* $\mathcal{M}_0$ *is a bounded subset of* $\mathcal{M}$ *then there exists a constant* $C > 0$ *with the following property: if* $u \in \mathcal{M}_0$, $\nabla J[u] = \tau + \sigma$ *with* $\tau \in T_u\mathcal{M}$ *and* $\sigma \perp_{\mathcal{H}} \tau$ *then*

$$\|\nabla J(u)\|_{\mathcal{H}} \leq C\|\tau\|_{\mathcal{H}}.$$

The next result provides the important boundedness of Palais-Smale sequences. It differs methodically in its proof from [56]. In the following proofs C denotes a constant that is independent of u but may change from line to line.

Lemma 2.2.4. *There exist constants* $C, \varepsilon > 0$ *such that*

$$\varepsilon \leq \|u\|_{\mathcal{H}} \leq CJ(u)^{\frac{p}{p+1}} \text{ for all } u \in \mathcal{M}. \tag{2.3}$$

Moreover, any Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ *of* $J : \mathcal{H} \to \mathbb{R}$ *is bounded.*

Proof. In the following we use the embedding of $\mathcal{H}$ into $L^{p+1}(\mathbb{R}\times\mathbb{T})$, cf. Proposition 2.3.1. For $u \in \mathcal{M}$ we find

$$\begin{aligned}
\|u^+\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}\times\mathbb{T}} u_x u_x^+ - V(x)u_t u_t^+ \\
&= \underbrace{J'(u)[u^+]}_{=J'(u)[u-u^-]=0} + \int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u|^{p-1}uu^+ \,\mathrm{d}(x,t) \\
&\leq \|\Gamma\|_\infty^{\frac{1}{p+1}} \left(\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u|^{p+1} \,\mathrm{d}(x,t)\right)^{\frac{p}{p+1}} \|u^+\|_{L^{p+1}} \\
&\leq C\|\Gamma\|_\infty^{\frac{1}{p+1}} \left(\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u|^{p+1} \,\mathrm{d}(x,t)\right)^{\frac{p}{p+1}} \|u^+\|_{\mathcal{H}}.
\end{aligned}$$

Together with a similar estimate for u^- and $J(u) = \frac{p-1}{p+1}\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u|^{p+1}\,\mathrm{d}(x,t)$ this implies the second inequality of (2.3). Since $J(u) \leq C\|u\|_{\mathcal{H}}^{p+1}$ for $u \in \mathcal{M}$ by Proposition 2.3.1 and $u \neq 0$ we also get the first inequality of (2.3).

It remains to show the boundedness of a Palais-Smale sequence. Let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence for J in $\mathcal{H}$. From the identities

$$-J'(u_n)[u_n^-] = \|u_n^-\|_{\mathcal{H}}^2 + \int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p-1}u_n u_n^- \,\mathrm{d}(x,t),$$
$$J'(u_n)[u_n^+] = \|u_n^+\|_{\mathcal{H}}^2 - \int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p-1}u_n u_n^+ \,\mathrm{d}(x,t)$$

we find

$$\|u_n^\pm\|_{\mathcal{H}}^2 \le \left(C\|\Gamma\|_\infty^{\frac{1}{p+1}}\left(\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p+1}\,\mathrm{d}(x,t)\right)^{\frac{p}{p+1}} + o(1)\right)\|u_n^\pm\|_{\mathcal{H}}$$

so that

$$\|u_n\|_{\mathcal{H}} \le C\left(\left(\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p+1}\,\mathrm{d}(x,t)\right)^{\frac{p}{p+1}} + 1\right). \tag{2.4}$$

Using the identity $J(u_n) - \frac{1}{2}J'(u_n)[u_n] = \frac{p-1}{p+1}\int_{\mathbb{R}\times\mathbb{T}}\Gamma(x)|u_n|^{p+1}\,\mathrm{d}(x,t)$ and (2.4) we obtain

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p+1}\,\mathrm{d}(x,t) &\le C(J(u_n) + \|u_n\|_{\mathcal{H}}) \\ &\le C\left(J(u_n) + \left(\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p+1}\,\mathrm{d}(x,t)\right)^{\frac{p}{p+1}} + 1\right).\end{aligned}$$

This implies

$$\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p+1}\,\mathrm{d}(x,t) \le C(J(u_n)+1)$$

and once again with the help of (2.4)

$$\|u_n\|_{\mathcal{H}}^{\frac{p+1}{p}} \le C(J(u_n)+1)$$

which proves boundedness of Palais-Smale sequences as claimed. □

The next lemma is a variant of P. L. Lions' concentration compactness result, cf. [99]. Since the norm in $\mathcal{H}$ is nonlocal in space, the result cannot be derived in the standard way by combining Hölder- and Sobolev-inequality. Instead, one uses the embedding from $\mathcal{H}$ into another space H defined by

$$H := \{u \in L^2(\mathbb{R}\times\mathbb{T}) : u(x,t) = \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \hat{u}_k(x)e_k(t),\ \hat{u}_k = \overline{\hat{u}_{-k}}\ \forall k\in\mathbb{Z}_{\mathrm{odd}},\ \|u\|_H < \infty\}$$

equipped with the norm $\|\cdot\|_H$ according to

$$\|u\|_H^2 := \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \frac{1}{|k|}\|\hat{u}_k'\|_{L^2}^2 + |k|\|\hat{u}_k\|_{L_V^2}^2.$$

The norm in H is local in space and is suited for standard concentration-compactness arguments. In Proposition 2.2.6 we show boundedness of the embeddings $\mathcal{H} \hookrightarrow H \hookrightarrow L^p(\mathbb{R}\times\mathbb{T})$ for $p\in[2,4)$. Next we state the concentration-compactness lemma. A proof using the intermediate space H and its embedding properties, interpolation arguments, and $\mathbb{R} = \bigcup_{x\in 2r\mathbb{Z}}[x-r,x+r]$ for $r>0$ can be found in [56].

Lemma 2.2.5. *Let $q \in [2,\infty)$ and $r > 0$ be given and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ such that*

$$\sup_{x\in 2r\mathbb{Z}} \int_{[x-r,x+r]\times\mathbb{T}} |u_n|^q \,\mathrm{d}(x,t) \to 0 \text{ as } n\to\infty.$$

Then $u_n \to 0$ in $L^{\tilde{q}}(\mathbb{R}\times\mathbb{T})$ as $n\to\infty$ for all $\tilde{q}\in(2,\infty)$.

Proposition 2.2.6. *Suppose that ($\mathcal{A}2.1$), ($\mathcal{A}2.2$), ($\mathcal{A}2.3$) are fulfilled. Then the following holds true:*

(i) *The embedding $\iota^\star : \mathcal{H} \to H$ is continuous.*
(ii) *For every $q \in [2,4)$ the embedding $\tilde{\iota} : H \to L^q(\mathbb{R}\times\mathbb{T})$ is continuous and locally compact, i.e., $\tilde{\iota} : H \to L^q(A\times\mathbb{T})$ is compact for every compact set $A \subseteq \mathbb{R}$.*

Proof. (i): By assumption ($\mathcal{A}2.3$) we have $\delta := \inf\{|\sqrt{\lambda} - k\omega| : \lambda\in\sigma(L), k\in\mathbb{N}_{\text{odd}}\} > 0$ and hence

$$|\lambda - k^2\omega^2| = |\sqrt{\lambda} - |k|\omega||\sqrt{\lambda} + |k|\omega| \geq \delta|k|\omega$$

for all $\lambda\in\sigma(L)$ and all $k\in\mathbb{Z}_{\text{odd}}$. Thus $|L_k| \geq \delta\omega|k| \operatorname{Id}$ on $H^1(\mathbb{R})$ and for $v\in H^1(\mathbb{R})$ we therefore have

$$\|v\|_{\mathcal{H}_k}^2 = \langle \sqrt{|L_k|}v, \sqrt{|L_k|}v\rangle_{L_V^2} \geq \delta\omega|k|\|v\|_{L_V^2}^2. \tag{2.5}$$

For $C > 0$ small enough and for $v\in\mathcal{H}_k^+$, $k\in\mathbb{Z}_{\text{odd}}$, we therefore find

$$\int_{\mathbb{R}} |v'|^2 - k^2\omega^2|v|^2 V(x)\,\mathrm{d}x = \|v\|_{\mathcal{H}_k}^2 \geq \frac{\omega^2k^2C}{|k|-C}\|v\|_{L_V^2}^2$$

which implies

$$\left(1-\frac{C}{|k|}\right)\int_R |v'|^2\,\mathrm{d}x \geq k^2\omega^2\|v\|_{L_V^2}^2$$

and hence by rearranging terms

$$\|v\|_{\mathcal{H}_k}^2 \geq \frac{C}{|k|}\|v'\|_{L^2}^2. \tag{2.6}$$

For $v\in\mathcal{H}_k^-$ we have $\int_{\mathbb{R}} |v'|^2 - k^2\omega^2|v|^2V(x)\,\mathrm{d}x \leq 0$ so that $\|v'\|_{L^2}^2 \leq k^2\omega^2\|v\|_{L_V^2}^2$ and hence in this case (2.6) follows from (2.5). Having established (2.6) separately for $v^+\in\mathcal{H}_k^+$ and $v^-\in\mathcal{H}_k^-$, we obtain the estimate (2.6) for $v\in\mathcal{H}_k$:

$$\frac{C}{2|k|}\|v'\|_{L^2}^2 \leq \frac{C}{|k|}(\|(v^+)'\|_{L^2}^2 + \|(v^-)'\|_{L^2}^2) \leq \|v^+\|_{\mathcal{H}_k}^2 + \|v^-\|_{\mathcal{H}_k}^2 = \|v\|_{\mathcal{H}_k}^2.$$

Then (2.5) and (2.6) for all $k\in\mathbb{Z}_{\text{odd}}$ imply the continuity of the embedding $\iota^\star : \mathcal{H}\to H$.

(ii): First we see that for $u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e_k(t)$ we get

$$\|u\|_{L^q(\mathbb{R}\times\mathbb{T})} \leq C_q\left(\sum_{k\in\mathbb{Z}_{\text{odd}}} \|\hat{u}_k\|_{L^q}^{q'}\right)^{1/q'}$$

for all $q\in[2,\infty]$ by standard Riesz-Thorin interpolation. With the Gagliardo-Nirenberg inequality for $2<q<4$ and $\theta = \frac{1}{2}-\frac{1}{q}$

$$\|v\|_{L^q} \leq C_{GN}\|v'\|_{L^2}^\theta\|v\|_{L^2}^{1-\theta}, \qquad v\in H^1(\mathbb{R})$$

and a triple Hölder inequality with exponents $\frac{4(q-1)}{q-2}$, $\frac{4(q-1)}{q+2}$, and $\frac{2(q-1)}{q-2}$ we obtain

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}\times\mathbb{T})}^{q'} &\leq C \sum_{k\in\mathbb{Z}_{\text{odd}}} \|u_k\|_{L^q}^{q'} \leq C \sum_{k\in\mathbb{Z}_{\text{odd}}} \|u_k'\|_{L^2}^{q'(\frac{1}{2}-\frac{1}{q})} \|u_k\|_{L^2}^{q'(\frac{1}{2}+\frac{1}{q})} \\
&= C \sum_{k\in\mathbb{Z}_{\text{odd}}} (|k|^{-\frac{1}{2}}\|u_k'\|_{L^2})^{q'(\frac{1}{2}-\frac{1}{q})} (|k|^{\frac{1}{2}}\|u_k\|_{L^2})^{q'(\frac{1}{2}+\frac{1}{q})} |k|^{\frac{-1}{q-1}} \\
&\leq C \left(\sum_{k\in\mathbb{Z}_{\text{odd}}} |k|^{-1}\|u_k'\|_{L^2}^2 \right)^{\frac{q-2}{4(q-1)}} \left(\sum_{k\in\mathbb{Z}_{\text{odd}}} |k|\|u_k\|_{L^2}^2 \right)^{\frac{q+2}{4(q-1)}} \left(\sum_{k\in\mathbb{Z}_{\text{odd}}} |k|^{\frac{-2}{q-2}} \right)^{\frac{q-2}{2(q-1)}} \\
&\leq C\|u\|_H^{q'}.
\end{aligned}$$

Note that $\sum_{k\in\mathbb{Z}_{\text{odd}}} |k|^{\frac{-2}{q-2}}$ converges due to $2 < q < 4$. This established the continuity of the embedding $\tilde{\iota} : H \to L^q(\mathbb{R}\times\mathbb{T})$. For $q = 2$ the embedding is clear. The local compactness can be seen as follows. First, we modify $\tilde{\iota}$ by setting $\tilde{\iota}_K := \tilde{\iota} \circ T_K$ for $K \in \mathbb{N}$, where T_K truncates the Fourier series to modes $k \in \mathbb{Z}_{\text{odd}}$ with $|k| \leq K$. Then $\tilde{\iota}_K \to \tilde{\iota}$ in the operator norm as $K \to \infty$. Since $\tilde{\iota}_N$ maps compactly into $L^q(A\times\mathbb{T})$ for every compact set $A \subseteq \mathbb{R}$, the same holds for $\tilde{\iota}$. □

We are now ready to give the proof of Theorem 2.1.1. We employ the abstract result Theorem 35 from [94], which provides a Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ for the functional J that belongs to $\mathcal{M}$ and is minimizing for $J|_{\mathcal{M}}$. This is possible since by Lemma 2.2.2 the conditions (B1), (B2) and (i) and (ii) of Theorem 35 in [94] are fulfilled. At this point, we do not claim that also (iii) of Theorem 35 in [94], which leads to the Palais-Smale condition, holds. But as we shall see this is indeed the case in case ($\mathcal{A}$2.5a) of Theorem 2.1.1 but not necessarily in case ($\mathcal{A}$2.5b). As a consequence of Theorem 35 in [94] we obtain a minimizing Palais-Smale $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ with $J'(u_n) \to 0$ as $n \to \infty$. By Lemma 2.2.4 we know that $(u_n)_{n\in\mathbb{N}}$ is bounded, and hence there is $u \in \mathcal{H}$ and a subsequence (again denoted by $(u_n)_{n\in\mathbb{N}}$) such that $u_n \rightharpoonup u$ as $n \to \infty$. Using that $u_n \to u$ in $L^{p+1}_{\text{loc}}(\mathbb{R}\times\mathbb{T})$ and that compactly supported functions are dense in $\mathcal{H}$ (cf. Remark 2.2.1) we deduce that $J'(u) = 0$.

It remains to shows that the limit function u belongs to $\mathcal{M}$ (which implies that it is non-zero) and that it is a minimizer of J on $\mathcal{M}$. For this purpose we give different proofs depending on which of the assumptions ($\mathcal{A}$2.5a) or ($\mathcal{A}$2.5b) of Theorem 2.1.1 holds.

Case ($\mathcal{A}$2.5a): By Lemma 2.2.2 (iv) also property (iii) of Theorem 35 in [94] holds and hence it provides a (necessarily nontrivial) ground state of $J|_{\mathcal{M}}$ and infinitely many bound states.

Case ($\mathcal{A}$2.5b) – purely periodic case: Let us first assume $\Gamma_{\text{loc}} \equiv 0$ so that we are in a purely periodic setting. In this case property (iii) of Theorem 35 in [94] does not hold. However, due to the periodic structure, concentration-compactness arguments can be used, and the proof is essentially the same as in [56]. Therefore, we only sketch the argument. First, since $u_n \in \mathcal{M}$ we have $J(u_n) = \frac{p-1}{p+1}\int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|u_n|^{p+1}\,\mathrm{d}(x,t)$, and using Lemma 2.2.4 it follows that 0 is not a limit point of $(u_n)_{n\in\mathbb{N}}$ in $L^{p+1}(\mathbb{R}\times\mathbb{T})$. Then, by Lemma 2.2.5 with $r = \frac{X}{2}$ there exist $x_n \in X\mathbb{Z}$ and $\delta > 0$ such that for $v_n(x,t) := u(x+x_n,t)$ we have

$$\int_{[-\frac{X}{2},\frac{X}{2}]\times\mathbb{T}} |v_n|^2\,\mathrm{d}(x,t) \geq \delta > 0.$$

The sequence $(v_n)_{n\in\mathbb{N}}$ is still a Palais-Smale sequence for J, and hence (by Lemma 2.2.4 and Proposition 2.3.1) has a weakly convergent subsequence $v_n \rightharpoonup v \neq 0$ as $n \to \infty$ with

$v \in \mathcal{H}$. Following our observations at the beginning of the proof, we find that $J'(v) = 0$ and due to $0 = J'(v)[v] = \|v^+\|_{\mathcal{H}}^2 - \|v^-\|_{\mathcal{H}}^2 - (p+1)J_1(v)$ we see that $v^+ \neq 0$. Hence, $v \in \mathcal{M}$ and by a Fatou-type argument we see that v is indeed a minimizer of J on $\mathcal{M}$.

Case ($\mathcal{A}$2.5b) – perturbed periodic case: Now we assume $\Gamma_{\mathrm{loc}} \not\equiv 0$. In this case let us consider two functionals: next to $J = J_0 - J_1$ we also consider $J^{\mathrm{per}} = J_0 - J_1^{\mathrm{per}}$ with

$$J_1^{\mathrm{per}}(u) = \tfrac{2}{p+1} \int_{\mathbb{R}\times\mathbb{T}} \Gamma_{\mathrm{per}}(x)|u|^{p+1} \,\mathrm{d}(x,t)$$

The only difference between the two functionals is that $J_1^{\mathrm{per}} \leq J_1$ due to the assumption that $\Gamma_{\mathrm{per}} \leq \Gamma = \Gamma_{\mathrm{per}} + \Gamma_{\mathrm{loc}}$. While the Hilbert space $\mathcal{H}$ and the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ stay the same for both functionals, the underlying Nehari-Pankov manifolds are different, i.e., next to $\mathcal{M}$ we also have

$$\mathcal{M}^{\mathrm{per}} = \{u \in \mathcal{H} \setminus \mathcal{H}^- : J^{\mathrm{per}\prime}(u)[w] = 0 \text{ for all } w \in [u] + \mathcal{H}^-\}$$

together with the two ground-state levels

$$c^{\mathrm{per}} = \inf\{J^{\mathrm{per}}(u) : u \in \mathcal{M}^{\mathrm{per}}\}, \quad c = \inf\{J(u) : u \in \mathcal{M}\}.$$

We already know from the previous case that c^{per} is attained for a minimizer u^{per} with $J^{\mathrm{per}\prime}(u^{\mathrm{per}}) = 0$. Let us show that $c \leq c^{\mathrm{per}}$. Associated to $\mathcal{M}$ and $\mathcal{M}^{\mathrm{per}}$ we have the two maps $m\colon \mathcal{H}\setminus\mathcal{H}^- \to \mathcal{M}$ and $m^{\mathrm{per}}\colon \mathcal{H}\setminus\mathcal{H}^- \to \mathcal{M}^{\mathrm{per}}$, cf. Lemma 2.2.2. Since $m^{\mathrm{per}}(u^{\mathrm{per}}) = u^{\mathrm{per}} \notin \mathcal{H}^-$ and $u := m(u^{\mathrm{per}}) \in [0,\infty)u^{\mathrm{per}} + \mathcal{H}^-$, Lemma 2.2.2 (v) shows $J^{\mathrm{per}}(u) \leq J^{\mathrm{per}}(u^{\mathrm{per}}) = c^{\mathrm{per}}$. On the other hand, $\Gamma_{\mathrm{loc}} \geq 0$ implies $J(u) \leq J^{\mathrm{per}}(u)$. Combining these, we have $c \leq J(u) \leq J^{\mathrm{per}}(u) \leq J^{\mathrm{per}}(u^{\mathrm{per}}) = c^{\mathrm{per}}$ as claimed.

Let us first consider the case where $c = c^{\mathrm{per}}$. For the above inequalities to be equalities, $\Gamma_{\mathrm{loc}} u = 0$ and $u = u^{\mathrm{per}}$ must hold. But then $u = u^{\mathrm{per}}$ is both a minimizer and a solution since $J'(u) = J^{\mathrm{per}\prime}(u^{\mathrm{per}}) = 0$. So let us assume in the following that $c < c^{\mathrm{per}}$ holds. Based on this inequality let us now verify that c is attained. As in the purely periodic case we start with the bounded Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ for J on $\mathcal{M}$ where 0 is not a limit point of $(u_n)_{n\in\mathbb{N}}$ in $L^{p+1}(\mathbb{R}\times\mathbb{T})$. By Lemma 2.2.5 this time for the exponent $q = p+1$ there exists a sequence $x_n \in X\mathbb{Z}$ and $\delta > 0$ such that

$$\int_{[x_n - \frac{X}{2}, x_n + \frac{X}{2}]\times\mathbb{T}} |u_n|^{p+1} \,\mathrm{d}(x,t) \geq \delta$$

for all $n \in \mathbb{N}$. We claim that due to $c < c^{\mathrm{per}}$ we have that 0 is not a limit point of $(u_n)_{n\in\mathbb{N}}$ in $L^{p+1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T})$, which is enough to conclude that a weakly convergent subsequence $u_n \rightharpoonup u$ provides a minimizer u of J on $\mathcal{M}$ (the multiplicity result is then the same as in the purely periodic case). So let us a assume for contradiction that a subsequence of $(u_n)_{n\in\mathbb{N}}$ (again denoted by $(u_n)_{\in\mathbb{N}}$) converges to 0 in $L^{p+1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T})$. Then necessarily $|x_n| \to \infty$ as $n \to \infty$. If we set $v_n(x,t) := u_n(x - x_n, t)$ then (up to a subsequence) $v_n \rightharpoonup v$ in $\mathcal{H}$ and $v_n \to v$ in $L^{p+1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T})$ and pointwise a.e. as $n \to \infty$ for some $v \in \mathcal{H} \setminus \{0\}$. Recalling Remark 2.2.1 let us take a function $\varphi \in \mathcal{M}$ with compact support and set $\varphi_n(x,t) := \varphi(x + x_n, t)$. Then

$$\begin{aligned} o(1) = J'(u_n)[\varphi_n] &= J_0'(u_n)[\varphi_n] - J_1^{\mathrm{per}\prime}(u_n)[\varphi_n] - 2\int_{\mathbb{R}\times\mathbb{T}} \Gamma_{\mathrm{loc}}(x)|u_n|^{p-1}u_n\varphi_n \,\mathrm{d}(x,t) \\ &= J_0'(v_n)[\varphi] - J_1^{\mathrm{per}\prime}(v_n)[\varphi] - 2\int_{\operatorname{supp}\varphi} \Gamma_{\mathrm{loc}}(x - x_n)|v_n|^{p-1}v_n\varphi \,\mathrm{d}(x,t) \\ &\to J^{\mathrm{per}\prime}(v)[\varphi] \end{aligned}$$

where we have used that $v_n^\pm \rightharpoonup v^\pm$, $v_n \to v$ in $L^{p+1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T})$, φ has compact support and $\Gamma_{\mathrm{loc}}(x) \to 0$ as $|x| \to \infty$. Hence v is a nontrivial critical point of J^{per} and belongs to $\mathcal{M}^{\mathrm{per}}$. Thus

$$\begin{aligned}
c^{\mathrm{per}} \le J^{\mathrm{per}}(v) &= J^{\mathrm{per}}(v) - \frac{1}{2} J^{\mathrm{per}\prime}(v)[v] \\
&= \frac{p+1}{p-1} \int_{\mathbb{R}\times\mathbb{T}} \Gamma^{\mathrm{per}}(x)|v|^{p+1}\,\mathrm{d}(x,t) \\
&\le \frac{p+1}{p-1} \liminf_{n\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{T}} \Gamma^{\mathrm{per}}(x)|v_n|^{p+1}\,\mathrm{d}(x,t) \\
&\le \frac{p+1}{p-1} \liminf_{n\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{T}} \Gamma(x)|v_n|^{p+1}\,\mathrm{d}(x,t) \\
&= \liminf_{n\in\mathbb{N}} \Big(J(u_n) - \frac{1}{2} J'(u_n)[u_n] \Big) \\
&= c
\end{aligned}$$

which contradicts the already established inequality $c < c^{\mathrm{per}}$. This contradiction completes the proof of the perturbed periodic case.

The multiplicity result can finally be obtained as follows: instead of restricting to $T/2$-antiperiodic functions, we can consider the set of $T/2m$-antiperiodic functions for $m \in \mathbb{N}_{\mathrm{odd}}$. Note that u being $T/2m$-antiperiodic implies that also $|u|^{p-1}u$ is $T/2m$-antiperiodic. Hence, $T/2m$-antiperiodicity is compatible with the underlying equation (2.1). Assuming $T/2m$-antiperiodicity amounts to allowing temporal Fourier-modes in the set $m\mathbb{Z}_{\mathrm{odd}}$ instead of $\mathbb{Z}_{\mathrm{odd}}$. We can now find a ground state u_m of J in the set of $T/2m$-antiperiodic functions for every $m \in \mathbb{N}_{\mathrm{odd}}$. The set $\{u_m : m \in \mathbb{N}_{\mathrm{odd}}\}$ cannot be finite since the (in absolute value) lowest index of all non-zero Fourier modes of u_m is at least m.

To finish the proof of Theorem 2.1.1 we show that our obtained critical point $u \in \mathcal{H}$ of J lies in $H^2(\mathbb{R}\times\mathbb{T})$ and satisfies the equation (2.1) pointwise almost everywhere. The proof is based on the ideas of Chapter 6 in [56]. We present in detail only the parts which differ from [56].

We start with analyzing the linear problem

$$V(x)\partial_t^2 w - \partial_x^2 w = f.$$

Lemma 2.2.7. *Let $f \in \mathcal{H}'$. Then there exists a unique solution $w \in \mathcal{H}$ to*

$$\sum_{k\in\mathbb{Z}_{\mathrm{odd}}} b_k(w_k, \varphi_k) = \langle f, \varphi \rangle_{\mathcal{H}'\times\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H} \tag{2.7}$$

and $\|w\|_{\mathcal{H}} = \|f\|_{\mathcal{H}'}$ holds. Further, if $f \in L^2_V(\mathbb{R}\times\mathbb{T})$ then $w \in H^1(\mathbb{R}\times\mathbb{T})$.

Proof. The first statement follows directly from Lemma 6.1 in [56]. For the proof of the second statement we use the spectral resolution $L = \int_{\mathbb{R}} \lambda \,\mathrm{d}P_\lambda$ to expand

$$f(x,t) = \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \mathrm{e}^{\mathrm{i}\omega k t} \Big(\int_{\mathbb{R}} \mathrm{d}P_\lambda[\hat{f}_k] \Big)(x)$$

and the solution w to (2.7) as

$$w(x,t) = \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \mathrm{e}^{\mathrm{i}\omega k t} \Big(\int_{\mathbb{R}} \frac{1}{\lambda - k^2\omega^2} \,\mathrm{d}P_\lambda[\hat{f}_k] \Big)(x)$$

so that

$$\partial_t w(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \mathrm{e}^{\mathrm{i}\omega k t}\Big(\int_{\mathbb{R}} \frac{\mathrm{i}\omega k}{\lambda - k^2\omega^2}\,\mathrm{d}P_\lambda[\hat f_k]\Big)(x)$$

By our spectral assumption ($\mathcal{A}2.3$) there exists a constant $c>0$ such that

$$\frac{|\omega k|}{|\lambda - k^2\omega^2|} = \frac{|\omega k|}{|\sqrt{\lambda} - |k||\omega|\big|\sqrt{\lambda}+|k|\omega|\big|} \le c\frac{|\omega k|}{\big|\sqrt{\lambda}+|k|\omega|\big|} \le c$$

for all $k\in\mathbb{Z}_{\text{odd}}$ and $\lambda\in\sigma(L)$. We conclude

$$\begin{aligned}\|\partial_t w\|^2_{L^2_V(\mathbb{R}\times\mathbb{T})} &= \sum_{k\in\mathbb{Z}_{\text{odd}}}\int_{\mathbb{R}} \frac{\omega^2k^2}{(\lambda-k^2\omega^2)^2}\,\mathrm{d}\Big\|P_\lambda \hat f_k\Big\|^2_{L^2_V(\mathbb{R})}\\ &\le \sum_{k\in\mathbb{Z}_{\text{odd}}}\int_{\mathbb{R}} c^2\,\mathrm{d}\Big\|P_\lambda \hat f_k\Big\|^2_{L^2_V(\mathbb{R})} = c^2\|f\|^2_{L^2_V(\mathbb{R}\times\mathbb{T})}.\end{aligned}$$

Next,

$$\sum_{k\in\mathbb{Z}_{\text{odd}}}\int_{\mathbb{R}} |\hat w_k'|^2 - k^2\omega^2 V(x)|\hat w_k|^2\,\mathrm{d}x = \Big\|w^+\Big\|^2_{\mathcal{H}} - \|w^-\|^2_{\mathcal{H}} \le \Big\|w^+\Big\|^2_{\mathcal{H}} + \|w^-\|^2_{\mathcal{H}} = \|w\|^2_{\mathcal{H}}$$

implies

$$\sum_{k\in\mathbb{Z}_{\text{odd}}}\int_{\mathbb{R}} |\hat w_k'|^2\,\mathrm{d}x \le \sum_{k\in\mathbb{Z}_{\text{odd}}}\int_{\mathbb{R}} k^2\omega^2V(x)|\hat w_k|^2\,\mathrm{d}x + \|w\|^2_{\mathcal{H}} = \|\partial_t w\|^2_{L^2_V(\mathbb{R}\times\mathbb{T})} + \|w\|^2_{\mathcal{H}}$$

and hence

$$\|\partial_x w\|^2_{L^2(\mathbb{R}\times\mathbb{T})} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \|\hat w_k'\|^2_{L^2(\mathbb{R})} < \infty.$$ □

Weak solutions to (2.1) are defined as follows.

Definition 2.2.8. *A time-periodic function $u\in H^1(\mathbb{R}\times\mathbb{T})$ is called a weak solution of* (2.1) *if*

$$\int_{\mathbb{R}\times\mathbb{T}}\Big(u_x\phi_x - V(x)u_t\phi_t - \Gamma(x)|u|^{p-1}u\phi\Big)\,\mathrm{d}(x,t) = 0 \tag{2.8}$$

holds for every time periodic $\phi\in H^1(\mathbb{R}\times\mathbb{T})$.

Next we transfer the result from Lemma 2.2.7 to the nonlinear case and show that critical points of J are weak solutions to (2.1).

Lemma 2.2.9. *Let $u\in\mathcal{H}$ be a critical point of J. Then $u\in H^1(\mathbb{R}\times\mathbb{T})$ is a weak solution to* (2.1) *in the sense of Definition 2.2.8.*

Proof. Since $u\in\mathcal{H}$ is a critical point of J it satisfies (2.7) with $f=\Gamma(x)|u|^{p-1}u$. Moreover, since $u\in L^q_V(\mathbb{R}\times\mathbb{T})$ for all $q\in[2,\infty)$ by Proposition 2.3.1 and $\Gamma\in L^\infty(\mathbb{R})$ by assumption ($\mathcal{A}2.1$) we see that $f\in L^2_V(\mathbb{R}\times\mathbb{T})$. Then Lemma 2.2.7 applies and we obtain $u\in H^1(\mathbb{R}\times\mathbb{T})$. Clearly, (2.8) holds for all $\phi\in\mathcal{H}$, which consists of $\frac{T}{2}$-antiperiodic functions. By density, (2.8) holds for all $\frac{T}{2}$-antiperiodic functions in $H^1(\mathbb{R}\times\mathbb{T})$. Finally, we note that (2.8) trivially holds for all $\frac{T}{2}$-periodic functions in $H^1(\mathbb{R}\times\mathbb{T})$ since then all three integrands are products of a $\frac{T}{2}$-antiperiodic function with a $\frac{T}{2}$-periodic function and thus integrate to 0. This finishes the proof of the lemma. □

Lemma 2.2.10. *Let $u \in \mathcal{H}$ be a weak solution to* (2.1). *Then $u \in H^2(\mathbb{R} \times \mathbb{T})$ satisfies* (2.1) *pointwise almost everywhere.*

Proof. By using difference quotients as is Lemma 6.3 and Lemma 6.4 in [56] we infer $\partial_t^2 u, \partial_x \partial_t u \in L^2(\mathbb{R}\times\mathbb{T})$. As in the proof of Lemma 6.5 in [56] we conclude $\partial_x^2 u \in L^2(\mathbb{R}\times\mathbb{T})$. □

2.3. L^p-EMBEDDINGS

This section is devoted to the proof of the following embedding property for the Hilbert space $\mathcal{H}$.

Proposition 2.3.1. *Let ($\mathcal{A}$2.1), ($\mathcal{A}$2.2), ($\mathcal{A}$2.3), and ($\mathcal{A}$2.4) be fulfilled. Then for every $p \in [2, \infty)$ the embedding $\mathcal{H} \hookrightarrow L^p(\mathbb{R}\times\mathbb{T})$ is bounded and locally compact, i.e., $\mathcal{H} \hookrightarrow L^p(A\times\mathbb{T})$ is compact for every compact set $A \subseteq \mathbb{R}$.*

As the space $\mathcal{H}$ depends strongly on the spectral projections of the operator L, we first develop a functional calculus for the operator L. In Section 2.3.1 we give a general description of the functional calculus, cf. Theorem 2.3.6. In Section 2.3.2 we develop a description of the associated spectral measure. This ends in Theorem 2.3.23, where we calculate the density of the spectral measure with respect to the sum of the Lebesgue measure (for the essential spectrum) and the counting measure (for the point spectrum). Lastly, in Section 2.3.3 we consider L^p-embeddings in a slightly more general setting. Using uniform bounds on (generalized) eigenfunctions and estimates on the spectrum, we show the embedding result Theorem 2.3.27, of which Proposition 2.3.1 is a special case (with $\alpha = \beta = 0$). Throughout this section, we will always assume that the potential V satisfies ($\mathcal{A}$2.1) and ($\mathcal{A}$2.2).

2.3.1. A FUNCTIONAL CALCULUS FOR L

We describe a functional calculus for the spectral problem

$$-u'' = \lambda V(x) u \qquad \text{for} \quad x \in \mathbb{R}. \tag{2.9}$$

Recall that $V \in L^\infty(\mathbb{R};\mathbb{R})$ satisfies $\operatorname{ess\,inf}_{\mathbb{R}} V > 0$. We follow [25] and begin with some preliminary notation.

Notation 2.3.2. *We denote the upper half-plane by $\mathbb{H} := \{z \in \mathbb{C} \colon \operatorname{Im}[z] > 0\}$, the unit circle by $\mathbb{S} := \{z \in \mathbb{C} \colon |z| = 1\}$, the unit circle except two points by $\mathbb{S}_* = \mathbb{S} \setminus \{-1, 1\}$. By a* solution *to* (2.9) *we mean a function $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$ solving* (2.9) *pointwise almost everywhere.*

Definition 2.3.3. *For $\lambda \in \mathbb{C}$, we denote by $\Psi_1(x;\lambda), \Psi_2(x;\lambda)$ the solutions to* (2.9) *with initial data*

$$\Psi_1(0;\lambda) = 1, \quad \Psi_1'(0;\lambda) = 0, \qquad \Psi_2(0;\lambda) = 0, \quad \Psi_2'(0;\lambda) = 1$$

and write $\Psi(x;\lambda) = \begin{pmatrix} \Psi_1(x;\lambda) \\ \Psi_2(x;\lambda) \end{pmatrix}$. We further define the Wronskian

$$W[f,g] := fg' - f'g \qquad \text{so that} \qquad W[f,g]\big|_a^b = \int_a^b (Lf \cdot g - f \cdot Lg)\, V \mathrm{d}x.$$

In particular, $W[f,g]$ is constant if f, g both solve $Lu = \lambda u$.

The following three results about the solutions of (2.9) can be found in [25, Chapter 9] for the case $V = 1$ and with $-\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)$ in place of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, but proofs can easily be adapted to the weighted setting. More precisely, the following three results correspond to Theorem 2.4, Theorem 2.3, and Section 5 in [25]. Related is also the work [11] where the authors treat the weighted setting on the half-line.

Theorem 2.3.4. *L is of "limit-point type" at $+\infty$ and $-\infty$. That is, for any $\lambda \in \mathbb{C}$ at most one linearly independent solution of* (2.9) *lies in $L^2_V([0,\infty))$, and the same holds for $L^2_V((-\infty,0])$.*

Theorem 2.3.5. *There exist holomorphic functions $m_\pm \colon \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ such that for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the solution*

$$u = \Psi_1(\,\cdot\,;\lambda) + m_\pm(\lambda)\Psi_2(\,\cdot\,;\lambda)$$

lies in $L^2(\pm[0,\infty))$. Moreover, we have the identity

$$\operatorname{Im}[m_\pm(\lambda)] = \operatorname{Im}[\lambda] \int_0^{\pm\infty} |u|^2 \, V \mathrm{d}x.$$

Next we present the definition of a functional calculus for the operator L. It uses the Hilbert space $L^2(\mu)$ consisting of functions $\mathbb{R} \to \mathbb{C}^2$ which are square-integrable with respect to a matrix-valued function μ defined on $\mathbb{R}$. See Section B.1 for a definition and basic properties of $L^2(\mu)$.

Theorem 2.3.6. *There exists an increasing function $\mu \colon \mathbb{R} \to \mathbb{R}^{2\times 2}$, called the* spectral measure*, such that the map $T \colon L^2_V(\mathbb{R};\mathbb{C}) \to L^2(\mu)$ given by*

$$T[f](\lambda) := \int_{\mathbb{R}} f(x)\Psi(x;\lambda) \, V \mathrm{d}x$$

for compactly supported $f \in L^2_V(\mathbb{R};\mathbb{C})$ is an isometric isomorphism, with inverse given by

$$T^{-1}[g](x) = \int_{\mathbb{R}} g_i(\lambda)\Psi_j(x;\lambda) \, \mathrm{d}\mu_{ij}(\lambda)$$

for compactly supported $g \in L^2(\mu)$, where we use Einstein summation convention. Moreover, at points $\lambda_1, \lambda_2 \in \mathbb{R}$ where μ is continuous the increment of the spectral measure can be computed as

$$\mu(\lambda_2) - \mu(\lambda_1) = \lim_{\varepsilon \to 0+} \int_{\lambda_1}^{\lambda_2} M(s + \mathrm{i}\varepsilon) \, \mathrm{d}s$$

where

$$M := \frac{1}{\pi} \operatorname{Im}\Big[(m_- - m_+)^{-1} \begin{pmatrix} 1 & \frac{1}{2}(m_- + m_+) \\ \frac{1}{2}(m_- + m_+) & m_- m_+ \end{pmatrix}\Big].$$

Let us show that T as above diagonalizes the differential operator L.

Lemma 2.3.7. *Let $f \in L^2_V(\mathbb{R};\mathbb{C})$. Then $f \in H^2(\mathbb{R};\mathbb{C})$ if and only if $\mathrm{Id}_{\mathbb{R}} \cdot T[f] \in L^2(\mu)$. In this case, $T[Lf] = \mathrm{Id}_{\mathbb{R}} \cdot T[f]$ holds.*

Proof. Part 1: First let $f \in H^2(\mathbb{R};\mathbb{C})$ be compactly supported. Then

$$\begin{aligned} T[Lf](\lambda) &= \int_{\mathbb{R}} Lf(x) \cdot \Psi(x;\lambda)\, V \mathrm{d}x = \int_{\mathbb{R}} f(x) \cdot L\Psi(x;\lambda)\, V \mathrm{d}x \\ &= \lambda \int_{\mathbb{R}} f(x)\Psi(x;\lambda)\, V \mathrm{d}x = \lambda T[f](\lambda). \end{aligned}$$

Thus $\mathrm{Id}_{\mathbb{R}} \cdot T[f] \in L^2(\mu)$ and $T[Lf] = \mathrm{Id}_{\mathbb{R}} \cdot T[f]$. For general f, we argue by approximation.

Part 2: Now assume that $\lambda T[f] \in L^2(\mu)$, and let $g \in H^2(\mathbb{R};\mathbb{C})$. Then we have

$$\begin{aligned} \int_{\mathbb{R}} f \cdot Lg\, V \mathrm{d}x &= \int_{\mathbb{R}} T_i[f](\lambda) \cdot \overline{T_j[Lg](\lambda)}\, \mathrm{d}\mu_{ij}(\lambda) = \int_{\mathbb{R}} T_i[f](\lambda) \cdot \overline{\lambda T_j[g](\lambda)}\, \mathrm{d}\mu_{ij}(\lambda) \\ &= \int_{\mathbb{R}} \lambda T_i[f](\lambda) \cdot \overline{T_j[g](\lambda)}\, \mathrm{d}\mu_{ij}(\lambda) = \int_{\mathbb{R}} T^{-1}[\mathrm{Id}_{\mathbb{R}} \cdot T[f]] \cdot g\, V \mathrm{d}x. \end{aligned}$$

Since L is self-adjoint, we have $f \in H^2(\mathbb{R};\mathbb{C})$ and $Lf = T^{-1}[\mathrm{Id}_{\mathbb{R}} \cdot T[f]]$. □

2.3.2. Description of the spectral measure

Recall that V is periodic on $[R^+,\infty)$ with period X^+ and also on $(-\infty, R^-]$ with period X^-. In this section, we use this property to give a better description of the spectral measure μ in Theorem 2.3.6.

Remark 2.3.8. One can also give a description of the spectral measure in the way we do below when V is asymptotically periodic at $\pm\infty$ provided that the defect from the periodic limiting profiles is integrable. We avoid this because it creates additional difficulties when considering L^p-embeddings.

Definition 2.3.9. *Define the propagation matrix*

$$P(y,x;\lambda) = \begin{pmatrix} \Psi_1(y;\lambda) & \Psi_2(y;\lambda) \\ \Psi_1'(y;\lambda) & \Psi_2'(y;\lambda) \end{pmatrix} \cdot \begin{pmatrix} \Psi_1(x;\lambda) & \Psi_2(x;\lambda) \\ \Psi_1'(x;\lambda) & \Psi_2'(x;\lambda) \end{pmatrix}^{-1}$$

so that any solution u of (2.9) *satisfies*

$$\begin{pmatrix} u(y) \\ u'(y) \end{pmatrix} = P(y,x;\lambda) \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}$$

for all $x, y \in \mathbb{R}$. Further define the monodromy matrices

$$P^{\pm}(\lambda) = P(0,x;\lambda)P(x \pm X^{\pm}, 0;\lambda),$$

where $x \gtrless R^{\pm}$.

Remark 2.3.10. The monodromy matrix $P^+(\lambda)$ is a propagation matrix along one period expressed in terms of values $\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$ at $x = 0$:

$$P^+(\lambda) = P(0,x;\lambda)P(x + X^+, x;\lambda)P(0,x;\lambda)^{-1}.$$

The same is true for P^-, except we move one period towards $-\infty$.

Lemma 2.3.11. *The propagation and monodromy matrices have determinant* 1*. For each* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*,* $P^{\pm}(\lambda)$ *has an eigenvalue* ρ *with* $|\rho| < 1$*, and the eigenspace is* $\operatorname{span}\{\begin{pmatrix} 1 \\ m_{\pm}(\lambda) \end{pmatrix}\}$*.*

Proof. We only consider "+". First, P^+ is well-defined since for $y, x > R^+$ and omitting λ we have

$$\begin{aligned} P(0,y)P(y+X^+,0) &= P(0,x)P(x,y)P(y+X^+,x+X^+)P(x+X^+,0) \\ &= P(0,x)P(x,y)P(y,x)P(x+X^+,0) = P(0,x)P(x+X^+,0) \end{aligned}$$

where $P(y+X^+, x+X^+) = P(y,x)$ because the differential operator is X^+-periodic on $[R^+, \infty)$.

Next P, and therefore also P^+, has determinant 1 since

$$\frac{\mathrm{d}}{\mathrm{d}x} \det \begin{pmatrix} \Psi_1(x;\lambda) & \Psi_2(x;\lambda) \\ \Psi_1'(x;\lambda) & \Psi_2'(x;\lambda) \end{pmatrix} = \Psi_1(x;\lambda)\Psi_2''(x;\lambda) - \Psi_1''(x;\lambda)\Psi_2(x;\lambda) = 0$$

and at $x = 0$ the determinant is 1 due to the definition of Ψ_1, Ψ_2.

Lastly let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We write $v := (1, m_+(\lambda))^\top$ and $u := \Psi_1(\,\cdot\,;\lambda) + m_+(\lambda)\Psi_2(\,\cdot\,;\lambda) = \Psi(\,\cdot\,;\lambda)\cdot v$, which is square integrable near $+\infty$ by Theorem 2.3.5. By definition of $P^+(\lambda)$ we further have for $n \in \mathbb{N}$ and $x > R^+$

$$\begin{aligned} \begin{pmatrix} u(x+nX^+) \\ u'(x+nX^+) \end{pmatrix} &= P(x+nX^+,0)v = P(x,0)P(0,x)P(x+nX^+,x)P(x,0)v \\ &= P(x,0)P(0,x)P(x+X^+,x)^n P(x,0)v = P(x,0)P^+(\lambda)^n v \end{aligned}$$

and therefore

$$u(x+nX^+) = \Psi(x;\lambda) \cdot P^+(\lambda)^n v.$$

Since the function u is L^2-localized, we have that $P^+(\lambda)^n v \to 0$ as $n \to \infty$. As $P^+(\lambda)$ has determinant 1 and is 2×2, we conclude that v must be an eigenvector of $P^+(\lambda)$ to an eigenvalue ρ with $|\rho| < 1$. □

Lemma 2.3.12. *The monodromy matrices* $P^{\pm}(\lambda)$ *are holomorphic on* $\mathbb{C}$*. As a consequence, the "singular sets"*

$$S_{\pm} := \{\lambda \in \mathbb{C} \colon P^{\pm}(\lambda) \text{ has eigenvalue } -1 \text{ or } 1\}$$

are discrete subsets of $\mathbb{R}$*. Moreover, there exist open neighborhoods* $D_{\pm}$ *of* $\overline{\mathbb{H}} \setminus S_{\pm}$ *and continuous functions* $\rho_{\pm} \colon D_{\pm} \cup S_{\pm} \to \mathbb{C}$ *that are holomorphic on* $D_{\pm}$ *with* $|\rho_{\pm}| < 1$ *on* $\mathbb{H}$ *and such that* $\rho_{\pm}(\lambda)$ *is an eigenvalue of* $P^{\pm}(\lambda)$ *for* $\lambda \in D_{\pm} \cup S_{\pm}$*. If* I *is a connected component of* $\mathbb{R} \setminus S_{\pm}$*, then one of the following alternatives holds:*

1. $|\rho_{\pm}| < 1$ *on* I*.*
2. $|\rho_{\pm}| = 1$ *on* I *and* $\rho_{\pm}'(\lambda)\overline{\rho_{\pm}(\lambda)} \in \mathrm{i}(0,\infty)$*. If* I *is bounded,* $\rho_{\pm}$ *is a diffeomorphism from* I *to one of the sets* $\mathbb{S} \cap \mathbb{H}$ *or* $\mathbb{S} \cap (-\mathbb{H})$*.*

Remark 2.3.13. Comparing $\rho_{\pm}$ of Lemma 2.3.12 with ρ of Lemma 2.3.11 (which depends on $\pm$), we have $\rho_{\pm} = \rho$ on $\mathbb{H}$, whereas $\rho_{\pm}$ can be either ρ or ρ^{-1} on $-\mathbb{H}$, depending on the behaviour of $\rho_{\pm}$ on $\mathbb{R}$.

Remark 2.3.14. As we will see in Proposition 2.3.20 below, intervals of type (a) and (b) correspond to spectral gaps and spectral bands of L, respectively.

Proof of Lemma 2.3.12. We only present the "+" case. By [98, §13, Theorem III] and arguments therein, $\Psi_j(x;\lambda)$ are holomorphic functions of λ for fixed x, and in particular P^+ are holomorphic. From Lemma 2.3.11 we know that $P^+(\lambda)$ does not have eigenvalues of modulus 1 for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. As a consequence of the identity theorem for holomorphic functions, S_+ must be discrete.

For $\lambda \in \mathbb{H}$, let $\rho_+(\lambda)$ be the unique eigenvalue of $P^+(\lambda)$ with $|\rho_+| < 1$. Then ρ_+ is continuous and can be continuously extended to $\rho_+ \colon \overline{\mathbb{H}} \to \mathbb{C}$. As $\rho_+(\lambda)$ is a simple eigenvalue of $P^+(\lambda)$ for all $\lambda \in \overline{\mathbb{H}} \setminus S_+$, the implicit function theorem applied to $\det(P^+(\lambda) - \rho I) = 0$ shows that ρ_+ can be extended to a holomorphic function in a neighborhood D_+ of $\overline{\mathbb{H}} \setminus S_+$.

Denote by C the connected component of $D_+ \setminus \overline{\mathbb{H}}$ touching all of I. Then either $|\rho_+| < 1$ on C or $|\rho_+| > 1$ on C. In the first case, the maximum principle shows that $|\rho_+| < 1$ on I, whereas in the second case we have $|\rho_+| = 1$ by continuity.

Let us assume $|\rho_+| = 1$ on I, take $\lambda_0 \in I$, and write

$$\rho_+(\lambda) = \rho_+(\lambda_0) + (\lambda - \lambda_0)^n f(\lambda)$$

where f is holomorphic with $f(\lambda_0) \neq 0$. For all $h \in \mathbb{H}$ and $\varepsilon > 0$ we have

$$1 > |\rho_+(\lambda_0 + \varepsilon h)|^2 = 1 + 2\varepsilon^n \operatorname{Re}[h^n f(\lambda_0)\overline{\rho_+(\lambda_0)}] + \mathcal{O}(\varepsilon^{n+1}).$$

This shows $\operatorname{Re}[h^n f(\lambda_0)\rho_+(\lambda_0)] \leq 0$ for all $h \in \mathbb{H}$, which is only possible when $n = 1$ and $f(\lambda_0)\rho_+(\lambda_0) \in \mathrm{i}[0,\infty)$, and therefore $f(\lambda_0) = \rho_+'(\lambda_0) \neq 0$.

As ρ_+' vanishes nowhere on I, ρ_+ is a diffeomorphism from I to its image $\rho_+(I) \subseteq \mathbb{S}_*$. If $I = (\lambda_1, \lambda_2)$ is bounded, we have $\lambda_1, \lambda_2 \in S_+$ and therefore $\rho_+(\lambda_1), \rho_+(\lambda_2) \in \{-1, 1\}$. This is only possible when $\rho_+(I)$ is either $\mathbb{S} \cap \mathbb{H}$ or $\mathbb{S} \cap (-\mathbb{H})$. ☐

As the eigenvalues $\rho_\pm$, and in particular the associated eigenfunctions, are central to the functional calculus, we introduce symbols for them and show a useful identity.

Definition 2.3.15. *For $\lambda \in D_\pm$, we denote by $v_\pm(\lambda)$ the nonzero eigenvector of $P^\pm(\lambda)$ to the eigenvalue $\rho_\pm(\lambda)$, and let $\phi_\pm(x;\lambda) \coloneqq v_\pm(\lambda) \cdot \Psi(x;\lambda)$ be the associated eigenfunction.*

Remark 2.3.16. Note that $\phi_\pm$ solves (2.9) and satisfies $\phi_\pm(x \pm X^\pm;\lambda) = \rho_\pm(\lambda)\phi_\pm(x;\lambda)$ for $x \gtrless R^\pm$, For $\lambda \in \mathbb{H}$ we have by $|\rho_\pm| < 1$ that $\phi_\pm \in L^2(\pm[0,\infty))$. Moreover, $v_\pm$ can locally be chosen holomorphic. In order to keep our notation short we omit the λ-dependency for $\phi_\pm$.

Lemma 2.3.17. *Let $\lambda \in \mathbb{R}$ with $\rho_\pm(\lambda) \in \mathbb{S}_*$ and $\phi_\pm(x) = v_\pm(\lambda) \cdot \Psi(x;\lambda)$. Then we have*

$$W[\phi_+, \overline{\phi_+}] = \frac{\rho_+(\lambda)}{\rho_+'(\lambda)} \int_{R^+}^{R^+ + X^+} |\phi_+|^2 \, V \mathrm{d}x \quad \text{or} \quad W[\phi_-, \overline{\phi_-}] = -\frac{\rho_-(\lambda)}{\rho_-'(\lambda)} \int_{R^- - X^-}^{R^-} |\phi_-|^2 \, V \mathrm{d}x.$$

Proof. We only consider the "+" case. Recall that

$$L\phi_+ = \lambda \phi_+, \qquad \phi_+(x + X^+;\lambda) = \rho_+(\lambda)\phi_+(x;\lambda)$$

so in particular $L\partial_\lambda\phi_+ = \phi_+ + \lambda\partial_\lambda\phi_+$ holds. Since $L\overline{\phi_+} = \lambda\overline{\phi_+}$ and $|\rho_+(\lambda)| = 1$, it follows that

$$\begin{aligned}\int_{R^+}^{R^++X^+} |\phi_+|^2\, V\mathrm{d}x &= \int_{R^+}^{R^++X^+} (L\partial_\lambda\phi_+ - \lambda\partial_\lambda\phi_+)\overline{\phi_+}\, V\mathrm{d}x \\ &= \int_{R^+}^{R^++X^+} \left(L\partial_\lambda\phi_+ \cdot \overline{\phi_+} - \partial_\lambda\phi_+ \cdot L\overline{\phi_+}\right) V\mathrm{d}x \\ &= W[\partial_\lambda\phi_+, \overline{\phi_+}]\Big|_{R^+}^{R^++X^+} \\ &= W[\partial_\lambda(\rho_+\phi_+), \overline{\rho_+\phi_+})(R^+] - W[\partial_\lambda\phi_+, \overline{\phi_+}](R^+) \\ &= \partial_\lambda\rho_+\overline{\rho_+}W[\phi_+, \overline{\phi_+}](R^+) + (\left|\rho_+^2\right| - 1)W[\partial_\lambda\phi_+, \overline{\phi_+}](R^+) \\ &= \partial_\lambda\rho_+\overline{\rho_+}W[\phi_+, \overline{\phi_+}]. \qquad \square\end{aligned}$$

We introduce two coefficients and a singular set to describe the interaction between ϕ_+ and ϕ_-.

Definition 2.3.18. *Let $\lambda \in D_\pm$ with $\rho_\pm(\lambda) \in \mathbb{S}_*$. Then, as $\phi_\pm$ is not real-valued, the functions $\phi_\pm, \overline{\phi_\pm}$ span the space of solutions to* (2.9)*. Therefore, there exist constants r, t such that*

$$\phi_\mp = r\phi_\pm + t\overline{\phi_\pm}.$$

We call the number $r = r(\phi_\mp; \phi_\pm)$ the reflection coefficient *and $t = t(\phi_\mp; \phi_\pm)$ the* transmission coefficient*. Also we define S_0 as the singular set*

$$S_0 \coloneqq \{\lambda \in D_+ \cap D_- \colon\ \mathrm{span}\{v_+(\lambda)\} = \mathrm{span}\{v_-(\lambda)\}\}$$

on which ϕ_+, ϕ_- are linearly dependent, and denote the "large singular set" by $S \coloneqq S_+ \cup S_- \cup S_0$.

Remark 2.3.19. Let $\lambda \in \mathbb{H}$. Then, as λ is not an eigenvalue of L, the $L^2(\pm[0,\infty))$-functions $\phi_\pm$ (cf. Remark 2.3.16) must be linearly independent. That is, $\mathbb{H} \cap S_0 = \emptyset$ holds. By the identity theorem, S_0 has no accumulation points in $D_+ \cap D_-$.

Recalling Theorem 2.3.6, we now calculate the density $\lim_{\varepsilon\to 0} M(\lambda + \mathrm{i}\varepsilon)$ of the spectral measure μ for λ away from the singularities S. With this we describe the essential spectrum; the point spectrum is considered separately afterwards.

Proposition 2.3.20. *The limit*

$$\lim_{\varepsilon\to 0+} M(\lambda + \mathrm{i}\varepsilon)$$

converges uniformly for λ in compact subsets of $\mathbb{R} \setminus S$. If we denote the limiting matrix by $M(\lambda)$ then

$$v_i\overline{v_j}M_{ij}(\lambda) = \frac{1}{2\pi}\left(\frac{\mathbb{1}_{\{|\rho_+(\lambda)|=1\}}|v \cdot v_-(\lambda)|^2|\rho_+'(\lambda)|}{|t(\phi_-;\phi_+)|^2\|\phi_+\|^2_{L^2_V(R^+,R^++X^+)}} + \frac{\mathbb{1}_{\{|\rho_-(\lambda)|=1\}}|v \cdot v_+(\lambda)|^2|\rho_-'(\lambda)|}{|t(\phi_+;\phi_-)|^2\|\phi_-\|^2_{L^2_V(R^--X^-,R^-)}}\right).$$

holds for $\lambda \in \mathbb{R} \setminus S$ and $v \in \mathbb{C}^2$.

Proof. Part 1: First, let $\lambda \in \mathbb{H}$. By Lemma 2.3.11 we can write $v_\pm(\lambda) = c_\pm(\lambda)(1, m_\pm(\lambda))^\top$ for some $c_\pm(\lambda) \in \mathbb{C} \setminus \{0\}$, and we set $w_\pm(\lambda) \coloneqq c_\mp(\lambda)^{-1}(m_\pm(\lambda), -1)^\top$. Using the definition of $M(\lambda)$ in Theorem 2.3.6 we calculate

$$w_{+,i}\overline{w_{+,j}}M_{ij}(\lambda) = \frac{\operatorname{Im}[m_+]}{\pi|c_-(\lambda)|^2}, \qquad w_{+,i}\overline{w_{-,j}}M_{ij}(\lambda) = 0,$$
$$w_{-,i}\overline{w_{+,j}}M_{ij}(\lambda) = 0, \qquad w_{-,i}\overline{w_{-,j}}M_{ij}(\lambda) = -\frac{\operatorname{Im}[m_-]}{\pi|c_+(\lambda)|^2}$$

Note that the identity

$$v = \frac{1}{m_+ - m_-}((v \cdot v_-)w_+ - (v \cdot v_+)w_-)$$

holds for any $v \in \mathbb{C}^2$. Therefore, for $\lambda \in \mathbb{H}$ we obtain

$$v_i\overline{v_j}M_{ij}(\lambda) = \frac{1}{\pi|m_+ - m_-|^2}\left(|v \cdot v_-(\lambda)|^2\frac{\operatorname{Im}[m_+]}{|c_-(\lambda)|^2} - |v \cdot v_+(\lambda)|^2\frac{\operatorname{Im}[m_-]}{|c_+(\lambda)|^2}\right) \tag{2.10}$$

Before taking the limit, we express (2.10) in simpler terms. For this, we calculate

$$\begin{aligned}\int_0^\infty |\phi_+|^2\, V\mathrm{d}x &= \frac{1}{\lambda - \overline{\lambda}}\int_0^\infty \left(L\phi_+ \cdot \overline{\phi_+} - \phi_+ \cdot L\overline{\phi_+}\right) V\mathrm{d}x = \frac{1}{\lambda - \overline{\lambda}}W[\phi_+, \overline{\phi_+}]\Big|_0^\infty \\ &= \frac{1}{\lambda - \overline{\lambda}}\left(\phi_+'(0)\overline{\phi_+(0)} - \phi_+(0)\overline{\phi_+'(0)}\right) = |c_+(\lambda)|^2\frac{\operatorname{Im}[m_+]}{\operatorname{Im}[\lambda]},\end{aligned}$$

and in the same way also obtain

$$\int_{-\infty}^0 |\phi_-|^2\, V\mathrm{d}x = -|c_-(\lambda)|^2\frac{\operatorname{Im}[m_-]}{\operatorname{Im}[\lambda]}.$$

Using this,

$$W[\phi_+, \phi_-] = \phi_+(0;\lambda)\phi_-'(0;\lambda) - \phi_+'(0;\lambda)\phi_-(0;\lambda) = c_+(\lambda)c_-(\lambda)(m_-(\lambda) - m_+(\lambda))$$

as well as Lemma 2.3.17, we can express (2.10) as

$$v_i\overline{v_j}M_{ij}(\lambda) = \frac{\operatorname{Im}[\lambda]}{\pi|W[\phi_+, \phi_-]|^2}\left(|v \cdot v_-(\lambda)|^2\int_0^\infty |\phi_+|^2 V\,\mathrm{d}x + |v \cdot v_+(\lambda)|^2\int_{-\infty}^0 |\phi_-|^2 V\,\mathrm{d}x\right) \tag{2.11}$$

Part 2: For the limit, let $\lambda_0 \in \mathbb{R} \setminus S$ and $\lambda \in \mathbb{H}$. Since $\phi_+(x + X^+) = \rho_+(x)\phi_+(x)$ we have

$$\int_0^\infty |\phi_+(x;\lambda)|^2\, V\mathrm{d}x = \int_0^{R^+} |\phi_+(x;\lambda)|^2\, V\mathrm{d}x + \frac{1}{1 - |\rho_+(\lambda)|^2}\int_{R^+}^{R^+ + X^+} |\phi_+(x;\lambda)|^2\, V\mathrm{d}x$$

If $|\rho_+(\lambda_0)| = 1$, then by Lemma 2.3.12 we have $\rho_+'(\lambda_0)\overline{\rho_+(\lambda_0)} = \mathrm{i}|\rho_+'(\lambda_0)|$ and thus

$$\begin{aligned}1 - |\rho_+(\lambda)|^2 &= -2\operatorname{Re}[(\lambda - \lambda_0)\rho_+'(\lambda_0)\overline{\rho_+(\lambda_0)}] + \mathcal{O}(|\lambda - \lambda_0|^2) \\ &= 2|\rho_+'(\lambda_0)|\operatorname{Im}[\lambda] + \mathcal{O}(|\lambda - \lambda_0|^2)\end{aligned}$$

as $\lambda \to \lambda_0$. It follows that

$$\operatorname{Im}[\lambda]\int_0^\infty |\phi_+(x;\lambda)|^2\, V\mathrm{d}x \to \begin{cases}\dfrac{1}{2|\rho_+'(\lambda_0)|}\displaystyle\int_{R^+}^{R^+ + X^+} |\phi_+(x;\lambda_0)|^2\, V\mathrm{d}x, & |\rho_+(\lambda_0)| = 1, \\ 0, & |\rho_+(\lambda_0)| < 1\end{cases}$$

as $\lambda \to \lambda_0$ along curves with $|\lambda - \lambda_0|^2 = \mathrm{o}(\mathrm{Im}[\lambda])$. As $\lambda_0 \notin S_0$, we have

$$W[\phi_+(\,\cdot\,;\lambda_0), \phi_-(\,\cdot\,;\lambda_0)] \neq 0.$$

So the remaining terms in the right-hand side of (2.11) are continuous at λ_0. Thus we so far have shown

$$\begin{aligned}\lim_{\varepsilon\to 0+} v_i\overline{v_j}M(\lambda+\mathrm{i}\varepsilon) = &\frac{\mathbb{1}_{\{|\rho_+(\lambda)|=1\}}|v\cdot v_-(\lambda)|^2\|\phi_+\|^2_{L^2_V(R^+,R^++X^+)}}{2\pi|W[\phi_+,\phi_-]|^2|\rho_+'(\lambda)|} \\ &+\frac{\mathbb{1}_{\{|\rho_-(\lambda)|=1\}}|v\cdot v_+(\lambda)|^2\|\phi_-\|^2_{L^2_V(R^--X^-,R^-)}}{2\pi|W[\phi_+,\phi_-]|^2|\rho_-'(\lambda)|}\end{aligned} \tag{2.12}$$

for $\lambda \in \mathbb{R}\setminus S$, and one easily sees that this convergence is locally uniform. Using Lemma 2.3.17 we have

$$\begin{aligned}W[\phi_+,\phi_-] &= W[\phi_+, r(\phi_-;\phi_+]\phi_+ + t(\phi_-;\phi_+)\overline{\phi_+}) \\ &= t(\phi_-;\phi_+)W[\phi_+,\overline{\phi_+}] = t(\phi_-;\phi_+)\frac{\rho_+(\lambda)}{\rho_+'(\lambda)}\|\phi_+\|^2_{L^2_V(R^+,R^++X^+)}\end{aligned}$$

as well as its analogue

$$W[\phi_+,\phi_-] = -W[\phi_-,\phi_+] = t(\phi_+;\phi_-)\frac{\rho_-(\lambda)}{\rho_-'(\lambda)}\|\phi_-\|^2_{L^2_V(R^--X^-,R^-)}.$$

The claim follows from this and (2.12). □

We complete the description of the spectral measure by considering the point spectrum.

Lemma 2.3.21. *Let $\lambda \in \mathbb{R}$. Then μ is discontinuous at λ if and only if $\lambda \in \sigma_p(L)$. In this case, letting $\phi(x;\lambda) = v\cdot\Psi(x;\lambda)$ be an eigenfunction, we have*

$$\mu(\lambda+) - \mu(\lambda-) = \frac{(v_i\overline{v_j})^2_{i,j=1}}{\|\phi\|^2_{L^2_V(\mathbb{R})}}.$$

Note that $v \in \mathbb{C}v_+(\lambda) = \mathbb{C}v_-(\lambda)$ holds for $\lambda \in \sigma_p(L)$, and in particular $\sigma_p(L) \subseteq S_0$.

Proof. We write $\Delta\mu(\lambda) = \mu(\lambda+) - \mu(\lambda-)$.

Part 1: Let μ be discontinuous at λ_0. For $w \in \mathbb{C}^2$ we have

$$f = T^{-1}[\mathbb{1}_{\{\lambda_0\}}w] = w_i\Psi_j(\,\cdot\,;\lambda_0)\Delta\mu_{ij}(\lambda_0) \in L^2_V(\mathbb{R}).$$

Lemma 2.3.7 yields $f \in H^2(\mathbb{R})$ with $Lf = \lambda_0 f$. Since there is at most 1 linearly independent eigenfunction ϕ, $\Delta\mu(\lambda_0)$ has rank 1. As $\Delta\mu(\lambda_0)$ is positive semidefinite and $w_i\Delta\mu_{ij}(\lambda_0)e_j \in \mathbb{C}v$ there exists $r > 0$ such that $\Delta\mu_{ij}(\lambda_0) = r\overline{v_i}v_j$ holds. It remains to check the value of r:

$$r|v|^4 = v_i\overline{v_j}\Delta\mu_{ij}(\lambda_0) = \left\|\mathbb{1}_{\{\lambda_0\}}v\right\|^2_{L^2(\mu)} = \left\|T^{-1}[\mathbb{1}_{\{\lambda_0\}}v]\right\|^2_{L^2_V} = \left\|r|v|^2\phi\right\|^2_{L^2_V} = r^2|v|^4\|\phi\|^2_{L^2_V}.$$

Note that ϕ is a multiple of a real-valued function since it is an eigenfunction to a simple real eigenvalue. Hence v is a multiple of a real vector and in particular $\overline{v_i}v_j = v_i\overline{v_j}$ holds.

Part 2: Now let $\lambda_0 \in \sigma_p(L)$ with eigenfunction ϕ. Then $\lambda_0 T[\phi](\lambda) = T[L\phi](\lambda) = \lambda T[\phi](\lambda)$ for $\lambda \in \mathbb{R}$ by Lemma 2.3.7 and therefore $T[\phi] = \mathbb{1}_{\{\lambda_0\}}w$ for some $w \in \mathbb{C}^2$. As

$$0 < \|\phi\|^2_{L^2_V} = \|T[\phi]\|^2_{L^2(\mu)} = w_i\overline{w_j}\Delta\rho_{ij}(\lambda_0),$$

μ must be discontinuous at λ_0. □

Notation 2.3.22. *In the following, for $\lambda \in \sigma_p(L)$ we denote by ϕ_0 and v_0 the eigenfunction and eigenvector from Lemma 2.3.21.*

Let us conclude this subsection by explicitly writing down the $L^2(\mu)$-norm.

Theorem 2.3.23. *Let $f\colon \mathbb{R} \to \mathbb{C}^2$ be measurable. Then*

$$\|f\|^2_{L^2(\mu)} = \frac{1}{2\pi}\int_{\mathbb{R}\setminus S} \frac{\mathbb{1}_{\{|\rho_+(\lambda)|=1\}}|f(\lambda)\cdot v_-(\lambda)|^2|\rho_+'(\lambda)|}{|t(\phi_-;\phi_+)|^2\|\phi_+\|^2_{L^2_V(R^+,R^++X^+)}} + \frac{\mathbb{1}_{\{|\rho_-(\lambda)|=1\}}|f(\lambda)\cdot v_+(\lambda)|^2|\rho_-'(\lambda)|}{|t(\phi_+;\phi_-)|^2\|\phi_-\|^2_{L^2_V(R^--X^-,R^-)}}\,\mathrm{d}\lambda + \sum_{\lambda\in\sigma_p(L)} \frac{|f(\lambda)\cdot v_0(\lambda)|^2}{\|\phi_0\|^2_{L^2_V(\mathbb{R})}}.$$

Proof. By Lemma 2.3.12 and Remark 2.3.19 the set S is at most countable and its complement has at most countably many connected components. Let us write[1] $S = \{\lambda_n : n \in \mathbb{N}\}$, $\mathbb{R}\setminus S = \cup_{n\in\mathbb{N}} I_n$. It follows that

$$\|f\|^2_{L^2(\mu)} = \sum_{n\in\mathbb{N}} \|f\mathbb{1}_{I_n}\|^2_{L^2(\mu)} + \sum_{n\in\mathbb{N}} \left\|f(\lambda_n)\mathbb{1}_{\{\lambda_n\}}\right\|^2_{L^2(\mu)}.$$

Lemma 2.3.21 shows that the second series is equal to

$$\sum_{\lambda\in\sigma_p(L)} \frac{|f(\lambda)\cdot v_0(\lambda)|^2}{\|\phi_0\|^2_{L^2_V(\mathbb{R})}}$$

For the first series, let $n \in \mathbb{N}$ and $[a, b] \subseteq I_n$. By Theorem 2.3.6 and Proposition 2.3.20 we have

$$\int_a^b M(\lambda)\,\mathrm{d}\lambda = \lim_{\varepsilon\to 0+}\int_a^b M(\lambda + \mathrm{i}\varepsilon)\,\mathrm{d}\lambda = \mu(b) - \mu(a).$$

As a, b were arbitrary, it follows that

$$\int_{I_n} f_i(\lambda)\overline{f_j(\lambda)}M_{ij}(\lambda)\,\mathrm{d}\lambda = \int_{I_n} f_i(\lambda)\overline{f_j(\lambda)}\,\mathrm{d}\mu_{ij}(\lambda)$$

holds. Summing over n and using the formula for $M(\lambda)$ given in Proposition 2.3.20 completes the proof. □

2.3.3. Embeddings

Let us generalize the torus $\mathbb{T}$ to a measure space Ω and $-\partial_t^2$ to a formal symmetric operator $\mathcal{L}$ on Ω satisfying $\mathcal{L}e_k = \nu_k e_k$, where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal system in $L^2(\Omega)$ and $\nu_k \geq 0$. For $k \in \mathbb{N}$, set $\hat{u}_k(x) = \int_\Omega u(x,t)\overline{e_k(t)}\,\mathrm{d}t$. We consider the differential operator $L - \mathcal{L}$ on $\mathbb{R}\times\Omega$, with L acting on $x \in \mathbb{R}$ and $\mathcal{L}$ on $t \in \Omega$. We associate to it the sesquilinear form

$$b(u,v) := \int_{\mathbb{R}\times\Omega} ((L-\mathcal{L})u\cdot\overline{v})\,V(x)\mathrm{d}(x,t) := \sum_{k\in\mathbb{N}}\int_{\mathbb{R}} \left((\lambda-\nu_k)T_i[\hat{u}_k](\lambda)\cdot\overline{T_j[\hat{v}_k](\lambda)}\right)\mathrm{d}\mu_{ij}(\lambda)$$

Next we give the domain of the above form.

[1] For simplicity of notation, we assume that both sets are countably infinite.

Definition 2.3.24. *Using the closed subspace*

$$L^2_{(e_k)} := \{u \in L^2_V(\mathbb{R} \times \Omega) \colon u(x,\,\cdot\,) \in \overline{\operatorname{span}\{e_k \colon k \in \mathbb{N}\}}\}$$
$$= \{u \in L^2_V(\mathbb{R} \times \Omega) \colon u(x,t) = \sum_{k\in\mathbb{N}} \hat{u}_k(x) e_k(t)\}$$

of $L^2_V(\mathbb{R} \times \Omega)$, we define the domain $\mathcal{H}$ of b by

$$\mathcal{H} = \left\{u \in L^2_{(e_k)} \colon \|u\|^2_{\mathcal{H}} = \langle u, u\rangle_{\mathcal{H}} < \infty\right\}$$

where

$$\langle u, v\rangle_{\mathcal{H}} := \sum_{k\in\mathbb{N}} \int_{\mathbb{R}} (|\lambda - \nu_k| T_i[\hat{u}_k](\lambda) \cdot \overline{T_j[\hat{v}_k](\lambda)}) \,\mathrm{d}\mu_{ij}(\lambda).$$

Our main goal is to investigate embeddings $\mathcal{H} \hookrightarrow L^p_V(\mathbb{R} \times \Omega)$. We use interpolation in p. The end point $p = 2$ will be trivial, and we prepare $p = \infty$ by first showing L^∞-bounds for the spectral density as well as eigenfunctions.

Lemma 2.3.25. *There exists a constant $C > 0$ (independent of λ) such that*

$$\frac{\|\phi_-\|_{L^\infty(\mathbb{R})}}{|t(\phi_-;\phi_+)| \|\phi_+\|_{L^2_V(R^+,R^++X^+)}} \le C$$

holds for all $\lambda \in \mathbb{R} \setminus S$ with $\rho_+(\lambda) \in \mathbb{S}_*$, analogously

$$\frac{\|\phi_+\|_{L^\infty(\mathbb{R})}}{|t(\phi_+;\phi_-)| \|\phi_-\|_{L^2_V(R^--X^-,R^-)}} \le C$$

holds for $\lambda \in \mathbb{R} \setminus S$ with $\rho_-(\lambda) \in \mathbb{S}_*$, and lastly

$$\frac{\|\phi_0\|_{L^\infty(\mathbb{R})}}{\|\phi_0\|_{L^2_V(\mathbb{R})}} \le C$$

holds for $\lambda \in \sigma_p(L)$.

Proof. We consider the first inequality, and calculate

$$W[\phi_-, \overline{\phi_-}] = W[r\phi_+ + t\overline{\phi_+}, \overline{r}\overline{\phi_+} + \overline{t}\phi_+] = (|r|^2 - |t|^2) W[\phi_+, \overline{\phi_+}].$$

Let us show that $|r(\phi_-;\phi_+)| \le |t(\phi_-;\phi_+)|$. For this, by Lemmas 2.3.12 and 2.3.17 we first have

$$W[\phi_+, \overline{\phi_+}] = \frac{\rho_+(\lambda)}{\rho_+'(\lambda)} \|\phi_+\|_{L^2_V(R^+,R^++X^+)^2} \in \mathrm{i}(-\infty, 0).$$

Case 1: Assume $\rho_-(\lambda) \in \mathbb{S}_*$. Then these lemmas also show $W[\phi_-, \overline{\phi_-}] \in \mathrm{i}(0,\infty)$, so we even have $|r| < |t|$.

Case 2: Assume $|\rho_-(\lambda)| < 1$. Recall that $\phi_-(x - X^-) = \rho_-(\lambda)\phi_-(x)$ holds. As this equation has at most 1 linearly independent solution and $\rho_-(\lambda) \in \mathbb{R}$, we see that ϕ_- is a real-valued function up to multiplication with a complex scalar. Therefore $W[\phi_-, \overline{\phi_-}] = 0$, and in particular $|r| = |t|$.

Thus $|r| \le |t|$ holds. Using semiperiodicity of both ϕ_+ and ϕ_- in their respective regions as well as Proposition 2.A.1, we then estimate

$$\begin{aligned}\|\phi_-\|_\infty = \|\phi_-\|_{L^\infty(R^- - X^-, \infty)} &= \left\| r\phi_+ + t\overline{\phi_+} \right\|_{L^\infty(R^- - X^-, \infty)} \\ &\le 2|t| \|\phi_+\|_{L^\infty(R^- - X^-, \infty)} = 2|t| \|\phi_+\|_{L^\infty(R^- - X^-, R^+ + X^+)} \lesssim |t| \|\phi_+\|_{L^2(R^+, R^+ + X^+)}\end{aligned}$$

uniformly in λ.

The second inequality can be shown in analogy to the first. So let us consider the third inequality. Here, using semiperiodicity of ϕ_0 on both $(-\infty, R^-)$ and (R^+, ∞) as well as Proposition 2.A.1 we find

$$\|\phi_0\|_\infty = \|\phi_0\|_{L^\infty(R^- - X^-, R^+ + X^+)} \lesssim \|\phi_0\|_{L^2_V(R^- - X^-, R^+ + X^+)} \le \|\phi_0\|_{L^2_V(\mathbb{R})} \qquad \square$$

Using the bounds of Lemma 2.3.25 we obtain a sufficient condition for boundedness of L^p-embeddings.

Lemma 2.3.26. *Let $p \in (2, \infty]$, let $(I_n^\pm)_{n\in\mathbb{N}}$ enumerate the connected components of $\mathbb{R} \setminus S_\pm$ on which $|\rho_\pm| = 1$ holds, and assume that*

$$C := \sum_{k\in\mathbb{N}} \|e_k\|_\infty^2 \left(\sum_{n\in\mathbb{N}} \operatorname{dist}(\nu_k, I_n^+)^{-s} + \sum_{n\in\mathbb{N}} \operatorname{dist}(\nu_k, I_n^-)^{-s} + \sum_{\lambda\in\sigma_p(L)} |\nu_k - \lambda|^{-s} \right) < \infty \qquad (2.13)$$

where $s := \frac{p}{p-2}$. Then the embedding $\mathcal{H} \hookrightarrow L^p_V(\mathbb{R} \times \Omega)$ is bounded.

Proof. Observe that the map

$$E \colon \mathcal{H} \to L^2_{(e_k)}, \; u \mapsto \sum_{k\in\mathbb{N}} e_k(t) T^{-1} \left[|\lambda - \nu_k|^{\frac{1}{2}} T[\hat{u}_k](\lambda) \right](x)$$

is an isometric isomorphism. Set $m_k(\lambda) = |\lambda - \nu_k|^{-\frac{s}{2}}$ and for $\theta \in [0,1]$, $q := \frac{2}{1-\theta}$ we consider the map

$$\iota_\theta \colon L^2_{(e_k)} \to L^q_V(\mathbb{R} \times \Omega), \quad u \mapsto \sum_{k\in\mathbb{N}} e_k(t) T^{-1}[m_k^\theta T[\hat{u}_k](\lambda)](x).$$

Our goal is to show that $\iota_\theta \circ E : \mathcal{H} \to L^q_V(\mathbb{R} \times \Omega)$ is bounded for $\theta \in [0,1]$ and $q = \frac{2}{1-\theta}$ using interpolation. If we then set $\theta = \frac{1}{s}$ we get $p = q$, $\iota_\theta \circ E = \mathrm{Id}$ and thus the proof will be finished.

Part 1: First, let $\theta = 1$, $q = \infty$ and assume that $u \in L^2_{(e_k)}$ has finitely many nonzero modes $\hat{u}_k$ and that $T[\hat{u}_k]$ is compactly supported for all k. Then we calculate

$$\begin{aligned}|\iota_1[u](x,t)| &= \left| \sum_{k\in\mathbb{N}} e_k(t) \int_{\mathbb{R}} m_k(\lambda) T_i[\hat{u}_k](\lambda) \Psi_j(x;\lambda) \, \mathrm{d}\mu_{ij}(\lambda) \right| \\ &\le \left(\sum_{k\in\mathbb{N}} \|T[\hat{u}_k]\|^2_{L^2(\mu)} \right)^{1/2} \left(\sum_{k\in\mathbb{N}} |e_k(t)|^2 \|m_k(\,\cdot\,)\Psi(x;\,\cdot\,)\|^2_{L^2(\mu)} \right)^{1/2}\end{aligned}$$

Note that the first term is

$$\sum_{k\in\mathbb{N}}\|T[\hat{u}_k]\|^2_{L^2(\mu)}=\sum_{k\in\mathbb{N}}\|\hat{u}_k\|^2_{L^2_V(\mathbb{R})}=\|u\|^2_{L^2_{(e_k)}}.$$

Now we consider the second term. Using the identities $\phi_\pm(x;\lambda)=\Psi(x;\lambda)\cdot v_\pm(\lambda), \phi_0(x;\lambda)=\Psi(x;\lambda)\cdot v(\lambda)$ as well as Theorem 2.3.23 and Lemma 2.3.25 we obtain

$$\begin{aligned}
&\|m_k(\,\cdot\,)\Psi(x;\,\cdot\,)\|^2_{L^2(\mu)}\\
&\quad=\frac{1}{2\pi}\int_{\mathbb{R}\setminus S}\frac{\mathbb{1}_{\{|\rho_+(\lambda)|=1\}}|m_k(\lambda)|^2|\phi_-(x)|^2|\rho_+'(\lambda)|}{|t(\phi_-;\phi_+)|^2\|\phi_+\|^2_{L^2_V(R^+,R^++X^+)}}+\frac{\mathbb{1}_{\{|\rho_-(\lambda)|=1\}}|m_k(\lambda)|^2|\phi_+(x)|^2|\rho_-'(\lambda)|}{|t(\phi_+;\phi_-)|^2\|\phi_-\|^2_{L^2_V(R^--X^-,R^-)}}\,\mathrm{d}\lambda\\
&\qquad+\sum_{\lambda\in\sigma_p(L)}\frac{|m_k(\lambda)|^2|\phi_0(x)|^2}{\|\phi_0\|^2_{L^2_V(\mathbb{R})}}\\
&\quad\lesssim\int_{\mathbb{R}\setminus S}|m_k(\lambda)|^2\Big(\mathbb{1}_{\{|\rho_+(\lambda)|=1\}}|\rho_+'(\lambda)|+\mathbb{1}_{\{|\rho_-(\lambda)|=1\}}|\rho_-'(\lambda)|\Big)\,\mathrm{d}\lambda+\sum_{\lambda\in\sigma_p(L)}|m_k(\lambda)|^2
\end{aligned}$$

uniformly in k, x. Using Lemma 2.3.12 we can further estimate

$$\begin{aligned}
\int_{\mathbb{R}\setminus S}\mathbb{1}_{\{|\rho_+(\lambda)|=1\}}|m_k(\lambda)|^2|\rho_+'(\lambda)|\,\mathrm{d}\lambda&=\sum_{n\in\mathbb{N}}\int_{I_n^+}|m_k(\lambda)|^2|\rho_+'(\lambda)|\,\mathrm{d}\lambda\\
&\le\sum_{n\in\mathbb{N}}\mathrm{dist}(\nu_k,I_n^+)^{-s}\int_{I_n^+}|\rho_+'(\lambda)|\,\mathrm{d}\lambda\\
&\le\pi\sum_{n\in\mathbb{N}}\mathrm{dist}(\nu_k,I_n^+)^{-s},
\end{aligned}$$

with similar estimates for the other two terms. Recalling the definition of C, we so far have shown

$$\|\iota_1[u]\|_\infty\lesssim C^{\frac{1}{2}}\|u\|_{L^2_{(e_k)}} \tag{2.14}$$

for $u\in L^2_{(e_k)}$ with compact frequency support. By density, (2.14) holds for all $u\in L^2_{(e_k)}$.

Part 2: For $\theta=0$ and $q=2$, $\iota_0\colon L^2_{(e_k)}\to L^2_V$ is the identity map. By interpolation (cf. [53, Chapter 2]) the map $\iota_\theta\colon L^2_{(e_k)}\to L^q_V(\mathbb{R}\times\Omega)$, $q=\frac{2}{1-\theta}$, is bounded for all $\theta\in[0,1]$. The claim of Lemma 2.3.26 follows by setting $\theta=\frac{1}{s}$ since for this choice $\iota_\theta E$ is the identity map and $p=q$ holds. □

Theorem 2.3.27. *Let $N\in\mathbb{N}$ and assume for the eigenvalues ν_k of $\mathcal{L}$ that $\#\{k\colon\sqrt{\nu_k}\in B\}\le N$ for any interval B of length 1, which implies that (ν_k) grows at least quadratically. Generalizing ($\mathcal{A}$2.3) we assume additionally*

$$\inf\Big\{\Big|\sqrt{\lambda}-\sqrt{\nu_k}\Big|\colon k\in\mathbb{N},\lambda\in\sigma(L)\Big\}>0.$$

Moreover, assume $\alpha,\beta\ge0$ exist such that the eigenfunctions of $\mathcal{L}$ satisfy $\|e_k\|_\infty\lesssim\nu_k^{\frac{\alpha}{2}}$, and generalizing ($\mathcal{A}$2.4) assume for the point spectrum of L that

$$\sum_{\lambda\in\sigma_p(L)}\lambda^{-r}<\infty$$

holds for $r>\frac{1}{2}+\beta$. Then the embedding $\mathcal{H}\hookrightarrow L^p_V(\mathbb{R}\times\Omega)$ is bounded and $\mathcal{H}\hookrightarrow L^p_V(A\times\Omega)$ is compact for all $p\in[2,2+\frac{1}{\alpha+\beta})$ and $A\subseteq\mathbb{R}$ compact.

Proof. Set $\delta := \inf\left\{\left|\sqrt{\lambda}-\sqrt{\nu_k}\right| : k\in\mathbb{N}, \lambda\in\sigma(L)\right\}$ and $s := \frac{p}{p-2} > 1+2\alpha+2\beta$. We show that the assumptions of Lemma 2.3.26 are satisfied.

Part 1: We estimate the spectral bands I_n^+. By definition, for $\lambda\in\{\min I_n^+, \max I_n^+\}$ we have $\rho_+(\lambda)\in\{-1,1\}$. Therefore there exists an X-periodic or X-antiperiodic solution ϕ of the problem

$$-\frac{1}{V_{\mathrm{per}}^+(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi = \lambda\phi, \qquad \phi(x+2X)=\phi(x). \tag{2.15}$$

If we enumerate the eigenvalues of (2.15) in increasing order by $(\lambda_j^+)_{j\in\mathbb{N}}$, we get in particular

$$\sum_{n\in\mathbb{N}} \operatorname{dist}(\nu_k, I_n^+)^{-s} \le 2\sum_{j\in\mathbb{N}}\left|\nu_k-\lambda_j^+\right|^{-s}.$$

Moreover, using that the eigenvalues of the $2X$-periodic Laplacian are $0, \frac{\pi}{X}, \frac{\pi}{X}, 2\frac{\pi}{X}, 2\frac{\pi}{X}, \dots$ (with eigenfunctions $1, \sin(\frac{\pi x}{X}), \cos(\frac{\pi x}{X}), \sin(2\frac{\pi x}{X}), \cos(2\frac{\pi x}{X}), \dots$), by the Min-Max-Principle (cf. [85, Chapter 11.2, Theorem 1]) we have

$$\begin{aligned}\lambda_j^+ &= \inf_{\substack{Y\subseteq H^1_{\mathrm{per}}(\mathbb{R})\\ \dim(Y)=j}} \sup_{\substack{u\in Y\\ u\neq 0}} \frac{\int_0^{2X}|u'|^2\,\mathrm{d}x}{\int_0^{2X}|u|^2\,V_{\mathrm{per}}^+(x)\mathrm{d}x}\\ &\ge \frac{1}{\left\|V_{\mathrm{per}}^+\right\|_\infty}\inf_{\substack{Y\subseteq H^1_{\mathrm{per}}(\mathbb{R})\\ \dim(Y)=j}} \sup_{\substack{u\in Y\\ u\neq 0}} \frac{\int_0^{2X}|u'|^2\,\mathrm{d}x}{\int_0^{2X}|u|^2\,\mathrm{d}x} = \frac{1}{\left\|V_{\mathrm{per}}^+\right\|_\infty}\left(\frac{\pi\left\lfloor\frac{j}{2}\right\rfloor}{X}\right)^2.\end{aligned}$$

Part 2: The following estimates are inspired by [56]. First, we estimate the double sum via

$$\begin{aligned}\sum_{k\in\mathbb{N}}\|e_k\|_\infty^2\sum_{n\in\mathbb{N}}\operatorname{dist}(\nu_k, I_n^+)^{-s} &\lesssim \sum_{k\in\mathbb{N}}|\nu_k|^\alpha\sum_{j\in\mathbb{N}}\left|\nu_k-\lambda_j^+\right|^{-s}\\ &= \sum_{k\in\mathbb{N}}|\nu_k|^\alpha\sum_{j\in\mathbb{N}}\left|\sqrt{\nu_k}-\sqrt{\lambda_j^+}\right|^{-s}\left|\sqrt{\nu_k}+\sqrt{\lambda_j^+}\right|^{-s}\\ &\le \sum_{j\in\mathbb{N}}\max\{\sqrt{\lambda_j^+},\delta\}^{-s}\sum_{k\in\mathbb{N}}|\nu_k|^\alpha\left|\sqrt{\nu_k}-\sqrt{\lambda_j^+}\right|^{-s}.\end{aligned}$$

Next, observe that for each $k\in\mathbb{N}$ we have $\sqrt{\nu_k}\in\bigcup_{m=0}^\infty[m,m+1)$ as well as $\sqrt{\nu_k}-\sqrt{\lambda_j^+}\in\bigcup_{m=-\infty}^\infty[m,m+1)$. Together with the assumption on the number of eigenvalues per interval of length 1, we use this (combined with a separate consideration for $m=-1,0$) to estimate for fixed j

$$\sum_{\substack{k\in\mathbb{N}\\ |\nu_k|\le 2\lambda_j^+}}|\nu_k|^\alpha\left|\sqrt{\nu_k}-\sqrt{\lambda_j^+}\right|^{-s} \le 2^\alpha(\lambda_j^+)^\alpha\sum_{k\in\mathbb{N}}\left|\sqrt{\nu_k}-\sqrt{\lambda_j^+}\right|^{-s} \le 2^\alpha(\lambda_j^+)^\alpha\cdot 2N\left(\delta^{-s}+\sum_{m=1}^\infty m^{-s}\right),$$

$$\sum_{\substack{k\in\mathbb{N}\\ |\nu_k|> 2\lambda_j^+}}|\nu_k|^\alpha\left|\sqrt{\nu_k}-\sqrt{\lambda_j^+}\right|^{-s} \le \left(1-\tfrac{1}{\sqrt{2}}\right)^{-s}\sum_{k\in\mathbb{N}}\left|\sqrt{\nu_k}\right|^{2\alpha-s} \le \left(1-\tfrac{1}{\sqrt{2}}\right)^{-s}2N\left(\delta^{2\alpha-s}+\sum_{m=1}^\infty m^{2\alpha-s}\right).$$

Therefore we obtain

$$\sum_{j\in\mathbb{N}}\max\{\sqrt{\lambda_j^+},\delta\}^{-s}\sum_{k\in\mathbb{N}}|\nu_k|^\alpha\left|\sqrt{\nu_k}-\sqrt{\lambda_j^+}\right|^{-s} \le C\sum_{j\in\mathbb{N}}\max\{\sqrt{\lambda_j^+},\delta\}^{-s}(1+(\lambda_j^+)^\alpha)$$

where the right-hand side is finite since $\lambda_j^+ \gtrsim (j-1)^2$ by part 1 and $s-2\alpha>1$ by assumption.

Part 3: So far, we have shown

$$\sum_{k\in\mathbb{N}} \|e_k\|_\infty^2 \sum_{n\in\mathbb{N}} \operatorname{dist}(\nu_k, I_n^+)^{-s} < \infty.$$

By the arguments above, this also holds for I_n^- and we have the estimate

$$\sum_{k\in\mathbb{N}} \|e_k\|_\infty^2 \sum_{\lambda\in\sigma_p(L)} |\nu_k-\lambda|^{-s} \le C \sum_{\lambda\in\sigma_p(L)} \max\{\sqrt{\lambda},\delta\}^{-s}(1+\lambda^\alpha)$$

which is finite by assumption.

Part 4: It remains to show local compactness of the embedding $E\colon \mathcal{H}\to L^p(\mathbb{R}\times\Omega)$. We only consider $p>2$ and take $A\subseteq\mathbb{R}$ compact. For $K\in\mathbb{N}$ consider

$$E_K\colon \mathcal{H}\to L^p_V(\mathbb{R}\times\Omega), u\mapsto \sum_{k=1}^K e_k(t)\hat{u}_k(x).$$

As the map $\mathcal{H}\to H^1(\mathbb{R}), u\mapsto \hat{u}_k$ is bounded for each k, E_K is a compact operator. A small modification of Lemma 2.3.26 shows that $\|E-E_K\| \lesssim C_K^{\frac{1}{2s}}$ where

$$C_K := \sum_{\substack{k\in\mathbb{N}\\ k>K}} \|e_k\|_\infty^2 \left(\sum_{n\in\mathbb{N}} \operatorname{dist}(\nu_k, I_n^+)^{-s} + \sum_{n\in\mathbb{N}} \operatorname{dist}(\nu_k, I_n^+)^{-s} + \sum_{\lambda\in\sigma_p(L)} |\nu_k-\lambda|^{-s} \right)$$

As the series in (2.13) converges, $C_K\to 0$ as $K\to\infty$, and therefore the limit $E\colon\mathcal{H}\to L^p_V(A\times\Omega)$ is also compact. □

2.A. Appendix: Eigenfunction bounds

In this section we discuss uniform estimates on functions u solving

$$-u'' = \lambda V(x)u, \qquad \text{for} \quad x\in I \tag{2.16}$$

on a compact interval I and with λ, V positive. For Schrödinger operators, uniform eigenfunction bounds with respect to λ of the type $\|u\|_\infty \lesssim \|u\|_2$ are known, cf. [50]. These can be transferred to the weighted eigenvalue problem (2.16) using the Liouville transform if V is twice differentiable. However, here we show a generalization of this inequality under the weaker assumption that V is of bounded variation.

Proposition 2.A.1. *Let I, J be bounded intervals of positive length with $J\subseteq I$, and $V\in BV(I)$ with $\operatorname{ess\,inf}_I V>0$. Then there exists a constant $C=C(I,J,V)>0$ such that*

$$\|u\|_{L^\infty(I)} \le C\|u\|_{L^2(J)} \tag{2.17}$$

for all $\lambda\ge 0$ and all solutions u to (2.16).

Note that the reverse inequality $\|u\|_{L^2(J)} \le \|u\|_{L^2(I)} \le |I|^{1/2}\|u\|_{L^\infty(I)}$ always holds.

Proof. Part 1: For fixed $\lambda \in [0,\infty)$, inequality (2.17) holds since the space of solutions to (2.16) has dimension $2 < \infty$. As the space of solutions to (2.16) depends continuously on λ, so does the optimal constant C in (2.17). Therefore it suffices to show that C can be bounded uniformly as $\lambda \to \infty$.

Part 2: We use a change of coordinates similar to the Liouville transform, replacing (u, u') by (ϕ_1, ϕ_2). For this, let $a := \inf I, \mu := \sqrt{\lambda}$, and for $x \in I$ set $t(x) := \int_a^x \sqrt{V(s)}\,\mathrm{d}s$ as well as

$$\phi(x) := \begin{pmatrix} \cos(\mu t(x)) & -\sin(\mu t(x)) \\ \sin(\mu t(x)) & \cos(\mu t(x)) \end{pmatrix} \begin{pmatrix} u(x) \\ \frac{1}{\mu\sqrt{V(x)}} u'(x) \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \begin{pmatrix} \cos(\mu t(x)) & \sin(\mu t(x)) \\ -\mu\sqrt{V(x)}\sin(\mu t(x)) & \mu\sqrt{V(x)}\cos(\mu t(x)) \end{pmatrix} \phi(x).$$

We assume w.l.o.g. that I, J are open.

Let $x, y \in I$ with $y \geq x$. We calculate

$$\begin{aligned} \phi(y) - \phi(x) &= \left[\frac{1}{\mu\sqrt{V(s)}} u'(s) \begin{pmatrix} -\sin(\mu t(s)) \\ \cos(\mu t(s)) \end{pmatrix} + u(s) \begin{pmatrix} \cos(\mu t(s)) \\ \sin(\mu t(s)) \end{pmatrix} \right]_x^y \\ &= \int_x^y u'(s) \begin{pmatrix} -\sin(\mu t(s)) \\ \cos(\mu t(s)) \end{pmatrix} \mathrm{d}\big(\tfrac{1}{\mu\sqrt{V(s)}}\big) \\ &\quad + \int_x^y \frac{u''(s)}{\mu\sqrt{V(s)}} \begin{pmatrix} -\sin(\mu t(s)) \\ \cos(\mu t(s)) \end{pmatrix} + u'(s) \begin{pmatrix} -\cos(\mu t(s)) \\ -\sin(\mu t(s)) \end{pmatrix} \mathrm{d}s \\ &\quad + \int_x^y u'(s) \begin{pmatrix} \cos(\mu t(s)) \\ \sin(\mu t(s)) \end{pmatrix} + \mu\sqrt{V(s)} u(s) \begin{pmatrix} -\sin(\mu t(s)) \\ \cos(\mu t(s)) \end{pmatrix} \mathrm{d}s \\ &= \int_x^y \frac{u'(s)}{\mu} \begin{pmatrix} -\sin(\mu t(s)) \\ \cos(\mu t(s)) \end{pmatrix} \mathrm{d}\big(\tfrac{1}{\sqrt{V(s)}}\big). \end{aligned}$$

Applying the triangle inequality to the above integral we obtain

$$|\phi(y)|_2 \leq |\phi(x)|_2 + \int_{[x,y]} \frac{|u'(s)|}{\mu} \,\mathrm{d}\nu \tag{2.18}$$

where ν is the total variation of the Lebesgue-Stieltjes measure associated to $\frac{1}{\sqrt{V}}$. Note that ν is finite with $\nu(I) \leq \mathrm{Var}(\frac{1}{\sqrt{V}}, I)$. Since u' is continuous and

$$\frac{|u'(s)|}{\mu} = \sqrt{V(s)} \left| \begin{pmatrix} -\sin(\mu t(s)) \\ \cos(\mu t(s)) \end{pmatrix} \cdot \phi(s) \right| \leq \|V\|_\infty^{1/2} |\phi(s)|_2 \tag{2.19}$$

holds for $s \in I$, we can apply the Grönwall inequality from Lemma 2.A.2 to (2.18) and obtain

$$|\phi(y)|_2 \leq |\phi(x)|_2 \exp\Big(\|V\|_\infty^{1/2} \nu([x,y])\Big)$$

for $y \geq x$. Setting $E := \exp\Big(\|V\|_\infty^{1/2} \nu(I)\Big)$, we have in particular

$$|\phi(y)|_2 \leq E |\phi(x)|_2 \tag{2.20}$$

for $y \geq x$. A similar argument shows that (2.20) also holds for $y \leq x$.

Part 3: Let us now estimate the L^∞ and L^2-norm. For this, choose some $\xi \in I$. First, we have

$$\|u\|_{L^\infty(I)} \leq \max_{x\in I}|\phi(x)|_2 \leq E|\phi(\xi)|_2.$$

Now consider the L^2-norm. For $\varepsilon > 0$, define an increasing sequence of points $x_n \in \overline{J}$ by $x_0 := \inf J$ and $x_n := \sup\{x \in J : \nu((x_{n-1}, x)) \leq \varepsilon\}$ for $n \in \mathbb{N}$. If $x_n < \sup J$ holds for fixed $n \in \mathbb{N}$, we have $\nu((x_{n-1}, x_n)) \leq \varepsilon \leq \nu((x_{n-1}, x_n])$ and therefore

$$\nu(I) \geq \sum_{j=1}^{n} \nu((x_{j-1}, x_j]) \geq n\varepsilon.$$

Hence the above iteration terminates after $N \leq \left\lceil \frac{\nu(I)}{\varepsilon} \right\rceil$ steps, and thus yields a partition $\inf J = x_0 < x_1 < \cdots < x_N = \sup J$ of J with $\nu((x_{n-1}, x_n)) \leq \varepsilon$ for all n. For arbitrary $\xi_n \in (x_{n-1}, x_n)$ we calculate

$$\begin{aligned}
&\int_J |u(s)|^2 \,\mathrm{d}s \\
&\quad = \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} \left| \begin{pmatrix} \cos(\mu t(s)) \\ \sin(\mu t(s)) \end{pmatrix} \cdot \phi(s) \right|^2 \mathrm{d}s \\
&\quad \geq \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} \left| \begin{pmatrix} \cos(\mu t(s)) \\ \sin(\mu t(s)) \end{pmatrix} \cdot \phi(\xi_n) \right|^2 \mathrm{d}s - \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} (|\phi(s)|_2 + |\phi(\xi_n)|_2)|\phi(s) - \phi(\xi_n)|_2 \,\mathrm{d}s.
\end{aligned}$$

We estimate the terms separately. For $n \in \{1, \ldots, N\}$, using integration by parts we have

$$\begin{aligned}
&\int_{x_{n-1}}^{x_n} \left| \begin{pmatrix} \cos(\mu t(s)) \\ \sin(\mu t(s)) \end{pmatrix} \cdot \phi(\xi_n) \right|^2 - \tfrac{1}{2}|\phi(\xi_n)|_2^2 \,\mathrm{d}s \\
&= \frac{1}{4\mu} \left[\frac{1}{\sqrt{V(s)}} \Big(\sin(2\mu t(s)) \big(|\phi_1(\xi_n)|^2 - |\phi_2(\xi_n)|^2 \big) + 4\sin(\mu t(s))^2 \operatorname{Re}[\phi_1(\xi_n)\overline{\phi_2(\xi_n)}] \Big) \right]_{x_{n-1}}^{x_n} \\
&\quad - \frac{1}{4\mu} \int_{x_{n-1}}^{x_n} \Big(\sin(2\mu t(s)) \big(|\phi_1(\xi_n)|^2 - |\phi_2(\xi_n)|^2 \big) + 4\sin(\mu t(s))^2 \operatorname{Re}[\phi_1(\xi_n)\overline{\phi_2(\xi_n)}] \Big) \,\mathrm{d}\big(\tfrac{1}{\sqrt{V(s)}}\big).
\end{aligned}$$

This allows us to estimate

$$\begin{aligned}
\int_{x_{n-1}}^{x_n} \left| \begin{pmatrix} \cos(\mu t(s)) \\ \sin(\mu t(s)) \end{pmatrix} \cdot \phi(\xi_n) \right|^2 &\geq \frac{x_n - x_{n-1}}{2} |\phi(\xi_n)|_2^2 - \frac{3|\phi(\xi_n)|_2^2}{4\mu} \Big(2\big\| \tfrac{1}{\sqrt{V}} \big\|_\infty + \nu(J) \Big) \\
&\geq \frac{x_n - x_{n-1}}{2E^2} |\phi(\xi)|_2^2 - \frac{3E^2|\phi(\xi)|_2^2}{4\mu} \Big(2\big\| \tfrac{1}{\sqrt{V}} \big\|_\infty + \nu(J) \Big).
\end{aligned}$$

For the second set of terms, we use (2.19) to estimate

$$\begin{aligned}
&\int_{x_{n-1}}^{x_n} (|\phi(s)|_2 + |\phi(\xi_n)|_2)|\phi(s) - \phi(\xi_n)|_2 \,\mathrm{d}s \\
&\quad \leq \int_{x_{n-1}}^{x_n} \|\phi\|_\infty \left| \int_{\xi_n}^{s} \frac{u'(s)}{\mu} \begin{pmatrix} -\sin(\mu t(s)) \\ \cos(\mu t(s)) \end{pmatrix} \mathrm{d}\big(\tfrac{1}{\sqrt{V(s)}}\big) \right|_2 \mathrm{d}s \\
&\quad \leq 2|\phi(\xi)|_2 (x_n - x_{n-1}) \|V\|_\infty^{1/2} \|\phi\|_\infty^2 \nu((x_{n-1}, x_n)) \\
&\quad \leq 2\varepsilon \|V\|_\infty^{1/2} E^2 |\phi(\xi)|^2 (x_n - x_{n-1}).
\end{aligned}$$

Summing up all estimates over n, we get

$$\int_I |u(s)|^2 \,\mathrm{d}s \geq \left(\frac{|J|}{2E^2} - E^2 \Big(\frac{3N}{4\mu} \Big(2\big\| \tfrac{1}{\sqrt{V}} \big\|_\infty + \nu(J) \Big) + 2\varepsilon |J| \|V\|_\infty^{1/2} \Big) \right) |\phi(\xi)|_2^2$$

The partition of J (and therefore N) depends on ε but not on $\mu = \sqrt{\lambda}$. Therefore, choosing ε sufficiently small, the constant appearing above is positive for large λ. This shows

$$\|u\|_{L^\infty(I)} \lesssim |\phi(\xi)|_2 \lesssim \|u\|_{L^2(I)}. \qquad \square$$

for large λ, completing the proof.

Lemma 2.A.2. *Let I be an interval, $f, g\colon I \to [0, \infty)$ be maps, ν be a locally finite Borel measure on I, and $C > 0$. Assume that g is continuous with $g \le Cf$ and*

$$f(y) \le f(x) + \int_{[x,y]} g \,\mathrm{d}\nu$$

holds for all $y \ge x$. Then we have

$$f(y) \le f(x) \exp(C\nu([x, y]))$$

for all $y \ge x$.

Proof. Fix $y \ge x$ and $\varepsilon > 0$. As g is uniformly continuous on $[x, y]$, we find $\delta > 0$ such that $|a - b| \le \delta$ implies $|g(a) - g(b)| \le \varepsilon$ for $a, b \in [x, y]$. Further choose a partition $x = x_0 < x_1 < \cdots < x_n = y$ such that $|x_m - x_{m-1}| \le \delta$ for $m \in \{1, \ldots, n\}$ and $\nu(\{x_m\}) = 0$ for $m \in \{1, \ldots, n - 1\}$. We then calculate

$$\begin{aligned} f(x_m) &\le f(x_{m-1}) + \int_{[x_{m-1},x_m]} g \,\mathrm{d}\nu \\ &\le f(x_{m-1}) + (g(x_{m-1}) + \varepsilon)\nu([x_{m-1}, x_m]) \\ &\le f(x_{m-1})(1 + C\nu([x_{m-1}, x_m])) + \varepsilon\nu([x_{m-1}, x_m]). \end{aligned}$$

Inserting this inequality into itself for $m = n, \ldots, 1$, we obtain

$$\begin{aligned} f(y) &\le f(x) \prod_{m=1}^{n} (1 + C\nu([x_{m-1}, x_m])) + \varepsilon \sum_{m=1}^{n} \nu([x_{m-1}, x_m]) \prod_{j=m+1}^{n} (1 + C\nu([x_{j-1}, x_j])) \\ &\le f(x) \exp(C\nu([x, y])) + \varepsilon\nu([x, y]) \exp(C\nu([x, y])) \end{aligned}$$

and the claim follows by letting $\varepsilon \to 0$. $\square$

Remark 2.A.3. A similar, more general result, which does not require continuity of g but uses half-open integration intervals can be found in [38, Theorem 5.1].

2.B. Appendix: Examples

We analyze the spectrum of $L = -\frac{1}{V(x)}\frac{d^2}{dx^2}$ in more detail and present examples of V such that our assumptions ($\mathcal{A}$2.1)–($\mathcal{A}$2.4) hold. First we observe a qualitative result for the spectrum of L.

Lemma 2.B.1.
1. $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(-\frac{1}{V_{\text{per}}^-(x)}\frac{d^2}{dx^2}) \cup \sigma_{\text{ess}}(-\frac{1}{V_{\text{per}}^+(x)}\frac{d^2}{dx^2})$
2. *The spectral bands $\sigma_{\text{ess}}(L)$ consist of purely absolutely continuous spectrum of L and edges of the spectral bands are no eigenvalues of L.*

3. *Every gap of* $\sigma_{\mathrm{ess}}(L)$ *contains at most finitely many eigenvalues of* L.

Proof. W.l.o.g. assume $R^- < 0 < R^+$. We follow the ideas of [9] and introduce the following notation: Let L^+_{per} be a self-adjoint realization of $-\frac{1}{V^+_{\mathrm{per}}(x)}\frac{d^2}{dx^2}$ in $L^2((0,+\infty);V^+_{\mathrm{per}})$ and let L^-_{per} be a self-adjoint realization of $-\frac{1}{V^-_{\mathrm{per}}(x)}\frac{d^2}{dx^2}$ in $L^2((-\infty,0);V^-_{\mathrm{per}})$. We denote by L^+ a self-adjoint realization of $-\frac{1}{V(x)}\frac{d^2}{dx^2}|_{(0,+\infty)}$ in $L^2((0,+\infty);V)$ and by L^- a self-adjoint realization of $-\frac{1}{V(x)}\frac{d^2}{dx^2}|_{(-\infty,0)}$ in $L^2((-\infty,0);V)$.

1. By Theorem 2.1 in [9] we have $\sigma_{\mathrm{ess}}(L^\pm_{\mathrm{per}}) = \sigma_{\mathrm{ess}}(L^\pm)$ and $L^\pm$ are bounded from below. Next we show that the resolvent difference of L and $L^-\oplus L^+$ is an operator of rank at most two. Since L^+, L^- and L are bounded from below, resolvent operators exist for some constant $\kappa > 0$ large enough and we can write
$$((L+\kappa\,\mathrm{Id})^{-1} - (L^-\oplus L^+ + \kappa\,\mathrm{Id})^{-1})f = w.$$
We set $(L+\kappa\,\mathrm{Id})u = f$ and $(L^-\oplus L^+ + \kappa\,\mathrm{Id})v = f$ and hence $w = u - v$. For the restriction on $(0,\infty)$ we have $(L+\kappa\,\mathrm{Id})w|_{(0,\infty)} = 0$, therefore $w|_{(0,\infty)}$ lies in the kernel of $(L+\kappa\,\mathrm{Id})|_{(0,\infty)}$ which has dimension at most one since $L+\kappa\,\mathrm{Id}$ is of limit-point type at $+\infty$, cf. Theorem 2.3.4. The same holds true for the restriction on $(-\infty,0)$. By Corollary 11.2.3 in [27] we obtain $\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(L^-\oplus L^+) = \sigma_{\mathrm{ess}}(L^-_{\mathrm{per}}) \cup \sigma_{\mathrm{ess}}(L^+_{\mathrm{per}})$.
2. See the proof of Theorems 1.1-1-3 in [9].
3. By Theorem 2.3 from [9] we know that every gap of $\sigma_{\mathrm{ess}}(L^\pm)$ contains at most finitely many eigenvalues of $L^\pm$ and therefore also every gap of $\sigma_{\mathrm{ess}}(L^-\oplus L^+)$ contains at most finitely many eigenvalues of $L^-\oplus L^+$. Since the resolvent difference of L and $L^-\oplus L^+$ is an operator of rank at most two we conclude by Theorem 3, Chapter 9.3 in [13] that in each gap of $\sigma_{\mathrm{ess}}(L)$ the operator L gains at most two more eigenvalues compared to the finitely many eigenvalues of $L^-\oplus L^+$ in each gap of $\sigma_{\mathrm{ess}}(L^-\oplus L^+)$. □

2.B.1. Purely periodic case

As potential V we consider a positive periodic step function V_{per} given as follows: take a partition $0 = \theta_0 < \theta_1 < \ldots < \theta_N = 1$ of the interval $[0,1]$ and positive values $a_1,\ldots,a_N > 0$ to define

$$V_{\mathrm{per}}(x) = a_i \text{ for } x \in [\theta_{i-1}X, \theta_i X) \text{ and } i = 1,\ldots,N \tag{2.21}$$

and extend V_{per} periodically to the real line with period X. First, we note that our assumptions $(\mathcal{A}2.1)$ and $(\mathcal{A}2.2)$ are satisfied by definition of V_{per}. Moreover, by Theorem 5.3.1 in [36], $\sigma_p(L) = \emptyset$ and hence $(\mathcal{A}2.4)$ is fulfilled.

Lemma 2.B.2. *Let* $q_i := \sqrt{a_i}(\theta_i - \theta_{i-1})X$ *for* $i = 1,\ldots,N$. *Assume that there is* $T > 0$ *such that*

$$4q_i \in T\mathbb{N}, \quad i = 1,\ldots,N$$

and suppose that

$$4q_{i_j} \in T\mathbb{N}_{\mathrm{odd}}$$

is satisfied for an even number of indices $1 \le i_1 < i_2 < \ldots < i_{2m} \le N$ *and no others. If moreover*

$$\alpha := \frac{a_{i_1}a_{i_3}\cdot\ldots\cdot a_{i_{2m-1}}}{a_{i_2}a_{i_4}\cdot\ldots\cdot a_{i_{2m}}} \neq 1 \tag{2.22}$$

then $(\mathcal{A}2.3)$ *is satisfied for* $\omega = \frac{2\pi}{T}$ *and* $V = V_{\mathrm{per}}$ *from above.*

Proof. We denote by $P_i(\lambda)$ the weighted monodromy matrix such that any solution of $-u'' = \lambda a_i u$ on $[\theta_{i-1}X, \theta_i X]$ satisfies

$$\begin{pmatrix} \sqrt{\lambda}u(\theta_i X) \\ u'(\theta_i X) \end{pmatrix} = P_i(\lambda) \begin{pmatrix} \sqrt{\lambda}u(\theta_{i-1} X) \\ u'(\theta_{i-1} X) \end{pmatrix}. \tag{2.23}$$

It is given by

$$P_i(\lambda) = \begin{pmatrix} \cos(\sqrt{\lambda}q_i) & \frac{1}{\sqrt{a_i}}\sin(\sqrt{\lambda}q_i) \\ -\sqrt{a_i}\sin(\sqrt{\lambda}q_i) & \cos(\sqrt{\lambda}q_i) \end{pmatrix} \tag{2.24}$$

and as a function of $\sqrt{\lambda}$ it is $\frac{2\pi}{q_i}$-periodic. Since $q_i \in \frac{T}{4}\mathbb{N}$ it is in particular also $\frac{8\pi}{T}$-periodic. The weighted monodromy matrix $P(\lambda; V_{\text{per}})$ of the full problem $-u'' = \lambda V_{\text{per}}(x)u$ on $[0, X]$ is then given by

$$P(\lambda; V_{\text{per}}) = P_N(\lambda) \cdot \ldots \cdot P_1(\lambda),$$

which is $4\omega = \frac{8\pi}{T}$-periodic as a function of $\sqrt{\lambda}$. Following Chapter 1 and Section 2.1 in [36] we know that

$$\sigma\left(-\frac{1}{V_{\text{per}}(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) = \{\lambda \in \mathbb{R} \colon |\operatorname{tr} P(\lambda; V_{\text{per}})| \le 2\}$$

where we use that the weighted monodromy matrix $P(\lambda; V_{\text{per}})$ and the standard monodromy matrix are similar and therefore have the same eigenvalues and the same trace. In order to check that ($\mathcal{A}$2.3) holds, the 4ω-periodicity w.r.t. $\sqrt{\lambda}$ implies that it suffices to check that $|\operatorname{tr} P(\omega^2; V_{\text{per}})|, |\operatorname{tr} P(9\omega^2; V_{\text{per}})| > 2$ since $\omega, 3\omega$ are the only odd multiples of ω in the periodicity cell $[0, 4\omega]$. If we insert $k^2\omega^2$, $k \in \mathbb{N}_{\text{odd}}$ into $P_i(\lambda)$ then the assumptions yield that

$$P_i(k^2\omega^2) \in \left\{\pm\begin{pmatrix} 0 & \frac{1}{\sqrt{a_i}} \\ -\sqrt{a_i} & 0 \end{pmatrix}\right\} \text{ if } 4q_i \in T\mathbb{N}_{\text{odd}} \quad \text{and} \quad P_i(k^2\omega^2) = \{\pm \operatorname{Id}\} \text{ if } 4q_i \in T\mathbb{N}_{\text{even}}.$$

By definition of the indices $i_1, \ldots, i_{2m}$ we have

$$P_{i_{j+1}}(k^2\omega^2)P_{i_j}(k^2\omega^2) \in \left\{\begin{pmatrix} \sqrt{\frac{a_{i_j}}{a_{i_{j+1}}}} & 0 \\ 0 & \sqrt{\frac{a_{i_{j+1}}}{a_{i_j}}} \end{pmatrix}\right\}$$

and obtain

$$P(k^2\omega^2; V_{\text{per}}) \in \left\{\pm\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} \end{pmatrix}\right\} \tag{2.25}$$

so that $|\operatorname{tr} P(k^2\omega^2; V_{\text{per}})| > 2$ by (2.22). Therefore assumption ($\mathcal{A}$2.3) holds. □

Lemma 2.B.3. *Let* $q_i := \sqrt{a_i}(\theta_i - \theta_{i-1})X$ *for* $i = 1, \ldots, N$ *be pairwise rational multiples of one another with greatest common divisor* $q := \gcd(q_1, \ldots, q_N)$ *defined as the largest positive number such that all* q_i *are integer multiples of* q. *If* $\frac{q_i}{q} \in \mathbb{N}_{\text{odd}}$ *is satisfied for an even number of indices* $1 \le i_1 < i_2 < \ldots < i_{2m} \le N$ *and no others, and additionally*

$$\frac{a_{i_1}a_{i_3} \cdot \ldots \cdot a_{i_{2m-1}}}{a_{i_2}a_{i_4} \cdot \ldots \cdot a_{i_{2m}}} \neq 1$$

holds then the assumptions of Lemma 2.B.2 hold with $T = 4\frac{q}{k}$ *for any* $k \in \mathbb{N}_{\text{odd}}$.

Remark 2.B.4. One can check that the conditions of Lemma 2.B.3 are not only sufficient but also necessary for Lemma 2.B.2.

Proof of Lemma 2.B.3. The condition $4q_i \in T\mathbb{N}$ for all $\mathbb{N}$ means that $\frac{T}{4}$ is a common divisor of the q_i, and is therefore equivalent to $\frac{T}{4} = \frac{q}{k}$ for some $k \in \mathbb{N}$. We now have to check condition (2.22), where we have

$$4q_i \in T\mathbb{N}_{\text{odd}} \iff k\frac{q_i}{q} \in \mathbb{N}_{\text{odd}} \iff k \in \mathbb{N}_{\text{odd}} \text{ and } \frac{q_i}{q} \in \mathbb{N}_{\text{odd}}.$$

Thus, for even k we have $m = 0$ in Lemma 2.B.2 so that (2.22) is false, whereas for odd k the indices $i_1, \ldots, i_{2m}$ of Lemma 2.B.2 coincide with the indices of the current lemma, so that (2.22) holds by assumption. □

Remark 2.B.5 (Two-step and three-step potentials). Let $N = 2$ and assume that $a_1, a_2 > 0$, $0 = \theta_0 < \theta_1 < \theta_2 = 1$ satisfy the assumptions from Lemma 2.B.3. This means that $\frac{q_2}{q_1} = \frac{r}{s}$ with $r, s \in \mathbb{N}_{\text{odd}}$ coprime and $T \in \frac{4q}{\mathbb{N}_{\text{odd}}}$ with the greatest common divisor $q = \frac{q_1}{s} = \frac{q_2}{r}$. These conditions coincide with those stated in [49] on admissible ranges of breather frequencies and material parameters for the case $X = 2\pi$.

In the case $N = 3$, assuming $a_1, a_2, a_3 > 0$ and $0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 = 1$ satisfying the conditions from Lemma 2.B.3 means (up to a permutation of the q_i) that $\frac{q_2}{q_1} = \frac{r}{s}$ with $r, s \in \mathbb{N}_{\text{odd}}$ coprime, $\frac{q_3}{q_1} = \frac{\tilde{r}}{\tilde{s}}$ with $\tilde{r} \in \mathbb{N}_{\text{even}}$, $\tilde{s} \in \mathbb{N}_{\text{odd}}$ coprime and $a_1 \neq a_2$. Further, the greatest common divisor is given by $q = \gcd(q_1, q_2, q_3) = \frac{q_1}{\operatorname{lcm}(s, \tilde{s})}$ where lcm denotes the least common multiple.

2.B.2. Perturbed periodic case

Next, we want to analyze the spectrum of L when V is a perturbed periodic potential and focus on two different cases.

2.B.2.1. Interface of dislocated periodic potentials

Let V_{per} be given by (2.21) and for $V_0, d > 0$ consider

$$V(x) = \begin{cases} V_{\text{per}}(x), & x < 0, \\ V_0, & 0 \le x < d, \\ V_{\text{per}}(x - d), & d \le x. \end{cases}$$

We observe that our assumptions ($\mathcal{A}$2.1) and ($\mathcal{A}$2.2) hold by definition.

Lemma 2.B.6. *Assume that V_{per} satisfies the assumptions of Lemma 2.B.2 and that for the same value of T we have $4q_0 \in T\mathbb{N}_{\text{even}}$ with $q_0 := \sqrt{V_0}d$. Then ($\mathcal{A}$2.3) and ($\mathcal{A}$2.4) hold.*

We state the proof together with the proof of Lemma 2.B.7 since they are based on the same idea.

2.B.2.2. Interface of periodic potentials

Let V_{per}^+ and V_{per}^- be different periodic potentials given by (2.21). Consider now the case

$$V(x) = \begin{cases} V_{\text{per}}^-(x), & x < 0, \\ V_{\text{per}}^+(x), & x \geq 0. \end{cases}$$

Note that assumptions ($\mathcal{A}2.1$) and ($\mathcal{A}2.2$) hold by definition.

Lemma 2.B.7. *Assume that $V_{\text{per}}^{\pm}$ both satisfy the assumptions of Lemma 2.B.2 for the same value of T with values $\alpha^{\pm}$ from* (2.22). *If additionally $\alpha^+, \alpha^- > 1$ or $\alpha^+, \alpha^- < 1$ then ($\mathcal{A}2.3$) and ($\mathcal{A}2.4$) hold.*

Proof. Observe first that according to Lemma 2.B.1 and Lemma 2.B.2 we know that ($\mathcal{A}2.3$) holds for the essential spectrum of L and hence it remains to verify ($\mathcal{A}2.3$) and ($\mathcal{A}2.4$) for the eigenvalues of

$$-u'' = \lambda V(x) u \text{ for } x \in \mathbb{R}.$$

In the following we use from Lemma 2.B.2 that the square root of the spectrum of L_{per} is periodic with period $4\omega = \frac{8\pi}{T}$.

Interface of dislocated periodic potentials. In analogy to (2.23) we define the weighted propagation matrix $P_0(\lambda)$ for solutions of $-u'' = \lambda V_0 u$ on $[0, d]$. It takes the form as in (2.24) and hence, as a function of $\sqrt{\lambda}$, it is periodic with period $\frac{2\pi}{q_0}$. By the assumption $4q_0 \in T\mathbb{N}_{\text{even}}$ it is co-periodic to the propagation matrix $P(\lambda; V_{\text{per}})$ of the periodic potential V_{per}, which has period $4\omega = \frac{8\pi}{T}$.

Let us consider a value $\lambda \in \sigma_p(L)$. By Lemma 2.B.1 we have $\lambda \notin \sigma_{\text{ess}}(L)$ and hence $|\operatorname{tr} P(\lambda; V_{\text{per}})| > 2$. Then, $P(\lambda; V_{\text{per}})$ has two distinct real eigenvalues $\rho(\lambda), \tilde{\rho}(\lambda)$ with $|\rho(\lambda)| < 1 < |\tilde{\rho}(\lambda)|$ and corresponding eigenvectors $v(\lambda), \tilde{v}(\lambda) \in \mathbb{R}^2$. We note that $\rho(\lambda)$, $\tilde{\rho}(\lambda)$, $\mathbb{R}v(\lambda)$, $\mathbb{R}\tilde{v}(\lambda)$ are 4ω-periodic as functions of $\sqrt{\lambda}$ (inside the resolvent set of L_{per}). If $\lambda \in \sigma_p(L)$ is an eigenvalue with $L^2(\mathbb{R})$-eigenfunction ϕ, we necessarily have $(\sqrt{\lambda}\phi(0), \phi'(0)) \in \mathbb{R}\tilde{v}(\lambda)$ and $(\sqrt{\lambda}\phi(d), \phi'(d)) \in \mathbb{R}v(\lambda)$ and therefore $P_0(\lambda)\tilde{v}(\lambda) \in \mathbb{R}v(\lambda)$. Since this condition is also 4ω-periodic as a function of $\sqrt{\lambda}$ we see that $\sigma_p(L)$ has the same finite number of eigenvalues in every interval of length 4ω. This already implies ($\mathcal{A}2.4$). For ($\mathcal{A}2.3$) we only need to check that $k^2\omega^2 \notin \sigma_p(L)$ for $k \in \mathbb{N}_{\text{odd}}$. From (2.25) we see by the structure of the propagation matrices that

$$\alpha > 1 \implies v(k^2\omega^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{v}(k^2\omega^2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\alpha < 1 \implies v(k^2\omega^2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{v}(k^2\omega^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

up to rescaling of the eigenvectors. Since $P_0(k^2\omega^2) \in \{\pm \operatorname{Id}\}$, we get $k^2\omega^2 \notin \sigma_p(L)$ and hence also ($\mathcal{A}2.3$) holds.

Interface of periodic potentials. The considerations are similar to the previous case. As before, we denote by $P(\lambda; V_{\text{per}}^{\pm})$ the propagation matrix for $V_{\text{per}}^{\pm}$. By our assumption, both have the same period $4\omega = \frac{8\pi}{T}$. For $\lambda \notin \sigma_{\text{ess}}(L)$, the matrix $P(\lambda; V_{\text{per}}^{\pm})$ has eigenvalues $\rho^{\pm}(\lambda), \tilde{\rho}^{\pm}(\lambda)$ with eigenvectors $v^{\pm}(\lambda), \tilde{v}^{\pm}(\lambda)$. The eigenpair $(\rho^+(\lambda), v^+(\lambda))$ generates a solution on $[0, \infty)$

decaying to 0 at $+\infty$ and $(\tilde{\rho}^-(\lambda), \tilde{v}^-(\lambda))$ generates a solution on $(-\infty, 0]$ decaying to 0 at $-\infty$. Therefore, the eigenvalue condition is given by

$$\lambda \in \sigma_p(L) \iff \tilde{v}^-(\lambda) \in \mathbb{R}v^+(\lambda).$$

Recall that at $\lambda = k^2\omega^2$ we have

$$\alpha^+ > 1 \implies v^+(k^2\omega^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \alpha^+ < 1 \implies v^+(k^2\omega^2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\alpha^- > 1 \implies \tilde{v}^-(k^2\omega^2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha^- < 1 \implies \tilde{v}^-(k^2\omega^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using our assumption that $\alpha^+, \alpha^- > 1$ or $\alpha^+, \alpha^- < 1$ we conclude $k^2\omega^2 \notin \sigma_p(L)$ and hence ($\mathcal{A}$2.3) and ($\mathcal{A}$2.4) holds as seen in the previous case. □

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CHAPTER 3

Breather solutions to nonlinear Maxwell equations with retarded material laws

Abstract. We show existence of breather solutions to nonlinear Maxwell's equations using variational methods. We consider Kerr-type optical materials in slab waveguide geometries, which are magnetically inactive while the electric response is the sum of a linear and a cubic term. Both terms are inhomogeneous and have bounded coefficients. In addition, the cubic term is temporally retarded while the linear term has instantaneous and retarded contributions. We consider breather solutions which are time-periodic, real-valued, transverse electric polarized waves traveling along one direction of the slab waveguide. Moreover, they are localized in the direction perpendicular to the slab and are polychromatic functions.

For carefully chosen material coefficients, the indefinite linear operator of the second-order formulation of Maxwell's equations has a spectral gap about 0. This allows us to find breather solutions variationally using the mountain pass method for the dual problem. Our method works for the following two types of materials: (1) a periodic linear response and perturbed periodic nonlinear response, or (2) an interface between two materials, each with periodic linear response, and a nonlinearity that decays away from the interface.

3.1. Introduction and main results

We consider Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} &= -\mathbf{B}_t, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} &= \mathbf{D}_t \end{aligned} \tag{3.1}$$

in $\mathbb{R}^3$ without changes and currents, and for materials of a slab waveguide geometry. We investigate existence of breather solutions traveling parallel to the waveguide. For the underlying material, we assume the constitutive relations

$$\mathbf{B} = \mu_0 \mathbf{H}, \qquad \mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \mathbf{P}(\mathbf{E}) \tag{3.2}$$

where $\mu_0, \epsilon_0 > 0$ denote vacuum permeability and vacuum permittivity. So we consider a material that is magnetically inactive and electrically active, with an electric displacement field $\mathbf{D}$ that depends nonlinearly on the electric field $\mathbf{E}$ through the polarization $\mathbf{P}(\mathbf{E})$. We consider Kerr optical materials modelled by $\mathbf{P}(\mathbf{E})$ consisting of a linear plus a cubic term of $\mathbf{E}$: the quadratic term is zero for silica glasses, and higher-order terms are omitted (cf. [3]). More precisely, we assume that the polarization is given either by

$$\mathbf{P}(\mathbf{E})(x, y, z, t) = \int_0^\infty g(x, \tau)\mathbf{E}(x, y, z, t - \tau)\,\mathrm{d}\tau + h(x) \int_0^\infty \nu(\tau)\mathbf{E}(x, y, z, t - \tau)^3\,\mathrm{d}\tau \tag{3.3.i}$$

where we abbreviated $\mathbf{E}^3 := |\mathbf{E}|^2\mathbf{E}$, or by

$$\mathbf{P}(\mathbf{E})(x,y,z,t) = \int_0^\infty g(x,\tau)\mathbf{E}(x,y,z,t-\tau)\,\mathrm{d}\tau + h(x)\Big(\int_0^\infty \nu(\tau)\mathbf{E}(x,y,z,t-\tau)\,\mathrm{d}\tau\Big)^3. \tag{3.3.ii}$$

Taking the curl of Faraday's law $\nabla\times\mathbf{E} = -\mathbf{B}_t$, one obtains from (3.1) and (3.2) the second order Maxwell's equation

$$\nabla\times\nabla\times\mathbf{E} + \epsilon_0\mu_0\partial_t^2(\mathbf{E}+\mathbf{P}(\mathbf{E})) = 0. \tag{3.4}$$

There are many results on breathers for nonlinear wave-type equations like (3.4) in the literature. Monochromatic breathers, which are given by $\mathbf{E}(x,y,z,t) = \mathrm{Re}[\mathcal{E}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}]$ with frequency $\omega > 0$ and profile $\mathcal{E}\colon \mathbb{R}^3 \to \mathbb{C}^3$, reduce (3.4) to the elliptic problem

$$\nabla\times\nabla\times\mathcal{E} + (\chi_1(x,y,z) + \chi_3(x,y,z)|\mathcal{E}|^2)\mathcal{E} = 0, \tag{3.5}$$

with appropriate functions χ_1, χ_3 derived from (3.4), by neglecting higher order harmonics, i.e., terms proportional to $\mathrm{e}^{\pm 3\mathrm{i}\omega t}$ in $\mathbf{P}(\mathbf{E})$. Alternatively, higher order harmonics vanish if the nonlinear part of the polarization is given by the time-average

$$\mathbf{P}_{\mathrm{NL}}(\mathbf{E})(x,y,z,t) = h(x,y,z)\int_0^T |\mathbf{E}(x,y,z,\tau)|^2\,\mathrm{d}\tau\mathbf{E}(x,y,z,t),$$

where $T := \frac{2\pi}{\omega}$. Saturated nonlinearities

$$\nabla\times\nabla\times\mathcal{E} + \chi(x,y,z,|\mathcal{E}|^2)\mathcal{E} = 0,$$

which are asymptotically linear as $|\mathcal{E}| \to \infty$, are also of interest. These were considered in a series of papers [87–93] by Stuart and Zhou. The authors considered transverse electric (TE) or transverse magnetic (TM) polarized waves in cylindrically symmetric waveguides, which reduce (3.5) to a one-dimensional scalar equation that can, e.g., be treated variationally. More general nonlinearities χ, including also power nonlinearities were investigated in [5, 8, 10, 42] for cylindrically or spherically symmetric solutions. The restriction to symmetric solutions can be overcome using a Helmholtz decomposition to deal with the kernel of the curl-curl-operator. This was investigated by Mederski et al. in a series of papers [63–66]. Mandel combined the dual variational method with Helmholtz decomposition in [57], and considered a spatially nonlocal nonlinearity in [58]. In [31, 32] Dohnal and Romani obtained breathers by bifurcation from a simple eigenvalue of the linear problem. For further results on monochromatic Maxwell equations we refer to the survey paper [7].

We move to the topic of polychromatic breathers, which have multiple (usually infinitely many) supported frequencies and are in general given by

$$\mathbf{E}(x,y,z,t) = \sum_{k\in\mathbb{Z}} \mathcal{E}_k(x,y,z)\mathrm{e}^{\mathrm{i}k\omega t}.$$

Let us first discuss the instantaneous polarization $\mathbf{P}_{\mathrm{NL}}(E) = h(x,y,z)\mathbf{E}^3$. Here, the authors of [34] considered breathers at an interface between two dielectrics and showed that these can be approximated on large but finite time scales by solutions of an amplitude equation, the nonlinear Schrödinger equation. Existence of true time-periodic solutions was shown in [49] and [19] for materials where the linear or nonlinear part of the polarization consists of Dirac measures in space and using variational methods or bifurcation theory, respectively. In

[71] we considered bounded material coefficients with spatially localized nonlinear interaction, and showed existence of breathers variationally for instantaneous as well as time-averaged polarizations. Breathers for the temporally retarded polarizations (3.3.i) and (3.3.ii) were obtained variationally in our recent paper [72] using the mountain pass method.

The main difference of this paper compared with [72] is that we consider breathers traveling at high velocities, and we only work with materials of slab geometry.

We solve (3.4) using the ansatz

$$\mathbf{E}(x,y,z,t) = w(x, t - \frac{1}{c}z) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3.6}$$

of a TE-polarized wave traveling with speed c in z-direction. The ansatz is divergence-free, so the curl-curl operator simplifies to $\nabla \times \nabla \times \mathbf{E} = -\Delta \mathbf{E}$. Normalizing the speed of light in vacuum to $c_0 \coloneqq (\epsilon_0 \mu_0)^{-\frac{1}{2}} = 1$ and inserting into (3.4), we obtain

$$-\partial_x^2 w - \frac{1}{c^2}\partial_t^2 w + \partial_t^2 (w + P(w)) = 0 \tag{3.7}$$

where the scalar polarization $P(w)$ is given by

$$P(w)(x,t) = \int_0^\infty g(x,\tau) w(x, t-\tau) \,\mathrm{d}\tau + h(x) \int_0^\infty \nu(\tau) w(x,t-\tau)^3 \,\mathrm{d}\tau \tag{3.8.i}$$

or

$$P(w)(x,t) = \int_0^\infty g(x,\tau) w(x, t-\tau) \,\mathrm{d}\tau + h(x) \Big(\int_0^\infty \nu(\tau) w(x,t-\tau) \,\mathrm{d}\tau \Big)^3, \tag{3.8.ii}$$

corresponding to (3.3.i) and (3.3.ii), respectively. To include instantaneous linear material responses, we assume a decomposition

$$g(x,\tau) = g_0(x)\delta_0(\tau) + g_1(x,\tau) \tag{3.9}$$

with δ_0 being the Dirac measure at 0 and g_0, g_1 bounded. On the other hand, we assume that the nonlinear material response has no instantaneous contribution.

We show existence of breather solutions which solve Maxwell's equations pointwise and are infinitely differentiable in time. The precise definition is given next.

Definition 3.1.1. *We call* $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H} \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ breather solutions *to Maxwell's equations with polarization* (3.3.i) [*or* (3.3.ii)] *with time-period* $T > 0$ *traveling with speed* c *in* z*-direction if each field* $\mathbf{F} \in \{\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}\}$ *satisfies*

$$\mathbf{F}(x,y,z,t+T) = \mathbf{F}(x,y,z,t) = \mathbf{F}(x,y,z+c\tau,t+\tau)$$

and if for all domains of the form $\Omega = \mathbb{R} \times [y_1,y_2] \times [z_1,z_2] \times [t_1,t_2]$ *it has the regularity*

$$\partial_t^n \partial_x^m \mathbf{F} \in L^2(\Omega;\mathbb{R}^3) \cap L^\infty(\Omega;\mathbb{R}^3)$$

for $n \in \mathbb{N}_0$ *and* $m \in \{0, \dots, \overline{m}(\mathbf{F})\}$ *where* $\overline{m}(\mathbf{E}) = 2, \overline{m}(\mathbf{D}) = 0, \overline{m}(\mathbf{B}) = \overline{m}(\mathbf{H}) = 1$. *Moreover, we require* (3.1), (3.2), *and* (3.3.i) [*or* (3.3.ii)] *to hold pointwise almost everywhere.*

Next we give two examples of material parameters g_0, g_1, h, ν for which we can show existence of breather solutions. In the first example, we consider a spatially periodic linear material response, i.e., g_0 and g_1 are periodic in x.

Theorem 3.1.2. *Let $c \in (0,\infty), \theta \in (0,1)\setminus\{\frac{1}{2}\}, T, X > 0$, let $g_1^{\mathrm{per}}, h^{\mathrm{per}}, h^{\mathrm{loc}} \in L^\infty(\mathbb{R};\mathbb{R})$ such that $g_1^{\mathrm{per}}, h^{\mathrm{per}}$ are X-periodic, h^{per} is positive almost everywhere, $h^{\mathrm{loc}} \geq 0$ and $h^{\mathrm{loc}}(x) \to 0$ as $|x| \to \infty$. We set $\omega := \frac{2\pi}{T}$ and define potentials g_0, g_1, h, ν by*

$$\begin{aligned} g_0(x) := g_0(x;\theta,X) &:= \frac{1}{c^2} - 1 + \begin{cases} \frac{T^2}{16\theta^2 X^2}, & x \in (0,\theta X) + X\mathbb{Z}, \\ \frac{T^2}{16(1-\theta)^2 X^2}, & x \in (\theta X, X) + X\mathbb{Z}, \end{cases} \\ g_1(x,t) &:= g_1^{\mathrm{per}}(x)\cos(\omega t)|\cos(\omega t)|\mathbb{1}_{[0,T]}(t), \\ h(x) &:= h^{\mathrm{per}}(x) + h^{\mathrm{loc}}(x), \\ \nu(t) &:= \operatorname{dist}(t, T\mathbb{Z})\mathbb{1}_{[0,T]}(t) \end{aligned}$$

for $x, t \in \mathbb{R}$. Then for polarization (3.3.i) *as well as polarization* (3.3.ii)*, there exist infinitely many distinct breather solutions with period T and speed c in the sense of Definition 3.1.1.*

In the second example, we consider a linear material response that is spatially periodic on each halfspace, and a localized nonlinear response.

Theorem 3.1.3. *Let $c \in (0,\infty), \theta^-, \theta^+ \in (0,\frac{1}{2})$ and $T, X^-, X^+ > 0$, let $g_1^{\mathrm{per}}, h^{\mathrm{loc}} \in L^\infty(\mathbb{R};\mathbb{R})$ such that g_1^{per} is X^--periodic on $(-\infty,0)$ and X^+-periodic on X^+, h^{loc} is almost everywhere positive and $h^{\mathrm{loc}}(x) \to 0$ as $|x| \to \infty$. Set $\omega := \frac{2\pi}{T}$ and define g_0, g_1, h, ν by*

$$\begin{aligned} g_0(x) &= \begin{cases} g_0(x;\theta^-,X^-), & x < 0, \\ g_0(x;\theta^+,X^+), & x > 0, \end{cases} \\ g_1(x,t) &= g_1^{\mathrm{per}}(x)\cos(\omega t)|\cos(\omega t)|\mathbb{1}_{[0,T]}(t), \\ h(x) &= h^{\mathrm{loc}}(x), \\ \nu(t) &= \operatorname{dist}(t, T\mathbb{Z})\mathbb{1}_{[0,T]}(t) \end{aligned}$$

for $x, t \in \mathbb{R}$, where $g_0(x;\theta,X)$ is defined in Theorem 3.1.2. Then for (3.3.i) *as well as* (3.3.ii) *there exist infinitely many distinct breather solutions with period T and speed c in the sense of Definition 3.1.1.*

Let us prepare the main theorem. Recall (3.7), which using (3.8.i), (3.8.ii), and (3.9) we rewrite as

$$\left[-\partial_x^2 + (1 - \tfrac{1}{c^2} + g_0(x))\partial_t^2\right] w + \partial_t^2(g_1 * w + P_{\mathrm{NL}}(w)) = 0 \tag{3.10}$$

where $*$ denotes convolution in time and P_{NL} is the nonlinear part of the scalar polarization. We consider velocities c that are so large that the potential

$$V(x) := 1 - \frac{1}{c^2} + g_0(x) \tag{3.11}$$

is positive, hence the linear operator $-\partial_x^2 + V(x)\partial_t^2$ is hyperbolic. It is convenient to divide this operator by $V(x)$ and instead consider $L + \partial_t^2$ with weighted Sturm-Liouville operator

$$L := -\frac{1}{V(x)}\partial_x^2. \tag{3.12}$$

The advantage of this representation is that the spectrum of $L + \partial_t^2$ restricted to time-periodic functions is easier to compute since L acts only on x, and therefore we simply have $\sigma(L + \partial_t^2) = \overline{\sigma(L) + \sigma(\partial_t^2)}$.

Let us fix the period $T > 0$ of the breather and consider as time-domain the set $\mathbb{T} := \mathbb{R}/_{T\mathbb{Z}}$. On it, ∂_t^2 has discrete spectrum $\sigma(\partial_t^2) = \{-\omega^2 k^2 \colon k \in \mathbb{Z}\}$ with $\omega := \frac{2\pi}{T}$ denoting the base frequency. Our analysis strongly uses the assumption that L has a spectral gap about $\omega^2 k^2$ for each $k \in \mathbb{Z}_{\text{odd}}$. The restriction to odd frequencies means we consider functions that are $\frac{T}{2}$-antiperiodic in time. It is helpful since for $k = 0$ we always have $0 \in \sigma(L)$, $\frac{T}{2}$-antisymmetry is compatible with (3.10), it still yields a variational problem, and it allows us to invert the operator $L + \partial_t^2$ after restricting to $\frac{T}{2}$-antiperiodic functions in time. The operator L having countably many specific spectral gaps requires a careful choice of the potential g_0. In Theorems 3.1.2 and 3.1.3 this is satisfied, and moreover the size of the spectral gaps grows linearly in $|k|$.

Remark 3.1.4. In Appendix 2.B there are further examples of potentials g_0, for which the results of Theorems 3.1.2 and 3.1.3 remain valid. As an example, in Theorem 3.1.2 one can consider

$$g_0(x) = \frac{1}{c^2} - 1 + \begin{cases} \frac{m^2 T^2}{16\theta^2 X^2}, & x \in (0, \theta X) + X\mathbb{Z}, \\ \frac{n^2 T^2}{16(1-\theta)^2 X^2}, & x \in (\theta X, X) + X\mathbb{Z}, \end{cases}$$

for $m, n \in \mathbb{N}_{\text{odd}}$ and $\theta \in (0,1)$, where instead of $\theta \neq \frac{1}{2}$ we require $g_0 \neq \text{const}$. More generally, one can consider a periodic arrangement of three or more step potentials under similar assumptions on the step heights and widths. There we also give examples that fit to Theorem 3.1.3.

Lastly, let us fix some notation for the torus $\mathbb{T}$. It is equipped with the Haar measure $\mathrm{d}t$ normalized such that $\int_{\mathbb{T}} 1 \,\mathrm{d}t = 1$. We denote the standard orthonormal basis on $\mathbb{T}$ by $e_k(t) := \mathrm{e}^{\mathrm{i}k\omega t}$. Accordingly, the Fourier coefficients of a function $\varphi \colon \mathbb{T} \to \mathbb{C}$ are given by $\hat{\varphi}_k = \mathcal{F}_k[\varphi] = \int_{\mathbb{T}} \varphi \overline{e_k} \,\mathrm{d}t$, and the inverse is $\varphi(t) = \mathcal{F}^{-1}[\hat{\varphi}_k](t) = \sum_{k \in \mathbb{Z}} \hat{\varphi}_k e_k(t)$. If φ depends on space and time, $\hat{\varphi}$ will always denote its temporal Fourier transform. Moreover, given a function f on $\mathbb{R}$, we define its periodization by $\text{Per}[f](t) = T \sum_{k \in \mathbb{Z}} f(t + kT)$ for $t \in \mathbb{T}$. Note that $f *_{\mathbb{R}} \varphi := \text{Per}[f] *_{\mathbb{T}} \varphi$ holds whenever φ is T-periodic.

Finally, we present our main existence result on breather solutions to Maxwell's equations (3.1) and (3.2) with polarization (3.3.i) or (3.3.ii).

Theorem 3.1.5. *Let $T > 0$ be the period of the breather, $\omega := \frac{2\pi}{T}$ be its frequency and $c \in (0, \infty)$ be its speed. Assume that for constants $\alpha > 1$, $\frac{1}{2} \leq \beta < 2, \gamma \leq 1$, $0 < d < \delta$ we have:*

- ($\mathcal{A}3.1$) $h \in L^\infty(\mathbb{R}; (0, \infty))$.
- ($\mathcal{A}3.2$) $\nu \in L^1(\mathbb{R}; \mathbb{R})$ *and its periodization* $\mathcal{N} := \text{Per}[\nu]$ *is even. Denoting its Fourier support restricted to odd frequencies by* $\mathfrak{R} := \left\{k \in \mathbb{Z}_{\text{odd}} \colon \hat{\mathcal{N}}_k \neq 0\right\}$, *we have* $\mathfrak{R} \neq \emptyset$ *and* $|\hat{\mathcal{N}}_k| \lesssim |k|^{-\alpha}$ *for all* $k \in \mathfrak{R}$.
- ($\mathcal{A}3.3$) $g_0 \in L^\infty(\mathbb{R}; \mathbb{R})$ *satisfies* $\operatorname{ess\,inf} g_0 > \frac{1}{c^2} - 1$ *and is locally of bounded variation.*
- ($\mathcal{A}3.4$) *For the spectrum of the operator* $L \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ *defined in* (3.12) *we have:*
  - $(\omega^2 k^2 - \delta|k|^\gamma, \omega^2 k^2 + \delta|k|^\gamma) \subseteq \rho(L)$ *holds for all* $k \in \mathfrak{R}$.
  - *The point spectrum* $\sigma_p(L)$ *satisfies* $\sum_{\lambda \in \sigma_p(L)} \lambda^{-\beta-\varepsilon} < \infty$ *for all* $\varepsilon > 0$.
- ($\mathcal{A}3.5$) $\alpha + \gamma - 2 > \beta$.
- ($\mathcal{A}3.6$) $g_1 \in L^\infty_x(\mathbb{R}; L^1_t(\mathbb{R}; \mathbb{R}))$ *and the periodization* $\mathcal{G}(x) := \text{Per}[g_1(x; \cdot)]$ *is even in* t *and satisfies* $|\hat{\mathcal{G}}_k(x)| \leq \frac{d}{\omega^2 |k|^{2-\gamma}} V(x)$ *for all* $x \in \mathbb{R}$, $k \in \mathfrak{R}$.

($\mathcal{A}$3.7) *If the polarization is* (3.3.ii)*, assume that* $|\hat{\mathcal{N}}_k| \gtrsim |k|^{-s}$ *holds for* $k \in \mathfrak{R}$ *and for some* $s \in \mathbb{R}$*. In addition, similar to (*$\mathcal{A}$*3.4) and (*$\mathcal{A}$*3.6) we then require that there exist constants* $\tilde{\gamma} \le 1$, $0 < \tilde{d} < \tilde{\delta}$ *with*

$$(\omega^2 k^2 - \tilde{\delta}|k|^{\tilde{\gamma}}, \omega^2 k^2 + \tilde{\delta}|k|^{\tilde{\gamma}}) \subseteq \rho(L), \qquad |\hat{\mathcal{G}}_k(x)| \le \frac{\tilde{d}}{\omega^2 |k|^{2-\tilde{\gamma}}} V(x)$$

for all $k \in \mathbb{Z}_{\text{odd}} \setminus \mathfrak{R}$ *(note that* $\alpha + \tilde{\gamma} - 2 > \beta$ *is not required).*

Further let one of the two following assumptions on the spatial geometry of g_0, g_1, h *hold.*

($\mathcal{A}$3.8a) $h(x) \to 0$ *as* $|x| \to \infty$*. In addition there exist* $R^\pm \in \mathbb{R}$ *such that* g_0 *is periodic on* $[R^+, \infty)$ *with period* $X^+ > 0$*, and also on* $(-\infty, R^-]$ *with period* X^-*.*

($\mathcal{A}$3.8b) $h = h^{\text{loc}} + h^{\text{per}}$ *where* $h^{\text{loc}}(x) \ge 0$ *satisfies* $h^{\text{loc}}(x) \to 0$ *as* $|x| \to \infty$*, and* h^{per}, g_0, g_1 *are periodic in* x *with common period.*

Then there exists a nonzero breather solution $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$ *to Maxwell's equations with period* T *and speed* c *in the sense of Definition 3.1.1.*

If moreover the set $\mathfrak{R}$ *is infinite, there exists infinitely many distinct breather solutions.*

Remark 3.1.6. Let us comment on Theorem 3.1.5 and its assumptions.

- The constitutive relations (3.3.i) and (3.3.ii) are translation invariant in time. That is, if $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ solve the Maxwell system (3.1) and (3.2), then the shifted functions $\mathbf{E}(\,\cdot\,, \cdot - \tau), \mathbf{D}(\,\cdot\,, \cdot - \tau), \mathbf{B}(\,\cdot\,, \cdot - \tau), \mathbf{H}(\,\cdot\,, \cdot - \tau)$ also are solutions. In Theorem 3.1.5, for $\#\mathfrak{R} = \infty$ we state existence of infinitely many distinct solution. By this, we mean infinitely many solutions that are not shifts of one another.
- For ($\mathcal{A}$3.1), it is also possible to treat a negative potential h in front of the nonlinearity. More precisely, Theorem 3.1.5 remains valid when h is replaced by $-h$, and we discuss changes to the proof in Remark 3.4.1. However, with our method it is not possible to treat nonlinear potentials h that change sign, or those that vanish on a set of nonzero measure.
- The evenness assumption in ($\mathcal{A}$3.2) and ($\mathcal{A}$3.6) is needed for the variational structure, since the linear part of the variational problem, cf. (3.17), is nonsymmetric without this assumption: Recall that the adjoint of a convolution operator is the convolution with the even reflection of the convolution kernel. Evenness is equivalent to time reversal symmetry of the time-periodic Maxwell equations.

  The growth assumptions on the Fourier coefficients $\hat{\mathcal{N}}_k$ in ($\mathcal{A}$3.2) together with ($\mathcal{A}$3.4) and ($\mathcal{A}$3.5) ensure that the variational problem can be treated using semilinear methods. For example, we show that the nonlinear terms are well-defined and finite on the form domain of the linear operator.
- Assumption ($\mathcal{A}$3.7) lets us control the linear operator also along frequencies $k \in \mathbb{Z}_{\text{odd}} \setminus \mathfrak{R}$. It is not needed for (3.8.i) since there the nonlinearity contains a convolution with ν, which projects onto the frequencies $\mathfrak{R}$. The lower bound $|k|^{-s}$ on the Fourier coefficients $\hat{\mathcal{N}}_k$ gives us control over the inverse of this convolution operator.
- The geometry assumptions ($\mathcal{A}$3.8a) and ($\mathcal{A}$3.8b) are used to overcome noncompactness issues for the variational functional: The decay of h in ($\mathcal{A}$3.8a) ensures complete continuity of the nonlinearity, whereas the periodic structure in ($\mathcal{A}$3.8b) allows us to use concentration-compactness arguments.

  Under ($\mathcal{A}$3.8b) the assumption on the point spectrum in ($\mathcal{A}$3.4) is trivially satisfied since the differential operator L is periodic, and therefore by Floquet-Bloch theory (cf. [36]) the spectrum $\sigma(L)$ consists of pure essential spectrum.

Outline

In Section 3.2 we formally convert the Maxwell problem into an Euler-Lagrange equation and investigate the appearing symmetric linear operator $\mathcal{L}$. Section 3.3 deals with the form domain $\mathcal{H}$ of the operator $\mathcal{L}$, i.e., the natural domain of the bilinear form associated to $\mathcal{L}$. We show boundedness as well as local compactness of embeddings $\mathcal{H} \hookrightarrow L^p(\mathbb{R} \times \mathbb{T})$. This then allows us to define the Lagrangian functional on $\mathcal{H}$. In Section 3.4 we consider the dual problem to the Euler-Lagrange equation, and solve it using the mountain pass method. We also discuss multiplicity of solutions claimed in Theorem 3.1.5. Next, in Section 3.5 we discuss regularity of solutions of the Euler-Lagrange equation as well as the Maxwell system, proving Theorem 3.1.5. Throughout these sections, we always assume that ($\mathcal{A}$3.1)–($\mathcal{A}$3.7) and one of ($\mathcal{A}$3.8a) or ($\mathcal{A}$3.8b) are fulfilled. Lastly, Appendix 3.A contains the proofs of Theorems 3.1.2 and 3.1.3 as well as some auxiliary results.

3.2. Variational formulation

We begin by transforming the scalar Maxwell problem (3.10) into a variational problem for an auxiliary variable u. We follow Chapter 4 for the formal derivation.

First we consider the polarization (3.8.i). We denote by $*$ the convolution on the time-domain $\mathbb{T}$ and use

$$V(x) = 1 - \frac{1}{c^2} + g_0(x), \qquad \mathcal{G}(x) = \operatorname{Per}[g_1(x,\,\cdot\,)], \qquad \mathcal{N} = \operatorname{Per}[\nu]$$

to rephrase (3.10) as

$$\Big(-\partial_x^2 + V(x)\partial_t^2\Big)w + \partial_t^2\mathcal{G}(x) * w + h(x)\partial_t^2\mathcal{N} * w^3 = 0. \tag{3.13}$$

Observe that V is bounded, strictly positive, and locally of bounded variation due to ($\mathcal{A}$3.3). The nonlinearity $\partial_t^2\mathcal{N} * w^3$ is supported only on frequencies $k \in \mathfrak{R}$, so it is reasonable to assume that w is also supported only on such frequencies. We denote the projection onto frequencies $k \in \mathfrak{R}$ by $P_{\mathfrak{R}}$, i.e., $P_{\mathfrak{R}}[\varphi] := \mathcal{F}^{-1}[\mathbb{1}_{k\in\mathfrak{R}}\mathcal{F}_k[\varphi]]$. On these frequencies, the operators $-\partial_t^2$, $\mathcal{N}*$ are invertible as Fourier multipliers with nonzero symbol, which allows us to rewrite (3.13) as

$$\Big(-\partial_t^2\mathcal{N}*\Big)^{-1}\Big(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*\Big)w - h(x)P_{\mathfrak{R}}[w^3] = 0. \tag{3.14}$$

We abbreviate (3.14) to $\mathcal{L}w - hP_{\mathfrak{R}}[w^3] = 0$ by introducing the linear operator

$$\mathcal{L} := \Big(-\partial_t^2\mathcal{N}*\Big)^{-1}\Big(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*\Big). \tag{3.15}$$

Note that ∂_t^2, $\mathcal{N}*$, and $-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*$ mutually commute since they act on time as Fourier multipliers. Since $\mathcal{N}, \mathcal{G}(x)$ are even in time, the convolution operators $\mathcal{N}*, \mathcal{G}(x)*$ are symmetric and hence $\mathcal{L}$ is symmetric.

Let us now consider polarization (3.8.ii) so that (3.10) becomes

$$\Big(-\partial_x^2 + V(x)\partial_t^2\Big)w + \partial_t^2\mathcal{G}(x) * w + h(x)\partial_t^2(\mathcal{N} * w)^3 = 0. \tag{3.16}$$

We substitute $u := \mathcal{N} * w$ in (3.16) and apply $P_{\mathfrak{R}}$ to see that u solves

$$(-\partial_t^2)^{-1}\Big(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*\Big)(\mathcal{N}*)^{-1}u - h(x)P_{\mathfrak{R}}[u^3] = 0,$$

whereas by projecting with $\mathrm{Id} - P_{\mathfrak{R}}$ onto (3.16) we obtain

$$\left(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2 \mathcal{G}(x)*\right)(\mathrm{Id} - P_{\mathfrak{R}})w = -h(x)\partial_t^2(\mathrm{Id} - P_{\mathfrak{R}})[u^3].$$

The first of the two equations reads

$$\mathcal{L}u - h(x)P_{\mathfrak{R}}[u^3] = 0, \tag{3.17}$$

and the second together with $P_{\mathfrak{R}}[w] = (\mathcal{N}*)^{-1}u$ allows us to reconstruct the wave profile w via

$$\begin{aligned} w &= P_{\mathfrak{R}}[w] + (\mathrm{Id} - P_{\mathfrak{R}})[w] \\ &= (\mathcal{N}*)^{-1}u + \left(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2 \mathcal{G}(x)*\right)^{-1}[-h(x)\partial_t^2(\mathrm{Id} - P_{\mathfrak{R}})[u^3]]. \end{aligned} \tag{3.18}$$

In particular, for both choices of the polarization we have to solve the problem (3.17). Then, we have $w = u$ for polarization (3.8.i) whereas w is given by (3.18) for polarization (3.8.ii).

We want to study the form domain $\mathcal{H}$ of $\mathcal{L}$. For this, we use a functional calculus for the Sturm-Liouville operator L on a weighted L^2-space. We introduce both before defining $\mathcal{H}$ in Definition 3.2.4.

Definition 3.2.1. *We define the V-weighted space $L^2_V(\mathbb{R};\mathbb{C}) := L^2(\mathbb{R};\mathbb{C}; V\mathrm{d}x)$. Uniform boundedness and positivity of V show $L^2_V(\mathbb{R};\mathbb{C}) = L^2(\mathbb{R};\mathbb{C})$ with equivalent norms.*

Theorem 3.2.2 (cf. Theorem 2.3.6). *Let $\Psi(x;\lambda) = (\Psi_1(x;\lambda), \Psi_2(x;\lambda))^\top$ be the fundamental system of solutions of $L\varphi = \lambda\varphi$ on $\mathbb{R}$ with initial data $(\Psi, \partial_x\Psi)|_{x=0} = I_{2\times 2}$. Then there exists a measure μ which is defined on the bounded Borel subsets of $\mathbb{R}$ and maps to positive semidefinite $\mathbb{R}^{2\times 2}$ matrices such that*

$$T\colon L^2_V(\mathbb{R};\mathbb{C}) \to L^2(\mu), \quad T[f](\lambda) = \int_{\mathbb{R}} f(x)\Psi(x;\lambda)\, V\mathrm{d}x$$

is an isometric isomorphism with inverse

$$T^{-1}[g](x) = \int_{\mathbb{R}} g_i(\lambda)\Psi_j(x;\lambda)\,\mathrm{d}\mu_{ij}(\lambda).$$

Here we use the Einstein summation convention. A definition of the Hilbert space $L^2(\mu)$ consisting of (equivalence classes of) $\mathbb{C}^2$-valued measurable functions can be found in [35, Definition XIII.5.8]. Its norm is given by $\|g\|^2_{L^2(\mu)} = \int_{\mathbb{R}} g_i\overline{g_j}\,\mathrm{d}\mu_{ij}(\lambda)$. We note that the integral defining T exists for compactly supported f, and T is defined by approximation for general f. The same holds for T^{-1}.

Let us point out some basic properties of the transform T. A proof is given in Appendix 3.A.

Lemma 3.2.3. *Let $f \in L^2(\mathbb{R};\mathbb{C})$. Then the following hold:*

1. *$f \in H^2(\mathbb{R};\mathbb{C})$ if and only if $\lambda T[f](\lambda) \in L^2(\mu)$, and we have*

$$Lf = T^{-1}[\lambda T[f](\lambda)]$$

2. *$f \in H^1(\mathbb{R};\mathbb{C})$ if and only if $\sqrt{\lambda}T[f](\lambda) \in L^2(\mu)$, and we have*

$$\int_{\mathbb{R}} |f'|^2\,\mathrm{d}x = \int_{\mathbb{R}} \lambda T_i[f](\lambda)\overline{T_j[f](\lambda)}\,\mathrm{d}\mu_{ij}(\lambda)$$

3. *Moreover, the support of μ satisfies*

$$\operatorname{supp}(\mu) := \bigcup_{i,j=1}^{2} \operatorname{supp}(\mu_{ij}) = \sigma(L)$$

Now we rigorously define the bilinear form $b_{\mathcal{L}}$ associated to the operator $\mathcal{L}$ of (3.15) and its domain $\mathcal{H}$. In the following, we identify the bilinear form $b_{\mathcal{L}}\colon \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ with the weak formulation of the operator $\mathcal{L}$ via $\mathcal{L}[u][\varphi] = b_{\mathcal{L}}[u,\varphi]$ for $u,\varphi\in\mathcal{H}$, i.e. $\mathcal{L}\colon\mathcal{H}\to\mathcal{H}'$ where $\mathcal{H}'$ is the dual space.

Definition 3.2.4. *We define the form domain $\mathcal{H}$ by*

$$\mathcal{H} := \Big\{u\in L^2(\mathbb{R}\times\mathbb{T};\mathbb{R})\colon \hat{u}_k = 0 \text{ for } k\in\mathbb{Z}\setminus\mathfrak{R}, \langle u,u\rangle_{\mathcal{H}} < \infty\Big\}$$

where

$$\langle u,v\rangle_{\mathcal{H}} := \sum_{k\in\mathfrak{R}} \int_{\mathbb{R}} \left|\frac{\lambda-\omega^2k^2}{\omega^2k^2\hat{\mathcal{N}}_k}\right| T_i[\hat{u}_k](\lambda)\overline{T_j[\hat{v}_k](\lambda)}\,\mathrm{d}\mu_{ij}(\lambda).$$

Next, we define the operators $\mathcal{L},\mathcal{L}_0,\mathcal{L}_1\colon\mathcal{H}\to\mathcal{H}'$ by $\mathcal{L} := \mathcal{L}_0+\mathcal{L}_1$ and

$$\begin{aligned}\mathcal{L}_0[u][\varphi] &:= \sum_{k\in\mathfrak{R}} \int_{\mathbb{R}} \frac{\lambda-\omega^2k^2}{\omega^2k^2\hat{\mathcal{N}}_k} T_i[\hat{u}_k](\lambda)\overline{T_j[\hat{\varphi}_k](\lambda)}\,\mathrm{d}\mu_{ij}(\lambda),\\ \mathcal{L}_1[u][\varphi] &:= \sum_{k\in\mathfrak{R}} \int_{\mathbb{R}} \frac{\hat{\mathcal{G}}_k(x)}{\hat{\mathcal{N}}_k}\hat{u}_k(x)\overline{\hat{\varphi}_k(x)}\,\mathrm{d}x\end{aligned}$$

for $u,\varphi\in\mathcal{H}$. We call a function u weak solution *to* (3.17) *if $u\in\mathcal{H}$ and*

$$\mathcal{L}[u][\varphi] - \int_{\mathbb{R}\times\mathbb{T}} h(x)u^3\varphi\,\mathrm{d}(x,t) = 0$$

holds for all $\varphi\in\mathcal{H}$. We show below in Lemma 3.2.7 and Proposition 3.3.2 that the above integrals and sums converge and that embedding $\mathcal{H}\hookrightarrow L^4(\mathbb{R}\times\mathbb{T})$ and the maps $\mathcal{L}_0,\mathcal{L}_1\colon\mathcal{H}\to\mathcal{H}'$ are bounded.

We continue by investigating the operator $\mathcal{L}$, its domain $\mathcal{H}$, and their properties. The following estimate on the symbol $|\lambda-\omega^2k^2|$ will be useful.

Remark 3.2.5. For $k\in\mathfrak{R}$ and $\lambda\in\sigma(L)$ we have

$$\left|\lambda-\omega^2k^2\right| \geq \delta|k|^\gamma \qquad\text{and}\qquad \left|\lambda-\omega^2k^2\right| \geq \frac{\delta|k|^\gamma}{\omega^2k^2+\delta|k|^\gamma}\lambda.$$

The first estimate follows directly from ($\mathcal{A}$3.4). For the second estimate, we fix k and consider $\frac{|\lambda-\omega^2k^2|}{\lambda}$ on $\lambda\in\sigma(L)\subseteq[0,\omega^2k^2-\delta|k|^\gamma]\cup[\omega^2k^2+\delta|k|^\gamma,\infty) := I$. We estimate the quotient from below by its minimum value on I, which is attained at $\lambda = \omega^2k^2+\delta|k|^\gamma$.

We have the following density result for $\mathcal{H}$.

Lemma 3.2.6. *The set $\mathcal{D} := \{u\in\mathcal{H}\cap C_c^\infty(\mathbb{R}\times\mathbb{T};\mathbb{R})\colon \hat{u}_k = 0$ for almost all $k\in\mathfrak{R}\}$ is dense in $\mathcal{H}$.*

Proof. Note for $u \in \mathcal{H}$ that

$$\sum_{\substack{k \in \mathfrak{R} \\ |k| \leq_k K}} \hat{u}_k(x) e_k(t) \to u$$

in $\mathcal{H}$ as $K \to \infty$. Moreover, for fixed k, using Remark 3.2.5 we have

$$\left| \frac{\lambda - \omega^2 k^2}{\omega^2 k^2 \hat{\mathcal{N}}_k} \right| \eqsim_k \lambda + 1$$

which combined with Lemma 3.2.3 shows that $\hat{u}_k \in H^1(\mathbb{R};\mathbb{C})$ and that the norms $\|\hat{u}_k\|_{H^1}$ and $\|\hat{u}_k(x)e_k(t)\|_{\mathcal{H}}$ are equivalent. As $C_c^\infty(\mathbb{R};\mathbb{C}) \hookrightarrow H^1(\mathbb{R};\mathbb{C})$ is dense, the result follows by approximating $\hat{u}_k$ for all $|k| \leq K$. □

Lastly, we show that $\mathcal{L}$ is invertible and indefinite. Both properties are essential for the dual variational method which we use to solve (3.17).

Lemma 3.2.7. *$\mathcal{L}, \mathcal{L}_0, \mathcal{L}_1$ are symmetric, $\mathcal{L}_0$ is an isometric isomorphism and $\|\mathcal{L}_1\| < 1$. By the Neumann series, $\mathcal{L}$ is an isomorphism.*

Proof. From the definitions of $\mathcal{H}$ and $\mathcal{L}_0$ we see that $\mathcal{L}_0$ is an isometric isomorphism, and it is clearly symmetric. By ($\mathcal{A}$3.6) the potential $\mathcal{G}(x)$ is even in time, so its Fourier coefficients are real-valued and hence $\mathcal{L}_1$ is symmetric. It remains to show the bound on $\mathcal{L}_1$. For this, recall that by Lemma 3.2.3 the spectral measure μ is supported on $\sigma(L)$. Using Remark 3.2.5 we have

$$\|u\|_{\mathcal{H}}^2 \geq \sum_{k \in \mathfrak{R}} \int_{\mathbb{R}} \frac{\delta |k|^\gamma}{\omega^2 k^2 |\hat{\mathcal{N}}_k|} T_i[\hat{u}_k](\lambda) \overline{T_j[\hat{u}_k](\lambda)} \, \mathrm{d}\mu_{ij}(\lambda) = \sum_{k \in \mathfrak{R}} \frac{\delta}{\omega^2 |k|^{2-\gamma} |\hat{\mathcal{N}}_k|} \|\hat{u}_k\|_{L_V^2}^2.$$

Next, using assumption ($\mathcal{A}$3.6), the Cauchy-Schwarz inequality and the above we find

$$\begin{aligned} |\mathcal{L}_1[u][\varphi]| &\leq \sum_{k \in \mathfrak{R}} \frac{1}{|\hat{\mathcal{N}}_k|} \left\| \frac{\hat{\mathcal{G}}_k}{V} \right\|_\infty \|\hat{u}_k\|_{L_V^2(\mathbb{R})} \|\hat{\varphi}_k\|_{L_V^2} \leq \sum_{k \in \mathfrak{R}} \frac{d}{\omega^2 |k|^{2-\gamma} |\hat{\mathcal{N}}_k|} \|\hat{u}_k\|_{L_V^2} \|\hat{\varphi}_k\|_{L_V^2} \\ &\leq \left(\sum_{k \in \mathfrak{R}} \frac{d}{\omega^2 |k|^{2-\gamma} |\hat{\mathcal{N}}_k|} \|\hat{u}_k\|_{L_V^2}^2 \right)^{\frac{1}{2}} \left(\sum_{k \in \mathfrak{R}} \frac{d}{\omega^2 |k|^{2-\gamma} |\hat{\mathcal{N}}_k|} \|\hat{\varphi}_k\|_{L_V^2}^2 \right)^{\frac{1}{2}} \leq \frac{d}{\delta} \|u\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}}, \end{aligned}$$

showing $\|\mathcal{L}_1\| \leq \frac{d}{\delta} < 1$. □

Lemma 3.2.8. *$\mathcal{L}$ is an indefinite bilinear form.*

Proof. Clearly, the spectrum $\sigma(L)$ of the Sturm-Liouville operator L contains 0 and is not bounded from above. Fix $k \in \mathfrak{R}$. By ($\mathcal{A}$3.4) and Lemma 3.2.3 we find a function $g \in L^2(\mu) \backslash \{0\}$ that is supported on $[\omega^2 k^2 + \delta |k|^\gamma, N]$ for some $N > 0$. The function

$$u(x,t) = T^{-1}[g](x) e_k(t) + T^{-1}[\overline{g}](x) e_{-k}(t).$$

satisfies

$$\mathcal{L}_0[u][u] = 2 \int_{\mathbb{R}} \frac{\lambda - \omega^2 k^2}{\omega^2 k^2 \hat{\mathcal{N}}_k} g_i(\lambda) \overline{g_j(\lambda)} \, \mathrm{d}\mu_{ij}(\lambda),$$

which is nonzero with $\operatorname{sign}(\mathcal{L}_0[u][u]) = \operatorname{sign}(\hat{\mathcal{N}}_k)$. Since $\|u\|_{\mathcal{H}}^2 = |\mathcal{L}_0[u][u]|$, Lemma 3.2.7 shows that $\mathcal{L}[u][u]$ is nonzero with $\operatorname{sign}(\mathcal{L}[u][u]) = \operatorname{sign}(\mathcal{L}_0[u][u]) = \operatorname{sign}(\hat{\mathcal{N}}_k)$.

If above we instead choose g supported on $[0, \omega^2 k^2 - \delta |k|^\gamma]$, then by the same argument we find u satisfying $\operatorname{sign}(\mathcal{L}[u][u]) = -\operatorname{sign}(\hat{\mathcal{N}}_k)$. □

3.3. EMBEDDINGS

We investigate the embeddings $\mathcal{H} \hookrightarrow L^p(\mathbb{R} \times \mathbb{T})$, and discuss their boundedness in Proposition 3.3.2, as well as a concentration-compactness result in Proposition 3.3.5. We use some ideas and results from Chapter 2.

Let us fix some notation. Recall that the potential g_0 (and therefore also V) is X^+-periodic on $[R^+, \infty)$ as well as X^--periodic on $(-\infty, R^-]$: In case ($\mathcal{A}$3.8a) this is part of the assumption, and in ($\mathcal{A}$3.8b) g_0 is periodic so we may choose $R^\pm$ arbitrarily and $X^+ = X^-$.

We denote by V^+ the X^+-periodic extension of $V|_{[R^+,\infty)}$ to $\mathbb{R}$ and similarly by V^- the X^--periodic extension of $V|_{(-\infty,R^-]}$ to $\mathbb{R}$. We define the periodic Sturm-Liouville operators

$$L^\pm \coloneqq -\frac{1}{V^\pm(x)}\partial_x^2.$$

According to Floquet-Bloch theory (cf. [76]), for the spectra of $L^\pm \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ we have

$$\sigma(L^\pm) = \bigcup_{n\in\mathbb{N}} I_n^\pm$$

where $I_n^\pm$ are compact intervals with $\min I_n^\pm \xrightarrow{n\to\infty} \infty$, called *spectral bands*. We assume that they are enumerated in the standard way for Floquet-Block theory: $I_n^\pm$ are increasing, i.e., $\min I_{n+1}^\pm \geq \max I_n^+$, and the boundary points $\{\min I_n^\pm, \max I_n^\pm : n \in \mathbb{N}\}$ consist precisely of those $\lambda \in \mathbb{R}$ where $L^\pm f = \lambda f$ admits nonzero $X^\pm$-periodic or $X^\pm$-antiperiodic solutions.

The operators $L^\pm$ are useful in the study of L together with information on its point spectrum. For example, $\sigma(L) = \sigma_p(L) \cup \sigma_{\text{ess}}(L) = \sigma_p(L) \cup \sigma(L^+) \cup \sigma(L^-)$ holds (cf. Lemma 2.B.1). Information on L allows us to better understand $\mathcal{H}$ and characterize its embeddings. We begin with the following sufficient condition for boundedness of the L^p-embeddings.

Lemma 3.3.1. *Let $p \in (2, \infty]$, $s \coloneqq \frac{p}{p-2}$ with*

$$C \coloneqq \sum_{k\in\mathfrak{R}} |k^2\hat{\mathcal{N}}_k|^s \left(\sum_{n\in\mathbb{N}} \operatorname{dist}(\omega^2k^2, I_n^+)^{-s} + \sum_{n\in\mathbb{N}} \operatorname{dist}(\omega^2k^2, I_n^-)^{-s} + \sum_{\lambda\in\sigma_p(L)} \left|\omega^2k^2 - \lambda\right|^{-s} \right) < \infty.$$

Then the embedding $\mathcal{H} \hookrightarrow L^p(\mathbb{R} \times \mathbb{T})$ is continuous.

Proof. We consider the isometry

$$E \colon \mathcal{H} \to L_V^2(\mathbb{R} \times \mathbb{T}; \mathbb{R}),\ u \mapsto \sum_{k\in\mathfrak{R}} T^{-1}\Big[\left|\frac{\lambda - \omega^2k^2}{\omega^2k^2\hat{\mathcal{N}}_k}\right|^{\frac{1}{2}} T[\hat{u}_k](\lambda)\Big](x)e_k(t)$$

and the family of operators

$$\iota_\theta \colon L_V^2(\mathbb{R} \times \mathbb{T}; \mathbb{R}) \to L^{\frac{2}{1-\theta}}(\mathbb{R} \times \mathbb{T}; \mathbb{R}), u \mapsto \sum_{k\in\mathfrak{R}} T^{-1}\Big[\left|\frac{\lambda - \omega^2k^2}{\omega^2k^2\hat{\mathcal{N}}_k}\right|^{-\frac{\theta s}{2}} T[\hat{u}_k](\lambda)\Big](x)e_k(t)$$

where θ varies over $[0, 1]$. Then $\iota_0 = \mathrm{Id} \colon L^2(\mathbb{R} \times \mathbb{T}; \mathbb{R}) \to L^2(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ is bounded, and boundedness of ι_1 follows from $C < \infty$ as in Lemma 2.3.26. For this, we note that the enumeration of the spectral bands $I_n^\pm$ used in Chapter 2 coincides with ours.

By interpolation, each ι_θ is bounded. Setting $\theta = \frac{1}{s}$, it follows that

$$\iota_\theta E = \mathrm{Id} \colon \mathcal{H} \to L^{\frac{2}{1-\theta}}(\mathbb{R} \times \mathbb{T}; \mathbb{R}) = L^p(\mathbb{R} \times \mathbb{T}; \mathbb{R})$$

is bounded. ☐

We are now ready to state our central embedding result, where we use information on the spectrum of L given in assumption ($\mathcal{A}3.4$), in particular the existence and size of the spectral gaps about $\omega^2 k^2$ for $k \in \mathfrak{R}$ as well as estimates on the point spectrum, to verify the condition in Lemma 3.3.1.

Proposition 3.3.2. *Let $p \in [2, p^\star)$ with $p^\star := \min\left\{\frac{2\beta}{\beta-1}, \frac{4\beta}{2\beta-\alpha-\gamma+2}\right\}$. Then the embedding $\mathcal{H} \hookrightarrow L^p(\mathbb{R}\times\mathbb{T})$ is continuous and locally compact.*

In the above quotients we set $\frac{a}{b} = \infty$ for $b \le 0$. Note that $p^\star > 4$ holds by assumption ($\mathcal{A}3.5$).

Proof. We only consider $p > 2$ and begin by showing continuity of the embedding. First we show that the sum corresponding to the point spectrum

$$\sum_{k\in\mathfrak{R}} |k^2 \hat{\mathcal{N}}_k|^s \sum_{\lambda\in\sigma_p(L)} \left|\omega^2 k^2 - \lambda\right|^{-s} \tag{3.19}$$

with $s := \frac{p}{p-2}$ is finite. Recall that $|\hat{\mathcal{N}}_k| \lesssim |k|^{-\alpha}$ due to assumption ($\mathcal{A}3.2$). We use the estimate

$$\sum_{k=m}^{n} k^r \eqsim_r \int_m^n k^r = \frac{n^{r+1} - m^{r+1}}{r+1} \lesssim_r n^{r+1} + m^{r+1}$$

on integer sums with $r \in \mathbb{R}\setminus\{-1\}$ and $m, n \in \mathbb{N}$ with $m < n$. To keep notation simple, below we assume that we are in the generic case $r \neq -1$. For $r = -1$ we can use $\sum_{k=m}^{n} k^{-1} \lesssim_\varepsilon n^\varepsilon$ instead, which leads to the same results provided ε is chosen sufficiently small.

We use separate estimates for (3.19) in the three cases $\omega^2k^2 \ll \lambda$, $\omega^2k^2 \approx \lambda$, and $\omega^2k^2 \gg \lambda$. First, we calculate

$$\begin{aligned}\sum_{\lambda\in\sigma_p(L)} \sum_{\substack{k\in\mathfrak{R}\\ \omega^2k^2<\frac{1}{2}\lambda}} |k|^{(2-\alpha)s}\left|\omega^2k^2-\lambda\right|^{-s} &\eqsim \sum_{\lambda\in\sigma_p(L)} \sum_{\substack{k\in\mathfrak{R}\\ \omega^2k^2<\frac{1}{2}\lambda}} |k|^{(2-\alpha)s}\lambda^{-s} \\ &\lesssim \sum_{\lambda\in\sigma_p(L)} \left(1+\lambda^{\frac{1+(2-\alpha)s}{2}}\right)\lambda^{-s}.\end{aligned}$$

Second, we have

$$\begin{aligned}\sum_{\lambda\in\sigma_p(L)} \sum_{\substack{k\in\mathfrak{R}\\ \omega^2k^2\ge 2\lambda}} |k|^{(2-\alpha)s}\left|\omega^2k^2-\lambda\right|^{-s} &\eqsim \sum_{\lambda\in\sigma_p(L)} \sum_{\substack{k\in\mathfrak{R}\\ \omega^2k^2\ge 2\lambda}} |k|^{(2-\alpha)s}|k|^{-2s} \\ &\lesssim \sum_{\lambda\in\sigma_p(L)} \lambda^{\frac{1-\alpha s}{2}}\end{aligned}$$

For the third and last sum we use $|\omega^2k^2-\lambda| = (\omega|k|+\sqrt{\lambda})\left|\omega|k|-\sqrt{\lambda}\right| \ge \sqrt{\lambda}\omega n$ where $n = \left\lfloor\left||k| - \frac{1}{\omega}\sqrt{\lambda}\right|\right\rfloor$. Observe that each $n \in \mathbb{N}_0$ is attained as a value for at most four k. By ($\mathcal{A}3.4$) we also have $|\omega^2k^2-\lambda| \ge \delta|k|^\gamma$, which we use instead when $n = 0$. Therefore

$$\begin{aligned}\sum_{\lambda\in\sigma_p(L)} \sum_{\substack{k\in\mathfrak{R}\\ \frac{1}{2}\lambda\le\omega^2k^2<2\lambda}} |k|^{(2-\alpha)s}\left|\omega^2k^2-\lambda\right|^{-s} &\eqsim \sum_{\lambda\in\sigma_p(L)} \lambda^{\frac{(2-\alpha)s}{2}} \sum_{\substack{k\in\mathfrak{R}\\ \frac{1}{2}\lambda\le\omega^2k^2<2\lambda}} \left|\omega^2k^2-\lambda\right|^{-s} \\ &\lesssim \sum_{\lambda\in\sigma_p(L)} \lambda^{\frac{(2-\alpha)s}{2}}\left(\delta\lambda^{-\frac{\gamma s}{2}} + \sum_{n=1}^{\infty}\lambda^{-\frac{s}{2}}n^{-s}\right)\end{aligned}$$

By ($\mathcal{A}3.4$) all the above sums are finite provided

$$\min\left\{s, \frac{\alpha s-1}{2}, \frac{(\alpha+\gamma-2)s}{2}, \frac{(\alpha-1)s}{2}\right\} > \beta. \tag{3.20}$$

Using $\frac{\alpha s-1}{2} > \frac{(\alpha-1)s}{2} \geq \frac{(\alpha+\gamma-2)s}{2}$, a direct calculation shows that (3.20) holds for $p < p^\star$.

For the remaining sums appearing in the constant C of Lemma 3.3.1, we first estimate

$$\sum_{n\in\mathbb{N}} \operatorname{dist}(\omega^2k^2, I_n^\pm)^{-s} \leq \sum_{\lambda\in S^\pm} \left|\omega^2k^2 - \lambda\right|^{-s} \tag{3.21}$$

with $S^\pm := \{\partial I_n^\pm \colon n \in \mathbb{N}\} \setminus \{0\}$. By the proof of Theorem 2.3.27 the spectral bands $I_n^\pm$ grow quadratically, so that $\sum_{\lambda\in S^\pm} \lambda^{-\frac{1}{2}-\varepsilon} < \infty$ for all $\varepsilon > 0$. We can therefore estimate the right-hand side of (3.21) as we did for the point spectrum, except β is replaced by $\frac{1}{2}$ in (3.20). As we required $\beta \geq \frac{1}{2}$ in Theorem 3.1.5, this generates no additional requirements.

Finally, local compactness of the embedding follows by a frequency cutoff approximation argument as in the proof of Theorem 2.3.27. □

For considerations of regularity, we use an improved embedding result introduced next, showing that low order temporal derivatives of a function $u \in \mathcal{H}$ still lie in $L^4(\mathbb{R} \times \mathbb{T})$.

Definition 3.3.3. *We define the fractional temporal derivative $|\partial_t|^s f$ of a function $f \colon \mathbb{T} \to \mathbb{C}$ for $s \in \mathbb{R}$ (with $\hat{f}_0 = 0$ if $s < 0$) as the Fourier multiplier with symbol $|\omega k|^s$, i.e., by $|\partial_t|^s f = \mathcal{F}^{-1}[|\omega k|^s \hat{f}_k]$.*

Remark 3.3.4. As in the proof of Proposition 3.3.2 we see that

$$|\partial_t|^\varepsilon \colon \mathcal{H} \to L^p(\mathbb{R} \times \mathbb{T}; \mathbb{R})$$

is bounded and locally compact for $p \in [2, p_\varepsilon^\star)$ with $p_\varepsilon^\star := \min\left\{\frac{2\beta}{\beta-1}, \frac{4\beta}{2\beta-\alpha+2\varepsilon-\gamma+2}\right\}$. Assumption ($\mathcal{A}3.5$) implies $p_\varepsilon^\star > 4$ for sufficiently small $\varepsilon > 0$.

We now prove a variant of the concentration-compactness principle of Lions.

Proposition 3.3.5. *Let $p \in [2, p^\star)$ where $p^\star$ is given by Proposition 3.3.2, $r > 0$, $w \colon \mathbb{R} \to [0, \infty)$ be a bounded and measurable weight function, and (u_n) be a bounded sequence in $\mathcal{H}$ with*

$$\sup_{x\in\mathbb{R}} \int_{[x-r,x+r]\times\mathbb{T}} |u_n|^p \, w(x) \mathrm{d}(x,t) \xrightarrow{n\to\infty} 0. \tag{3.22}$$

Then $u_n \xrightarrow{n\to\infty} 0$ in $L^q_w(\mathbb{R} \times \mathbb{T}) = L^q(\mathbb{R} \times \mathbb{T}; w(x)\mathrm{d}(x,t))$ for all $q \in (2, p^\star)$.

Proof. Part 1: By Hölder's inequality it suffices to give the proof for $p = 2$ and one $q \in (2, p^\star)$, which we choose later. Inspired by [56], we consider an auxiliary Hilbert space H defined by

$$H = \left\{u \in L^2(\mathbb{R} \times \mathbb{T}; \mathbb{R}) \colon \hat{u}_k = 0 \text{ for } k \in \mathbb{Z} \setminus \mathfrak{R}, \|u\|_H < \infty\right\},$$
$$\|u\|_H^2 = \sum_{k\in\mathfrak{R}} |k|^{\alpha+\gamma-4} \int_{\mathbb{R}} |\hat{u}_k'|^2 + V(x)k^2|\hat{u}_k|^2 \,\mathrm{d}x.$$

The H-norm is local in x which will allow us to get additional information on the embedding $\mathcal{H} \hookrightarrow L^q_w(\mathbb{R}\times\mathbb{T})$ by considering $\mathcal{H} \hookrightarrow H \hookrightarrow L^q_w(\mathbb{R}\times\mathbb{T})$.

Using assumption ($\mathcal{A}$3.2), Lemma 3.2.3 and the estimate $|\lambda-\omega^2k^2| \gtrsim |k|^\gamma + \lambda|k|^{\gamma-2}$ (see Remark 3.2.5) we find

$$\begin{aligned}\|u\|_{\mathcal{H}}^2 &\gtrsim \sum_{k\in\mathfrak{R}} |k|^{\alpha-2}\int_{\mathbb{R}}\Big|\lambda-\omega^2k^2\Big| T_i[\hat{u}_k](\lambda)\overline{T_j[\hat{u}_k](\lambda)}\,\mathrm{d}\mu_{ij}(\lambda)\\ &\gtrsim \sum_{k\in\mathfrak{R}} |k|^{\alpha+\gamma-4}\int_{\mathbb{R}}\Big(k^2+\lambda\Big) T_i[\hat{u}_k](\lambda)\overline{T_j[\hat{u}_k](\lambda)}\,\mathrm{d}\mu_{ij}(\lambda)\\ &= \sum_{k\in\mathfrak{R}} |k|^{\alpha+\gamma-4}\int_{\mathbb{R}}|\hat{u}_k'|^2+V(x)k^2|\hat{u}_k|^2\,\mathrm{d}x = \|u\|_H^2,\end{aligned}$$

so that $\mathcal{H}\hookrightarrow H$ is bounded.

Part 2: We now consider the embedding $H \hookrightarrow L^q(\mathbb{R}\times\mathbb{T})$. For this, let $I\subseteq\mathbb{R}$ be an interval of length $2r$. On I, let $\varphi_n(x) = \frac{1}{\sqrt{2r}}\mathrm{e}^{\mathrm{i}n\frac{\pi}{r}x}$ and define the spatial Fourier transform $F[\phi]$ of $\phi\colon I\to\mathbb{C}$ by

$$F_n[\phi] = \int_I \phi(x)\overline{\varphi_n(x)}\,\mathrm{d}x \qquad \text{for} \quad n\in\mathbb{Z}.$$

Fix some $s>8$ and define $q>2>q'$ by $\frac{1}{q'} = 1-\frac{1}{q} = \frac{1}{2}+\frac{1}{s}$. We calculate

$$\begin{aligned}\left\|\Big(|k|^{\alpha+\gamma-4}(n^2+k^2)\Big)^{-\frac12}\right\|_{\ell^s(\mathbb{Z}\times\mathfrak{R})}^s &= \sum_{k\in\mathfrak{R}}|k|^{-\frac{(\alpha+\gamma-4)s}{2}}\sum_{n\in\mathbb{Z}}(n^2+k^2)^{-\frac{s}{2}}\\ \lesssim \sum_{k\in\mathfrak{R}}|k|^{-\frac{(\alpha+\gamma-4)s}{2}}\cdot|k|^{1-s} &\le \sum_{k\in\mathfrak{R}}|k|^{1-\frac{\beta s}{2}} < \infty,\end{aligned}$$

where we used $\alpha+\gamma-2>\beta\ge\frac12$. From this, for $u\in H$ we obtain

$$\begin{aligned}\|u\|_{L^q_w(I\times\mathbb{T})}^2 &\lesssim \|u\|_{L^q(I\times\mathbb{T})}^2 \lesssim \|F_n[\hat{u}_k]\|_{\ell^{q'}(\mathbb{Z}\times\mathfrak{R})}^2\\ &\le \left\|\Big(|k|^{\alpha+\gamma-4}(n^2+k^2)\Big)^{\frac12}F_n[\hat{u}_k]\right\|_{\ell^2(\mathbb{Z}\times\mathfrak{R})}^2\left\|\Big(|k|^{\alpha+\gamma-4}(n^2+k^2)\Big)^{-\frac12}\right\|_{\ell^s(\mathbb{Z}\times\mathfrak{R})}^2\\ &\lesssim \sum_{k\in\mathfrak{R}}|k|^{\alpha+\gamma-4}\sum_{n\in\mathbb{Z}}\Big(n^2+k^2\Big)|F_n[\hat{u}_k]|^2\,\mathrm{d}x\\ &\lesssim \sum_{k\in\mathfrak{R}}|k|^{\alpha+\gamma-4}\int_I|\hat{u}_k'|^2+V(x)k^2|\hat{u}_k|^2\,\mathrm{d}x.\end{aligned}$$

We now choose intervals $I_j \coloneqq [(2j-1)r,(2j+1)r]$, and define the norm

$$\|u\|_{\ell^pL^q_w} \coloneqq \left\|\Big(\|u\|_{L^q_w(I_j\times\mathbb{T})}\Big)_j\right\|_{\ell^p(\mathbb{Z})}.$$

Then, using the above for $I=I_j$ and summing over j we obtain

$$\|u\|_{\ell^2L^q_w} \lesssim \|u\|_H \lesssim \|u\|_{\mathcal{H}}$$

By Hölder interpolation we have

$$\|u_n\|_{\ell^{p_\theta}L^{q_\theta}_w} \le \|u_n\|_{\ell^2L^q_w}^\theta\|u_n\|_{\ell^\infty L^2_w}^{1-\theta}$$

with $\frac{1}{p_\theta} = \frac{\theta}{2} + \frac{1-\theta}{\infty}$, $\frac{1}{q_\theta} = \frac{\theta}{q} + \frac{1-\theta}{2}$ for all $\theta \in [0,1]$. Now fix $\theta \in (0,1)$ to be the unique solution to $p_\theta = q_\theta$, and let (u_n) be a bounded sequence in $\mathcal{H}$ satisfying (3.22) with $p = 2$, i.e., $\|u_n\|_{\ell^\infty L^2_w} \to 0$ as $n \to \infty$. This shows

$$\|u_n\|_{L^{q_\theta}_w(\mathbb{R}\times\mathbb{T})} = \|u_n\|_{\ell^{p_\theta} L^{q_\theta}_w} \lesssim \|u_n\|_{\mathcal{H}}^{\theta} \|u_n\|_{\ell^\infty L^2}^{1-\theta} \to 0$$

as $n \to \infty$, completing the proof. □

3.4. The dual problem

We solve (3.17) variationally with the dual variational method. Using that h is positive and bounded by ($\mathcal{A}$3.1), we formally substitute $v := h^{\frac{3}{4}} u^3$ in (3.17) and multiply with $h^{\frac{1}{4}}\mathcal{L}^{-1}$ to obtain the dual problem

$$\mathcal{L}u - hP_{\mathfrak{R}}[u^3] = 0 \iff \mathcal{L}h^{-\frac{1}{4}} v^{\frac{1}{3}} - h^{\frac{1}{4}} P_{\mathfrak{R}}[v] = 0 \iff v^{\frac{1}{3}} - h^{\frac{1}{4}}\mathcal{L}^{-1} h^{\frac{1}{4}} P_{\mathfrak{R}}[v] = 0$$

where $v^{\frac{1}{3}}$ denotes the real cube root of the real-valued function v. We abbreviate the weighted inverse operator by $\mathcal{L}_h^{-1} = h^{\frac{1}{4}}\mathcal{L}^{-1}h^{\frac{1}{4}}P_{\mathfrak{R}}$, and thus consider the problem

$$v^{\frac{1}{3}} - \mathcal{L}_h^{-1} v = 0. \tag{3.23}$$

Having solved (3.23), we can formally recover a solution u of (3.17) by setting

$$u = h^{-\frac{1}{4}} v^{\frac{1}{3}} = \mathcal{L}^{-1} h^{\frac{1}{4}} P_{\mathfrak{R}}[v].$$

Remark 3.4.1. As stated in Remark 3.1.6, we can also consider (3.17) for negative h, for which the dual problem is given by

$$\mathcal{L}u - hP_{\mathfrak{R}}[u^3] = 0 \iff (-\mathcal{L})u - (-h)P_{\mathfrak{R}}[u^3] = 0 \iff v^{\frac{1}{3}} - (-h)^{\frac{1}{4}}(-\mathcal{L})^{-1}(-h)^{\frac{1}{4}} P_{\mathfrak{R}}[v] = 0$$

with $v := (-h)^{\frac{3}{4}} u^3$. The properties of $\mathcal{L}$ that we use below (symmetry, invertibility, indefiniteness) are also satisfied by $-\mathcal{L}$. Therefore our results on (3.17) and (3.23) can be transferred to the negative case by the substitution $h \rightsquigarrow -h$, $\mathcal{L} \rightsquigarrow -\mathcal{L}$.

Next we properly define the operator $\mathcal{L}_h^{-1}$.

Definition 3.4.2. *Let $\iota\colon \mathcal{H} \hookrightarrow L^4(\mathbb{R}\times\mathbb{T};\mathbb{R})$ be the bounded embedding of Proposition 3.3.2 and ι' be its adjoint. Then, using that $\mathcal{L}\colon \mathcal{H} \to \mathcal{H}'$ is invertible according to Lemma 3.2.7, we define the h-weighted inverse by*

$$\mathcal{L}_h^{-1} := h^{\frac{1}{4}} \iota \mathcal{L}^{-1} \iota' h^{\frac{1}{4}} \colon L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R}) \to L^4(\mathbb{R}\times\mathbb{T};\mathbb{R})$$

Next, we call a function v a solution *to* (3.23) *if $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$ is a critical point of the energy functional*

$$J\colon L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R}) \to \mathbb{R},\ J(v) := \int_{\mathbb{R}\times\mathbb{T}} \frac{3}{4}|v|^{\frac{4}{3}} - \frac{1}{2}\mathcal{L}_h^{-1} v \cdot v \,\mathrm{d}(x,t),$$

or equivalently if $v^{\frac{1}{3}} - \mathcal{L}_h^{-1} v = 0$ in $L^4(\mathbb{R}\times\mathbb{T};\mathbb{R})$.

Remark 3.4.3. In Definition 3.4.2, note that $P_{\mathfrak{R}}\iota = \iota$ holds by definition of $\mathcal{H}$, hence we can omit $P_{\mathfrak{R}}$ in the definition of $\mathcal{L}_h^{-1}$. By Proposition 3.3.2 the map $\mathcal{L}_h^{-1}$ is locally compact.

In Theorem 3.4.8 we show that there exists a nonzero solution to the dual problem (3.23). More precisely, we show that there exists a ground state as defined below.

Definition 3.4.4. *We call the energy level*

$$\mathfrak{c}_{\mathrm{gs}} := \inf_{\substack{v \in L^{4/3}(\mathbb{R}\times\mathbb{T};\mathbb{R})\setminus\{0\} \\ J'(v)=0}} J(v)$$

the ground state energy level*, and any nonzero critical point v of J with $J(v) = \mathfrak{c}_{\mathrm{gs}}$ is called a* ground state*.*

Remark 3.4.5. The substitution $v = h^{\frac{3}{4}} u^3$, which gives a one-to-one correspondence between solutions u of (3.17) and solutions v to (3.23), also links the ground states of the two problems. Indeed, if u is a solution of (3.17), which is the Euler-Lagrange equation of

$$\tilde{J}(u) = \int_{\mathbb{R}\times\mathbb{T}} \tfrac{1}{2}\mathcal{L}u \cdot u - \tfrac{1}{4} h u^4 \,\mathrm{d}(x,t),$$

we have

$$\tilde{J}(u) = \tilde{J}(u) - \tfrac{1}{2}\tilde{J}'(u)[u] = \tfrac{1}{4}\int_{\mathbb{R}\times\mathbb{T}} h u^4 \,\mathrm{d}(x,t) = \tfrac{1}{4}\int_{\mathbb{R}\times\mathbb{T}} |v|^{\frac{4}{3}} \,\mathrm{d}(x,t) = J(v) - \tfrac{1}{2}J'(v)[v] = J(v).$$

To show existence of ground states, we first use the mountain pass method to obtain a Palais-Smale sequence for J.

Proposition 3.4.6. *There exists $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})\setminus\{0\}$ with $J(v) \le 0$. For such v we define the mountain pass energy level by*

$$\mathfrak{c}_{\mathrm{mp}} := \mathfrak{c}_{\mathrm{mp}}(v) := \inf_{\substack{\gamma \in C([0,1];L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})) \\ \gamma(0)=0,\gamma(1)=v}} \sup_{t\in[0,1]} J(\gamma(t)).$$

Then $\mathfrak{c}_{\mathrm{mp}} > 0$ and there exists a Palais-Smale sequence for J at level $\mathfrak{c}_{\mathrm{mp}}$.

Proof. We first show that there exists $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$ with $\int_{\mathbb{R}\times\mathbb{T}} \mathcal{L}_h^{-1} v \cdot v \,\mathrm{d}(x,t) > 0$. Assume for a contradiction that

$$\int_{\mathbb{R}\times\mathbb{T}} \mathcal{L}_h^{-1} v \cdot v \,\mathrm{d}(x,t) = \int_{\mathbb{R}\times\mathbb{T}} \iota\mathcal{L}^{-1}\iota' h^{\frac{1}{4}} v \cdot h^{\frac{1}{4}} v \,\mathrm{d}(x,t) \le 0$$

for all $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$. As h vanishes almost nowhere, $h^{\frac{1}{4}} L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T}) \subseteq L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})$ is dense. By approximation it follows that

$$\int_{\mathbb{R}\times\mathbb{T}} \iota\mathcal{L}^{-1}\iota' v \cdot v \,\mathrm{d}(x,t) \le 0$$

holds for all $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$. Next, let φ lie in the dense subset $\mathcal{D} \subseteq \mathcal{H}$ of Lemma 3.2.6. Then $v := \mathcal{L}\varphi \in L^\infty(\mathbb{R}\times\mathbb{T})$ is compactly supported, and we have

$$\int_{\mathbb{R}\times\mathbb{T}} \iota\mathcal{L}^{-1}\iota' v \cdot v \,\mathrm{d}(x,t) = \mathcal{L}[\varphi][\varphi] \le 0.$$

Using density of $\mathcal{D} \subseteq \mathcal{H}$ we conclude that $\mathcal{L}$ is negative semidefinite, contradicting Lemma 3.2.8.

So our assumption was false, and therefore we find $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$ with $\int_{\mathbb{R}\times\mathbb{T}} \mathcal{L}_h^{-1} v \cdot v \,\mathrm{d}(x,t) > 0$. Then $J(sv) \leq 0$ for sufficiently large s.

Now let $0 < r < \left(\frac{3}{2\|\mathcal{L}_h^{-1}\|}\right)^{\frac{3}{2}}$ and $\tilde{v} \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})$ with $\|v\|_{\frac{4}{3}} = r$. Then

$$J(\tilde{v}) \geq \frac{3}{4} r^{\frac{4}{3}} - \frac{1}{2}\left\|\mathcal{L}_h^{-1}\right\| r^2 > 0$$

and therefore also $\mathfrak{c}_{\mathrm{mp}} \geq \frac{3}{4} r^{\frac{4}{3}} - \frac{1}{2}\left\|\mathcal{L}_h^{-1}\right\| r^2 > 0$. By the mountain pass theorem (see Theorem 1.2.2) there exists a Palais-Smale sequence (v_n) for J at level $\mathfrak{c}_{\mathrm{mp}}$. □

Remark 3.4.7. Any Palais-Smale sequence for J is bounded. Indeed, if (v_n) is a Palais-Smale sequence at level $\mathfrak{c}$, then

$$2\mathfrak{c} + \mathrm{o}(1) + \mathrm{o}(\|v_n\|_{\frac{4}{3}}) = 2J(v_n) - J'(v_n)[v_n] = \frac{1}{2}\|v_n\|_{\frac{4}{3}}^{\frac{4}{3}},$$

shows that (v_n) is bounded and moreover that $\|v_n\|_{\frac{4}{3}} \to (4\mathfrak{c})^{\frac{3}{4}}$ as $n \to \infty$.

Next we show an existence result for the dual problem.

Theorem 3.4.8. *There exists a ground state of* (3.23)*.*

The proof of Theorem 3.4.8 differs depending on the choice of assumption: ($\mathcal{A}$3.8a) or ($\mathcal{A}$3.8b). The case ($\mathcal{A}$3.8a) is simpler since J satisfies the Palais-Smale condition, and the proof is carried out in Proposition 3.4.9. Case ($\mathcal{A}$3.8b) is investigated in Lemma 3.4.10 for purely periodic coefficients g_0, g_1, h using the concentration-compactness principle of Proposition 3.3.5, and in Proposition 3.4.11 for the general case using energy comparison arguments.

Proposition 3.4.9. *Assume ($\mathcal{A}$3.1)–($\mathcal{A}$3.6) and ($\mathcal{A}$3.8a). Then there exists a ground state of* (3.23)*.*

Proof. Let (v_n) be the Palais-Smale sequence from Proposition 3.4.6. By Remark 3.4.7, up to a subsequence which we again label by v_n, there exists $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$ with $v_n \rightharpoonup v$ in $L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})$.

By Proposition 3.3.2 the embedding $\iota\colon \mathcal{H} \hookrightarrow L^4(\mathbb{R}\times\mathbb{T};\mathbb{R})$ is locally compact. Then $h^{\frac{1}{4}}\iota$ is compact since h decays to 0 at $\pm\infty$ by assumption ($\mathcal{A}$3.8a), and in particular $\mathcal{L}_h^{-1}$ is compact and thus $\mathcal{L}_h^{-1} v_n \to \mathcal{L}_h^{-1} v$ in L^4. Using

$$J'(v_n) = v_n^{\frac{1}{3}} - \mathcal{L}_h^{-1} v_n \to 0 \text{ in } L^4$$

we see that $v_n^{\frac{1}{3}}$ converges to $\mathcal{L}_h^{-1} v$ in L^4, which implies $v_n \to (\mathcal{L}_h^{-1} v)^3$ in $L^{\frac{4}{3}}$.

Since also $v_n \rightharpoonup v$, we have $v = (\mathcal{L}_h^{-1} v)^3$ and $v_n \to v$ in $L^{\frac{4}{3}}$, so $v^{\frac{1}{3}} - \mathcal{L}_h^{-1} v = 0$ in L^4. By continuity of J we have $J(v_n) \to J(v)$, i.e., $J(v) = \mathfrak{c}_{\mathrm{mp}}$.

Thus far, we have shown existence of a nonzero critical point of J, and thus $\mathfrak{c}_{\mathrm{gs}} \neq \infty$. Now let (v_n) be a sequence of critical points of J with $J(v_n) \to \mathfrak{c}_{\mathrm{gs}}$. Then the above arguments shows $v_n \to v$ in $L^{\frac{4}{3}}$ up to a subsequence, and hence v is a ground state. □

Lemma 3.4.10. *Assume (A3.1)–(A3.6) and (A3.8b) with* $h^{\text{loc}} = 0$. *Then there exists a ground state of* (3.23).

Proof. Part 1: Denote the common period of $h = h^{\text{per}}, g_0, g_1$ by X, so V is also X-periodic by its definition (3.11). Let us investigate the shift τ in x by X, i.e., $\tau[f](x) = f(x - X)$. With the spectral transform T and fundamental solution Ψ given in (3.2.2), we calculate for compactly supported $f \in L^2(\mathbb{R}; \mathbb{C})$

$$T[\tau f](\lambda) = \int_{\mathbb{R}} f(x - X)\Psi(x; \lambda)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\Psi(x + X; \lambda)\,\mathrm{d}x = M(\lambda)T[f](\lambda)$$

where the matrix $M(\lambda) \in \mathbb{R}^{2\times 2}$ is given by

$$M(\lambda)\Psi(x; \lambda) = \Psi(x + X; \lambda).$$

for all $x \in \mathbb{R}$, and it exists since $\Psi(\,\cdot\,; \lambda)$ solves an X-periodic differential equation. As τ is an isometric isomorphism on $L^2_V(\mathbb{R}; \mathbb{C})$, multiplication with $M(\lambda)$ is an isometric isomorphism of $L^2(\mu)$. For $u \in \mathcal{H}$ we have

$$\begin{aligned}
\|\tau u\|_{\mathcal{H}}^2 &= \sum_{k\in\mathfrak{R}} \left\| \left| \frac{\lambda - \omega^2 k^2}{\omega^2 k^2 \hat{\mathcal{N}}_k} \right|^{\frac{1}{2}} T[\tau \hat{u}_k](\lambda) \right\|_{L^2(\mu)}^2 \\
&= \sum_{k\in\mathfrak{R}} \left\| M(\lambda) \left| \frac{\lambda - \omega^2 k^2}{\omega^2 k^2 \hat{\mathcal{N}}_k} \right|^{\frac{1}{2}} T[\hat{u}_k](\lambda) \right\|_{L^2(\mu)}^2 \\
&= \sum_{k\in\mathfrak{R}} \left\| \left| \frac{\lambda - \omega^2 k^2}{\omega^2 k^2 \hat{\mathcal{N}}_k} \right|^{\frac{1}{2}} T[\hat{u}_k](\lambda) \right\|_{L^2(\mu)}^2 = \|u\|_{\mathcal{H}}^2,
\end{aligned}$$

i.e., τ is an isometric isomorphism of $\mathcal{H}$. A similar calculation shows $\mathcal{L}_0[\tau u][\tau u] = \mathcal{L}_0[u][u]$, and $\mathcal{L}_1[\tau u][\tau u] = \mathcal{L}_1[u][u]$ follows directly from periodicity of $\mathcal{G}$. Thus we have $\mathcal{L}[\tau u][\tau u] = \mathcal{L}[u][u]$. Let us define τ on $\mathcal{H}'$ by $\tau|_{\mathcal{H}'} = \left((\tau|_{\mathcal{H}})^{-1}\right)'$, which is an isometric isomorphism of $\mathcal{H}'$. Then by the above, $\tau\mathcal{L} = \mathcal{L}\tau$, $\tau\mathcal{L}^{-1} = \mathcal{L}^{-1}\tau$ and $\tau\mathcal{L}_h^{-1} = \mathcal{L}_h^{-1}\tau$ hold.

Part 2: Let (v_n) be the Palais-Smale sequence given by Proposition 3.4.6. We apply Proposition 3.3.5 with $r = X$, $p = q = 4$, $w = h$ and $u_n := \mathcal{L}^{-1}\iota' h^{\frac{1}{4}} v_n = h^{-\frac{1}{4}}\mathcal{L}_h^{-1} v_n$ to obtain a sequence of points $x_n \in \mathbb{R}$ with

$$\limsup_{n\to\infty} \|u_n\|_{L^4_h([x_n - X, x_n + X]\times\mathbb{T})} = \limsup_{n\to\infty} \left\|\mathcal{L}_h^{-1} v_n\right\|_{L^4([x_n - X, x_n + X]\times\mathbb{T})} > 0.$$

since

$$\|u_n\|_{L^4_h(\mathbb{R}\times\mathbb{T})} = \left\|\mathcal{L}_h^{-1} v_n\right\|_{L^4(\mathbb{R}\times\mathbb{T})} = \left\|v_n^3\right\|_{L^4(\mathbb{R}\times\mathbb{T})} + \mathrm{o}(1) \to (4\mathfrak{c}_{\mathrm{mp}})^{\frac{9}{4}} > 0$$

by Remark 3.4.7.

W.l.o.g. we may assume $x_n = k_n X$ for some $k_n \in \mathbb{Z}$. Then, $\tilde{v}_n := \tau^{k_n} v_n$ satisfies

$$\left\|\mathcal{L}_h^{-1} v_n\right\|_{L^4([x_n - X, x_n + X]\times\mathbb{T})} = \left\|\tau^{-k_n}\mathcal{L}_h^{-1}\tau^{k_n} v_n\right\|_{L^4([x_n - X, x_n + X]\times\mathbb{T})} = \left\|\mathcal{L}_h^{-1}\tilde{v}_n\right\|_{L^4([-X, X]\times\mathbb{T})}.$$

Now choose a subsequence, again denoted by $\tilde{v}_n$ such that $\tilde{v}_n \rightharpoonup v$ in $L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})$ and

$$\left\|\mathcal{L}_h^{-1}\tilde{v}_n\right\|_{L^4([-X, X]\times\mathbb{T})} \to s > 0$$

By local compactness of $\mathcal{L}_h^{-1}$ (see Proposition 3.3.2) we have $\left\|\mathcal{L}_h^{-1}v\right\|_{L^4([-X,X]\times\mathbb{T})} = s$, and thus $v \neq 0$, as well as $\mathcal{L}_h^{-1}\tilde{v}_n \to \mathcal{L}_h^{-1}v$ in $L^4_{\text{loc}}(\mathbb{R}\times\mathbb{T})$. In addition,

$$\int_{\mathbb{R}\times\mathbb{T}}\left(\tilde{v}_n^{\frac{1}{3}} - \mathcal{L}_h^{-1}\tilde{v}_n\right)\varphi\,\mathrm{d}(x,t) = \int_{\mathbb{R}\times\mathbb{T}}\left(v_n^{\frac{1}{3}} - \mathcal{L}_h^{-1}v_n\right)\tau^{-k_n}\varphi\,\mathrm{d}(x,t) = J'(v_n)[\tau^{-k_n}\varphi] = \mathrm{o}(\|\varphi\|_{\frac{4}{3}})$$

as $n\to\infty$ for $\varphi \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})$. This shows $\tilde{v}_n^{\frac{1}{3}} \to \mathcal{L}_h^{-1}v$ in $L^4_{\text{loc}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$ and thus $\tilde{v}_n \to (\mathcal{L}_h^{-1}v)^3$ in $L^{\frac{4}{3}}_{\text{loc}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$. Therefore $v = (\mathcal{L}_h^{-1}v)^3$ holds, that is, v is a critical point of J. In particular, we have $\mathfrak{c}_{\text{gs}} \neq \infty$.

Part 3: Now let (v_n) be a sequence of critical points of J with $J(v_n)\to\mathfrak{c}_{\text{gs}}$. Arguing as in part 2, up to a subsequence and shifts $x_n = k_nX \in \mathbb{R}$ we have $\tilde{v}_n := \tau^{k_n}v_n \rightharpoonup v$, where v is a nonzero critical point of J. For the energy level of v we calculate

$$J(v) = J(v) - \frac{1}{2}J'(v)[v] = \frac{1}{4}\|v\|_{\frac{4}{3}}^{\frac{4}{3}} \le \liminf_{n\to\infty}\frac{1}{4}\|\tilde{v}_n\|_{\frac{4}{3}}^{\frac{4}{3}} = \liminf_{n\to\infty} J(\tilde{v}_n) - \frac{1}{2}J'(\tilde{v}_n)[\tilde{v}_n] = \mathfrak{c}_{\text{gs}}.$$

Since also $J(v) \ge \mathfrak{c}_{\text{gs}}$ by definition of the ground state energy, we see that v is a ground state. □

Proposition 3.4.11. *Assume (A3.1)–(A3.6) and (A3.8b). Then there exists a ground state of* (3.23).

Proof. Part 1: We consider the *periodic* functional

$$J^{\text{per}}(v) = \int_{\mathbb{R}\times\mathbb{T}} \tfrac{3}{4}|v|^{\frac{4}{3}} - \tfrac{1}{2}\mathcal{L}_{h^{\text{per}}}^{-1}v\cdot v\,\mathrm{d}(x,t)$$

on $L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})$ which has a ground state v^{per} due to Lemma 3.4.10. We denote its energy by $\mathfrak{c}_{\text{gs}}^{\text{per}} := J^{\text{per}}(u^{\text{per}})$. Recall that $0 < h^{\text{per}} \le h^{\text{per}} + h^{\text{loc}} = h$ holds by assumption (A3.8b). Setting $v := \left(\frac{h^{\text{per}}}{h}\right)^{\frac{1}{4}}v^{\text{per}}$, we estimate

$$\begin{aligned} J(sv) &= \int_{\mathbb{R}\times\mathbb{T}} \frac{3}{4}s^{\frac{4}{3}}|v|^{\frac{4}{3}} - \frac{1}{2}s^2\mathcal{L}_h^{-1}v\cdot v\,\mathrm{d}(x,t) \\ &= \int_{\mathbb{R}\times\mathbb{T}} \frac{3}{4}s^{\frac{4}{3}}\left(\tfrac{h^{\text{per}}}{h}\right)^{\frac{1}{3}}|v^{\text{per}}|^{\frac{4}{3}} - \frac{1}{2}s^2\mathcal{L}_{h^{\text{per}}}^{-1}v^{\text{per}}\cdot v^{\text{per}}\,\mathrm{d}(x,t) \\ &\le J^{\text{per}}(sv^{\text{per}}) \end{aligned}$$

for any $s > 0$. For the particular value $s_0 := \left(\frac{3}{2}\right)^{\frac{3}{2}}$ we calculate

$$J^{\text{per}}(s_0v^{\text{per}}) = \frac{27}{16}(J^{\text{per}})'(v^{\text{per}})[v^{\text{per}}] = 0.$$

In particular, we can estimate the mountain pass energy via

$$\mathfrak{c}_{\text{mp}} := \mathfrak{c}_{\text{mp}}(s_0v) \le \max_{s\in[0,s_0]} J(sv) \le \max_{s\in[0,s_0]} J^{\text{per}}(sv^{\text{per}}) = J^{\text{per}}(v^{\text{per}}) =: \mathfrak{c}_{\text{gs}}^{\text{per}}.$$

Part 2: Let us first consider the case $\mathfrak{c}_{\text{mp}} = \mathfrak{c}_{\text{gs}}^{\text{per}}$. From the above (in)equality we conclude that $v^{\text{per}} = 0$ holds almost everywhere on $\{h \neq h^{\text{per}}\}$. In particular, we have $v = v^{\text{per}}$, and from

$$(v^{\text{per}})^{\frac{1}{3}} - \mathcal{L}_{h^{\text{per}}}^{-1}v^{\text{per}} = 0$$

we see that $\mathcal{L}_{h^{\text{per}}}^{-1} v^{\text{per}}$ also vanishes on $\{h \neq h^{\text{per}}\}$. Therefore

$$(v)^{\frac{1}{3}} - \mathcal{L}_h^{-1} v = (v^{\text{per}})^{\frac{1}{3}} - \mathcal{L}_{h^{\text{per}}}^{-1} = 0,$$

and $v = v^{\text{per}}$ is a ground state of both J and J^{per}.

Part 3: Let us now consider the case $\mathfrak{c}_{\text{mp}} < \mathfrak{c}_{\text{gs}}^{\text{per}}$. Let v_n be a Palais-Smale sequence for J with $J(v_n) \to \mathfrak{c}_{\text{mp}}$. Then up to a subsequence $v_n \rightharpoonup v \in L^{\frac{4}{3}}(\mathbb{R} \times \mathbb{T})$ by Remark 3.4.7. In order to show $v \neq 0$, let us assume for a contradiction that $v = 0$ so that $\mathcal{L}_h^{-1} v_n \to 0$ in L^4_{loc} by Proposition 3.3.2. As in the proof of Lemma 3.4.10 there exists a subsequence, again denoted by v_n, such that suitable shifts $\tilde{v}_n = \tau^{k_n} v_n$ weakly converge to some $0 \neq \tilde{v} \in L^{\frac{4}{3}}(\mathbb{R} \times \mathbb{T}; \mathbb{R})$. From $v_n^3 = \mathcal{L}_h^{-1} v_n + \mathrm{o}(1) \to 0$ in L^4_{loc} we conclude $|k_n| \to \infty$. For compactly supported $\varphi \in L^{\frac{4}{3}}(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ we calculate

$$\begin{aligned}
&\left|(J^{\text{per}})'(v_n)[\tau^{-k_n}\varphi] - J'(v_n)[\tau^{-k_n}\varphi]\right| \\
&\quad = \left| \int_{\mathbb{R}\times\mathbb{T}} \left(\mathcal{L}_h^{-1} - \mathcal{L}_{h^{\text{per}}}^{-1}\right) v_n \cdot \tau^{-k_n}\varphi \, \mathrm{d}(x,t) \right| \\
&\quad = \left| \int_{\mathbb{R}\times\mathbb{T}} \left((h^{\frac{1}{4}} - (h^{\text{per}})^{\frac{1}{4}}) \iota \mathcal{L}^{-1} \iota' h^{\frac{1}{4}} + (h^{\text{per}})^{\frac{1}{4}} \iota \mathcal{L}^{-1} \iota' (h^{\frac{1}{4}} - (h^{\text{per}})^{\frac{1}{4}}) \right) v_n \cdot \tau^{-k_n}\varphi \, \mathrm{d}(x,t) \right| \\
&\quad \leq \|\iota\|^2 \left\|\mathcal{L}^{-1}\right\| \left(\left\|h^{\frac{1}{4}} v_n\right\|_{\frac{4}{3}} \left\|(h^{\frac{1}{4}} - (h^{\text{per}})^{\frac{1}{4}}) \tau^{-k_n}\varphi\right\|_{\frac{4}{3}} + \left\|(h^{\frac{1}{4}} - (h^{\text{per}})^{\frac{1}{4}}) v_n\right\|_{\frac{4}{3}} \left\|(h^{\text{per}})^{\frac{1}{4}} \tau^{-k_n}\varphi\right\|_{\frac{4}{3}} \right) \\
&\quad \lesssim \|v_n\|_{\frac{4}{3}} \left\|(h^{\text{loc}})^{\frac{1}{4}} \tau^{-k_n}\varphi\right\|_{\frac{4}{3}} + \left\|(h^{\text{loc}})^{\frac{1}{4}} v_n\right\|_{\frac{4}{3}} \|\varphi\|_{\frac{4}{3}} \to 0
\end{aligned}$$

as $n \to \infty$ since h^{loc} is localized by ($\mathcal{A}$3.8b). Thus we have

$$\int_{\mathbb{R}\times\mathbb{T}} \tilde{v}_n^{\frac{1}{3}} \varphi - \mathcal{L}_{h^{\text{per}}}^{-1} \tilde{v}_n \cdot \varphi \, \mathrm{d}(x,t) = (J^{\text{per}})'(\tilde{v}_n)[\varphi] = (J^{\text{per}})'(v_n)[\tau^{-k_n}\varphi] \to 0.$$

Using $\mathcal{L}_{h^{\text{per}}}^{-1} \tilde{v}_n \to \mathcal{L}_{h^{\text{per}}}^{-1} \tilde{v}$ in L^4_{loc}, the above shows $\tilde{v}_n \to (\mathcal{L}_{h^{\text{per}}}^{-1} \tilde{v})^3$ in $L^{\frac{4}{3}}_{\text{loc}}$, so that $\tilde{v}^{\frac{1}{3}} - \mathcal{L}_{h^{\text{per}}}^{-1} \tilde{v} = 0$. As $\tilde{v}$ is a nonzero solution to the periodic problem, we have

$$\begin{aligned}
\mathfrak{c}_{\text{gs}}^{\text{per}} &\leq J^{\text{per}}(\tilde{v}) = J^{\text{per}}(\tilde{v}) - 2(J^{\text{per}})'(\tilde{v})[\tilde{v}] = \frac{1}{4} \|\tilde{v}\|_{\frac{4}{3}}^{\frac{4}{3}} \\
&\leq \liminf_{n\to\infty} \frac{1}{4} \|\tilde{v}_n\|_{\frac{4}{3}}^{\frac{4}{3}} = \liminf_{n\to\infty} J(v_n) - 2J'(v_n)[v_n] = \mathfrak{c}_{\text{mp}},
\end{aligned}$$

a contradiction.

So we have shown $v_n \rightharpoonup v \neq 0$. By testing with compactly supported φ, arguing as above we see that v solves $v^{\frac{1}{3}} - \mathcal{L}_h^{-1} v = 0$ and $J(v) \leq \liminf_{n\to\infty} J(v_n) = \mathfrak{c}_{\text{mp}}$ holds. So we have found a critical point v with $J(v) \leq \mathfrak{c}_{\text{mp}} < \mathfrak{c}_{\text{gs}}^{\text{per}}$.

Part 4: By parts 2 or 3 we have $\mathfrak{c}_{\text{gs}} \leq \mathfrak{c}_{\text{gs}}^{\text{per}}$. We then choose v_n to be a Palais-Smale sequence with $J(v_n) \to \mathfrak{c}_{\text{gs}}$. Repeating the arguments of parts 2 or 3 we obtain a nonzero critical point v of J satisfying $J(v) \leq \mathfrak{c}_{\text{gs}}$. By definition of the ground state energy, v is a ground state. □

Lastly, we discuss a multiplicity result for solutions.

Proposition 3.4.12. *Assume in addition that the set $\mathfrak{R}$ is infinite. Then there exist infinitely many solutions of* (3.23) *that are not spatiotemporal shifts of each other.*

Proof. For fixed $m \in \mathbb{N}_{\text{odd}}$, we seek $\frac{T}{2m}$-antiperiodic in time solutions to (3.23), or equivalently functions with frequency support on $\mathfrak{R} \cap m\mathbb{Z}_{\text{odd}}$. These solutions are precisely critical points of J restricted to the space of $\frac{T}{2m}$-antiperiodic functions.

For infinitely many m we have $\mathfrak{R} \cap m\mathbb{Z}_{\text{odd}} \neq \emptyset$, and for these m as in Theorem 3.4.8 we obtain existence of a nonzero $\frac{T}{m}$-periodic solution v_m to (3.23). Each v_m has a minimal temporal period $T_m > 0$ which is a divisor of $\frac{T}{m}$. From $0 < T_m \to 0$ as $n \to \infty$ we see that there exist infinitely many distinct minimal periods, and clearly the corresponding solutions v_m are not shifts of one another. □

3.5. Regularity

In this section, we discuss differentiability and integrability properties of a solution v to (3.23) as well as for corresponding solutions u to (3.17), w to (3.10) and $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ to Maxwell's equations.

Lemma 3.5.1. *Let $v \in L^{\frac{4}{3}}(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ be a solution to (3.23). Then $u := h^{-\frac{1}{4}} v^{\frac{1}{3}}$ lies in $\mathcal{H}$ and is a weak solution to (3.17) with $\partial_t^l \partial_x^m u \in L^2 \cap L^\infty(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ for $l \in \mathbb{N}_0$ and $m \in \{0, 1, 2\}$.*

Proof. Part 1: As v solves (3.23), the function u satisfies

$$u = h^{-\frac{1}{4}} \mathcal{L}_h^{-1} v = \iota \mathcal{L}^{-1} \iota' h^{\frac{1}{4}} v = \iota \mathcal{L}^{-1} \iota' h u^3,$$

so that $u = \mathcal{L}^{-1} \iota' h^{\frac{1}{4}} v \in \mathcal{H}$. Then for $\varphi \in \mathcal{H}$ we have

$$0 = \mathcal{L}[u - \mathcal{L}^{-1} \iota' h u^3][\varphi] = \mathcal{L}[u][\varphi] - \int_{\mathbb{R} \times \mathbb{T}} h u^3 \varphi \, \mathrm{d}(x, t),$$

i.e., u is a weak solution to (3.17).

Part 2: According to Remark 3.3.4, we fix $\varepsilon > 0$ such that $|\partial_t|^\varepsilon \colon \mathcal{H} \to L^4(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ is bounded, therefore $|\partial_t|^\varepsilon u \in L^4(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ holds. From the fractional Leibniz rule (cf. [12]) we obtain $|\partial_t|^\varepsilon [u^3] \in L^{\frac{4}{3}}(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ with norm $\| |\partial_t|^\varepsilon [u^3] \|_{\frac{4}{3}} \lesssim (\|u\|_4 + \| |\partial_t|^\varepsilon u \|_4)^3$. Thus

$$|\partial_t|^\varepsilon u = |\partial_t|^\varepsilon \mathcal{L}^{-1} \iota' h u^3 = \mathcal{L}^{-1} \iota' h |\partial_t|^\varepsilon [u^3] \in \mathcal{H}$$

holds, and from the embedding $|\partial_t|^{2\varepsilon} u \in L^4$ follows. Iterating the above argument, we get $|\partial_t|^{n\varepsilon} \in \mathcal{H}$ for all $n \in \mathbb{N}_0$. Recall that $\mathcal{H}$ is supported on frequencies $k \in \mathfrak{R}$ and that $0 \notin \mathfrak{R}$. Therefore $|\partial_t|^s \colon \mathcal{H} \to \mathcal{H}$ is bounded for all $s \leq 0$, and the above shows $|\partial_t|^s u \in \mathcal{H}$ for all $s \in \mathbb{R}$.

Part 3: From boundedness of the embedding $\mathcal{H} \hookrightarrow L^2$ (see Proposition 3.3.2) it follows that $|\partial_t|^s u \in L^2$ for all $s \in \mathbb{R}$. We next calculate

$$\| |\partial_t|^s u \|_\infty^2 \leq \left(\sum_{k \in \mathfrak{R}} |\omega k|^s \|\hat{u}_k\|_\infty \right)^2 \leq \sum_{k \in \mathfrak{R}} \frac{1}{\omega^2 k^2} \cdot \sum_{k \in \mathfrak{R}} |\omega k|^{2s+2} \|\hat{u}_k\|_\infty^2 \lesssim \sum_{k \in \mathfrak{R}} |\omega k|^{2s+2} \|\hat{u}_k\|_{H^1}^2.$$

Using Lemma 3.2.3 and Remark 3.2.5 and assumption ($\mathcal{A}3.2$) we further estimate

$$\sum_{k \in \mathfrak{R}} |\omega k|^{2s+2} \|\hat{u}_k\|_{H^1}^2 \lesssim \sum_{k \in \mathfrak{R}} |\omega k|^{2s+2} \int_{\mathbb{R}} (1 + \lambda) T_i[\hat{u}_k] \overline{T_j[\hat{u}_k]} \, \mathrm{d}\mu_{ij}(\lambda)$$

$$
\begin{aligned}
&\lesssim \sum_{k\in\mathfrak{R}} |k|^{2s+4-\gamma} \int_{\mathbb{R}} \left|\lambda-\omega^2k^2\right| T_i[\hat{u}_k]\overline{T_j[\hat{u}_k]}\, \mathrm{d}\mu_{ij}(\lambda) \\
&\lesssim \sum_{k\in\mathfrak{R}} |\omega k|^{2s+6-\gamma-\alpha} \int_{\mathbb{R}} \left|\frac{\lambda-\omega^2k^2}{\omega^2k^2\hat{\mathcal{N}}_k}\right| T_i[\hat{u}_k]\overline{T_j[\hat{u}_k]}\, \mathrm{d}\mu_{ij}(\lambda) \\
&= \left\| |\partial_t|^{\frac{2s+6-\gamma-\alpha}{2}} u \right\|_{\mathcal{H}}^2 < \infty
\end{aligned}
$$

for any $s\in\mathbb{R}$. In particular, we have $\partial_t^l u \in L^2\cap L^\infty$ for all $l\in\mathbb{N}_0$.

Part 4: For regularity of spatial derivatives, from (3.14) we obtain the identity

$$\partial_x^2 u = V(x)\partial_t^2 u + \partial_t^2 \mathcal{G} * u + h\partial_t^2 \mathcal{N} * P_{\mathfrak{R}}[u^3]. \tag{3.24}$$

Observe that the right-hand side of (3.24) is infinitely differentiable in time with values in $L^2\cap L^\infty$ since u, u^3 are and the convolution operators $\mathcal{G}*, \mathcal{N}*, P_{\mathfrak{R}}$ are regularity preserving (see Lemma 3.5.2 below), so the same holds for $\partial_x^2 u$. □

Lemma 3.5.2. *Let M be a Fourier multiplier with symbol $\hat{M}_k$ of at most polynomial growth, i.e., $\left|\hat{M}_k\right| \lesssim |k|^r$ for some $r>0$ and all $k\in\mathbb{Z}$. Further let $p\in[1,\infty]$ and $f\colon \mathbb{T}\to\mathbb{C}$ be a function with $\partial_t^n f\in L^p(\mathbb{T};\mathbb{C})$ for all $n\in\mathbb{N}_0$ and $\hat{f}_0=0$. Then $\partial_t^n Mf\in L^p(\mathbb{T};\mathbb{C})$ for all $n\in\mathbb{N}_0$.*

Proof. The symbol $(\mathrm{i}\omega k)^{-\lceil r+1\rceil}\hat{M}_k$ is square-summable and therefore

$$m(t) := \sum_{k\in\mathbb{Z}\setminus\{0\}} (\mathrm{i}\omega k)^{-\lceil r+1\rceil}\hat{M}_k e_k(t)$$

converges in $L^2(\mathbb{T};\mathbb{C})$. The claim follows from this and

$$\|\partial_t^n Mf\|_p = \left\|\partial_t^{n+\lceil r+1\rceil} m * f\right\|_p \le \|m\|_1 \left\|\partial_t^{n+\lceil r+1\rceil} f\right\|_p. \qquad \square$$

After having shown regularity of the solution u to (3.17), we now consider the profile w of the electric field. For polarization (3.8.i) there is nothing to do since $w=u$. So let us discuss polarization (3.8.ii) where w is defined by (3.18).

Lemma 3.5.3. *Let $v\in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$ be a solution to (3.23). We define $u := h^{-\frac{1}{4}}v^{\frac{1}{3}}$ as in Lemma 3.5.1, and w by (3.18). Then $\partial_t^l\partial_x^m w\in L^2\cap L^\infty(\mathbb{R}\times\mathbb{T};\mathbb{R})$ for all $l\in\mathbb{N}_0$ and $m\in\{0,1,2\}$.*

Proof. Part 1: By Lemmas 3.5.1 and 3.5.2 we already know that the functions

$$\partial_t^l P_{\mathfrak{R}} w = (\mathcal{N}*)^{-1}\partial_t^l u \qquad \text{and} \qquad \partial_t^{l+2}(\mathrm{Id} - P_{\mathfrak{R}})[u^3]$$

lie in $L^2\cap L^\infty$ for any $l\in\mathbb{N}_0$. Hence it remains to show

$$\partial_t^l(\mathrm{Id} - P_{\mathfrak{R}})w = \left(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*\right)^{-1}[-h(x)\partial_t^{l+2}(\mathrm{Id} - P_{\mathfrak{R}})[u^3]] \in L^2\cap L^\infty,$$

and in particular that $-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*$ is invertible on suitable spaces of functions f satisfying $P_{\mathfrak{R}} f = 0$.

Part 2: Taking the Fourier series in time decomposes the linear operator $-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2 \mathcal{G}(x)*$ into the sequence of operators $(L_k)_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}}$ with

$$L_k := -\partial_x^2 - \omega^2 k^2 V(x) - \omega^2 k^2 \hat{\mathcal{G}}_k(x)$$

which we split into the main part $L_{k,0}$ and the perturbation $L_{k,1}$ via

$$L_{k,0} := -\partial_x^2 - \omega^2 k^2 V(x), \qquad L_{k,1} := -\omega^2 k^2 \hat{\mathcal{G}}_k(x).$$

First, for $\varphi \in H^2(\mathbb{R};\mathbb{C})$ using Lemma 3.2.3 and Remark 3.2.5 we calculate

$$\begin{aligned}
\int_{\mathbb{R}} |L_{k,0}\varphi|^2 \frac{1}{V} \mathrm{d}x &= \int_{\mathbb{R}} (\lambda - \omega^2 k^2)^2 T_i[\varphi]\overline{T_j[\varphi]} \,\mathrm{d}\mu_{ij}(\lambda) \\
&\geq \int_{\mathbb{R}} \frac{(\tilde{\delta}|k|^{\tilde{\gamma}})^2}{2} \left(1 + \frac{\lambda^2}{(\omega^2 k^2 + \delta|k|^{\tilde{\gamma}})^2}\right) T_i[\varphi]\overline{T_j[\varphi]} \,\mathrm{d}\mu_{ii}(\lambda) \\
&\gtrsim |k|^{2\tilde{\gamma}-4} \int_{\mathbb{R}} (1+\lambda^2) T_i[\varphi]\overline{T_j[\varphi]} \,\mathrm{d}\mu_{ij}(\lambda) \\
&= |k|^{2\tilde{\gamma}-4} \int_{\mathbb{R}} \Big(\Big|\frac{1}{V}\varphi''\Big|^2 + |\varphi|^2 \Big) V \mathrm{d}(x),
\end{aligned}$$

and it follows that $L_{k,0}\colon H^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ is invertible and $\left\|L_{k,0}^{-1}\right\| \lesssim |k|^{2-\tilde{\gamma}}$ holds. For $L_{k,1}$, using ($\mathcal{A}$3.7) and Remark 3.2.5 we estimate

$$\begin{aligned}
\int_{\mathbb{R}} |L_{k,1}\varphi|^2 \frac{1}{V} \mathrm{d}x &\leq (\omega^2 k^2)^2 \left(\frac{\tilde{d}}{\omega^2 |k|^{2-\tilde{\gamma}}}\right)^2 \int_{\mathbb{R}} |\varphi|^2 V \mathrm{d}x = \left(\tilde{d}|k|^{\tilde{\gamma}}\right)^2 \int_{\mathbb{R}} T_i[\varphi]\overline{T_j[\varphi]} \,\mathrm{d}\mu_{ij}(\lambda) \\
&\leq \left(\tilde{d}|k|^{\tilde{\gamma}}\right)^2 \int_{\mathbb{R}} \left(\frac{|\lambda - \omega^2 k^2|}{\tilde{\delta}|k|^{\tilde{\gamma}}}\right)^2 T_i[\varphi]\overline{T_j[\varphi]} \,\mathrm{d}\mu_{ij}(\lambda) = \frac{\tilde{d}^2}{\tilde{\delta}^2} \int_{\mathbb{R}} |L_{k,0}\varphi|^2 \frac{1}{V} \mathrm{d}x.
\end{aligned}$$

Since $\tilde{d} < \tilde{\delta}$, the Neumann series shows that $L_k = L_{k,0} + L_{k,1}\colon H^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C};\frac{1}{V}\mathrm{d}x)$ is invertible with

$$\left\|L_k^{-1}\right\| \leq \frac{\left\|L_{k,0}^{-1}\right\|}{1 - \left\|L_{k,1}L_{k,0}^{-1}\right\|} \leq \frac{\left\|L_{k,0}^{-1}\right\|}{1 - \frac{\tilde{d}}{\tilde{\delta}}} \lesssim |k|^{2-\tilde{\gamma}}.$$

Part 3: We define the inverse of the linear operator by

$$\left(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*\right)^{-1} \varphi := \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} L_k^{-1}[\hat{\varphi}_k](x) e_k(t)$$

for φ such that the series on the right-hand side converges in L^2. Let us abbreviate

$$f := -h(x)\partial_t^{l+2}(\operatorname{Id} - P_{\mathfrak{R}})[u^3],$$

which lies in $L^2 \cap L^\infty$ by Lemma 3.5.2 and boundedness of h, and recall

$$\partial_t^l(\operatorname{Id} - P_{\mathfrak{R}})w = \left(-\partial_x^2 + V(x)\partial_t^2 + \partial_t^2\mathcal{G}(x)*\right)^{-1} f = \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} L_k^{-1}[\hat{f}_k](x) e_k(t).$$

We show that this term lies in $L^2 \cap L^\infty$. First we estimate

$$\left\|\partial_t^l(\operatorname{Id} - P_{\mathfrak{R}})w\right\|_2^2 = \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} \left\|L_k^{-1}\hat{f}_k\right\|_2^2 \lesssim \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} \left\|L_k^{-1}\hat{f}_k\right\|_{H^2}^2$$

$$\lesssim \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} |\omega k|^{2\lceil 2-\tilde{\gamma}\rceil} \left\|\hat{f}_k\right\|_2^2 = \left\|\partial_t^{\lceil 2-\tilde{\gamma}\rceil} f\right\|_2^2 < \infty.$$

Next for L^∞ we have

$$\begin{aligned}\left\|\partial_t^l(\mathrm{Id} - P_{\mathfrak{R}})w\right\|_\infty^2 &\le \left(\sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} \left\|L_k^{-1}\hat{f}_k\right\|_\infty\right)^2 \lesssim \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} k^2 \left\|L_k^{-1}\hat{f}_k\right\|_\infty^2 \\ &\lesssim \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} k^2 \left\|L_k^{-1}\hat{f}_k\right\|_{H^2}^2 \lesssim \sum_{k\in\mathbb{Z}_{\text{odd}}\setminus\mathfrak{R}} |\omega k|^{2\lceil 3-\tilde{\gamma}\rceil} \left\|\hat{f}_k\right\|_2^2 = \left\|\partial_t^{\lceil 3-\tilde{\gamma}\rceil} f\right\|_2^2 < \infty.\end{aligned}$$

Combined, we have shown that $\partial_t^l w = \partial_t^l P_{\mathfrak{R}} w + \partial_t^l (1 - P_{\mathfrak{R}})w$ lies in $L^2 \cap L^\infty$.

Part 4: Using (3.16) we see

$$\partial_x^2 w = V(x)\partial_t^2 w + \partial_t^2 \mathcal{G} * w + h\partial_t^2 u^3.$$

where the right-hand side (and therefore $\partial_x^2 w$) is infinitely time-differentiable by the above and Lemma 3.5.2. □

We are now ready to prove the main theorem.

Proof of Theorem 3.1.5. Let $v \in L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T};\mathbb{R})$ be a nonzero solution to (3.23), which exists by Theorem 3.4.8. Then $u := h^{-\frac{1}{4}} v^{\frac{1}{3}}$ is the corresponding solution of (3.17). Define $w := u$ for polarization (3.3.i), or define w by (3.18) for polarization (3.3.ii). Then $\partial_t^l \partial_x^m w \in L^2 \cap L^\infty(\mathbb{R}\times\mathbb{T};\mathbb{R})$ for $l \in \mathbb{N}_0$ and $m \in \{0, 1, 2\}$ by Lemma 3.5.1 or Lemma 3.5.3, and w solves (3.10).

We define ∂_t^{-1} on $\mathbb{T}$ as the Fourier multiplier with symbol $\frac{1}{\mathrm{i}\omega k}$. Then, recalling the ansatz (3.6) and Maxwell's equations (3.1),(3.2) we obtain the following formula for $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ corresponding to the scalar field w:

$$\begin{aligned}\mathbf{E}(x,y,z,t) &= w(x, t - \tfrac{1}{c}z)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix},\\ \mathbf{B}(x,y,z,t) &= -\partial_t^{-1}\nabla\times\mathbf{E} = -\begin{pmatrix}\frac{1}{c}w(x,t-\frac{1}{c}z)\\0\\W_x(x,t-\frac{1}{c}z)\end{pmatrix},\\ \mathbf{H}(x,y,z,t) &= \frac{1}{\mu_0}\mathbf{B}(x,y,z,t),\\ \mathbf{D}(x,y,z,t) &= \epsilon_0(\mathbf{E} + \mathbf{P}(\mathbf{E}))\end{aligned}$$

where we have set $W := \partial_t^{-1} w$. From the regularity of w it follows that

$$\partial_t^l\mathbf{E} \in H^2 \cap W^{2,\infty}(\Omega;\mathbb{R}^3), \qquad \partial_t^l\mathbf{B}, \partial_t^l\mathbf{H} \in H^1 \cap W^{1,\infty}(\Omega;\mathbb{R}^3), \qquad \partial_t^l\mathbf{D} \in L^2 \cap L^\infty(\Omega;\mathbb{R}^3)$$

for $l \in \mathbb{N}_0$ and domains of the form $\Omega = \mathbb{R}\times[y_1, y_2]\times[z_2, z_2]\times[t_1, t_2]$. Note that $\mathbf{D}$ may not be differentiable in space because the material coefficients g_0, g_1, h of $\mathbf{P}$ were not assumed to be smooth. □

3.A. Appendix

Here we give a proof of Lemma 3.2.3 on fundamental properties of the transform T. We also prove Theorems 3.1.2 and 3.1.3.

Proof of Lemma 3.2.3. (a): See Lemma 2.3.7.

(b): If $f \in H^2(\mathbb{R};\mathbb{C})$, using (a) and that T is an isometry we have

$$\int_{\mathbb{R}} |f'|^2 \,\mathrm{d}x = \int_{\mathbb{R}} Lf \cdot \overline{f}\, V \mathrm{d}x = \int_{\mathbb{R}} T^{-1}[\lambda T[f](\lambda)] \cdot \overline{f}\, V \mathrm{d}x = \int_{\mathbb{R}} \lambda T_i[f](\lambda)\overline{T_j[f](\lambda)}\,\mathrm{d}\mu_{ij}(\lambda).$$

For general f, one can argue by approximation.

(c): First let $\lambda_0 \in \mathbb{C} \setminus \operatorname{supp}(\mu)$. Then $\frac{1}{\lambda-\lambda_0}$ is a bounded symbol on $\operatorname{supp}(\mu)$, thus

$$(L-\lambda_0)^{-1} = T^{-1}\Big[\frac{1}{\lambda-\lambda_0} T[\,\cdot\,](\lambda)\Big]$$

is bounded, i.e., $\lambda_0 \in \rho(L)$ holds.

Now let $\lambda_0 \in \operatorname{supp}(\mu)$, i.e., $\lambda_0 \in \operatorname{supp}(\mu_{ij})$ for some $i,j \in \{1,2\}$. By definition we have

$$|\mu_{ij}|(B_\varepsilon(\lambda_0)) > 0$$

for all $\varepsilon > 0$, with $|\mu_{ij}|$ denoting the total variation of μ_{ij}. Split $B_\varepsilon(\lambda_0) = E^+ \cup E^-$ with $E^+ \cap E^- = \emptyset$ into positive and negative part according to the measure μ_{ij}. Then we have

$$0 < |\mu_{ij}|(B_\varepsilon(\lambda_0)) = \mu_{ij}(E^+) - \mu_{ij}(E^-) \le \sqrt{\mu_{ii}(E^+)\mu_{jj}(E^+)} + \sqrt{\mu_{ii}(E^-)\mu_{jj}(E^-)}$$

since $\mu(E^\pm)$ are positive semidefinite matrices. This implies

$$\mu_{ii}(B_\varepsilon(\lambda_0)) = \mu_{ii}(E^+) + \mu_{ii}(E^-) > 0, \qquad \mu_{jj}(B_\varepsilon(\lambda_0)) = \mu_{jj}(E^+) + \mu_{jj}(E^-) > 0$$

since μ_{ii}, μ_{jj} are a nonnegative measures. This shows $\lambda_0 \in \operatorname{supp}(\mu_{ii}) \cap \operatorname{supp}(\mu_{jj})$.

We denote by δ_i the i-th unit vector, and consider the function $f := T^{-1}[\mathbb{1}_{B_\varepsilon(\lambda_0)}\delta_i]$. Then since

$$\frac{\|(L-\lambda_0)f\|_{L^2_V}}{\|f\|_{L^2_V}} = \frac{\left\|(\lambda-\lambda_0)\mathbb{1}_{B_\varepsilon(\lambda_0)}\right\|_{L^2(\mu_{ii})}}{\left\|\mathbb{1}_{B_\varepsilon(\lambda_0)}\right\|_{L^2(\mu_{ii})}} \le \varepsilon$$

for arbitrary $\varepsilon > 0$, $L - \lambda_0$ has no bounded inverse, i.e., $\lambda_0 \in \sigma(L)$ holds. □

Proof of Theorems 3.1.2 and 3.1.3. For both theorems, we check that the assumptions of Theorem 3.1.5 are satisfied. First ($\mathcal{A}$3.1), ($\mathcal{A}$3.3) hold by definition, and the same is true for ($\mathcal{A}$3.8b) in Theorem 3.1.2, and for ($\mathcal{A}$3.8a) in Theorem 3.1.3. Next we calculate the Fourier coefficients

$$\hat{\mathcal{N}}_k = \int_{\mathbb{T}} T \operatorname{dist}(t, T\mathbb{Z})\mathrm{e}^{-\mathrm{i}\omega k t}\,\mathrm{d}t = \int_0^T \operatorname{dist}(t, T\mathbb{Z})\mathrm{e}^{-\mathrm{i}\omega k t}\,\mathrm{d}t = \begin{cases} \frac{T^2}{4}, & k = 0, \\ 0, & k \neq 0 \text{ even}, \\ -\frac{T^2}{2k^2\pi^2}, & k \text{ odd}, \end{cases}$$

and see that ($\mathcal{A}3.2$) holds with $\alpha = 2$. The spectrum of the operator L is investigated in Appendix 2.B. There, we show that L has a spectral gap about $\omega^2 k^2$ for each $k \in \mathbb{Z}_{\text{odd}}$, that the size of the gap grows linearly in $|k|$, and moreover that the point spectrum grows quadratically, i.e., ($\mathcal{A}3.4$) holds with $\gamma = 1$, $\beta = \frac{1}{2}$ and some $\delta > 0$. Next ($\mathcal{A}3.5$) holds since $\alpha + \gamma - 2 = 1 > \frac{1}{2} = \beta$.

We calculate

$$\hat{\mathcal{G}}_k(x) = g_1^{\text{per}}(x) \int_0^T \cos(\omega t)|\cos(\omega t)|\mathrm{e}^{-\mathrm{i}\omega k t}\,\mathrm{d}t = \begin{cases} 0, & k \text{ even}, \\ \frac{4T(-1)^n g_1^{\text{per}}(x)}{(4k-k^3)\pi}, & k = 2n+1 \text{ odd}, \end{cases}$$

and find $\left|\hat{\mathcal{G}}_k(x)\right| \leq \frac{C}{k^2} \frac{1}{\omega^2 |k|} V(x)$. Since in ($\mathcal{A}3.6$) we require $d < \delta$, this only shows ($\mathcal{A}3.6$) for sufficiently large k. Similarly, ($\mathcal{A}3.7$) holds with $s = 2$ but only for sufficiently large k.

This restriction to large frequencies is not an issue: Because $\mathfrak{N} = \left\{k \in \mathbb{Z}_{\text{odd}} : \hat{\mathcal{N}}_k \neq 0\right\} = \mathbb{Z}_{\text{odd}}$ is infinite, by Proposition 3.4.12 and its proof there exist infinitely many distinct breathers that are supported only on large frequencies. $\square$

CHAPTER 4

Travelling breather solutions in waveguides for cubic nonlinear Maxwell equations with retarded material laws

This chapter consists of the preprint [72], which is joint work with Wolfgang Reichel. The contents were edited to better fit with the thesis.

——————— Start of Preprint ———————

Abstract. For Maxwell's equations with nonlinear polarization we prove the existence of time-periodic breather solutions travelling along slab or cylindrical waveguides. The solutions are TE-modes which are localized in one (slab case) or both (cylindrical case) space directions orthogonal to the direction of propagation. We assume a magnetically inactive and electrically nonlinear material law with a linear $\chi^{(1)}$- and a cubic $\chi^{(3)}$-contribution to the polarization. The $\chi^{(1)}$-contribution may be retarded in time or instantaneous whereas the $\chi^{(3)}$-contribution is always assumed to be retarded in time. We consider two different cubic nonlinearities which provide a variational structure under suitable assumptions on the retardation kernels. By choosing a sufficiently small propagation speed along the waveguide the second order formulation of the Maxwell system becomes essentially elliptic for the **E**-field so that solutions can be constructed by the mountain pass theorem. The compactness issues arising in the variational method are overcome by either the cylindrical geometry itself or by extra assumptions on the linear and nonlinear parts of the polarization in case of the slab geometry. Our approach to breather solutions in the presence of time-retardation is systematic in the sense that we look for general conditions on the Fourier-coefficients in time of the retardation kernels. Our main existence result is complemented by concrete examples of coefficient functions and retardation kernels.

4.1. Introduction

We show existence and regularity of spatially localized, real-valued and time-periodic solutions (called breathers) to Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\mathbf{B}_t, \\ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{D}_t, \end{aligned} \tag{4.1}$$

without charges and currents. (4.1) is posed on all of $\mathbb{R}^3$ with an underlying material that is either a slab waveguide or a cylindrically symmetric waveguide. We look for solutions that are travelling parallel to the direction of the waveguide, and which are transverse-electric, i.e., the electric field **E** is orthogonal to the direction of travel. We assume that the material satisfies the constitutive relations

$$\mathbf{B} = \mu_0 \mathbf{H}, \qquad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}) \tag{4.2}$$

where $\mu_0, \epsilon_0 \in (0, \infty)$ are the vacuum permeability and permittivity, respectively. This means that the material is magnetically inactive. However, the displacement field **D** depends nonlinearly on the electric field $\mathbf{E}$ through the polarization field $\mathbf{P}(\mathbf{E})$, which is modeled as a sum of a linear and a cubic function of $\mathbf{E}$. Both parts are local in space but nonlocal in time (cf. [3] for a physical motivation) and are given by

$$\begin{aligned}\mathbf{P}(\mathbf{E})(x,t) &= \epsilon_0 \int_0^\infty \chi^{(1)}(\mathbf{x},\tau)[\mathbf{E}(\mathbf{x},t-\tau)]\,\mathrm{d}\tau \\ &+ \epsilon_0 \int_0^\infty \int_0^\infty \int_0^\infty \chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{E}(\mathbf{x},t-\tau_1),\mathbf{E}(\mathbf{x},t-\tau_2),\mathbf{E}(\mathbf{x},t-\tau_3)]\,\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3. \quad (4.3)\end{aligned}$$

Here $\mathbf{x}=(x,y,z)$ denotes the spatial variable, the susceptibility tensor $\chi^{(1)}(\mathbf{x},\tau)\colon \mathbb{R}^3 \to \mathbb{R}^3$ is linear and $\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)\colon \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$ is trilinear.

By taking the curl of Faraday's law $\nabla\times\mathbf{E} = -\mathbf{B}_t$, we obtain from (4.1),(4.2) the second order form of Maxwell's equations

$$\nabla\times\nabla\times\mathbf{E} + \epsilon_0\mu_0\partial_t^2\mathbf{E} + \mu_0\partial_t^2\mathbf{P}(\mathbf{E}) = 0. \quad (4.4)$$

While (4.4) is an equation only for $\mathbf{E}$, the other electromagnetic fields can be recovered if (4.4) holds: $\mathbf{B}$ is obtained from $\nabla\times\mathbf{E}=-\mathbf{B}_t$ by time-integration, and $\mathbf{H},\mathbf{D}$ are then determined by the material laws (4.2). Next, $\mathbf{B}$ is divergence-free if it is divergence-free at time 0 since $\partial_t\nabla\cdot\mathbf{B} = -\nabla\cdot(\nabla\times\mathbf{E}) = 0$. Lastly, $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}(\mathbf{E})$ will be divergence-free because of the choices of $\mathbf{E},\mathbf{P}$ made later.

We assume that the material is either a slab waveguide or a cylindrical waveguide. In the first case, the susceptibility tensors $\chi^{(j)}$ remain constant as $\mathbf{x}$ moves parallel to the slab. Assuming that the slab is given by $\{x=0\}$, this means that

$$\chi^{(1)}(\mathbf{x},\tau)=\chi^{(1)}(x,\tau), \qquad \chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)=\chi^{(3)}(x,\tau_1,\tau_2,\tau_3). \quad (4.5.1)$$

If instead the underlying material has a cylindrical waveguide geometry, we assume that the susceptibility tensors $\chi^{(j)}$ depend only on the distance from $\mathbf{x}$ to the cylinder core which we assume to be given by $\{x=y=0\}$, so that

$$\chi^{(1)}(\mathbf{x},\tau)=\chi^{(1)}(r,\tau), \qquad \chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)=\chi^{(3)}(r,\tau_1,\tau_2,\tau_3). \quad (4.5.2)$$

where $r=\sqrt{x^2+y^2}$.

We assume that the materials are isotropic, i.e.,

$$\chi^{(1)}(\mathbf{x},\tau)[O\mathbf{v}] = O\chi^{(1)}(\mathbf{x},\tau)[\mathbf{v}], \quad \chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[O\mathbf{u},O\mathbf{v},O\mathbf{w}] = O\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{u},\mathbf{v},\mathbf{w}]$$

holds for $O\in\mathrm{SO}(3)$. This means that $\chi^{(1)}(\mathbf{x},\tau)\in\mathbb{R}I_{3\times3}$. For $\chi^{(3)}$ a variety of isotropic scenarios are possible, but in this paper we only consider two kinds of nonlinear material responses: either

$$\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{u},\mathbf{v},\mathbf{w}] = h(\mathbf{x})\nu(\tau_1)\delta(\tau_2-\tau_1)\delta(\tau_3-\tau_1)\langle\mathbf{u},\mathbf{v}\rangle\mathbf{w} \quad (4.6.\mathrm{i})$$

or

$$\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{u},\mathbf{v},\mathbf{w}] = h(\mathbf{x})\nu(\tau_1)\nu(\tau_2)\nu(\tau_3)\langle\mathbf{u},\mathbf{v}\rangle\mathbf{w} \quad (4.6.\mathrm{ii})$$

where δ denotes the Dirac measure at 0, $\langle\,\cdot\,,\,\cdot\,\rangle$ is the standard inner product on $\mathbb{R}^3$, and h,ν are given real-valued functions.

For these material laws, we will see that (4.4) can be viewed as a variational problem, and we will use a simple mountain pass method in order to construct breather solutions to (4.1),(4.2).

We deal with the kernel of the curl operator in (4.4) by looking for breather solutions in special divergence-free ansatz spaces that we discuss next. For the slab geometry (4.5.1) we make the TE-polarized traveling wave ansatz

$$\mathbf{E}(\mathbf{x},t) = w(x, t - \tfrac{1}{c}z) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{4.7.1}$$

where $w\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is periodic in the second variable, which we again denote by t. For the cylindrical geometry (4.5.2) we instead consider the circular TE-polarized traveling wave ansatz

$$\mathbf{E}(\mathbf{x},t) = w(r, t - \tfrac{1}{c}z) \cdot \begin{pmatrix} -y/r \\ x/r \\ 0 \end{pmatrix} \tag{4.7.2}$$

with $r = \sqrt{x^2+y^2}$ and $w\colon (0,\infty) \times \mathbb{R} \to \mathbb{R}$ being periodic in t. Both ansatzes for the electric field are divergence-free, so that $\nabla \times \nabla \times \mathbf{E} = -\Delta \mathbf{E}$ holds, and are of a simple, essentially two-dimensional form, which greatly simplifies the discussion. More specifically, for the slab ansatz (4.7.1) problem (4.4) reduces to

$$(-\partial_x^2 - \tfrac{1}{c^2}\partial_t^2)w + \epsilon_0\mu_0\partial_t^2 w + \mu_0\partial_t^2 P(w) = 0 \tag{4.8.1}$$

and for the cylindrical ansatz (4.7.2) to

$$(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} - \tfrac{1}{c^2}\partial_t^2)w + \epsilon_0\mu_0\partial_t^2 w + \mu_0\partial_t^2 P(w) = 0. \tag{4.8.2}$$

Here, depending on the choice of the nonlinear susceptibility tensor, the scalar polarization P is given either by

$$P(w)(\mathbf{x},t) = \epsilon_0 \int_0^\infty \chi^{(1)}(\mathbf{x},\tau)w(\mathbf{x},t-\tau)\,\mathrm{d}\tau + \epsilon_0 h(\mathbf{x}) \int_0^\infty w(\mathbf{x},t-\tau)^3 \nu(\tau)\,\mathrm{d}\tau \tag{4.9.i}$$

or by

$$P(w)(\mathbf{x},t) = \epsilon_0 \int_0^\infty \chi^{(1)}(\mathbf{x},\tau)w(\mathbf{x},t-\tau)\,\mathrm{d}\tau + \epsilon_0 h(\mathbf{x}) \Big(\int_0^\infty w(\mathbf{x},t-\tau)\nu(\tau)\,\mathrm{d}\tau\Big)^3 \tag{4.9.ii}$$

for susceptibilities (4.6.i) and (4.6.ii), respectively. The simple form of the nonlinearity in $P(w)$, especially that the variables $\mathbf{x}$ and τ appear separated, are needed in order to obtain a variational problem. We denote by $*$ the convolution in time and normalize the speed of light to $c_0^2 = (\epsilon_0\mu_0)^{-1} = 1$. Then problem (4.8.1) with polarization (4.9.i), which we discuss as an example, becomes

$$\Big(-\partial_x^2 + \Big(1 - \tfrac{1}{c^2} + \chi^{(1)}*\Big)\partial_t^2\Big)w + h(\mathbf{x})(\nu * \partial_t^2)w^3 = 0.$$

Inverting the convolution operator $\nu * \partial_t^2$ formally[1], we then obtain

$$\Big(\nu * \partial_t^2\Big)^{-1}\Big(-\partial_x^2 + \Big(1 - \tfrac{1}{c^2} + \chi^{(1)}*\Big)\partial_t^2\Big)w + h(\mathbf{x})w^3 = 0. \tag{4.10}$$

Given our assumptions, we can ensure that the linear operator is symmetric when restricted to suitable spaces of time-periodic functions. Hence solutions formally are critical points of the functional

$$J(w) = \int \Big(\tfrac{1}{2}w \cdot \Big(\nu * \partial_t^2\Big)^{-1}\Big(-\partial_x^2 + \Big(1 - \tfrac{1}{c^2} + \chi^{(1)}*\Big)\partial_t^2\Big)w + \tfrac{1}{4}h(\mathbf{x})w^4\Big)\,\mathrm{d}(\mathbf{x},t). \tag{4.11}$$

[1] rigorous considerations are given later

Using the mountain pass method, we will find critical points and then show that they are breather solutions to Maxwell's equations (4.1),(4.2).

In the literature, there are several papers treating existence of breather solutions of (4.4). Many authors have considered monochromatic solutions, i.e., solutions of the form $\mathbf{E}(\mathbf{x},t) = \mathcal{E}(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t} + c.c.$, where the complex conjugate is necessary in order to make the $\mathbf{E}$-field real valued. This is a viable approach if one ignores the higher-order harmonics $\mathrm{e}^{\pm 3\mathrm{i}\omega t}$ coming from the nonlinear part of the polarization, or if one considers a nonlinear susceptibility tensor given by

$$\chi^{(3)}(\mathbf{x},\tau_1,\tau_2,\tau_3)[\mathbf{u},\mathbf{v},\mathbf{w}] = \tfrac{1}{2T}h(\mathbf{x})\mathbb{1}_{[0,T]}(\tau_1)\delta(\tau_2-\tau_1)\delta(\tau_3)\langle\mathbf{u},\mathbf{v}\rangle\mathbf{w}, \tag{4.12}$$

where $T = \frac{2\pi}{\omega}$ is the period of the breather $\mathbf{E}$. Both approaches lead to the nonlinear curl-curl problem

$$\nabla\times\nabla\times\mathcal{E} - \omega^2 g(\mathbf{x})\mathcal{E} - \omega^2 h(\mathbf{x})|\mathcal{E}|^2\mathcal{E} = 0 \tag{4.13}$$

which is variational provided $g(\mathbf{x}) = \int_0^\infty \chi^{(1)}(\mathbf{x},\tau)\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau$ is real valued. Instead of the cubic nonlinearity $h(\mathbf{x})|\mathcal{E}|^2\mathcal{E}$, saturated nonlinearities $h(\mathbf{x},|\mathcal{E}|^2)\mathcal{E}$, which grow linearly as $|\mathcal{E}|\to\infty$, are also of interest and were first investigated by Stuart et al. [62, 87–93]. In these papers divergence-free, traveling, TE- or TM-polarized ansatz functions similar to (4.7.2) were used to reduce the Maxwell problem to an elliptic one-dimensional problem and to solve it via variational methods. An extension of Stuart's approach to more general wave-guide profiles was given in [64]. Standing monochromatic breathers composed of axisymmetric divergence free ansatz functions were considered in [5, 8, 10]. The next step forward to overcome special divergence free ansatz functions was accomplished by Mederski et al. [7, 63, 65, 67] who considered the full curl-curl problem (4.13), also for more general nonlinearities $\partial_{\mathcal{E}}h(\mathbf{x},\mathcal{E})$. The difficulties arising from the infinite-dimensional kernel of $\nabla\times$ were overcome by a Helmholtz decomposition and suitable profile decompositions for Palais-Smale sequences. Alternative approaches used limiting absorption principles [59], dual variational approaches [57, 58], approximations near gap edges of photonic crystals [33], and monochromatic time-decaying solutions at interfaces of metals and dielectrics [30–32]. In the latter series of papers, also time-periodic solutions can be found if one additionally assumes $\mathcal{PT}$-symmetry of the materials.

If one does not want to rely on very specific retardation kernels as in (4.12) or if one wants to take higher harmonics into account then one is naturally led to polychromatic breather solutions, i.e., solutions which have multiple supported frequencies in time-domain. In the context of instantaneous material laws they have recently received increasing attention. As a model problem consider

$$\nabla\times\nabla\times\mathcal{E} + g(\mathbf{x})\partial_t^2\mathcal{E} + h(\mathbf{x})\partial_t^2(|\mathcal{E}|^2\mathcal{E}) = 0$$

For this problem, rigorous existence result for travelling breathers in the slab geometry (4.5.1) where either g or h contains delta distributions are given in [49] by variational methods and in [19] via bifurcation theory. Even earlier in [75] the authors used a combination of local bifurcation theory and continuation methods in a partly analytical and partly numerical study on traveling wave solutions where the linear coefficient g is a periodic arrangement of delta potentials. Another rigorous existence results for breathers on finite but large time scales can be found in [34] for a set-up of Kerr-nonlinear dielectrics occupying two different halfspaces. In our recent paper [71] we proved the first (to the best of our knowledge) existence result for polychromatic breathers in the context of nonlinear Maxwell's equations without presence of any delta-potentials. The $\chi^{(1)}$-part of the polarization was instantaneous and the $\chi^{(3)}$-part was compactly supported in space and either instantaneous or retarded. Due to the compact support in space both variants of the nonlinearity could be treated with the same variational method. Beyond this result we are not aware of any rigorous treatment of polychromatic breathers in the context of nonlinear Maxwell's equations with time retarded material laws.

4.1.1. Examples

In two theorems below, we give examples of susceptibility tensors $\chi^{(1)}, \chi^{(3)}$ for which breather solutions of Maxwell's equations (4.1),(4.2),(4.3) exist. For simplicity, we choose an instantaneous linear response while the nonlinear response has to be retarded. These examples are special cases of a general existence result given later in this chapter, cf. Theorem 4.1.3. Let us note that in contrast to some of the previously mentioned results, our breather solutions are generally polychromatic in nature and the potentials considered are bounded functions. Since our breathers lie in suitable Sobolev spaces they are sufficiently differentiable to solve Maxwell's equations pointwise, and they decay at infinity in an L^p-sense. They may have higher-order space-derivatives depending on smoothness of the material coefficients in space. They are also infinitely differentiable in time because the material properties do not change over time.

We begin with an exemplary result for the slab geometry (4.5.1).

Theorem 4.1.1. *Let $T > 0$ denote the temporal period, $\omega := \frac{2\pi}{T}$ the associated frequency, and $c \in (0, \infty)$ the speed of travel of the breather solution. Assume that the linear susceptibility tensor is given by $\chi^{(1)}(\mathbf{x}, \tau) = g(x)\delta(\tau)I_{3\times3}$, and the nonlinear susceptibility tensor $\chi^{(3)}$ is given by* (4.6.i) *or* (4.6.ii) *with*

$$h(\mathbf{x}) = h(x), \qquad \nu(\tau) = (2 - |\sin(\omega\tau)|)\mathbb{1}_{[0,T]}(\tau).$$

Moreover, assume that the potentials $g, h \in L^\infty(\mathbb{R})$ have periodic backgrounds $g^{\text{per}}, h^{\text{per}} \in L^\infty(\mathbb{R})$ with a common period, i.e.,

$$g(x) - g^{\text{per}}(x) \to 0, \qquad h(x) - h^{\text{per}}(x) \to 0 \quad \text{as } x \to \pm\infty,$$

and that the inequalities

$$g^{\text{per}} \le g, \qquad \operatorname*{ess\,sup}_{\mathbb{R}} g < \tfrac{1}{c^2} - 1, \qquad h^{\text{per}} \le h, \qquad h^{\text{per}} \not\le 0$$

are satisfied. Then there exist nonzero time-periodic solutions $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H}$ of Maxwell's equations (4.1),(4.2),(4.3) *where $\mathbf{E}$ is of the form* (4.7.1)*. They satisfy*

$$\partial_t^n \mathbf{E} \in W^{2,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{B}, \partial_t^n \mathbf{H} \in W^{1,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{D} \in L^p(\Omega; \mathbb{R}^3)$$

for all $n \in \mathbb{N}_0$, $p \in [2, \infty]$ and all domains $\Omega = \mathbb{R} \times [y, y+1] \times [z, z+1] \times [t, t+1]$, with norm bounds independent of y, z, t.

The potentials g, h describe the spatial dependency of the polarization field. In the above theorem we have required them to be asymptotically periodic at $\pm\infty$. This periodic structure helps us to overcome noncompactness of embeddings on $\mathbb{R}$. The assumption on the ordering $g \ge g^{\text{per}}, h \ge h^{\text{per}}$ is a standard tool to resolve noncompactness issues also for the nonperiodic problem. The upper bound $\frac{1}{c^2} - 1$ on g and the choice of ν ensure that (4.4) is elliptic.

One aspect of the choice of ν is that its Fourier coefficients are positive. This aspect will become very important in the general result of Theorem 4.1.3. Ellipticity will ensure that the associated energy has a mountain pass geometry, and a mountain pass method will be used to construct breather solutions. Note also that breathers are localized in the x-direction (in the L^p-sense stated above), but not in y, z, or t, which is due to the ansatz (4.7.1), since all solutions satisfying this ansatz necessarily are independent of y and periodic in both z and t.

Similar to Theorem 4.1.1 for the slab geometry, below we give an exemplary result with cylindrical geometry (4.5.2).

Theorem 4.1.2. *Let $T > 0$ be the period of the breather, $c \in (0, \infty)$ be its speed, and $g, h \in L^\infty([0, \infty))$ be material coefficients. Define the linear susceptibility by $\chi^{(1)}(\mathbf{x}, \tau) := g(r)\delta(\tau)I_{3\times3}$ and let the nonlinear susceptibility $\chi^{(3)}$ be given by* (4.6.i) *or* (4.6.ii) *with $h(\mathbf{x}) = h(r), \nu(\tau) = (2 - |\sin(\omega\tau)|)\mathbb{1}_{[0,T]}(\tau)$ where $r := \sqrt{x^2 + y^2}$, $\omega := \frac{2\pi}{T}$. Further let*

$$\operatorname*{ess\,sup}_{\mathbb{R}} g < \tfrac{1}{c^2} - 1, \qquad h \not\leq 0.$$

Then there exist nonzero time-periodic solutions $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H}$ of Maxwell's equations (4.1), (4.2), (4.3) *where $\mathbf{E}$ is of the form* (4.7.2). *They satisfy*

$$\partial_t^n \mathbf{E} \in W^{2,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{B}, \partial_t^n \mathbf{H} \in W^{1,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{D} \in L^p(\Omega; \mathbb{R}^3)$$

for all $n \in \mathbb{N}_0$, $p \in [2, \infty]$ and all domains $\Omega = \mathbb{R}^2 \times [z, z+1] \times [t, t+1]$, with norm bounds independent of z, t.

In contrast to Theorem 4.1.1, in Theorem 4.1.2 we do not need any asymptotics for the potentials g, h. This is because the cylindrical setting itself comes with compactness, as we discuss in Section 4.5. To illustrate this, recall that the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in (2, \infty)$ becomes compact when restricted to radially symmetric functions. Lastly, the ansatz (4.7.2) is periodic in both z and t, so breather solutions in the cylindrical setting decay in the x and y directions orthogonal to the direction of propagation.

4.1.2. Main theorem

Before stating the main theorem, we fix some notation.

4.1.2.1. Measures on torus and real line, periodization of a measure

Since breathers are time-periodic, the natural time domain is the torus $\mathbb{T} := \mathbb{R}/_{T\mathbb{Z}}$ with period T, and we denote the canonical projection by $P_\mathbb{T} \colon \mathbb{R} \to \mathbb{T}$. With $\mathcal{M}(\mathbb{T})$, $\mathcal{M}(\mathbb{R})$ we denote the set of all $\mathbb{R}$-valued measures λ on the Borel σ-algebra of $\mathbb{T}$ and $\mathbb{R}$, respectively, and we equip it with the total variation norm $\|\lambda\| = |\lambda|(\mathbb{T})$ or $\|\lambda\| = |\lambda|(\mathbb{R})$. The push-forward along $P_\mathbb{T}$, or periodization, $\operatorname{Per} : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{T})$ is defined as follows: for $\lambda \in \mathcal{M}(\mathbb{R})$ we set $\operatorname{Per}[\lambda] \in \mathcal{M}(\mathbb{T})$ by $\operatorname{Per}[\lambda](E) = \lambda(P_\mathbb{T}^{-1}(E))$ for any Borel subset $E \subseteq \mathbb{T}$. The new measure $\operatorname{Per}[\lambda] \in \mathcal{M}(\mathbb{T})$ is called the periodization of λ. In this way, the torus is equipped with the measure $\mathrm{d}t = \frac{1}{T}\operatorname{Per}[\mathbb{1}_{[0,T]}\,\mathrm{d}\tau]$, where $\mathrm{d}\tau$ denotes the Lebesgue measure on $\mathbb{R}$.

4.1.2.2. Instantaneous vs. retarded $\chi^{(1)}$-contribution

While the nonlinear susceptibility tensor $\chi^{(3)}$ necessarily represents a retarded material response, cf. (4.6.i) or (4.6.ii), the $\chi^{(1)}$-contribution to the material response may be instantaneous or retarded. The first case is given by $\chi^{(1)}(\mathbf{x},\tau) = g(x)\delta(\tau)I_{3\times3}$ or $\chi^{(1)}(\mathbf{x},\tau) = g(r)\delta(\tau)I_{3\times3}$ from Section 4.1.1. The second case may be written in the form $\chi^{(1)}(\mathbf{x},\tau)\,\mathrm{d}\tau = \mathrm{d}G(\mathbf{x})(\tau)I_{3\times3}$ where for fixed $\mathbf{x} \in \mathbb{R}^3$ we have that $G(\mathbf{x}) \in \mathcal{M}(\mathbb{R})$ is an $\mathbb{R}$-valued Borel measure. Mathematically, the second case comprises the first and hence in the following an instantaneous $\chi^{(1)}$-contribution is subsumed in the retarded case.

4.1.2.3. Fourier transform

Let us fix a convention for Fourier series and Fourier transform. For a time-periodic function $v\colon \mathbb{T} \to \mathbb{C}$ we define its Fourier coefficients by $\hat{v}_k = \mathcal{F}_k[v] = \int_{\mathbb{T}} v\overline{e_k}\,\mathrm{d}t$ with $e_k(t) := \mathrm{e}^{\mathrm{i}k\omega t}$, $\omega := \frac{2\pi}{T}$. Thus the inverse is $v(t) = \mathcal{F}_t^{-1}[\hat{v}_k] = \sum_{k\in\mathbb{Z}} \hat{v}_k e_k(t)$. For a function v depending on space and T-periodically on time, $\hat{v}$ will always denote the (discrete) Fourier transform in time. In the same way we define the discrete Fourier transform in time $\hat{\lambda}$ of a measure $\lambda \in \mathcal{M}(\mathbb{T})$. Finally, a function $v : \mathbb{T} \to \mathbb{R}$ or a measure $\lambda \in \mathcal{M}(\mathbb{T})$ is called positive semidefinite if its Fourier coefficients $\hat{v}_k$ or $\hat{\lambda}_k$ are all nonnegative.

Similarly we fix the notion of the spatial (continuous) Fourier transform of a space-dependent function $v\colon \mathbb{R}^d \to \mathbb{C}$, where we write

$$\mathcal{F}_\xi[v] = \int_{\mathbb{R}^d} v(x)\overline{\mathrm{e}^{\mathrm{i}x\cdot\xi}}\,\frac{\mathrm{d}x}{(2\pi)^{d/2}}, \qquad \mathcal{F}_x^{-1}[v] = \int_{\mathbb{R}^d} v(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\frac{\mathrm{d}\xi}{(2\pi)^{d/2}}$$

The spatial (continuous) Fourier transform of a function depending on both space and time is defined analogously, and we omit indices of $\mathcal{F}, \mathcal{F}^{-1}$ when they are clear from the context.

4.1.2.4. Cylindrical and slab geometry

We say that a map $A : \mathbb{R}^3 \to Y$ possesses cylindrical symmetry if $A(\mathbf{x}) = A(\tilde{\mathbf{x}})$ for all $\mathbf{x} = (x,y,z), \tilde{\mathbf{x}} = (\tilde{x},\tilde{y},\tilde{z}) \in \mathbb{R}^3$ with $x^2+y^2 = \tilde{x}^2+\tilde{y}^2$. In this case we write $A(\mathbf{x}) = A(r)$ with $r = \sqrt{x^2+y^2}$. Likewise we say that a map $A : \mathbb{R}^3 \to Y$ possesses slab symmetry if $A(\mathbf{x}) = A(\tilde{\mathbf{x}})$ for all $\mathbf{x} = (x,y,z), \tilde{\mathbf{x}} = (x,\tilde{y},\tilde{z}) \in \mathbb{R}^3$ and write $A(\mathbf{x}) = A(x)$ in this case.

Having clarified our notation, we now present the main theorem of this paper.

Theorem 4.1.3. *Let $T > 0$ denote the temporal period, $\omega := \frac{2\pi}{T}$ the associated frequency, and $c \in (0,\infty)$ the speed of travel of the breather solution. We make the following assumptions:*

- ($\mathcal{A}$4.1) *The linear susceptibility tensor $\chi^{(1)}$ is given by $\chi^{(1)}(\mathbf{x},\tau)\,\mathrm{d}\tau = \mathrm{d}G(\mathbf{x})(\tau)I_{3\times3}$ where $G\colon \mathbb{R}^3 \to \mathcal{M}(\mathbb{R})$ is measurable. The nonlinear susceptibility tensor $\chi^{(3)}$ is given by* (4.6.i) *or* (4.6.ii) *where $h \in L^\infty(\mathbb{R}^3)$ and $\nu \in \mathcal{M}(\mathbb{R})$.*
- ($\mathcal{A}$4.2) *G and h both have either cylindrical or slab geometry.*
- ($\mathcal{A}$4.3) $\sup_{\mathbf{x}\in\mathbb{R}^3}\|G(\mathbf{x})\|_{\mathcal{M}(\mathbb{R})} < \infty$ *and* $h \not\leq 0$.
- ($\mathcal{A}$4.4) *The periodization $\mathcal{G}(\mathbf{x})$ of $G(\mathbf{x})$ is even in time for all $\mathbf{x} \in \mathbb{R}^3$ and satisfies*

$$\sup_{\mathbf{x}\in\mathbb{R}^3, k\in\mathbb{Z}} \mathcal{F}_k[\mathcal{G}(\mathbf{x})] < \tfrac{1}{c^2} - 1.$$

($\mathcal{A}$4.5) *The periodization* $\mathcal{N}$ *of* ν *is even in time,* $\neq 0$*, and* $|k|^{-\beta} \lesssim \mathcal{F}_k[\mathcal{N}] \lesssim |k|^{-\alpha}$ *for all* $k \in \mathbb{Z} \setminus \{0\}$ *with* $\mathcal{F}_k[\mathcal{N}] \neq 0$ *and some* $\beta \geq \alpha > \alpha^\star$ *where* $\alpha^\star = 1$ *in the slab geometry and* $\alpha^\star = \frac{3}{2}$ *in the cylindrical geometry.*

($\mathcal{A}$4.6) *In case of the slab geometry, one of the following holds in addition:*

- ($\mathcal{A}$4.6a) $h(x) \to 0$ *as* $x \to \pm\infty$,
- ($\mathcal{A}$4.6b) $\mathcal{G}(x) = \mathcal{G}^{\text{per}}(x) + \mathcal{G}^{\text{loc}}(x)$ *and* $h(x) = h^{\text{per}}(x) + h^{\text{loc}}(x)$ *where* $\mathcal{G}^{\text{per}}(x), h(x)$ *are periodic with common period, and we have* $\left\|\mathcal{G}^{\text{loc}}(x)\right\|_{\mathcal{M}(\mathbb{T})} \to 0$ *and* $h^{\text{loc}}(x) \to 0$ *as* $x \to \pm\infty$*. Moreover,* $\mathcal{G}^{\text{loc}}(x)$ *is positive semidefinite for all* $x \in \mathbb{R}$ *and* $h^{\text{loc}} \geq 0, h^{\text{per}} \not\leq 0$ *hold.*

Under these assumptions, there exists a nontrivial breather solution $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H}$ *of Maxwell's equations* (4.1),(4.2)*. It satisfies*

$$\partial_t^n \mathbf{E} \in W^{2,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{B}, \partial_t^n \mathbf{H} \in W^{1,p}(\Omega; \mathbb{R}^3), \qquad \partial_t^n \mathbf{D} \in L^p(\Omega; \mathbb{R}^3)$$

for all $n \in \mathbb{N}_0$*,* $p \in [2, \infty]$ *and all domains* Ω *that are of the form* $\Omega = \mathbb{R} \times [y, y+1] \times [z, z+1] \times [t, t+1]$ *in the slab case and* $\Omega = \mathbb{R}^2 \times [z, z+1] \times [t, t+1]$ *in the cylindrical case, with norm bounds independent of* y, z, t.

Remark 4.1.4. Our assumptions ($\mathcal{A}$4.1)–($\mathcal{A}$4.5) on the structure of the linear and nonlinear retardation kernels can be seen as a systematic attempt to find out what can be done in a variational setting. The main assumptions are expressed via the Fourier coefficients of $\mathcal{G}(\mathbf{x})$ and $\mathcal{N}$.

(a) The fact that both $\mathcal{G}(\mathbf{x})$ and $\mathcal{N}$ have real Fourier coefficients stems from their evenness in time which could be understood physically as a balance of loss and gain within one period of time.

(b) In assumption ($\mathcal{A}$4.4) the upper bound on $\mathcal{F}_k[\mathcal{G}(\mathbf{x})]$ can be achieved by taking a sufficiently small propagation speed c. It is exactly this assumption which makes the linear operator in (4.10) elliptic, and combined with positive definiteness of $\mathcal{N}$ it makes the quadratic part in (4.11) positive definite.

Remark 4.1.5. If in the setting of Theorem 4.1.3 the set $\mathfrak{R} := \{k \in \mathbb{Z} \setminus \{0\} : \mathcal{F}_k[\mathcal{N}] \neq 0\}$ is infinite then we moreover have existence of infinitely many nontrivial breathers with the stated properties.

Remark 4.1.6. The sign assumptions on $\mathcal{G}^{\text{loc}}, h^{\text{loc}}$ in ($\mathcal{A}$4.6b) of Theorem 4.1.3 yield a strict relation between the mountain pass energy level of the problem compared with the energy of the "periodic" problem, (i.e., with $\mathcal{G}, h$ replaced by $\mathcal{G}^{\text{per}}, h^{\text{per}}$), see Lemma 4.3.7 for a precise formulation. This energy inequality gives us some compactness which is crucial for showing existence of breathers. The examples Theorems 4.1.1 and 4.1.2 satisfy ($\mathcal{A}$4.1)–($\mathcal{A}$4.6). For ($\mathcal{A}$4.5) this is true because

$$\mathcal{F}_k[\mathcal{N}] = \mathcal{F}_k[2 - |\sin(\omega t)|] = \begin{cases} 2 - \frac{2}{\pi}, & k = 0, \\ 0, & k \text{ odd}, \\ \frac{2}{\pi(k^2-1)}, & k \neq 0 \text{ even}. \end{cases}$$

Breather solutions are more regular when the material coefficients $\mathcal{G}, h$ have higher regularity. For $\Omega \subseteq \mathbb{R}^4$ we denote by $C_b^j(\Omega; \mathbb{R}^3)$ the space of j-times differentiable functions mapping into $\mathbb{R}^3$ with bounded derivatives, and abbreviate $\tilde{C}_b^j(\Omega; \mathbb{R}^3) := W^{j,2}(\Omega; \mathbb{R}^3) \cap C_b^j(\Omega; \mathbb{R}^3)$.

Corollary 4.1.7. *If in the context of Theorem 4.1.3 we additionally have*

(R) $g \in C_b^l(\mathbb{R}^3; \mathcal{M}(\mathbb{R})), h \in C_b^l(\mathbb{R}^3)$ *for some* $l \in \mathbb{N}_0$*,*

then the regularity improves to

$$\partial_t^n \mathbf{E} \in \tilde{C}_b^{2+l}(\Omega; \mathbb{R}^3), \quad \partial_t^n \mathbf{B}, \partial_t^n \mathbf{H} \in \tilde{C}_b^{1+l}(\Omega; \mathbb{R}^3), \quad \partial_t^n \mathbf{D} \in \tilde{C}_b^l(\Omega; \mathbb{R}^3)$$

with norm bounds independent of y, z, t*.*

4.1.3. Outline of paper

We begin by investigating the slab geometry (4.5.1). In Section 4.2 we convert Maxwell's equations into the Euler-Lagrange equation of a suitable Lagrangian functional, and show that this functional has mountain pass geometry. Using the mountain pass theorem, in Section 4.3 we show that the Euler-Lagrange equation admits a ground state solution. The convergence of Palais-Smale sequences approaching the ground state level is unclear in general because the spatial domain is unbounded, and thus our arguments depend on the particular form of the potentials in ($\mathcal{A}$4.6). For ($\mathcal{A}$4.6a), the nonlinearity is compact which makes this the easiest case. For ($\mathcal{A}$4.6b) we first rely on translation arguments in space for the purely periodic case. Then we use comparison arguments for the perturbed periodic case. After having shown existence and multiplicity of breathers, we investigate their regularity in Section 4.4. Finally, Section 4.5 details the arguments for the cylindrical geometry (4.5.2) and highlights the differences to the slab geometry.

4.2. Variational problem

From now on, we always assume that the assumptions of Theorem 4.1.3 are satisfied. We transform (4.8.1) into a problem for a surrogate variable u, which we then treat using the mountain pass method. We only consider the slab problem (4.8.1) as the cylindrical problem (4.8.2) can be treated similarly. In Section 4.5 we discuss the differences between the slab and the cylindrical problem, and how to treat the latter.

Using the periodizations $\mathcal{G}(x), \mathcal{N}$ of $g(x), \nu$ we can rewrite the scalar polarization (4.9.i) as

$$\begin{aligned} P(w)(x,t) &= \epsilon_0 \int_0^\infty w(x, t-\tau)\,\mathrm{d}g(x)(\tau) + \epsilon_0 h(x) \int_0^\infty w(x,t-\tau)^3\,\mathrm{d}\nu(\tau) \\ &= \epsilon_0 \int_{\mathbb{T}} w(x,t-\tau)\,\mathrm{d}\mathcal{G}(x)(\tau) + \epsilon_0 h(x) \int_{\mathbb{T}} w(x,t-\tau)^3\,\mathrm{d}\mathcal{N}(\tau) \end{aligned}$$

since w is T-periodic in t. We abbreviate this by writing

$$P(w) = \epsilon_0(\mathcal{G} * w) + \epsilon_0 h(\mathcal{N} * w^3)$$

where $*$ denotes convolution of a measure with a function on $\mathbb{T}$. Similarly, the polarization (4.9.ii) can be written in the form

$$P(w) = \epsilon_0(\mathcal{G} * w) + \epsilon_0 h(\mathcal{N} * w)^3,$$

Next we define the projection $P_{\mathfrak{R}}$ onto the set $\mathfrak{R} := \{k \in \mathbb{Z} \setminus \{0\} \colon \mathcal{F}_k[\mathcal{N}] \neq 0\}$ of "regular" frequency indices by

$$P_{\mathfrak{R}}[v] = \mathcal{F}^{-1}\left[\mathbb{1}_{k\in\mathfrak{R}}\mathcal{F}_k[v]\right],$$

as well as the projection onto the "singular" frequency indices $\mathfrak{S} := \mathbb{Z} \setminus \mathfrak{R}$ by $P_{\mathfrak{S}}[v] := \mathcal{F}^{-1}[\mathbb{1}_{k\in\mathfrak{S}}\mathcal{F}_k[v]] = (\mathrm{Id} - P_{\mathfrak{R}})[v]$. Note that $\partial_t, \mathcal{G}*, P_{\mathfrak{R}}$, and $P_{\mathfrak{S}}$ mutually commute since they all act on t as Fourier multipliers. We apply both to (4.8.1) for time-periodic w to obtain the two problems

$$\Big(-\partial_x^2 + \partial_t^2(1 - \tfrac{1}{c^2} + \mathcal{G}*)\Big)P_{\mathfrak{R}}[w] + h\partial_t^2 P_{\mathfrak{R}}[N(w)] = 0,$$
$$\Big(-\partial_x^2 + \partial_t^2(1 - \tfrac{1}{c^2} + \mathcal{G}*)\Big)P_{\mathfrak{S}}[w] + h\partial_t^2 P_{\mathfrak{S}}[N(w)] = 0,$$

where the cubic nonlinearity $N(w)$ is given by

$$N(w) = \mathcal{N} * w^3 \qquad \text{or} \qquad N(w) = (\mathcal{N} * w)^3$$

corresponding to (4.6.i) and (4.6.ii), respectively.

Let us first consider the nonlinearity $N(w) = \mathcal{N} * w^3$. Using $P_{\mathfrak{S}}(\mathcal{N}*) = 0$ and that the linear operator $\Big(-\partial_x^2 + \partial_t^2(1 - \frac{1}{c^2} + \mathcal{G}*)\Big)$ is injective,[2] we can further simplify this to

$$\Big(-\partial_x^2 + \partial_t^2(1 - \tfrac{1}{c^2} + \mathcal{G}*)\Big)w + h\partial_t^2(\mathcal{N} * w^3) = 0, \qquad P_{\mathfrak{S}}[w] = 0.$$

Observe that the convolution operator $\mathcal{N}*$ is formally invertible on $\ker P_{\mathfrak{S}}$ since $\mathcal{F}_k[\mathcal{N}] \neq 0$ for $k \in \mathfrak{R}$. Therefore we may rephrase this problem as

$$(-\partial_t^2\mathcal{N}*)^{-1}\Big(-\partial_x^2 + \partial_t^2(1 - \tfrac{1}{c^2} + \mathcal{G}*)\Big)u - hP_{\mathfrak{R}}[u^3] = 0, \qquad P_{\mathfrak{S}}[u] = 0 \tag{4.14}$$

with $u := w$.

For the second nonlinearity $N(w) = (\mathcal{N} * w)^3$ we set $u := \mathcal{N} * w$ and therefore get

$$\begin{aligned} &(-\partial_t^2\mathcal{N}*)^{-1}\Big(-\partial_x^2 + \partial_t^2(1 - \tfrac{1}{c^2} + \mathcal{G}*)\Big)u - hP_{\mathfrak{R}}[u^3] = 0, \qquad P_{\mathfrak{S}}[u] = 0, \\ &\Big(-\partial_x^2 + \partial_t^2(1 - \tfrac{1}{c^2} + \mathcal{G}*)\Big)P_{\mathfrak{S}}[w] + h\partial_t^2 P_{\mathfrak{S}}[u^3] = 0. \end{aligned} \tag{4.15}$$

Note that the first of the two equations above is (4.14). Hence, also for the second nonlinearity, it is sufficient to solve (4.14) for u and then use the second equation to determine the missing values of $\mathcal{F}_k[w]$ for $k \in \mathfrak{S}$.

We first focus our attention on investigating the "effective problem" (4.14), which using

$$V(x) := \tfrac{1}{c^2} - 1 - \mathcal{G}*$$

we can write as

$$(-\partial_t^2\mathcal{N}*)^{-1}\Big(-\partial_x^2 - V(x)\partial_t^2\Big)u - hP_{\mathfrak{R}}[u^3] = 0, \qquad P_{\mathfrak{S}}[u] = 0. \tag{4.16}$$

Since $\mathcal{G}, \mathcal{N}$ are even in time, the differential operator above is symmetric, and therefore solutions of (4.16) formally are critical points of the functional

$$J(u) = \int_{\mathbb{R}\times\mathbb{T}} \Big(\tfrac{1}{2}u \cdot (-\partial_t^2\mathcal{N}*)^{-1}\Big(-\partial_x^2 - V(x)\partial_t^2\Big)u - \tfrac{1}{4}h(x)u^4\Big)\,\mathrm{d}(x,t), \qquad P_{\mathfrak{S}}[u] = 0.$$

Next we properly define the domain H of the functional J sketched above, and we investigate embeddings $H \hookrightarrow L^p$.

[2] The operator is uniformly elliptic due to assumption ($\mathcal{A}$4.4).

Definition 4.2.1. *We define the space*

$$H := \left\{u \in L^2(\mathbb{R}\times\mathbb{T})\colon \hat{u}_k \equiv 0 \text{ for } k \in \mathbb{Z}\setminus\mathfrak{R}, \|u\|_H^2 := \langle\!\langle u,u\rangle\!\rangle_H < \infty\right\}$$

where

$$\langle\!\langle u,v\rangle\!\rangle_H = \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}\int_{\mathbb{R}}\left(\hat{u}_k'\overline{\hat{v}_k'} + \omega^2k^2\hat{u}_k\overline{\hat{v}_k}\right)\mathrm{d}x$$

Note that $V_k(x) := \frac{1}{c^2} - 1 - \mathcal{F}_k[\mathcal{G}(x)]$ is bounded and strictly positive by assumption ($\mathcal{A}$4.4), so that

$$\langle u,v\rangle_H := \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}\int_{\mathbb{R}}(\hat{u}_k'\overline{\hat{v}_k'} + \omega^2k^2V_k(x)\hat{u}_k\overline{\hat{v}_k})\,\mathrm{d}x$$

defines an equivalent inner product on H.

Assumption ($\mathcal{A}$4.5) on the decay of the Fourier coefficients of $\mathcal{N}$ ensures that H continuously embeds into $L^p(\mathbb{R}\times\mathbb{T})$ for all $p\in[2,p^\star)$ where $p^\star > 4$, as we show below in Lemma 4.2.4. Boundedness of $H \hookrightarrow L^2(\mathbb{R}\times\mathbb{T})$ helps to shows that H is a Hilbert space. Indeed, if $(u^{(n)})$ is a Cauchy sequence in H, then from the embedding we have $u^{(n)}\to u$ in $L^2(\mathbb{R}\times\mathbb{T})$ and from the definition of H also $\mathcal{F}_k[u^{(n)}]\to v_k$ in $H^1(\mathbb{R})$ for some $v_k\in H^1(\mathbb{R})$ and each $k\in\mathfrak{R}$. Then $v_k = \hat{u}_k$ holds from which one finds $u\in H$ and $u^{(n)}\to u$ as $n\to\infty$.

The Hilbert space H allows us to write

$$J(u) = \tfrac{1}{2}\langle u,u\rangle_H - \tfrac{1}{4}\int_{\mathbb{R}\times\mathbb{T}} h(x)u^4\,\mathrm{d}(x,t) \quad\text{for}\quad u\in H,$$

and to define solutions $u\in H$ of (4.14) in the following way.

Definition 4.2.2 (weak solution). *A function $u\colon \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ is called a* weak solution *to* (4.14) *if $u\in H$ and*

$$\langle u,v\rangle_H - \int_{\mathbb{R}\times\mathbb{T}} h(x)u^3v\,\mathrm{d}(x,t) = 0$$

for all $v\in H$. This is equivalent to $J'(u)=0$.

It is standard to verify the validity of the following density result for H, which will prove very useful for some approximation arguments.

Lemma 4.2.3. *The set $\{u\in C_c^\infty(\mathbb{R}\times\mathbb{T})\colon \hat{u}_k\equiv 0 \text{ for almost all } k\in\mathbb{Z}\}\cap H$ is dense in H.*

Lemma 4.2.4. *For any $p\in[2,p^\star)$ with $p^\star := \frac{4}{2-\alpha}$ ($p^\star=\infty$ if $\alpha\geq 2$) and $\alpha>1$ from ($\mathcal{A}$4.5), the embedding $H\hookrightarrow L^p(\mathbb{R}\times\mathbb{T})$ is continuous and the embedding $H\hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T})$ is compact.*

Proof. Let us first show continuity. For this, we calculate

$$\|u\|_p \lesssim \|\mathcal{F}_{\xi,k}[u]\|_{L^{p'}(\mathbb{R}\times\mathfrak{R})} \lesssim \left\|\sqrt{\frac{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}{\xi^2+\omega^2k^2}}\right\|_{L^r(\mathbb{R}\times\mathfrak{R})}\cdot\left\|\sqrt{\frac{\xi^2+\omega^2k^2}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}}\mathcal{F}_{\xi,k}[u]\right\|_{L^2(\mathbb{R}\times\mathfrak{R})} \lesssim \|u\|_H$$

where $\frac{1}{r} = \frac{1}{2} - \frac{1}{p} < \frac{\alpha}{4}$ and the first factor is finite since by ($\mathcal{A}$4.5) we have

$$\begin{aligned}\left\| \sqrt{\frac{\omega^2 k^2 \mathcal{F}_k[\mathcal{N}]}{\xi^2 + \omega^2 k^2}} \right\|_{L^r(\mathbb{R}\times\mathfrak{R})}^r &= \sum_{k\in\mathfrak{R}} \int_{\mathbb{R}} \left(\frac{\omega^2 k^2 \mathcal{F}_k[\mathcal{N}]}{\xi^2 + \omega^2 k^2} \right)^{r/2} \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \left(\frac{1}{\xi^2 + 1} \right)^{r/2} \mathrm{d}\xi \cdot \sum_{k\in\mathfrak{R}} |\omega k| \mathcal{F}_k[\mathcal{N}]^{r/2} \lesssim \sum_{k\in\mathfrak{R}} |k|^{1-\frac{\alpha r}{2}} < \infty.\end{aligned}$$

In order to show compactness, define for $K \in \mathbb{N}_{\text{odd}}$ the projection onto "small" frequencies $P_K \colon H \to H$ by $P_K[u] = \mathcal{F}_t^{-1}\left[\mathbb{1}_{|k|\le K} \mathcal{F}_k[u]\right]$. Then on $P_K H$ the norm $\|u\|_H$ is equivalent to

$$|||u||| = \sum_{\substack{k\in\mathfrak{R} \\ |k|\le K}} \|\hat{u}_k\|_{H^1(\mathbb{R})}.$$

Since the embedding $H^1(\mathbb{R}) \to L^p_{\text{loc}}(\mathbb{R})$ is compact for any $p \in [2, \infty]$ and the sum above is finite, it follows that $P_K \colon H \to L^p_{\text{loc}}(\mathbb{R} \times \mathbb{T})$ is compact. Next, the calculations above show for $u \in H$ that

$$\|u - P_K[u]\|_p \le C \sum_{\substack{k\in\mathfrak{R} \\ |k|>K}} |k|^{1-\frac{\alpha r}{2}} \|u\|_H$$

for some $C > 0$ independent of K, so that $P_K \to \mathrm{Id}$ in $\mathcal{B}(H; L^p(\mathbb{R} \times \mathbb{T}))$ as $K \to \infty$. Thus H embeds compactly into $L^p_{\text{loc}}(\mathbb{R} \times \mathbb{T})$. □

We show that low-order (fractional) time-derivatives of u also lie in L^p in Corollary 4.2.6 as a generalization of the embedding of Lemma 4.2.4. This regularity gain will be used in Section 4.4 for a bootstrapping argument. We first give a definition of these derivatives.

Definition 4.2.5. *For $s \in \mathbb{R}$ we define the fractional time-derivative $|\partial_t|^s$ as the Fourier multiplier $|\partial_t|^s v(t) = \mathcal{F}_t^{-1}[|\omega k|^s \hat{v}_k]$.*

Corollary 4.2.6. *As in the proof of Lemma 4.2.4 we see that for $p \in [2, p^\star)$ and $\varepsilon > 0$ sufficiently small (depending on α, p), the map $|\partial_t|^\varepsilon \colon H \to L^p(\mathbb{R} \times \mathbb{T})$ is bounded and $|\partial_t|^\varepsilon \colon H \to L^p_{\text{loc}}(\mathbb{R} \times \mathbb{T})$ is compact.*

In the following lemma we show a variant of the concentration-compactness principle that will be a useful tool for extracting a nonzero limit from Palais-Smale sequences.

Lemma 4.2.7. *Let (u_n) be a bounded sequence in H, $r > 0$ and $\tilde{p} \in [2, p^\star)$ such that*

$$\sup_{x\in\mathbb{R}} \|u_n\|_{L^{\tilde{p}}([x-r,x+r]\times\mathbb{T})} \to 0$$

as $n \to \infty$. Then $u_n \to 0$ in $L^p(\mathbb{R} \times \mathbb{T})$ for all $p \in (2, p^\star)$.

Proof. By Hölder's inequality and Lemma 4.2.4 it suffices to show the result for $\tilde{p} = 2$ and one $p \in (2, p^\star)$, which we shall choose so close to 2 that $2 < q := \frac{4}{4-p} < p^\star$. Let $\phi_m \colon \mathbb{R} \to [0, 1]$ be a smooth partition of unity with $\operatorname{supp}\phi_m \subseteq [(m-1)r,(m+1)r]$, $\|\phi_m'\|_\infty \le \frac{2}{r}$. Using that at any point of $\mathbb{R}$ at most 2 of the ϕ_m are nonzero, we calculate

$$\begin{aligned}
\|u_n\|_p^p &= \int_{\mathbb{R}\times\mathbb{T}} \left| \sum_{m\in\mathbb{Z}} \phi_m u_n \right|^p \mathrm{d}(x,t) \\
&\le 2^{p-1} \int_{\mathbb{R}\times\mathbb{T}} \sum_{m\in\mathbb{Z}} |\phi_m u_n|^p \,\mathrm{d}(x,t) \\
&= 2^{p-1} \sum_{m\in\mathbb{Z}} \|\phi_m u_n\|_p^p \\
&\le 2^{p-1} \sum_{m\in\mathbb{Z}} \|\phi_m u_n\|_q^2 \|\phi_m u_n\|_2^{p-2} \\
&\lesssim \sup_{x\in\mathbb{R}} \|u_n\|_{L^2([x-r,x+r]\times\mathbb{T})}^{p-2} \sum_{m\in\mathbb{Z}} \|\phi_m u_n\|_H^2.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\|\phi_m u_n\|_H^2 &= \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2 k^2 \mathcal{F}_k[\mathcal{N}]} \int_{\mathbb{R}} \left(|\phi_m' \hat{u}_k + \phi_m \hat{u}_k'|^2 + \omega^2 k^2 |\phi_m \hat{u}_k|^2 \right) \mathrm{d}x \\
&\le C \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2 k^2 \mathcal{F}_k[\mathcal{N}]} \int_{(m-1)r}^{(m+1)r} \left(|\hat{u}_k'|^2 + \omega^2 k^2 |\hat{u}_k|^2 \right) \mathrm{d}x,
\end{aligned}$$

it follows that $\sum_{m\in\mathbb{Z}} \|\phi_m u_n\|_H^2 \le 2C\|u\|_H^2$. Thus, from the assumptions we obtain $\|u_n\|_p \to 0$ as $n \to \infty$. □

4.3. Existence of ground states

In the following, let J be given by Definition 4.2.2. We call the energy level

$$c_{\mathrm{gs}} := \inf_{\substack{u\in H\setminus\{0\} \\ J'(u)=0}} J(u)$$

the *ground state energy level*, and any solution $u \in H \setminus \{0\}$ of $J'(u) = 0$ with $J(u) = c_{\mathrm{gs}}$ a *ground state* of J. Note that $c_{\mathrm{gs}} = +\infty$ if there are no nonzero critical points of J. Next we present the main result of this section. The rest of this section is dedicated to its proof.

Theorem 4.3.1. *There exists a ground state of J.*

We first note that the following necessary condition for existence of ground states holds.

Lemma 4.3.2. $c_{\mathrm{gs}} > 0$.

Proof. By Lemma 4.2.4 with $p = 4$ we have $J'(u)[u] = \langle u, u\rangle_H + \mathcal{O}(\|u\|_H^4)$ as $u \to 0$. In particular, for a critical point $u \in H$ of J we have $0 = \|u\|_H^2 + \mathcal{O}(\|u\|_H^4)$, and therefore there exists $c > 0$ such that $\|u\|_H \ge c$ for every critical point in $H \setminus \{0\}$. The claim follows from this since every critical point u of J satisfies $J(u) = J(u) - \frac{1}{4}J'(u)[u] = \frac{1}{4}\langle u, u\rangle_H$. □

We will extract the ground state as a limit of a suitable Palais-Smale sequence. Next we use the mountain pass theorem to obtain a particular Palais-Smale sequence.

Proposition 4.3.3. *There exists $u_0 \in H$ with $J(u_0) < 0$. For such u_0, the mountain pass energy level*

$$c_{\mathrm{mp}} := \inf_{\substack{\gamma \in C([0;1];H) \\ \gamma(0)=0, \gamma(1)=u_0}} \sup_{s \in [0,1]} J(\gamma(s))$$

is positive and there exists a Palais-Smale sequence for J at level c_{mp}.

Proof. For the construction of a suitable u_0 we choose $\varphi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} h\varphi^4 \,\mathrm{d}x > 0$, which exists since $h \not\leq 0$ and $C_c^\infty(\mathbb{R})$ is dense in $L^4(\mathbb{R})$. We then choose $u_0(x,t) = r \operatorname{Re}[\varphi(x) e_{k_0}(t)]$ for some $k_0 \in \mathfrak{R}$ and $r > 0$. This implies that

$$J(u_0) = \tfrac{1}{2} r^2 \langle \operatorname{Re}[\varphi(x) e_{k_0}(t)], \operatorname{Re}[\varphi(x) e_{k_0}(t)] \rangle_H - \tfrac{3}{32} r^4 \int_{\mathbb{R}} h\varphi^4 \,\mathrm{d}x$$

is negative, provided r has been chosen sufficiently large. By the embedding $H \hookrightarrow L^4$ we moreover have

$$J(u) = \tfrac{1}{2} \langle u, u \rangle_H - \mathcal{O}\Big(\|u\|_H^4 \Big)$$

as $u \to 0$. Thus $c_{\mathrm{mp}} > 0$ and by the mountain pass theorem, cf. [99], there exists a Palais-Smale sequence (u_n) at level c_{mp}. □

Lemma 4.3.4. *Any Palais-Smale sequence for J is bounded.*

Proof. Let (u_n) be a Palais-Smale sequence at level c. Then

$$\langle u_n, u_n \rangle_H = 4J(u_n) - J'(u_n)[u_n] = 4c + \mathrm{o}(1) + \mathrm{o}(\|u_n\|_H)$$

as $n \to \infty$, which shows that (u_n) is bounded in H. □

Next we show the following result on weakly convergent Palais-Smale sequences in our setting.

Lemma 4.3.5. *Let (u_n) be a Palais-Smale sequence for J with $J(u_n) \to c$ and $u_n \rightharpoonup u$ in H. Then u is a critical point of J and $J(u) \leq c$. Moreover, if $u \neq 0$ and $c = c_{\mathrm{gs}}$ then u is a ground state and $u_n \to u$ in H.*

Proof. By Lemmas 4.2.4 and 4.3.4 we have $u_n \to u$ in L^4_{loc}. Thus for compactly supported $v \in H$ it follows that

$$J'(u_n)[v] = \langle u_n, v \rangle_H - \int_{\mathbb{R}\times\mathbb{T}} h(x) u_n^3 v \,\mathrm{d}(x,t) \to \langle u, v \rangle_H - \int_{\mathbb{R}\times\mathbb{T}} h(x) u^3 v \,\mathrm{d}(x,t) = J'(u)[v]$$

so that $J'(u)[v] = 0$. By a density argument (cf. Lemma 4.2.3) it follows that u is a critical point of J. Next we calculate

$$J(u) = J(u) - \tfrac{1}{4} J'(u)[u] = \tfrac{1}{4} \langle u, u \rangle_H \leq \tfrac{1}{4} \lim_{n\to\infty} \langle u_n, u_n \rangle_H = \lim_{n\to\infty} J(u_n) - \tfrac{1}{4} J'(u_n)[u_n] = c.$$

If $c = c_{\mathrm{gs}}$, then we have $J(u) \geq c_{\mathrm{gs}}$ since $u \neq 0$ by assumption, and thus from the above inequality we find $J(u) = c_{\mathrm{gs}}$ and in addition $\langle u_n, u_n \rangle_H \to \langle u, u \rangle_H$. Combined with $u_n \rightharpoonup u$ in H this shows $u_n \to u$ in H. □

In many situations, e.g., in a translation-invariant setting, there are always Palais-Smale sequences converging weakly to 0. Therefore the main task in the following will be to find a Palais-Smale sequence with $u_n \rightharpoonup u \neq 0$. The arguments for this (and the proof of Theorem 4.3.1) differ between the types of nonlinearity, and are split into subsections accordingly.

4.3.1. Proof of Theorem 4.3.1 for ($\mathcal{A}$4.6a) and the purely periodic case of ($\mathcal{A}$4.6b)

First we show how to extract a nonzero limit from a given Palais-Smale sequence.

Lemma 4.3.6. *Assume ($\mathcal{A}$4.6a) or ($\mathcal{A}$4.6b) with $\mathcal{G}^{\mathrm{loc}}, h^{\mathrm{loc}} \equiv 0$. Let (u_n) be a Palais-Smale sequence for J at level $c > 0$. Then there exists a critical point $u \in H \setminus \{0\}$ of J with $J(u) \leq c$.*

Proof. Part 1: We consider ($\mathcal{A}$4.6a). Up to a subsequence we have $u_n \rightharpoonup u$ in H and $u_n \to u$ in L^4_{loc} by Lemmas 4.2.4 and 4.3.4, where Lemma 4.3.5 guarantees that u is a critical point of J. Moreover, since h is bounded and $h(x) \to 0$ as $x \to \pm\infty$ by ($\mathcal{A}$4.1) and ($\mathcal{A}$4.6a), we have $h(x)u_n^3 \to h(x)u^3$ in $L^{4/3}(\mathbb{R} \times \mathbb{T})$. This implies for $v \in H$ that

$$\langle u_n - u, v\rangle_H = J'(u_n)[v] - J'(u)[v] + \int_{\mathbb{R}\times\mathbb{T}} h(x)(u^3 - u_n^3)v \,\mathrm{d}(x,t) = \mathrm{o}(\|v\|_H)$$

as $n \to \infty$. So $u_n \to u$ in H, and in particular $J(u) = c$ and $u \neq 0$ hold.

Part 2: We now consider ($\mathcal{A}$4.6b) with $\mathcal{G}^{\mathrm{loc}}, h^{\mathrm{loc}} \equiv 0$. Since

$$\int_{\mathbb{R}\times\mathbb{T}} h(x)u_n^4 \,\mathrm{d}(x,t) = 4J(u_n) - 2J'(u_n)[u_n] \to 4c,$$

we have $u_n \not\to 0$ in $L^4(\mathbb{R} \times \mathbb{T})$. Let $X > 0$ denote the period of $\mathcal{G}$ and h. By Lemma 4.2.7 there exist $x_n \in \mathbb{R}$ with

$$\liminf_{n\to\infty} \|u_n\|_{L^4([x_n - X, x_n + X]\times\mathbb{T})} > 0 \tag{4.17}$$

and w.l.o.g. we may assume $x_n \in X\mathbb{Z}$. Let us define a new sequence $\tilde{u}_n$ by $\tilde{u}_n(x,t) = u_n(x - x_n, t)$, so that $J(\tilde{u}_n) = J(u_n) \to c$ and $J'(\tilde{u}_n) \to 0$. Up to a subsequence we have $\tilde{u}_n \rightharpoonup u$ in H where $u \neq 0$ by (4.17). The claim now follows from Lemma 4.3.5 applied to $(\tilde{u}_n)$. ☐

Proof of Theorem 4.3.1 for ($\mathcal{A}$4.6a) and ($\mathcal{A}$4.6b) with $\mathcal{G}^{\mathrm{loc}}, h^{\mathrm{loc}} \equiv 0$. Combining Proposition 4.3.3 and Lemma 4.3.6 we see that there exists a nonzero critical point of J. Thus $c_{\mathrm{gs}} < \infty$ and by definition of c_{gs} there exists a sequence (u_n) of critical points of J with $J(u_n) \to c_{\mathrm{gs}}$. Since $c_{\mathrm{gs}} > 0$ by Lemma 4.3.2, applying Lemma 4.3.6 to (u_n) we find a ground state of J. ☐

4.3.2. Proof of Theorem 4.3.1 for ($\mathcal{A}$4.6b)

We call the problem with $\mathcal{G}, h$ replaced by $\mathcal{G}^{\mathrm{per}}, h^{\mathrm{per}}$ the *periodic* problem and denote it with the superscript "per". The previous subsection guarantees the existence of a periodic ground state u^{per} of J^{per}.

Note that both J and J^{per} are defined on the same Hilbert space H. The assumption ($\mathcal{A}$4.6b) implies that $J \leq J^{\mathrm{per}}$ on H and, assuming $(\mathcal{G}^{\mathrm{loc}}, h^{\mathrm{loc}}) \neq 0$, the inequality is even strict on functions which do not have zero sets of positive measure. For our nonlocal problem (4.14) we do not know whether or not a unique continuation theorem holds, which is why we cannot rule out that a critical point of J or J^{per} could have a zero set of positive measure. Nevertheless, the subsequent arguments work without a unique continuation theorem and are based on the comparison of energy levels between our current and the periodic problem.

Lemma 4.3.7. *Assume that no ground state of J^{per} is a critical point of J. Then there exists $u_0 \in H$ with $J(u_0) < 0$ such that the mountain pass energy level*

$$c_{\text{mp}} := \inf_{\substack{\gamma \in C([0,1];H) \\ \gamma(0)=0, \gamma(1)=u_0}} \sup_{s \in [0,1]} J(\gamma(s))$$

satisfies $0 < c_{\text{mp}} < c_{\text{gs}}^{\text{per}}$.

Proof. Let u^{per} be a ground state of J^{per}. As u^{per} is not a critical point of J, we have $\mathcal{G}^{\text{loc}} * u^{\text{per}} \neq 0$ or $h^{\text{loc}}(u^{\text{per}})^3 \neq 0$. By the assumptions on the signs of $\mathcal{G}^{\text{loc}}, h^{\text{loc}}$ we moreover have

$$\langle u^{\text{per}}, u^{\text{per}} \rangle_H \leq \langle u^{\text{per}}, u^{\text{per}} \rangle_H^{\text{per}} \quad \text{and} \quad \int_{\mathbb{R} \times \mathbb{T}} h(x)(u^{\text{per}})^4 \, \mathrm{d}(x,t) \geq \int_{\mathbb{R} \times \mathbb{T}} h^{\text{per}}(x)(u^{\text{per}})^4 \, \mathrm{d}(x,t)$$

where at least one inequality is strict. In particular, $J(su^{\text{per}}) < J^{\text{per}}(su^{\text{per}})$ holds for $s \neq 0$. Now set $u_0 := \sqrt{2} u^{\text{per}}$. Then $J(u_0) < J^{\text{per}}(u_0) = 0$ and

$$c_{\text{mp}} \leq \max_{s \in [0,1]} J(su_0) < \max_{s \in [0,1]} J^{\text{per}}(su_0).$$

Moreover, the degree 4 polynomial $s \mapsto J^{\text{per}}(su^{\text{per}})$ has the critical points $0, 1, -1$ since $0, u^{\text{per}}, -u^{\text{per}}$ are critical points of J^{per}. Together with $J(0) < J^{\text{per}}(u^{\text{per}}) = J^{\text{per}}(-u^{\text{per}})$ we see that the latter two points are the global maxima of the polynomial. Therefore

$$c_{\text{mp}} < \max_{s \in [0,1]} J^{\text{per}}(su_0) = J^{\text{per}}(u^{\text{per}}) = c_{\text{gs}}^{\text{per}}.$$

Positivity of c_{mp} was already shown in Proposition 4.3.3. □

Similar to Lemma 4.3.6 of the previous subsection, we require a result on convergence of a given Palais-Smale sequence, which we present next.

Lemma 4.3.8. *Assume (A4.6b). Let u_n be a Palais-Smale sequence for J at level $c \in (0, c_{\text{gs}}^{\text{per}})$. Then there also exists a critical point $u \in H \setminus \{0\}$ of J with $J(u) \leq c$.*

Proof. We denote by X the spatial period of $\mathcal{G}^{\text{per}}, h^{\text{per}}$. As in the proof of Lemma 4.3.6, Part 2, we have that $u_n \not\to 0$ in $L^4(\mathbb{R} \times \mathbb{T})$ and that a sequence $x_n \in X\mathbb{Z}$ exists such that

$$\liminf_{n \to \infty} \|u_n\|_{L^4([x_n - X, x_n + X] \times \mathbb{T})} > 0.$$

We claim that $u_n \not\to 0$ in L^4_{loc} along any subsequence.

Assume for a contradiction that there exists a subsequence of (u_n), which we again denote by (u_n), such that $u_n \to 0$ in L^4_{loc}. Since $u_n \not\to 0$ in L^4, we necessarily have $|x_n| \to \infty$. We define $\tilde{u}_n$ by $\tilde{u}_n(x,t) = u_n(x - x_n, t)$. Then up to a subsequence we have $\tilde{u}_n \rightharpoonup u$ in H and $\tilde{u}_n \to u$ in L^4_{loc} for some $u \in H \setminus \{0\}$. For compactly supported $v \in H$ we set $v_n(x,t) = v(x + x_n, t)$ and calculate

$$\begin{aligned}
J'(u_n)[v_n] &= \langle u_n, v_n \rangle_H - \int_{\mathbb{R} \times \mathbb{T}} h(x) u_n^3 v_n \, \mathrm{d}(x,t) \\
&= \langle u_n, v_n \rangle_H^{\text{per}} - \int_{\mathbb{R} \times \mathbb{T}} h^{\text{per}}(x) u_n^3 v_n \, \mathrm{d}(x,t) \\
&\quad - \sum_{k \in \mathfrak{R}} \int_{\mathbb{R}} \frac{\mathcal{F}_k[\mathcal{G}^{\text{loc}}(x)]}{\mathcal{F}_k[\mathcal{N}]} \left(\mathcal{F}_k[u_n] \overline{\mathcal{F}_k[v_n]} \right) \mathrm{d}x - \int_{\mathbb{R} \times \mathbb{T}} h^{\text{loc}}(x) u_n^3 v_n \, \mathrm{d}(x,t)
\end{aligned}$$

$$
\begin{aligned}
&= \langle \tilde u_n, v\rangle_H^{\mathrm{per}} - \int_{\mathbb{R}\times\mathbb{T}} h^{\mathrm{per}}(x)\tilde u_n^3 v \,\mathrm{d}(x,t) \\
&\qquad - \sum_{k\in\mathfrak{R}} \int_{\mathbb{R}} \frac{\mathcal{F}_k[\mathcal{G}^{\mathrm{loc}}(x-x_n)]}{\mathcal{F}_k[\mathcal{N}]} \big(\mathcal{F}_k[\tilde u_n]\overline{\mathcal{F}_k[v]}\big)\,\mathrm{d}x - \int_{\mathbb{R}\times\mathbb{T}} h^{\mathrm{loc}}(x-x_n)\tilde u_n^3 v\,\mathrm{d}(x,t) \\
&\to \langle u, v\rangle_H^{\mathrm{per}} - \int_{\mathbb{R}\times\mathbb{T}} h^{\mathrm{per}}(x) u^3 v\,\mathrm{d}(x,t) = (J^{\mathrm{per}})'(u)[v]
\end{aligned}
$$

where we used $|x_n| \to \infty$ and $\mathcal{G}^{\mathrm{loc}}(x) \to 0, h^{\mathrm{loc}}(x) \to 0$ as $x \to \pm\infty$. This shows that $u \neq 0$ is a critical point of J^{per}, and in particular $J^{\mathrm{per}}(u) \geq c_{\mathrm{gs}}^{\mathrm{per}}$ holds. However, for fixed $R > 0$ we have

$$
\begin{aligned}
&\sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]} \int_{-R}^{R} \big(|\mathcal{F}_k[u]'|^2 + \omega^2k^2V_k^{\mathrm{per}}(x)|\mathcal{F}_k[u]|^2\big)\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty} \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]} \int_{-R}^{R} \big(|\mathcal{F}_k[\tilde u_n]'|^2 + \omega^2k^2V_k^{\mathrm{per}}(x)|\mathcal{F}_k[\tilde u_n]|^2\big)\,\mathrm{d}x \\
&= \liminf_{n\to\infty} \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]} \int_{x_n-R}^{x_n+R} \big(|\mathcal{F}_k[u_n]'|^2 + \omega^2k^2V_k^{\mathrm{per}}(x)|\mathcal{F}_k[u_n]|^2\big)\,\mathrm{d}x \\
&= \liminf_{n\to\infty} \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]} \int_{x_n-R}^{x_n+R} \big(|\mathcal{F}_k[u_n]'|^2 + \omega^2k^2V_k(x)|\mathcal{F}_k[u_n]|^2\big)\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty} \langle u_n, u_n\rangle_H
\end{aligned}
$$

from which $\langle u, u\rangle_H^{\mathrm{per}} \leq \liminf_{n\to\infty}\langle u_n, u_n\rangle_H$ follows in the limit $R \to \infty$. This implies

$$
\begin{aligned}
c < c_{\mathrm{gs}}^{\mathrm{per}} &\leq J^{\mathrm{per}}(u) = J^{\mathrm{per}}(u) - \tfrac14 (J^{\mathrm{per}})'(u)[u] = \tfrac14\langle u, u\rangle_H^{\mathrm{per}} \\
&\leq \tfrac14 \liminf_{n\to\infty}\langle u_n, u_n\rangle_H = \liminf_{n\to\infty} J(u_n) - \tfrac14 J'(u_n)[u_n] = c,
\end{aligned}
$$

a contradiction.

Thus we have shown the claim. By Lemmas 4.2.4 and 4.3.4 up to a subsequence we have $u_n \rightharpoonup u$ in H and $u_n \to u$ in L^4_{loc}, where we now know $u \neq 0$. Applying Lemma 4.3.5 completes the proof. □

Proof of Theorem 4.3.1 for (A4.6b). Assume first that $c_{\mathrm{gs}} < c_{\mathrm{gs}}^{\mathrm{per}}$ holds. Let u_n be a sequence of critical points of J with $J(u_n) \to c_{\mathrm{gs}}$. From Lemmas 4.3.2 and 4.3.8 it follows that there exists a ground state of J. In the general situation, we distinguish between two cases.

Case 1: If there exists a ground state u^{per} of J^{per} which also is a critical point of J then clearly $c_{\mathrm{gs}} \leq c_{\mathrm{gs}}^{\mathrm{per}}$ holds. If $c_{\mathrm{gs}} < c_{\mathrm{gs}}^{\mathrm{per}}$ there is nothing left to show, and when $c_{\mathrm{gs}} = c_{\mathrm{gs}}^{\mathrm{per}}$ then u^{per} is a ground state of J.

Case 2: If no ground state of J^{per} solves $J'(u) = 0$, then by Lemma 4.3.7 there exists a Palais-Smale sequence u_n for J at some level $c_{\mathrm{mp}} \in (0, c_{\mathrm{gs}}^{\mathrm{per}})$. Since $c_{\mathrm{gs}} \leq c_{\mathrm{mp}}$ by Lemma 4.3.8, this shows $c_{\mathrm{gs}} \leq c_{\mathrm{mp}} < c_{\mathrm{gs}}^{\mathrm{per}}$. □

4.4. Regularity

So far we have shown existence of a ground state to (4.14) in the slab geometry. In this section, we discuss its regularity properties. The necessary modifications for the cylindrical geometry are explained in Section 4.5.2.

We proceed in two steps. First, we show regularity for the solution u to (4.16): It is infinitely differentiable in time, twice differentiable in space, and derivatives lie in $L^2 \cap L^\infty$. We also show that if the material parameters are l times continuously differentiable, then u is $l+2$ time differentiable in space and derivatives lie in $L^2 \cap C_b$.

Then we transfer this regularity from the function u to the electromagnetic fields $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H}$ since these can be expressed as functions of u.

We begin by showing infinite time differentiability in the space H, see Lemma 4.4.2, which we prepare with an auxiliary result.

Lemma 4.4.1. *Let $s > 0$ and $u, |\partial_t|^s u \in L^p(\mathbb{R} \times \mathbb{T})$ where $p \in [3, \infty]$. Then $|\partial_t|^s (u^3) \in L^{p/3}(\mathbb{R} \times \mathbb{T})$.*

Proof. By [12, Proposition 1] the estimate

$$\||\partial_t|^s vw\|_{L^r(\mathbb{T})} \lesssim \||\partial_t|^s v\|_{L^{p_1}(\mathbb{T})} \|w\|_{L^{q_1}(\mathbb{T})} + \|v\|_{L^{p_2}(\mathbb{T})} \||\partial_t|^s w\|_{L^{q_2}(\mathbb{T})}$$

holds for all $r, p_j, q_j \in [1, \infty]$ with $\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}$ and $v \in C^\infty(\mathbb{T})$. By a density argument we obtain

$$\begin{aligned}
\left\| |\partial_t|^s (u^3) \right\|_{L^{p/3}(\mathbb{R}\times\mathbb{T})} &= \left\| \left\| |\partial_t|^s (u^3) \right\|_{L^{p/3}(\mathbb{T})} \right\|_{L^{p/3}(\mathbb{R})} \\
&\lesssim \left\| \||\partial_t|^s u\|_{L^p(\mathbb{T})} \left\| u^2 \right\|_{L^{p/2}(\mathbb{T})} + \|u\|_{L^p(\mathbb{T})} \left\| |\partial_t|^s (u^2) \right\|_{L^{p/2}(\mathbb{T})} \right\|_{L^{p/3}(\mathbb{R})} \\
&\lesssim \left\| \||\partial_t|^s u\|_{L^p(\mathbb{T})} \|u\|_{L^p(\mathbb{T})}^2 \right\|_{L^{p/3}(\mathbb{R})} \\
&\le \left\| \||\partial_t|^s u\|_{L^p(\mathbb{T})} \right\|_{L^p(\mathbb{R})} \left\| \|u\|_{L^p(\mathbb{T})} \right\|_{L^p(\mathbb{R})}^2 .
\end{aligned}$$

□

Lemma 4.4.2. *Let $u \in H$ be a critical point of J. Then $|\partial_t|^s u \in H$ for all $s \in \mathbb{R}$.*

Proof. Since $u \in H$, $|\partial_t|^s u \in H$ holds for $s \le 0$. Moreover, if $|\partial_t|^s u \in H$ then $|\partial_t|^\sigma u \in H$ for all $\sigma \le s$. By Corollary 4.2.6 there exists $\varepsilon > 0$ such that $|\partial_t|^\varepsilon \colon H \to L^4(\mathbb{R} \times \mathbb{T})$ is bounded. We show by induction that $|\partial_t|^{n\varepsilon} u \in H$ holds for $n \in \mathbb{N}_0$. So assume $|\partial_t|^{n\varepsilon} u \in H$ for fixed $n \in \mathbb{N}_0$. Let $v \in H$ with $|\partial_t|^{(n+1)\varepsilon} v \in H$. Then we have

$$\begin{aligned}
0 &= J'(u)[|\partial_t|^{(n+1)\varepsilon} v] \\
&= \left\langle u, |\partial_t|^{(n+1)\varepsilon} v \right\rangle_H - \int_{\mathbb{R}\times\mathbb{T}} h(x) u^3 \cdot |\partial_t|^{(n+1)\varepsilon} v \,\mathrm{d}(x,t) \\
&= \langle |\partial_t|^{n\varepsilon} u, |\partial_t|^\varepsilon v \rangle_H - \int_{\mathbb{R}\times\mathbb{T}} h(x) |\partial_t|^{n\varepsilon} (u^3) \cdot |\partial_t|^\varepsilon v \,\mathrm{d}(x,t).
\end{aligned}$$

By Lemma 4.4.1 with $p = 4$ and a density argument (cf. Lemma 4.2.3) we see that the map $v \mapsto \int_{\mathbb{R}\times\mathbb{T}} h(x) |\partial_t|^{n\varepsilon} (u^3) \cdot |\partial_t|^\varepsilon v \,\mathrm{d}(x,t)$ extends to a bounded linear functional on H. Hence there exists $w \in H$ with

$$\langle w, v \rangle_H = \int_{\mathbb{R}\times\mathbb{T}} h(x) |\partial_t|^{n\varepsilon} (u^3) \cdot |\partial_t|^\varepsilon v \,\mathrm{d}(x,t) = \langle |\partial_t|^{n\varepsilon} u, |\partial_t|^\varepsilon v \rangle_H$$

for $v \in H$ with $|\partial_t|^{(n+1)\varepsilon} v \in H$. Again by density we get $|\partial_t|^{(n+1)\varepsilon} u = w$. □

In order to proceed we need the following little result on the mapping properties of Fourier multiplier operators.

Lemma 4.4.3. *Let $Mv = \mathcal{F}^{-1}[m_k \hat{v}_k]$ be a Fourier multiplier with symbol $|m_k| \lesssim |k|^\sigma$ of polynomial growth and let u be a function with $|\partial_t|^s u \in H$ for all $s \in \mathbb{R}$. Then $|\partial_t|^s Mu \in H$ for all $s \in \mathbb{R}$. The same holds for H replaced by $L^p(\mathbb{R} \times \mathbb{T})$ with $p \in [1, \infty]$ if we require $\hat{u}_0 = 0$.*

Proof. In the Hilbert space setting we have

$$\||\partial_t|^s Mu\|_H \leq (\sup_{k \in \mathfrak{R}} |m_k||\omega k|^{-\sigma}) \left\| |\partial_t|^{s+\sigma} u \right\|_H < \infty$$

In the $L^p(\mathbb{R} \times \mathbb{T})$ case, the series $\mu(t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} m_k |\omega k|^{-\sigma-1} e_k(t)$ converges in $L^1(\mathbb{T})$. Hence

$$\||\partial_t|^s Mu\|_p = \left\| \mu * |\partial_t|^{s+\sigma+1} u \right\|_p \leq \|\mu\|_1 \left\| |\partial_t|^{s+\sigma+1} u \right\|_p < \infty. \qquad \square$$

Note that Lemma 4.4.3 applies to the multipliers $\mathcal{N}*, \mathcal{G}(x)*$ and by ($\mathcal{A}$4.5) also to $(\mathcal{N}*)^{-1}$.

Continuing our regularity analysis we show that u and its derivatives lie in $L^2 \cap L^\infty$. This also shows that u satisfies (4.14) strongly.

Proposition 4.4.4. *Let $u \in H$ be a critical point of J. Then $|\partial_t|^s u \in W^{2,p}(\mathbb{R} \times \mathbb{T})$ for all $s \in \mathbb{R}$ and $p \in [2, \infty]$, and it satisfies the equation*

$$-u_{xx} - V(x)\partial_t^2 u + h(x)\partial_t^2 \left(\mathcal{N} * u^3\right) = 0. \tag{4.18}$$

If (R) holds, we moreover have $|\partial_t|^s u \in W^{2+l,2}(\mathbb{R} \times \mathbb{T}) \cap C_b^{2+l}(\mathbb{R} \times \mathbb{T})$.

Proof. We remark that equation (4.18) formally follows by applying $-\partial_t^2 \mathcal{N}*$ to (4.14).

Part 1: We first show $|\partial_t|^s u \in L^p(\mathbb{R} \times \mathbb{T})$. Because of boundedness of the embedding $H \hookrightarrow L^2(\mathbb{R} \times \mathbb{T})$ and interpolation, it suffices to give the result for $p = \infty$. Similarly as in the proof of Lemma 4.2.4 we calculate

$$\begin{aligned}
\||\partial_t|^s u\|_{L^\infty(\mathbb{R}\times\mathbb{T})} &\lesssim \||\omega k|^s \mathcal{F}_{\xi,k}[u]\|_{L^1(\mathbb{R}\times\mathfrak{R})} \\
&\leq \left\| |\omega k|^{-1/2} \sqrt{\frac{\omega^2 k^2 \mathcal{F}_k[\mathcal{N}]}{\xi^2 + \omega^2 k^2}} \right\|_{L^2(\mathbb{R}\times\mathfrak{R})} \cdot \left\| |\omega k|^{s+1/2} \sqrt{\frac{\xi^2 + \omega^2 k^2}{\omega^2 k^2 \mathcal{F}_k[\mathcal{N}]}} \mathcal{F}_{\xi,k}[u] \right\|_{L^2(\mathbb{R}\times\mathfrak{R})}
\end{aligned}$$

where the first factor is finite since $0 \leq \mathcal{F}_k[\mathcal{N}] \leq |k|^{-\alpha}$, $\alpha > 1$, and the second factor is equivalent to $\left\| |\partial_t|^{s+1/2} u \right\|_H$ and thus finite by Lemma 4.4.2.

Part 2: Next we show $|\partial_t|^s u_x \in L^2(\mathbb{R} \times \mathbb{T})$:

$$\begin{aligned}
\||\partial_t|^s u_x\|^2_{L^2(\mathbb{R}\times\mathbb{T})} &= \sum_{k \in \mathfrak{R}} \int_{\mathbb{R}} |\omega k|^{2s} |\hat{u}_k'|^2 \,\mathrm{d}x \\
&\lesssim \sum_{k \in \mathfrak{R}} \int_{\mathbb{R}} \frac{|\omega k|^{2s+2-\alpha}}{\omega^2 k^2 \mathcal{F}_k[\mathcal{N}]} |\hat{u}_k'|^2 \,\mathrm{d}x \leq \left\| |\partial_t|^{s+1-\alpha/2} u \right\|_H^2 < \infty.
\end{aligned}$$

Now let $v \in H$ with $|\partial_t|^{s+2} \mathcal{N} * v \in H$. As $\mathcal{N}$ is even by ($\mathcal{A}$4.5), we have

$$\begin{aligned}
0 &= J'(u)[|\partial_t|^{s+2} \mathcal{N} * v] \\
&= \sum_{k \in \mathfrak{R}} |\omega k|^s \int_{\mathbb{R}} \hat{u}_k' \overline{\hat{v}_k'} + \omega^2 k^2 V_k(x) \hat{u}_k \overline{\hat{v}_k} \,\mathrm{d}x - \int_{\mathbb{R}\times\mathbb{T}} h(x) u^3 \cdot |\partial_t|^{s+2} \mathcal{N} * v \,\mathrm{d}(x,t) \\
&= \int_{\mathbb{R}\times\mathbb{T}} |\partial_t|^s u_x \cdot v_x \,\mathrm{d}(x,t) + \int_{\mathbb{R}\times\mathbb{T}} |\partial_t|^{2+s} \left(V(x)u - h(x)\mathcal{N} * u^3\right) \cdot v \,\mathrm{d}(x,t).
\end{aligned}$$

Since v was arbitrary, by density it follows that

$$|\partial_t|^s u_{xx} = |\partial_t|^{2+s}\Big(V(x)u - h(x)P_{\mathfrak{R}}[\mathcal{N} * u^3]\Big) \tag{4.19}$$

holds. The term on the right-hand side lies in $L^p(\mathbb{R}\times\mathbb{T})$ by Lemmas 4.4.1 and 4.4.3 and the first part of the proof. Thus, $|\partial_t|^s u_{xx} \in L^p(\mathbb{R}\times\mathbb{T})$ and $|\partial_t|^s u \in W^{2,p}(\mathbb{R}\times\mathbb{T})$.

Part 3: Assume (R), i.e., $\mathcal{G} \in C_b^l(\mathbb{R};\mathcal{M}(\mathbb{T}))$, $h \in C_b^l(\mathbb{R})$. First, we have $|\partial_t|^s u \in C_b(\mathbb{R}\times\mathbb{T})$ by Part 1 and Sobolev's embedding. Continuity of $\mathcal{G}, h$ shows that the right-hand side of (4.19) is continuous, so $|\partial_t|^s u_{xx} \in C_b(\mathbb{R}\times\mathbb{T})$ holds, and in particular $|\partial_t|^s u \in C_b^2(\mathbb{R}\times\mathbb{T})$.

For $l > 0$ we argue by induction over $k = 0,\dots,l$. We use that by (4.19) we have

$$\partial_x^{k+2}|\partial_t|^s u = \partial_x^k|\partial_t|^{s+2}\Big(V(x)u - h(x)\mathcal{N} * u^3\Big)$$

where the right-hand side lies in $L^2(\mathbb{R}\times\mathbb{T}) \cap C_b(\mathbb{R}\times\mathbb{T})$ by the product rule and the induction hypothesis. This allows us to conclude $|\partial_t|^s u \in W^{2+k,2}(\mathbb{R}\times\mathbb{T}) \cap C_b^{k+2}(\mathbb{R}\times\mathbb{T})$. □

Recall for the first type of nonlinearity (4.6.i) that the profile w of the electric field is given by $w = u$. For the second type of nonlinearity (4.6.ii), by (4.15) the profile satisfies $P_{\mathfrak{R}}[w] = (\mathcal{N}*)^{-1}u$ where $P_{\mathfrak{S}}[w]$ solves a differential equation. Therefore we need to discuss next the regularity of w for the second type of nonlinearity, which is done in the following analogue to Proposition 4.4.4.

Proposition 4.4.5. *Let $u \in H$ be a critical point of J. Define $w = w_1 + w_2$ where*

$$w_1 = P_{\mathfrak{R}}[w] = (\mathcal{N}*)^{-1}u, \quad w_2 = P_{\mathfrak{S}}[w] = (-\partial_x^2 - V(x)\partial_t^2)^{-1}(h(x)\partial_t^2 P_{\mathfrak{S}}[u^3]).$$

Then w satisfies $|\partial_t|^s w \in W^{2,p}(\mathbb{R}\times\mathbb{T})$ for all $s \in \mathbb{R}$, $p \in [2,\infty]$ and solves

$$(-\partial_x^2 - V(x)\partial_t^2)w + h(x)\partial_t^2(\mathcal{N} * w)^3 = 0.$$

If (R) holds, we moreover have $|\partial_t|^s w \in W^{2+l,2}(\mathbb{R}\times\mathbb{T}) \cap C_b^{2+l}(\mathbb{R}\times\mathbb{T})$.

Proof. First, by Lemma 4.4.3 and Proposition 4.4.4 we see that the function $w_1 := (\mathcal{N}*)^{-1}u := \sum_{k\in\mathfrak{R}} \frac{1}{\mathcal{F}_k[\mathcal{N}]}\hat{u}_k(x)e_k(t)$ satisfies $|\partial_t|^s w_1 \in W^{2,p}(\mathbb{R}\times\mathbb{T})$ for $s \in \mathbb{R}, p \in [2,\infty]$, with additional regularity if $\mathcal{G}, h$ fulfills (R). Applying $(\mathcal{N}*)^{-1}$ to (4.18) we see that w_1 solves

$$(-\partial_x^2 - V(x)\partial_t^2)w_1 + h(x)\partial_t^2 P_{\mathfrak{R}}[u^3] = 0.$$

Let us now turn our attention to w_2 and define the space

$$H_2 := \Big\{v \in H^1(\mathbb{R}\times\mathbb{T})\colon \hat{v}_k \equiv 0 \text{ for } k \in \mathfrak{R}\cup\{0\}\Big\}.$$

Since $V_k(x)$ is bounded and positive, the Riesz representation theorem provides $w_2 \in H_2$ with

$$\int_{\mathbb{R}\times\mathbb{T}} \partial_x w_2 \cdot \partial_x v + V(x)\partial_t w_2 \cdot \partial_t v \,\mathrm{d}(x,t) = -\int_{\mathbb{R}\times\mathbb{T}} h(x)\partial_t^2 P_{\mathfrak{S}}[u^3]\cdot v \,\mathrm{d}(x,t) \tag{4.20}$$

for all $v \in H_2$.

Similarly as for solutions u of $J'(u)[v] = 0$ for $v \in H$, we obtain regularity for the solution w_2 to (4.20). In contrast to J, where critical points satisfy a truly nonlinear equation, the right-hand side of (4.20) is independent of w_2 and its regularity properties have been established in Proposition 4.4.4. Let us sketch the arguments:

As in Lemma 4.4.2 we find $|\partial_t|^s w_2 \in H_2$ for $s \in \mathbb{R}$. Using that the 0-th Fourier mode of $h(x)\partial_t^2 P_{\mathfrak{S}}[u^3]$ vanishes, w_2 satisfies

$$-\partial_x^2|\partial_t|^s w_2 - V(x)\partial_t^2|\partial_t|^s w_2 = -h(x)\partial_t^2|\partial_t|^s P_{\mathfrak{S}}[u^3] \tag{4.21}$$

By the fractional Leibniz rule from Lemma 4.4.1, the regularity properties of u from Proposition 4.4.4 and the boundedness of the Fourier symbol of $P_{\mathfrak{S}}$ we find that the right-hand side of (4.21) lies in L^2. Therefore $|\partial_t|^s w_2 \in W^{2,2}(\mathbb{R}\times\mathbb{T}) \subseteq L^\infty(\mathbb{R}\times\mathbb{T})$ holds for $s \in \mathbb{R}$ and (4.21) shows $|\partial_t|^s w_2 \in W^{2,p}(\mathbb{R}\times\mathbb{T})$ for $s\in\mathbb{R}$, $p\in[2,\infty]$. The additional regularity when $\mathcal{G}, h$ satisfy (R) can then be shown as in Proposition 4.4.4 by iteratively applying space-derivatives to (4.21). $\square$

Lastly, we discuss the regularity of the corresponding electromagnetic fields.

Proof of Theorem 4.1.3 for slab geometries. Let u be a nontrivial critical point of J. If the nonlinearity is given by $N(w) = \mathcal{N} * w^3$ then we set $w := u$. If otherwise $N(w) = (\mathcal{N} * w)^3$ then let w be from Proposition 4.4.5. Then we define $W := \partial_t^{-1} w$ and reconstruct the electromagnetic fields by

$$\mathbf{E}(\mathbf{x},t) = w(x, t - \tfrac{1}{c}z)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \mathbf{B}(\mathbf{x},t) = -\begin{pmatrix}\frac{1}{c}w(x,t-\frac{1}{c}z)\\0\\W_x(x,t-\frac{1}{c}z)\end{pmatrix}$$

$$\mathbf{D}(\mathbf{x},t) = \epsilon_0(w + \mathcal{G}*w + h(x)N(w))\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \mathbf{H}(\mathbf{x},t) = \tfrac{1}{\mu_0}\mathbf{B}(\mathbf{x},t).$$

Due to Proposition 4.4.4 and Proposition 4.4.5 we have the inclusions

$$\partial_t^n\mathbf{E} \in W^{2,p}(\Omega;\mathbb{R}^3), \qquad \partial_t^n\mathbf{B}, \partial_t^n\mathbf{H} \in W^{1,p}(\Omega;\mathbb{R}^3), \qquad \partial_t^n\mathbf{D} \in L^p(\Omega;\mathbb{R}^3),$$

and assuming (R) we moreover have

$$\partial_t^n\mathbf{E} \in \tilde{C}_b^{2+l}(\Omega;\mathbb{R}^3), \qquad \partial_t^n\mathbf{B}, \partial_t^n\mathbf{H} \in \tilde{C}_b^{1+l}(\Omega;\mathbb{R}^3), \qquad \partial_t^n\mathbf{D} \in \tilde{C}_b^{l}(\Omega;\mathbb{R}^3)$$

for any domain $\Omega = \mathbb{R}\times[y,y+1]\times[z,z+1]\times[t,t+1]$ and all $n\in\mathbb{N}$, $p\in[2,\infty]$ with norm bounds independent of y,z,t. By direct calculation one checks that the fields $\mathbf{E},\mathbf{D},\mathbf{B},\mathbf{H}$ solve Maxwell's equations (4.1), (4.2). $\square$

Proof of Remark 4.1.5. To show that there exist infinitely many solutions, we search for breather solutions with time period $\frac{T}{n}$ instead of T. If we define the corresponding time-domain $\mathbb{T}_n := \mathbb{R}/_{\frac{T}{n}\mathbb{Z}}$ then the k-th Fourier coefficient of a $\frac{T}{n}$-periodic function f, when understood as a $\mathbb{T}$-periodic function, satisfies $\mathcal{F}_k[f\colon \mathbb{T}_n \to \mathbb{R}] = \mathcal{F}_{nk}[f\colon\mathbb{T}\to\mathbb{R}]$. Therefore, by the above arguments there exists a $\frac{T}{n}$-periodic breather solution $(\mathbf{E}_n,\mathbf{D}_n,\mathbf{B}_n,\mathbf{H}_n)$ to (4.1), (4.2) with minimal period $T_n > 0$ for all such $n\in\mathbb{N}$ where $\mathfrak{R}\cap n\mathbb{Z}\neq\emptyset$. By assumption, the set $\{n\in\mathbb{N}\colon \mathfrak{R}\cap n\mathbb{Z}\neq\emptyset\}$ is infinite. Since T_n is a divisor of $\frac{T}{n}$ the minimal period T_n goes to 0 for $n\to\infty$. Hence infinitely many among the breather solutions $(\mathbf{E}_n,\mathbf{D}_n,\mathbf{B}_n,\mathbf{H}_n)$ must be mutually distinct. $\square$

4.5. Modifications for the cylindrical geometry

In this section we discuss the cylindrical problem, that is, we consider equation (4.8.2) instead of (4.8.1). The only difference between the two problems is in the spatial differential operator, where we now work with $-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}$ on the domain $r \in [0,\infty)$ instead of $-\partial_x^2$ for $x \in \mathbb{R}$. The differential operator $-\partial_r^2 - \frac{1}{r}\partial_r$ is the 2d Laplacian for radially symmetric functions, and $\frac{1}{r^2}$ is an additional positive term. Hence it is natural to equip the domain $[0,\infty)$ with the measure $r\mathrm{d}r$, and to identify functions on it with radially symmetric functions of the variables $(x,y) \in \mathbb{R}^2$ via $r = \sqrt{x^2+y^2}$. We use the subscript "rad" to denote spaces of functions that are radially symmetric in (x,y). Since the term $\int_0^\infty \frac{u^2}{r^2}\, r\mathrm{d}r$ cannot be controlled by the H^1_{rad}-Sobolev norm of u (recall that Hardy's inequality fails in two dimensions) we need to add this term in the form domain of the differential operator.

We will discuss how the arguments from Sections 4.2 to 4.4 have to be adapted to treat the cylindrical problem. We use the same structure as in these sections. In order to not repeat the previous chapters, we discuss in detail only results that require new techniques to adapt them to the cylindrical geometry and roughly sketch the other results.

4.5.1. Modifications for Sections 4.2 and 4.3

In analogy to Definitions 4.2.1 and 4.2.2, we define the functional of interest $\tilde{J}$ and its domain $\tilde{H}$ (replacing J and H).

Definition 4.5.1. *We define the space*

$$\tilde{H} := \left\{ u \in L^2([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t)) \colon \hat{u}_k = 0 \text{ for } k \in \mathbb{Z}_{\mathrm{even}} \cup \mathfrak{S}, \|u\|_{\tilde{H}}^2 := \langle\!\langle u,u\rangle\!\rangle_{\tilde{H}} < \infty \right\}$$

with the two equivalent inner products

$$\langle\!\langle u,v\rangle\!\rangle_{\tilde{H}} := \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]} \int_0^\infty \Big(\hat{u}_k'\overline{\hat{v}_k'} + \big(\tfrac{1}{r^2} + \omega^2k^2\big)\hat{u}_k\overline{\hat{v}_k}\Big)\, r\mathrm{d}r,$$
$$\langle u,v\rangle_{\tilde{H}} := \sum_{k\in\mathfrak{R}} \frac{1}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]} \int_0^\infty \big(\hat{u}_k'\overline{\hat{v}_k'} + \big(\tfrac{1}{r^2} + \omega^2k^2\tilde{V}_k(r)\big)\hat{u}_k\overline{\hat{v}_k}\big)\, r\mathrm{d}r$$

where $\tilde{V}_k(r) := \frac{1}{c^2} - 1 - \mathcal{F}_k[\mathcal{G}(r)]$*. On* $\tilde{H}$*, we define the functional*

$$\tilde{J}(u) := \tfrac{1}{2}\langle u,u\rangle_{\tilde{H}} - \tfrac{1}{4}\int_{[0,\infty)\times\mathbb{T}} h(x)u^4\, r\mathrm{d}(r,t) \quad \text{for} \quad u \in \tilde{H}$$

so that its critical points $u \in \tilde{H}$ *satisfy*

$$\tilde{J}'(u)[v] = \langle u,v\rangle_{\tilde{H}} - \int_{[0,\infty)\times\mathbb{T}} h(x)u^3v\, r\mathrm{d}(r,t) = 0 \quad \text{for} \quad v \in \tilde{H}$$

As a first step, we discuss the embedding properties of $\tilde{H}$.

Lemma 4.5.2. *For any* $p \in (2,p^\star)$ *with* $p^\star = \frac{6}{3-\alpha}$ *(*$p^\star = \infty$ *if* $\alpha \geq 3$*), the embedding* $\tilde{H} \hookrightarrow L^p([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))$ *is compact. Moreover,* $\tilde{H} \hookrightarrow L^2([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))$ *is continuous and* $\tilde{H} \hookrightarrow L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))$ *is compact.*

Proof. We interpret a function $u\colon [0,\infty)\times\mathbb{T}\to\mathbb{R}$ as a function of the three variables (x,y,t) which is radially symmetric in (x,y) via $r=\sqrt{x^2+y^2}$. Let σ_ρ be the surface measure of the sphere $S_\rho\subseteq\mathbb{R}^2$ of radius ρ centered at 0, normalized such that $\sigma_\rho(S_\rho)=1$, and continued by 0 to a Borel measure on $\mathbb{R}^2$. Using the Hansen-Bessel formula (cf. [95, equation 9.19]), we see that its Fourier transform is given by

$$\mathcal{F}^{-1}_{(x,y)}(\sigma_\rho)=\frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}r\rho\sin(\varphi)}\,\mathrm{d}\varphi=\frac{1}{2\pi}J_0(r\rho)$$

where J_0 is the Bessel function of first kind. Using $|J_0(s)|\lesssim s^{-\theta}$ for $\theta\in[0,\frac12]$ (cf. [39]) we obtain

$$\left\|r^\theta u\right\|_{L^\infty(\mathbb{R}^2\times\mathbb{T})}\lesssim\left\||\xi|^{-\theta}\mathcal{F}_{\xi,k}[u]\right\|_{L^1(\mathbb{R}^2\times\mathbb{Z})},$$

where we used that $\mathcal{F}_{\xi,k}[u]$ is radially symmetric in ξ. Note also that we have $\|u\|_{L^2(\mathbb{R}^2\times\mathbb{T})}=\|\mathcal{F}_{\xi,k}[u]\|_{L^2(\mathbb{R}^2\times\mathbb{Z})}$. This allows us to use the Riesz-Thorin interpolation theorem (cf. [53]) and get (with # denoting the counting measure) that the map

$$T:\begin{cases} L^{p'}_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{Z};|\xi|^{-2\theta}\mathrm{d}\xi\otimes\#) & \to\quad L^p_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T};r^{-2\theta}\mathrm{d}(x,t)),\\ \qquad\qquad v & \mapsto\quad r^\theta\mathcal{F}^{-1}_{(x,y),t}(|\xi|^{-\theta}v)\end{cases}$$

is bounded for all $p\in[2,\infty]$. To see this note that with $v=|\xi|^{-\theta}\mathcal{F}_{\xi,k}[u]$ we have

$$\begin{aligned}\|v\|_{L^{p'}(\mathbb{R}^2\times\mathbb{Z};|\xi|^{-2\theta}\mathrm{d}\xi\otimes\#)}&=\|\mathcal{F}_{\xi,k}[u]|\xi|^{-\theta_0}\|_{L^{p'}(\mathbb{R}^2\times\mathbb{Z};\mathrm{d}\xi\otimes\#)}\\ \|Tv\|_{L^p(\mathbb{R}^2\times\mathbb{T};r^{-2\theta}\mathrm{d}(x,t))}&=\|r^{\theta_0}u\|_{L^p(\mathbb{R}^2\times\mathbb{T};\mathrm{d}(x,t))}\end{aligned}$$

where $\theta_0=\theta\frac{(p-2)}{p}$ ranges through $[0,\frac12-\frac1p]$ as θ runs through $[0,\frac12]$. Thus we have

$$\begin{aligned}\left\|r^{\theta_0}u\right\|_{L^p(\mathbb{R}^2\times\mathbb{T})}&\lesssim\left\||\xi|^{-\theta_0}\mathcal{F}_{\xi,k}[u]\right\|_{L^{p'}(\mathbb{R}^2\times\mathfrak{R})}\\ &\le\left\||\xi|^{-\theta_0}\sqrt{\frac{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}{|\xi|^2+\omega^2k^2}}\right\|_{L^r(\mathbb{R}^2\times\mathfrak{R})}\left\|\sqrt{\frac{|\xi|^2+\omega^2k^2}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}}\mathcal{F}_{\xi,k}[u]\right\|_{L^2(\mathbb{R}^2\times\mathfrak{R})}\end{aligned}$$

where $\frac1r=\frac12-\frac1p<\frac{\alpha}{6}$. By the choice of $p^\star$ and assumption ($\mathcal{A}$4.5), the L^r-norm is finite provided θ_0 is chosen sufficiently small, and the L^2-norm can be estimated against $\|u\|_{\tilde H}$.

For the particular choice $\theta_0=0$, this shows that the embedding $\tilde H\hookrightarrow L^p_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$ is continuous. Moreover, we can argue similarly as in the proof of Lemma 4.2.4 to verify that the local embedding $\tilde H\hookrightarrow L^p_{\mathrm{rad,loc}}(\mathbb{R}^2\times\mathbb{T})$ is compact.

It remains to show that $\tilde H\hookrightarrow L^p_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$ is compact for $p\neq2$. For $R>0$ consider the compact map $E_R\colon\tilde H\to L^p_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T}), u\mapsto u\mathbb{1}_{B_R(0)\times\mathbb{T}}$. Using the above inequality we have

$$\|E_Ru-u\|_p\le\left\|\left(\tfrac{r}{R}\right)^{\theta_0}u\right\|_p\lesssim R^{-\theta_0}\|u\|_{\tilde H}$$

so by choosing any admissible $\theta_0>0$ and taking the limit $R\to\infty$ we see that the embedding $I\colon\tilde H\to L^p_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$ is compact as the uniform limit of a sequence of compact operators. □

Notice that, unlike in the slab setting (cf. Lemma 4.2.4), the embedding of Lemma 4.5.2 is compact for $p>2$. This is why we do not require additional assumptions ($\mathcal{A}$4.6a) or ($\mathcal{A}$4.6b) in the cylindrical setting. One can then show existence of ground states similar to the "compact" case ($\mathcal{A}$4.6a) of Section 4.3. The only difference is that existence of a convergent subsequence of $h(r)u_n^3$ in $L^{4/3}_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$ is guaranteed by the compact embedding instead of decay properties of h.

4.5.2. Modifications for Section 4.4

In the following, we show in Propositions 4.5.3 and 4.5.4 two regularity results that are the cylindrical counterparts to Propositions 4.4.4 and 4.4.5.

Here, arguments will get more difficult since the cylindrical geometry is effectively 2-dimensional in space (compared with 1d for the slab problem). For some arguments it will be advantageous to view the $\frac{1}{r^2}$ not as an additional order 0 term, but as part of the differential operator. From [8] we use the identity

$$\partial_r^2 + \tfrac{1}{r}\partial_r - \tfrac{1}{r^2} = \tfrac{1}{r^2}\partial_r r^3 \partial_r \tfrac{1}{r}, \tag{4.22}$$

which means that up to the multiplicative factors $r, \frac{1}{r}$ we are dealing with $\frac{1}{r^3}\partial_r r^3 \partial_r$, which is the Laplacian of a radially symmetric function in 4 dimensions.

Similar to Proposition 4.4.4 we show that u and its derivatives lie in $L^2 \cap L^\infty$.

Proposition 4.5.3. *Let $u \in \tilde{H}$ be a critical point of $\tilde{J}$. Then the terms*

$$\max\{r,1\}|\partial_t|^s(\tfrac{u}{r}), \qquad \max\{r,1\}|\partial_t|^s\partial_r(\tfrac{u}{r}), \qquad r|\partial_t|^s\partial_r^2(\tfrac{u}{r})$$

lie in $L^p([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))$ for all $s \in \mathbb{R}$ and $p \in [2,\infty]$, and u solves pointwise

$$(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} - V(x)\partial_t^2)u + h(x)\partial_t^2(\mathcal{N} * u^3) = 0.$$

If (R) holds, then the terms

$$\max\{r,1\}|\partial_t|^s\partial_r^n(\tfrac{u}{r}) \quad \text{for} \quad 0 \le n \le l+1 \qquad \text{as well as} \qquad r|\partial_t|^s\partial_r^{l+2}(\tfrac{u}{r})$$

lie in $L^2([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t)) \cap C_b([0,\infty)\times\mathbb{T})$. Moreover, the second term vanishes at $r=0$, and the same holds for the first term when n is odd.

Proof. Part 1: First, following the proof of Lemma 4.4.2 we obtain $|\partial_t|^s u \in \tilde{H}$ for all $s \in \mathbb{R}$.

Next, for $p \in [2,\infty)$ we calculate

$$\begin{aligned}\||\partial_t|^s u\|_p &\lesssim \||\omega k|^s \mathcal{F}_{\xi,k}\|_{L^{p'}(\mathbb{R}^2\times\mathfrak{R})} \\ &\le \left\| |\omega k|^{-1}\sqrt{\frac{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}{|\xi|^2+\omega^2k^2}}\right\|_{L^r(\mathbb{R}^2\times\mathfrak{R})} \left\| |\omega k|^{s+1}\sqrt{\frac{|\xi|^2+\omega^2k^2}{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}}\mathcal{F}_{\xi,k}[u]\right\|_{L^2(\mathbb{R}^2\times\mathfrak{R})} \lesssim \left\||\partial_t|^{s+1}u\right\|_{\tilde{H}}.\end{aligned}$$

Here $1-\frac{1}{p} = \frac{1}{p'} = \frac{1}{r}+\frac{1}{2}$, and the L^r-norm is finite since $r>2$ and therefore

$$\begin{aligned}&\sum_{k\in\mathbb{R}} |\omega k|^{-r}\int_{\mathbb{R}^2}\left(\frac{\omega^2k^2\mathcal{F}_k[\mathcal{N}]}{|\xi|^2+\omega^2k^2}\right)^{\frac{r}{2}}\mathrm{d}\xi \\ &\quad = \int_{\mathbb{R}^2}\left(\frac{1}{|\xi|^2+1}\right)^{r/2}\mathrm{d}\xi \cdot \sum_{k\in\mathbb{R}}|\omega k|^{2-r}\mathcal{F}_k[\mathcal{N}]^{\frac{r}{2}} \lesssim \sum_{k\in\mathfrak{R}}|k|^{2-\frac{7}{4}r} < \infty.\end{aligned}$$

Part 2: Arguing as in part 2 of the proof of Proposition 4.4.4 we have that $|\partial_t|^s u_r, \frac{1}{r}|\partial_t|^s u \in L^2([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))$ for $s\in\mathbb{R}$ and

$$|\partial_t|^s u_{rr} + \tfrac{1}{r}|\partial_t|^s u_r - \tfrac{1}{r^2}|\partial_t|^s u = |\partial_t|^{s+2}\Big(V(r)u - h(r)P_{\mathfrak{R}}[\mathcal{N} * u^3]\Big) \tag{4.23}$$

holds pointwise. From now on arguments differ depending on if r is large or small, and we discuss these cases in part 3 and part 4, respectively.

Part 3a: Let $r > R_1$ for fixed $R_1 > 0$. Then part 1 combined with (4.23) shows that $\Delta_r |\partial_t|^s u := (\partial_r^2 + \frac{1}{r}\partial_r)|\partial_t|^s u \in L^p([R_1,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))$ for $p \in [2,\infty)$. Now choose a cutoff $\psi_1 \in C^\infty([0,\infty))$ with $\operatorname{supp}\psi_1 \subseteq (R_1,\infty)$ and $\psi_1 \equiv 1$ on $[R_2,\infty)$ for $R_2 > R_1$. Then, interpreting $v_1 := \psi_1(r)u$ as a function of the three variables (x,y,t) via $r = \sqrt{x^2+y^2}$ and continuing by zero, we have $|\partial_t|^s v_1 \in L^p_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$ and

$$\Delta_{(x,y)}|\partial_t|^s v_1 = \Delta_r\psi_1 \cdot |\partial_t|^s u + 2\partial_r\psi_1 \cdot \partial_r|\partial_t|^s u + \psi_1 \cdot \Delta_r|\partial_t|^s u \in L^2(\mathbb{R}^2\times\mathbb{T}).$$

This shows $|\partial_t|^s v_1 \in H^2_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$, and by Sobolev's embedding we in particular have $|\partial_t|^s u \in L^\infty([R_2,\infty)\times\mathbb{T})$, $|\partial_t|^s u_r \in L^6([R_2,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))$.

Similar to above, but now with a smooth cutoff ψ_2 such that $\operatorname{supp}\psi_2 \subseteq (R_2,\infty)$, $\psi_2 \equiv 1$ on $[R_3,\infty)$ for $R_3 > R_2$, we see that $v_2 := \psi_2(r)u$ satisfies $\Delta_{(x,y)}|\partial_t|^s v_2 \in L^6_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$ where again (4.23) was used. Thus $|\partial_t|^s v_2 \in W^{2,6}_{\mathrm{rad}}(\mathbb{R}^2\times\mathbb{T})$ by L^p-boundedness of the Riesz transform, cf. [40, Corollary 5.2.8]. By Sobolev's embedding we have $\partial_r|\partial_t|^s u \in L^\infty([R_3,\infty)\times\mathbb{T})$, and then $\Delta_r|\partial_t|^s u \in L^\infty([R_3,\infty)\times\mathbb{T})$ by (4.23). So far we have shown

$$|\partial_t|^s u, |\partial_t|^s u_r, |\partial_t|^s u_{rr} \in L^2([R_3,\infty)\times\mathbb{T}; r\mathrm{d}(r,t)) \cap L^\infty([R_3,\infty)\times\mathbb{T}).$$

This shows the first part of Proposition 4.5.3 for $r > R_3$, where $R_3 > 0$ can be chosen arbitrarily.

Part 3b: We assume (R), i.e., $\mathcal{G}, h \in C_b^l$, and still consider large r. From *Part 3a* we obtain $|\partial_t|^s u, |\partial_t|^s u_r, \in C_b([R,\infty)\times\mathbb{T})$ from the high Sobolev regularity and therefore also $|\partial_t|^s u_{rr} \in C_b([R,\infty)\times\mathbb{T})$ by applying (4.23). Now $|\partial_t|^s\partial_r^n u \in L^2([R,\infty)\times\mathbb{T}; r\mathrm{d}(r,t)) \cap C_b([R,\infty)\times\mathbb{T})$ for $2 < n \le l+2$ can be shown iteratively by applying space-derivatives to (4.23) and using that all terms except the highest order space-derivative term lie in $L^2([R,\infty)\times\mathbb{T}; r\mathrm{d}(r,t)) \cap C_b([R,\infty)\times\mathbb{T})$ by the induction hypothesis.

Part 4a: Let us now consider small r. We use the representation via the 4d Laplacian, i.e., we consider $U\colon \mathbb{R}^4\times\mathbb{T}\to\mathbb{R}$, $U(X,t) = U(X_1,X_2,X_3,X_4,t) = \frac{1}{|X|}u(|X|,t)$. Notice that the L^2-norms are equivalent, i.e.,

$$\left\| \tfrac{1}{|X|} f(|X|,t) \right\|_{L^2(B_R\times\mathbb{T})} = \sqrt{2}\pi \|f\|_{L^2([0,R]\times\mathbb{T}; r\mathrm{d}(r,t))}$$

holds with $B_R \subseteq \mathbb{R}^4$ denoting the ball of radius R centered at 0. Multiplying (4.23) by $\frac{1}{r}$, setting $r = |X|$, and recalling (4.22) we have

$$\Delta_X|\partial_t|^s U = V(r)|\partial_t|^{s+2}U - \tfrac{h(r)}{r}|\partial_t|^{s+2}P_{\mathfrak{R}}[\mathcal{N} * u^3]. \tag{4.24}$$

By part 1 the right-hand side of (4.24) lies in $L^2(\mathbb{R}^4\times\mathbb{T})$. Therefore $|\partial_t|^s U \in H^2(\mathbb{R}^4\times\mathbb{T})$, which by Sobolev's embeddings shows $|\partial_t|^s\partial_r U \in L^{10/3}(\mathbb{R}^4\times\mathbb{T}), |\partial_t|^s U \in L^{10}(\mathbb{R}^4\times\mathbb{T})$. Now let $R_1 > 0$ and $\psi_1 \in C_c^\infty([0,R_1))$ be a smooth cutoff function with $\psi_1 \equiv 1$ on $[0,R_2]$ for some $0 < R_2 < R_1$ and set $v_1 := \psi_1(|X|)U$. Then, since

$$\Delta_X|\partial_t|^s v_1 = \Delta_X\psi_1 \cdot |\partial_t|^s U + \partial_r\psi_1 \cdot \partial_r|\partial_t|^s U + \psi_1 \cdot \Delta_X|\partial_t|^s U$$

and since we may write (4.24) as

$$\Delta_X|\partial_t|^s U = V(r)|\partial_t|^{s+2}U - r^2 h(r)|\partial_t|^{s+2}P_{\mathfrak{R}}[\mathcal{N} * U^3], \tag{4.25}$$

we have $\Delta_X |\partial_t|^s v_1 \in L^{10/3}(\mathbb{R}^4 \times \mathbb{T})$ which by L^p-boundedness of the Riesz transform implies $|\partial_t|^s v_1 \in W^{2,10/3}(\mathbb{R}^4 \times \mathbb{T})$. From Sobolev's embedding we have $|\partial_t|^s U \in L^\infty(B_{R_2} \times \mathbb{T})$, $\nabla_X |\partial_t|^s U \in L^{10}(B_{R_2} \times \mathbb{T})$. Repeating this argument with $0 < R_3 < R_2$ and a cutoff function $\psi_2 \in C_c^\infty([0, R_2))$, $\psi_j \equiv 1$ on $[0, R_3]$ shows $\nabla_X |\partial_t|^s U \in L^\infty(B_{R_3} \times \mathbb{T})$. Using (4.25), regularity of the terms in the claim of Proposition 4.5.3 follows since

$$|\partial_t|^s \tfrac{u}{r} = |\partial_t|^s U \in L^\infty, \qquad |\partial_t|^s \partial_r(\tfrac{u}{r}) = \tfrac{X}{r} \cdot \nabla_X |\partial_t|^s U \in L^\infty,$$
$$r|\partial_t|^s \partial_r^2(\tfrac{u}{r}) = r\Delta_X |\partial_t|^s U - 3\tfrac{X}{r} \cdot \nabla_X |\partial_t|^s U \in L^\infty$$

The L^p-estimates follow from these since $B_{R_3} \times \mathbb{T}$ has finite volume.

Part 4b: Assume (R), and again consider small r. First, $|\partial_t|^s U, \nabla_X |\partial_t|^s U$ are continuous by Sobolev's embedding, and continuity of $\Delta_X |\partial_t|^s U$ follows from this by (4.25). Existence and continuity of higher derivatives

$$\nabla_X^n \Delta_X |\partial_t|^s U, \qquad \nabla_X^{n+1} |\partial_t|^s U, \qquad \nabla_X^n |\partial_t|^s U$$

for $0 < n \leq l$ can again be shown using induction and repeatedly applying ∇_X to (4.25). This implies continuity of all terms except the highest order one in Proposition 4.5.3 since $|\partial_t|^s \partial_r^n(\frac{u}{r}) = (\nabla_X^n |\partial_t|^s U)[\frac{X}{r}, \ldots, \frac{X}{r}]$. Moreover, odd r-derivatives of $\frac{u}{r}$ vanish at $r = 0$ since U is radially symmetric. For the highest order term we have

$$(\nabla_X^l \Delta_X |\partial_t|^s U)[\tfrac{X}{r}, \ldots, \tfrac{X}{r}] = |\partial_t|^s \partial_r^{l+2}(\tfrac{u}{r}) + |\partial_t|^s \partial_r^l(\tfrac{3}{r}\partial_r(\tfrac{u}{r}))$$

which shows that $|\partial_t|^s \partial_r^{l+2}(\frac{u}{r})$ is continuous away from 0. To see the behaviour of the highest order term near $r = 0$ we use the differentiability properties of U and a Taylor expansion of $|\partial_t|^s(\frac{u}{r})$ about $r = 0$ as follows. Let $|\partial_t|^s(\frac{u}{r}) = T_{l+1}(|\partial_t|^s(\frac{u}{r}); 0) + f$ be the Taylor expansion of $|\partial_t|^s(\frac{u}{r})$ of degree $l + 1$ about $r = 0$ with remainder f. Then we have

$$|\partial_t|^s \partial_r^l(\tfrac{3}{r}\partial_r(\tfrac{u}{r})) = \partial_r^l\Big(\tfrac{3}{r}\partial_r\left[T_{l+1}(|\partial_t|^s \tfrac{u}{r}; 0)\right]\Big) + \partial_r^l\Big(\tfrac{3}{r}\partial_r f\Big) = \frac{3(-1)^l}{l!} \frac{|\partial_t|^s(\frac{u}{r})_r(0)}{r^{l+1}} + \mathrm{o}(\tfrac{1}{r}) = \mathrm{o}(\tfrac{1}{r})$$

as $r \to 0$ since $|\partial_t|^s(\frac{u}{r})_r(0) = 0$ by radial symmetry. This shows that $r|\partial_t|^s \partial_r^{l+2}(\frac{u}{r}) \to 0$ as $r \to 0$. □

Next, in Proposition 4.5.4, similar to Proposition 4.4.5 we discuss the second nonlinearity (4.6.ii).

Proposition 4.5.4. *Let $u \in \tilde{H}$ be a critical point of $\tilde{J}$ and let the nonlinearity be given by $N(w) = \mathcal{N} * w^3$. Define $w = w_1 + w_2$ where*

$$w_1 = P_{\mathfrak{R}}[w] = (\mathcal{N}*)^{-1}u, \quad w_2 = P_{\mathfrak{S}}[w] = (-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} - V(x)\partial_t^2)^{-1}(h(x)\partial_t^2 P_{\mathfrak{S}}[u^3])$$

Then the functions

$$\max\{r, 1\}|\partial_t|^s \tfrac{w}{r}, \qquad \max\{r, 1\}|\partial_t|^s \partial_r(\tfrac{w}{r}), \qquad r|\partial_t|^s \partial_r^2(\tfrac{w}{r})$$

lie in $L^p([0, \infty) \times \mathbb{T}; r\mathrm{d}(r, t))$ for all $s \in \mathbb{R}$ and $p \in [2, \infty]$, and w solves

$$(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} - V(x)\partial_t^2)w + h(x)\partial_t^2(\mathcal{N} * w)^3 = 0.$$

If (R) holds, then the terms

$$\max\{r, 1\}|\partial_t|^s \partial_r^n(\tfrac{w}{r}) \quad \textit{for} \quad 0 \leq n \leq l + 1 \qquad \textit{as well as} \qquad r|\partial_t|^s \partial_r^{l+2}(\tfrac{w}{r})$$

lie in $L^2([0, \infty) \times \mathbb{T}; r\mathrm{d}(r, t)) \cap C_b([0, \infty) \times \mathbb{T})$. Moreover, the second term vanishes at $r = 0$, and the same holds for the first term when n is odd.

Proof. We follow Proposition 4.4.5, and choose $w_2 \in \tilde{H}_2$, where

$$\tilde{H}_2 := \Big\{v \in H^1([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t))\colon \tfrac{1}{r}v \in L^2([0,\infty)\times\mathbb{T}; r\mathrm{d}(r,t)), \hat{v}_k \equiv 0 \text{ for } k \in \mathfrak{R}\cup\{0\}\Big\},$$

such that

$$\begin{aligned}\int_{[0,\infty)\times\mathbb{T}} &\Big(\partial_r w_2\cdot\partial_r v + \tfrac{1}{r^2}w_2\cdot v + V(r)\partial_t w_2\cdot\partial_t v\Big)\, r\mathrm{d}(r,t)\\ &= -\int_{[0,\infty)\times\mathbb{T}} \Big(h(r)\partial_t^2 P_{\mathfrak{S}}[u^3]\cdot v\Big)\, r\mathrm{d}(r,t)\end{aligned}$$

for all $v \in \tilde{H}_2$. Regularity of w_1 follows from Proposition 4.5.3, and the arguments therein can also be used to show regularity of w_2. □

As the last part of this chapter, we discuss regularity of the electromagnetic wave profiles.

Proof of Theorem 4.1.3 for cylindrical geometries. Part 1: Let $u \in \tilde{H}$ be a nontrivial critical point of $\tilde{J}$. For the nonlinearity $N(w) = \mathcal{N} * w^3$ set $w := u$, else let w be from Proposition 4.5.4. Define $W := \partial_t^{-1} w$ and the electromagnetic fields by

$$\mathbf{D}(\mathbf{x},t) = \epsilon_0(w + \mathcal{G}*w + h(r)N(w))\cdot\begin{pmatrix}-y/r\\ x/r\\ 0\end{pmatrix},\qquad \mathbf{E}(\mathbf{x},t) = w(r,t-\tfrac{1}{c}z)\cdot\begin{pmatrix}-y/r\\ x/r\\ 0\end{pmatrix},$$

$$\mathbf{B}(\mathbf{x},t) = -\tfrac{1}{c}w\cdot\begin{pmatrix}x/r\\ y/r\\ 0\end{pmatrix} - (\tfrac{1}{r}W + W_r)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix},\qquad \mathbf{H}(\mathbf{x},t) = \tfrac{1}{\mu_0}\mathbf{B}(\mathbf{x},t)$$

By a straightforward calculation one sees that $\mathbf{E},\mathbf{D},\mathbf{B},\mathbf{H}$ solve Maxwell's equations (4.1), (4.2), so it remains to show their regularity. For simplicity we only consider $\mathbf{E}$ and only discuss spatial derivatives. Abbreviating $p(\mathbf{x}) := (-y,x,0)$, denoting the Euclidean scalar product in $\mathbb{R}^3$ by $\langle\,\cdot\,,\,\cdot\,\rangle$ and the space derivative by $D_{\mathbf{x}}$, we have

$$\begin{aligned}\mathbf{E} &= \frac{w}{r}p\\ &= r\frac{w}{r}\frac{p}{r},\\ D_{\mathbf{x}}\mathbf{E}[h] &= \frac{w}{r}D_{\mathbf{x}}p[h] + \tfrac{1}{r}\partial_r\Big(\frac{w}{r}\Big)\langle\mathbf{x},h\rangle p\\ &= \frac{w}{r}D_{\mathbf{x}}p[h] + r\partial_r\Big(\frac{w}{r}\Big)\langle\frac{\mathbf{x}}{r},h\rangle\frac{p}{r}\\ D_{\mathbf{x}}^2\mathbf{E}[h_1,h_2] &= \tfrac{1}{r}\partial_r\Big(\frac{w}{r}\Big)[\langle\mathbf{x},h_1\rangle D_{\mathbf{x}}p[h_2] + \langle\mathbf{x},h_2\rangle D_{\mathbf{x}}p[h_1] + \langle h_1,h_2\rangle p]\\ &\quad + \Big(\tfrac{1}{r}\partial_r\Big)^2\Big(\frac{w}{r}\Big)\langle\mathbf{x},h_1\rangle\langle\mathbf{x},h_2\rangle p,\\ &= \partial_r\Big(\frac{w}{r}\Big)\Big[\langle\frac{\mathbf{x}}{r},h_1\rangle D_{\mathbf{x}}p[h_2] + \langle\frac{\mathbf{x}}{r},h_2\rangle D_{\mathbf{x}}p[h_1] + \langle h_1,h_2\rangle\frac{p}{r}\Big]\\ &\quad + \Big[r\partial_r^2\Big(\frac{w}{r}\Big) - \partial_r\Big(\frac{w}{r}\Big)\Big]\langle\frac{\mathbf{x}}{r},h_1\rangle\langle\frac{\mathbf{x}}{r},h_2\rangle\frac{p}{r},\end{aligned}$$

so Propositions 4.5.3 and 4.5.4 show that these terms lie in L^p for $p \in [2,\infty]$.

Part 2: Let us now assume (R). We need to show that higher order derivatives exist, are continuous and square-integrable. Away from $r = 0$, this is clear by Propositions 4.5.3

and 4.5.4, so it remains to show continuity of derivatives in $r = 0$. First, by induction one can show that for $0 \leq n \leq l+2$ the derivative $D^n_{\mathbf{x}}\mathbf{E}$ can be written as a sum

$$D^n_{\mathbf{x}}\mathbf{E} = \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n} (\tfrac{1}{r}\partial_r)^j \left(\frac{w}{r}\right) \cdot p_{n,j},$$

where $p_{n,j}(\mathbf{x})\colon (\mathbb{R}^3)^n \to \mathbb{R}^3$ is symmetric, n-multilinear, and its coefficients are homogeneous polynomials of degree $2j+1-n$ in $\mathbf{x}$. We use Taylor approximation and write $\frac{w}{r} = T_{n-1}(\frac{w}{r};0)+f$ with Taylor polynomial $T_{n-1}(\frac{w}{r};0)$ and remainder f.

Let us next consider summands with $j < n$. Recall that all odd Taylor coefficients are zero, so $q_{n,j} \coloneqq (\frac{1}{r}\partial_r)^j T_{n-1}(\frac{w}{r};0)$ is an even polynomial. In addition, we can estimate the remainder via $(\frac{1}{r}\partial_r)^j f = \mathrm{o}(r^{n-1-2j})$ as $r \to 0$. Thus

$$(\tfrac{1}{r}\partial_r)^j \left(\frac{w}{r}\right) \cdot p_{n,j} = (q_{n,j}(r) + \mathrm{o}(r^{n-1-2j}))p_{n,j} \to q_{n,j}(0)p_{n,j}(0)$$

as $r \to 0$. Now let $j = n$. Similar to the above arguments, one can show

$$\left[(\tfrac{1}{r}\partial_r)^n - \frac{1}{r^n}\partial_r^n\right]\left(\frac{w}{r}\right) = q_{n,n}(r) + \mathrm{o}(r^{-n-1})$$

as $r \to 0$ for some polynomial $q_{n,n}$. Thus

$$D^n_{\mathbf{x}}\mathbf{E} = \frac{1}{r^n}\partial_r^n(\tfrac{w}{r}) \cdot p_{n,n} + \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n} q_{n,j}(r)p_{n,j}(\mathbf{x}) + \mathrm{o}(1) \to \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n} q_{n,j}(0)p_{n,j}(0)$$

as $r \to 0$ by Propositions 4.5.3 and 4.5.4. Since the argument for existence of infinitely many solutions is the same as for slab geometries at the end of Section 4.4, this completes the proof. Observe that $p_{n,j}(0) = 0$ for $j \neq \frac{n+1}{2}$, so in particular all even derivatives of $\mathbf{E}$ vanish at 0. □

——— End of Preprint ———

CHAPTER 5

Existence of traveling breather solutions to cubic nonlinear Maxwell equations in waveguide geometries

This chapter consists of the preprint [71], which is written together with Wolfgang Reichel. We moved the results of [71, Appendix A] on fractional Sobolev spaces into Section B.2 since these spaces are also used in Chapter 6. We also adjusted the notation to align with the rest of this thesis.

——————— Start of Preprint ———————

Abstract. We consider the full set of Maxwell equations in a slab or cylindrical waveguide with a cubically nonlinear material law for the polarization of the electric field. The nonlinear polarization may be instantaneous or retarded, and we assume it to be confined inside the core of the waveguide. We prove existence of infinitely many spatially localized, real-valued and time-periodic solutions (breathers) propagating inside the waveguide by applying a variational minimization method to the resulting scalar quasilinear elliptic-hyperbolic equation for the profile of the breathers. The temporal period of the breathers has to be carefully chosen depending on the linear properties of the waveguide. As an example, our results apply if a two-layered linear axisymmetric waveguide is enhanced by a third core region with low refractive index where also the nonlinearity is located. In this case we can also connect our existence result with a bifurcation result. We illustrate our results with numerical simulations. Our solutions are polychromatic functions in general, but for some special models of retarded nonlinear material laws, also monochromatic solutions can exist. In this case the numerical simulations raise an interesting open question: are the breather solutions with minimal energy monochromatic or polychromatic?

5.1. Introduction and exemplary results

Our results show the existence of spatially localized, real-valued and time-periodic solutions (called breathers) to the full set of Maxwell's equations. We consider two types of waveguide geometries: the slab waveguide and the axially symmetric waveguide. Our breathers travel inside the waveguide and are periodic in the direction of travel. In the axially symmetric waveguide they decay to zero in all directions orthogonal to the waveguide, whereas in the slab waveguide they are independent of one direction orthogonal to the waveguide and decay to zero in the remaining direction. The nonlinear properties of the material are confined to the waveguide and are built according to a Kerr-law which may be instantaneous or retarded (temporally averaged). Before we summarize the main literature contributions we first introduce the physical problem. Towards the end of this introduction we comment on the physical consequences of our main theorems.

As our underlying physical model we consider Maxwell's equations

$$\begin{aligned} \nabla\cdot\mathbf{D}=0, \quad \nabla\times\mathbf{E}=-\mathbf{B}_t,\\ \nabla\cdot\mathbf{B}=0, \quad \nabla\times\mathbf{H}=\mathbf{D}_t, \end{aligned} \tag{5.1}$$

in the absence of charges and currents. Constitutive relations between the electric field $\mathbf{E}$ and electric displacement $\mathbf{D}$ as well as the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ are formulated by the following material laws

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}(\mathbf{E}), \qquad \mathbf{B}=\mu_0\mathbf{H}, \tag{5.2}$$

where $\epsilon_0>0$ is the vacuum permittivity, $\mu_0>0$ the vacuum permeability and $c_0={}^1/\sqrt{\epsilon_0\mu_0}$ the vacuum speed of light. The relation $\mathbf{B}=\mu_0\mathbf{H}$ reflects that the we assume no interaction of the magnetic field with the material. The interaction of the electric field with the material, however, is described by the polarization field $\mathbf{P}(\mathbf{E})$ which we assume to take the form

$$\mathbf{P}(\mathbf{E})=\epsilon_0\chi_1(\mathbf{x})\mathbf{E}+\epsilon_0\chi_3(\mathbf{x})\mathbf{N}(\mathbf{E}) \tag{5.3}$$

with $\mathbf{x}=(x,y,z)$ being the spatial variable, cf. [3, 6, 68]. Moreover, we assume that the cubic nonlinearity $\mathbf{N}(\mathbf{E})$ is isotropic, of Kerr-type, and retarded (temporally averaged) of the form

$$\mathbf{N}(\mathbf{E})(\mathbf{x},t)=\int_0^\infty \tilde{\kappa}(\tau)|\mathbf{E}(\mathbf{x},t-\tau)|^2\,\mathrm{d}\tau\,\mathbf{E}(\mathbf{x},t) \tag{5.4}$$

which includes the case of an instantaneous nonlinearity

$$\mathbf{N}(\mathbf{E})=|\mathbf{E}|^2\mathbf{E} \tag{5.5}$$

if we allow $\tilde{\kappa}=\delta_0$ to be the delta-distribution supported at time 0. A physical discussion of these material laws is given in [20, 37, 68] where also higher-order dependencies and anisotropy are discussed. Since we are looking for time-periodic fields $\mathbf{E}(\mathbf{x},t+T)=\mathbf{E}(\mathbf{x},t)$ with period $T>0$, the nonlinearity may be re-written as

$$\mathbf{N}(\mathbf{E})(\mathbf{x},t)=\frac{1}{T}\int_0^T \kappa(\tau)|\mathbf{E}(\mathbf{x},t-\tau)|^2\,\mathrm{d}\tau\,\mathbf{E}(\mathbf{x},t)=\Big(\kappa*|\mathbf{E}(\mathbf{x},\cdot)|^2\Big)(t)\mathbf{E}(\mathbf{x},t) \tag{5.6}$$

with the T-periodic function $\kappa(\tau)\coloneqq T\sum_{k\in\mathbb{Z}}\tilde{\kappa}(\tau+kT)$ and where we understand $\tilde{\kappa}\mid_{(-\infty,0)}\equiv 0$. Moreover we have used the convolution notation $(\kappa*v)(t)=\frac{1}{T}\int_0^T\kappa(\tau)v(t-\tau)\,\mathrm{d}\tau$ for the weighted temporal average of a measurable function v (which still includes the instantaneous case where $\kappa=\delta_0^{\mathrm{per}}$). From these equations we obtain the following second-order quasilinear equation for the electric field $\mathbf{E}$:

$$\nabla\times\nabla\times\mathbf{E}+\epsilon_0\mu_0\big((1+\chi_1(\mathbf{x}))\mathbf{E}+\chi_3(\mathbf{x})\mathbf{N}(\mathbf{E})\big)_{tt}=0. \tag{5.7}$$

We will show as part of our results how to recover the full set of Maxwell's equations from (5.7). Under suitable assumptions on the convolution kernel κ, cf. (5.15), we will show that (5.7) has a variational structure. Examples are given in Section 5.1.1.

We are interested in breather solutions of (5.7) which are moving with speed $c\in(0,\infty)$. Our results depend on the choice of the coefficients χ_1, χ_3, the retardation function κ, the propagation speed c, and the desired period $T>0$. We denote the frequency associated to T by $\omega\coloneqq\frac{2\pi}{T}$.

In the literature there are several treatments of the existence of breather solutions of (5.7). The first sequence of papers deals with monochromatic breathers, i.e., breathers of the type

$\mathbf{E}(\mathbf{x},t) = E(\mathbf{x})\cos(k_0\omega t + t_0)$. Such breathers are not compatible with the instantaneous nonlinearity but with the retarded nonlinearity, e.g., in the case $\kappa(t) = 1$ which may occur when $\tilde{\kappa}(t) = \sum_{n\in\mathbb{N}_0} \alpha_k \mathbb{1}_{[nT,(n+1)T)}$ with $\alpha_n \geq 0$, $\sum_{n=0}^{\infty} \alpha_n = T^{-1}$. Monochromatic breathers have the advantage that (5.7) reduces to the stationary elliptic problem

$$\nabla \times \nabla \times E - \epsilon_0\mu_0 k_0^2\omega^2((1+\chi_1(\mathbf{x}))E + \frac{\chi_3(\mathbf{x})}{2}|E|^2)E = 0. \tag{5.8}$$

Instead of a cubic nonlinearity $\frac{\chi_3(\mathbf{x})}{2}|E|^2E$, also saturated nonlinearities $g(\mathbf{x},|E|^2)E$ with a bounded function g naturally appear. The cases of saturated nonlinearities were first elaborated by Stuart et al. [62, 87–93] in the case of traveling breathers in an axisymmetric waveguide. Using divergence free, TE- or TM-polarized ansatz fields, (5.8) was reduced to a one-dimensional nonlinear elliptic problem which can, e.g., be solved variationally. In the follow-up result [64] the assumption of strict axisymmetry is dropped and more general two-dimensional waveguide profiles are considered, also allowing pure power nonlinearities. The case of standing monochromatic breathers also originates from Stuart's work and leads to the elliptic nonlinear curl-curl problem (5.8) in the vector-valued case. First works [5, 8, 10] considered axisymmetric divergence free ansatz functions, which allowed to reduce $\nabla\times\nabla\times$ to $-\Delta$. Using Helmholtz decomposition and suitable profile decompositions for Palais-Smale sequences, this restriction has been overcome by Mederski et al. [63, 65, 67], see also the survey [7] and references therein, with the isotropic cubic Kerr-nonlinearity still being left as an open problem. A different approach using limiting absorption principles [59] or dual variational approaches was carried out by Mandel [57], cf. also [58] where a spatially nonlocal variant of the stationary curl-curl problem was solved. Still within the area of monochromatic breathers, Dohnal et al. considered in [32] breathers at interfaces between (lossy) metals and dielectrics including retardation and in [33] they rigorously approximated breathers in photonic crystals when the frequency parameter is near a band edge.

In the second, much smaller sequence of papers, truly polychromatic breathers are considered for instantaneous nonlinearities. The first approach which we are aware of, is [75] where spatially localized traveling wave solutions of the 1+1-dimensional version of the quasi-linear Maxwell problem (5.7) were investigated. The authors treat the case where the linear coefficient χ_1 is a periodic arrangement of delta potentials. Using local bifurcation methods the authors solve a related system which is homotopically linked to the Maxwell problem written as an infinite coupled system arising from a multiple scale ansatz. It is analytically not clear whether the bifurcation branch ever reaches the original Maxwell system but numerical results support the existence of spatially localized traveling waves. A fully rigorous treatment for the existence of breathers on finite large time scales was given in [34] for a set-up of Kerr-nonlinear dielectrics occupying two different halfspaces. Two further rigorous treatments of exact polychromatic breather solutions occurred in [19] and [49] where either the linear or the nonlinear coefficients take the form of delta-distributions and the existence of travelling breathers was accomplished by using bifurcation theory and variational methods, respectively. We are not aware of any treatment of polychromatic breathers in the presence of retarded nonlinearities.

5.1.1. Examples of our results

We first describe our results on the level of examples. General results will be given in Section 5.2. Breather solutions are rare phenomena, and hence the fact that our examples contain rather specific assumptions on the material coefficients and do not leave much leeway for perturbations should not be surprising. The main difference to the previous results may be summarized as follows: while we allow both instantaneous and retarded material laws, our traveling breather solutions are generally polychromatic and hence not limited to monochromatic ansatz functions. Moreover, our solutions satisfy the full set of Maxwell's equations exactly, the material coefficients χ_1, χ_3 are bounded, and our solutions can be numerically approximated with little effort.

In the following, the speed of light is assumed to be $1 = 1/\sqrt{\epsilon_0\mu_0}$. Breather solutions will be time-periodic with period T and are propagating along the z-axis with speed $c \in (0,\infty)$. We consider two geometries for breathers: the cylindrical geometry where $\chi_1(\mathbf{x}) = \tilde{\chi}_1(r), \chi_3(\mathbf{x}) = \tilde{\chi}_3(r)$ only depend on $r = \sqrt{x^2+y^2}$, and the slab geometry where $\chi_1(\mathbf{x}) = \tilde{\chi}_1(x), \chi_3(\mathbf{x}) = \tilde{\chi}_3(x)$ only depend on x. In the cylindrical geometry we consider electric fields of the form

$$\mathbf{E}(\mathbf{x},t) = W(r, t - \tfrac{1}{c}z) \cdot (-\tfrac{y}{r}, \tfrac{x}{r}, 0)^\top$$

and in the slab geometry the electric field takes the form

$$\mathbf{E}(\mathbf{x},t) = (0, W(x, t - \tfrac{1}{c}z), 0)^\top$$

where in both settings W is a real-valued profile which is localized in the first variable (r-direction in the cylindrical case and x-direction in the slab case) and T-periodic in the second variable. In both geometries the electric field is a divergence-free TE-mode which means that $\mathbf{E}$ is orthogonal to the direction of propagation.

Definition 5.1.1. *The fields* $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H} \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^3)$ *weakly solve Maxwell's equations provided*

$$\int_{\mathbb{R}^4} \mathbf{D} \cdot \nabla\phi \,\mathrm{d}(\mathbf{x},t) = 0, \qquad \int_{\mathbb{R}^4} \mathbf{E} \cdot \nabla \times \Phi \,\mathrm{d}(\mathbf{x},t) = \int_{\mathbb{R}^4} \mathbf{B} \cdot \partial_t \Phi \,\mathrm{d}(\mathbf{x},t),$$
$$\int_{\mathbb{R}^4} \mathbf{B} \cdot \nabla\phi \,\mathrm{d}(\mathbf{x},t) = 0, \qquad \int_{\mathbb{R}^4} \mathbf{H} \cdot \nabla \times \Phi \,\mathrm{d}(\mathbf{x},t) = -\int_{\mathbb{R}^4} \mathbf{D} \cdot \partial_t \Phi \,\mathrm{d}(\mathbf{x},t)$$

holds for all $\phi \in C_c^\infty(\mathbb{R}^4;\mathbb{R})$ *and* $\Phi \in C_c^\infty(\mathbb{R}^4;\mathbb{R}^3)$.

The following theorem can be read as an explicit recipe for the construction of materials which support breathers. For the kernel $\tilde{\kappa}$ we generally assume (5.15). Explicit examples include, e.g., $\tilde{\kappa}(t) = \mathbb{1}_{[0,\infty)}(T^4 + 4t^4)^{-1} t$ or $\tilde{\kappa}(t) = \sum_{n\in\mathbb{N}_0} \alpha_n \mathbb{1}_{[nT,(n+1)T]}(t)$ where $\alpha_n \geq 0$ with $\sum_{n\in\mathbb{N}_0} \alpha_n = T^{-1}$, cf. Remark 5.A.2 for details. The material coefficients $\tilde{\chi}_1$, $\tilde{\chi}_3$ are assumed fixed and positive and take the form

$$\tilde{\chi}_1(x) = \begin{cases} d, & |x| < R, \\ \tilde{\chi}_1^*(|x| - R), & |x| > R, \end{cases} \qquad \tilde{\chi}_3(x) = \begin{cases} -\gamma, & |x| < R, \\ 0, & |x| > R \end{cases}$$

where either $\tilde{\chi}_1^* = \tilde{\chi}_1^{\text{per}} : \mathbb{R} \to \mathbb{R}$ is a P-periodic function defined on one periodicity cell by

$$\tilde{\chi}_1^{\text{per}}(x) = \begin{cases} a & |x| < \frac{1}{2}\theta P, \\ b, & \frac{1}{2}\theta P < |x| < \frac{1}{2}P \end{cases}$$

or $\tilde{\chi}_1^* = \tilde{\chi}_1^{\text{step}} : \mathbb{R} \to \mathbb{R}$ is a step function defined by

$$\tilde{\chi}_1^{\text{step}}(x) = \begin{cases} a, & |x| < \rho, \\ b, & |x| > \rho \end{cases}$$

with $a, b, d, P, R, \gamma, \rho > 0, \theta \in (0,1)$.

We are also using a sign-dependent distance function for a point $p \in \mathbb{R}$ and a set $M \subseteq \mathbb{R}$:

$$\operatorname{dist}^+(p, M) = \inf\{d^+(p,m) : m \in M\} \text{ with } d^+(p,m) = \begin{cases} |p - m| & \text{if } m \geq p, \\ \infty & \text{if } m < p. \end{cases}$$

Theorem 5.1.2. *Suppose that the nonlinearity N is given by either* (5.5) *or by* (5.6) *where κ satisfies* (5.15)*. Then there exists a (nonzero) T-periodic real-valued weak solution of the Maxwell problem* (5.1)*,* (5.2)*,* (5.3) *in the sense of Definition 5.1.1 both for the slab and the cylindrical case*[1]*, and for the following two choices of the polarization coefficient $\tilde{\chi}_1^*$:*

(i) *If $\tilde{\chi}_1^* = \tilde{\chi}_1^{\mathrm{per}}$ then we assume that the propagation speed $c \in (0,\infty)$ is chosen such that $0 < d < c^{-2} - 1 < \min\left\{a, b, \frac{a+d}{2}\right\}$ and*

$$\frac{\sqrt{a+1-c^{-2}} \cdot \theta}{\sqrt{b+1-c^{-2}} \cdot (1-\theta)} = \frac{m}{n} \in \frac{\mathbb{N}_{\mathrm{odd}}}{\mathbb{N}_{\mathrm{odd}}} \tag{5.9}$$

and define

$$T := \frac{4\sqrt{a+1-c^{-2}}\theta P}{m} = \frac{4\sqrt{b+1-c^{-2}}(1-\theta)P}{n}.$$

(ii) *If $\tilde{\chi}_1^* = \tilde{\chi}_1^{\mathrm{step}}$ then we assume that the propagation speed $c \in (0,\infty)$ is chosen such that $0 < \min\{b,d\} \le \max\{b,d\} < c^{-2} - 1 < a$. Moreover, there are $m, n \in \mathbb{N}$ coprime with*

$$0 < \xi < \arctan\sqrt{\tfrac{a+1-c^{-2}}{-d-1+c^{-2}}} \;\; \textit{where } \xi := \mathrm{dist}^+\left(\arctan\sqrt{\tfrac{a+1-c^{-2}}{-b-1+c^{-2}}}, \frac{m\pi}{2n} + \frac{\pi}{n}\mathbb{Z}\right) \tag{5.10}$$

and

$$T := 4\sqrt{a+1-c^{-2}}\rho\frac{n}{m}.$$

Additionally, the solution has at all times finite and uniformly bounded electromagnetic energy per unit square in y, z (slab case), or per unit segment in z (cylindrical case).

5.1.2. Discussion of the examples

Let us explain the reason behind the particular choices of the coefficients in a physical context. The parameters a, b are properties of a linear waveguide (without any nonlinear effect) whose profile is given either by the purely periodic profile $\tilde{\chi}_1^{\mathrm{per}}$ or the pure step profile $\tilde{\chi}_1^{\mathrm{step}}$. Then the conditions on a, b, c have the nature of a nonresonance condition, i.e., there are no guided waves $\mathbf{E}(\mathbf{x}, t) = \tilde{W}(r)e^{ik\omega(t-z/c)} \cdot (-\frac{y}{r}, \frac{x}{r}, 0)^\top$ with time period $T = \frac{2\pi}{\omega}$ propagating with speed c along the linear waveguide. Mathematically, this is expressed by a property of the operator $(1+\chi_1(\mathbf{x}))^{-1} \cdot \nabla \times \nabla \times$ appearing in (5.7): namely all multiples $k^2\omega^2$ with $k \in \mathbb{Z}_{\mathrm{odd}}$ are required to stay away from the spectrum of this weighted operator when restricted to suitable TE-modes propagating with speed c along the waveguide. This requirement is quite restrictive and its fulfillment can be guaranteed if $\omega = \frac{2\pi}{T}$ is chosen in the particular way and the parameters a, b, c satisfy either (5.9) or (5.10).

The remaining conditions on d may be described as follows: by inserting a new material of width $2R$ at the center of the waveguide the purely periodic or pure step waveguide is perturbed. On the linear level the new material has a low refractive index d and on the level of the nonlinear refractive index it contributes a defocusing effect. The quantitative strength of the nonlinear effect plays no role in the sense that $\gamma > 0$ may be arbitrary small. The value d always satisfies a two-sided condition: on one hand $0 < d < c^{-2} - 1$ and on the other hand

$$c^{-2} - 1 < \frac{a+d}{2} \quad \text{or} \quad 0 < \xi < \arctan\sqrt{\tfrac{a+1-c^{-2}}{-d-1+c^{-2}}}.$$

We note that these conditions are always satisfied if d is below but sufficiently close to $c^{-2} - 1$. On the linear level, the presence of the new (linear) material at the core of the waveguide

[1] In the cylindrical case, write r instead of x, and restrict $\tilde{\chi}_1, \tilde{\chi}_3$ to the half-line $[0,\infty)$.

still does not allow for guided waves of time period T and wave speed c. However, at a different value $d_* < d$ such a linear guided mode exists. Moreover, for all values $\tilde{d} \in (d_*, d)$ a solution of the nonlinear equation (5.7) exists, which bifurcates from 0 as $\tilde{d} \to d_*$. In other words, the solution of Theorem 5.1.2 is part of a bifurcation phenomenon with d as a bifurcation parameter. In a nutshell: the nonlinear equation allows for guided modes in the waveguide at parameter values for which there are no linear guided modes. We comment on this phenomenon in Section 5.7.

5.1.3. Outline of paper

In Section 5.2 we state the general form of our results (Theorem 5.2.1 and Theorem 5.2.4) of which Theorem 5.1.2 is a special case. For particular choices of the parameters compatible with Theorem 5.1.2, illustrations of approximate breathers can be found at the end of Section 5.2. Our main results are stated both for the cylindrical geometry and the slab geometry. For the proofs we discuss in detail only the cylindrical geometry, as the slab geometry can be treated similarly with less difficulties. Sections 5.3–5.5 contain the proof of our main results. In Section 5.3 we show how the problem (5.7) on $\mathbb{R}^3 \times \mathbb{R}$ can be reduced to a problem on the bounded domain $[0, R] \times [0, T]$. We then treat this reduced problem using a simple variational minimization method. In Section 5.4 we study a regularization of the bounded domain problem and in this way obtain an improved regularity result for the solutions of both the regularized and the original problem. Section 5.5 closes the proof of the main results. Adaptations for the slab geometry are discussed in Section 5.6. Moreover, in Section 5.7 we show the further regularity result that $\|\mathbf{E}\|_{L^\infty(\operatorname{supp}\chi_3; L^2([0,T]))}$ is finite and we explain what this has to do with the dielectric character of the waveguide. Finally, in the same section, we comment on the bifurcation phenomenon w.r.t. the parameter d.

The appendices contain important technical tools. In Appendix 5.A we show a basic convexity result for our variational approach, lower bounds on integrated versions of the nonlinearity, and two trace inequalities. Then, in Appendix 5.B we verify that the examples given in Theorem 5.1.2 satisfy the conditions of the general existence results. Lastly, Appendix 5.C details the numerical methods used to obtain approximations to the breather solutions that appear in the following section in the images in Figures 5.1 and 5.2 as illustrations of Theorem 5.1.2.

Let us finish this introduction by pointing out some observations and open questions, cf. Section 5.2 for details. In all our results we allow the breathers to be a polychromatic superposition of Fourier modes of arbitrary multiples of the basic frequency ω. In case of an instantaneous nonlinearity, necessarily infinitely many Fourier modes are non-zero. For time-averaged nonlinearities there is the possibility of monochromatic breathers and indeed (under suitable assumptions on κ) such monochromatic breathers exists. As our numerical simulations suggest, they appear to be more smooth than their polychromatic counterparts, and moreover, for the slab geometry, it seems that only monochromatic ground states exist. This is not the case for the cylindrical geometry. These findings based on numerical observations are analytically still open, but they do shed new light onto the a-priori choice of a monochromatic ansatz by Stuart et al. [62, 87–93] and later by others [5, 7, 8, 10, 63, 65, 67].

5.2. Main results and numerical illustration

After having given examples we now state our main results in more general form. We divide this into two subsections: one for the cylindrical geometry and one for the slab geometry. We define $\mathbb{T} := \mathbb{R}/_{T\mathbb{Z}}$ as the torus of length T which is our time domain equipped with the measure $\mathrm{d}t = \frac{1}{T}\mathrm{d}\lambda$ where $\mathrm{d}\lambda$ is the Lebesgue measure on $[0, T]$.

5.2.1. Cylindrical geometry

First we consider a cylindrical material, where $\chi_1(\mathbf{x}) = \tilde{\chi}_1(r)$ and $\chi_3(\mathbf{x}) = \tilde{\chi}_3(r)$ with $r := \sqrt{x^2 + y^2}$. For $\mathbf{E}$ we consider a wave which is radial in the (x, y)-directions, travels with speed $c > 0$ in z-direction, and which has the form

$$\mathbf{E}(\mathbf{x}, t) = w_t(r, t - \tfrac{1}{c}z) \cdot (-\tfrac{y}{r}, \tfrac{x}{r}, 0)^\top \tag{5.11}$$

with a real-valued profile function $w_t(r, t)$. Inserting the ansatz (5.11) into (5.7) and integrating once w.r.t. t yields

$$-w_{rr} - \tfrac{1}{r}w_r + \tfrac{1}{r^2}w + \big(\tilde{\chi}_1(r) + 1 - c^{-2}\big)w_{tt} + \tilde{\chi}_3(r)N(w_t)_t = 0, \qquad r \in [0, \infty), t \in \mathbb{R} \tag{5.12}$$

with

$$N(w_t) = N_{\mathrm{ins}}(w_t) = w_t^3 \tag{5.13}$$

or

$$N(w_t) = N_{\mathrm{av}}(w_t) = (\kappa * w_t^2)w_t \tag{5.14}$$

corresponding to (5.5) and (5.6), respectively. If the nonlinearity is given by (5.14), we require κ to satisfy the following assumptions:

$$\begin{cases} \kappa \in C^\alpha(\mathbb{T}) \text{ for some } \alpha > 0, \\ \kappa(t) = \kappa(-t) > 0 \text{ for } t \in \mathbb{T}, \\ L^4(\mathbb{T}) \to \mathbb{R}, v \mapsto \int_{\mathbb{T}} (\kappa * v^2)v^2 \,\mathrm{d}t \text{ is convex} \end{cases} \tag{5.15}$$

where the convexity assumption is satisfied if, e.g., $\max\kappa \le 2\min\kappa$ or if the Fourier transform of κ is non-negative, cf. Lemma 5.A.1 and Remark 5.A.2 for further concrete examples. In the following, N will always denote either N_{ins} or N_{av}. Under assumptions (5.15) on κ, we will show that (5.12) has a variational structure that is crucial in our study.

In the context of radial symmetry it is important to see the relation between a radially symmetric function $f_\sharp : \mathbb{R}^2 \setminus B_R(0) \to \mathbb{R}$, $R \ge 0$, and its radial profile function $f\colon [R, \infty) \to \mathbb{R}$ via the map $f_\sharp\colon \mathbb{R}^2 \setminus B_R(0) \to \mathbb{R}, (x, y) \mapsto f(\sqrt{x^2 + y^2})$. For $1 \le p < \infty$ this gives rise to the function spaces

$$L^p_{\mathrm{rad}}([R, \infty)) := \Big\{ f \in L^1_{\mathrm{loc}}((R, \infty))\colon f_\sharp \in L^p(\mathbb{R}^2 \setminus B_R(0)) \Big\}$$

with norm

$$\|f\|_{L^p_{\mathrm{rad}}([R,\infty))} := \frac{1}{\sqrt[p]{2\pi}} \|f_\sharp\|_{L^p(\mathbb{R}^2 \setminus B_R(0))} = \|f\|_{L^p([R,\infty), r\mathrm{d}r)}$$

For functions depending on radius and time we define

$$L^p_{\text{rad}}([R,\infty)\times\mathbb{T}) := \Big\{ f \in L^1_{\text{loc}}((R,\infty)\times\mathbb{T}) \colon f_\sharp \in L^p(\mathbb{R}^2 \setminus B_R(0)\times\mathbb{R}) \Big\}.$$

Other spaces of radially symmetric functions based on $L^2_{\text{rad}}([R,\infty)\times\mathbb{T})$, such as $H^k_{\text{rad}}([R,\infty)\times\mathbb{T})$, are defined analogously.

For time-periodic functions $w\colon [0,\infty)\times\mathbb{T}\to\mathbb{C}$ we consider the temporal Fourier transform $\mathcal{F}$ and denote for $k\in\mathbb{Z}$ the k-th Fourier coefficient of w by $\hat{w}_k = \mathcal{F}_k[w] = ⨍_{\mathbb{T}} w\overline{e_k}\,\mathrm{d}t$ where $e_k(t) := \mathrm{e}^{\mathrm{i}k\omega t}$. For the linear part of the differential equation (5.12)

$$Lw = -w_{rr} - \tfrac{1}{r}w_r + \tfrac{1}{r^2}w + \big(\tilde{\chi}_1 + 1 - c^{-2}\big)w_{tt}$$

we can apply the Fourier transform and obtain $\mathcal{F}_k[Lw] = L_k\hat{w}_k$ with

$$L_k := -\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} - k^2\omega^2\big(\tilde{\chi}_1 + 1 - c^{-2}\big).$$

We make the following assumptions on the nonlinearity N, the potentials $\tilde{\chi}_1,\tilde{\chi}_3$ and the operators L_k. Denote by $\mathbb{N}_{\text{odd}} := 2\mathbb{N}-1 = \{1,3,5,\ldots\}$.

($\mathcal{A}5.1$) $\tilde{\chi}_1,\tilde{\chi}_3 \in L^\infty([0,\infty),\mathbb{R})$ and $\operatorname{supp}(\tilde{\chi}_3) = [0,R]$ where $R>0$.

($\mathcal{A}5.2$) N is given either by (5.13), or by (5.14) where κ satisfies (5.15).

($\mathcal{A}5.3$) $\operatorname{ess\,sup}_{[0,R]}\tilde{\chi}_1 \le c^{-2}-1$, $\operatorname{ess\,sup}_{[0,R]}\tilde{\chi}_3 < 0$.

($\mathcal{A}5.4$) There exists a solution $\phi_k \in H^2_{\text{rad}}([R,\infty))\setminus\{0\}$ of $L_k\phi_k = 0$ for each $k\in\mathbb{N}_{\text{odd}}$.

($\mathcal{A}5.5$) The following inequalities hold for ϕ_k, $k\in\mathbb{N}_{\text{odd}}$:

$$\liminf_{k\to\infty} \frac{|\phi_k(R)|}{\|\phi_k\|_{L^2_{\text{rad}}([R,\infty))}} > 0, \qquad \sup_k \frac{|\phi_k'(R)|}{k\|\phi_k\|_{L^2_{\text{rad}}([R,\infty))}} < \infty.$$

($\mathcal{A}5.6$) With I_α denoting the modified Bessel function of first kind, there exists $k_0\in\mathbb{N}_{\text{odd}}$ such that $\phi_{k_0}(R)\neq 0$ and the following inequality holds:

$$\frac{\phi_{k_0}'(R)}{\phi_{k_0}(R)} > \frac{\lambda k_0 I_1'(\lambda k_0 R)}{I_1(\lambda k_0 R)} \quad \text{where} \quad \lambda := \omega\big(c^{-2} - 1 - \operatorname*{ess\,inf}_{[0,R]}\tilde{\chi}_1\big)^{1/2}.$$

We call ϕ_k a fundamental solution for L_k. Since $L_k = L_{-k}$ we define $\phi_{-k} := \phi_k$ for all $k\in\mathbb{N}_{\text{odd}}$. The reason for considering $k\in\mathbb{N}_{\text{odd}}$ instead of $k\in\mathbb{N}_0$ is that $\ker(L_0) = \operatorname{span}\big\{r,\tfrac{1}{r}\big\}$ does not contain nonzero $L^2_{\text{rad}}([R,\infty))$-functions. The restriction to $\mathbb{N}_{\text{odd}}$ amounts to considering $T/2$-antiperiodic functions which is compatible with the cubic nonlinearity in (5.12).

Assumption ($\mathcal{A}5.6$) is in place to ensure existence of nontrivial solutions to (5.12). Since $\frac{I_1'(z)}{I_1(z)} \to 1$ as $z\to\infty$ (see [39]), a sufficient condition for ($\mathcal{A}5.6$) to hold is

($\mathcal{A}5.6$') $$\limsup_{k\to\infty} \frac{\phi_k'(R)}{k\phi_k(R)} > \omega\big(c^{-2} - 1 - \operatorname*{ess\,inf}_{[0,R]}\tilde{\chi}_1\big)^{1/2},$$

which additionally ensures that ($\mathcal{A}5.6$) holds for infinitely many k_0.

Next we state our main theorem for the cylindrical geometry.

Theorem 5.2.1. *Assume ($\mathcal{A}5.1$)–($\mathcal{A}5.6$) hold for given $N,\kappa,\tilde{\chi}_1,\tilde{\chi}_3$ and T. Then there exists a (nonzero) T-periodic real-valued weak solution of the Maxwell problem* (5.1), (5.2), (5.3) *in the sense of Definition 5.1.1. Furthermore, localization orthogonal to the direction of propagation is expressed by the fact that at all times $t_0 \in \mathbb{R}$ the electromagnetic energy per unit segment along the z-direction*

$$\int_{\mathbb{R}\times\mathbb{R}\times[z_0,z_0+1]} (\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H})\,\mathrm{d}(x,y,z)$$

is finite for all $z_0 \in \mathbb{R}$ and uniformly bounded.

Remark 5.2.2. Let us explain why our assumptions ($\mathcal{A}$5.3), ($\mathcal{A}$5.6) enforce $\tilde{\chi}_1$ to take values both below and above $c^{-2}-1$. Suppose for contradiction that $\tilde{\chi}_1 \le c^{-2}-1$ everywhere on $[0,\infty)$. If w is a weak T-periodic solution to (5.12), we see that $w=0$ must hold by multiplying (5.12) with w and integrating on $[0,\infty)\times\mathbb{T}$ with respect to the measure $r\,\mathrm{d}r\mathrm{d}t$. Hence, non-trivial solutions do not exist. In fact, the assumption ($\mathcal{A}$5.6) conflicts with $\tilde{\chi}_1(r) \le c^{-2}-1$ everywhere on $[0,\infty)$. Namely, in this case ϕ_k satisfies $(r\phi_k')' = (\frac{1}{r}+rk^2\omega^2(c^{-2}-1-\tilde{\chi}_1(r))\phi_k$. Multiplication with ϕ_k and integration from R to ∞ yields $R\phi_k'(R)\phi_k(R) = -\int_R^\infty r|\phi_k'|^2 + (\frac{1}{r}+rk^2\omega^2(c^{-2}-1-\tilde{\chi}_1(r))|\phi_k|^2\,dr \le 0$. Thus, $\phi_k'(R)$ and $\phi_k(R)$ have opposite sign, contradicting ($\mathcal{A}$5.6) and the fact that I_1, I_1' are positive on $(0,\infty)$.

We end this subsection with a multiplicity result. For this, we first explain what kind of multiplicity we consider. Given a solution w of (5.12), any time-shift $(x,t)\mapsto w(x,t+\tau)$ for $\tau\in\mathbb{T}$ also solves (5.12). Moreover, if $N=N_{\text{av}}$ with $\kappa\equiv 1$ one can shift the individual frequencies separately, i.e. $(x,t)\mapsto \sum_{k\in\mathbb{Z}}\hat{w}_k(x)e_k(t+\tau_k)$ solves (5.12) for all $\tau_k\in\mathbb{T}$ with $\tau_k=\tau_{-k}$. By *distinct* solutions we mean solutions that are not shifts of one another.

Theorem 5.2.3. *Assume ($\mathcal{A}$5.1)–($\mathcal{A}$5.5) hold for given $N,\kappa,\tilde{\chi}_1,\tilde{\chi}_3,T$. If ($\mathcal{A}$5.6) holds for infinitely many $k_0\in\mathbb{N}_{\text{odd}}$ (e.g., if ($\mathcal{A}$5.6') is true) then there exist infinitely many distinct T-periodic real-valued weak solutions of the Maxwell problem* (5.1), (5.2), (5.3) *in the sense of Definition 5.1.1 with finite and uniformly bounded electromagnetic energy per unit segment along the z-direction.*

5.2.2. Slab geometry

In our second setting, we consider slab materials that extend infinitely in the (y,z)-directions. Here $\chi_1(\mathbf{x})=\tilde{\chi}_1(x)$, $\chi_3(\mathbf{x})=\tilde{\chi}_3(x)$ and we look for traveling polarized waves moving at speed $c>0$ in y-direction and being constant along the z-direction. More precisely, we consider fields $\mathbf{E}$ given by the ansatz

$$\mathbf{E}(\mathbf{x},t) = \left(0,0,w_t(x,t-\tfrac{1}{c}y)\right)^\top. \tag{5.16}$$

Inserting into (5.7) and integrating once w.r.t. t leads to the equation

$$-w_{xx} + \left(\tilde{\chi}_1(x)+1-c^{-2}\right)w_{tt} + \tilde{\chi}_3(x)N(w_t)_t = 0 \tag{5.17}$$

for the profile function $w_t(x,t)$. Similar to the radial setting we define the operators

$$\widetilde{L} := -\partial_x^2 + \left(\tilde{\chi}_1(x)+1-c^{-2}\right)\partial_t^2, \qquad \widetilde{L}_k := -\partial_x^2 - k^2\omega^2\left(\tilde{\chi}_1(x)+1-c^{-2}\right),$$

so that $\mathcal{F}_k\widetilde{L} = \widetilde{L}_k\mathcal{F}_k$ holds for the temporal Fourier transform $\mathcal{F}$. We require the following assumptions on $\tilde{\chi}_1, \tilde{\chi}_3$ and $\widetilde{L}_k$:

($\tilde{\mathcal{A}}$5.1) $\tilde{\chi}_1, \tilde{\chi}_3 \in L^\infty(\mathbb{R},\mathbb{R})$ are even with $\mathrm{supp}(\tilde{\chi}_3) = [-R,R]$ where $R>0$.

($\tilde{\mathcal{A}}$5.2) N is given either by (5.13), or by (5.14) where κ satisfies (5.15).

($\tilde{\mathcal{A}}$5.3) $\operatorname{ess\,sup}_{[-R,R]} \tilde{\chi}_1 \le c^{-2} - 1$, $\operatorname{ess\,sup}_{[-R,R]} \chi_3 < 0$.

($\tilde{\mathcal{A}}$5.4) There exists a solution $\widetilde{\phi}_k \in H^2([R,\infty)) \setminus \{0\}$ of $\widetilde{L}_k \widetilde{\phi}_k = 0$ for each $k \in \mathbb{N}_{\mathrm{odd}}$.

($\tilde{\mathcal{A}}$5.5) The following inequalities hold for $\widetilde{\phi}_k$, $k \in \mathbb{N}_{\mathrm{odd}}$:

$$\liminf_{k\to\infty} \frac{|\widetilde{\phi}_k(R)|}{\|\widetilde{\phi}_k\|_{L^2([R,\infty))}} > 0, \qquad \sup_k \frac{|\widetilde{\phi}_k'(R)|}{k\|\widetilde{\phi}_k\|_{L^2([R,\infty))}} < \infty.$$

($\tilde{\mathcal{A}}$5.6) There exists $k_0 \in \mathbb{N}_{\mathrm{odd}}$ such that $\widetilde{\phi}_{k_0}(R) \neq 0$ and the following inequality holds:

$$\frac{\widetilde{\phi}_{k_0}'(R)}{\widetilde{\phi}_{k_0}(R)} > \lambda k_0 \tanh(\lambda k_0 R) \quad \text{with} \quad \lambda := \omega\big(c^{-2} - 1 - \operatorname*{ess\,inf}_{[-R,R]} \tilde{\chi}_1\big)^{1/2}.$$

Again, a sufficient condition for ($\tilde{\mathcal{A}}$5.6) to hold is

($\tilde{\mathcal{A}}$5.6') $\displaystyle \limsup_{k\to\infty} \frac{\widetilde{\phi}_k'(R)}{k\widetilde{\phi}_k(R)} > \omega\big(c^{-2} - 1 - \operatorname*{ess\,inf}_{[-R,R]} \tilde{\chi}_1\big)^{1/2}.$

We can now formulate our main theorems for the slab geometry.

Theorem 5.2.4. *Assume ($\tilde{\mathcal{A}}$5.1)–($\tilde{\mathcal{A}}$5.6) hold for given $N, \tilde{\chi}_1, \tilde{\chi}_3$ and T. Then there exists a (nonzero) T-periodic real-valued weak solution of the Maxwell problem* (5.1), (5.2), (5.3) *in the sense of Definition 5.1.1. Furthermore, localization in the x-direction is expressed by the fact that at all times $t_0 \in \mathbb{R}$ the electromagnetic energy per unit square in the y, z-direction*

$$\int_{\mathbb{R}\times[y_0,y_0+1]\times[z_0,z_0+1]} (\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H})\,\mathrm{d}(x,y,z)$$

is finite for all $y_0, z_0 \in \mathbb{R}$ and uniformly bounded w.r.t. t_0, z_0.

Theorem 5.2.5. *Assume ($\tilde{\mathcal{A}}$5.1)–($\tilde{\mathcal{A}}$5.5) hold for given $N, \tilde{\chi}_1, \tilde{\chi}_3, T$. If ($\mathcal{A}$5.6) holds for infinitely many $k_0 \in \mathbb{N}_{\mathrm{odd}}$ (e.g., if ($\mathcal{A}$5.6') is true) then there exist infinitely many distinct T-periodic real-valued weak solutions of the Maxwell problem* (5.1), (5.2), (5.3) *in the sense of Definition 5.1.1 with finite and uniformly bounded electromagnetic energy per unit square along the y, z-direction.*

5.2.3. Numerical illustrations, discussion, and some open questions

In the following we apply the numerical scheme outlined in Appendix 5.C and show results for the profile w_t of the electric field, cf. (5.11) or (5.16). The breathers we obtain analytically are ground states in the sense that they are minimizers of the energy functional E discussed in Section 5.3. Here we show approximations to these ground states. We consider particular potentials $\tilde{\chi}_1$ and $\tilde{\chi}_1$ which are compatible with the parameter choices of Theorem 5.1.2. For the periodic case $\tilde{\chi}_1^* = \tilde{\chi}_1^{\mathrm{per}}$ we show in Figure 5.1 four images which cover both choices of the nonlinearity (time-averaged and instantaneous) and both choices of the geometry (cylindrical and slab). For the step case $\tilde{\chi}_1^* = \tilde{\chi}_1^{\mathrm{step}}$ also four images covering both types of nonlinearities and both types of geometries are shown in Figure 5.2.

The following observations can be made leading to open questions or conjectures:

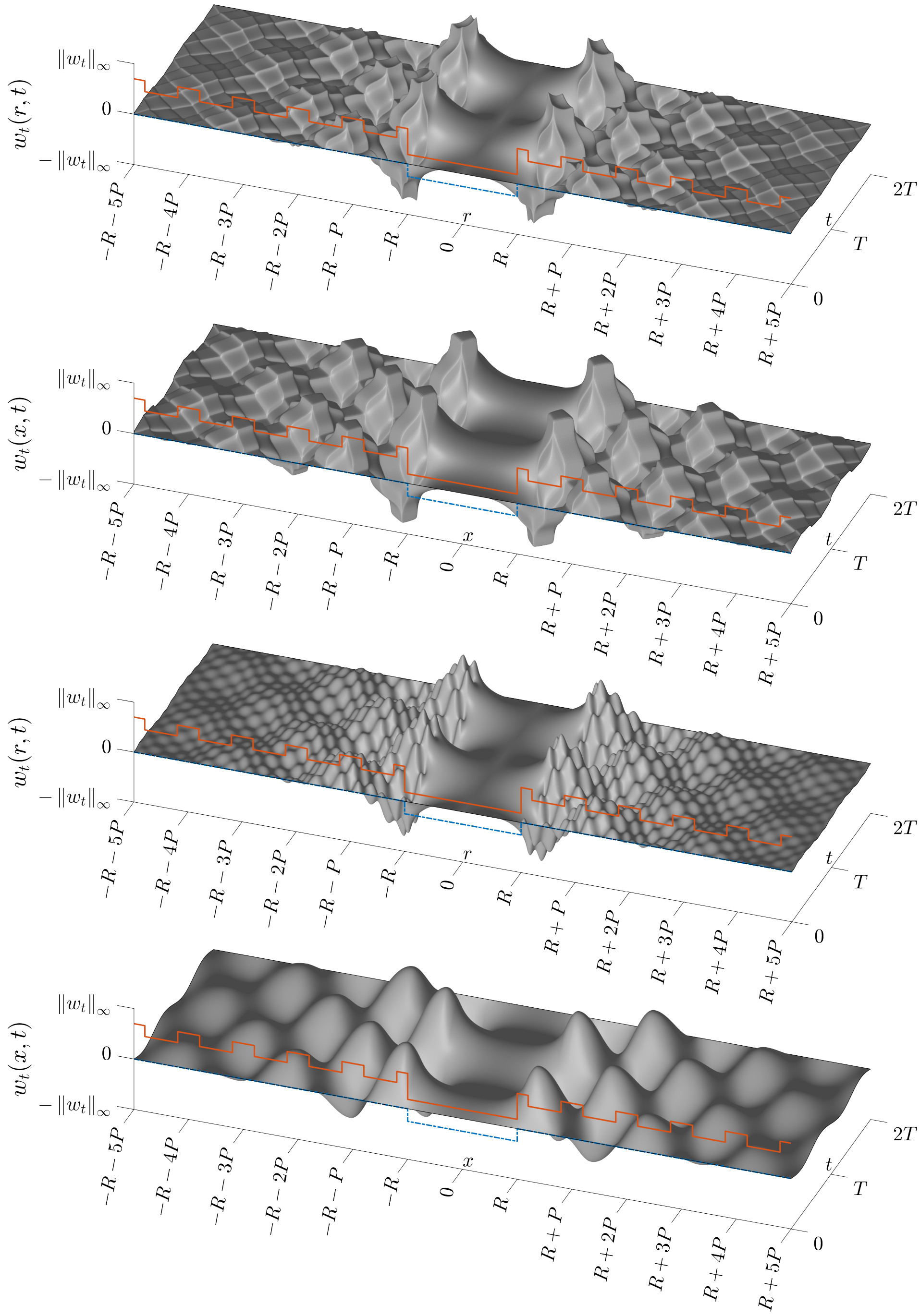

Figure 5.1.: Periodic potential outside $[-R, R]$: intensity (approximated) of electric field of breather solutions to Theorem 5.1.2 in reduced coordinates (cf. (5.11) and (5.16)) over 2 time periods, with potentials $\tilde{\chi}_1$ (orange) and $\tilde{\chi}_3$ (blue). Parameters are $T = 4, \omega = \frac{\pi}{2}, c = \frac{2}{3}, a = \frac{45}{16}, b = \frac{35}{18}, d = \frac{3}{4}, R = P = 2, \theta = \frac{2}{5}, \gamma = m = n = 1$, $\kappa \equiv 1$. Top to bottom: N_{ins} and cylindrical geometry; N_{ins} and slab geometry; N_{av} and cylindrical geometry with $R = \frac{43}{20}$ instead; N_{av} and slab geometry.

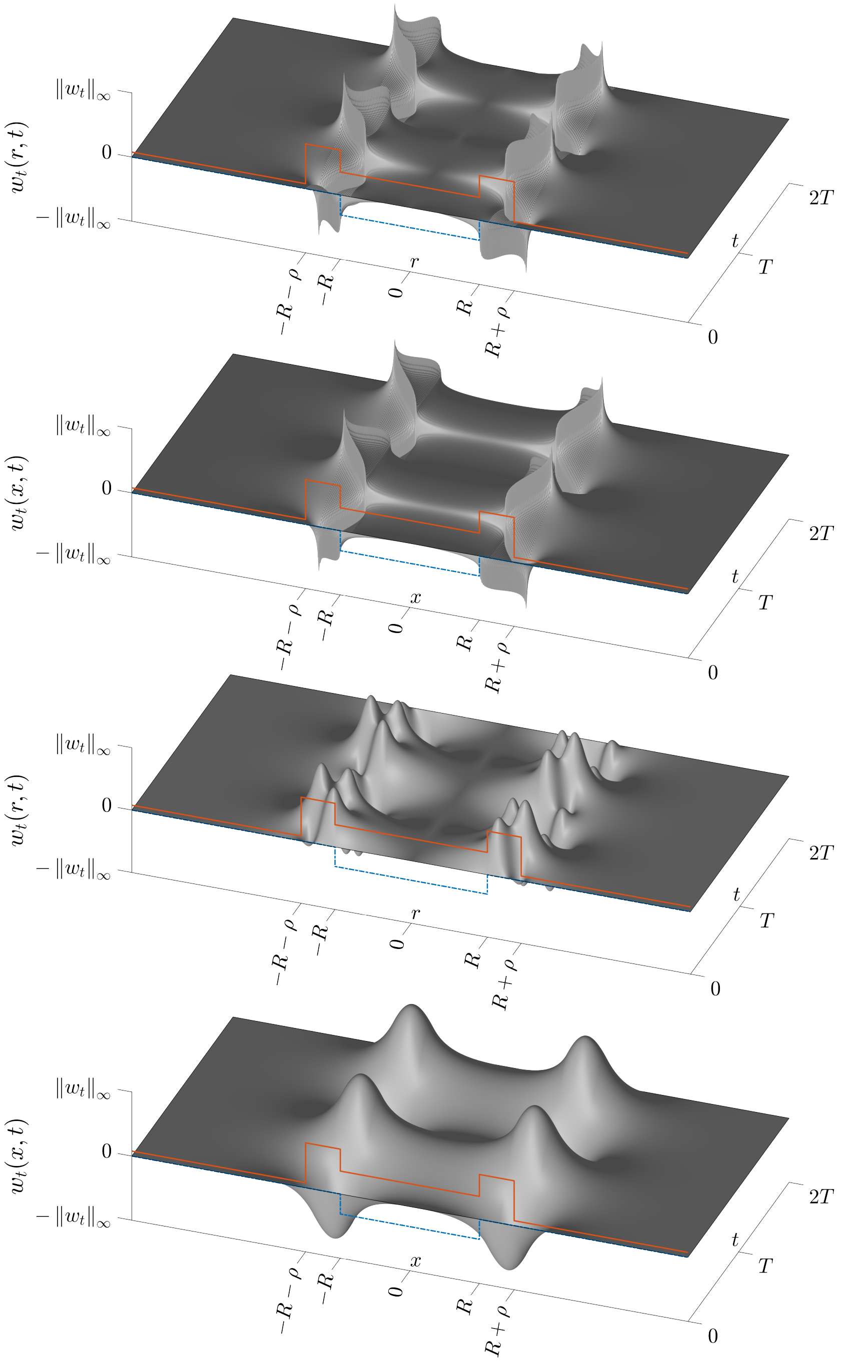

Figure 5.2.: Step potential outside $[-R, R]$: intensity (approximated) of electric field of breather solutions to Theorem 5.1.2 in reduced coordinates (cf. (5.11) and (5.16)) over 2 time periods, with potentials $\tilde{\chi}_1$ (orange) and $\tilde{\chi}_3$ (blue). Parameters are $T = 4, \omega = \frac{\pi}{2}, c = \frac{2}{3}, a = \frac{9}{4}, b = \frac{1}{4}, d = \frac{23}{20}, R = 2, \rho = \gamma = m = n = 1, \kappa \equiv 1$. Top to bottom: N_{ins} and cylindrical geometry; N_{ins} and slab geometry; N_{av} and cylindrical geometry with $R = \frac{9}{4}$ instead; N_{av} and slab geometry.

- Although it is in general impossible to tell whether a computed solution is a global or just a local minimizer, the numerical minimization scheme in the instantaneous case always ends up in the same state (up to time shifts) independently of the initial state. One may therefore conjecture that ground states are unique up to shifts in time. Moreover, they seem to be even in time.
- Ground states for time-averaged nonlinearities seem to be more smooth than for instantaneous nonlinearities. Can one show improved regularity of ground states for time-averaged nonlinearities?
- For time-averaged nonlinearities one can consider monochromatic solutions with frequencies $k\omega$ provided $\hat{\kappa}_{2k} = 0$ (see discussion below). In the cylindrical setting we found both monochromatic and polychromatic breathers (depending on the chosen parameters), whereas in the slab setting we only found monochromatic breathers. Can one prove that in the slab setting ground states are monochromatic? Under which parameter conditions in the cylindrical setting are ground states monochromatic/polychromatic?

A monochromatic breather has a profile w of the form

$$w(r,t) = \mathrm{Re}[v(r)e_k(t)] = \tfrac{1}{2}v(r)e_k(t) + \tfrac{1}{2}\overline{v(r)}e_{-k}(t)$$

for some function v. It is compatible with the nonlinearity in the time-averaged case if $\hat{\kappa}_{2k} = 0$, since then the nonlinearity

$$N_{\mathrm{av}}(w) = \tfrac{1}{4}\,\mathrm{Re}[\hat{\kappa}_{2k}(v^3 e_{3k} + |v|^2 v e_k) + 2\hat{\kappa}_0|v|^2 v e_k] = \tfrac{\hat{\kappa}_0}{2}\,\mathrm{Re}[|v|^2 v e_k]$$

is also monochromatic along monochromatic functions. The bottom images in Figures 5.1 and 5.2 always depict monochromatic breathers (for the slab geometry, time-averaged nonlinearity with $\kappa \equiv 1$, and frequency index $k = 1$). All other images show polychromatic breathers. Furthermore, for the time-averaged nonlinearity one can state that if there exists a nontrivial breather w then there also exists a monochromatic breather with frequency index $k \in \mathbb{N}_{\mathrm{odd}}$ provided $\hat{\kappa}_{2k} = 0$ and $\int_0^\infty L_k\hat{w}_k \cdot \hat{w}_k \, r\mathrm{d}r < 0$.

The instantaneous nonlinearity $N = N_{\mathrm{ins}}$ however is not compatible with monochromatic breathers, hence all breathers for $N = N_{\mathrm{ins}}$ are necessarily polychromatic, and they have infinitely many excited frequency indices k.

5.3. Reduction to a bounded domain problem

From now on we assume that assumptions ($\mathcal{A}$5.1)–($\mathcal{A}$5.6) are satisfied, and we set

$$V(r) \coloneqq -(\tilde{\chi}_1(r) + 1 - c^{-2}) \quad \text{and} \quad \Gamma(r) \coloneqq -\tilde{\chi}_3(r), \tag{5.18}$$

allowing us to write (5.12) as

$$-w_{rr} - \tfrac{1}{r}w_r + \tfrac{1}{r^2}w - V(r)w_{tt} - \Gamma(r)N(w_t)_t = 0, \quad r \in [0,\infty), t \in \mathbb{T} \tag{5.19}$$

where $V \geq 0, \Gamma \geq 0$ on $[0, R]$ due to ($\mathcal{A}$5.3). We will show that (5.19) can be reduced to a variational problem (5.22) below, where the conditions (5.15) on κ are essential.

We consider functions w which are $T/2$–antiperiodic in time. This is compatible with the structure of (5.19), in particular with the cubic nonlinearity, and we use the suffix “anti” to denote spaces consisting of functions which are $T/2$–antiperiodic in time.

Using the fundamental solutions ϕ_k given by ($\mathcal{A}$5.4) we can further make the ansatz

$$w(r,t) = \begin{cases} u(r,t), & 0 \le r < R, \\ \sum_{k\in\mathbb{Z}_{\text{odd}}} \alpha_k \phi_k(r) e_k(t), & r > R. \end{cases}$$

where $\alpha_k \in \mathbb{C}$ and $u \in H^1_{\text{rad,anti}}([0,R]\times\mathbb{T})$ are to be determined. Note that $\alpha_{-k} = \overline{\alpha_k}$ since w, ϕ_k being real-valued together with $\phi_{-k} = \phi_k$ imply $\alpha_{-k}\phi_{-k} = \overline{\alpha_k \phi_k}$.

We want to ensure that w and w_r taken from inside and outside match at $r = R$. This leads to the following conditions:

$$u(R,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \alpha_k \phi_k(R) e_k(t), \qquad u_r(R,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \alpha_k \phi_k'(R) e_k(t). \tag{5.20}$$

By assumption ($\mathcal{A}$5.5) we have $\phi_k(R) \neq 0$ for almost all $k \in \mathbb{Z}_{\text{odd}}$. Let

$$\mathfrak{F} \coloneqq \{k \in \mathbb{Z}_{\text{odd}} \colon \phi_k(R) = 0\} \subseteq \mathbb{Z}_{\text{odd}}$$

denote the finite exclusion set. The exceptional indices $k \in \mathfrak{F}$ have to be treated differently than the regular indices $k \in \mathfrak{R} \coloneqq \mathbb{Z}_{\text{odd}} \setminus \mathfrak{F}$. Note also that due to assumption ($\mathcal{A}$5.5) there exist constants $c^\star, C^\star > 0$ such that

$$|\phi_k(R)| \ge c^\star \|\phi_k\|_{L^2_{\text{rad}}([R,\infty))}, \quad |\phi_k'(R)| \le C^\star |k| \|\phi_k\|_{L^2_{\text{rad}}([R,\infty))}, \quad \frac{|\phi_k'(R)|}{|\phi_k(R)|} \le \frac{C^\star}{c^\star}|k| \tag{5.21}$$

hold for all $k \in \mathfrak{R}$.

Let us show the difference between $\mathfrak{F}$ and $\mathfrak{R}$: for $k \in \mathfrak{F}$ equations (5.20) reduce to

$$\hat{u}_k(R) = 0 \quad \text{and} \quad \alpha_k = \frac{\hat{u}_k'(R)}{\phi_k'(R)},$$

whereas for $k \in \mathfrak{R}$ we have

$$\alpha_k = \frac{\hat{u}_k(R)}{\phi_k(R)} \quad \text{and} \quad \hat{u}_k'(R) = \frac{\phi_k'(R)}{\phi_k(R)} \hat{u}_k(R).$$

Thus we formally obtain the following boundary value problem for u:

$$\begin{cases} -u_{rr} - \frac{1}{r}u_r + \frac{1}{r^2}u - V(r)u_{tt} - \Gamma(r)N(u_t)_t = 0 \text{ in } [0,R]\times\mathbb{T}, \\ \hat{u}_k'(R) = \frac{\phi_k'(R)}{\phi_k(R)}\hat{u}_k(R) \text{ for } k \in \mathfrak{R}, \\ \hat{u}_k(R) = 0 \text{ for } k \in \mathfrak{F}. \end{cases} \tag{5.22}$$

The formal calculation will be justified in the proof of Theorem 5.2.1 when we establish the weak-solution property. Problem (5.22) again is variational and solutions are critical points of the functional E given by

$$\begin{aligned} &E(u) = E_I(u) - E_B(u) \quad \text{where} \\ &E_I(u) \coloneqq \int_{[0,R]\times\mathbb{T}} \left(\tfrac{1}{2}u_r^2 + \tfrac{1}{2}\left(\tfrac{1}{r}u\right)^2 + \tfrac{1}{2}V(r)u_t^2 + \tfrac{1}{4}\Gamma(r)N(u_t)u_t \right) r \,\mathrm{d}(r,t) \\ &E_B(u) \coloneqq \frac{R}{2} \sum_{k\in\mathfrak{R}} \frac{\phi_k'(R)}{\phi_k(R)} |\hat{u}_k(R)|^2 \end{aligned} \tag{5.23}$$

subject to the constraints $\hat{u}_k(R) = 0$ for $k \in \mathfrak{F}$. Indeed, for a (sufficiently regular) solution u and a sufficiently smooth function $\varphi\colon [0,R] \times \mathbb{T} \to \mathbb{R}$ we have

$$\begin{aligned}
0 &= \int_{[0,R]\times\mathbb{T}} \Big(-u_{rr} - \tfrac{1}{r} u_r + \tfrac{1}{r^2} u - V(r) u_{tt} - \Gamma(r) N(u_t)_t\Big) \varphi \, r \mathrm{d}(r,t) \\
&= \int_{[0,R]\times\mathbb{T}} \Big(u_r \varphi_r + \tfrac{1}{r^2} u\varphi + V(r) u_t \varphi_t + \Gamma(r) N(u_t)\varphi_t\Big) \, r\mathrm{d}(r,t) - R \int_{\mathbb{T}} u_r(R,t)\varphi(R,t)\,\mathrm{d}t \\
&= \int_{[0,R]\times\mathbb{T}} \Big(u_r \varphi_r + \tfrac{1}{r^2} u\varphi + V(r) u_t \varphi_t + \Gamma(r) N(u_t)\varphi_t\Big) \, r\mathrm{d}(r,t) - R \sum_{k\in\mathfrak{R}} \frac{\phi_k'(R)}{\phi_k(R)} \hat{u}_k(R) \overline{\hat{\varphi}_k(R)} \\
&= E'(u)[\varphi].
\end{aligned}$$

Here we used $\hat{\varphi}_k(R) = 0$ for $k \in \mathfrak{F}$ and that by Plancherel

$$R \int_{\mathbb{T}} u_r(R,t)\varphi(R,t)\,\mathrm{d}t = R \int_{\mathbb{T}} u_r(R,t)\overline{\varphi}(R,t)\,\mathrm{d}t = R \sum_{k\in\mathfrak{R}} \frac{\phi_k'(R)}{\phi_k(R)} \hat{u}_k(R)\overline{\hat{\varphi}_k(R)}$$

so that this quantity is real and thus coincides with $E_B'(u)[\varphi] = \mathrm{Re}\Big[R \sum_{k\in\mathfrak{R}} \frac{\phi_k'(R)}{\phi_k(R)} \hat{u}_k(R)\overline{\hat{\varphi}_k(R)}\Big]$. We further used that $E_N(u) \coloneqq \int_{[0,R]\times\mathbb{T}} \Big(\tfrac{1}{4}\Gamma(r) N(u_t) u_t\Big)\, r\mathrm{d}(r,t)$ satisfies

$$E_N'(u)[\varphi] = \int_{[0,R]\times\mathbb{T}} (\Gamma(r) N(u_t)\varphi_t)\, r\mathrm{d}(r,t).$$

Indeed, for $N = N_{\mathrm{ins}}$ we have

$$E_N(u) = \tfrac{1}{4} \int_{[0,R]\times\mathbb{T}} \Big(\Gamma(r) u_t^4\Big)\, r\mathrm{d}(r,t), \quad \text{hence} \quad E_N'(u)[\varphi] = \int_{[0,R]\times\mathbb{T}} \Big(\Gamma(r) u_t^3 \varphi_t\Big)\, r\mathrm{d}(r,t).$$

If $N = N_{\mathrm{av}}$, using that κ is even by (5.15) one has

$$\begin{aligned}
\int_{\mathbb{T}} (\kappa * (u_t\varphi_t)) u_t^2 \,\mathrm{d}t &= \int_{\mathbb{T}}\int_{\mathbb{T}} \kappa(t-\tau) u_t(\tau)\varphi_t(\tau) u_t(t)^2 \,\mathrm{d}\tau\mathrm{d}t \\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \kappa(\tau-t) u_t(\tau)\varphi_t(\tau) u_t(t)^2 \,\mathrm{d}t\mathrm{d}\tau = \int_{\mathbb{T}} (\kappa * u_t^2) u_t \varphi_t \,\mathrm{d}\tau,
\end{aligned}$$

and therefore $E_N(u) = \frac{1}{4} \int_{[0,R]\times\mathbb{T}} (\Gamma(r)(\kappa * u_t^2) u_t^2)\, r\mathrm{d}(r,t)$ does satisfy

$$\begin{aligned}
E_N'(u)[\varphi] &= \tfrac{1}{2} \int_{[0,R]\times\mathbb{T}} \Big(\Gamma(r)(\kappa * u_t\varphi_t) u_t^2 + \Gamma(r)(\kappa * u_t^2) u_t \varphi_t\Big)\, r\mathrm{d}(r,t) \\
&= \int_{[0,R]\times\mathbb{T}} \Big(\Gamma(r)(\kappa * u_t^2) u_t\varphi_t\Big)\, r\mathrm{d}(r,t).
\end{aligned}$$

As a next step we properly define the functional E and investigate its properties.

Definition 5.3.1. *Define the norm $\|\|_N$ depending on the nonlinearity N by*

$$\|v\|_{N_{\mathrm{ins}}} \coloneqq \left(\int_{[0,R]\times\mathbb{T}} v^4 \, r\mathrm{d}(r,t)\right)^{1/4} = \|v\|_{L^4_{\mathrm{rad}}([0,R]\times\mathbb{T})}$$

and

$$\|v\|_{N_{\mathrm{av}}} \coloneqq \left(\int_0^R \left(\int_{\mathbb{T}} v^2 \,\mathrm{d}t\right)^2 r\mathrm{d}r\right)^{1/4} = \|v\|_{L^4_{\mathrm{rad}}([0,R];L^2(\mathbb{T}))}.$$

Remark 5.3.2. We have

$$\|v\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \leq \tfrac{\sqrt{R}}{\sqrt[4]{2}}\|v\|_{N_{\mathrm{av}}}, \quad \|v\|_{N_{\mathrm{av}}} \leq \|v\|_{N_{\mathrm{ins}}}, \quad \text{and} \quad \int_{[0,R]\times\mathbb{T}} \Gamma(r)N(v)v\,r\mathrm{d}(r,t) \eqsim \|v\|_N^4.$$

The first two estimates immediately follow from Hölder's inequality. The last estimate is clear for $N = N_{\mathrm{ins}}$ since $N(v)v = v^4$ and Γ is bounded and strictly positive by assumption ($\mathcal{A}5.1$). For $N = N_{\mathrm{av}}$ we have

$$\int_{\mathbb{T}} N(v)v\,\mathrm{d}(t) = \int_{\mathbb{T}}\int_{\mathbb{T}} \kappa(t-\tau)v^2(r,\tau)v^2(r,t)\,\mathrm{d}\tau\mathrm{d}t$$

so that

$$\operatorname*{ess\,inf}_{[0,R]}\Gamma\cdot\min\kappa\cdot\|v\|_N^4 \leq \int_{[0,R]\times\mathbb{T}}\Gamma(r)N(v)v\,r\mathrm{d}(r,t) \leq \operatorname*{ess\,sup}_{[0,R]}\Gamma\cdot\max\kappa\cdot\|v\|_N^4.$$

Proposition 5.3.3. *The functionals E, E_I, E_B given by* (5.23) *are well-defined and differentiable on the reflexive Banach space*

$$Y_N := \left\{ u\in W^{1,1}_{\mathrm{loc,anti}}([0,R]\times\mathbb{T}) \,\middle|\, \begin{array}{l} u_r, \tfrac{1}{r}u \in L^2_{\mathrm{rad}}([0,R]\times\mathbb{T}), \\ \|u_t\|_N < \infty, \hat{u}_k(R) = 0 \text{ for } k\in\mathfrak{F} \end{array} \right\}$$

with norm

$$\|u\|_{Y_N} := \|u_r\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} + \left\|\tfrac{1}{r}u\right\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} + \|u_t\|_N.$$

The derivative is given by

$$\begin{aligned} E'(u)[\varphi] = &\int_{[0,R]\times\mathbb{T}}\Big(u_r\varphi_r + \tfrac{1}{r^2}u\varphi + V(r)u_t\varphi_t + \Gamma(r)N(u_t)\varphi_t\Big)\,r\mathrm{d}(r,t) \\ &- R\sum_{k\in\mathfrak{R}}\frac{\phi_k'(R)}{\phi_k(R)}\hat{u}_k(R)\overline{\hat{\varphi}_k(R)}. \end{aligned}$$

Furthermore, E_I is sequentially weakly lower semicontinuous, E_B is sequentially weakly continuous, and E is sequentially weakly lower semicontinuous as well as coercive. Therefore E attains its minimum $E^\star := \inf E = E(u^\star)$ and $u^\star$ is a critical point of E.

Proof. Using assumption ($\mathcal{A}5.1$) and Remark 5.3.2 one can show in a standard way that E_I is well-defined and differentiable. The formula for the derivative follows from the calculations above. Since $V \geq 0$ the quadratic terms of E_I are convex, and the same holds for the remaining part E_N since $\Gamma \geq 0$ together with assumption 5.15 in the time-averaged case. Therefore E_I is (sequentially) weakly lower semicontinuous.

With (5.21) we obtain $|E_B(u)| \leq C_0\|u(R,\cdot)\|^2_{H^{1/2}(\mathbb{T})}$, so from compactness of the trace (see Lemma 5.A.5) it follows that E_B is sequentially weakly continuous and in particular continuous.

It remains to show that E is coercive. Using Remark 5.3.2 and Lemma 5.A.5 with $\varepsilon := \frac{1}{4C_0}$ we estimate

$$\begin{aligned} E(u) &\geq \tfrac{1}{2}\|u_r\|^2_{L^2_{\mathrm{rad}}} + \tfrac{1}{2}\left\|\tfrac{1}{r}u\right\|^2_{L^2_{\mathrm{rad}}} + \tfrac{c_1}{4}\|u_t\|_N^4 - C_0\Big(\varepsilon\|u_r\|^2_{L^2_{\mathrm{rad}}} + C(\varepsilon)\|u_t\|_N^2\Big) \\ &= \tfrac{1}{4}\|u_r\|^2_{L^2_{\mathrm{rad}}} + \tfrac{1}{2}\left\|\tfrac{1}{r}u\right\|^2_{L^2_{\mathrm{rad}}} + \tfrac{c_1}{4}\|u_t\|_N^4 - C_0C(\varepsilon)\|u_t\|_N^2 \end{aligned}$$

for some $c_1 > 0$. Thus $E(u) \to \infty$ as $\|u\|_{Y_N} \to \infty$. Using [86, Chapter I, Theorem 1.2] we find that E attains its infimum at a critical point, which completes the proof. □

Next we show that assumption ($\mathcal{A}5.6$) is a sufficient condition for the solution $u^\star$ obtained above to be nontrivial.[2]

Proposition 5.3.4. *The minimal energy level of E satisfies $E^\star < 0$ and hence $u^\star \neq 0$.*

Proof. Let $k_0 \in \mathfrak{R}$ be as in ($\mathcal{A}5.6$) and recall that $\lambda = \omega \|V\|_{L^\infty([0,R])}^{1/2}$. Define

$$f(r) := I_1(\lambda k_0 r), \qquad u(r,t) := \varepsilon f(r)(e_{k_0}(t) + e_{-k_0}(t))$$

where I_1 is the modified Bessel functions of first kind, i.e., it satisfies

$$\Big(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} + 1\Big) I_1 = 0, \qquad I_1(0) = 0.$$

We calculate

$$\begin{aligned}
&E(u)\\
&= \int_{(0,R)\times\mathbb{T}} \Big(\tfrac12 u_r^2 + \tfrac12(\tfrac1r u)^2 + \tfrac12 V(r)u_t^2 + \tfrac14\Gamma(r)N(u_t)u_t\Big)\, r\mathrm{d}(r,t) - \frac{R}{2}\sum_{k\in\mathbb{Z}_{\text{odd}}} \frac{\phi_k'(R)}{\phi_k(R)}|\hat{u}_k(R)|^2\\
&= \varepsilon^2\left(\int_0^R \Big((f')^2 + (\tfrac1r f)^2 + \omega^2 k_0^2 V(r) f^2\Big)\, r\mathrm{d}r - \frac{R\phi_{k_0}'(R)}{\phi_{k_0}(R)} f(R)^2\right) + \mathcal{O}(\varepsilon^4)\\
&\le \varepsilon^2\left(\int_0^R f\Big(-f'' - \tfrac1r f' + \tfrac{1}{r^2} f + \lambda^2 k_0^2 f\Big)\, r\mathrm{d}r + \big[rff'\big]_0^R - \frac{R\phi_{k_0}'(R)}{\phi_{k_0}(R)} f(R)^2\right) + \mathcal{O}(\varepsilon^4)\\
&= \varepsilon^2 R f(R)^2 \left(\frac{f'(R)}{f(R)} - \frac{\phi_{k_0}'(R)}{\phi_{k_0}(R)}\right) + \mathcal{O}(\varepsilon^4).
\end{aligned}$$

We have $f(R) > 0$ and by assumption ($\mathcal{A}5.6$) also

$$\frac{f'(R)}{f(R)} - \frac{\phi_{k_0}'(R)}{\phi_{k_0}(R)} = \frac{\lambda k_0 I_1'(\lambda k_0 R)}{I_1(\lambda k_0 R)} - \frac{\phi_{k_0}'(R)}{\phi_{k_0}(R)} < 0.$$

Thus $E(u) < 0$ for $\varepsilon > 0$ sufficiently small, which completes the proof. □

5.4. Approximation by finitely many harmonics

In this section we discuss approximations of the minimizers of E by finitely many harmonics

$$u(r,t) \approx \sum_{\substack{k\in\mathbb{Z}_{\text{odd}}\\ |k|\le K}} \hat{u}_k(r) e_k(t),$$

that is we consider E on the subspace Y_N^K of Y_N defined next.

Definition 5.4.1. *Let $K \in \mathbb{N}_{\text{odd}}$. Then we define*

$$Y_N^K := \{u \in Y_N \colon \hat{u}_k \equiv 0 \text{ for } |k| > K\}.$$

First we discuss the canonical projection from Y_N to Y_N^K

[2] One can show that ($\mathcal{A}5.6$) is also necessary for $u^\star \neq 0$ in the case where V is constant on $[0, R]$.

Lemma 5.4.2. *For $K \in \mathbb{N}_{\text{odd}}$, define the operator*

$$S^K : Y_N \to Y_N^K, \quad S^K[u](r,t) = \sum_{\substack{k \in \mathbb{Z}_{\text{odd}} \\ |k| \le K}} \hat{u}_k(r) e_k(t).$$

Then the operators S^K are uniformly bounded in $\mathcal{B}(Y_N)$ and $S^K u \to u$ in Y_N as $K \to \infty$ for all $u \in Y_N$.

Proof. For $p \in (1, \infty)$ the Fourier cutoff operators S^K defined by

$$S^K : L^p_{\text{anti}}(\mathbb{T}) \to L^p_{\text{anti}}(\mathbb{T}), \quad S^K[f](t) = \sum_{\substack{k \in \mathbb{Z}_{\text{odd}} \\ |k| \le K}} \hat{f}_k e_k(t).$$

are uniformly bounded and $S^K f \to f$ in $L^p_{\text{anti}}(\mathbb{T})$ as $K \to \infty$ (see [40, Theorem 4.1.8 and Corollary 4.1.3]). By acting on the time variable only, S^K extend to uniformly bounded operators

$$S^K : L^q_{\text{rad}}([0,R]; L^p_{\text{anti}}(\mathbb{T})) \to L^q_{\text{rad}}([0,R]; L^p_{\text{anti}}(\mathbb{T}))$$

with $S^K u \to u$ in $L^q_{\text{rad}}([0,R]; L^p_{\text{anti}}(\mathbb{T}))$ as $K \to \infty$. Then from $S^K[u_r] = (S^K u)_r, S^K[u_t] = (S^K u)_t$, and $S^K[\frac{1}{r} u] = \frac{1}{r}(S^K u)$ it follows that $S^K : Y_N \to Y_N^K$ are also uniformly bounded operators and $S^K u \to u$ in Y_N as $K \to \infty$. □

Next we show that the minimal energy level $E^\star$ can be approximated from within Y_N^K.

Lemma 5.4.3. *For every $K \in \mathbb{N}_{\text{odd}}$ there exists $u^{K,\star} \in Y_N^K$ such that $E^{K,\star} := \inf E|_{Y_N^K} = E(u^{K,\star})$. Furthermore $\lim\limits_{K \to \infty} E^{K,\star} = E^\star$ holds.*

Proof. Arguing as in Proposition 5.3.3, one can show that there exists a minimizer $u^{K,\star} \in Y_N^K$ of $E|_{Y_N^K}$. Setting $u^K := S^K(u^\star)$ we find

$$E(u^\star) = E^\star \le E^{K,\star} \le E(u^K). \tag{5.24}$$

Using $u^K \to u^\star$ as $K \to \infty$ and that E is continuous, the second claim follows from (5.24) in the limit $K \to \infty$. □

As a next step we establish uniform estimates on the minimizers $u^{K,\star}$. First, we introduce the fractional time derivative $|\partial_t|^s$ and a quantity Q_N that behaves like a norm stronger than $\|\|\|_N$.

Definition 5.4.4. *For $s \in \mathbb{R}$ we define the fractional time derivative $|\partial_t|^s$ as the Fourier multiplier with symbol $|\omega k|^s$, i.e., $\mathcal{F}_k |\partial_t|^s = |\omega k|^s \mathcal{F}_k$.*

Definition 5.4.5. *For $N = N_{\text{ins}}$ we define the quantity*

$$Q_{N_{\text{ins}}}(v) := \left(\int_{[0,R] \times \mathbb{T}} \left(|\partial_t|^{1/2} (v|v|) \right)^2 r \mathrm{d}(r,t) \right)^{1/4}.$$

For $N = N_{\text{av}}$ we define the quantity

$$Q_{N_{\text{av}}}(v) := \left(\int_0^R \|v(r,\cdot)\|^2_{L^2(\mathbb{T})} \| |\partial_t|^{1/2} v(r,\cdot)\|^2_{L^2(\mathbb{T})} \, r \mathrm{d}r \right)^{1/4}.$$

Remark 5.4.6. For $N = N_{\text{ins}}$ by Lemma 5.A.3 we have

$$\int_{[0,R]\times\mathbb{T}} \Gamma(r) N_{\text{ins}}(v)|\partial_t| v\, r\mathrm{d}(r,t) \geq c^* Q_{N_{\text{ins}}}(v)^4,$$

with $c^* = \frac{1}{2}\operatorname{ess\,inf}_{[0,R]}\Gamma > 0$ whereas for $N = N_{\text{av}}$ using Lemma 5.A.4 (with constants c_1, C_2) we have

$$\int_{[0,R]\times\mathbb{T}} \Gamma(r) N_{\text{av}}(v)|\partial_t| v\, r\mathrm{d}(r,t) \geq c^* Q_{N_{\text{av}}}(v)^4 - C^*\|v\|_{N_{\text{av}}}^4$$

with $c^* = c_1 \operatorname{ess\,inf}_{[0,R]}\Gamma$ and $C^* = C_2 \operatorname{ess\,sup}_{[0,R]}\Gamma$. In particular,

$$\int_{[0,R]\times\mathbb{T}} N(v)|\partial_t| v\, r\mathrm{d}(r,t) \geq c^* Q_N(v)^4 - C^*\|v\|_N^4$$

holds for both choices of N.

The minimizers $u^{K,\star}$ formally are solutions of

$$\begin{cases} S^K[-u_{rr} - \frac{1}{r}u_r + \frac{1}{r^2}u - V(r)u_{tt} - \Gamma(r)N(u_t)_t] = 0 \text{ in } [0,R]\times\mathbb{T}, \\ \hat{u}_k'(R) = \frac{\phi_k'(R)}{\phi_k(R)}\hat{u}_k(R) \text{ for } k \in \mathfrak{R}, |k| \leq K, \\ \hat{u}_k(R) = 0 \text{ for } k \in \mathfrak{F}, |k| \leq K. \end{cases}$$

Here the main part $-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} - V(r)\partial_t^2 - \Gamma(r)\partial_t N(\partial_t\,\cdot\,)$ is elliptic by ($\mathcal{A}$5.3), which is why we expect the solution u to have increased regularity. Often this is shown by testing the problem against derivatives of the solution. In Proposition 5.4.7, we obtain improved regularity by testing the problem against $|\partial_t| u^{K,\star}$. However, with this method it is impossible to obtain even more regularity because when testing against $|\partial_t|^s u^{K,\star}$ with $s > 1$ one can no longer control the appearing boundary terms.

Proposition 5.4.7. *There exist constants $C_1, \ldots, C_5 > 0$ independent of K such that the following holds:*

1. $\left\|u^{K,\star}\right\|_{Y_N} \leq C_1$,
2. $\left\||\partial_t|^{1/2} u_r^{K,\star}\right\|_{L^2_{\text{rad}}([0,R]\times\mathbb{T})} \leq C_2$,
3. $\left\|\frac{1}{r}|\partial_t|^{1/2} u^{K,\star}\right\|_{L^2_{\text{rad}}([0,R]\times\mathbb{T})} \leq C_3$,
4. $Q_N\left(u_t^{K,\star}\right) \leq C_4$,
5. $\left\|u^{K,\star}(R,\cdot)\right\|_{H^1(\mathbb{T})} \leq C_5$.

Proof. Since E is coercive (see Proposition 5.3.3), there exists $C_1 > 0$ such that $E(u) > 0$ holds for all $u \in Y_N$ with $\|u\|_{Y_N} > C_1$. Using $E(u^{K,\star}) = \min E|_{Y_N^K} \leq E(0) = 0$ we conclude $\left\|u^{K,\star}\right\|_{Y_N} \leq C_1$, so that 1 holds.

For 2–5 we first note that

$$|||u||| = \sum_{\substack{k\in\mathbb{Z}_{\text{odd}}\\ |k|\leq K}} \left(\|\hat{u}_k'\|_{L^2_{\text{rad}}([0,R])} + \left\|\tfrac{1}{r}\hat{u}_k\right\|_{L^2_{\text{rad}}([0,R])} + \|\hat{u}_k\|_{L^4_{\text{rad}}([0,R])} \right)$$

defines an equivalent norm on Y_N^K. Thus the operators $|\partial_t|^s$ are bounded on Y_N^K for all $s \in \mathbb{R}$. In particular, $|\partial_t| u^{K,\star} \in Y_N^K$. Using $V \geq 0$ on $[0,R]$ and (5.21) we calculate

$$\begin{aligned}
0 &= E'(u^{K,\star})[|\partial_t| u^{K,\star}] \\
&= \int_{[0,R]\times\mathbb{T}} \Big(u_r^{K,\star} |\partial_t| u_r^{K,\star} + \tfrac{1}{r^2} u^{K,\star} |\partial_t| u^{K,\star} + V(r) u_t^{K,\star} |\partial_t| u_t^{K,\star} + \Gamma(r) N(u_t^{K,\star}) |\partial_t| u_t^{K,\star} \Big) r \mathrm{d}(r,t) \\
&\quad - R \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \frac{\phi_k'(R)}{\phi_k(R)} \omega |k| \Big| \mathcal{F}_k[u^{K,\star}](R) \Big|^2 \\
&\geq \int_{[0,R]\times\mathbb{T}} \Big(\Big(|\partial_t|^{1/2} u_r^{K,\star} \Big)^2 + \Big(\tfrac{1}{r} |\partial_t|^{1/2} u^{K,\star} \Big)^2 + \Gamma(r) N(u_t^{K,\star}) |\partial_t| u_t^{K,\star} \Big) r \mathrm{d}(r,t) \\
&\quad - C_0 R \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \omega k^2 \Big| \mathcal{F}_k[u^{K,\star}](R) \Big|^2.
\end{aligned}$$

Using further

$$C_0 R \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \omega k^2 \Big| \mathcal{F}_k[u^{K,\star}](R) \Big|^2 \leq \tilde{C}_0 \Big\| u^{K,\star}(R,\cdot) \Big\|_{H^1(\mathbb{T})}^2,$$

Remark 5.4.6, Lemma 5.A.7 with $\varepsilon = \frac{1}{2\tilde{C}_0}$ as well as $aX^2 - bX \geq X - \frac{(b+1)^2}{4a}$, we obtain

$$\begin{aligned}
0 &\geq \tfrac{1}{2} \Big\| |\partial_t|^{1/2} u_r^{K,\star} \Big\|_{L^2_{\mathrm{rad}}}^2 + \Big\| \tfrac{1}{r} |\partial_t|^{1/2} u^{K,\star} \Big\|_{L^2_{\mathrm{rad}}}^2 + c^* Q_N\Big(u_t^{K,\star}\Big)^4 - C^* \Big\| u_t^{K,\star} \Big\|_N^4 - \widetilde{C}_0 C(\varepsilon) Q_N \Big(u_t^{K,\star}\Big)^2 \\
&\geq \tfrac{1}{2} \Big\| |\partial_t|^{1/2} u_r^{K,\star} \Big\|_{L^2_{\mathrm{rad}}}^2 + \Big\| \tfrac{1}{r} |\partial_t|^{1/2} u^{K,\star} \Big\|_{L^2_{\mathrm{rad}}}^2 + Q_N\Big(u_t^{K,\star}\Big)^2 - \frac{(\widetilde{C}_0 C(\varepsilon) + 1)^2}{4c^*} - C^* C_1^4.
\end{aligned} \tag{5.25}$$

With $C := \frac{(\widetilde{C}_0 C(\varepsilon)+1)^2}{4c^*} + C^* C_1^4$ the estimates 2–4 follow from (5.25) where

$$C_2 := \sqrt{2C}, \quad C_3 := C_4 := \sqrt{C},$$

and lastly 5 follows from 2 and 4 using Lemma 5.A.7 again. □

The following result is the most important result in this section. It shows how a minimizer u of E gains additional regularity via the approximation by finitely many harmonics. This will be the key to establish regularity of the solutions of (5.19) across the boundary at $r = R$.

Proposition 5.4.8. *Up to a subsequence, the limit $u = \lim_{K\to\infty} u^{K,\star}$ exists in Y_N. The function u is a minimizer of E and satisfies*

1. $\|u\|_{Y_N} \leq C_1$,
2. $\Big\| |\partial_t|^{1/2} u_r \Big\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \leq C_2$,
3. $\Big\| \tfrac{1}{r} |\partial_t|^{1/2} u \Big\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \leq C_3$,
4. $Q_N(u_t) \leq C_4$,
5. $\|u(R,\cdot)\|_{H^1(\mathbb{T})} \leq C_5$,

where the constants $C_1, \ldots, C_5$ are the same as in Proposition 5.4.7.

Proof. We only consider the case $N = N_{\text{av}}$, as for $N = N_{\text{ins}}$ one can argue similarly. Then due to Proposition 5.4.7 and the definition of $Q_{N_{\text{av}}}$ the weak limits

$$\begin{aligned} u^{K,\star} &\rightharpoonup u \text{ in } Y_N, \\ |\partial_t|^{1/2} u_r^{K,\star} &\rightharpoonup f \text{ in } L^2_{\text{rad}}([0,R]\times\mathbb{T}), \\ \tfrac{1}{r}|\partial_t|^{1/2} u^{K,\star} &\rightharpoonup g \text{ in } L^2_{\text{rad}}([0,R]\times\mathbb{T}), \\ \|u_t^{K,\star}\|_{L^2(\mathbb{T})}|\partial_t|^{1/2} u_t^{K,\star} &\rightharpoonup h \text{ in } L^2_{\text{rad}}([0,R]\times\mathbb{T}), \\ u^{K,\star}(R,\,\cdot\,) &\rightharpoonup b \text{ in } H^1(\mathbb{T}) \end{aligned}$$

exist for $K \to \infty$ up to a subsequence and satisfy $\|u\|_{Y_N} \le C_1$, $\|f\|_{L^2_{\text{rad}}} \le C_2$, $\|g\|_{L^2_{\text{rad}}} \le C_3$, $\|h\|_{L^2_{\text{rad}}} \le C_4$, $\|b\|_{H^1} \le C_5$. Using the properties of the functional E from Proposition 5.3.3 and Lemma 5.4.3 we further obtain

$$E^\star \le E(u) \le \lim_{K\to\infty} E(u^{K,\star}) = \lim_{K\to\infty} E^{K,\star} = E^\star,$$

so that $E(u) = E^\star = \lim_{K\to\infty} E(u^{K,\star})$. In particular u is a minimizer of E.

Also, since $E_B(u^{K,\star}) \to E_B(u)$ for $K \to \infty$, we obtain $E_I(u^{K,\star}) \to E_I(u)$ as $K \to \infty$. From this it follows that $u^{K,\star} \to u$ in Y_N as $K \to \infty$ as we show next. Since $u^{K,\star} \rightharpoonup u$ we see that

$$\tfrac{1}{r}u^{K,\star} \rightharpoonup \tfrac{1}{r}u,\ u_r^{K,\star} \rightharpoonup u_r \text{ in } L^2_{\text{rad}}([0,R]\times\mathbb{T}), \qquad u_t^{K,\star} \rightharpoonup u_t \text{ in } \|\|_N.$$

Moreover, by weak sequential lower semicontinuity we have

$$\begin{aligned} &E_I(u) \\ &= \tfrac12 \|u_r\|^2_{L^2_{\text{rad}}} + \tfrac12 \left\|\tfrac1r u\right\|^2_{L^2_{\text{rad}}} + \tfrac12 \left\|V^{1/2}u_t\right\|^2_{L^2_{\text{rad}}} + \tfrac14 \left\|\Gamma^{1/4}u_t\right\|^4_{N_{\text{av}}} \\ &\le \tfrac12 \liminf_{K\to\infty}\left\|u_r^{K,\star}\right\|^2_{L^2_{\text{rad}}} + \tfrac12 \liminf_{K\to\infty}\left\|\tfrac1r u^{K,\star}\right\|^2_{L^2_{\text{rad}}} + \tfrac12 \liminf_{K\to\infty}\left\|V^{1/2}u_t^{K,\star}\right\|^2_{L^2_{\text{rad}}} + \tfrac14 \liminf_{K\to\infty}\left\|\Gamma^{1/4}u_t^{K,\star}\right\|^4_{N_{\text{av}}} \\ &\le \tfrac12 \limsup_{K\to\infty}\left\|u_r^{K,\star}\right\|^2_{L^2_{\text{rad}}} + \tfrac12 \liminf_{K\to\infty}\left\|\tfrac1r u^{K,\star}\right\|^2_{L^2_{\text{rad}}} + \tfrac12 \liminf_{K\to\infty}\left\|V^{1/2}u_t^{K,\star}\right\|^2_{L^2_{\text{rad}}} + \tfrac14 \liminf_{K\to\infty}\left\|\Gamma^{1/4}u_t^{K,\star}\right\|^4_{N_{\text{av}}} \\ &\le \limsup_{K\to\infty} E_I(u^{K,\star}) = E_I(u). \end{aligned}$$

Notice that in the second inequality we have replaced one lim inf by a lim sup and in the last inequality we used that $\limsup_{n\to\infty} a_n + \sum_{i=1}^p \liminf_{n\to\infty} b_n^i \le \limsup_{n\to\infty}(a_n + \sum_{i=1}^p b_n^i)$ which follows from $\sup_{n\in\mathbb{N}} a_n + \sum_{i=1}^p \inf_{n\in\mathbb{N}} b_n^i \le \sup_{n\in\mathbb{N}}(a_n + \sum_{i=1}^p b_n^i)$. It follows that $\|u_r\|^2_{L^2_{\text{rad}}([0,R]\times\mathbb{T})} = \liminf_{K\to\infty}\left\|u_r^{K,\star}\right\|^2_{L^2_{\text{rad}}([0,R]\times\mathbb{T})} = \limsup_{K\to\infty}\left\|u_r^{K,\star}\right\|^2_{L^2_{\text{rad}}([0,R]\times\mathbb{T})}$. Combining weak convergence $u_r^{K,\star} \rightharpoonup u_r$ with convergence of the norms $\left\|u_r^{K,\star}\right\|_{L^2_{\text{rad}}} \to \|u_r\|_{L^2_{\text{rad}}}$, we find that $u_r^{K,\star} \to u_r$ in $L^2_{\text{rad}}([0,R]\times\mathbb{T})$ as $K\to\infty$. With a similar argument we find $\frac1r u^{K,\star} \to \frac1r u$ in $L^2_{\text{rad}}([0,R]\times\mathbb{T})$ and $u_t^{K,\star} \to u_t$ in $\|\|_{N_{\text{av}}}$ as $K\to\infty$. Together, this shows $u^{K,\star} \to u$ in Y_N as $K\to\infty$.

It remains to show the estimates 2–5. These follow from the identities

$$f = |\partial_t|^{1/2}u_r, \quad g = \tfrac1r|\partial_t|^{1/2}u, \quad h = \|u_t\|_{L^2(\mathbb{T})}|\partial_t|^{1/2}u_t, \quad b = u(R,\,\cdot\,),$$

where we only discuss $h = \|u_t\|_{L^2(\mathbb{T})}|\partial_t|^{1/2}u_t$ as an example. First, by definition of Q_N and convergence $u^{K,\star} \to u$ in Y_N we have $\left\|u^{K,\star}\right\|_{L^2(\mathbb{T})} u^{K,\star} \to \|u\|_{L^2(\mathbb{T})} u$ in $L^2_{\text{rad}}([0,R]\times\mathbb{T})$. Taking the Fourier transform, for $k \in \mathbb{Z}_{\text{odd}}$ we find

$$\mathcal{F}_k[\|u_t^{K,\star}\|_{L^2(\mathbb{T})} u_t^{K,\star}] \to \mathcal{F}_k[\|u_t\|_{L^2(\mathbb{T})} u_t]$$

and also

$$\sqrt{\omega|k|}\mathcal{F}_k[\|u_t^{K,\star}\|_{L^2(\mathbb{T})}u_t^{K,\star}] = \mathcal{F}_k[\|u_t^{K,\star}\|_{L^2(\mathbb{T})}|\partial_t|^{1/2}u_t^{K,\star}] \rightharpoonup \mathcal{F}_k[h]$$

in $L^2_{\mathrm{rad}}([0,R])$ as $K \to \infty$. Thus $\mathcal{F}_k[h] = \sqrt{\omega|k|}\mathcal{F}_k[\|u_t\|_{L^2(\mathbb{T})}u_t]$, i.e., $h = |\partial_t|^{1/2}(\|u_t\|_{L^2(\mathbb{T})}u_t) = \|u_t\|_{L^2(\mathbb{T})}|\partial_t|^{1/2}u_t$. □

5.5. Proof of Theorems 5.2.1 and 5.2.3

The proof of Theorem 5.2.1 is split into two parts. First, using results from Sections 5.3–5.4, we show in Proposition 5.5.2 that there exists a weak solution to the problem (5.19) in the sense of Definition 5.5.1 below. In Proposition 5.5.4 we show that from the solution of (5.19), one can reconstruct a solution of Maxwell's equations (5.1)–(5.3), and that this solution has finite electromagnetic energy per unit segment in z-direction.

Definition 5.5.1. *A function $w\colon (0,\infty)\times\mathbb{T}\to\mathbb{R}$ is called a T-periodic weak solution to* (5.19) *if w lies in*

$$X := \left\{w \in W^{1,1}_{\mathrm{loc}}((0,\infty)\times\mathbb{T})\colon \tfrac{1}{r}w, w_r, w_t \in L^2_{\mathrm{rad}}([0,\infty)\times\mathbb{T}), \left\|w_t|_{[0,R]\times\mathbb{T}}\right\|_N < \infty\right\}.$$

and satisfies the equation

$$\int_{[0,\infty)\times\mathbb{T}}\left(w_r\varphi_r + \tfrac{1}{r^2}w\varphi + V(r)w_t\varphi_t + \Gamma(r)N(w_t)\varphi_t\right) r\,\mathrm{d}(r,t) = 0$$

for all $\varphi \in X$.

Proposition 5.5.2. *There exists a nontrivial weak solution to* (5.19) *in the sense of Definition 5.5.1.*

We prepare the proof of Proposition 5.5.2 with an estimate on the fundamental solutions ϕ_k.

Lemma 5.5.3. *There exists a constant $C > 0$ such that $\|\phi_k'\|_{L^2_{\mathrm{rad}}([R,\infty))} \le C|k|\|\phi_k\|_{L^2_{\mathrm{rad}}([R,\infty))}$ holds for all $k \in \mathbb{Z}_{\mathrm{odd}}$.*

Proof. By assumption we have

$$\left\|\phi_k'' + \tfrac{1}{r}\phi_k'\right\|_{L^2_{\mathrm{rad}}([R,\infty))} = \left\|\tfrac{1}{r^2}\phi_k + k^2\omega^2 V\phi_k\right\|_{L^2_{\mathrm{rad}}([R,\infty))} \le k^2\left(\tfrac{1}{R^2} + \omega^2\|V\|_\infty\right)\|\phi_k\|_{L^2_{\mathrm{rad}}([R,\infty))}$$

Due to [2, Lemma 5.5] the inequality

$$\|\phi_k'\|_{L^2_{\mathrm{rad}}([R,\infty))} \le C_0\left(\varepsilon\left\|\phi_k'' + \tfrac{1}{r}\phi_k'\right\|_{L^2_{\mathrm{rad}}([R,\infty))} + \tfrac{1}{\varepsilon}\|\phi_k\|_{L^2_{\mathrm{rad}}([R,\infty))}\right)$$

holds for some $C_0 > 0$. Choosing $\varepsilon = \frac{1}{|k|}$, the claim follows with $C = C_0(\frac{1}{R^2} + \omega^2\|V\|_\infty + 1)$. □

Proof of Proposition 5.5.2. Let u denote the minimizer of E obtained through Proposition 5.4.8. Then u is nonzero by Proposition 5.3.4. As motivated in Section 5.3 we define

$$w(r,t) := \begin{cases} u(r,t), & r < R, \\ \sum_{k\in\mathfrak{F}}\frac{\hat{u}_k'(R)}{\phi_k'(R)}\phi_k(r)e_k(t) + \sum_{k\in\mathfrak{R}}\frac{\hat{u}_k(R)}{\phi_k(R)}\phi_k(r)e_k(t) & r > R. \end{cases} \tag{5.26}$$

First we show that $\hat{u}_k'(R)$ exists for all $k \in \mathbb{Z}_{\mathrm{odd}}$. To do this, let $\varepsilon \in (0,R)$. Then for $\psi \in C_c^\infty((\varepsilon,R);\mathbb{C})$ we have

$$\begin{aligned} 0 &= E'(u)[\mathrm{Re}[\psi(r)e_k(t)]] \\ &= \mathrm{Re}\left[\int_0^R \Big(\hat{u}_k'(r)\overline{\psi'(r)} + \Big[\tfrac{1}{r^2}\hat{u}_k(r) + k^2\omega^2 V(r)\hat{u}_k(r) - \mathrm{i}k\omega\Gamma(r)\mathcal{F}_k[N(u_t)](r)\Big]\overline{\psi(r)}\Big)\, r\mathrm{d}r\right]. \end{aligned}$$

Since ψ was arbitrary, this shows that $\hat{u}_k \in H^1([\varepsilon,R])$ is a weak solution to

$$\hat{u}_k'' = -\tfrac{1}{r}\hat{u}_k' + \tfrac{1}{r^2}\hat{u}_k + k^2\omega^2 V\hat{u}_k - \mathrm{i}k\omega\Gamma\mathcal{F}_k[N(u_t)] \quad \text{on} \quad [\varepsilon,R]. \tag{5.27}$$

Note that the right-hand side of (5.27) lies in $L^{4/3}([\varepsilon,R])$. Thus $\hat{u}_k \in W^{2,4/3}([\varepsilon,R])$ and solves (5.27) pointwise. In particular, we have $\hat{u}_k \in C^1([\varepsilon,R])$ and therefore $\hat{u}_k'(R)$ exists.

Next we show that w lies in X. Clearly, w is real-valued, and $\frac{1}{r}w, w_r, w_t \in L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})$ and $N(w_t)w_t \in L^1_{\mathrm{rad}}([0,R]\times\mathbb{T})$. Since the antiperiodicity of w forces the zero-th Fourier mode to vanish, we see that $\|w\|_{L^2_{\mathrm{rad}}([R,\infty)\times\mathbb{T})}$ and hence $\|\frac{1}{r}w\|_{L^2_{\mathrm{rad}}([R,\infty)\times\mathbb{T})}$ are bounded by $\|w_t\|_{L^2_{\mathrm{rad}}([R,\infty)\times\mathbb{T})}$. Therefore, it remains to show that $w_r, w_t \in L^2_{\mathrm{rad}}([R,\infty)\times\mathbb{T})$ since the function values at $r=R$ match according to the construction of w.

Using (5.21), Proposition 5.4.8, and Lemma 5.5.3 we find

$$\begin{aligned} \sum_{k\in\mathfrak{R}} \|\hat{w}_k'\|^2_{L^2_{\mathrm{rad}}([R,\infty))} &= \sum_{k\in\mathfrak{R}} \left|\frac{\hat{u}_k(R)}{\phi_k(R)}\right|^2 \|\phi_k'\|^2_{L^2_{\mathrm{rad}}([R,\infty))} \\ &\lesssim \sum_{k\in\mathfrak{R}} k^2|\hat{u}_k(R)|^2 \lesssim \|u(R,\cdot)\|^2_{H^1(\mathbb{T})} < \infty, \\ \sum_{k\in\mathfrak{R}} \omega^2 k^2 \|\hat{w}_k\|^2_{L^2_{\mathrm{rad}}([R,\infty))} &= \sum_{k\in\mathfrak{R}} \omega^2 k^2 \left|\frac{\hat{u}_k(R)}{\phi_k(R)}\right|^2 \|\phi_k\|^2_{L^2_{\mathrm{rad}}([R,\infty))} \\ &\lesssim \sum_{k\in\mathfrak{R}} k^2|\hat{u}_k(R)|^2 \lesssim \|u(R,\cdot)\|^2_{H^1(\mathbb{T})} < \infty. \end{aligned}$$

Since the finite sum $\sum_{k\in\mathfrak{F}} \frac{\hat{u}_k'(R)}{\phi_k'(R)}\phi_k(r)e_k(t)$ belongs to $H^1_{\mathrm{rad}}([R,\infty)\times\mathbb{T})$ this shows that the sum $w(r,t) = \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \hat{w}_k(r)e_k(t)$ converges in $H^1_{\mathrm{rad}}([R,\infty)\times\mathbb{T})$. It remains to show that w is a weak T-periodic solution to (5.19) in the sense of Definition 5.5.1. That is, we need to verify

$$I[\varphi] \coloneqq \int_{[0,\infty)\times\mathbb{T}} \Big(w_r\varphi_r + \tfrac{1}{r^2}w\varphi + V(r)w_t\varphi_t + \Gamma(r)N(w_t)\varphi_t\Big)\, r\mathrm{d}(r,t) = 0$$

for all $\varphi \in X$. Since $w_r, w, w_t, N(w_t)$ are $T/2$-antiperiodic in time, it follows that $I[\varphi]=0$ for $T/2$-periodic φ. So from now on let φ be $T/2$-antiperiodic in time. We calculate

$$\begin{aligned} I[\varphi] &= \int_{[0,R]\times\mathbb{T}} \Big(u_r\varphi_r + \tfrac{1}{r^2}u\varphi + V(r)u_t\varphi_t + \Gamma(r)N(u_t)\varphi_t\Big)\, r\mathrm{d}(r,t) \\ &\quad + \sum_{k\in\mathfrak{F}} \frac{\hat{u}_k'(R)}{\phi_k'(R)} \int_R^\infty \Big(\phi_k'\overline{\hat{\varphi}_k'} + \tfrac{1}{r^2}\phi_k\overline{\hat{\varphi}_k} + k^2\omega^2 V(r)\phi_k\overline{\hat{\varphi}_k}\Big)\, r\mathrm{d}r \\ &\quad + \sum_{k\in\mathfrak{R}} \frac{\hat{u}_k(R)}{\phi_k(R)} \int_R^\infty \Big(\phi_k'\overline{\hat{\varphi}_k'} + \tfrac{1}{r^2}\phi_k\overline{\hat{\varphi}_k} + k^2\omega^2 V(r)\phi_k\overline{\hat{\varphi}_k}\Big)\, r\mathrm{d}r \qquad (5.28) \\ &= \int_{[0,R]\times\mathbb{T}} \Big(u_r\varphi_r + \tfrac{1}{r^2}u\varphi + V(r)u_t\varphi_t + \Gamma(r)N(u_t)\varphi_t\Big)\, r\mathrm{d}(r,t) \\ &\quad - \sum_{k\in\mathfrak{F}} \frac{\hat{u}_k'(R)}{\phi_k'(R)} \cdot R\phi_k'(R)\overline{\hat{\varphi}_k(R)} - \sum_{k\in\mathfrak{R}} \frac{\hat{u}_k(R)}{\phi_k(R)} \cdot R\phi_k'(R)\overline{\hat{\varphi}_k(R)} \end{aligned}$$

If in addition $\hat{\varphi}_k(R) = 0$ holds for all $k \in \mathfrak{F}$, then

$$\begin{aligned} I[\varphi] &= \int_{[0,R]\times\mathbb{T}} \Big(u_r\varphi_r + \tfrac{1}{r^2}u\varphi + V(r)u_t\varphi_t + \Gamma(r)u_t^3\varphi_t \Big)\, r\mathrm{d}(r,t) - R\sum_{k\in\mathfrak{R}} \frac{\phi_k'(R)}{\phi_k(R)}\hat{u}_k(R)\overline{\hat{\varphi}_k(R)} \\ &= E'(u)[\varphi|_{[0,R]\times\mathbb{T}}] = 0 \end{aligned}$$

where we have used $\varphi|_{[0,R]\times\mathbb{T}} \in Y_N$. Now we want to conclude $I[\varphi] = 0$ in the general case where $\varphi \in X$ but $\hat{\varphi}_k(R) \neq 0$ for some $k \in \mathfrak{F}$. Note that since φ is real-valued we have the decomposition

$$X = \{\varphi \in X : \hat{\varphi}_k(R) = 0 \text{ for all } k \in \mathfrak{F}, k > 0\} \oplus \mathrm{lin}_{\mathbb{R}}\{\mathrm{Re}[\psi e_k], \mathrm{Re}[\mathrm{i}\psi e_k] \colon k \in \mathfrak{F}, k > 0\}$$

for any $\psi \in C_c^\infty((0,\infty))$ with $\psi(R) \neq 0$. By linearity it suffices to show the identity $I[\mathrm{Re}[\psi(r)e_k(t)]] = 0 = I[\mathrm{Re}[\mathrm{i}\psi(r)e_k(t)]]$ for all $k \in \mathfrak{F}$. Using (5.28) we calculate

$$\begin{aligned} &I[\mathrm{Re}[\psi(r)e_k(t)]] \\ &\quad = \mathrm{Re}\Bigg[\int_0^R \Big(\hat{u}_k'\overline{\psi'} + \Big[\tfrac{1}{r^2}\hat{u}_k + k^2\omega^2V(r)\hat{u}_k - \mathrm{i}k\omega\Gamma(r)\mathcal{F}_k[N(u_t)](r)\Big]\overline{\psi}\Big)\, r\mathrm{d}r - R\hat{u}_k'(R)\overline{\psi(R)}\Bigg] \\ &\quad = \mathrm{Re}\Bigg[\int_0^R \Big(-\hat{u}_k'' - \tfrac{1}{r}\hat{u}_k' + \tfrac{1}{r^2}\hat{u}_k + k^2\omega^2V(r)\hat{u}_k - \mathrm{i}k\omega\Gamma(r)\mathcal{F}_k[N(u_t)]\Big)\overline{\psi}\, r\mathrm{d}r\Bigg] = 0, \end{aligned}$$

where the last equality follows from (5.27) with $\varepsilon := \min\operatorname{supp}\psi$. Replacing ψ by $\mathrm{i}\psi$ in the above calculation, we obtain also $I[\mathrm{Re}[\mathrm{i}\psi(r)e_k(t)]] = 0$. □

Proposition 5.5.4. *Let w be a T-periodic weak solution to* (5.19) *in the sense of Definition 5.5.1. Then the fields* $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H}$ *given by*

$$\begin{aligned} \mathbf{D}(\mathbf{x},t) &= \epsilon_0\Big((1+\chi_1(\mathbf{x}))w_t(r,t-\tfrac{1}{c}z) + \chi_3(\mathbf{x})N(w_t)(r,t-\tfrac{1}{c}z)\Big)\cdot(-\tfrac{y}{r},\tfrac{x}{r},0)^\top, \\ \mathbf{E}(\mathbf{x},t) &= w_t(r,t-\tfrac{1}{c}z)\cdot(-\tfrac{y}{r},\tfrac{x}{r},0)^\top, \\ \mathbf{B}(\mathbf{x},t) &= -\Big(\tfrac{1}{r}w(r,t-\tfrac{1}{c}z) + w_r(r,t-\tfrac{1}{c}z)\Big)\cdot(0,0,1)^\top - \tfrac{1}{c}w_t(r,t-\tfrac{1}{c}z)\cdot(\tfrac{x}{r},\tfrac{y}{r},0)^\top, \\ \mathbf{H}(\mathbf{x},t) &= \tfrac{1}{\mu_0}\mathbf{B}(\mathbf{x},t), \end{aligned}$$

where $\mathbf{x} = (x,y,z)$ and $r = \sqrt{x^2+y^2}$ are weak solutions to Maxwell's equations (5.1)–(5.3) *in the sense of Definition 5.1.1. Furthermore, the electromagnetic energy is finite orthogonal to the direction of propagation, i.e.,*

$$\int_{\mathbb{R}\times\mathbb{R}\times[z_0,z_0+1]} (\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H})\,\mathrm{d}(x,y,z)$$

is uniformly bounded w.r.t. z_0, t_0.

Proof. We use cylindrical coordinates $(x,y,z) = (r\cos(\theta), r\sin(\theta), z)$. We abbreviate

$$e_r = (\tfrac{x}{r},\tfrac{y}{r},0)^\top, \quad e_\theta = (-\tfrac{y}{r},\tfrac{x}{r},0)^\top, \quad e_z = (0,0,1)^\top$$

and use the representations

$$\begin{aligned} \nabla\phi &= \partial_r\phi\cdot e_r + \tfrac{1}{r}\partial_\theta\phi\cdot e_\theta + \partial_z\phi\cdot e_z, \\ \nabla\times\Phi &= \Big(\tfrac{1}{r}\partial_\theta\Phi^z - \partial_z\Phi^\theta\Big)e_r + (\partial_z\Phi^r - \partial_r\Phi^z)e_\theta + \tfrac{1}{r}\Big(\partial_r(r\Phi^\theta) - \partial_\theta\Phi^r\Big)e_z \end{aligned}$$

where $\Phi = \Phi^r e_r + \Phi^\theta e_\theta + \Phi^z e_z$. For better readability, we omit the domain $[0,\infty)\times[0,2\pi]\times\mathbb{R}\times\mathbb{R}$ when integrating with cylindrical coordinates as well as arguments, so $w = w(r, t - \frac{1}{c}z)$. In particular, $\partial_z w = -\frac{1}{c}\partial_t w = -\frac{1}{c}w_t$ holds, which we use below. Now let $\phi \in C_c^\infty(\mathbb{R}^4;\mathbb{R})$ and $\Phi \in \mathbb{C}_c^\infty(\mathbb{R}^4;\mathbb{R}^3)$. Identities (5.2) and (5.3) hold by definition, so it remains to check the four integral identities of Definition 5.1.1, beginning with

$$\int_{\mathbb{R}^4} (\mathbf{D}\cdot\nabla\phi)\,\mathrm{d}(x,y,z,t) = \int (\mathbf{D}^\theta \partial_\theta \phi)\, r\mathrm{d}(r,\theta,z,t) = 0,$$

where the integral above is zero because $\mathbf{D}$ is independent of θ. Next,

$$\begin{aligned}\int_{\mathbb{R}^4} (\mathbf{B}\cdot\nabla\phi)\,\mathrm{d}(x,y,z,t) &= \int \big(-(\tfrac{1}{r}w + w_r)\partial_z\phi - \tfrac{1}{c}w_t\partial_r\phi\big)\, r\mathrm{d}(r,\theta,z,t)\\ &= \int \big(-\tfrac{1}{r}\partial_r(rw)\partial_z\phi + \partial_z w\partial_r\phi\big)\, r\mathrm{d}(r,\theta,z,t) = \int \big(w(\partial_r\partial_z\phi - \partial_z\partial_r\phi)\big)\, r\mathrm{d}(r,\theta,z,t) = 0.\end{aligned}$$

For the third integral we have

$$\begin{aligned}&\int_{\mathbb{R}^4} (\mathbf{E}\cdot\nabla\times\Phi - \mathbf{B}\cdot\partial_t\Phi)\,\mathrm{d}(x,y,z,t)\\ &\quad= \int \big(w_t(\partial_z\Phi^r - \partial_r\Phi^z) + (\tfrac{1}{r}w + w_r)\partial_t\Phi^z + \tfrac{1}{c}w_t\partial_t\Phi^r\big)\, r\mathrm{d}(r,\theta,z,t)\\ &\quad= \int \big(\partial_t w(\partial_z\Phi^r - \partial_r\Phi^z) + \tfrac{1}{r}\partial_r(rw)\partial_t\Phi^z - \partial_z w\partial_t\Phi^r\big)\, r\mathrm{d}(r,\theta,z,t)\\ &\quad= \int \big(w(-\partial_t\partial_z\Phi^r + \partial_t\partial_r\Phi^z - \partial_r\partial_t\Phi^z + \partial_z\partial_t\Phi^r)\big)\, r\mathrm{d}(r,\theta,z,t) = 0.\end{aligned}$$

For the last identity, using integration by parts, that integrals with ∂_θ vanish, and the definitions of V, Γ in (5.18), we have

$$\begin{aligned}&\int_{\mathbb{R}^4} -\mathbf{H}\cdot\nabla\times\Phi - \mathbf{D}\cdot\partial_t\Phi\,\mathrm{d}(\mathbf{x},t)\\ &\quad= \tfrac{1}{\mu_0}\int \big((\tfrac{1}{r}w + w_r)\tfrac{1}{r}(\partial_r(r\Phi^\theta) - \partial_\theta\Phi^r) + \tfrac{1}{c}w_t(\tfrac{1}{r}\partial_\theta\Phi^z - \partial_z\Phi^\theta\big)\, r\mathrm{d}(r,\theta,z,t)\\ &\qquad- \int \epsilon_0\big((1+\chi_1)w_t + \chi_3 N(w_t))\partial_t\Phi^\theta\big)\, r\mathrm{d}(r,\theta,z,t)\\ &\quad= \tfrac{1}{\mu_0}\int \big(\partial_r w\partial_r\Phi^\theta + \tfrac{1}{r}(\partial_r w\Phi^\theta + w\partial_r\Phi^\theta) + \tfrac{1}{r^2}w\Phi^\theta - \tfrac{1}{c}\partial_z w\partial_t\Phi^\theta\big)\, r\mathrm{d}(r,\theta,z,t)\\ &\qquad- \tfrac{1}{\mu_0}\int \big(\epsilon_0\mu_0(1+\chi_1)\partial_t w\partial_t\Phi^\theta + \epsilon_0\mu_0\chi_3 N(\partial_t w)\partial_t\Phi^\theta\big)\, r\mathrm{d}(r,\theta,z,t)\\ &\quad= \tfrac{1}{\mu_0}\int \big(\partial_r w\partial_r\Phi^\theta + \tfrac{1}{r^2}w\Phi^\theta + V(r)\partial_t w\partial_t\Phi^\theta + \Gamma(r)N(w_t)\partial_t\Phi^\theta\big)\, r\mathrm{d}(r,\theta,z,t)\\ &\quad= \tfrac{1}{\mu_0}\int_{[0,\infty)\times\mathbb{T}} \big(w_r\varphi_r + \tfrac{1}{r^2}w\varphi + V(r)w_t\varphi_t + \Gamma(r)N(w_t)\varphi_t\big)\, r\mathrm{d}(r,t) = 0.\end{aligned}$$

where in the last line $w = w(r,t)$ is no longer in traveling coordinates, φ is given by

$$\varphi(r,t) := T\sum_{k\in\mathbb{Z}} \int_{[0,2\pi]\times\mathbb{R}} \Phi^\theta(r,\theta,z,t+kT+\tfrac{1}{c}z)\,\mathrm{d}(\theta,z),$$

and the last equality holds due to Definition 5.5.1. To show finiteness of the energy, using

$$\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H} = \epsilon_0(1+\chi_1)w_t^2 + \varepsilon\chi_3 N(w_t)w_t + \tfrac{1}{\mu_0}(\tfrac{1}{r}w + w_r)^2 + \tfrac{1}{c^2\mu_0}w_t^2$$

we calculate

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{R}\times[z_0,z_0+1]}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H})\,\mathrm{d}(x,y,z)\\&=\tfrac{2\pi c}{\mu_0}\int_{[0,\infty)\times[t_0-(z_0+1)/c,t_0-z_0/c]}\Big((-V(r)+\tfrac{2}{c^2})w_t^2-\Gamma(r)N(w_t)w_t+(\tfrac{1}{r}w+w_r)^2\Big)\,r\mathrm{d}(r,t),\end{aligned}$$

which is uniformly bounded w.r.t. t_0 and z_0 because V,Γ are bounded and w lies in X. □

Now that we have completed the proof of Theorem 5.2.1 it remains to show the multiplicity result of Theorem 5.2.3.

Proof of Theorem 5.2.3. Let $\mathfrak{K}$ denote the (infinite) set of numbers $k_0\in\mathbb{N}_{\text{odd}}$ for which ($\mathcal{A}$5.6) holds. For $k_0\in\mathfrak{K}$ we consider the subspace

$$Y_{N,k_0}:=\{u\in Y_N \mid u \text{ is } \tfrac{T}{2k_0}\text{-antiperiodic in time}\}\subseteq Y_N.$$

Similarly to the proof of Proposition 5.5.2 one can show that E attains a minimum value on Y_{N,k_0} and that from the minimizer, one can construct a weak solution of (5.19) using (5.26). Here we use that problem (5.19) is compatible with considering $\frac{T}{2k_0}$-antiperiodic in time functions, i.e., $N(w_t)$ is $\frac{T}{2k_0}$-antiperiodic in time if w_t has this property. The solution of (5.19) gives rise to a solution of Maxwell's equations by Proposition 5.5.4.

Repeating this for all $k_0\in\mathfrak{K}$, we obtain a family $\{(\mathbf{D}_{k_0},\mathbf{E}_{k_0},\mathbf{B}_{k_0},\mathbf{H}_{k_0},)\colon k_0\in\mathfrak{K}\}$ of solutions to Maxwell's equations. Each solution has a minimal nonzero time-period that is a divisor of $\frac{T}{k_0}$. Thus, this family has minimal periods becoming arbitrarily small and therefore infinitely many among the solutions must be mutually distinct. □

5.6. Modifications in the slab setting

Here we sketch modifications that have to be done in Sections 5.3 to 5.5 in order to prove Theorems 5.2.4 and 5.2.5. First our solution ansatz becomes

$$w(x,t)=\begin{cases}u(x,t), & |x|<R,\\ \sum_{k\in\mathbb{Z}_{\text{odd}}}\alpha_k\widetilde{\phi}_k(|x|)e_k(t), & |x|>R\end{cases}$$

where $u\in H^1_{\text{anti,even}}([-R,R]\times\mathbb{T})$ is to be determined and

$$\begin{aligned}\alpha_k&=\frac{\hat u_k'(R)}{\widetilde{\phi}_k'(R)},\quad \hat u_k(R)=0 && \text{for } k\in\mathfrak{F},\\ \alpha_k&=\frac{\hat u_k(R)}{\widetilde{\phi}_k(R)},\quad \hat u_k'(R)=\frac{\widetilde{\phi}_k'(R)}{\widetilde{\phi}_k(R)}\hat u_k(R) && \text{for } k\in\mathfrak{R}.\end{aligned}$$

We use the subscript "even" to denote functions that are even in space.

The restriction to even functions is done in order to shorten this chapter, but it is not necessary. For example, one could instead look for functions u that are odd in space, or not impose any spatial symmetry. In the latter case one need not make any symmetry assumptions on V,Γ (see assumption ($\tilde{\mathcal{A}}$5.1)) if instead one requires fundamental solutions to exist both on $[R,\infty)$ and $(-\infty,-R]$.

Going back to the problem, we (formally) obtain the boundary value problem

$$\begin{cases} -u_{xx} - V(x)u_{tt} - \Gamma(x)N(u_t)_t = 0 \text{ in } [0,R]\times\mathbb{T}, \\ \hat{u}_k'(R) = \frac{\widetilde{\phi}_k'(R)}{\widetilde{\phi}_k(R)}\hat{u}_k(R) \text{ for } k \in \mathfrak{R}, \\ \hat{u}_k(R) = 0 \text{ for } k \in \mathfrak{F} \\ u_x(0,\,\cdot\,) = 0 \end{cases}$$

for u, where the last condition comes from u being even in space. This problem has variational structure as solutions are critical points of

$$\widetilde{E}(u) = \int_{[0,R]\times\mathbb{T}} \tfrac{1}{2}u_x^2 + \tfrac{1}{2}V(x)u_t^2 + \tfrac{1}{4}\Gamma(x)N(u_t)u_t \,\mathrm{d}(x,t) - \tfrac{1}{2}\sum_{k\in\mathfrak{R}} \frac{\widetilde{\phi}_k'(R)}{\widetilde{\phi}_k(R)}|\hat{u}_k(R)|^2$$

subject to the constraints $\hat{u}_k(R) = 0$ for $k \in \mathfrak{F}$. We can proceed like in Sections 5.3 to 5.5 in order to prove existence, regularity, and multiplicity of some minimizers of $\widetilde{E}$. The main differences to the radial setting are the following: First, we do not work in radially weighted Sobolev spaces, so $r\mathrm{d}r$ is replaced by $\mathrm{d}x$ and L^p_{rad} by L^p. Further, the radial Laplacian $\partial_r^2 + \frac{1}{r}\partial_r$ is replaced by the 1d Laplacian ∂_x^2. In addition, the term $\frac{1}{r^2}w$ is absent in problem (5.17), so that this term (and related terms, e.g., $\frac{1}{r}u$ in E and part 3 of Proposition 5.4.7) do not appear in the slab setting.

So we define $\|\|\|_N^{\sim}$ and $\widetilde{Q}_N$ like $\|\|\|_N$ and Q_N but without the radial weight. Notice that $\widetilde{E}$ is well-defined on the reflexive Banach space

$$\widetilde{Y}_N \coloneqq \Big\{u \in H^1_{\mathrm{anti,even}}([-R,R]\times\mathbb{T})\colon \|u_t\|_N^{\sim} < \infty, \hat{u}_k(R) = 0 \text{ for } k \in \mathfrak{F}\Big\}.$$

More noticeable changes have to be made in the proof of Proposition 5.3.4. There we made the ansatz

$$u(r,t) = \varepsilon I_1(\lambda k_0 r)(e_{k_0}(t) + e_{-k_0}(t))$$

in order to show that $\inf E < 0$, and I_1 was a solution of

$$(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} + 1)I_1 = 0.$$

For the slab setting the natural ansatz is

$$u(x,t) = \varepsilon \cosh(\lambda k_0 x)(e_{k_0}(t) + e_{-k_0}(t))$$

since $(-\partial_x^2 + 1)\cosh = 0$, which also explains the way we formulated assumption ($\tilde{\mathcal{A}}$5.6).

We note that the trace embeddings can be adapted to the slab setting, i.e., the trace map $\mathrm{tr}\colon \widetilde{Y}_N \to H^{1/2}(\mathbb{T}), v \mapsto v(R,\,\cdot\,)$ is compact and the estimates appearing in Lemmas 5.A.5 and 5.A.7 also hold with $L^p_{\mathrm{rad}}, \|\|\|_N, Q_N$ replaced by $L^p, \|\|\|_N^{\sim}, \widetilde{Q}_N$. This is because the trace of v only depends on the function v in a small neighborhood of $x = R$, and the radial weight is not singular at $x = R$.

Lastly, the electromagnetic waves reconstructed from the profile w for the slab geometry are given by

$$\begin{aligned} \mathbf{D}(\mathbf{x},t) &= \epsilon_0\Big((1+\chi_1(\mathbf{x}))w_t(x,t-\tfrac{1}{c}y) + \chi_3(\mathbf{x})N(w_t)(x,t-\tfrac{1}{c}y)\Big)\cdot(0,0,1)^\top, \\ \mathbf{E}(\mathbf{x},t) &= w_t(r,t-\tfrac{1}{c}y)\cdot(0,0,1)^\top, \\ \mathbf{B}(\mathbf{x},t) &= (\tfrac{1}{c}w_t(x,t-\tfrac{1}{c}y), w_x(x,t-\tfrac{1}{c}y), 0)^\top, \\ \mathbf{H}(\mathbf{x},t) &= \tfrac{1}{\mu_0}\mathbf{B}(\mathbf{x},t), \end{aligned}$$

which can be shown similar to Proposition 5.5.4 for the cylindrical geometry.

5.7. Further regularity estimate and bifurcation phenomenon

Checking the assumptions ($\mathcal{A}$5.1)–($\mathcal{A}$5.6) and ($\tilde{\mathcal{A}}$5.1)–($\tilde{\mathcal{A}}$5.6) one sees that they depend not directly on χ_1 but on $\chi_1^c := \chi_1 - c^{-2}$. As we show next, for every solution of Theorem 5.2.1 or Theorem 5.2.3 the $L^\infty([0,R];L^2(\mathbb{T}))$-norm of the $\mathbf{E}$-field is finite and can be bounded by a constant depending only on χ_1^c (as well as on χ_3 and κ). A possible physical interpretation of this result is described below.

Proposition 5.7.1. *Let* $\mathbf{D},\mathbf{E},\mathbf{B},\mathbf{H}$ *be a solution of Maxwell's equation as in Theorem 5.2.1 or Theorem 5.2.3. Then* $\|\mathbf{E}\|_{L^\infty([0,R];L^2(\mathbb{T}))}$ *is finite. The same holds true in the slab setting for solutions from Theorem 5.2.4 or Theorem 5.2.5.*

Proof. We focus on the radial setting and time-averaged nonlinearity. As in Section 5.5 let $\mathbf{D},\mathbf{E},\mathbf{B},\mathbf{H}$ be a solution of Maxwell's equations such that $|\mathbf{E}|^2 = w_t^2$ where w is a weak solution of (5.12) in the sense of Definition 5.5.1. We begin by formally multiplying (5.19) with $-w_{tt}$ and integrating w.r.t. t to obtain

$$\begin{aligned}0 &= \int_{\mathbb{T}} \Big(-w_{rr} - \tfrac{1}{r}w_r + \tfrac{1}{r^2}w - V(r)w_{tt} - \Gamma(r)((\kappa * w_t^2)w_t)_t\Big)(-w_{tt})\,\mathrm{d}t \\ &= \int_{\mathbb{T}} -w_{trr}w_t - \tfrac{1}{r}w_{tr}w_t + \frac{1}{r^2}w_t^2 + V(r)w_{tt}^2 + 2\Gamma(r)(\kappa * w_t w_{tt})w_t w_{tt} + \Gamma(r)(\kappa * w_t^2)w_{tt}^2\,\mathrm{d}t.\end{aligned}$$

Writing $f(r) := \frac{1}{2}\int_{\mathbb{T}} w_t^2\,\mathrm{d}t$, we have

$$0 = -f'' - \tfrac{1}{r}f' + \left[\int_{\mathbb{T}} \tfrac{1}{r^2}w_t^2 + w_{tr}^2 + V(r)w_{tt}^2\,\mathrm{d}t + \Gamma(r)J''(w_t)[w_{tt},w_{tt}]\right]$$

where $J(v) := \frac{1}{4}\int_{\mathbb{T}}(\kappa * v^2)v^2\,\mathrm{d}t$ is convex by assumption (5.15) and therefore all terms in the square bracket are non-negative. This combined with $f(0) = 0, f(R) \geq 0$ shows that f is increasing on $[0,R]$. Thus $\|w_t\|_{L^2(\mathbb{T})}$ is bounded on $[0,R]$ by $\|w_t(R,\cdot)\|_{L^2(\mathbb{T})}$, which is finite by Proposition 5.4.8.

To justify this formal calculation, we argue as in Section 5.4: since $w|_{[0,R]\times\mathbb{T}}$ was obtained as the limit in Y_N of a sequence $u^{K,\star}$ defined in Lemma 5.4.3, we set $f^K(r) := \frac{1}{2}\int_{\mathbb{T}}(u_t^{K,\star})^2\,\mathrm{d}t$ and get that $f^K \to f$ in $L^2_{\mathrm{rad}}([0,R])$. Since $u^{K,\star} \in Y_N^K$ and time-derivatives are bounded on Y_N^K, we have $(f^K)', \frac{1}{r}f^K \in L^1([0,R])$ so that f^K is continuous and it indeed satisfies $f^K(0) = 0$. The formal argument above can therefore be applied to f^K and yields that f^K is monotone increasing on $[0,R]$. Thus f is monotone increasing and hence bounded by the constant $\frac{1}{2}C_5$ from Proposition 5.4.8, completing the proof.

The proof for the slab setting is similar; the main difference is that at zero we have a Neumann condition $w_x(0,t) = 0$ instead of a Dirichlet condition. The proof with the instantaneous nonlinearity follows by setting $\kappa = \delta_0$ above. □

Recall the constitutive relation

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}(\mathbf{E}) = \epsilon_0(1+\chi_1(\mathbf{x}))\mathbf{E} + \epsilon_0\chi_3(\mathbf{x})(\kappa * |\mathbf{E}|^2)\mathbf{E}$$

for the time-averaged nonlinearity. The quantity

$$\epsilon_0(1+\chi_1(\mathbf{x})) + \epsilon_0\chi_3(\mathbf{x})(\kappa * |\mathbf{E}|^2)$$

may be called the effective permittivity and can be estimated from below by

$$\begin{aligned}&\epsilon_0(1+\chi_1(\mathbf{x}))+\epsilon_0\chi_3(\mathbf{x})(\kappa * |\mathbf{E}|^2)\\&\quad\geq \epsilon_0(1+c^{-2}-\|\chi_1^c\|_{L^\infty(\mathbb{R}^3)})-\epsilon_0\|\chi_3\|_{L^\infty(\mathbb{R}^3)}\|\kappa\|_{L^\infty(\mathbb{T})}\|\mathbf{E}\|^2_{L^\infty([0,R];L^2(\mathbb{T}))}.\end{aligned}$$

As described above, the existence of $\mathbf{E}$ hinges on $\chi_1^c=\chi_1-c^{-2}$ and the norm $\|\mathbf{E}\|^2_{L^\infty([0,R];L^2(\mathbb{T}))}$ only depends on χ_1^c and not on χ_1. Hence, if $c>0$ is sufficiently small then the effective permittivity is positive, which gives the waveguide the character of a dielectric. In other words, for time-averaged nonlinearities and for sufficiently small propagation speed $c>0$, the fields are not strong enough to change the dielectric character of the waveguide. It is open if the same holds for instantaneous nonlinearities.

Finally, we comment on the bifurcation phenomenon outlined in Section 5.1.2 in the context of the cylindrical geometry. We consider $V_d(r)=-(\tilde{\chi}_{1,d}(r)+1-c^{-2})$, $\Gamma=-\tilde{\chi}_3(r)$, where the material parameters $\tilde{\chi}_{1,d}$, $\tilde{\chi}_3$ are as in Section 5.1.2 and where we emphasize the d-dependence of $\tilde{\chi}_1$ and V by adding a lower index d. In fact, d will be seen as a bifurcation parameter. Due to the ansatz $\mathbf{E}(\mathbf{x},t)=w_t(r,t-\frac{1}{c}z)\cdot(-\frac{y}{r},\frac{x}{r},0)^\top$ and the fact that $u(\cdot,t)=w(\cdot,t)\mid_{[0,R]}$ solves the boundary value problem (5.22), the bifurcation phenomenon can be explained on the level of u as a solution of the d-dependent boundary value problem (5.22) on $[0,R]\times\mathbb{T}$. Recall that on $[0,R]$ the function $V_d(r)=-(d+1-c^{-2})$ is just a positive constant.

Let us first fix a value d^* as in Theorem 5.1.2 so that assumptions ($\mathcal{A}$5.1)–($\mathcal{A}$5.6) hold. Then we consider the linear eigenvalue problem

$$\begin{cases}-u_{rr}-\frac{1}{r}u_r+\frac{1}{r^2}u+\underbrace{(d^*+1-c^{-2})}_{-V_{d^*}(r)}u_{tt}=\lambda u_{tt} \text{ in } [0,R]\times\mathbb{T},\\ \hat{u}_k'(R)=\frac{\phi_k'(R)}{\phi_k(R)}\hat{u}_k(R) \text{ for } k\in\mathfrak{R},\\ \hat{u}_k(R)=0 \text{ for } k\in\mathfrak{F}.\end{cases}\tag{5.29}$$

The smallest eigenvalue λ can be obtained by minimizing

$$E_{d^*,\mathrm{lin}}(u)=\int_{[0,R]\times\mathbb{T}}\left(u_r^2+\left(\tfrac{1}{r}u\right)^2+V_{d^*}(r)u_t^2\right)r\,\mathrm{d}(r,t)-2E_B(u)$$

subject the constraint

$$\int_{[0,R]\times\mathbb{T}}u_t^2\,r\,\mathrm{d}(r,t)=1$$

on the space

$$Y_{\mathrm{lin}}=\left\{u\in W^{1,1}_{\mathrm{loc,anti}}((0,R]\times\mathbb{T})\mid u_r,\tfrac{1}{r}u,u_t\in L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})\right\}.$$

Since assumptions ($\mathcal{A}$5.1)–($\mathcal{A}$5.6) hold for d^*, the negative minimum $\lambda<0$ is attained. It appears as a Lagrange multiplier which coincides with the smallest eigenvalue. Moreover, the minimizer u_{lin} satisfies (5.29) so that

$$\begin{cases}-u_{\mathrm{lin},rr}-\frac{1}{r}u_{\mathrm{lin},r}+\frac{1}{r^2}u_{\mathrm{lin}}+\underbrace{(d+1-c^{-2})}_{-V_d(r)}u_{\mathrm{lin},tt}=(d-d_*)u_{\mathrm{lin},tt} \text{ in } [0,R]\times\mathbb{T},\\ \hat{u}_{\mathrm{lin},k}'(R)=\frac{\phi_k'(R)}{\phi_k(R)}\hat{u}_{\mathrm{lin},k}(R) \text{ for } k\in\mathfrak{R},\\ \hat{u}_{\mathrm{lin},k}(R)=0 \text{ for } k\in\mathfrak{F}.\end{cases}$$

where we have set $d_* = d^* + \lambda$. In particular, for the bifurcation parameter $d \in (d_*, d^*]$ we find that

$$E_{d,\mathrm{lin}}(u_{\mathrm{lin}}) = \int_{[0,R]\times\mathbb{T}} \left(\tfrac{1}{2}u_{\mathrm{lin},r}^2 + \tfrac{1}{2}\left(\tfrac{1}{r}u_{\mathrm{lin}}\right)^2 + \tfrac{1}{2}V_d(r)u_{\mathrm{lin},t}^2\right) r\,\mathrm{d}(r,t) - E_B(u_{\mathrm{lin}}) < 0.$$

Hence, for a sufficiently small multiple $\varepsilon > 0$ we can insert $\varepsilon u_{\mathrm{lin}}$ into the functional E_d for the nonlinear problem and get $E_d(\varepsilon u_{\mathrm{lin}}) < 0$. This shows that $E_d^\star = \inf_{Y_N} E_d < 0$ and it is therefore the substitute for ($\mathcal{A}5.6$) which we do not verify for $d \in (d_*, d^*)$. Since ($\mathcal{A}5.1$)–($\mathcal{A}5.5$) continue to hold for all $d \in (d_*, d^*]$ we conclude that the nonlinear problem

$$\begin{cases} -u_{rr} - \frac{1}{r}u_r + \frac{1}{r^2}u - V_d(r)u_{tt} - \Gamma(r)N(u_t)_t = 0 \text{ in } [0,R]\times\mathbb{T}, \\ \hat{u}_k'(R) = \frac{\phi_k'(R)}{\phi_k(R)}\hat{u}_k(R) \text{ for } k \in \mathfrak{R}, \\ \hat{u}_k(R) = 0 \text{ for } k \in \mathfrak{F}. \end{cases}$$

has a nontrivial ground state u^d. Let us now show that indeed $u^d \to 0$ in suitable norms as $d \to d_*$, which shows bifurcation from the zero-solution at $d = d_*$ and continuation of solutions as d runs from d_* up to the primarily chosen value d^*.

Lemma 5.7.2. *For $d \in [d_*, d^*]$ any minimizer u^d of E_d satisfies*

$$\left\|u^d\right\|_{Y_N} = \mathcal{O}((d - d_*)^{1/4})$$

as $d \searrow d_*$.

Proof. We first show that $\left\|u_t^d\right\|_N$ is uniformly bounded for $d \in [d_*, d^*]$. As in the proof of Proposition 5.3.3 we find

$$\begin{aligned} E_d(u) &= \int_{[0,R]\times\mathbb{T}} \left(\tfrac{1}{2}u_r^2 + \tfrac{1}{2}\left(\tfrac{1}{r}u\right)^2 + \tfrac{1}{2}V_d(r)u_t^2 + \tfrac{1}{4}\Gamma(r)N(u_t)u_t\right) r\,\mathrm{d}(r,t) - E_B(u) \\ &\geq \int_{[0,R]\times\mathbb{T}} \left(\tfrac{1}{2}u_r^2 + \tfrac{1}{4}\Gamma(r)N(u_t)u_t\right) r\,\mathrm{d}(r,t) - C_0\|u(R,\cdot)\|_{H^{1/2}(\mathbb{T})}^2 \\ &\geq \int_{[0,R]\times\mathbb{T}} \left(\tfrac{1}{2}u_r^2 + \tfrac{1}{4}\Gamma(r)N(u_t)u_t\right) r\,\mathrm{d}(r,t) - \frac{1}{4}\|u_r\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})}^2 - C_0C(\tfrac{1}{4C_0})\|u_t\|_N^2 \\ &\gtrsim \|u_t\|_N^2(\|u_t\|_N^2 - C_0C(\tfrac{1}{4C_0})). \end{aligned}$$

If we insert u^d and use $E_d^\star = E_d(u^d) \leq 0$ the claim on the uniform boundedness of $\left\|u_t^d\right\|_N$ follows.

Next we claim that $E_d^\star = \mathcal{O}(d - d_*)$. To see this, we find

$$\begin{aligned} E_d(u) \geq E_{d,\mathrm{lin}}(u) &= \int_{[0,R]\times\mathbb{T}} \left(\tfrac{1}{2}u_r^2 + \tfrac{1}{2}\left(\tfrac{1}{r}u\right)^2 + \tfrac{1}{2}V_d(r)\right)u_t^2\, r\,\mathrm{d}(r,t) - E_B(u) \\ &\geq (d_* - d)\int_{[0,R]\times\mathbb{T}} u_t^2\, r\,\mathrm{d}(r,t) \\ &\gtrsim (d_* - d)\|u_t\|_N^2 \end{aligned}$$

by Remark 5.3.2. The claim follows by inserting $u = u^d$.

Now we can use the equality

$$0 > E_d^\star = E_d(u^d) - \frac{1}{2}E_d'(u^d)[u^d] = -\tfrac{1}{4}\int_{[0,R]\times\mathbb{T}} \Gamma(r)N(u_t^d)u_t^d\, r\,\mathrm{d}(r,t)$$

and the previous step to conclude $\left\|u_t^d\right\|_N^4 = \mathcal{O}(E_d^*) = \mathcal{O}(d-d_*)$. Finally,

$$\begin{aligned}
0 &= E_d'(u^d)[u^d] \\
&= \int_{[0,R]\times\mathbb{T}} \Big((u_r^d)^2 + \left(\tfrac{1}{r}u^d\right)^2 + V_d(r)(u_t^d)^2 + \Gamma(r)N(u_t^d)u_t^d \Big)\, r\mathrm{d}(r,t) - 2E_B(u^d) \\
&\geq \tfrac{1}{2}\left\|u_r^d\right\|^2_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} + \left\|\tfrac{u^d}{r}\right\|^2_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \\
&\quad + (c^{-2}-1-d^*)\left\|u_t^d\right\|^2_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} - 2C_0C(\tfrac{1}{4C_0})\left\|u_t^d\right\|^2_N
\end{aligned}$$

implies that

$$\left\|u_r^d\right\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})}, \left\|\tfrac{u^d}{r}\right\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})}, \left\|u_t^d\right\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} = \mathcal{O}(\left\|u_t^d\right\|_N) = \mathcal{O}((d-d_*)^{1/4})$$

as claimed. □

5.A. Appendix: Properties of the nonlinearities

For the instantaneous nonlinearity, it is clear that the function E_N is convex. In the time-averaged case this follows from assumption (5.15) together with $\Gamma \geq 0$. Next we discuss two conditions that are sufficient for convexity in the time-averaged setting, i.e., (5.15).

Lemma 5.A.1. *The convexity assumption of* (5.15) *on κ is satisfied for example if the other assumptions hold and either* $\max\kappa \leq 2\min\kappa$ *or* $\hat{\kappa}_k \geq 0$ *for all $k\in\mathbb{Z}$, or more generally if κ is a sum of functions satisfying these conditions.*

Proof. Set $f\colon L^4(\mathbb{T})\to\mathbb{R}, f(v) = ⨍_{\mathbb{T}}(\kappa * v^2)v^2$. Then using that κ is even we calculate

$$f''(v)[u,u] = 4\int_{\mathbb{T}}(\kappa * v^2)u^2\,\mathrm{d}t + 8\int_{\mathbb{T}}(\kappa * uv)uv\,\mathrm{d}t.$$

Part 1: If $\max\kappa \leq 2\min\kappa$, with $c := (\min\kappa + \max\kappa)/2$ we can estimate

$$\begin{aligned}
f''(v)[u,u] &= 4\int_{\mathbb{T}}(\kappa * v^2)u^2\,\mathrm{d}t + 8c\Big(\int_{\mathbb{T}} uv\,\mathrm{d}t\Big)^2 + 8\int_{\mathbb{T}}((\kappa-c)*uv)uv\,\mathrm{d}t \\
&\geq 4\min\kappa\|uv\|_2^2 - 8\|\kappa-c\|_\infty\|uv\|_2^2 \geq 0.
\end{aligned}$$

Part 2: If instead $\hat{\kappa}_k \geq 0$, we can estimate

$$f''(v)[u,u] \geq 8\int_{\mathbb{T}}(\kappa * uv)uv\,\mathrm{d}t = 8\sum_{k\in\mathbb{Z}}\hat{\kappa}_k|\mathcal{F}_k(uv)|^2 \geq 0.$$ □

Next we aim at lower bounds for $E'(u)[|\partial_t|u]$. Using integration by parts, one sees that the quadratic terms appearing in $E'(u)[|\partial_t|u]$ are L^2-norms of suitable fractional derivatives of u. In the next two lemmas, we investigate the remaining non-quadratic term $⨍ N(u_t)|\partial_t|u_t$. We begin with the instantaneous nonlinearity.

Remark 5.A.2. Let us give a few examples of kernels $\tilde{\kappa}$ describing the nonlinear polarization (cf. (5.4)) that lead via $\kappa(t) = T\sum_{k\in\mathbb{Z}}\tilde{\kappa}(t+kT)$ to admissible potentials κ for (5.15).

(a) First, we consider

$$\tilde{\kappa}(t) = \begin{cases} 0, & t < 0, \\ (T^4 + 4t^4)^{-1} t, & t \geq 0 \end{cases}$$

where $c > 0$. Let us show that the resulting κ is admissible. To do this, we write

$$\tilde{\kappa}(t) = \frac{1}{2T}\left(\frac{1}{T^2 + (2t-T)^2} - \frac{1}{T^2 + (2t+T)^2}\right) \quad \text{for } t \geq 0,$$

so $\kappa(t)$ is a telescoping series with value

$$\kappa(t) = \frac{1}{2T(T^2 + (2t-T)^2)} \quad \text{for } t \in [0, T).$$

We see that κ is even about $\frac{T}{2}$, and by periodicity also even about 0, and that $\min \kappa = \kappa(0) = \frac{1}{4T^3}, \max \kappa = \kappa(\frac{T}{2}) = \frac{1}{2T^3}$ hold. Since κ is Lipschitz continuous, this combined with Lemma 5.A.1 show that κ satisfies (5.15).

(b) More generally, $\tilde{\kappa}(t) = \mathbb{1}_{t \geq 0}[g(2t - T) - g(2t + T)]$ with even, Hölder-continuous $g \colon \mathbb{R} \to \mathbb{R}$ is an admissible example if $\max_{[0,T]} g \leq 2 \min_{[0,T]} g$ holds.

(c) Let us give another example: Consider

$$\tilde{\kappa}(t) = \sum_{n \in \mathbb{N}_0} \alpha_n \mathbb{1}_{[nT,(n+1)T)}(t)$$

where $(\alpha_n) \in \ell^1$ with $\sum_{n \in \mathbb{N}_0} \alpha_n = \frac{1}{T}$. Then $\kappa \equiv 1$ and therefore it satisfies (5.15).

(d) Finally, using a Debye-type exponential decay in the kernel function cf. [28], let us consider $\tilde{\kappa}(t) = \alpha \mathrm{e}^{-\beta t} \mathbb{1}_{t \geq 0}$ and its discretized version $\tilde{\kappa}_d(t) = \alpha \sum_{n=0}^{\infty} \mathrm{e}^{-\beta n T} \mathbb{1}_{[nT,(n+1)T)}$ with $\alpha, \beta > 0$. Subject to the choice $\alpha = (1 - \mathrm{e}^{-\beta T})/T$ the discretized version clearly falls into the category (c) whereas for the continuous version we get $\kappa(t) = \mathrm{e}^{-\beta t}$ for $t \in [0, T)$ so that κ is neither even nor continuous on $\mathbb{T}$ and hence does not satisfy (5.15). Therefore our results do not apply to $\tilde{\kappa}$, but can be used for $\tilde{\kappa}_d$. Clearly, the smaller $T > 0$ the better $\tilde{\kappa}_d$ approximates $\tilde{\kappa}$, and our results provide existence of breathers with frequencies tending to infinity as $T \searrow 0$. This, however, does not allow for any conclusion about breathers for nonlinear Maxwell equations with Debye-type exponential decay kernel.

Lemma 5.A.3. *The inequality*

$$2 \int_{\mathbb{T}} v^3 \cdot |\partial_t| v \, \mathrm{d}t \geq \int_{\mathbb{T}} \left(|\partial_t|^{1/2} (v|v|)\right)^2 \mathrm{d}t \tag{5.30}$$

holds for all $v \in C^\infty(\mathbb{T})$.

Proof. We first encountered an estimate similar to (5.30) in [26, Proposition 2.3], and we prove (5.30) in a similar fashion. Note that $v|v| \in C^{1,1}(\mathbb{T}) \subseteq H^{1/2}(\mathbb{T})$. Thus both sides of (5.30) are well-defined and we may use symmetry to obtain

$$\int_{\mathbb{T}} \left(|\partial_t|^{1/2} (v|v|)\right)^2 \mathrm{d}t = \int_{\mathbb{T}} v|v| \cdot |\partial_t| (v|v|) \, \mathrm{d}t.$$

Using the representation of Lemma B.2.8, we calculate

$$2 \int_{\mathbb{T}} v^3 \cdot |\partial_t| v \, \mathrm{d}t - \int_{\mathbb{T}} v|v| \cdot |\partial_t| (v|v|) \, \mathrm{d}t$$

$$\begin{aligned}
&= C \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{2}{h^2} v(t)^3 (2v(t) - v(t+h) - v(t-h)) \\
&\qquad\qquad - \frac{1}{h^2} v(t)|v(t)| (2v(t)|v(t)| - v(t+h)|v(t+h)| - v(t-h)|v(t-h)|) \, \mathrm{d}h \mathrm{d}t \\
&= C \int_{\mathbb{R}} \frac{1}{h^2} \int_{\mathbb{T}} 2v(t)^3 \big(v(t) - v(t+h)\big) - v(t)|v(t)| \big(v(t)|v(t)| - v(t+h)|v(t+h)|\big) \\
&\qquad\qquad + 2v(t)^3 \big(v(t) - v(t-h)\big) - v(t)|v(t)| \big(v(t)|v(t)| - v(t-h)|v(t-h)|\big) \, \mathrm{d}t \mathrm{d}h \\
&= C \int_{\mathbb{R}} \frac{1}{h^2} \int_{\mathbb{T}} 2v(t)^3 \big(v(t) - v(t+h)\big) - v(t)|v(t)| \big(v(t)|v(t)| - v(t+h)|v(t+h)|\big) \\
&\qquad\qquad + 2v(t+h)^3 \big(v(t+h) - v(t)\big) - v(t+h)|v(t+h)| \big(v(t+h)|v(t+h)| - v(t)|v(t)|\big) \, \mathrm{d}t \mathrm{d}h \\
&= C \int_{\mathbb{R}} \frac{1}{h^2} \int_{\mathbb{T}} v(t)^4 + v(t+h)^4 - 2v(t)^3 v(t+h) - 2v(t) v(t+h)^3 + 2v(t)|v(t)| v(t+h)|v(t+h)| \, \mathrm{d}t \mathrm{d}h
\end{aligned}$$

Next we claim that the last integrand is everywhere non-negative. To see this, abbreviate $a := v(t)$, $b := v(t+h)$. If a and b have the same sign, we find

$$a^4 + b^4 - 2a^3 b - 2ab^3 + 2a|a|b|b| = \Big(a^2 + b^2\Big)(a-b)^2 \geq 0.$$

If a and b have opposite signs, we instead calculate

$$a^4 + b^4 - 2a^3 b - 2ab^3 + 2a|a|b|b| = \Big(a^2 - b^2\Big)^2 - 2ab(a^2 + b^2) \geq 0.$$

This completes the proof. □

The counterpart for the temporally averaged nonlinearity reads as follows. Its proof is very different from the proof of the previous lemma.

Lemma 5.A.4. *There exists constants $c_1, C_2 > 0$ such that*

$$\int_{\mathbb{T}} (\kappa * v^2) v |\partial_t| v \, \mathrm{d}t \geq c_1 \|v\|^2_{L^2(\mathbb{T})} \||\partial_t| v\|^2_{L^2(\mathbb{T})} - C_2 \|v\|^4_{L^2(\mathbb{T})}$$

holds for all $v \in C^\infty(\mathbb{T})$.

Proof. By 5.15, $\kappa \in C^\alpha(\mathbb{T})$. Inspired by the famous Kenig-Ponce-Vega inequality [48], we define the Leibniz-defect for the fractional half-derivative as

$$\delta = |\partial_t|^{1/2} ((\kappa * v^2) v) - v |\partial_t|^{1/2} (\kappa * v^2) - (\kappa * v^2) |\partial_t|^{1/2} v.$$

Using Lemma B.2.4 and Lemma B.2.10 we estimate

$$\|\delta\|_2 \lesssim \Big[\kappa * v^2\Big]_{C^\alpha} [v]_{H^{1/2 - \alpha/2}} \leq [\kappa]_{C^\alpha} \|v\|_2^2 [v]_{H^{1/2-\alpha/2}} \lesssim [\kappa]_{C^\alpha} \|v\|_2^{2+\alpha} \|v\|^{1-\alpha}_{H^{1/2}}.$$

We further have

$$|\partial_t|^{1/2} (\kappa * v^2) = (|\partial_t|^{\alpha/2} \kappa) * (|\partial_t|^{1/2 - \alpha/2} v^2) = |\partial_t|^{\alpha/2} \kappa * (2v |\partial_t|^{1/2-\alpha/2} v + \tilde{\delta})$$

with Leibniz-defect $\tilde{\delta}$ given by

$$\tilde{\delta} = |\partial_t|^{1/2-\alpha/2} (v^2) - 2v |\partial_t|^{1/2-\alpha/2} v.$$

By applying Lemma B.2.10, Lemma B.2.5, and Lemma B.2.4 for p close to 1 we obtain the estimate

$$\|\tilde{\delta}\|_p \lesssim [v]^2_{W^{2p,1/4-\alpha/6}} \lesssim \|v\|^2_{H^{1/4-\alpha/8}} \lesssim \|v\|_2^{1+\alpha/2}\|v\|_{H^{1/2}}^{1-\alpha/2},$$

so that

$$\begin{aligned}\||\partial_t|^{1/2}(\kappa * v^2)\|_\infty &\le 2\||\partial_t|^{\alpha/2}\kappa\|_\infty\|v\|_2\||\partial_t|^{1/2-\alpha/2}v\|_2 + \||\partial_t|^{\alpha/2}\kappa\|_{p'}\|\tilde{\delta}\|_p \\ &\lesssim \|\kappa\|_{C^\alpha}\Big(\|v\|_2^{1+\alpha}\|v\|_{H^{1/2}}^{1-\alpha} + \|v\|_2^{1+\alpha/2}\|v\|_{H^{1/2}}^{1-\alpha/2}\Big)\end{aligned}$$

where we have used [78, Theorem 2.6] for the estimates on $|\partial_t|^{\alpha/2}\kappa$.

Next we estimate the quantity appearing in the claim:

$$\begin{aligned}\int_{\mathbb{T}}(\kappa * v^2)v\cdot|\partial_t|v\,\mathrm{d}t &= \int_{\mathbb{T}}|\partial_t|^{1/2}((\kappa * v^2)v)\cdot|\partial_t|^{1/2}v\,\mathrm{d}t \\ &= \int_{\mathbb{T}}\Big((\kappa * v^2)|\partial_t|^{1/2}v + v|\partial_t|^{1/2}(\kappa * v^2) + \delta\Big)\cdot|\partial_t|^{1/2}v\,\mathrm{d}t \\ &\ge \int_{\mathbb{T}}(\kappa * v^2)(|\partial_t|^{1/2}v)^2\,\mathrm{d}t \\ &\quad - C\|v\|_2\|\kappa\|_{C^\alpha}\Big(\|v\|_2^{1+\alpha}\|v\|_{H^{1/2}}^{1-\alpha} + \|v\|_2^{1+\alpha/2}\|v\|_{H^{1/2}}^{1-\alpha/2}\Big)\Big\||\partial_t|^{1/2}v\Big\|_2 \\ &\quad - C[\kappa]_{C^\alpha}\|v\|_2^{2+\alpha}\|v\|_{H^{1/2}}^{1-\alpha}\Big\||\partial_t|^{1/2}v\Big\|_2.\end{aligned}$$

The claim now follows using

$$\int_{\mathbb{T}}(\kappa * v^2)(|\partial_t|^{1/2}v)^2\,\mathrm{d}t \ge \min\kappa\cdot\|v\|_2^2\||\partial_t|^{1/2}v\|_2^2$$

and Young's inequality for products. □

Next we prove two important trace inequalities that are adapted to the terms appearing in our functional. In Lemma 5.A.5 we estimate the trace in $H^{1/2}(\mathbb{T})$ against $\|\|_N$, and the "regularized" embedding Lemma 5.A.7 estimates the trace in $H^1(\mathbb{T})$ against Q_N.

Lemma 5.A.5. *The trace map*

$$\mathrm{tr}\colon Y_N \to H^{1/2}(\mathbb{T}), u \mapsto u(R,\,\cdot\,)$$

is well-defined and compact. Furthermore, for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|\mathrm{tr}\,u\|^2_{H^{1/2}(\mathbb{T})} \le \varepsilon\|u_r\|^2_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} + C(\varepsilon)\|u_t\|_N^2. \tag{5.31}$$

holds for all $u \in Y_N$.

Remark 5.A.6. By Remark 5.3.2 we have the continuous embedding $\iota\colon Y_N \hookrightarrow H^1_{\mathrm{rad}}([0,R]\times\mathbb{T})$, and it is well known that the trace maps $H^1([0,R]\times\mathbb{T})$ into $H^{1/2}(\mathbb{T})$, and the same holds for $H^1_{\mathrm{rad}}([0,R]\times\mathbb{T})$. However, both the embedding ι and the trace map $\mathrm{tr}\colon H_{\mathrm{rad}}([0,R]\times\mathbb{T}) \to H^{1/2}(\mathbb{T})$ are noncompact maps. Their composition $\mathrm{tr}\circ\iota$ however is compact, as we show below. This is true because of the temporal decay in the embedding ι.

Proof of Lemma 5.A.5. Since $\|\|_{N_{\mathrm{av}}} \lesssim \|\|_{N_{\mathrm{ins}}}$ by Remark 5.3.2 and thus $Y_{N_{\mathrm{ins}}} \hookrightarrow Y_{N_{\mathrm{av}}}$, if suffices to consider the case $N = N_{\mathrm{av}}$.

Let $u \in H^2_{\mathrm{anti}}([0,R]\times\mathbb{T})$. Fix some $\psi \in C^\infty([0,R])$ with $\psi = 0$ on $[0, \frac{1}{2}R]$ and $\psi(R) = 1$. With $v(r,t) \coloneqq \psi(r)u(r,t)$ we calculate

$$\begin{aligned}
\|\mathrm{tr}\, u\|^3_{H^{1/2}(\mathbb{T})} &= \|\mathrm{tr}\, v\|^3_{H^{1/2}(\mathbb{T})} \le C_0 \||\partial_t|^{1/2}\,\mathrm{tr}\, v\|^3_{L^2(\mathbb{T})} \\
&= 3C_0 \int_0^R \Big(\||\partial_t|^{1/2} v\|_{L^2(\mathbb{T})} \int_{\mathbb{T}} |\partial_t|^{1/2} v \cdot |\partial_t|^{1/2} v_r \,\mathrm{d}t \Big)\,\mathrm{d}r \\
&= 3C_0 \int_0^R \Big(\||\partial_t|^{1/2} v\|_{L^2(\mathbb{T})} \int_{\mathbb{T}} |\partial_t| v \cdot v_r \,\mathrm{d}t \Big)\,\mathrm{d}r \\
&\le 3C_0 \Big(\int_{[0,R]\times\mathbb{T}} v_r^2 \, r\mathrm{d}(r,t) \cdot \int_0^R \||\partial_t|^{1/2} v\|^2_{L^2(\mathbb{T})} \||\partial_t| v\|^2_{L^2(\mathbb{T})}\, r\mathrm{d}r \Big)^{1/2}
\end{aligned}$$

where the factor r can be introduced since v is supported on $[\frac{1}{2}R, R]\times\mathbb{T}$. Using Lemma B.2.11, Remark 5.3.2 and $\inf_{\mathbb{T}} \kappa > 0$, $\inf_{[0,R]} \Gamma > 0$ we continue the estimate

$$\begin{aligned}
\|\mathrm{tr}\, u\|^3_{H^{1/2}(\mathbb{T})} &\le C_1 \|v_r\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \|v_t\|^2_{N_{\mathrm{av}}} \\
&\le C_2 \Big(\|u_t\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} + \|u_r\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \Big) \|u_t\|^2_{N_{\mathrm{av}}} \\
&\le C_3 \Big(\|u_t\|_{N_{\mathrm{av}}} + \|u_r\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \Big) \|u_t\|^2_{N_{\mathrm{av}}} \\
&\le C_3 \|u\|^3_{Y_N}.
\end{aligned}$$

By approximation, the inequality

$$\|\mathrm{tr}\, u\|^3_{H^{1/2}(\mathbb{T})} \le C_3 \Big(\|u_t\|_{N_{\mathrm{av}}} + \|u_r\|_{L^2_{\mathrm{rad}}([0,R]\times\mathbb{T})} \Big) \|u_t\|^2_{N_{\mathrm{av}}}$$

can be shown to hold for all $u \in Y_N$, so that tr is a well-defined and bounded operator on Y_N. The inequality (5.31) now follows immediately using Young's inequality for products. It remains to show compactness of the trace operator. To do this, we consider the operator

$$\mathrm{tr}^K \coloneqq \mathrm{tr} \circ S^K$$

for $K \in \mathbb{N}_{\mathrm{odd}}$, cf. Lemma 5.4.2 for a definition of the projection operators S^K. Then tr^K is a compact operator since it is bounded and has finite-dimensional range. Since $\mathcal{F}_k[u - S^K u] = 0$ for $|k| < K + 2$, using the improved estimate from Lemma B.2.11 in our calculation above, we find

$$\Big\|\mathrm{tr}(u - S^K u)\Big\|^3_{H^{1/2}(\mathbb{T})} \le \frac{C_3}{\sqrt{K+2}} \Big\|u - S^K u\Big\|^3_{Y_N},$$

so that in particular

$$\Big\|\mathrm{tr}\, u - \mathrm{tr}^K u\Big\|^3_{H^{1/2}(\mathbb{T})} = \Big\|\mathrm{tr}(u - S^K u)\Big\|^3_{H^{1/2}(\mathbb{T})} \le \frac{C_2\Big(1 + \Big\|S^K\Big\|\Big)^3}{\sqrt{K+2}} \|u\|^3_{Y_N}$$

holds. Using Lemma 5.4.2 it follows that $\mathrm{tr}^K \to \mathrm{tr}$ in $\mathcal{B}(Y_N; H^{1/2}(\mathbb{T}))$, which shows that tr is compact. □

Next we show in Lemma 5.A.7 the "regularized" trace inequality, which is the main tool used to obtain improved regularity in Section 5.4.

Lemma 5.A.7. *For all $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\|\operatorname{tr} u\|_{H^1(\mathbb{T})}^2 \leq \varepsilon \left\| |\partial_t|^{1/2} u_r \right\|_{L^2_{\text{rad}}([0,R]\times\mathbb{T})}^2 + C(\varepsilon) Q_N(u_t)^2 \tag{5.32}$$

holds for all $u \in Y_N^K$ and $K \in \mathbb{N}_{\text{odd}}$ and where $C(\varepsilon)$ does not depend on K.

Proof. Part 1: Let $N = N_{\text{ins}}$. Fix $\psi \in C^\infty([0,R])$ with $\psi = 0$ on $[0, \frac{1}{2}R]$, $\psi(R) = 1$ and set $v(r,t) \coloneqq \psi(r) u(r,t)$. Further let H denote the Hilbert transform in time, which is given by $\mathcal{F}_k H = \mathrm{i}\,\mathrm{sign}(k) \mathcal{F}_k$. Using $\partial_t = H|\partial_t|$ we calculate

$$\begin{aligned}
&\|\operatorname{tr} u\|_{H^1(\mathbb{T})}^3 \\
&\quad = \|\operatorname{tr} v\|_{H^1(\mathbb{T})}^3 \\
&\quad \lesssim \|\operatorname{tr} v_t\|_{L^2(\mathbb{T})}^3 \\
&\quad \lesssim \int_{\mathbb{T}} |v_t(R,\,\cdot\,)|^3 \,\mathrm{d}t \\
&\quad = 3 \int_{[0,R]\times\mathbb{T}} v_{tr} v_t |v_t| \,\mathrm{d}(r,t) \\
&\quad = 3 \int_{[0,R]\times\mathbb{T}} H|\partial_t|^{1/2} v_r \cdot |\partial_t|^{1/2}(v_t|v_t|) \,\mathrm{d}(r,t) \\
&\quad \leq 3 \left(\int_{[0,R]\times\mathbb{T}} \Big(H|\partial_t|^{1/2} v_r \Big)^2 \mathrm{d}(r,t) \cdot \int_{[0,R]\times\mathbb{T}} \Big(|\partial_t|^{1/2}(v_t|v_t|) \Big)^2 \mathrm{d}(r,t) \right)^{1/2} \\
&\quad = 3 \left(\int_{[0,R]\times\mathbb{T}} \Big(\psi|\partial_t|^{1/2} u_r + \psi'|\partial_t|^{1/2} u \Big)^2 \mathrm{d}(r,t) \cdot \int_{[0,R]\times\mathbb{T}} \psi^4 \Big(|\partial_t|^{1/2}(u_t|u_t|) \Big)^2 \mathrm{d}(r,t) \right)^{1/2} \\
&\quad \lesssim \left(\left\| |\partial_t|^{1/2} u_r \right\|_{L^2_{\text{rad}}} + \left\| |\partial_t|^{1/2} u \right\|_{L^2_{\text{rad}}} \right) Q_{N_{\text{ins}}}(u_t)^2
\end{aligned}$$

where in the last inequality we have estimated $\psi^2, \psi'^2, \psi^4 \lesssim r$. From Lemma B.2.11 we further have

$$\left\| |\partial_t|^{1/2} u \right\|_{L^2_{\text{rad}}} \lesssim \|u_t\|_{L^2_{\text{rad}}} \lesssim \|u_t\|_{L^4_{\text{rad}}} = \|u_t|u_t|\|_{L^2_{\text{rad}}}^{1/2} \lesssim \left\| |\partial_t|^{1/2}(u_t|u_t|) \right\|_{L^2_{\text{rad}}}^{1/2} = Q_{N_{\text{ins}}}(u_t).$$

Combining both inequalities with Young's inequality for products, the estimate (5.32) follows.

Part 2: Here we consider $N = N_{\text{av}}$. We define v as above, but now we estimate

$$\begin{aligned}
\|\operatorname{tr} u\|_{H^1(\mathbb{T})}^3 &\lesssim \|\operatorname{tr} u_t\|_{L^2(\mathbb{T})}^3 \\
&= \|\operatorname{tr} v_t\|_{L^2(\mathbb{T})}^3 \\
&= 3 \int_0^R \left(\|v_t\|_{L^2(\mathbb{T})} \int_{\mathbb{T}} v_t v_{tr} \,\mathrm{d}t \right) \mathrm{d}r \\
&= 3 \int_0^R \left(\|v_t\|_{L^2(\mathbb{T})} \int_{\mathbb{T}} |\partial_t|^{1/2} v_t \cdot H|\partial_t|^{1/2} v_r \,\mathrm{d}t \right) \mathrm{d}r \\
&\leq 3 \left(\int_{[0,R]\times\mathbb{T}} \Big(H|\partial_t|^{1/2} v_r \Big)^2 \mathrm{d}(r,t) \cdot \int_0^R \|v_t\|_{L^2(\mathbb{T})}^2 \left\| |\partial_t|^{1/2} v_t \right\|_{L^2(\mathbb{T})}^2 \mathrm{d}r \right)^{1/2} \\
&\lesssim \left(\left\| |\partial_t|^{1/2} u_r \right\|_{L^2_{\text{rad}}} + \left\| |\partial_t|^{1/2} u \right\|_{L^2_{\text{rad}}} \right) Q_{N_{\text{av}}}(u_t)^2
\end{aligned}$$

where again $\operatorname{supp}\psi \subseteq [\frac{1}{2}R, R]$ has been used. Using Lemma B.2.11 we further obtain

$$\begin{aligned}\left\| |\partial_t|^{1/2} u \right\|_{L^2_{\mathrm{rad}}} &\lesssim \|u_t\|_{L^2_{\mathrm{rad}}} \lesssim \|u_t\|_{L^4_{\mathrm{rad}}([0,R];L^2(\mathbb{T}))} = \left(\int_0^R \|u_t(r,\cdot)\|^4_{L^2(\mathbb{T})} \, r\mathrm{d}r \right)^{1/4} \\ &\leq \left(\int_0^R \|u_t(r,\cdot)\|^2_{L^2(\mathbb{T})} \left\| |\partial_t|^{1/2} u_t(r,\cdot) \right\|^2_{L^2(\mathbb{T})} r\mathrm{d}r \right)^{1/4} = Q_{N_{\mathrm{av}}}(u_t).\end{aligned}$$

Combining both estimates above with Young's inequality for products, the estimate (5.32) follows. □

5.B. Appendix: Examples

In this section we prove Theorem 5.1.2 by verifying the assumptions of Theorem 5.2.1 or Theorem 5.2.4. We prepare the proof with a lemma on convergence of infinite matrix products.

Lemma 5.B.1. *Let*

$$A_n = \begin{pmatrix} 1+\alpha_n & \beta_n \\ \gamma_n & \lambda+\delta_n \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

where $|\lambda| < 1$, and $|\alpha_n| \leq \frac{C}{n^2}$ as well as $|\beta_n|, |\gamma_n|, |\delta_n| \leq \frac{C}{n}$ hold for all $n \in \mathbb{N}$. Then the product

$$\prod_{n=1}^{\infty} A_n := \lim_{m\to\infty} (A_1 \cdot A_2 \cdot \ldots \cdot A_m)$$

converges against a matrix of the form $\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$. If all A_n are invertible, then $\prod_{n=1}^{\infty} A_n \neq 0$. Further, there exists a function $f\colon (0,\infty) \to (0,\infty)$ with $f(0+) = 0$ such that

$$\left\| \prod_{n=1}^{\infty} A_n - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq f(C).$$

Proof. First we consider the product

$$\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} := \prod_{n=N}^{m-1} A_n$$

where we choose $N \in \mathbb{N}$ so large that denominators appearing in the following four constants C_a, C_b, C_c, C_d with

$$\begin{aligned} C_a &:= 2CC_b, \\ C_b &:= \max\left\{ \frac{1}{\frac{2N}{N+1} - 1 - \frac{2C}{N}}, \frac{C}{\frac{N}{N+1} - |\lambda| - \frac{C+2C^2}{N}} \right\}, \\ C_c &:= 2CC_d, \\ C_d &:= \max\left\{ \frac{\frac{C}{N(1-|\lambda|)}}{\frac{2N}{N+1} - 1 - \frac{2C}{N}}, \frac{C + \frac{C^2}{N(1-|\lambda|)}}{\frac{N}{N+1} - |\lambda| - \frac{C+2C^2}{N}} \right\}, \end{aligned}$$

are positive and $\frac{C_a}{N} < 1$ holds. We show by induction that the following estimates hold for $m \geq N$:

$$\begin{aligned} |a_m - 1| &\leq C_a(\tfrac{1}{N} - \tfrac{1}{m}), & |b_m| &\leq \tfrac{C_b}{m}, \\ |c_m| &\leq C_c\Big(\tfrac{1}{N} - \tfrac{1}{m}\Big) + \tfrac{C}{N}\tfrac{1-|\lambda|^{m-N}}{1-|\lambda|}, & \Big|d_m - \lambda^{m-N}\Big| &\leq \tfrac{C_d}{m}. \end{aligned} \tag{5.33}$$

We will moreover show for the differences that

$$|a_{m+1} - a_m| \leq \tfrac{C_a}{m(m+1)} \qquad |c_{m+1} - c_m| \leq \tfrac{C_c}{m(m+1)} + \tfrac{C}{N}|\lambda|^{m-N} \tag{5.34}$$

holds. First, for $m = N$ the estimates (5.33) hold since $a_N = 1, b_N = 0, c_N = 0, d_N = 1$. For the induction step, let us assume that (5.33) holds for fixed $m \geq N$. Using $a_{m+1} = (1+\alpha_m)a_m + \gamma_m b_m$ as well as $|a_m| \leq C_a^+ := 1 + \frac{C_a}{N}$ we find

$$|a_{m+1} - a_m| \leq \frac{C}{m^2}C_a^+ + \frac{C}{m}\frac{C_b}{m} \leq \frac{(N+1)C}{N}\Big(C_a^+ + C_b\Big)\frac{1}{m(m+1)} \leq \frac{C_a}{m(m+1)}.$$

This in turn implies that $|a_{m+1} - 1| \leq |a_{m+1} - a_m| + |a_m - 1| \leq C_a\Big(\frac{1}{N} - \frac{1}{m+1}\Big)$. Next, from $b_{m+1} = \beta_m a_m + (\lambda + \delta_m)b_m$ we obtain

$$|b_{m+1}| \leq \frac{C}{m}C_a^+ + |\lambda|\frac{C_b}{m} + \frac{C}{m}\frac{C_b}{m} \leq \frac{N+1}{N}\bigg(CC_a^+ + |\lambda|C_b + \frac{CC_b}{N}\bigg)\frac{1}{m+1} \leq \frac{C_b}{m+1}.$$

Then we use $c_{m+1} = (1+\alpha_m)c_m + \gamma_m d_m$ as well as $|c_m| \leq C_c^+ := \frac{C_c}{N} + \frac{C}{N(1-|\lambda|)}$ to obtain

$$\begin{aligned} |c_{m+1} - c_m| &\leq \frac{C}{m^2}C_c^+ + \frac{C}{m}\Big(|\lambda|^{m-N} + \tfrac{C_d}{m}\Big) \\ &\leq \frac{(N+1)C}{N}\Big(C_c^+ + C_d\Big)\frac{1}{m(m+1)} + \frac{C}{N}|\lambda|^{m-N}, \\ &\leq \frac{C_c}{m(m+1)} + \frac{C}{N}|\lambda|^{m-N}, \end{aligned}$$

from which the desired estimate on $|c_{m+1}|$ follows as before. From $d_{m+1} = \beta_m c_m + (\lambda + \delta_m)d_m$ we obtain

$$\begin{aligned} \Big|d_{m+1} - \lambda^{m+1-N}\Big| &\leq \frac{C}{m}C_c^+ + |\lambda|\frac{C_d}{m} + \frac{C}{m}\bigg(|\lambda|^{m-N} + \frac{C_d}{m}\bigg) \\ &\leq \frac{N+1}{N}\bigg(CC_c^+ + |\lambda|C_d + C + \frac{CC_d}{N}\bigg)\frac{1}{m+1} \leq \frac{C_d}{m+1}. \end{aligned}$$

This shows the estimates (5.33), (5.34). It follows that $b_m, d_m \to 0$ and that a_m, c_m converge as $m \to \infty$. Thus we have shown that the product $\prod_{n=N}^{\infty} A_n$ converges against a matrix of the form $\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$. This implies convergence of the product $\prod_{n=1}^{\infty} A_n$ with the limit being given by

$$\prod_{n=1}^{\infty} A_n = [A_1 \cdot \ldots \cdot A_{N-1}] \cdot \begin{pmatrix} \lim_{m\to\infty} a_m & 0 \\ \lim_{m\to\infty} c_m & 0 \end{pmatrix},$$

which has the specified form of vanishing second column. From (5.33) we get $|a_m - 1| < \frac{C_a}{N} < 1$ so that $\lim_{m\to\infty} a_m \neq 0$. Thus $\prod_{n=1}^{\infty} A_n \neq 0$ if we assume that $A_1, \ldots, A_{N-1}$ are invertible.

It remains to show the estimate

$$\left\| \prod_{n=1}^{\infty} A_n - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq f(C).$$

We choose $\|\cdot\|$ to be the column sum norm. To emphasize the dependence of $C_a, \dots, C_d$ on the constant C, in the following we write $C_a(C), \dots, C_d(C)$. Let $C^\star > 0$ and observe that there exists a sufficiently large $N \in \mathbb{N}$ such that the denominators appearing in $C_a(C), \dots, C_d(C)$ are positive for all $C \in (0, C^\star]$. Then (5.33) shows that

$$\left\| \prod_{n=N}^{\infty} A_n - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq \frac{C_a(C) + C_c(C)}{N} + \frac{C}{N(1 - |\lambda|)} =: f_N(C)$$

holds, where $f_N\colon (0, C^\star] \to (0, \infty)$ with $f_N(0+) = 0$. Then

$$\begin{aligned} \left\| \prod_{n=N-1}^{\infty} A_n - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| &\leq \|A_{N-1}\| \left\| \prod_{n=N}^{\infty} A_n - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| + \left\| A_{N-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\ &\leq \max\left\{ 1 + \tfrac{NC}{(N-1)^2}, \tfrac{2C}{N-1} + |\lambda| \right\} f_N(C) + \tfrac{NC}{(N-1)^2} =: f_{N-1}(C) \end{aligned}$$

where $f_{N-1}(0+) = 0$. Repeating this $N-2$ times, we find a function f_1 such that $f_1(0+) = 0$ and

$$\left\| \prod_{n=1}^{\infty} A_n - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq f_1(C). \qquad \square$$

The proof of Theorem 5.1.2 is split into four parts, because the fundamental solutions ϕ_k strongly depend on both the chosen geometry (radial problem or slab problem) and on the linear potential $\tilde{\chi}_1^*$ (step potential $\tilde{\chi}_1^{\text{step}}$ or periodic step potential $\tilde{\chi}_1^{\text{per}}$).

For the proof, we introduce the following (non-negative) variables to denote the values of the piecewise constant potential $V = -(\tilde{\chi}_1 + 1 - c^{-2})$:

(i) If $\tilde{\chi}_1^* = \tilde{\chi}_1^{\text{per}}$, we set

$$\alpha := a + 1 - c^{-2}, \quad \beta := b + 1 - c^{-2}, \quad \delta := -(d + 1 - c^{-2}),$$

where by the assumptions of Theorem 5.1.2 we have $\alpha, \beta, \delta > 0$ and $\delta < \alpha$.

(ii) If $\tilde{\chi}_1^* = \tilde{\chi}_1^{\text{step}}$, let

$$\alpha := a + 1 - c^{-2}, \quad \beta := -(b + 1 - c^{-2}), \quad \delta := -(d + 1 - c^{-2}),$$

where again $\alpha, \beta, \delta > 0$ by assumption.

Proof of Theorem 5.1.2, Part 1. First, we consider the periodic step potential $\tilde{\chi}_1^{\text{per}}$ with cylindrical geometry, i.e., (5.12). We verify the assumptions ($\mathcal{A}$5.1)–($\mathcal{A}$5.5) and ($\mathcal{A}$5.6') in order to apply Theorem 5.2.3. Firstly, assumptions ($\mathcal{A}$5.1), ($\mathcal{A}$5.2), and ($\mathcal{A}$5.3) hold by definition.

Step 1. Here we construct the fundamental solutions ϕ_k based on the following idea: we define propagation matrices $M_{L_k}(r, r')$ with the property: $\binom{\phi_k(r)}{\phi_k'(r)} := M_{L_k}(r, r')\binom{a}{b}$ provides the solution of $L_k\phi_k = 0$ at r with initial values $\binom{a}{b}$ at r'. On subintervals where the potential V takes constant values, M_{L_k} can be explicitly computed. Iterating the propagation from $r_n := R + nP + \frac{1}{2}\theta P$ back to r with prescribed decay $r_n^{-1/2}\tau^n$ at r_n, $\tau = \min\{\sqrt{\frac{\alpha}{\beta}}, \sqrt{\frac{\beta}{\alpha}}\} < 1$ and sending $n \to \infty$ will provide the fundamental solution.

Now we start with the propagation matrices on intervals where V is constant. The general solution of

$$\Big(-\partial_r^2 - \tfrac{1}{r}\partial_r + \tfrac{1}{r^2} - k^2\omega^2\alpha\Big)f = 0 \tag{5.35}$$

is given by

$$f(r) = AJ_1(\alpha_k r) + BY_1(\alpha_k r)$$

where J_ν, Y_ν are the Bessel functions of first (resp. second) kind and $\alpha_k := k\omega\sqrt{\alpha}$. Thus the propagation matrix for (5.35) is given by

$$\begin{pmatrix} f(r) \\ f'(r) \end{pmatrix} = M_{\alpha,k}(r,r') \cdot \begin{pmatrix} f(r') \\ f'(r') \end{pmatrix}$$

with

$$M_{\alpha,k}(r,r') = \begin{pmatrix} J_1(\alpha_k r) & Y_1(\alpha_k r) \\ \alpha_k J_1'(\alpha_k r) & \alpha_k Y_1'(\alpha_k r) \end{pmatrix} \cdot \begin{pmatrix} J_1(\alpha_k r') & Y_1(\alpha_k r') \\ \alpha_k J_1'(\alpha_k r') & \alpha_k Y_1'(\alpha_k r') \end{pmatrix}^{-1}.$$

Next we calculate the asymptotic expansion of $M_{\alpha,k}(r,r')$. With the asymptotics (cf. [39])

$$\begin{aligned}
J_1(z) &= \sqrt{\frac{2}{\pi z}}\Big(\sin(z-\tfrac{\pi}{4}) + \tfrac{3}{8z}\cos(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z^2}\big)\Big),\\
Y_1(z) &= \sqrt{\frac{2}{\pi z}}\Big(-\cos(z-\tfrac{\pi}{4}) + \tfrac{3}{8z}\sin(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z^2}\big)\Big),\\
J_1'(z) &= \sqrt{\frac{2}{\pi z}}\Big(\cos(z-\tfrac{\pi}{4}) - \tfrac{7}{8z}\sin(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z^2}\big)\Big),\\
Y_1'(z) &= \sqrt{\frac{2}{\pi z}}\Big(\sin(z-\tfrac{\pi}{4}) + \tfrac{7}{8z}\cos(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z^2}\big)\Big)
\end{aligned}$$

as $z \to \infty$, we find

$$\begin{aligned}
M_{\alpha,k}(r,r') = \sqrt{\frac{r'}{r}}\begin{pmatrix} 1 & 0 \\ 0 & \alpha_k \end{pmatrix}\Big[\begin{pmatrix} \frac{3}{8z} & -1 \\ 1 & \frac{7}{8z} \end{pmatrix} + \mathcal{O}\big(\tfrac{1}{z^2}\big)\Big]\begin{pmatrix} \cos(z-z') & \sin(z-z') \\ -\sin(z-z') & \cos(z-z') \end{pmatrix} \\
\cdot \Big[\begin{pmatrix} \frac{7}{8z'} & 1 \\ -1 & \frac{3}{8z'} \end{pmatrix} + \mathcal{O}\big(\tfrac{1}{z'^2}\big)\Big]\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_k} \end{pmatrix}
\end{aligned} \tag{5.36}$$

as $z, z' \to \infty$, where $z = \alpha_k r, z' = \alpha_k r'$. If, in particular, $r' - r = \theta P$, then since

$$z - z' = -k\omega\theta P\sqrt{\alpha} \in \tfrac{\pi}{2}\mathbb{Z}_{\text{odd}},$$

we have $\cos(z-z') = 0$ and $\sin(z-z') \in \{\pm 1\}$, so we can further simplify

$$\begin{aligned}
M_{\alpha,k}(r,r') &\in \Big\{\pm\sqrt{\frac{r'}{r}}\begin{pmatrix} 1 & 0 \\ 0 & \alpha_k \end{pmatrix}\Big[\begin{pmatrix} \frac{7}{8z'} - \frac{3}{8z} & 1 \\ -1 & \frac{3}{8z'} - \frac{7}{8z} \end{pmatrix} + \mathcal{O}\big(\tfrac{1}{z^2} + \tfrac{1}{z'^2}\big)\Big]\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_k} \end{pmatrix}\Big\} \\
&= \Big\{\pm\sqrt{\frac{r'}{r}}\begin{pmatrix} 1 & 0 \\ 0 & k\omega \end{pmatrix}\Big[\begin{pmatrix} \frac{1}{2k\omega\sqrt{\alpha}r} & \frac{1}{\sqrt{\alpha}} \\ -\sqrt{\alpha} & -\frac{1}{2k\omega\sqrt{\alpha}r} \end{pmatrix} + \mathcal{O}\big(\tfrac{1}{kr^2}\big)\Big]\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k\omega} \end{pmatrix}\Big\}
\end{aligned}$$

as $kr \to \infty$. If we denote by $r_n := R + nP + \frac{1}{2}\theta P$, $r_n' := R + nP + \big(1 - \frac{1}{2}\theta\big)P$ for $n \in \mathbb{N}_0$ the points where V changes from one constant value to the other, then we have

$$\begin{aligned}
M_{L_k}(r_n, r_{n+1}) &= M_{\beta,k}(r_n, r_n') \cdot M_{\alpha,k}(r_n', r_{n+1}) \\
&= \sigma\sqrt{\frac{r_{n+1}}{r_n}}\begin{pmatrix} 1 & 0 \\ 0 & k\omega \end{pmatrix}\Bigg[\begin{pmatrix} \sqrt{\frac{\alpha}{\beta}} & 0 \\ \big(\sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{\alpha}{\beta}}\big)\frac{1}{2k\omega r_n} & \sqrt{\frac{\beta}{\alpha}} \end{pmatrix} + \mathcal{O}\big(\tfrac{1}{kn^2}\big)\Bigg]\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k\omega} \end{pmatrix}
\end{aligned}$$

as $kn \to \infty$ where $\sigma = -\sin(k\omega\theta P\sqrt{\alpha})\sin(k\omega(1-\theta)P\sqrt{\beta}) \in \{\pm 1\}$ does not depend on k. To simplify the asymptotics, we introduce the rescaling

$$S_k(r) := \sqrt{r}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\omega k} \end{pmatrix}$$

and define the rescaled propagation matrices

$$\begin{aligned} M^S_{L_k}(r,r') &:= S_k(r) M_{L_k}(r,r') S_k(r')^{-1}, \\ M^S_{\alpha,k}(r,r') &:= S_k(r) M_{\alpha,k}(r,r') S_k(r')^{-1}, \\ M^S_{\beta,k}(r,r') &:= S_k(r) M_{\beta,k}(r,r') S_k(r')^{-1}. \end{aligned}$$

In the following we assume $\alpha > \beta$, the case $\beta > \alpha$ can be treated similarly. We then define

$$\Psi_k(r) := \lim_{m\to\infty}\left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}}\right)^m M^S_{L_k}(r, r_m). \tag{5.37}$$

This definition is according to the idea introduced at the beginning of the proof: $\Psi_k(r)$ contains decaying fundamental solutions for a rescaled version of the operator L_k, where the geometric decay factor $\sqrt{\frac{\beta}{\alpha}}$ has been built into the solution. As we shall see, the second column of Ψ_k vanishes, which reflects the fact that there can only be one solution which decays at infinity. Next we note that

$$\begin{aligned} \Psi_k(r) &= \lim_{m\to\infty} M^S_{L_k}(r, r_0)\prod_{n=0}^{m-1}\left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1})\right) \\ &= M^S_{L_k}(r, r_0)\prod_{n=0}^{\infty}\left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1})\right). \end{aligned}$$

Using the asymptotics

$$\sigma^{-1}\sqrt{\frac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1}) = \begin{pmatrix} 1 & 0 \\ \mathcal{O}\left(\frac{1}{kn}\right) & \frac{\beta}{\alpha} \end{pmatrix} + \mathcal{O}\left(\tfrac{1}{kn^2}\right)$$

as $kn \to \infty$ and Lemma 5.B.1, the limit in (5.37) exists and is nonzero. In particular, the limit matrix Ψ_k has a vanishing second column so that we can define

$$\begin{pmatrix} \psi_k^{(1)}(r) & 0 \\ \psi_k^{(2)}(r) & 0 \end{pmatrix} := \Psi_k(r).$$

If we also undo the rescaling and define

$$\begin{pmatrix} \phi_k^{(1)}(r) \\ \phi_k^{(2)}(r) \end{pmatrix} := S_k(r)^{-1}\begin{pmatrix} \psi_k^{(1)}(r) \\ \psi_k^{(2)}(r) \end{pmatrix}$$

then

$$\begin{pmatrix} \phi_k^{(1)}(r) \\ \phi_k^{(2)}(r) \end{pmatrix} = M_{L_k}(r,r')\begin{pmatrix} \phi_k^{(1)}(r') \\ \phi_k^{(2)}(r') \end{pmatrix}.$$

since $\psi_k^{(1)}, \psi_k^{(2)}$ satisfy the identity

$$\begin{pmatrix} \psi_k^{(1)}(r) \\ \psi_k^{(2)}(r) \end{pmatrix} = M^S_{L_k}(r,r')\begin{pmatrix} \psi_k^{(1)}(r') \\ \psi_k^{(2)}(r') \end{pmatrix}.$$

This shows that $\phi_k(r) := \phi_k^{(1)}(r)$ satisfies $L_k\phi_k = 0$ and $\phi_k' = \phi_k^{(2)}$ so that ϕ_k is the sought fundamental solution. Its properties will be verified in the next step.

Step 2. We are left with verifying assumptions ($\mathcal{A}$5.4), ($\mathcal{A}$5.5), and ($\mathcal{A}$5.6') for the functions ϕ_k obtained in Step 1. We have already seen as a result of Lemma 5.B.1 that $\phi_k \neq 0$. Next we show that $\phi_k \in L^2_{\mathrm{rad}}([R,\infty)\times\mathbb{T})$. From the asymptotics (5.36) we see that there exists a constant $C_1 > 0$ such that

$$\left\| M^S_{\alpha,k}(r,r') \right\|, \left\| M^S_{\beta,k}(r,r') \right\| \le C_1$$

holds for all $k \in \mathbb{N}_{\mathrm{odd}}$ and all $r, r' \ge R$. By Lemma 5.B.1 there further exists a constant $C_2 > 0$ such that

$$\left\| \prod_{n=n_0}^{\infty} \left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1}) \right) \right\| \le C_2$$

holds for all $n_0 \in \mathbb{N}_0$ and all $k \in \mathbb{N}_{\mathrm{odd}}$.

For every $r \in [R,\infty)$ there exists a unique $n_0 \in \mathbb{N}_0$ such that $r \in (r'_{n_0-1}, r'_{n_0}]$. From

$$\begin{aligned} \sqrt{r}\phi_k(r) = \psi_k^{(1)}(r) &= \left(\lim_{m\to\infty} \left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}}\right)^m M^S_{L_k}(r, r_{n_0}) M^S_{L_k}(r_{n_0}, r_m) \right)_{1,1} \\ &= \left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}}\right)^{n_0} \left(M^S_{L_k}(r, r_{n_0}) \prod_{n=n_0}^{\infty} \left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1}) \right) \right)_{1,1}, \end{aligned} \tag{5.38}$$

and

$$M^S_{L_k}(r, r_{n_0}) = \begin{cases} M^S_{\alpha,k}(r, r_{n_0}), & r \in (r'_{n_0-1}, r_{n_0}] \\ M^S_{\beta,k}(r, r_{n_0}), & r \in (r_{n_0}, r'_{n_0}] \end{cases}$$

we obtain

$$r|\phi_k(r)|^2 \le \left(\tfrac{\beta}{\alpha}\right)^{n_0} C_1^2 C_2^2$$

and therefore also

$$\|\phi_k\|^2_{L^2_{\mathrm{rad}}([R,\infty))} = \int_R^\infty |\phi_k(r)|^2\, r\mathrm{d}r \le \sum_{n_0=0}^{\infty} \left(\tfrac{\beta}{\alpha}\right)^{n_0} C_1^2 C_2^2 P < \infty,$$

where we used $r'_{n_0} - r'_{n_0-1} = P$. While we obtained an L^2-bound on ϕ_k which is uniform in k, with the help of the equation $L_k\phi_k = 0$ one can easily show that $\phi_k \in H^2_{\mathrm{rad}}([R,\infty))$, but the H^2-bound will be k-dependent. Thus assumption ($\mathcal{A}$5.4) holds.

Next we discuss the asymptotics of ϕ_k. We use

$$M^S_{\alpha,k}(r,r') = \begin{pmatrix} \cos(k\omega\sqrt{\alpha}(r-r')) & \frac{1}{\sqrt{\alpha}}\sin(k\omega\sqrt{\alpha}(r-r')) \\ -\sqrt{\alpha}\sin(k\omega\sqrt{\alpha}(r-r')) & \cos(k\omega\sqrt{\alpha}(r-r')) \end{pmatrix} + \mathcal{O}\left(\tfrac{1}{k}\right)$$

and likewise for $M^S_{\beta,k}(r,r')$ as well as

$$\prod_{n=n_0}^{\infty} \left(\sigma^{-1}\sqrt{\tfrac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1}) \right) \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

as $k \to \infty$, cf. Lemma 5.B.1. Thus, together with (5.38) we obtain

$$\begin{aligned}
&\int_{r'_{n_0-1}}^{r'_{n_0}} |\phi_k(r)|^2 \, r\mathrm{d}r \\
&= \left(\tfrac{\beta}{\alpha}\right)^{n_0} \left(\int_{r'_{n_0-1}}^{r_{n_0}} \cos(k\omega\sqrt{\alpha}(r - r_{n_0}))^2 \, \mathrm{d}r + \int_{r_{n_0}}^{r'_{n_0}} \cos(k\omega\sqrt{\beta}(r - r_{n_0}))^2 \, \mathrm{d}r + \mathrm{o}(1) \right) \\
&= \left(\tfrac{\beta}{\alpha}\right)^{n_0} \left(\tfrac{P}{2} + \mathrm{o}(1) \right)
\end{aligned}$$

as $k \to \infty$. In particular, we have

$$\liminf_{k\to\infty} \|\phi_k\|^2_{L^2_{\mathrm{rad}}([R,\infty))} \geq \sum_{n_0=1}^{\infty} \left(\tfrac{\beta}{\alpha}\right)^{n_0} \tfrac{P}{2} > 0.$$

In order to verify assumptions ($\mathcal{A}$5.5) and ($\mathcal{A}$5.6') we calculate the asymptotics of $\phi_k(R)$ and $\phi_k'(R)$. Setting

$$m_\alpha := 4\omega\sqrt{\alpha}\theta\tfrac{P}{2\pi} \in \mathbb{N}_{\mathrm{odd}},$$

we have

$$\begin{aligned}
\sqrt{R}\phi_k(R) = \psi_k^{(1)}(R) &= \left(M^S_{\alpha,k}(R, r_0) \prod_{n=0}^{\infty} \left(\sigma^{-1} \sqrt{\tfrac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1}) \right) \right)_{1,1} \\
&= \cos(k\omega\sqrt{\alpha}(R - r_0)) + \mathrm{o}(1) = \cos(km_\alpha \tfrac{\pi}{4}) + \mathrm{o}(1) \in \left\{ \pm\tfrac{1}{\sqrt{2}} \right\} + \mathrm{o}(1)
\end{aligned}$$

as $k \to \infty$. Combined with the estimates on $\|\phi_k\|_{L^2_{\mathrm{rad}}}$ this shows the first part of assumption ($\mathcal{A}$5.5). Next we have

$$\begin{aligned}
\frac{\sqrt{R}}{k\omega}\phi_k'(R) = \psi_k^{(2)}(R) &= \left(M^S_{\alpha,k}(R, r_0) \prod_{n=0}^{\infty} \left(\sigma^{-1} \sqrt{\tfrac{\beta}{\alpha}} M^S_{L_k}(r_n, r_{n+1}) \right) \right)_{2,1} \\
&= -\sqrt{\alpha}\sin(k\omega\sqrt{\alpha}(R - r_0)) + \mathrm{o}(1) = \sqrt{\alpha}\sin(km_\alpha\tfrac{\pi}{4}) + \mathrm{o}(1) \in \left\{ \pm\tfrac{\sqrt{\alpha}}{\sqrt{2}} \right\} + \mathrm{o}(1),
\end{aligned}$$

which shows the second part of assumption ($\mathcal{A}$5.5). Finally, we have

$$\frac{\phi_k'(R)}{k\phi_k(R)} = \omega\sqrt{\alpha}\tan(km_\alpha\tfrac{\pi}{4}) + \mathrm{o}(1) = -\omega\sqrt{\alpha}(-1)^{\frac{m_\alpha+k}{2}} + \mathrm{o}(1)$$

as $k \to \infty$, so

$$\limsup_{k\to\infty} \frac{\phi_k'(R)}{k\phi_k(R)} = \omega\sqrt{\alpha} > \omega\sqrt{\delta} = \omega\|V\|^{1/2}_{L^\infty([0,R])}$$

since $\alpha > \delta$. Thus assumption ($\mathcal{A}$5.6') holds, and Theorem 5.2.3 yields the result. □

Proof of Theorem 5.1.2, Part 2. Now we consider the periodic step potential with the slab geometry, i.e., (5.17). We verify the assumptions of Theorem 5.2.5 in order to apply it.

By the set-up we have that ($\tilde{\mathcal{A}}$5.1), ($\tilde{\mathcal{A}}$5.2), and ($\tilde{\mathcal{A}}$5.3) hold. The determination of the fundamental solutions $\widetilde{\phi}_k$ follows the Floquet-Bloch theory for second-order periodic differential operators. Details can be found in [49, Appendix 6.2]. The main outcome is the following: there are two Floquet-multipliers

$$\rho_k \in \left\{ -\sqrt{\frac{\alpha}{\beta}}\sin(km'l\pi)\sin(km'\pi), -\sqrt{\frac{\beta}{\alpha}}\sin(km'l\pi)\sin(km'\pi) \right\}$$

where

$$l = \sqrt{\frac{\beta}{\alpha}}\frac{1-\theta}{\theta} \quad \text{and} \quad \left\{2m' = 4\sqrt{\alpha}\theta\omega\frac{P}{2\pi}, 2m'l = 4\sqrt{\beta}(1-\theta)\omega\frac{P}{2\pi}\right\} \subseteq \mathbb{N}_{\text{odd}}$$

and in our setting $m = 2m'$, $n = 2m'l$. This implies that $|\rho_k| \in \{\sqrt{\frac{\alpha}{\beta}}, \sqrt{\frac{\beta}{\alpha}}\}$, so that in modulus one of them is smaller than 1 and one of them larger than 1. For each Floquet exponent there is a solution of $L_k\widetilde{\phi}_k = 0$ with $\widetilde{\phi}_k(x+P) = \rho_k\widetilde{\phi}_k(x)$ for all $x \in \mathbb{R}$. If we choose the fundamental solution corresponding to the Floquet multiplier with modulus less than 1 then this leads to

$$\left\|\widetilde{\phi}_k\right\|^2_{L^2([R,\infty))} = \frac{1}{1-\rho_k}\left\|\widetilde{\phi}_k\right\|^2_{L^2([R,R+P))}.$$

Using the normalization $\widetilde{\phi}_k(R) = 1$ we have

$$\widetilde{\phi}_k'(R) = -k\omega\sqrt{\alpha}\tan(k\omega\sqrt{\alpha}\theta\tfrac{P}{2}) = -k\omega\sqrt{\alpha}\tan(mk\tfrac{\pi}{4}) = k\omega\sqrt{\alpha}(-1)^{\frac{k+m}{2}}$$

and

$$0 < \inf_{k\in\mathbb{N}_{\text{odd}}}\left\|\widetilde{\phi}_k\right\|^2_{L^2([R,R+P))} \leq \sup_{k\in\mathbb{N}_{\text{odd}}}\left\|\widetilde{\phi}_k\right\|^2_{L^2([R,R+P))} < \infty.$$

From these estimates it follows that also ($\tilde{\mathcal{A}}$5.4)–($\tilde{\mathcal{A}}$5.5) hold. Moreover, since $\frac{\widetilde{\phi}_k'(R)}{k\widetilde{\phi}_k(R)} = (-1)^{\frac{k+m}{2}}\omega\sqrt{\alpha}$ and $\alpha > \delta = \|V\|_{L^\infty([-R,R])}$ the final condition ($\tilde{\mathcal{A}}$5.6') is true and so Theorem 5.2.5 yields the result. □

Proof of Theorem 5.1.2, Part 3. Next we consider the step potential $\tilde{\chi}_1^{\text{step}}$ with cylindrical geometry, i.e., problem (5.12). We verify the assumptions ($\mathcal{A}$5.1)–($\mathcal{A}$5.5), ($\mathcal{A}$5.6') in order to apply Theorem 5.2.3. First, ($\mathcal{A}$5.1), ($\mathcal{A}$5.2), and ($\mathcal{A}$5.3) hold by definition. Let J_ν, Y_ν denote the Bessel functions of first (resp. second) kind and K_ν denote the modified Bessel function of second kind. For $k \in \mathbb{N}_{\text{odd}}$ and with $\alpha_k := k\omega\sqrt{\alpha}$, $\beta_k := k\omega\sqrt{\beta}$, the fundamental solution ϕ_k is (up to a constant) then given by

$$\phi_k(r) = \begin{cases} A_kJ_1(\alpha_kr) + B_kY_1(\alpha_kr), & R < r < R+\rho, \\ K_1(\beta_kr), & r > R+\rho. \end{cases}$$

with

$$\begin{pmatrix} A_k \\ B_k \end{pmatrix} = \begin{pmatrix} J_1(\alpha_k(R+\rho)) & Y_1(\alpha_k(R+\rho)) \\ \alpha_kJ_1'(\alpha_k(R+\rho)) & \alpha_kY_1'(\alpha_k(R+\rho)) \end{pmatrix}^{-1}\begin{pmatrix} K_1(\beta_k(R+\rho)) \\ \beta_kK_1'(\beta_k(R+\rho)) \end{pmatrix}.$$

We begin by estimating the functions ϕ_k. Using the asymptotics (cf. [40])

$$\begin{aligned} J_1(z) &= \sqrt{\tfrac{2}{\pi z}}\Big(\sin(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z}\big)\Big), & J_1'(z) &= \sqrt{\tfrac{2}{\pi z}}\Big(\cos(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z}\big)\Big), \\ Y_1(z) &= \sqrt{\tfrac{2}{\pi z}}\Big(-\cos(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z}\big)\Big), & Y_1'(z) &= \sqrt{\tfrac{2}{\pi z}}\Big(\sin(z-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{z}\big)\Big), \\ K_1(z) &= \sqrt{\tfrac{\pi}{2z}}\mathrm{e}^{-z}\Big(1 + \mathcal{O}\big(\tfrac{1}{z}\big)\Big), & K_1'(z) &= \sqrt{\tfrac{\pi}{2z}}\mathrm{e}^{-z}\Big(-1 + \mathcal{O}\big(\tfrac{1}{z}\big)\Big) \end{aligned}$$

as $z \to \infty$, we find

$$A_k = -\tfrac{\pi}{2}\mathrm{e}^{-\beta_k(R+\rho)}\left(\sqrt{\tfrac{\beta_k}{\alpha_k}}\cos(\alpha_k(R+\rho)-\tfrac{\pi}{4}) - \sqrt{\tfrac{\alpha_k}{\beta_k}}\sin(\alpha_k(R+\rho)-\tfrac{\pi}{4}) + \mathcal{O}\big(\tfrac{1}{k}\big)\right),$$

$$B_k = -\tfrac{\pi}{2}\mathrm{e}^{-\beta_k(R+\rho)}\Big(\sqrt{\tfrac{\beta_k}{\alpha_k}}\sin(\alpha_k(R+\rho)-\tfrac{\pi}{4})+\sqrt{\tfrac{\alpha_k}{\beta_k}}\cos(\alpha_k(R+\rho)-\tfrac{\pi}{4})+\mathcal{O}\big(\tfrac{1}{k}\big)\Big),$$

and thus

$$\begin{aligned}\|\phi_k\|^2_{L^2_{\mathrm{rad}}([R+\rho,\infty))} &= \tfrac{\pi}{4\beta_k^2}\mathrm{e}^{-2\beta_k(R+\rho)}\Big(1+\mathcal{O}(\tfrac{1}{k})\Big),\\ \|\phi_k\|^2_{L^2_{\mathrm{rad}}([R,R+\rho])} &= \tfrac{\pi}{4\alpha_k}\mathrm{e}^{-2\beta_k(R+\rho)}\Big(\tfrac{\alpha_k}{\beta_k}+\tfrac{\beta_k}{\alpha_k}\Big)\rho\Big(1+\mathcal{O}(\tfrac{1}{k})\Big)\end{aligned}$$

as $k\to\infty$. In particular we have

$$\|\phi_k\|_{L^2_{\mathrm{rad}}([R,\infty))} = \frac{\mathrm{e}^{-\beta_k(R+\rho)}}{\sqrt{k}}\Big(C+\mathcal{O}\big(\tfrac{1}{k}\big)\Big)$$

for some $C>0$. We further have

$$\begin{aligned}\phi_k(R) &= \sqrt{\tfrac{\pi}{2\alpha_k R}}\mathrm{e}^{-\beta_k(R+\rho)}\Big(\sqrt{\tfrac{\alpha_k}{\beta_k}}\cos(\alpha_k\rho)+\sqrt{\tfrac{\beta_k}{\alpha_k}}\sin(\alpha_k\rho)+\mathcal{O}\big(\tfrac{1}{k}\big)\Big)\\ \phi_k'(R) &= \sqrt{\tfrac{\alpha_k\pi}{2R}}\mathrm{e}^{-\beta_k R+\rho}\Big(\sqrt{\tfrac{\alpha_k}{\beta_k}}\sin(\alpha_k\rho)-\sqrt{\tfrac{\beta_k}{\alpha_k}}\cos(\alpha_k\rho)+\mathcal{O}\big(\tfrac{1}{k}\big)\Big).\end{aligned}$$

Note that $\frac{\alpha_k}{\beta_k}=\sqrt{\frac{\alpha}{\beta}}$ is constant.

As ϕ_k above is the fundamental solution, assumption ($\mathcal{A}$5.4) holds, and the second part of assumption ($\mathcal{A}$5.5) follows directly from the asymptotics. Let $\vartheta=\arctan\big(\frac{\alpha_k}{\beta_k}\big)$. Then

$$\begin{aligned}\phi_k(R) &= \sqrt{\tfrac{\pi}{2\alpha_k R}\Big(\tfrac{\alpha_k}{\beta_k}+\tfrac{\beta_k}{\alpha_k}\Big)}\mathrm{e}^{-\beta_k(R+\rho)}\Big(\sin(\alpha_k\rho+\vartheta)+\mathcal{O}\big(\tfrac{1}{k}\big)\Big).\\ \phi_k'(R) &= -\sqrt{\tfrac{\alpha_k\pi}{2R}\Big(\tfrac{\alpha_k}{\beta_k}+\tfrac{\beta_k}{\alpha_k}\Big)}\mathrm{e}^{-\beta_k(R+\rho)}\Big(\cos(\alpha_k\rho+\vartheta)+\mathcal{O}\big(\tfrac{1}{k}\big)\Big).\end{aligned}$$

By assumption of the theorem on the values T and ϑ we can write

$$\alpha_k\rho+\vartheta=\frac{km\pi}{2n}+\frac{m\pi}{2n}+\frac{l\pi}{n}-\xi$$

for some $l\in\mathbb{Z}$. Since m,n are co-prime, the expression $\mathbb{Z}_{\mathrm{odd}}\ni k\mapsto\alpha_k\rho+\vartheta \mod \pi$ is $2n$-periodic and attains the n values

$$\frac{\pi}{n}-\xi,\frac{2\pi}{n}-\xi,\dots,\pi-\xi$$

and no others. Further, none of these values are zeros of sine. This shows that also the first part of assumption ($\mathcal{A}$5.5) holds. In addition, we have

$$\frac{\phi_k'(R)}{\phi_k(R)}=-\alpha_k\Big(\cot\Big(\tfrac{((k+1)m+2l)\pi}{2n}-\xi\Big)+\mathcal{O}\big(\tfrac{1}{k}\big)\Big).$$

Therefore for $\varepsilon>0$ sufficiently small we find infinitely many $k\in\mathbb{N}_{\mathrm{odd}}$ such that

$$\alpha_k\rho+\vartheta=\pi-\xi \mod \pi \quad\text{and}\quad \frac{\phi_k'(R)}{k\phi_k(R)}=\omega\sqrt{\alpha}\cot(\xi)+\mathcal{O}\big(\tfrac{1}{k}\big)\geq\omega\sqrt{\delta}+\varepsilon$$

hold. This verifies ($\mathcal{A}$5.6'). Finally, we have checked all assumptions of Theorem 5.2.3 which provides existence of T-periodic solutions. □

Proof of Theorem 5.1.2, Part 4. Lastly we discuss the step potential with slab geometry, i.e., (5.17). Like in part 3, we set $\alpha_k := \omega k\sqrt{\alpha}$, $\beta_k := \omega k\sqrt{\beta}$. Then the fundamental solutions $\widetilde{\phi}_k$ are (up to a constant) given by

$$\widetilde{\phi}_k(x) = \begin{cases} \widetilde{A}_k \sin(\alpha_k x) + \widetilde{B}_k \cos(\alpha_k x), & R < x < R + \rho, \\ \mathrm{e}^{-\beta_k x}, & x > R + \rho \end{cases}$$

with

$$\begin{aligned} \begin{pmatrix} \widetilde{A}_k \\ \widetilde{B}_k \end{pmatrix} &= \begin{pmatrix} \sin(\alpha_k(R+\rho)) & \cos(\alpha_k(R+\rho)) \\ \alpha_k \cos(\alpha_k(R+\rho)) & -\alpha_k \sin(\alpha_k(R+\rho)) \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{e}^{-\beta_k(R+\rho)} \\ -\beta_k \mathrm{e}^{-\beta_k(R+\rho)} \end{pmatrix} \\ &= \mathrm{e}^{-\beta_k(R+\rho)} \begin{pmatrix} \sin(\alpha_k(R+\rho)) - \frac{\beta_k}{\alpha_k}\cos(\alpha_k(R+\rho)) \\ \cos(\alpha_k(R+\rho)) + \frac{\beta_k}{\alpha_k}\sin(\alpha_k(R+\rho)) \end{pmatrix}. \end{aligned}$$

Therefore ($\tilde{\mathcal{A}}$5.4) holds and

$$\begin{aligned} \left\|\widetilde{\phi}_k\right\|^2_{L^2([R+\rho,\infty))} &= \tfrac{1}{2\beta_k}\mathrm{e}^{-2\beta_k(R+\rho)}, \\ \left\|\widetilde{\phi}_k\right\|^2_{L^2([R,R+\rho])} &= \tfrac{\rho}{2}\mathrm{e}^{-2\beta_k(R+\rho)}\left(1 + \tfrac{\beta_k^2}{\alpha_k^2}\right)\left(1 + \mathcal{O}(\tfrac{1}{k})\right) \end{aligned}$$

so that

$$\left\|\widetilde{\phi}_k\right\|_{L^2([R,\infty))} = \mathrm{e}^{-\beta_k(R+\rho)}\left(C + \mathcal{O}\left(\tfrac{1}{k}\right)\right)$$

holds for some $C > 0$ as $k \to \infty$. In particular,

$$\widetilde{\phi}_k(R) = \mathrm{e}^{-\beta_k(R+\rho)}\left(\cos(\alpha_k\rho) + \tfrac{\beta_k}{\alpha_k}\sin(\alpha_k\rho)\right) = \mathrm{e}^{-\beta_k(R+\rho)}\sqrt{1 + \tfrac{\beta}{\alpha}}\sin(\alpha_k\rho + \vartheta)$$

and

$$\widetilde{\phi}'_k(R) = -\alpha_k \mathrm{e}^{-\beta_k(R+\rho)}\sqrt{1 + \tfrac{\beta}{\alpha}}\cos(\alpha_k\rho + \vartheta)$$

with $\vartheta = \arctan\left(\sqrt{\frac{\alpha}{\beta}}\right)$. From here on we can argue almost identically as in the proof of part 3 for the verification of the conditions ($\tilde{\mathcal{A}}$5.5) and ($\tilde{\mathcal{A}}$5.6'). □

5.C. Appendix: Numerical method

In this section we provide details on the generation of Figures 5.1 and 5.2. For simplicity, we only consider the radial geometry setting.

As discussed in Section 5.3, solutions w to (5.12) can be obtained from critical points u of the functional E, see (5.23), and in particular from the minimizer of E. We numerically minimize $E|_Z$ over a finite dimensional space Z: $E(u) \approx \min E|_Z$. Then from u we reconstruct an approximate breather w using the formula (5.26).

Motivated by Section 5.4 we choose the ansatz space

$$Z = \left\{u\colon u(x,t) = \sum_{\substack{k\in\mathbb{Z}_{\text{odd}} \\ |k|\le K}} f_k(x)e_k(t) \;\middle|\; f_k \in F, f_{-k} = \overline{f_k}\right\},$$

where F is a (complex-valued) 1d finite element space, which we have chosen to be the space of piecewise linear elements with equidistant nodes $0, \frac{R}{N}, \dots, \frac{(N-1)R}{N}, R$.

The illustrations from Figures 5.1 and 5.2 are then obtained by choosing $K = 64, N = 128$ and using a MATLAB built-in function to solve the minimization problem. The code to generate them can be found in [70].

——— End of Preprint ———

CHAPTER 6

Wellposedness for a $(1+1)$-dimensional wave equation with quasilinear boundary condition

This chapter consists of the paper [73], which is a joint work with Wolfgang Reichel and Roland Schnaubelt. We moved [73, Appendix B] to Section B.2, changed the notation to be consistent with the thesis, and made a small modification to the statement of Theorem 6.1.1.

Start of Paper

Abstract. We consider the linear wave equation $V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0$ on $[0,\infty) \times [0,\infty)$ with initial conditions and a nonlinear Neumann boundary condition $u_x(0,t) = (f(u_t(0,t)))_t$ at $x = 0$. This problem is an exact reduction of a nonlinear Maxwell problem in electrodynamics. In the case where $f\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism we study global existence, uniqueness and well-posedness of the initial value problem by the method of characteristics and fixed point methods. We also prove conservation of energy and momentum and discuss why there is no well-posedness in the case where f is a decreasing homeomorphism. Finally we show that previously known time-periodic, spatially localized solutions (breathers) of the wave equation with the nonlinear Neumann boundary condition at $x = 0$ have enough regularity to solve the initial value problem with their own initial data.

6.1. Introduction and main results

In this paper we study the initial value problem for the following 1+1-dimensional wave equation with quasilinear boundary condition:

$$\begin{cases} V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in [0,\infty), t \in [0,\infty), \\ u_x(0,t) = (f(u_t(0,t)))_t, & x = 0, t \in [0,\infty), \\ u(x,t_0) = u_0(x), u_t(x,0) = u_1(x), & x \in [0,\infty), t = 0. \end{cases} \tag{6.1}$$

This initial value problem has two main features: the wave equation on the half-axis $[0,\infty)$ is linear with a space-dependent speed of propagation and the boundary condition at $x = 0$ is a rather singular, quasilinear, 2nd-order in time Neumann-condition. We show well-posedness on all time intervals $[0,T]$ with $T > 0$, and preservation of energy and momentum.

Our interest in (6.1) stems from the fact that it appears in the context of electromagnetics as an exact reduction of a nonlinear Maxwell system. We recall the Maxwell equations in the absence of charges and currents

$$\begin{aligned} &\nabla \cdot \mathbf{D} = 0, && \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, && \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}), \\ &\nabla \cdot \mathbf{B} = 0, && \nabla \times \mathbf{H} = \partial_t \mathbf{D}, && \mathbf{B} = \mu_0 \mathbf{H} \end{aligned}$$

with the electric field $\mathbf{E}$, the electric displacement field $\mathbf{D}$, the polarization field $\mathbf{P}$, the magnetic field $\mathbf{B}$, and the magnetic induction field $\mathbf{H}$. Particular properties of the underlying material are modelled by the specification of the relations between $\mathbf{E}, \mathbf{D}, \mathbf{P}$ on one hand, and

$\mathbf{B}, \mathbf{H}$ on the other hand. Here, we assume a magnetically inactive material, i.e., $\mathbf{B} = \mu_0 \mathbf{H}$, but on the electric side we assume a material with a Kerr-type nonlinear behaviour, cf. [4], Section 2.3, given through

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi_1(\mathbf{x})\mathbf{E} + \varepsilon_0 \chi_{\mathrm{NL}}(\mathbf{x}) g(|\mathbf{E}|^2)\mathbf{E}$$

with $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and $|\cdot|$ the Euclidean norm on $\mathbb{R}^3$. For simplicity we assume that $\chi_1, \chi_{\mathrm{NL}}$ are given scalar valued functions instead of the more general situation where they are matrix valued. The scalar constants ε_0, μ_0 are such that $c = (\varepsilon_0 \mu_0)^{-1/2}$ is the speed of light in vacuum. Local existence, well-posedness and regularity results for the general nonlinear Maxwell system have been shown on $\mathbb{R}^3$ by Kato [47] and on domains by Spitz [83, 84].

In its second order formulation the Maxwell system becomes

$$0 = \nabla \times \nabla \times \mathbf{E} + \partial_t^2 \Big(\mu_0 \varepsilon_0 (1 + \chi_1(\mathbf{x}))\mathbf{E} + \mu_0 \varepsilon_0 \chi_{\mathrm{NL}}(\mathbf{x}) g(|\mathbf{E}|^2)\mathbf{E} \Big). \tag{6.2}$$

We assume additionally that $\chi_1(\mathbf{x}) = \chi_1(x)$, $\chi_{\mathrm{NL}}(\mathbf{x}) = \chi_{\mathrm{NL}}(x)$ and that $\mathbf{E}$ takes the form of a polarized traveling wave

$$\mathbf{E}(\mathbf{x}, t) = (0, 0, U(x, \kappa^{-1} y - t))^T. \tag{6.3}$$

Then the quasilinear vectorial wave-type equation (6.2) turns into the scalar equation

$$V(x) U_{tt} - U_{xx} + \Gamma(x)(g(U^2)U)_{tt} = 0 \tag{6.4}$$

for $U = U(x, t)$, where $V(x) = \mu_0 \varepsilon_0 (1 + \chi_1(x)) - \kappa^{-2}$ and $\Gamma(x) = \mu_0 \varepsilon_0 \chi_{\mathrm{NL}}(x)$. Note that (6.4) is an exact reduction of the Maxwell problem, from which all fields can be reconstructed. E.g., the magnetic induction $\mathbf{B}$ can be retrieved from $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ by time-integration and it will satisfy $\nabla \cdot \mathbf{B} = 0$ provided it does so at time $t = 0$. By assumption the magnetic field is given by $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$ and it satisfies $\nabla \times \mathbf{H} = \partial_t \mathbf{D}$. It remains to check that the displacement field $\mathbf{D}$ satisfies the Gauss law $\nabla \cdot \mathbf{D} = 0$ in the absence of external charges. This follows directly from the constitutive equation $\mathbf{D} = \varepsilon_0(1 + \chi_1(\mathbf{x}))\mathbf{E} + \varepsilon_0 \chi_{\mathrm{NL}}(\mathbf{x}) g(|\mathbf{E}|^2)\mathbf{E}$ and the assumption of the polarized form of the electric field in (6.3).

An extreme case for the potential Γ in front of the nonlinearity arises when $\Gamma(x) = 2\delta_0(x)$ is a multiple of the δ-distribution at 0, cf. [19, 49]. If additionally $V(x)$ is even and $U(x, t) = u_t(x, t)$ for an even function $u(x, t) = u(-x, t)$, by removing one time derivative (6.4) becomes

$$\begin{cases} V(x) u_{tt}(x, t) - u_{xx}(x, t) = 0, & x \in [0, \infty), t \in [0, \infty), \\ u_x(0, t) = (f(u_t(0, t)))_t, & x = 0, t \in [0, \infty) \end{cases} \tag{6.5}$$

with $f(s) := g(s^2)s$. Clearly (6.1) is the initial value problem for (6.5). This extreme model describes the concentration of the entire nonlinear behavior in a waveguide-like structure where the width of the waveguide has been shrunk to zero and the strength of the nonlinearity has been sent to infinity. A similar model, where the potential V in front of the linear term is taken as a multiple of a δ-distribution, has been considered in [75]. Clearly, δ-distributions are inserted purely for mathematical simplicity, and may be considered as a step towards physically more realistic models with L^∞-potentials.

From the point of view of time-periodic solutions, problem (6.5) with $f(s) = \pm s^3$ has been considered in [49]. Under specific assumptions on the linear potential V the existence of infinitely many breathers, i.e., real-valued, time-periodic, spatially localized solutions of (6.5), was shown. Typical examples of V were given in classes of piecewise continuous functions having jump discontinuities. Under different assumptions on V and Γ, but still including δ-distributions, problem (6.5) was considered in [19] and real-valued breathers were constructed. A series of works considering linear and nonlinear wave equations with Neumann boundary conditions emerged from [24, 97]. Attention was given to global existence and well-posedness as well as to blow-up phenomena arising from nonlinear terms in either the boundary condition or the equation. In [21, 24] decay and global attractors were obtained. We point out that in contrast to our work these papers consider nonlinear terms at the boundary which are only of first order in time and have a damping character. This may also be the reason why even higher-dimensional cases are by now well-understood in this first-order case. Perhaps closest to our set-up is the paper [96], where a linear wave equation in the domain is coupled to a linear wave equation at the boundary and the well-posedness is shown to be true exactly in dimension one – in complete accordance with our set-up.

Our goal is to study the initial value problem (6.1) from the point of view of well-posedness, to derive the conservation of momentum and energy, and to verify that known time-periodic solutions from [49] satisfy (6.1) with their own initial values. Note that the boundary condition in (6.1) becomes $u_x(0,t) = \pm 3u_t(0,t)^2 u_{tt}(0,t)$ in the model case $f(s) = \pm s^3$. Hence, (6.1) is a singular initial value problem which is not covered by typical theories like, e.g., energy methods or monotone operators. Instead, our approach will be to prove existence by making use of the method of characteristics. Uniqueness, well-posedness, global existence, and the conservation of energy and momentum will build upon this.

Our basic assumptions on the initial data u_0, u_1 are:

($\mathcal{A}$6.0) $u_0 \in C^1([0,\infty))$, $u_1 \in C([0,\infty))$.

Here $C^k([0,\infty)) = C^k([0,\infty),\mathbb{R})$, and in general all function spaces consist of real-valued functions unless the codomain is explicitly mentioned. Motivated by the results from [49] we are interested in the case where the coefficient V may have discontinuities. In particular, we consider piecewise C^1 functions V. We have chosen the setting $(u_0,u_1) \in C^1([0,\infty)) \times C([0,\infty))$ since it perfectly fits to our method which is inspired by the method of characteristics. We bridge the gap to Sobolev space-based weak solutions (introduced in Definition 6.1.6) by Proposition 6.5.2 in Section 6.5.

Let $I \subseteq \mathbb{R}$ be a closed interval. We call a function $\phi\colon I \to \mathbb{R}$ piecewise C^k if there exists a discrete set $D \subseteq I$ such that $\phi \in C^k(I \setminus D)$ and the limits $\phi^{(j)}(x-)$ and $\phi^{(j)}(x+)$ exist for all $x \in D$ and $0 \le j \le k$, although they do not need to coincide. If I is bounded from below (or above), in addition we require $\phi^{(j)}(\min I+)$ (or $\phi^{(j)}(\max I-)$) to exist for all $0 \le j \le k$. Let $PC^k(I)$ denote the set of piecewise C^k functions on I, and for $\phi \in PC(I) := PC^0(I)$ let us denote by $D(\phi)$ the set of discontinuities of ϕ.

For the coefficient V and the nonlinear function f we assume

($\mathcal{A}$6.1) $V \in PC^1([0,\infty)), V, V' \in L^\infty, \inf V > 0$,

($\mathcal{A}$6.2) $\inf\{|d_1 - d_2| \text{ with } d_1, d_2 \in D(V) \cup \{0\}, d_1 \neq d_2\} > 0$,

($\mathcal{A}$6.3) $f\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism.

The main theorem of this paper is given next.

Theorem 6.1.1. *Assume ($\mathcal{A}$6.0)–($\mathcal{A}$6.3). Then* (6.1) *admits a unique and global C^1-solution. Moreover,* (6.1) *is well-posed on every finite time interval* $[0,T]$ *with* $T > 0$.

In Proposition 6.6.1 our concept of continuous dependence on data is stated precisely. In the above result the assumption ($\mathcal{A}$6.3) is crucial. For a decreasing homeomorphism f the result of Theorem 6.1.1 does not hold, see Remark 6.1.7. Since we have already used the notion of a C^1-solution, we are going to explain it in detail next. As the notion of a C^1-solution will also be used for subdomains of $[0, \infty) \times [0, \infty)$ we first define the notion of an admissible domain.

Definition 6.1.2 (admissible domain). *We call a set* $\Omega \subseteq [0, \infty) \times [0, \infty)$ *an* admissible domain *if it is of the form*

$$\Omega = \{(x, t) \in [0, \infty) \times [0, \infty) \mid t \leq h(x)\}$$

where $h \equiv +\infty$ *or* $h \colon [0, \infty) \to \mathbb{R}$ *is Lipschitz with* $|h_x(x)| \leq \sqrt{V(x)}$ *for almost all* x. *We denote the relative interior of* Ω *by*

$$\Omega^\circ := \{(x, t) \in [0, \infty) \times [0, \infty) \mid t < h(x)\}.$$

In order to explain the notion of a C^1-solution let us first mention that we cannot expect that a solution of (6.1) has everywhere second derivatives u_{tt} or u_{xx}. This is essentially due to the nonlinear boundary condition and the discontinuities of second derivatives which propagate away from $x = 0$. However, if we denote by $c(x) := \frac{1}{\sqrt{V(x)}}$ the inverse of the x-dependent wave speed, then we can factorize the wave operator as

$$\partial_t^2 - c(x)^2 \partial_x^2 = (\partial_t - c(x)\partial_x)(\partial_t + c(x)\partial_x) + c(x)c_x(x)\partial_x.$$

It is then reasonable for a C^1-solution to have almost everywhere a mixed second directional derivative $\partial^2_{\nu,\mu}$ with directions $\nu = (1, -c(x))$ and $\mu = (1, c(x))$. This is the basis for the following definition.

Definition 6.1.3 (solution). *A function* $u \in C^1(\Omega)$ *on an admissible domain* Ω *is called a* C^1*-solution to* (6.1) *if the following hold:*

(i) *For all* $(x, t) \in \Omega \setminus (D(c) \cup D(c_x) \times \mathbb{R})$ *we have* $(\partial_t - c(x)\partial_x)(u_t + c(x)u_x)(x, t) = -c(x)c_x(x)u_x(x, t)$.

(ii) $(f(u_t(0, t)))_t = u_x(0, t)$ *for all* $(0, t) \in \Omega^\circ$.

(iii) $u(x, 0) = u_0(x)$ *for all* $(x, 0) \in \Omega$, $u_t(x, 0) = u_1(x)$ *for all* $(x, 0) \in \Omega^\circ$.

Problem (6.1) has a momentum given by

$$M(u, t) := \int_0^\infty V(x)u_t \, \mathrm{d}x + f(u_t(0, t)) \tag{6.6}$$

and an energy given by

$$E(u, t) := \tfrac{1}{2} \int_0^\infty \Big(V(x)u_t(x, t)^2 + u_x(x, t)^2\Big) \, \mathrm{d}x + F(u_t(0, t)) \tag{6.7}$$

where $F(s) := sf(s) - \int_0^s f(\sigma) \, \mathrm{d}\sigma$. If, e.g., f is continuously differentiable, then $F(s)$ is a primitive of $sf'(s)$. The conservation of momentum and energy is stated next.

Theorem 6.1.4. *Assume (A6.0)–(A6.3) and that* u *is a* C^1*-solution of* (6.1) *with initial data satisfying* $u_0'(x), u_1(x) \to 0$ *as* $x \to \infty$. *Then the momentum given by* (6.6) *and the energy given by* (6.7) *are time-invariant.*

Remark 6.1.5. Note that $F(s) = \int_0^s (f(s) - f(\sigma))\,\mathrm{d}\sigma$ tends to $+\infty$ as $s \to \pm\infty$, since by assumption ($\mathcal{A}6.3$) we have $f(s) \to \pm\infty$ as $s \to \pm\infty$. Therefore, due to Theorem 6.1.4, $u_x(\,\cdot\,, t)$ and $u_t(\,\cdot\,, t)$ are bounded in $L^2([0,\infty))$ and $u_t(0,t)$ is bounded as well.

Another common notion of solution for (6.1) is the notion of a weak solution, which we only give for $\Omega = [0,\infty)^2$. The fact that a C^1-solution to (6.1) is also a weak solution to (6.1) holds true and will be proven in Proposition 6.5.2 in Section 6.5.

Definition 6.1.6 (weak solution). *A function $u \in W^{1,1}_{loc}([0,\infty) \times [0,\infty))$ is called a weak solution to* (6.1) *if $f(u_t(0,\,\cdot\,)) \in L^1_{loc}([0,\infty))$, $u(\,\cdot\,,0) = u_0$, and u satisfies*

$$
\begin{aligned}
0 = &\int_0^\infty \int_0^\infty (V(x) u_t \varphi_t - u_x \varphi_x)\,\mathrm{d}x\mathrm{d}t + \int_0^\infty f(u_t(0,t))\varphi_t(0,t)\,\mathrm{d}t \\
&+ \int_0^\infty V(x) u_1(x) \varphi(x,0)\,\mathrm{d}x + f(u_1(0))\varphi(0,0)
\end{aligned}
$$

for all $\varphi \in C_c^\infty([0,\infty) \times [0,\infty))$.

Remark 6.1.7. Due to assumption ($\mathcal{A}6.3$) we have only considered increasing functions f. If we instead allow $f\colon \mathbb{R} \to \mathbb{R}$ to be a decreasing homeomorphism, then (6.1) will not be well-posed in general and can have multiple solutions. Consider for example the cubic term $f(y) = -y^3$ with constant potential $V = 1$ and homogeneous initial data:

$$
\begin{cases}
u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in [0,\infty), t \in [0,\infty),\\
u_x(0,t) = -(u_t(0,t)^3)_t, & x = 0, t \in [0,\infty),\\
u(x,t_0) = 0, u_t(x,t_0) = 0, & x \in [0,\infty), t = 0.
\end{cases}
\tag{6.8}
$$

By direct calculation one can show that the right-traveling wave

$$
u_p(x,t) = \begin{cases}
\left(\frac{2}{3}(t-x)\right)^{\frac{3}{2}}, & x < t,\\
0, & x \geq t
\end{cases}
$$

is a nontrivial solution to (6.8). In fact, u_p is a C^1-solution of $(\partial_x + \partial_t)u = 0$. But (6.8) also has the trivial solution $u = 0$, or $u(x,t) = \pm u_p(x, t-\tau)$ for any $\tau \geq 0$. However, due to the continuity of f^{-1}, one can still show existence of solutions to (6.1) in the case where f grows at least linearly, cf. ($\mathcal{A}6.4$). This follows from the arguments in Sections 6.3 and 6.4. Theorem 6.1.4 also holds when f is decreasing, but now the quantity $F(y)$ tends to $-\infty$ as $y \to \pm\infty$, so that (6.7) does not give rise to estimates on u. Lastly, also in this case C^1-solutions to (6.1) are weak solutions.

In addition to the problem being posed on the positive real half-line $x \in [0,\infty)$, we can also consider the same quasilinear problem posed on a bounded domain $x \in [0,L]$ where we impose a homogeneous Dirichlet condition at $x = L$:

$$
\begin{cases}
V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in [0,L], t \in [0,\infty),\\
u_x(0,t) = (f(u_t(0,t)))_t, & t \in [0,\infty),\\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in [0,L],\\
u(L,t) = 0, & t \in [0,\infty).
\end{cases}
\tag{6.9}
$$

Both Theorem 6.1.1 and conservation of energy (cf. Theorem 6.1.4) remain valid when making the obvious adaptations to this setting.

Remark 6.1.8. Assume $(\mathcal{A}6.0)$–$(\mathcal{A}6.3)$. Then (6.9) admits a unique and global C^1-solution u. Moreover, the energy given by

$$E(u,t) := \tfrac{1}{2}\int_0^L \Big(V(x)u_t(x,t)^2 + u_x(x,t)^2\Big)\,\mathrm{d}x + F(u_t(0,t)).$$

is time-invariant.

Remark 6.1.9. For Dirichlet boundary data, momentum is in general not conserved.

The paper is structured as follows. In Section 6.2 we provide a change of variables which turns the wave operator with variable wave speed in (6.1) into a constant coefficient operator with an additional first order term. The well-known constant coefficient operator is useful since it provides implicit solution formulas which are analyzed in Section 6.3. In Section 6.4 we prove the existence and uniqueness part of Theorem 6.1.1 under an extra assumption on the nonlinearity f. We use that the wave equation has finite speed of propagation to argue locally. Difficulties arise at the boundary $x = 0$ where the nonlinearity comes in to play, and near jumps of V where wave breaking occurs. We use the method of characteristics and Section 6.3 to obtain a reduced problem that is a retarded ordinary differential equation, which can be treated using fixed-point arguments. Since the ODE at the boundary is nonlinear we use the extra assumption on f to close the fixed-point argument. In Section 6.5 we prove energy and momentum conservation as stated in Theorem 6.1.4, and the fact that C^1-solutions of (6.1) in the sense of Definition 6.1.3 are also weak solutions, cf. Proposition 6.5.2. Using the conservation laws, we also obtain a priori bounds that allow us to remove the extra assumption on f from Section 6.4. The well-posedness part of Theorem 6.1.1 is shown in Section 6.6 using similar methods as in the existence and uniqueness parts. Finally, in Section 6.7 we verify that the breather solutions obtained in [49] satisfy (6.1) with their own initial values. This is mainly a problem of regularity, as we have to show that these breathers are of class C^1. To achieve this, we follow ideas from [49] and improve upon their bootstrapping argument. The Appendix 6.A contains some technical results used in the proofs of the main results.

6.2. A change of variables

It will be convenient to normalize the wave speed to 1. To achieve this, we introduce a new variable $z = \kappa(x) = \int_0^x \frac{1}{c(s)}\,\mathrm{d}s$, and thus a new coordinate system (z,t). Avoiding new notation we denote the functions V, c, u, u_0, u_1 transformed into this new coordinate system again by V, c, u, u_0, u_1. The relation between the two coordinate systems is given by

$$\frac{\partial z}{\partial x} = \frac{1}{c(x)} \quad \text{or} \quad c(x)\partial_x = \partial_z \quad \text{or} \quad \mathrm{d}x = c(x)\,\mathrm{d}z.$$

From now on until the end of Section 6.5, we will exclusively work with the coordinate system (z,t). As before we denote the points where c is discontinuous by $D(c)$ and the points where c_z is discontinuous by $D(c_z)$.

Formally the initial value problem (6.1) transforms into

$$\begin{cases} u_{tt}(z,t) - u_{zz}(z,t) = -\frac{c_z(z)}{c(z)}u_z(z,t), & z\in[0,\infty), t\in[0,\infty),\\ \frac{1}{c(0)}u_z(0,t) = (f(u_t(0,t)))_t, & t\in[0,\infty),\\ u(z,0)=u_0(z), u_t(z,0)=u_1(z), & z\in[0,\infty). \end{cases} \tag{6.10}$$

where we need to take into account that $u_x = \frac{1}{c} u_z$ is continuous (and not u_z itself) and that the differential equation does not hold at the discontinuities of c and c_z. A detailed definition of the solution concept is given below in Definition 6.2.3.

We begin by rephrasing Definitions 6.1.2 and 6.1.3 for the new coordinate system.

Definition 6.2.1 (admissible domain). *We call a set* $\Omega \subseteq [0,\infty) \times [0,\infty)$ *an* admissible domain *if it is of the form*

$$\Omega = \{(z,t) \in [0,\infty) \times [0,\infty) \mid t \leq h(z)\}$$

where $h \equiv +\infty$ *or* $h\colon [0,\infty) \to \mathbb{R}$ *is Lipschitz continuous with Lipschitz constant* 1*. We denote its relative interior by*

$$\Omega^\circ := \{(z,t) \in [0,\infty) \times [0,\infty) \mid t < h(z)\}.$$

Next we introduce function spaces that capture the condition of the continuity of $\frac{1}{c} u_z$.

Definition 6.2.2 (x-dependent function spaces). *Let the transformation between* (x,t) *and* (z,t)*-coordinates be given by* $\tilde{\kappa}(x,t) := (\kappa(x), t) = (z,t)$*. For* $\Omega \subseteq [0,\infty) \times [0,\infty)$ *we write*

$$C^1_{(x,t)}(\Omega) := \{u\colon \Omega \to \mathbb{R} \mid u \circ \tilde{\kappa} \in C^1(\tilde{\kappa}^{-1}(\Omega))\}$$

where we understand u *to be a function of* (z,t) *variables, and* $\tilde{u} := u \circ \tilde{\kappa}$ *is the* (x,t)*-dependent version of* u*, i.e.,* $\tilde{u}(x,t) = u(z,t)$ *holds. Note that* $u \in C^1_{(x,t)}(\Omega)$ *if and only if* $u, u_t, \frac{1}{c} u_z \in C(\Omega)$*.*

Similarly, for an interval $I \subseteq [0,\infty)$ *we define*

$$C^1_x(I) := \{v\colon I \to \mathbb{R} \mid v \circ \kappa \in C^1(\kappa^{-1}(I))\}.$$

where again we understand v *to be a function of* z*.*

Definition 6.2.3 (solution). *A function* $u \in C^1_{(x,t)}(\Omega)$ *on an admissible domain* Ω *is called a* C^1*-solution to* (6.10) *if the following hold:*

(i) *For all* $(z,t) \in \Omega \setminus (D(c) \cup D(c_z) \times \mathbb{R})$ *we have* $(\partial_t - \partial_z)(u_t + u_z)(z,t) = -\frac{c_z(z)}{c(z)} u_z(z,t)$*.*

(ii) $f(u_t(0,t))_t = \frac{1}{c(0)} u_z(0,t)$ *for all* $(0,t) \in \Omega^\circ$*.*

(iii) $u(z,0) = u_0(z)$ *for all* $(z,0) \in \Omega$*,* $u_t(z,0) = u_1(z)$ *for all* $(z,0) \in \Omega^\circ$*.*

Remark 6.2.4. Note that $u\colon \Omega \to \mathbb{R}$ is a C^1-solution to (6.1) in the (x,t)-coordinates if and only if it is a C^1-solution to (6.10) in the (z,t)-coordinates.

6.3. Auxiliary results on the linear part

In this section we gather some auxiliary results and estimates on the linear wave equation. These will prove useful for the study of the nonlinear initial boundary value problem (6.10). All results of this section hold under the assumptions ($\mathcal{A}$6.0)–($\mathcal{A}$6.3).

We first note that the wave equation has finite speed of propagation; if we know its behavior at time t_0 on an interval $[z_0 - r, z_0 + r]$, then we can defer its accurate behavior on the space-time triangle with corners $(z_0 - r, t_0)$, $(z_0 + r, t_0)$ and $(z_0, t_0 + r)$.

Definition 6.3.1. *For $(z_0, t_0) \in \mathbb{R}^2$ and $r > 0$ we denote the triangle with corners $(z_0 - r, t_0)$, $(z_0 + r, t_0)$ and $(z_0, t_0 + r)$ by*

$$\Delta(z_0, t_0, r) \coloneqq \{(z, t) \in \mathbb{R}^2 \mid t \geq t_0, |z - z_0| + |t - t_0| \leq r\},$$

its base projected onto the z-axis is given by $P_z\Delta(z_0, t_0, r) = [z_0 - r, z_0 + r]$ with projection $P_z(z, t) \coloneqq z$. Similarly, we define left and right half triangles

$$\Delta_-(z_0, t_0, r) \coloneqq \Delta(z_0, t_0, r) \cap \{z \leq z_0\}, \qquad \Delta_+(z_0, t_0, r) \coloneqq \Delta(z_0, t_0, r) \cap \{z \geq z_0\}$$

whose bases are given by

$$P_z\Delta_-(z_0, t_0, r) = [z_0 - r, z_0], \qquad P_z\Delta_-(z_0, t_0, r) = [z_0, z_0 + r].$$

Recall the solution formula for the 1-dimensional wave equation:

Theorem 6.3.2. *Let $(z_0, t_0) \in \mathbb{R}^2$, $r > 0$, $\Delta \coloneqq \Delta(z_0, t_0, r)$ and $B \coloneqq P_z\Delta$. Assume that $u_0 \in C^1(B)$, $u_1 \in C(B)$, and that $g \in L^\infty(\Delta)$ is continuous outside a set L consisting of finitely many lines of the form $\{z = const\}$. Then the function*

$$u(z, t) = \tfrac{1}{2}(u_0(z + t - t_0) + u_0(z - t + t_0)) + \tfrac{1}{2}\int_{z-t+t_0}^{z+t-t_0} u_1(y)\,\mathrm{d}y + \tfrac{1}{2}\int_{\Delta(z,t_0,t-t_0)} g(y, \tau)\,\mathrm{d}(y, \tau)$$

belongs to $C^1(\Delta)$ and is the unique C^1-solution of the problem

$$\begin{cases} (\partial_t - \partial_z)(u_t + u_z) = g, & (z, t) \in \Delta, \\ u(z, t_0) = u_0(z), \quad u_t(z, t_0) = u_1(z), & z \in B \end{cases}$$

in the following sense: $u(\cdot, t_0) = u_0(\cdot)$, $u_t(\cdot, t_0) = u_1(\cdot)$ on B and the directional derivative $(\partial_t - \partial_z)(u_t + u_z)$ exists and equals g on $\Delta^\circ \setminus L$.

Remark 6.3.3. For every C^1-solution u of $(\partial_t - \partial_z)(u_t + u_z) = g$ on a domain we have that $(\partial_t + \partial_z)(u_t - u_z) = (\partial_t - \partial_z)(u_t + u_z)$ wherever g is continuous, cf. Schwarz's theorem [79, Theorem 9.41]. As a consequence, any of the two factorizations of the wave operator $(\partial_t - \partial_z)(\partial_t + \partial_z)$ or $(\partial_t + \partial_z)(\partial_t - \partial_z)$ can be used and yield the same solution.

By combining the above Theorem 6.3.2 with a fixed point argument, we can treat the initial value problem for $(\partial_t - \partial_z)(u_t + u_z) = -\frac{c_z(z)}{c(z)}u_z$ on sufficiently small triangles Δ. In order to have a slightly more general situation available we work with a piecewise continuous function λ instead of $\frac{c_z}{c}$.

Corollary 6.3.4. *Let $(z_0, t_0) \in \mathbb{R}^2$ and $\Delta \coloneqq \Delta(z_0, t_0, r)$, $B \coloneqq P_z\Delta$ for $r > 0$. Assume $u_0 \in C^1(B)$, $u_1 \in C(B)$ and $\lambda \in PC(B)$ such that $r\|\lambda\|_\infty < 1$. Then*

$$\begin{cases} (\partial_t - \partial_z)(u_t + u_z) = -\lambda(z)u_z, & (z, t) \in \Delta, \\ u(z, t_0) = u_0(z), u_t(z, t_0) = u_1(z), & z \in B \end{cases} \tag{6.11}$$

has a unique solution $u \in C^1(\Delta)$ in the sense of Theorem 6.3.2 with $g = -\lambda u_z$ and $L = D(\lambda) \times \mathbb{R}$. We denote this solution by $\Phi(u_0, u_1) \coloneqq u$.

Remark 6.3.5. If additionally u_0, u_1, λ are odd around $z = z_0$, then the solution of (6.11) is again odd around $z = z_0$. To see this, notice that under these assumptions the odd reflection of the solution u of (6.11) again solves (6.11) – but with the opposite factorization of the wave operator. Hence, by Remark 6.3.3 and uniqueness of solutions, u coincides with its odd reflection.

Proof of Corollary 6.3.4. W.l.o.g. we assume $(z_0, t_0) = (0, 0)$. Let $u \in C^1(\Delta)$. Then by Theorem 6.3.2 u is a solution if and only if

$$u(z,t) = \tfrac{1}{2}(u_0(z+t) + u_0(z-t)) + \tfrac{1}{2}\int_{z-t}^{z+t} u_1(y)\,\mathrm{d}y - \tfrac{1}{2}\int_{\Delta(z,0,t)} \lambda(y)u_z(y,\tau)\,\mathrm{d}(y,\tau) \quad (6.12)$$

holds for $(z,t) \in \Delta$. Taking the derivative w.r.t. z we obtain

$$\begin{aligned} u_z(z,t) &= \tfrac{1}{2}(u_0'(z+t) + u_0'(z-t)) + \tfrac{1}{2}(u_1(z+t) - u_1(z-t)) \\ &\quad - \tfrac{1}{2}\int_0^t \lambda(z+t-s)u_z(z+t-s,s)\,\mathrm{d}s + \tfrac{1}{2}\int_0^t \lambda(z-t+s)u_z(z-t+s,s)\,\mathrm{d}s. \end{aligned} \quad (6.13)$$

We consider (6.13) as a fixed point problem for $u_z \in C(\Delta)$. If we denote the right-hand side of (6.13) by $T(u_z)(z,t)$, then clearly T maps $C(\Delta)$ into itself. Furthermore, one has

$$\begin{aligned} &\|T(u_z) - T(w_z)\|_\infty \\ &= \tfrac{1}{2}\sup_{(z,t)\in\Delta}\left| -\int_0^t \lambda(z+s)\cdot[u_z - w_z](z+s,t-s)\,\mathrm{d}s + \int_0^t \lambda(z-s)\cdot[u_z-w_z](z-s,t-s)\,\mathrm{d}s \right| \\ &\le \|\lambda\|_\infty r \cdot \|u_z - w_z\|_\infty \end{aligned}$$

so that by Banach's fixed-point theorem there exists a unique solution u_z of (6.13). With the help of u_z we define u as in (6.12) and thus get the claimed result. □

In the setting of the above proof, we can obtain estimates on the solution u. First, if we set $q := r\|\lambda\|_\infty$, then by Banach's fixed-point theorem we have

$$\|u_z - 0\|_\infty \le \frac{1}{1-q}\|T(0) - 0\|_\infty.$$

Using $\|T(0)\|_\infty \le \|u_0'\|_\infty + \|u_1\|_\infty$, we obtain

$$\|u_z\|_\infty \le \frac{1}{1-q}(\|u_0'\|_\infty + \|u_1\|_\infty)$$

From

$$\begin{aligned} u(z,t) &= \tfrac{1}{2}(u_0(z+t) + u_0(z-t)) + \tfrac{1}{2}\int_{z-t}^{z+t} u_1(y)\,\mathrm{d}y - \tfrac{1}{2}\int_0^t \int_{z-(t-\tau)}^{z+(t-\tau)} \lambda(y)u_z(y,\tau)\,\mathrm{d}y\mathrm{d}\tau, \\ u_t(z,t) &= \tfrac{1}{2}(u_0'(z+t) - u_0'(z-t)) + \tfrac{1}{2}(u_1(z+t) + u_1(z-t)) \\ &\quad - \tfrac{1}{2}\int_0^t \lambda(z+s)u_z(z+s,t-s)\,\mathrm{d}s - \tfrac{1}{2}\int_0^t \lambda(z-s)u_z(z-s,t-s)\,\mathrm{d}s \end{aligned}$$

we also obtain

$$\|u\|_\infty \le \|u_0\|_\infty + r\|u_1\|_\infty + \tfrac{1}{2}r^2\|\lambda\|_\infty\|u_z\|_\infty, \qquad \|u_t\|_\infty \le \|u_0'\|_\infty + \|u_1\|_\infty + r\|\lambda\|_\infty\|u_z\|_\infty.$$

Combining these estimates, we get the following result.

Corollary 6.3.6. *In the setting of Corollary 6.3.4, the following estimates hold with* $q := r\|\lambda\|_\infty$*:*

$$\begin{aligned}
\|u\|_\infty &\le \|u_0\|_\infty + \frac{rq}{2(1-q)}\|u_0'\|_\infty + \frac{r(1-\frac{1}{2}q)}{1-q}\|u_1\|_\infty,\\
\|u_z\|_\infty &\le \frac{1}{1-q}(\|u_0'\|_\infty + \|u_1\|_\infty),\\
\|u_t\|_\infty &\le \frac{1}{1-q}(\|u_0'\|_\infty + \|u_1\|_\infty).
\end{aligned}$$

In particular, there exists a constant $C = C(r, \|\lambda\|_\infty)$ *such that the operator-norm of the linear solution operator* $\Phi : C^1(B) \times C(B) \to C^1(\Delta)$*, which maps the data* $(u_0, u_1) \in C^1(B) \times C(B)$ *to the solution of* (6.11)*, satisfies*

$$\|\Phi\| \le C.$$

Recall that in Definition 6.2.3 we required $\frac{u_z}{c}$ to be continuous. Since c may have jumps, say, at z_0, we also need to treat the jump condition

$$\frac{u_z(z_0+,t)}{c(z_0+)} = \frac{u_z(z_0-,t)}{c(z_0-)}.$$

We prepare this in the following lemma by adding to (6.11) the inhomogeneous Dirichlet condition $u(z_0,t) \overset{!}{=} b(t)$ at the spatial boundary $z = z_0$.

Lemma 6.3.7. *Let* $(z_0,t_0) \in \mathbb{R}^2$ *and* $\Delta_+ := \Delta_+(z_0,t_0,r)$*,* $B_+ := P_z\Delta_+$ *for* $r > 0$*. Assume* $u_0 \in C^1(B_+)$*,* $u_1 \in C(B_+), b \in C^1([t_0,t_0+r])$ *with* $b(t_0) = u_0(z_0), b'(t_0) = u_1(z_0)$ *and* $\lambda \in PC(B_+)$ *such that* $r\|\lambda\|_\infty < 1$*. Then the problem*

$$\begin{cases}
(\partial_t - \partial_z)(u_t + u_z) = -\lambda(z)u_z, & (z,t) \in \Delta_+^\circ,\\
u(z_0,t) = b(t), & t \in [t_0, t_0+r],\\
u(z,t_0) = u_0(z), u_t(z,t_0) = u_1(z), & z \in B_+,
\end{cases} \tag{6.14}$$

has a unique C^1*-solution* $u\colon \Delta_+ \to \mathbb{R}$ *in the sense of Theorem 6.3.2 with* $g = -\lambda u_z$ *and* $L = D(\lambda) \times \mathbb{R}$*. We denote this solution by* $\Phi_+(b,u_0,u_1) := u$*. The assertion also holds for the left half triangle* $\Delta_- := \Delta_-(z_0,t_0,r)$ *with corresponding solution operator* Φ_-*.*

Proof. Note that the function G^b defined on Δ_+ by

$$G^b(z,t) = \begin{cases}
b(t_0) + (t-t_0)b'(t_0), & z - z_0 > t - t_0 \ge 0,\\
b(t+z_0-z) + (z-z_0)b'(t_0), & t - t_0 \ge z - z_0 \ge 0
\end{cases} \tag{6.15}$$

belongs to $C^1(\Delta_+)$, solves the homogenous wave equation $(\partial_t - \partial_z)(\partial_t + \partial_z)G^b = 0$ on Δ_+, and satisfies $G^b(z_0,t) = b(t)$. Setting $v := u - G^b$, problem (6.14) can be rewritten as

$$\begin{cases}
(\partial_t - \partial_z)(v_t + v_z) = -\lambda(z)\Big(v_z + G^b_z\Big), & (z,t) \in \Delta_+^\circ,\\
v(z_0,t) = 0, & t \in [t_0,t_0+r],\\
v(z,t_0) = u_0(z) - b(t_0) =: v_0(z), & z \in B_+,\\
v_t(z,t_0) = u_1(z) - b'(t_0) =: v_1(z), & z \in B_+.
\end{cases} \tag{6.16}$$

Note that $v_0(z_0) = v_1(z_0) = 0$ by assumption. If we extend the functions v_0, v_1, and λ in an odd way and G^b in an even way around $z = z_0$, we can consider the problem

$$\begin{cases} (\partial_t - \partial_z)(\tilde{v}_t + \tilde{v}_z) = -\lambda_{\text{odd}}(z) \cdot \left(\tilde{v}_z + G^b_{\text{even},z}\right) & (z,t) \in \Delta^\circ, \\ \tilde{v}(z,t_0) = v_{0,\text{odd}}(z), & z \in B, \\ \tilde{v}_t(z,t_0) = v_{1,\text{odd}}(z), & z \in B, \end{cases} \tag{6.17}$$

where $\Delta := \Delta(z_0, t_0, r)$ and $B := P_z\Delta$. Arguing as in the proof of Corollary 6.3.4, we see that due to the Banach fixed-point theorem, (6.17) has a unique solution, which must be odd, cf. Remark 6.3.5. Now, on one hand the solution of (6.17) solves (after restriction to Δ_+) (6.16) and, on the other hand, after odd extension around $z = z_0$ every solution of (6.16) solves (6.17). This shows existence and uniqueness for (6.16) and hence for (6.14). □

Remark 6.3.8. One can show that there exists a constant $C = C(r, \|\lambda\|_\infty)$ such that

$$\Phi_\pm \colon \mathcal{D}(\Phi_\pm) \subseteq C^1([t_0, t_0 + r]) \times C^1(B_\pm) \times C(B_\pm) \to C^1(\Delta_\pm)$$

satisfy $\|\Phi_\pm\| \le C$, where the domain $\mathcal{D}(\Phi_\pm)$ consists of those (b, u_0, u_1) that satisfy $b(t_0) = u_0(z_0)$ and $b'(t_0) = u_1(z_0)$.

When treating the nonlinear problem (6.1), the operators $\Phi_\pm$ play an important role and the estimate in Remark 6.3.8 will be used. However, we need to investigate the dependency of $\Phi_\pm$ on the datum b more precisely. This will be achieved next in the case where $u_0 = u_1 = 0$.

Lemma 6.3.9 (Estimate on $\Phi_\pm$ in the case $u_0 = u_1 = 0$). *Let $\Delta_\pm$ and λ be as in Lemma 6.3.7 with $q := r\|\lambda\|_\infty < 1$. Assume $b \in C^1([t_0, t_0 + r])$ and $b(t_0) = b'(t_0) = 0$. Then for $u := \Phi_\pm(b, 0, 0)$ one has*

$$|u_z(z,t) \pm b'(m)| \le \alpha|z - z_0||b'(m)| + \beta \int_{t_0}^{m} |b'(\tau)| \,\mathrm{d}\tau,$$

where $m := \max\{t_0, t - |z - z_0|\}$, $\alpha := \frac{2}{4-q}\|\lambda\|_\infty$, and $\beta := \frac{4}{(2-q)(4-q)}\|\lambda\|_\infty$.

Proof. We only give the proof in the "+"-case and for $(z_0, t_0) = (0, 0)$. We revisit the proof of Lemma 6.3.7 where Φ_+ is defined. From (6.13) we know that v_z satisfies

$$\begin{aligned} v_z(z,t) = &-\tfrac{1}{2}\int_0^t \lambda_{\text{odd}}(z+s) \cdot \left(G^b_{\text{even},z}(z+s, t-s) + v_z(z+s, t-s)\right) \mathrm{d}s \\ &+ \tfrac{1}{2}\int_0^t \lambda_{\text{odd}}(z-s) \cdot \left(G^b_{\text{even},z}(z-s, t-s) + v_z(z-s, t-s)\right) \mathrm{d}s. \end{aligned}$$

We denote the term on the right-hand side by $T(v_z)(z,t)$ and already know that T is Lipschitz continuous with constant $q < 1$. Therefore we may write the solution as $v_z := \lim_{n\to\infty} T^n(0)$ and thus have to study $v_z^{(n)} := T^n(0)$. The claimed inequality for u_z will follow once we have shown that

$$|v_z(z,t)| \le \alpha|z - z_0||b'(m)| + \beta \int_{t_0}^{m} |b'(\tau)| \,\mathrm{d}\tau.$$

Due to $v_z = \lim_{n\to\infty} T^n(0)$ it is sufficient to show that this estimate holds for all $v_z^{(n)}$. Since $v_z^{(0)} = 0$, there is nothing left to show for $n = 0$. Now assume that the estimate has been shown for some fixed n. Recalling the definition of G^b from (6.15), we have

$$G^b_{\text{even},z}(z,t) = -\operatorname{sign}(z)b'(\max\{t - |z|, 0\}).$$

Notice that $G^b_{\text{even},z}(z,t)$ vanishes for $|z| \geq t$. Therefore, if $v_z^{(n)}$ vanishes for $|z| \geq t$ then also $v_z^{(n+1)} = T(v_z^{(n)})$ vanishes on this set. So in the following we may assume $|z| < t$. We will only consider $z \geq 0$ as $z < 0$ can be treated similarly. For $z \geq 0$ and $t > z$ the expression $m = \max\{t - |z|, 0\}$ simplifies to $m = t - z$. We begin by estimating the terms which are independent of $v_z^{(n)}$:

$$\begin{aligned}
&\left|\int_0^t \lambda_{\text{odd}}(z+s) G^b_{\text{even},z}(z+s, t-s)\,\mathrm{d}s\right| \\
&\quad = \left|-\int_0^t \lambda_{\text{odd}}(z+s) b'(\max\{t-z-2s, 0\})\,\mathrm{d}s\right| \\
&\quad \leq \tfrac{1}{2}\|\lambda\|_\infty \int_0^{t-z} |b'(\tau)|\,\mathrm{d}\tau = \tfrac{1}{2}\|\lambda\|_\infty \int_0^m |b'(\tau)|\,\mathrm{d}\tau, \\
&\left|\int_0^t \lambda_{\text{odd}}(z-s) G^b_{\text{even},z}(z-s, t-s)\,\mathrm{d}s\right| \\
&\quad = \left|-\int_0^z \lambda_{\text{odd}}(z-s) b'(t-z)\,\mathrm{d}s + \int_z^t \lambda_{\text{odd}}(z-s) b'(\max\{t+z-2s, 0\})\,\mathrm{d}s\right| \\
&\quad \leq \|\lambda\|_\infty z|b'(t-z)| + \tfrac{1}{2}\|\lambda\|_\infty \int_0^{t-z} |b'(\tau)|\,\mathrm{d}\tau = \|\lambda\|_\infty z|b'(m)| + \tfrac{1}{2}\|\lambda\|_\infty \int_0^m |b'(\tau)|\,\mathrm{d}\tau.
\end{aligned}$$

The remaining two summands are treated by

$$\begin{aligned}
&\left|\int_0^t \lambda_{\text{odd}}(z+s) v_z^{(n)}(z+s, t-s)\,\mathrm{d}s\right| \\
&\quad \leq \|\lambda\|_\infty \int_0^t \left(\alpha(z+s)|b'(\max\{t-z-2s, 0\})| + \beta \int_0^{\max\{t-z-2s,0\}} |b'(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad = \|\lambda\|_\infty \int_0^{\frac{t-z}{2}} \left(\alpha(z+s)|b'(t-z-2s)| + \beta \int_0^{t-z-2s} |b'(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \leq \|\lambda\|_\infty \int_0^{\frac{t-z}{2}} \left(\alpha \frac{t+z}{2}|b'(t-z-2s)| + \beta \int_0^{t-z} |b'(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad = \|\lambda\|_\infty \left(\alpha \frac{t+z}{4} + \beta \frac{t-z}{2}\right) \int_0^m |b'(\tau)|\,\mathrm{d}\tau, \\
&\left|\int_0^t \lambda_{\text{odd}}(z-s) v_z^{(n)}(z-s, t-s)\,\mathrm{d}s\right| \\
&\quad \leq \|\lambda\|_\infty \int_0^t \left(\alpha|z-s||b'(\max\{t-s-|z-s|, 0\})| + \beta \int_0^{\max\{t-s-|z-s|,0\}} |b'(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad = \|\lambda\|_\infty \int_0^z \left(\alpha(z-s)|b'(t-z)| + \beta \int_0^{t-z} |b'(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\qquad + \|\lambda\|_\infty \int_z^{\frac{z+t}{2}} \left(\alpha(s-z)|b'(t+z-2s)| + \beta \int_0^{t+z-2s} |b'(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \leq \|\lambda\|_\infty \left(\alpha \frac{z^2}{2}|b'(m)| + \beta z \int_0^m |b'(\tau)|\,\mathrm{d}\tau\right) \\
&\qquad + \|\lambda\|_\infty \left(\alpha \frac{t-z}{4} + \beta \frac{t-z}{2}\right) \int_0^m |b'(\tau)|\,\mathrm{d}\tau.
\end{aligned}$$

Summing up all four estimates, we obtain

$$\begin{aligned}
&2\left|v_z^{(n+1)}(z,t)\right| \\
&\quad\le \tfrac{1}{2}\|\lambda\|_\infty \int_0^m |b'(\tau)|\,\mathrm{d}\tau \\
&\qquad + \|\lambda\|_\infty z|b'(m)| + \tfrac{1}{2}\|\lambda\|_\infty \int_0^m |b'(\tau)|\,\mathrm{d}\tau \\
&\qquad + \|\lambda\|_\infty \left(\alpha\frac{t+z}{4} + \beta\frac{t-z}{2}\right)\int_0^m |b'(\tau)|\,\mathrm{d}\tau \\
&\qquad + \|\lambda\|_\infty \left(\alpha\frac{z^2}{2}|b'(m)| + \beta z\int_0^m |b'(\tau)|\,\mathrm{d}\tau\right) \\
&\qquad + \|\lambda\|_\infty \left(\alpha\frac{t-z}{4} + \beta\frac{t-z}{2}\right)\int_0^m |b'(\tau)|\,\mathrm{d}\tau \\
&\quad= \|\lambda\|_\infty\left(1+\alpha\frac{z}{2}\right)z|b'(m)| \\
&\qquad + \|\lambda\|_\infty\left(\tfrac{1}{2}+\tfrac{1}{2}+\alpha\frac{t+z}{4}+\beta\frac{t-z}{2}+\beta z+\alpha\frac{t-z}{4}+\beta\frac{t-z}{2}\right)\int_0^m |b'(\tau)| \\
&\quad=: 2C_1 z|b'(m)| + 2C_2\int_0^m |b'(\tau)|\,\mathrm{d}\tau.
\end{aligned}$$

It remains to verify $C_1 \le \alpha$ and $C_2 \le \beta$. In fact, using $t, z \le r$, we obtain

$$\begin{aligned}
2C_1 &\le \|\lambda\|_\infty + \frac{q}{2}\alpha = 2\alpha, \\
2C_2 &\le \|\lambda\|_\infty + \frac{q}{2}\alpha + q\beta = 2\alpha + q\beta = 2\beta,
\end{aligned}$$

where the equalities hold by definition of α and β, respectively. □

6.4. MAIN PART OF PROOF OF THEOREM 6.1.1

In this section, we will prove the existence and uniqueness part of the main Theorem 6.1.1 under the additional assumption that f grows at least linearly, i.e., for some $a, A > 0$ we have

($\mathcal{A}6.4$) $|f(x)| \ge a|x| - A$ for $x \in \mathbb{R}$.

The assumption ($\mathcal{A}6.4$) will be used in Lemma 6.4.3 below as an upper bound on f^{-1} which helps in the construction of solutions to (6.10). In Section 6.5 we show that the argument of f, that is $u_t(0,t)$, is uniformly bounded on finite time intervals, and thereby eliminate the growth assumptions on f. The well-posedness part of Theorem 6.1.1 will be completed in Section 6.6.

We will again use that the wave equation has finite speed of propagation so that we may argue locally. To be more specific, we will work on the following types of triangular domains:

- A *jump triangle* is a triangle $\Delta = \Delta(z_0, 0, r)$ with base $B = P_z\Delta \subseteq (0,\infty)$, where $z_0 \in D(c)$ and B intersects $D(c)$ in no other point. These are useful for the study of the jump condition $\frac{u_z(z+,t)}{c(z+)} = \frac{u_z(z-,t)}{c(z-)}$.
- A *boundary triangle* is a half-triangle $\Delta_+ = \Delta_+(0,0,r)$ with base $B_+ = P_z\Delta_+ = [0,r]$ where B_+ does not intersect $D(c)$. These are used to study the nonlinear Neumann condition $\frac{u_z}{c(0)} = (f(u_t))_t$.

- A *plain triangle* is a triangle $\Delta = \Delta(z_0, 0, r)$ with base $B = P_z\Delta \subseteq (0, \infty)$ not intersecting $D(c)$. These are used to cover the remaining space.

In the next three Lemmata, we show that (6.10) is well-posed on all three types of domains.

Lemma 6.4.1. *Let Δ be a plain triangle with base B. Assume $r\|\frac{c_z}{c}\|_\infty < 1$. Then (6.10) has a unique C^1-solution u on Δ and there exists a constant $C = C(r, \|\frac{c_z}{c}\|_\infty)$ such that the solution operator $\Phi\colon C^1(B) \times C(B) \to C^1(\Delta), (u_0, u_1) \mapsto u$ satisfies $\|\Phi\| \le C$.*

Proof. Since Δ is disjoint from the spatial boundary $z = 0$, the boundary condition (ii) in Definition 6.2.3 is trivially satisfied on Δ. By Corollary 6.3.4 we have uniqueness of solutions, and the estimate holds by Corollary 6.3.6. □

Lemma 6.4.2. *Let Δ be a jump triangle with base B. Assume $r\|\frac{c_z}{c}\|_\infty < 1$. Then (6.10) has a unique C^1-solution u on Δ and there exists a constant $C = C(r, \|\frac{c_z}{c}\|_\infty)$ such that the solution operator $\Phi\colon C^1_x(B) \times C(B) \to C^1_{(x,t)}(\Delta), (u_0, u_1) \mapsto u$ satisfies $\|\Phi\| \le C$.*

Proof. As in Lemma 6.4.1, the boundary condition at $z = 0$ trivially holds. Now let $\Delta = \Delta(z_0, 0, r)$. If $u\colon \Delta \to \mathbb{R}$ is a solution of (6.10), then by defining $b\colon [0, r] \to \mathbb{R}, b(t) = u(z_0, t)$ and using Lemma 6.3.7 we have

$$u(z,t) = \begin{cases} \Phi_+(b, u_0, u_1)(z,t), & z \ge z_0, \\ \Phi_-(b, u_0, u_1)(z,t), & z \le z_0. \end{cases} \tag{6.18}$$

On the other hand, if $b \in C^1([0, r])$ with $b(0) = u_0(z_0)$ and $b'(0) = u_1(z_0)$ is given, then the function u defined by (6.18) satisfies $u, u_t \in C(\Delta)$ as $\Phi_\pm(b, u_0, u_1)$ and $\Phi_\pm(b, u_0, u_1)_t$ coincide with b resp. b' at the boundary $z = z_0$. Hence, u solves (6.10) if and only if u_x is continuous, i.e.,

$$\frac{u_z(z_0+, t)}{c(z_0+)} = \frac{u_z(z_0-, t)}{c(z_0-)} \tag{6.19}$$

holds for all $t \in [0, r]$. Using (6.18), we can write (6.19) as

$$\frac{1}{c(z_0-)}\Phi_-(b, u_0, u_1)_z(z_0, t) = \frac{1}{c(z_0+)}\Phi_+(b, u_0, u_1)_z(z_0, t)$$

or as

$$b'(t) = \gamma\Big(\frac{1}{c(z_0-)}(b'(t) - \Phi_-(b, u_0, u_1)_z(z_0, t)) + \frac{1}{c(z_0+)}(b'(t) + \Phi_+(b, u_0, u_1)_z(z_0, t))\Big) \tag{6.20}$$

with

$$\gamma := \Big(\frac{1}{c(z_0-)} + \frac{1}{c(z_0+)}\Big)^{-1}.$$

We denote the right-hand side of (6.20) by $T(b)(t)$ and show now that $\Psi\colon b \mapsto u_0(z_0) + \int_0^{(\cdot)} T(b)(\tau)\,\mathrm{d}\tau$ is a strict contraction in the space $X := \{b \in C^1([0, r]) \mid b(0) = u_0(z_0)\}$ with norm $\|b\|_X = \sup\{\mathrm{e}^{-\mu t}|b'(t)|\colon t \in [0, r]\}$, where $\mu > 0$ will be chosen later. So let $b, \tilde b \in X$ and write $\hat b := b - \tilde b$. Next we estimate

$$\begin{aligned} &\Big|\Psi(b)'(t) - \Psi(\tilde b)'(t)\Big| \\ &\quad = \gamma\Big|\frac{1}{c(z_0-)}\Big(\hat b'(t) - \Phi_-(\hat b, 0, 0)_z(z_0, t)\Big) + \frac{1}{c(z_0+)}\Big(\hat b'(t) + \Phi_+(\hat b, 0, 0)_z(z_0, t)\Big)\Big| \end{aligned}$$

$$\leq \gamma\Big(\frac{1}{c(z_0-)}\beta \int_0^t \Big|\hat{b}'(\tau)\Big| \,\mathrm{d}\tau + \frac{1}{c(z_0+)}\beta \int_0^t \Big|\hat{b}'(\tau)\Big| \,\mathrm{d}\tau\Big)$$
$$= \beta \int_0^t \Big|\hat{b}'(\tau)\Big| \,\mathrm{d}\tau \leq \beta \Big\|\hat{b}\Big\|_X \int_0^t \mathrm{e}^{\mu\tau} \,\mathrm{d}\tau \leq \frac{\beta}{\mu}\mathrm{e}^{\mu t}\Big\|\hat{b}\Big\|_X,$$

where β is the constant from Lemma 6.3.9. If we choose $\mu > \beta$, then Ψ is a strict contraction so that $b = \Psi(b)$ has a unique solution by Banach's fixed-point theorem. Using Remark 6.3.8, the fixed-point theorem also shows that b linearly and continuously depends on u_0 and u_1. Moreover, boundedness of the linear solution operator Φ then follows from (6.18). □

Next we discuss well-posedness on boundary triangles. Unlike for the other types of triangles, now the nonlinear boundary condition of (6.10) appears, and becomes the main object of our study.

Lemma 6.4.3. *Let Δ_+ be a boundary triangle with base B_+. Assume $r\|\frac{c_z}{c}\|_\infty < 1$. Then* (6.10) *has a unique C^1-solution on Δ_+.*

Let us give a motivation of this result. As in Lemma 6.4.2 it will be convenient to rephrase the problem as an ordinary differential equation. Again we write $b(t) = u(0,t)$ so that u is a solution on Δ_+ if and only if

$$u = \Phi_+(b, u_0, u_1) \quad \text{and} \quad \frac{\mathrm{d}f(u_t(0,t))}{\mathrm{d}t} = \frac{u_z(0,t)}{c(0)}$$

hold. We may rewrite the latter equation as

$$\frac{\mathrm{d}f(b'(t))}{\mathrm{d}t} = \frac{1}{c(0)}\Phi_+(b, u_0, u_1)_z(0,t),$$

eliminating u. We rewrite this as an equation in $d(t) \coloneqq f(b'(t))$, where b can be reconstructed from d via $b_d(t) \coloneqq u_0(0) + \int_0^t f^{-1}(d(\tau))\,\mathrm{d}\tau$. We are left with solving

$$d'(t) = \frac{1}{c(0)}\Phi_+(b_d, u_0, u_1)_z(0,t), \qquad d(0) = f(u_1(0)). \tag{6.21}$$

We have $\Phi_+(b, u_0, u_1)_z(0,t) = -b'(t) + g(t)$ where g depends (up to a small error) only on the initial data u_0, u_1, hence

$$d'(t) = \frac{1}{c(0)}\big[g(t) - b_d'(t)\big] = \frac{1}{c(0)}\Big[g(t) - f^{-1}(d(t))\Big]. \tag{6.22}$$

Ignoring the error, (6.22) would be an ODE with monotone decreasing right-hand side (in $d(t)$), which is known to be uniquely solvable. Lemma 6.3.9 gives us an estimate on this small error and is the main ingredient in the uniqueness proof, and we use the estimate ($\mathcal{A}$6.4) and a fixed-point argument to show existence.

Proof of Lemma 6.4.3. It suffices to show that (6.21) has a unique solution.

Uniqueness: Assume that $d, \tilde{d}$ are solutions to (6.21) that coincide up to time $t_\star \geq 0$, but not at time t_n for some $t_n \geq 0$ with $t_n \downarrow t_\star$ as $n \to \infty$. Define $\delta(t) \coloneqq \Big|f^{-1}(d(t)) - f^{-1}(\tilde{d}(t))\Big|$. For $\varepsilon > 0$ consider the function

$$h_\varepsilon(t) \coloneqq \varepsilon(1 + t - t_\star) + \frac{1}{c(0)}\int_{t_\star}^t \Big(-\delta(s) + \beta\int_{t_\star}^s \delta(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s,$$

where β is the constant from Lemma 6.3.9.

Claim: The inequality $\left|d(t) - \tilde{d}(t)\right| < h_\varepsilon(t)$ holds for all $t \geq t_\star$.

Clearly, the claim holds true for $t = t_\star$, and thus by continuity for t close to $t_\star$. Assume the claim is false. Then there exists some minimal $t_i > t_\star$ such that $\left|d(t_i) - \tilde{d}(t_i)\right| = h_\varepsilon(t_i)$. W.l.o.g. assume that $d(t_i) \geq \tilde{d}(t_i)$. Since $d(t) - \tilde{d}(t) < h_\varepsilon(t)$ for $t_\star \leq t < t_i$, we get $d'(t_i) - \tilde{d}'(t_i) \geq h'_\varepsilon(t_i)$ which implies

$$\frac{1}{c(0)}\Phi_+(b_d, 0, 0)_z(0, t_i) - \frac{1}{c(0)}\Phi_+(b_{\tilde{d}}, 0, 0)_z(0, t_i) \geq \varepsilon + \frac{1}{c(0)}\left(-\delta(t_i) + \beta\int_{t_\star}^{t_i}\delta(\tau)\,\mathrm{d}\tau\right)$$

and hence

$$\Phi_+(b_d - b_{\tilde{d}}, 0, 0)_z(0, t_i) + \delta(t_i) > \beta\int_{t_\star}^{t_i}\delta(\tau)\,\mathrm{d}\tau \geq 0. \tag{6.23}$$

On the other hand, setting $b := b_d - b_{\tilde{d}}$ we have

$$\left|\Phi_+(b, 0, 0)_z(0, t_i) + b'(t_i)\right| \leq \beta\int_{t_\star}^{t_i}|b'(\tau)|\,\mathrm{d}\tau$$

due to Lemma 6.3.9. Since $b'(t_i) = f^{-1}(d(t_i)) - f^{-1}(\tilde{d}(t_i))$ and since f^{-1} is increasing, we see that $b'(t_i) = \delta(t_i)$. Combining these facts, we find

$$|\Phi_+(b, 0, 0)_z(0, t_i) + \delta(t_i)| \leq \beta\int_{t_\star}^{t_i}\delta(\tau)\,\mathrm{d}\tau$$

which contradicts (6.23). So the claim holds.

Letting ε go to 0, we obtain

$$\left|d(t) - \tilde{d}(t)\right| \leq \frac{1}{c(0)}\int_{t_\star}^{t}\left(-\delta(s) + \beta\int_{t_\star}^{s}\delta(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s$$

for any $t \geq t_\star$. Fubini implies that the term on the right-hand side is negative for $t \in (t_\star, t_\star + \frac{1}{\beta})$, a contradiction.

Existence: Let $D, \mu > 0$. Consider the set

$$K := \{d \in W^{1,\infty}([0, r])\colon d(t_0) = f^{-1}(u_1(0)), |d(t)| \leq D\mathrm{e}^{\mu t}, |d'(t)| \leq D\mu\mathrm{e}^{\mu t} \text{ for } t \in [0, r]\},$$

which is a convex and compact subset of $C([0, r])$, as well as the operator

$$T\colon K \to C([0, r]), \quad T(d)(t) = f^{-1}(u_1(0)) + \frac{1}{c(0)}\int_{t_0}^{t}\Phi_+(b_d, u_0, u_1)_z(0, \tau)\,\mathrm{d}\tau.$$

We choose $D := \max\{|f^{-1}(u_1(0))|, 1\}$, so that K is nonempty as it contains the constant function $d \equiv f^{-1}(u_1(0))$. To see that T is continuous, let $d_n \in K$ with $d_n \to d$ in $C([0, r])$ as $n \to \infty$. As f^{-1} is uniformly continuous on $[-D\mathrm{e}^{\mu r}, D\mathrm{e}^{\mu r}]$, we have $f^{-1} \circ d_n \to f^{-1} \circ d$ in $C([0, r])$, from which it follows that

$$b_{d_n} = u_0(0) + \int_0^{(\cdot)} f^{-1}(d_n(\tau))\,\mathrm{d}\tau$$

converges to

$$b_d = u_0(0) + \int_0^{(\cdot)} f^{-1}(d(\tau))\,\mathrm{d}\tau.$$

in $C^1([0,r])$. Due to Remark 6.3.8, the operator $\Phi_+(\,\cdot\,,u_0,u_1)\colon C^1([0,r]) \to C^1(\Delta_+)$ is continuous. Hence $T(d_n) \to T(d)$ in $C([0,r])$ as $n \to \infty$.

To check that T maps into K, we need to verify that for any $d \in K$ one has

$$|T(d)'(t)| \leq D\mu e^{\mu t}. \tag{6.24}$$

Notice that $|d(t)| \leq De^{\mu t}$ follows from (6.24) by integration. By assumption ($\mathcal{A}$6.4) on the growth on f we have $|f^{-1}(y)| \leq \frac{|y|+A}{a}$, and in particular $|b_d'(t)| = |f^{-1}(d(t))| \leq \frac{De^{\mu t}+A}{a}$. We use this inequality, $|b_d(t)| \leq |u_0(0)| + t\|b_d'\|_\infty$ as well as Remark 6.3.8 to estimate

$$\begin{aligned}
|T(d)'(t)| &= \frac{1}{c(0)}|\Phi_+(b_d,u_0,u_1)_z(0,t)| \\
&\leq \frac{C}{c(0)}\Big(\|b_d\|_{[0,t],C^1} + \|u_0\|_{C^1} + \|u_1\|_\infty\Big) \\
&\leq \frac{C}{c(0)}\Big((1+t)\|b_d'\|_{[0,t],\infty} + 2\|u_0\|_{C^1} + \|u_1\|_\infty\Big) \\
&\leq \frac{C}{c(0)}\left((1+t)\frac{De^{\mu t}+A}{a} + 2\|u_0\|_{C^1} + \|u_1\|_\infty\right) \\
&\leq \frac{C}{c(0)}\left((1+r)\frac{D+A}{a} + 2\|u_0\|_{C^1} + \|u_1\|_\infty\right)e^{\mu t}.
\end{aligned}$$

Therefore T maps K into itself if we choose

$$\mu := \frac{C}{c(0)D}\left((1+r)\frac{D+A}{a} + 2\|u_0\|_{C^1} + \|u_1\|_\infty\right).$$

Hence existence follows by applying Schauder's fixed-point Theorem. □

Using the existence and uniqueness results on plain, jump, and boundary triangles shown above, next we prove existence and uniqueness on the whole space $[0,\infty)\times[0,\infty)$ by covering it with these specific triangles.

Proof of Theorem 6.1.1 with additional assumption ($\mathcal{A}$6.4). We show existence and uniqueness of the solution to (6.1) under the assumption ($\mathcal{A}$6.4). Well-posedness will be discussed in Section 6.6.

Existence: Denote by $\mathcal{C}$ the set containing all jump, boundary and plain triangles where the heights r have to satisfy $r\|\frac{c_z}{c}\|_\infty < 1$. As we have just shown in the previous three lemmata, (6.10) admits a unique solution on each $\Delta \in \mathcal{C}$. Since $\mathcal{C}$ is closed with respect to finite intersection, we obtain a solution u of (6.10) on $\cup_{\Delta\in\mathcal{C}}\Delta$. With

$$h := \tfrac{1}{2}\min\left\{\left\|\frac{c_z}{c}\right\|_\infty^{-1}, |z_1 - z_2|\colon z_1, z_2 \in D(c)\cup\{0\}, z_1 \neq z_2\right\} > 0$$

we have $[0,\infty)\times[0,h) \subseteq \cup_{\Delta\in T}\Delta$, see Figure 6.1 for an illustration of this covering property. By restriction, we therefore obtain a solution $u^{(1)}$ of (6.1) on $[0,\infty)\times[0,\tilde{h}]$ for any $0<\tilde{h}<h$. The same argument, used with initial data $u_0^{(2)}(z) := u^{(1)}(z,\tilde{h})$ and $u_1^{(2)}(z) := u_t^{(1)}(z,\tilde{h})$ instead of u_0, u_1, yields another solution $u^{(2)}$ on $[0,\infty)\times[0,\tilde{h}]$. Repeating this, we construct solutions $u^{(k)}$ on $[0,\infty)\times[0,\tilde{h}]$ with $u^{(k+1)}(z,0) = u^{(k)}(z,\tilde{h})$ and $u_t^{(k+1)}(z,0) = u_t^{(k)}(z,\tilde{h})$ for $k\in\mathbb{N}$.

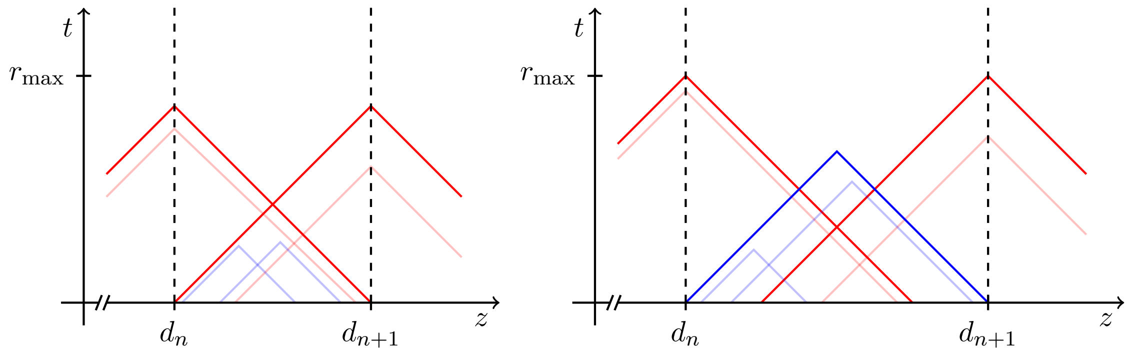

Figure 6.1.: Sketch: Covering of $[0,\infty) \times [0,\infty)$ by jump triangles (red) and plain triangles (blue), with $r_{\max} := \|\frac{c_z}{c}\|_\infty^{-1}$ being the maximum height of triangles. Black dashed lines indicate jumps of c. Left: $|d_{n+1} - d_n| < r_{\max}$ where covering has height $\frac{1}{2}|d_{n+1} - d_n|$, right: $|d_{n+1} - d_n| > r_{\max}$ where covering has height $\frac{1}{2}r_{\max}$.

Finally, we define the map $u\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $u(z, (k-1)\tilde{h} + \tau) = u^{(k)}(z,\tau)$ for $\tau \in [0,\tilde{h}]$, which solves (6.1).

Uniqueness: Assume that $u, \tilde{u}\colon \Omega \to \mathbb{R}$ are two different solutions to (6.10), where $\Omega = \{(z,t) \mid t \le h(z)\}$ is an admissible domain. So there exists $(z_0,t_0) \in \Omega$ with $u(z_0,t_0) \ne \tilde{u}(z_0,t_0)$. Consider the (possibly cut-off) triangle $\Delta := \Delta(z_0, 0, t_0) \cap \{z \ge 0\}$ and define the set $N := \{(z,t) \in \Delta \mid u(z,t) \ne \tilde{u}(z,t)\}$ and $t_{\inf} := \inf P_t(N)$, where P_t denotes the projection onto the second variable. Choose some sequence $(z_n,t_n) \in N$ with $t_n \to t_{\inf}$ and $z_n \to z_\infty \in [0,\infty)$.

For $\varepsilon > 0$ consider the (possibly cut-off) triangle $\Delta_\varepsilon := \Delta \cap \Delta(z_\infty, t_{\inf}, \varepsilon) \cap \{z \ge 0\}$ with base B_ε.

Claim: $u(z,t_{\inf}) = \tilde{u}(z,t_{\inf})$ and $u_t(z,t_{\inf}) = \tilde{u}_t(z,t_{\inf})$ hold for all $z \in B_\varepsilon$.

If $t_{\inf} = 0$, this holds because both u and $\tilde{u}$ satisfy the same initial conditions. If $t_{\inf} > 0$, by assumption we have $u(z,t) = \tilde{u}(z,t)$ for $z \in B_\varepsilon$ and $t < t_{\inf}$ as $(z,t) \in \Delta$ and therefore also $u_t(z,t) = \tilde{u}_t(z,t)$, so that the claim is obtained by taking the limit $t \to t_{\inf}$.

If we choose ε small enough, then Δ_ε is a jump (if $z_\infty \in D(c)$), boundary (if $z_\infty = 0$) or plain triangle (otherwise). By the previously established uniqueness results on these triangles, u and $\tilde{u}$ must coincide on Δ_ε. But since $t_n \ge t_{\inf}$ for all n, we have $(z_n,t_n) \in \Delta_\varepsilon$ for n sufficiently large, so that $u(z_n,t_n) = \tilde{u}(z_n,t_n)$. This cannot be since $(z_n,t_n) \in N$. $\square$

Remark 6.4.4 (Modifications for the bounded domain version). In order to capture the homogeneous Dirichlet boundary condition for the bounded domain version of the theorem, we also need to consider "Dirichlet" triangles Δ_- with center $z_0 = L$. Problem (6.1) is well-defined on the domain Δ_- assuming $r\|\frac{c_z}{c}\|_\infty < 1$. In fact the solution on "Dirichlet" triangles is simply given by $u = \Phi_-(0,u_0,u_1)$. We can then proceed as in the above proof to show existence and uniqueness of solutions, i.e., Remark 6.1.8. Conservation of energy can be shown as in Section 6.5.

6.5. Energy, Momentum, and Completion of Theorem 6.1.1

Using $V(x) = \frac{1}{c(x)^2}$, the energy (6.7) can be written as

$$\begin{aligned} E(u,t) &\coloneqq \tfrac{1}{2}\int_0^\infty \Big(V(x)u_t(x,t)^2 + u_x(x,t)^2\Big)\,\mathrm{d}x + F(u_t(0,t)) \\ &= \tfrac{1}{2}\int_0^\infty \left(\frac{1}{c(z)^2}u_t(z,t)^2 + \left(\frac{u_z(z,t)}{c(z)}\right)^2\right)\cdot c(z)\,\mathrm{d}z + F(u_t(0,t)) \\ &= \tfrac{1}{2}\int_0^\infty \frac{1}{c(z)}\Big(u_t(z,t)^2 + u_z(z,t)^2\Big)\,\mathrm{d}z + F(u_t(0,t)) \end{aligned}$$

where $F(y) = yf(y) - \int_0^y f(v)\,\mathrm{d}v$. In (z,t)–coordinates the momentum reads

$$M(u,t) = \int_0^\infty \frac{1}{c(z)}u_t(z,t)\,\mathrm{d}z + f(u_t(0,t)).$$

We now show that both quantities are time-invariant.

Proof of Theorem 6.1.4. Let $\Omega \subseteq [0,\infty)\times[0,\infty)$ be a Lipschitz domain such that c is C^1 on Ω. Recall that $(\partial_t \mp \partial_z)(u_t \pm u_z) + \frac{c_z}{c}u_z = 0$. In the following, for a term $a(\pm,\mp)$ which may have $\pm$ or $\mp$ signs, we write $\sum^{\pm} a(\pm,\mp) \coloneqq a(+,-) + a(-,+)$.

Part 1: Energy. With ν being the outer normal at $\partial\Omega$ we calculate

$$\begin{aligned} 0 &= \sum\nolimits^{\pm}\int_\Omega \left[(\partial_t \mp \partial_z)(u_t \pm u_z) + \frac{c_z}{c}u_z\right]\cdot\frac{1}{c}(u_t \pm u_z)\,\mathrm{d}(z,t) \\ &= \sum\nolimits^{\pm}\int_{\partial\Omega}(\nu_2 \mp \nu_1)\frac{1}{c}(u_t \pm u_z)^2\,\mathrm{d}\sigma \\ &\quad + \sum\nolimits^{\pm}\int_\Omega\left(\frac{c_z}{c^2}u_z(u_t \pm u_z) - \frac{1}{c}(u_t \pm u_z)\cdot(\partial_t \mp \partial_z)(u_t \pm u_z) \mp \frac{c_z}{c^2}(u_t \pm u_z)^2\right)\mathrm{d}(z,t). \end{aligned}$$

The sum $\sum^{\pm}$ over the boundary integrals can be simplified to

$$\sum\nolimits^{\pm}\int_{\partial\Omega}(\nu_2 \mp \nu_1)\frac{1}{c}(u_t \pm u_z)^2\,\mathrm{d}\sigma = \int_{\partial\Omega}\left(\frac{2}{c}\nu_2(u_t^2 + u_z^2) - \frac{4}{c}\nu_1 u_t u_z\right)\mathrm{d}\sigma.$$

The sum $\sum^{\pm}$ of the integrands in the integral over Ω vanishes as can be seen by the following calculation using once more the differential equation $(\partial_t \mp \partial_z)(u_t \pm u_z) + \frac{c_z}{c}u_z = 0$:

$$\begin{aligned} &\sum\nolimits^{\pm}\left(\frac{c_z}{c^2}u_z(u_t \pm u_z) - \frac{1}{c}(u_t \pm u_z)\cdot(\partial_t \mp \partial_z)(u_t \pm u_z) \mp \frac{c_z}{c^2}(u_t \pm u_z)^2\right) \\ &\quad = \sum\nolimits^{\pm}\left(\frac{c_z}{c^2}u_z(u_t \pm u_z) + \frac{1}{c}(u_t \pm u_z)\frac{c_z}{c}u_z \mp \frac{c_z}{c^2}(u_t \pm u_z)^2\right) \\ &\quad = \frac{c_z}{c^2}\sum\nolimits^{\pm}\Big(2u_z(u_t \pm u_z) \mp (u_t \pm u_z)^2\Big) = 0. \end{aligned}$$

Hence

$$\int_{\partial\Omega}\left(\frac{2}{c}\nu_2(u_t^2 + u_z^2) - \frac{4}{c}\nu_1 u_t u_z\right)\mathrm{d}\sigma = 0. \tag{6.25}$$

Since $D(c)$ and $D(c_z)$ are discrete sets, we find an increasing sequence $0 = a_1 < a_2 < a_3 < \ldots$ with $a_k \to \infty$ as $k \to \infty$ such that $D(c) \cup D(c_z) \subseteq \{a_k \colon k \in \mathbb{N}\}$.

Now let $t_1 < t_2 \in \mathbb{R}$ and $K \in \mathbb{N}$. We choose $\Omega = [a_k, a_{k+1}] \times [t_1, t_2]$ and sum (6.25) from $k = 1$ to K. As terms along common boundaries cancel, we obtain

$$0 = \int_{\partial([0,a_{K+1}]\times[t_1,t_2])} \left(\frac{2}{c}\nu_2(u_t^2 + u_z^2) - \frac{4}{c}\nu_1 u_t u_z \right) \mathrm{d}\sigma$$

or equivalently

$$\begin{aligned} &\tfrac{1}{2}\int_0^{a_{K+1}} \left(\frac{1}{c}u_t^2 + \frac{1}{c}u_z^2 \right) \mathrm{d}z \Big|_{t=t_2} \\ &\qquad = \tfrac{1}{2}\int_0^{a_{K+1}} \left(\frac{1}{c}u_t^2 + \frac{1}{c}u_z^2 \right) \mathrm{d}z \Big|_{t=t_1} - \int_{t_1}^{t_2} \frac{1}{c} u_t u_z \, \mathrm{d}t \Big|_{z=a_{K+1}} + \int_{t_1}^{t_2} \frac{1}{c} u_t u_z \, \mathrm{d}t \Big|_{z=0}. \end{aligned}$$

The estimates established in Corollary 6.3.6 and the assumptions on the initial conditions u_0, u_1 show that $u_t(z,t)$ and $u_z(z,t)$ converge to 0 as $z \to \infty$ uniformly on $[t_1, t_2]$. In the limit $K \to \infty$, we thus obtain

$$\tfrac{1}{2}\int_0^{\infty} \left(\frac{1}{c}u_t^2 + \frac{1}{c}u_z^2 \right) \mathrm{d}z \Big|_{t=t_2} = \tfrac{1}{2}\int_0^{\infty} \left(\frac{1}{c}u_t^2 + \frac{1}{c}u_z^2 \right) \mathrm{d}z \Big|_{t=t_1} + \int_{t_1}^{t_2} \frac{1}{c} u_t u_z \, \mathrm{d}t \Big|_{z=0}.$$

Switching back to (x,t)–coordinates, we infer

$$\begin{aligned} \int_{t_1}^{t_2} u_t u_x \, \mathrm{d}t \Big|_{x=0} &= \int_{t_1}^{t_2} u_t(0,t) u_x(0,t) \, \mathrm{d}t \\ &= \int_{t_1}^{t_2} u_t(0,t) f(u_t(0,t))_t \, \mathrm{d}t = F(u_t(0,t_2)) - F(u_t(0,t_1)) \end{aligned}$$

where the last equality is due to Lemma 6.A.1. This shows the claimed energy conservation:

$$\tfrac{1}{2}\int_0^{\infty} \left(V(x)u_t^2 + u_x^2 \right) \mathrm{d}x + F(u_t(0,t)) \Big|_{t=t_2} = \tfrac{1}{2}\int_0^{\infty} \left(V(x)u_t^2 + u_x^2 \right) \mathrm{d}x + F(u_t(0,t)) \Big|_{t=t_1}.$$

Part 2: Momentum. We calculate

$$\begin{aligned} 0 &= \sum\nolimits^{\pm} \int_\Omega \frac{1}{c} \left[(\partial_t \pm \partial_z)(u_t \mp u_z) + \frac{c_z}{c} u_z \right] \mathrm{d}(z,t) \\ &= \sum\nolimits^{\pm} \int_{\partial\Omega} (\nu_2 \pm \nu_1) \frac{1}{c} (u_t \mp u_z) \, \mathrm{d}\sigma \\ &\quad + \sum\nolimits^{\pm} \int_\Omega \left(\pm \frac{c_z}{c^2}(u_t \mp u_z) + \frac{c_z}{c^2} u_z \right) \mathrm{d}(z,t) \\ &= 2 \int_{\partial\Omega} \left(\nu_2 \frac{1}{c} u_t - \nu_1 \frac{1}{c} u_z \right) \mathrm{d}\sigma. \end{aligned} \tag{6.26}$$

Again we choose $\Omega = [a_k, a_{k+1}] \times [t_1, t_2]$, and sum (6.26) from $k = 1$ to K. As before all terms along common boundaries cancel, whence we obtain

$$\int_0^{a_{K+1}} \frac{1}{c} u_t \, \mathrm{d}z \Big|_{t=t_2} = \int_0^{a_{K+1}} \frac{1}{c} u_t \, \mathrm{d}z \Big|_{t=t_1} + \int_{t_1}^{t_2} \frac{1}{c} u_z \, \mathrm{d}t \Big|_{z=a_{K+1}} - \int_{t_1}^{t_2} \frac{1}{c} u_z \, \mathrm{d}t \Big|_{z=0}.$$

Since

$$\int_{t_1}^{t_2} \frac{1}{c} u_z \, \mathrm{d}t \Big|_{z=0} = \int_{t_1}^{t_2} f(u_t(0,t))_t \, \mathrm{d}t = f(u_t(0,t_2)) - f(u_t(0,t_1)),$$

in the limit $K \to \infty$ we find the claimed momentum conservation:

$$\int_0^{\infty} \frac{1}{c^2} u_t \, \mathrm{d}x + f(u_t(0,t)) \Big|_{t=t_2} = \int_0^{\infty} \frac{1}{c^2} u_t \, \mathrm{d}x + f(u_t(0,t)) \Big|_{t=t_1}. \qquad \square$$

In Section 6.4, we required an extra growth condition ($\mathcal{A}6.4$) on f in order to prove a first version of Theorem 6.1.1. We now discuss how to exploit the energy conservation to eliminate this extra growth assumption and prove Theorem 6.1.1 in full generality.

Lemma 6.5.1. *For $t > 0$ the estimate*

$$F(u_t(0,t)) \le F(u_1(0)) + \tfrac{1}{2}\int_0^{\kappa^{-1}(t)} \Big(V(x)u_1(x)^2 + u_{0,x}(x)^2\Big)\,\mathrm{d}x$$

holds, where $\kappa(x) = \int_0^x \frac{1}{c(s)}\,\mathrm{d}s = \int_0^x \sqrt{V(s)}\,\mathrm{d}s$.

Proof. Fix $t_1 > 0$, let $\varepsilon > 0$ and define modified initial data $\tilde{u}_0, \tilde{u}_1 \colon [0,\infty) \to \mathbb{R}$ by setting

$$\tilde{u}_0'(z) = \begin{cases} u_0'(z), & z \le t_1, \\ \frac{t_1+\varepsilon-z}{\varepsilon}u_0'(t_1), & t_1 \le z \le t_1+\varepsilon, \\ 0, & z \ge t_1+\varepsilon, \end{cases} \qquad \tilde{u}_1(z) = \begin{cases} u_1(z), & z \le t_1, \\ \frac{t_1+\varepsilon-z}{\varepsilon}u_1(t_1), & t_1 \le z \le t_1+\varepsilon, \\ 0, & z \ge t_1+\varepsilon, \end{cases}$$

and $\tilde{u}_0(0) = u_0(0)$. Denote the solution to (6.10) corresponding to these initial data by $\tilde{u}$. By uniqueness of the solution, $u(z,t) = \tilde{u}(z,t)$ for $|z| + |t| \le t_1$. In particular, $\tilde{u}_t(0,t_1) = u_t(0,t_1)$. This yields

$$\begin{aligned} &F(u_t(0,t_1)) \\ &\quad = F(\tilde{u}_t(0,t_1)) \le E(\tilde{u},t_1) = E(\tilde{u},0) \\ &\quad = F(\tilde{u}_t(0,0)) + \tfrac{1}{2}\int_0^\infty \Big(V(x)\tilde{u}_1(x)^2 + \tilde{u}_0'(x)^2\Big)\,\mathrm{d}x \\ &\quad = F(u_1(0)) + \tfrac{1}{2}\int_0^{\kappa^{-1}(t_1)} \Big(V(x)u_1(x)^2 + u_0'(x)^2\Big)\,\mathrm{d}x \\ &\qquad + \tfrac{1}{2}\int_{\kappa^{-1}(t_1)}^{\kappa^{-1}(t_1+\varepsilon)} \Big(V(x)\tilde{u}_1(x)^2 + \tilde{u}_0'(x)^2\Big)\,\mathrm{d}x. \end{aligned}$$

Letting $\varepsilon \to 0$, the last term goes to 0. ☐

Proof of Theorem 6.1.1 without additional assumption ($\mathcal{A}6.4$). We show existence and uniqueness of the solution to (6.1). Well-posedness will be discussed in Section 6.6. Fix $T > 0$ and let

$$C := F(u_1(0)) + \tfrac{1}{2}\int_0^{\kappa^{-1}(T)} \Big(V(x)u_1(x)^2 + u_{0,x}(x)^2\Big)\,\mathrm{d}x$$

Recall from Remark 6.1.5 that $F(y) \to \infty$ as $y \to \pm\infty$. Therefore the set $\{y \colon F(y) \le C\}$ is contained in the interval $[-K,K]$ for some $K > 0$. Now consider the cut-off version of f given by

$$f_K(y) = \begin{cases} y - K + f(K), & y \ge K, \\ f(y), & -K \le y \le K, \\ y + K + f(-K), & y \le -K, \end{cases}$$

which satisfies the growth condition ($\mathcal{A}6.4$). As we have shown in Section 6.4, there exists a unique solution u_K of (6.1) with f replaced by f_K. Then, using $F_K(y) = yf_K(y) - \int_0^y f_K(s)\,\mathrm{d}s$, Lemma 6.5.1 gives $F_K(u_{K,t}(0,t)) \le C$ for $t \le T$, so that $u_{K,t}(0,t)$ takes values in $[-K,K]$ where the functions f, F and f_k, F_k coincide. Hence u_K is the unique solution of the original problem (6.1) up to time T. ☐

Next, we verify that C^1-solutions to (6.1) are indeed weak solutions in the sense of Definition 6.1.6.

Proposition 6.5.2. *A C^1-solution to* (6.1) *is also a weak solution to* (6.1).

Proof. Let u be a C^1-solution to (6.1). We have to show that

$$\begin{aligned}0 = &\int_0^\infty \int_0^\infty (V(x)u_t\varphi_t - u_x\varphi_x)\,\mathrm{d}x\mathrm{d}t + \int_0^\infty f(u_t(0,t))\varphi_t(0,t)\,\mathrm{d}t\\ &+ \int_0^\infty V(x)u_1(x)\varphi(x,0)\,\mathrm{d}x + f(u_1(0))\varphi(0,0)\end{aligned}$$

holds for all $\varphi \in C_c^\infty([0,\infty)\times[0,\infty))$.

Let $\Omega \subseteq [0,\infty)\times[0,\infty)$ be a Lipschitz domain such that c is C^1 on Ω. Denoting the outer normal at $\partial\Omega$ by ν, we obtain

$$\begin{aligned}0 &= \int_\Omega \left[(\partial_t - \partial_z)(u_t+u_z) + \frac{c_z}{c}u_z\right]\cdot\frac{1}{c}\varphi\,\mathrm{d}(z,t)\\ &= \int_{\partial\Omega}\frac{1}{c}(u_t+u_z)\varphi\cdot(\nu_2-\nu_1)\,\mathrm{d}\sigma + \int_\Omega\left(\frac{c_z}{c^2}u_z\varphi - (u_t+u_z)(\partial_t-\partial_z)\left[\frac{1}{c}\varphi\right]\right)\mathrm{d}(z,t)\\ &= \int_{\partial\Omega}\left(\frac{1}{c}u_t\varphi\nu_2 - \frac{1}{c}u_z\varphi\nu_1\right)\mathrm{d}\sigma + \int_\Omega\left(\frac{1}{c}u_z\varphi_z - \frac{1}{c}u_t\varphi_t\right)\mathrm{d}(z,t)\\ &\quad+\int_{\partial\Omega}\left(\frac{1}{c}u_z\varphi\nu_2 - \frac{1}{c}u_t\varphi\nu_1\right)\mathrm{d}\sigma + \int_\Omega\left(u_t\partial_z\left[\frac{1}{c}\varphi\right] - u_z\partial_t\left[\frac{1}{c}\varphi\right]\right)\mathrm{d}(z,t).\end{aligned}$$

We next show that the sum of the last two integrals equals zero. First, we calculate

$$\begin{aligned}&\int_{\partial\Omega}\left(\frac{1}{c}u_z\varphi\nu_2 - \frac{1}{c}u_t\varphi\nu_1\right)\mathrm{d}\sigma + \int_\Omega\left(u_t\partial_z\left[\frac{1}{c}\varphi\right] - u_z\partial_t\left[\frac{1}{c}\varphi\right]\right)\mathrm{d}(z,t)\\ &\quad= \int_{\partial\Omega}\left(\frac{1}{c}u_z\varphi\nu_2 - \frac{1}{c}u_t\varphi\nu_1 + u\partial_z\left[\frac{1}{c}\varphi\right]\nu_2 - u\partial_t\left[\frac{1}{c}\varphi\right]\nu_1\right)\mathrm{d}\sigma\\ &\quad= \int_{\partial\Omega}(\nu_2\partial_z - \nu_1\partial_t)\left[\frac{1}{c}u\varphi\right]\mathrm{d}\sigma.\end{aligned}$$

Let $\gamma\colon [0,l]\to\mathbb{R}$ be a positively oriented parametrization of $\partial\Omega$ by arc length. As ν is the outer normal at $\partial\Omega$, the identity $\gamma' = (\nu_2,-\nu_1)^\top$ holds. Hence,

$$\int_{\partial\Omega}(\nu_2\partial_z - \nu_1\partial_t)\left[\frac{1}{c}u\varphi\right]\mathrm{d}\sigma = \int_{\partial\Omega}\begin{pmatrix}\nu_2\\-\nu_1\end{pmatrix}\cdot\nabla\left[\frac{1}{c}u\varphi\right]\mathrm{d}\sigma = \int_0^l \gamma'(s)\cdot\nabla\left[\frac{1}{c}u\varphi\right](\gamma(s))\,\mathrm{d}s = 0$$

as γ is closed. Thus we have shown

$$0 = \int_{\partial\Omega}\left(\frac{1}{c}u_t\varphi\nu_2 - \frac{1}{c}u_z\varphi\nu_1\right)\mathrm{d}\sigma + \int_\Omega\left(\frac{1}{c}u_z\varphi_z - \frac{1}{c}u_t\varphi_t\right)\mathrm{d}(z,t). \tag{6.27}$$

As in the proof of Theorem 6.1.4 we choose an increasing sequence $0 = a_1 < a_2 < a_3 < \dots$ with $a_k\to\infty$ as $k\to\infty$ such that $D(c)\cup D(c_z)\subseteq\{a_k\colon k\in\mathbb{N}\}$. We take $\Omega = [a_k,a_{k+1}]\times[n,n+1]$ in (6.27) and sum over $k\in\mathbb{N}$ and $n\in\mathbb{N}_0$. Using that boundary terms along common boundaries cancel out, the fact that φ has compact support, and (6.1), we obtain

$$\begin{aligned}0 &= \int_{\partial[0,\infty)^2}\left(\frac{1}{c}u_t\varphi\nu_2 - \frac{1}{c}u_z\varphi\nu_1\right)\mathrm{d}\sigma + \int_{[0,\infty)^2}\left(\frac{1}{c}u_z\varphi_z - \frac{1}{c}u_t\varphi_t\right)\mathrm{d}(z,t)\\ &= -\int_0^\infty\left[\frac{1}{c}u_t\varphi\right](z,0)\,\mathrm{d}z + \int_0^\infty\left[\frac{1}{c}u_z\varphi\right](0,t)\,\mathrm{d}t + \int_0^\infty\int_0^\infty\left(\frac{1}{c}u_z\varphi_z - \frac{1}{c}u_t\varphi_t\right)\mathrm{d}z\mathrm{d}t\end{aligned}$$

$$= -\int_0^\infty V(x)u_t(x,0)\varphi(x,0)\,\mathrm{d}x + \int_0^\infty u_x(0,t)\varphi(0,t)\,\mathrm{d}t + \int_0^\infty\int_0^\infty (u_x\varphi_x - V(x)u_t\varphi_t)\,\mathrm{d}x\mathrm{d}t$$
$$= -\int_0^\infty V(x)u_1(x)\varphi(x,0)\,\mathrm{d}x + \int_0^\infty (f(u_t(0,t)))_t\varphi(0,t)dert$$
$$+\int_0^\infty\int_0^\infty (u_x\varphi_x - V(x)u_t\varphi_t)\,\mathrm{d}x\mathrm{d}t$$
$$= -\int_0^\infty V(x)u_1(x)\varphi(x,0)\,\mathrm{d}x - \int_0^\infty f(u_t(0,t))\varphi_t(0,t)\,\mathrm{d}t - f(u_1(0))\varphi(0,0)$$
$$+\int_0^\infty\int_0^\infty (u_x\varphi_x - V(x)u_t\varphi_t)\,\mathrm{d}x\mathrm{d}t$$

which finishes the proof. □

6.6. Well-posedness

The section completes the proof of the well-posedness claim stated in Theorem 6.1.1. To be precise, (6.1) is well-posed in the following sense. The spaces $C_x([0,\infty)), C_x^1([0,\infty))$, and $C^1_{(x,t)}([0,\infty)\times[0,\infty))$ are endowed with uniform convergence on compact sets. Their subspaces consisting of bounded functions are denoted by the subscript b, and are endowed with uniform convergence.

Proposition 6.6.1. *Assume that $u_0^{(n)}, u_1^{(n)}$ are initial data with $u_0^{(n)} \to u_0$ in $C_x^1([0,\infty))$ and $u_1^{(n)} \to u_1$ in $C([0,\infty))$, and denote by $u^{(n)}$ and u the solutions of* (6.10) *corresponding to these initial data. Then we have $u^{(n)} \to u$ in $C^1_{(x,t)}([0,\infty)\times[0,\infty))$.*

Moreover, if the initial data are all bounded with $u_0^{(n)} \to u_0$ in $C^1_{b,x}([0,\infty))$ and $u_1^{(n)} \to u_0$ in $C_b([0,\infty))$ then $u^{(n)} \to u$ in $C^1_{b,(x,t)}([0,\infty)\times[0,T])$ for all $T>0$.

Sketch of proof. We proceed similar to the proof of Theorem 6.1.1. Choose some

$$0 < \bar{r} < \min\left\{\left(5-\sqrt{17}\right)\left\|\frac{c_z}{c}\right\|_\infty^{-1}, |z_1 - z_2| \colon z_1, z_2 \in D(c)\cup\{0\}, z_1 \neq z_2\right\}$$

and let β be as in Lemma 6.3.9 with $r = \bar{r}$ and $\lambda = \frac{c_z}{c}$. The choice of $\bar{r}$ implies $\beta\bar{r} < \frac{4(5-\sqrt{17})}{(-3+\sqrt{17})(-1+\sqrt{17})} = 1$ as well as $q := \bar{r}\|\frac{c_z}{c}\|_\infty < 1$.

Denote by $\mathcal{C}$ the set containing all triangles Δ that are of jump-type or plain-type and such that their base-radii r are at most $\bar{r}$. Then by Lemmas 6.4.1 and 6.4.2, there exists a constant $C>0$ such that

$$\left\|u^{(n)} - u\right\|_{C^1_{(x,t)}(\Delta)} \le C\max\left\{\left\|u_0^{(n)} - u_0\right\|_{C_x^1(B)}, \left\|u_1^{(n)} - u_1\right\|_{C(B)}\right\}$$

holds for each triangle $\Delta\in\mathcal{C}$ with base B.

We also consider a single boundary-type triangle Δ_+ with center $z_0 = 0$ and height $\bar{r}$. Writing $b(t) := u(0,t)$, $b^{(n)}(t) := u^{(n)}(0,t)$, $d(t) := f(u_t(0,t))$ as well as $d^{(n)}(t) := f(u_t^{(n)}(0,t))$, as in the proof of Lemma 6.4.3 we obtain

$$d'(t) = \frac{1}{c(0)}\Phi_+(b,u_0,u_1)_z(0,t), \qquad \left(d^{(n)}\right)'(t) = \frac{1}{c(0)}\Phi_+(b^{(n)},u_0^{(n)},u_1^{(n)})_z(0,t).$$

Setting $\hat{b}(t) := u_0^{(n)}(0) - u_0(0) + t\Big(u_1^{(n)}(0) - u_1(0)\Big)$, we find

$$\begin{aligned}
&c(0)\Big(\Big(d^{(n)}\Big)'(t) - d'(t)\Big)\\
&\quad = \Phi_+(b^{(n)} - b, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0,t)\\
&\quad = \Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0,t) + \Phi_+(b^{(n)} - b - \hat{b}, 0, 0)_z(0,t)\\
&\quad = \Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0,t)\\
&\qquad - \Big[f^{-1}(d^{(n)}(t)) - f^{-1}(d(t)) - \Big(u_1^{(n)}(0) - u_1(0)\Big)\Big] + \rho(n,t)
\end{aligned}$$

where Lemma 6.3.9 gives

$$|\rho(n,t)| \le \beta \int_0^t \Big|f^{-1}(d^{(n)}(\tau)) - f^{-1}(d(\tau)) - u_1^{(n)}(0) + u_1(0)\Big|\,\mathrm{d}\tau.$$

Multiplying with $\operatorname{sign}\Big(d^{(n)}(t) - d(t)\Big)$ and integrating, we obtain

$$\begin{aligned}
&c(0)\Big|d^{(n)}(t) - d(t)\Big|\\
&\le c(0)\Big|d^{(n)}(0) - d(0)\Big|\\
&\quad + \int_0^t \Big(\Big|\Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0,s)\Big| - \Big|f^{-1}(d^{(n)}(s)) - f^{-1}(d(s))\Big| + \Big|u_1^{(n)}(0) - u_1(0)\Big|\Big)\,\mathrm{d}s\\
&\quad + \beta \int_0^t \int_0^s \Big|f^{-1}(d^{(n)}(\tau)) - f^{-1}(d(\tau)) - u_1^{(n)}(0) + u_1(0)\Big|\,\mathrm{d}\tau\mathrm{d}s\\
&\le \int_0^t \Big(\Big|\Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0,s)\Big| - \Big|f^{-1}(d^{(n)}(s)) - f^{-1}(d(s))\Big| + \Big|u_1^{(n)}(0) - u_1(0)\Big|\Big)\,\mathrm{d}s\\
&\quad + \beta \int_0^{\bar{r}} \int_0^t \Big(\Big|f^{-1}(d^{(n)}(\tau)) - f^{-1}(d(\tau))\Big| + \Big|u_1^{(n)}(0) - u_1(0)\Big|\Big)\,\mathrm{d}\tau\mathrm{d}s\\
&= \int_0^t \Big|\Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0,s)\Big|\,\mathrm{d}s + (1 + \bar{r}\beta)t\Big|u_1^{(n)}(0) - u_1(0)\Big|\\
&\quad - (1 - \bar{r}\beta) \int_0^t \Big|f^{-1}(d^{(n)}(s)) - f^{-1}(d(s))\Big|\,\mathrm{d}s\\
&\le \int_0^t \Big|\Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0,s)\Big|\,\mathrm{d}s + (1 + \bar{r}\beta)t\Big|u_1^{(n)}(0) - u_1(0)\Big|\\
&\le \tilde{C}\Big(\bar{r}, \Big\|\frac{c_z}{c}\Big\|_\infty\Big) \max\Big\{\Big\|u_0^{(n)} - u_0\Big\|_{C_x^1([0,\infty))}, \Big\|u_1^{(n)} - u_1\Big\|_{C([0,\infty))}\Big\}.
\end{aligned}$$

This shows the uniform convergence of $d^{(n)}$ to d on $[0,\bar{r}]$ as $n \to \infty$. Since

$$b(t) = u_0(0) + \int_0^t f^{-1}(d(\tau))\,\mathrm{d}\tau, \qquad b^{(n)}(t) = u_0(0) + \int_0^t f^{-1}(d^{(n)}(\tau))\,\mathrm{d}\tau$$

for $t \in [0,\bar{r}]$, it follows that $b^{(n)} \to b$ in $C^1([0,\bar{r}])$ as $n \to \infty$, and therefore we see that $u^{(n)} = \Phi_+(b^{(n)}, u_0^{(n)}, u_1^{(n)}) \to \Phi_+(b, u_0, u_1) = u$ in $C^1(\Delta_+)$.

Combined, we find that that $u^{(n)} \to u$ in $C^1_{(x,t)}(\mathcal{D})$ where $\mathcal{D} := \cup_{\Delta\in\mathcal{C}}\Delta$. Note that $[0,\infty) \times [0,\frac{\bar{r}}{2}] \subseteq \mathcal{D}$, so in particular $u^{(n)} \to u$ in $C^1_{(x,t)}([0,\infty) \times [0,\frac{\bar{r}}{2}])$.

Moreover, the estimates on jump-type and plain-type triangles also show that $u^{(n)} \to u$ converges uniformly on $\mathcal{D}$ provided the initial data $u_0^{(n)}, u_1^{(n)}$ converge uniformly.

Applying this result repeatedly $k \in \mathbb{N}$ times, we find $u^{(n)} \to u$ in $C^1_{(x,t)}([0,\infty) \times [0,k\frac{\bar{r}}{2}])$ for all $k \in \mathbb{N}$, with uniform convergence for uniformly convergent initial data. ☐

6.7. Breather solutions and their regularity

One can also consider (6.1) in the context of breather solutions, where a *breather* is a time-periodic and spatially localized function. With time-period denoted by T, the time domain becomes the torus $\mathbb{T} := \mathbb{R}/_{T\mathbb{Z}}$ and after dropping the initial data, (6.1) reads

$$\begin{cases} V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in [0,\infty), t \in \mathbb{T}, \\ u_x(0,t) = (f(u_t(0,t)))_t, & t \in \mathbb{T}. \end{cases} \tag{6.28}$$

In [49] the case of a cubic boundary term $f(y) = \frac{1}{2}\gamma y^3$ ($\gamma \in \mathbb{R} \setminus \{0\}$) and a 2π-periodic step potential $V : \mathbb{R} \to \mathbb{R}$ given by

$$V(x) = \begin{cases} a, & |x| < \pi\theta, \\ b, & \theta\pi < |x| < \pi, \end{cases} \tag{6.29}$$

where $b > a > 0$ and $\theta \in (0,1)$ was discussed. It was shown that if V satisfies

$$4\sqrt{a}\theta\omega \in 2\mathbb{N}_0 + 1 \quad \text{and} \quad 4\sqrt{b}(1-\theta)\omega \in 2\mathbb{N}_0 + 1, \tag{6.30}$$

where $\omega := \frac{2\pi}{T}$ is the frequency, then there exist infinitely many weak breather solutions u of (6.28) with time-period T. A weak solution of (6.28) is defined next.

Definition 6.7.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing, odd homeomorphism. A* weak solution *of* (6.28) *is a function $u \in H^1([0,\infty) \times \mathbb{T})$ with $u(0,\cdot) \in W^{1,1}(\mathbb{T})$ and $f(u_t(0,\cdot)) \in L^1(\mathbb{T})$ which satisfies*

$$\int_{[0,\infty)\times\mathbb{T}} -V(x)u_t\varphi_t + u_x\varphi_x \,\mathrm{d}(x,t) - \int_{\mathbb{T}} f(u_t(0,t))\varphi_t(0,t)\,\mathrm{d}t = 0$$

for all test functions $\varphi \in C_c^\infty([0,\infty) \times \mathbb{T})$.

Remark 6.7.2. We require that the trace $u(0,\cdot)$ of u at $x = 0$ has an integrable weak first-order time derivative in order to give a pointwise meaning to $u_t(0,t)$ and, in particular, to define $f(u_t(0,t))$ pointwise almost everywhere.

In the setting of [49] where $f(y) = \frac{1}{2}\gamma y^3$, one requires $u_t(0,t) \in L^3(\mathbb{T})$ and

$$2\int_{[0,\infty)\times\mathbb{T}} -V(x)u_t\varphi_t + u_x\varphi_x \,\mathrm{d}(x,t) - \gamma\int_{\mathbb{T}} u_t(0,t)^3\varphi_t(0,t)\,\mathrm{d}t = 0.$$

In [49, Theorem 4] it was furthermore shown that weak solutions to (6.28) constructed in [49] lie in $H^{\frac{5}{4}-\varepsilon}(\mathbb{T}, L^2(0,\infty)) \cap H^{\frac{1}{4}-\varepsilon}(\mathbb{T}, H^1(0,\infty))$ for $\varepsilon > 0$. Here, the Bochner spaces $H^s(\mathbb{T}, X)$ are defined by

$$\|u\|^2_{H^s(\mathbb{T},X)} := \sum_{k\in\mathbb{Z}} (1+k^2)^s \|\hat{u}_k\|_X^2 < \infty.$$

In this section, we show the following improved regularity result for breather solutions of (6.28):

Theorem 6.7.3. *Let V be given by* (6.29) *and* (6.30) *Further assume ($\mathcal{A}$6.3), that f^{-1} is r-Hölder continuous with $r \in (0,1)$ and that u is a weak solution to* (6.28)*. Then u is $\frac{T}{2}$-antiperiodic, lies in $C^{1,r}([0,\infty) \times \mathbb{T})$ and is a C^1-solution to* (6.1) *with its own initial data, i.e., $u_0(x) = u(x,0)$ and $u_1(x) = u_t(x,0)$. In addition, there exists $C > 0$ such that $|u(x,t)| \leq C\mathrm{e}^{-\rho x}$ where $\rho := \frac{\log(b)-\log(a)}{4\pi}$.*

Note that in the setting of [49], the assumptions of Theorem 6.7.3 are satisfied with $r = \frac{1}{3}$. In the following, we are going to prove Theorem 6.7.3 and we will always assume the assumptions of Theorem 6.7.3. We begin with a discussion of the linear operator $V(x)\partial_t^2 - \partial_x^2$ appearing in (6.28).

6.7.1. Fourier decomposition of $V(x)\partial_t^2 - \partial_x^2$

We denote by $e_k(t) := \mathrm{e}^{\mathrm{i}k\omega t}$ the orthonormal Fourier base of $L^2(\mathbb{T})$ and decompose u in its Fourier series with respect to t:

$$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x) e_k(t) =: \mathcal{F}^{-1}(\hat{u})$$

with

$$\hat{u}_k(x) := \mathcal{F}_k(u) := \int_{\mathbb{T}} u(x,t)\overline{e_k(t)}\,\mathrm{d}t.$$

Writing $L := V(x)\partial_t^2 - \partial_x^2$ and $L_k := -\partial_x^2 - k^2\omega^2 V(x)$, we see that any solution u of (6.28) satisfies

$$0 = Lu$$

and therefore also

$$0 = \mathcal{F}_k L u = L_k \mathcal{F}_k u = L_k \hat{u}_k \tag{6.31}$$

for all $k \in \mathbb{Z}$. Since

$$\|u\|^2_{L^2([0,\infty)\times\mathbb{T})} + \|u_x\|^2_{L^2([0,\infty)\times\mathbb{T})} = \sum_{k\in\mathbb{Z}} \|\hat{u}_k\|^2_{L^2(0,\infty)} + \|(\hat{u}_k)_x\|^2_{L^2(0,\infty)},$$

each $\hat{u}_k$ is an $H^1((0,\infty),\mathbb{C})$-solution of (6.31). As V (and therefore also L_k) is given explicitly, we can characterize the space of solutions of (6.31) as follows.

Proposition 6.7.4. *If $k \in \mathbb{Z}$ is even, then the only solution $\hat{u}_k \in H^1((0,\infty),\mathbb{C})$ to* (6.31) *is $\hat{u}_k = 0$. If k is odd, there exists $\phi_k \in H^2((0,\infty),\mathbb{R})$ such that a function $\hat{u}_k \in H^1((0,\infty),\mathbb{C})$ solves* (6.31) *if and only if $\hat{u}_k = \lambda\phi_k$ for some $\lambda \in \mathbb{C}$. Furthermore, ϕ_k satisfies*

$$\phi_k(0) = 1, \quad \phi_k'(0) = Ck(-1)^{(k-1)/2}, \quad \phi_k(x+4\pi) = \frac{a}{b}\phi_k(x)$$

for $x > 0$, where $C = C(T,a) \in \mathbb{R}$ is a constant independent of k. The function ϕ_k is called fundamental Bloch mode of (6.31).

A proof of Proposition 6.7.4 for odd k can be found in [49, Appendix A2]. The nonexistence result for even k can be obtained using similar arguments: For $k \neq 0$ the monodromy matrix for L_k is the identity matrix so that (6.31) only has spatially periodic solutions. For $k = 0$, the solutions of (6.31) are affine.

Next we establish a bootstrapping argument that will be used to obtain the $C^{1,r}$ regularity in Theorem 6.7.3.

6.7.2. Bootstrapping argument

Assume that u is a weak solution to (6.28) in the sense of Definition 6.7.1. By Proposition 6.7.4, all even Fourier modes of u vanish and there exists a complex sequence $\hat{\alpha}_k$ such that

$$u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{\alpha}_k \phi_k(x) e_k(t). \tag{6.32}$$

In particular, u is $\frac{T}{2}$-antiperiodic. Since $\phi_k(0) = 1$, we have $\alpha(t) \coloneqq \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{\alpha}_k e_k(t) = u(0,t)$. In addition to α, we also consider the quantity $\beta(t) \coloneqq f(u_t(0,t))$. Thus we have

$$\alpha = \partial_t^{-1} f^{-1}(\beta). \tag{6.33}$$

Let us explain the idea of the proof on a formal level. First, by (6.32) we can express u, and terms derived from u, as a function of α. Setting $\Psi(\alpha)(t) \coloneqq u_x(0,t)$, the boundary condition of (6.28) can be written as $\Psi(\alpha) = f(u_t(0,\cdot\,))_t = \beta'$ or as

$$\beta = \partial_t^{-1}\Psi(\alpha). \tag{6.34}$$

As shown below, the maps $\beta \mapsto \partial_t^{-1} f^{-1}(\beta) = \alpha$ and $\alpha \mapsto \partial_t^{-1}\Psi(\alpha) = \beta$ are regularity improving. Bootstrapping this regularity improvement, we show in Lemma 6.7.7 that α (which by definition lies in $W^{1,1}(\mathbb{T})$) and β (which by definition lies in $L^1(\mathbb{T})$), both are in $C^{1,r}(\mathbb{T})$. From there, using $u(0,\cdot\,) = \alpha \in C^{1,r}(\mathbb{T})$, $u_x(0,\cdot\,) = \beta' \in \mathbb{C}^r(\mathbb{T})$ and that the wave equation preserves regularity, we show $u \in C^{1,r}([0,\infty)\times\mathbb{T})$.

Note that u satisfies the boundary condition of (6.28) in a weak sense, so that $u_x(0,\cdot\,) = \beta'$ is not clear a priori and will be shown as part of the proof of Theorem 6.7.3. Moreover, since u lies in $H^1([0,\infty)\times\mathbb{T})$, its derivative u_x does not admit traces in general so that $\Psi(\alpha)$ need not be defined. This is not an issue, because next we establish a (rigorous) identity that we replace (6.34) with.

Using Definition 6.7.1 with $\varphi(x,t) = \psi(x)\overline{e_k(t)}$ for $k \in \mathbb{Z}_{\text{odd}}$, where $\psi \in C_c^\infty([0,\infty))$ and $\psi(0) = 1$, we obtain

$$\begin{aligned}
0 &= \int_{[0,\infty)\times\mathbb{T}} \Big[-V(x)u_t\psi(x)\overline{e_k'(t)} + u_x\psi'(x)\overline{e_k(t)}\Big]\,\mathrm{d}(x,t) - \int_{\mathbb{T}} f(u_t(0,t))\psi(0)\overline{e_k'(t)}\,\mathrm{d}t \\
&= \int_0^\infty \Big[-V(x)\mathrm{i}k\omega\hat{\alpha}_k\phi_k(x)\overline{\mathrm{i}k\omega}\psi(x) + \hat{\alpha}_k\phi_k'(x)\psi'(x)\Big]\,\mathrm{d}x + \mathrm{i}k\omega\hat{\beta}_k \\
&= \int_0^\infty \Big[-\hat{\alpha}_k k^2\omega^2 V(x)\phi_k(x)\psi(x) - \hat{\alpha}_k\phi_k''(x)\psi(x)\Big]\,\mathrm{d}x - \hat{\alpha}_k\phi_k'(0)\psi(0) + \mathrm{i}k\omega\hat{\beta}_k \\
&= -\phi_k'(0)\hat{\alpha}_k + \mathrm{i}k\omega\hat{\beta}_k,
\end{aligned}$$

or

$$\hat{\beta}_k = \frac{\phi_k'(0)}{\mathrm{i}k\omega}\hat{\alpha}_k. \tag{6.35}$$

Since $u(0,\cdot\,)$ is $\frac{T}{2}$-antiperiodic, the even Fourier coefficients of $\alpha = u(0,\cdot\,)$ vanish, and since f is odd the even Fourier coefficients of $\beta = f(u_t(0,\cdot\,))$ also vanish. Hence from (6.35) we obtain

$$\beta = \mathcal{F}^{-1}\left(\left(\frac{\phi_k'(0)}{\mathrm{i}k\omega}\hat{\alpha}_k\right)_{k\in\mathbb{Z}_{\text{odd}}}\right), \tag{6.36}$$

assuming the sequence on the right-hand side lies in $\mathcal{F}(L^1(\mathbb{T}))$. In the following, we use (6.36) instead of the formal equation (6.34).

We next investigate the properties of the maps defined by (6.33) and (6.36), which we consider as maps between the fractional Sobolev-Slobodeckij spaces $W^{s,p}(\mathbb{T})$ or between Hölder spaces $C^s(\mathbb{T})$. The definition and all employed properties of the spaces $W^{s,p}(\mathbb{T})$ can be found in Section B.2. In the following we use the suffix "anti" to denote that the space consists of functions which are $\frac{T}{2}$-antiperiodic in time.

Lemma 6.7.5. *The map*

$$\beta \mapsto \partial_t^{-1} f^{-1}(\beta)$$

is well-defined from $W^{s,p}_{\mathrm{anti}}(\mathbb{T})$ *to* $W^{1+rs,p/r}_{\mathrm{anti}}(\mathbb{T})$ *for any* $s \in [0,1)$ *and* $p \in [1,\infty)$ *as well as from* $C^{0,s}_{\mathrm{anti}}(\mathbb{T})$ *to* $C^{1,rs}_{\mathrm{anti}}(\mathbb{T})$ *for any* $s \in [0,1]$.

Proof. If $\beta \in C^{0,s}_{\mathrm{anti}}(\mathbb{T})$, then $f^{-1}(\beta) \in C^{0,rs}_{\mathrm{anti}}(\mathbb{T})$ since f^{-1} is r-Hölder regular, and thus $\partial_t^{-1} f^{-1}(\beta) \in C^{1,rs}_{\mathrm{anti}}(\mathbb{T})$. If $\beta \in W^{s,p}_{\mathrm{anti}}(\mathbb{T})$, then $f^{-1}(\beta) \in W^{rs,p/r}_{\mathrm{anti}}(\mathbb{T})$ by Lemma B.2.7 and thus $\partial_t^{-1} f^{-1}(\beta) \in W^{1+rs,p/r}_{\mathrm{anti}}(\mathbb{T})$. □

Lemma 6.7.6. *The map*

$$\alpha \mapsto \mathcal{F}^{-1}\left(\left(\frac{\phi_k'(0)}{\mathrm{i}k\omega}\hat{\alpha}_k\right)_{k\in\mathbb{Z}_{\mathrm{odd}}}\right)$$

is well-defined from $W^{s,p}_{\mathrm{anti}}(\mathbb{T})$ *to* $W^{s,p}_{\mathrm{anti}}(\mathbb{T})$ *for all* $s \in (0,\infty)$ *and* $p \in [1,\infty)$ *as well as from* $C^{k,s}_{\mathrm{anti}}(\mathbb{T})$ *to* $C^{k,s}_{\mathrm{anti}}(\mathbb{T})$ *for all* $k \in \mathbb{N}_0$ *and* $s \in [0,1]$.

Proof. We begin by taking a closer look at the Fourier multiplier $\hat{M}_k := \frac{\phi_k'(0)}{\mathrm{i}k\omega}$ which is defined for $k \in \mathbb{Z}_{\mathrm{odd}}$ and extended by 0 to the whole of $\mathbb{Z}$. By Proposition 6.7.4 we have $\phi_k'(0) = Ck(-1)^{(k-1)/2}$ for a real constant C depending only on T and a. From this we obtain

$$\hat{M}_k = -\frac{\mathrm{i}C}{\omega}\operatorname{Im}\mathrm{i}^k$$

for all $k \in \mathbb{Z}$. Now, $\hat{M}_k$ are the Fourier coefficients of

$$M(t) := \frac{\sqrt{T}C}{2\omega}\Big(\delta_{T/4}(t) - \delta_{-T/4}(t)\Big)$$

where δ_x denotes the Dirac measure at x. In particular, M is a finite measure. For $\alpha \in L^1_{\mathrm{anti}}(\mathbb{T})$ we calculate

$$\begin{aligned}\mathcal{F}_k(M * \alpha) &= \int_{\mathbb{T}}\int_{\mathbb{T}} \alpha(t-s)\,\mathrm{d}M(s)\overline{e_k(t)}\,\mathrm{d}t \\ &= \int_{\mathbb{T}}\int_{\mathbb{T}} \alpha(t-s)\overline{e_k(t-s)}\,\mathrm{d}t\overline{e_k(s)}\,\mathrm{d}M(s) = \hat{M}_k\hat{\alpha}_k.\end{aligned}$$

so that $\mathcal{F}^{-1}\Big(k \mapsto \hat{M}_k\hat{\alpha}_k\Big)$ exists and equals $M * \alpha$. To see that $M * (\,\cdot\,)$ maps $W^{s,p}_{\mathrm{anti}}(\mathbb{T})$ into $W^{s,p}_{\mathrm{anti}}(\mathbb{T})$ and $C^{k,s}_{\mathrm{anti}}(\mathbb{T})$ into $C^{k,s}_{\mathrm{anti}}(\mathbb{T})$, let $\|\cdot\|$ be $\|\cdot\|_{W^{s,p}}$ or $\|\cdot\|_{C^{k,s}}$ (or any translation invariant norm). Then

$$\begin{aligned}\Big\|\mathcal{F}^{-1}\Big((\hat{M}_k\hat{\alpha}_k)_{k\in\mathbb{Z}_{\mathrm{odd}}}\Big)\Big\| &= \|M * \alpha\| = \Big\|\int_{\mathbb{T}} \alpha(\,\cdot\, - s)\,\mathrm{d}M(s)\Big\| \\ &\le \int_{\mathbb{T}} \|\alpha(\,\cdot\, - s)\|\,\mathrm{d}|M|(s) = |M|(\mathbb{T})\|\alpha\|.\end{aligned}$$

□

With the previous two lemmata, we can complete the bootstrapping argument stated next.

Lemma 6.7.7. *If the pair* (α,β) *satisfies* (6.33) *and* (6.36) *with* $\alpha,\beta \in L^1_{\text{anti}}(\mathbb{T})$*, then* $\alpha,\beta \in C^{1,r}_{\text{anti}}(\mathbb{T})$.

Proof. By Lemma 6.7.5 we have $\alpha \in W^{1,1/r}_{\text{anti}}(\mathbb{T})$, and therefore $\beta \in W^{1,1/r}_{\text{anti}}(\mathbb{T})$ by Lemma 6.7.6. Applying Lemmas 6.7.5 and 6.7.6 again, we get $\alpha,\beta \in W^{1+r-\varepsilon,1/r^2}_{\text{anti}}(\mathbb{T})$ for any $\varepsilon > 0$. Repeating this n times, we obtain $\alpha,\beta \in W^{1+r-\varepsilon,1/r^{2+n}}_{\text{anti}}(\mathbb{T})$. If $n \in \mathbb{N}$ is large enough, then $W^{1+r-\varepsilon,1/r^{2+n}}_{\text{anti}}(\mathbb{T})$ embeds continuously into $C^1_{\text{anti}}(\mathbb{T})$ by Lemma B.2.6, so in particular we have $\alpha,\beta \in C^1_{\text{anti}}(\mathbb{T})$. Now, applying Lemmas 6.7.5 and 6.7.6 one last time yields $\alpha,\beta \in C^{1,r}_{\text{anti}}(\mathbb{T})$. □

Next we prove the main theorem of this section, Theorem 6.7.3.

Proof of Theorem 6.7.3. Note that $\alpha \in W^{1,1}(\mathbb{T}), \beta \in L^1(\mathbb{T})$ hold by Definition 6.7.1, and both are $\frac{T}{2}$-antiperiodic as we have seen above. So Lemma 6.7.7 is applicable and yields $\alpha,\beta \in C^{1,r}_{\text{anti}}(\mathbb{T})$.

By $d_1 \coloneqq \theta\pi, d_2 \coloneqq (2-\theta)\pi, d_3 \coloneqq (2+\theta)\pi, \dots$ we label the discontinuities of V. We start by showing that $u \in C^{1,r}_{\text{anti}}([0,d_1]\times\mathbb{T})$. To do this, consider

$$w(x,t) \coloneqq \frac{1}{2}\big(\alpha(t+\sqrt{a}x)+\alpha(t-\sqrt{a}x)\big) + \frac{1}{2\sqrt{a}}\big(\beta(t+\sqrt{a}x)-\beta(t-\sqrt{a}x)\big). \tag{6.37}$$

Note that w is $\frac{T}{2}$-antiperiodic in time. The k-th Fourier coefficient of w is given by

$$\begin{aligned}\hat{w}_k(x) &= \frac{\hat{\alpha}_k}{2}\left(\mathrm{e}^{\mathrm{i}k\omega\sqrt{a}x}+\mathrm{e}^{-\mathrm{i}k\omega\sqrt{a}x}\right) + \frac{\hat{\beta}_k}{2\sqrt{a}}\left(\mathrm{e}^{\mathrm{i}k\omega\sqrt{a}x}-\mathrm{e}^{-\mathrm{i}k\omega\sqrt{a}x}\right)\\ &= \hat{\alpha}_k\cos(k\omega\sqrt{a}x) + \frac{\hat{\beta}_k\mathrm{i}}{\sqrt{a}}\sin(k\omega\sqrt{a}x).\end{aligned}$$

We see that $\hat{w}_k$ solves $L_k\hat{w}_k = 0$ on $[0,d_1]$ and at $x=0$ it satisfies

$$\hat{w}_k(0) = \hat{\alpha}_k = \hat{\alpha}_k\phi_k(0) \quad\text{and}\quad \hat{w}_k'(0) = \frac{\hat{\beta}_k\mathrm{i}}{\sqrt{a}}k\omega\sqrt{a} = \hat{\alpha}_k\phi_k'(0),$$

where we have used (6.36). So $\hat{w}_k(x) = \hat{\alpha}_k\phi_k(x)$ must hold, and from this we obtain

$$w(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}}\hat{w}_k(x)e_k(t) = \sum_{k\in\mathbb{Z}_{\text{odd}}}\hat{\alpha}_k\phi_k(x)e_k(t) = u(x,t).$$

As w is given by (6.37), $u = w \in C^{1,r}_{\text{anti}}([0,d_1]\times\mathbb{T})$ follows immediately.

Now assume that $u \in C^{1,r}_{\text{anti}}([0,d_n]\times\mathbb{T})$ holds for some $n\in\mathbb{N}$. We aim to show $u \in C^{1,r}_{\text{anti}}([0,d_{n+1}])$. Denote by $v\in\{a,b\}$ the value of V on (d_n,d_{n+1}) and define a function w by

$$w(x,t) = \frac{1}{2}\big(u(d_n,t+\sqrt{v}(x-d_n))+u(d_n,t-\sqrt{v}(x-d_n))\big) + \frac{1}{2\sqrt{v}}\int_{t-\sqrt{v}(x-d_n)}^{t+\sqrt{v}(x-d_n)} u_x(d_n,\tau)\,\mathrm{d}\tau \tag{6.38}$$

for $x\in[d_n,d_{n+1}]$ and $t\in\mathbb{T}$. Then $w \in C^{1,r}_{\text{anti}}([d_n,d_{n+1}]\times\mathbb{T})$ follows immediately from (6.38). Arguing as above, one can show $L_k\hat{w}_k(x) = 0$ for all $k\in\mathbb{Z}$. Since $\hat{w}_k(d_n) = \hat{u}_k(d_n) = \hat{\alpha}_k\phi_k(d_n)$ and $\hat{w}_k'(d_n) = \hat{\alpha}_k\phi_k'(d_n)$, we again get $\hat{w}_k(x) = \hat{\alpha}_k\phi_k(x)$ and thus $w=u$ on $[d_n,d_{n+1}]\times\mathbb{T}$.

Next we show the uniform bound $|u(x,t)| \le C\mathrm{e}^{-\rho x}$ with $\rho = \frac{\log(b)-\log(a)}{4\pi}$. By Proposition 6.7.4, u satisfies $u(x+4\pi,t) = \frac{a}{b}u(x,t)$ for all $x \in [0,\infty)$ and $t \in \mathbb{T}$. Hence we can choose

$$C := \max_{x\in[0,4\pi],t\in\mathbb{T}} \mathrm{e}^{\rho x}|u(x,t)|.$$

To show that u is a C^1-solution to (6.1), first from (6.37) it follows that the directional derivative

$$(\partial_t - c(x)\partial_x)(u_t + c(x)u_x)$$

exists and equals 0 for $x \in (0, d_1)$ as $c(x) = \frac{1}{\sqrt{a}}$ here. Similarly, using (6.38) we obtain

$$(\partial_t - c(x)\partial_x)(u_t + c(x)u_x) = 0$$

for $x \in (d_n, d_{n+1})$ as $c(x) = \frac{1}{\sqrt{v}}$. Lastly, due to (6.37) and the definitions of α, β we have

$$u_x(0,t) = w_x(0,t) = \beta'(t) = (f(\alpha'(t)))_t = (f(u_t(0,t)))_t.$$

for all $t \in \mathbb{T}$. This shows that u is a C^1-solution to (6.1) with its own initial data. □

6.A. Appendix

Recall that $F'(s) = sf'(s)$ formally holds, so that $(F\circ g)'(s) = g(s)(f\circ g)'(s)$. Integrating the second equality from t_0 to t_1, the resulting identity holds pointwise as we show next.

Lemma 6.A.1. *For $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ and $g \in C([t_0,t_1],\mathbb{R})$ with $f\circ g \in C^1([t_0,t_1],\mathbb{R})$, the equation*

$$F(g(t_1)) - F(g(t_0)) = \int_{t_0}^{t_1} g(t)\frac{\mathrm{d}f(g(t))}{\mathrm{d}t}\,\mathrm{d}t$$

holds.

Proof. Assume first that f and g are both C^1 in which case the definition $F(y) = yf(y) - \int_0^y f(s)\,\mathrm{d}s$ and integration by parts yield the result

$$\begin{aligned}\int_{t_0}^{t_1} g(t)\frac{\mathrm{d}f(g(t))}{\mathrm{d}t}\,\mathrm{d}t &= [g(t)f(g(t))]_{t=t_0}^{t_1} - \int_{t_0}^{t_1} g'(t)f(g(t))\,\mathrm{d}t \\ &= [g(t)f(g(t))]_{t=t_0}^{t_1} - \int_{g(t_0)}^{g(t_1)} f(v)\,\mathrm{d}v = F(g(t_1)) - F(g(t_0)).\end{aligned} \tag{6.39}$$

For the general case, choose a sequence of non-negative smooth mollifiers $\phi_n \colon \mathbb{R} \to [0,\infty)$ converging to δ_0, each with support in $[-\frac{1}{n}, \frac{1}{n}]$ and with average $\int_{\mathbb{R}} \phi_n(x)\,\mathrm{d}x = 1$. Since f is strictly increasing, so is $f_n := \phi_n * f$. In particular, f_n is bijective and we may define $g_n := (f_n)^{-1} \circ f \circ g$ so that $f_n \circ g_n = f \circ g$.

Clearly, $f_n \to f$ uniformly on compacts. To see that $g_n \to g$ uniformly on compacts, it suffices to show $\left\|(f_n)^{-1} - f^{-1}\right\|_\infty \le \frac{1}{n}$ for $n \in \mathbb{N}$. Note that

$$f_n(x-\tfrac{1}{n}) = \int_{x-\frac{2}{n}}^{x} f(y)\phi_n(x-\tfrac{1}{n}-y)\,\mathrm{d}y \le \int_{x-\frac{2}{n}}^{x} f(x)\phi_n(x-\tfrac{1}{n}-y)\,\mathrm{d}y = f(x).$$

If we choose $x := f^{-1}(y)$ for arbitrary $y \in \mathbb{R}$ and apply $(f_n)^{-1}$ to both sides of the above inequality, we get $f^{-1}(y) - \frac{1}{n} \leq (f_n)^{-1}(y)$. Similarly, $f^{-1}(y) + \frac{1}{n} \geq (f_n)^{-1}(y)$ holds so that the estimate $\left\|(f_n)^{-1} - f^{-1}\right\|_\infty \leq \frac{1}{n}$ is shown. Letting $F_n(s) := sf_n(s) - \int_0^s f_n(\sigma)\,\mathrm{d}\sigma$, by (6.39) we have

$$F_n(g_n(t_1)) - F_n(g_n(t_0)) = \int_{t_0}^{t_1} g_n(t)\frac{\mathrm{d}f_n(g_n(t))}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_0}^{t_1} g_n(t)\frac{\mathrm{d}f(g(t))}{\mathrm{d}t}\,\mathrm{d}t.$$

For $n \to \infty$, the desired result follows. $\square$

End of Paper

APPENDIX A

THE METHOD OF THE GENERALIZED NEHARI MANIFOLD

A.1. Introduction and main result

In this chapter, we discuss the generalized Nehari manifold and the corresponding variational method. This method can be used to find critical points of functionals with indefinite linear part and superlinear, signed nonlinearities. We use it to find breather solutions in Chapter 2. To the best of our knowledge, this method is originally due to Pankov [74] and was later explored systematically by Szulkin and Weth [94], who also gave the method its name. The motivation for our report is twofold: first, we present a slightly generalized setting that is adapted to our use case, and second we want to give a simple self-contained proof. Our arguments often follow ideas of [94].

To motivate the method of the generalized Nehari manifold, let us introduce the typical setting in which it is used. On our domain of interest, a measure space Ω, we seek solutions to the semilinear equation

$$\mathcal{L}u = f(x,u) \qquad \text{for} \quad x \in \Omega,\ u(x) \in \mathbb{R}, \tag{A.1}$$

where $\mathcal{L}\colon D(\mathcal{L}) \subseteq L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$ is a self-adjoint linear operator and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is the nonlinearity satisfying $f(x,0) = 0$. One may think of $\mathcal{L}$ as the self-adjoint realization of a formal symmetric operator with suitable boundary conditions.

Assuming $L^2(\Omega;\mathbb{R})$ is separable, by the spectral theorem (cf. [77]) there exists a projection-valued measure $P_{(\,\cdot\,)}$ on $\mathbb{R}$ such that $\mathcal{L} = \int_{\mathbb{R}} \lambda \, \mathrm{d}P_\lambda$. With this, we define the quadratic form $q_{\mathcal{L}}$ associated to $\mathcal{L}$ by

$$q_{\mathcal{L}}(u,v) = \int_{\mathbb{R}} \lambda \, \mathrm{d}\langle P_\lambda u, v\rangle_{L^2(\Omega)}$$

for

$$u, v \in \mathcal{H} := \left\{ u \in L^2(\Omega;\mathbb{R})\colon \int_{\mathbb{R}} |\lambda| \, \mathrm{d}\langle P_\lambda u, u\rangle_{L^2(\Omega)} < \infty \right\}.$$

If we additionally assume $0 \notin \sigma_{\mathrm{ess}}(\mathcal{L})$, then 0 is not an accumulation point of $\sigma(\mathcal{L})$ and therefore the formula

$$\langle u, v\rangle := \int_{\mathbb{R}} (\mathbb{1}_{\{0\}}(\lambda) + |\lambda|) \, \mathrm{d}\langle P_\lambda u, v\rangle_{L^2(\Omega)}$$

defines a complete inner product on $\mathcal{H}$. We next define three orthogonal projections on $\mathcal{H}$ by

$$u^+ := P^+ u := P_{(0,\infty)} u, \qquad u^0 := P^0 u := P_{\{0\}} u, \qquad u^- := P^- u := P_{(-\infty,0)} u,$$

which give rise to an orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^0 \oplus \mathcal{H}^-$. Importantly, for $u \in D(\mathcal{L})$ and $v \in \mathcal{H}$ we have $\langle \mathcal{L}u, v\rangle_{L^2(\Omega)} = q_{\mathcal{L}}(u,v) = \langle (P^+ - P^-)u, v\rangle$ which naturally leads to the definition that $u \in \mathcal{H}$ with

$$\langle (P^+ - P^-)u, \varphi\rangle = \int_\Omega f(x,u)\varphi \, \mathrm{d}x \tag{A.2}$$

for all $\varphi \in \mathcal{H}$ is a weak solution to (A.1). Setting

$$F(x,s) \coloneqq \int_0^s f(x,\sigma)\,\mathrm{d}\sigma, \quad I(u) \coloneqq \int_\Omega F(x,u(x))\,\mathrm{d}x, \quad J(u) \coloneqq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u)$$

we see that (A.2) is the Euler-Lagrange-equation of the energy functional J. All nontrivial critical points of J lie on the set

$$\mathcal{M} \coloneqq \{u \in \mathcal{H} \setminus \{0\} \colon J'(u)|_{\mathbb{R}u + \mathcal{H}^0 \oplus \mathcal{H}^-} = 0\},$$

which is called the *generalized Nehari manifold*.

The method of generalized Nehari manifold, which we present next, attempts to extract a critical point of J by finding a minimizer of $J|_{\mathcal{M}}$. In general, it only shows existence of a Cerami sequence (see Definition 1.4.1) for J, but not existence of a nonzero critical point. In order to obtain a critical point from the Cerami sequence, one needs further assumptions on the nonlinearity I. For example, in [94, Theorem 35] the authors assume complete continuity of I'. In Chapter 2 we also consider noncompact asymptotically periodic nonlinearities and employ concentration-compactness arguments to conclude existence.

Theorem A.1.1. *Let Ω be a measure space and for some $2 < p_\star \le p^\star < \infty$ we assume*

($\mathcal{A}$A.1) *The domain $\mathcal{H}$ is a real Hilbert space that admits an orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^0 \oplus \mathcal{H}^-$ such that $\mathcal{H}^+ \neq \{0\}$ and $\dim \mathcal{H}^0 < \infty$. We denote the decomposition of $u \in \mathcal{H}$ by $u = u^+ + u^0 + u^-$.*

($\mathcal{A}$A.2) *There exists a bounded embedding $\mathcal{H} \hookrightarrow L^{p_\star}(\Omega) \cap L^{p^\star}(\Omega)$. Moreover, if $u_n \rightharpoonup u$ in $\mathcal{H}$, then (u_n) converges pointwise a.e. to u along a subsequence.*

($\mathcal{A}$A.3) *$F(x,s) = \int_0^s f(x,\sigma)\,\mathrm{d}\sigma$ where $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies*

(i) *f is a Carathéodory function, i.e., $f(x,\,\cdot\,)$ is continuous for a.e. $x \in \Omega$ and $f(\,\cdot\,,s)$ is measurable for all $s \in \mathbb{R}$.*

(ii) *$|f(x,s)| \lesssim |s|^{p_\star - 1} + |s|^{p^\star - 1}$ uniformly in $x \in \Omega$ and $s \in \mathbb{R}$.*

(iii) *$s \mapsto f(x,s)/|s|$ is strictly increasing on $\mathbb{R}$ with limits $\pm\infty$ as $s \to \pm\infty$ for all $x \in \Omega$.*

Then there exists a Cerami sequence for J in $\mathcal{H}$ at level $\mathfrak{c} \coloneqq \inf_{u \in \mathcal{M}} J(u) > 0$, where

$$J(u) \coloneqq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_\Omega F(x,u(x))\,\mathrm{d}x. \tag{A.3}$$

Remark A.1.2.

- In Theorem A.1.1 we do not explicitly assume that $\mathcal{H}$ is the form domain of a selfadjoint operator $\mathcal{L}$, as motivated in the introduction. A critical point u of (A.3) solves

$$\langle u^+, \varphi^+ \rangle - \langle u^-, \varphi^- \rangle - \int_\Omega f(x,u(x))\varphi(x)\,\mathrm{d}x$$

for $\varphi \in \mathcal{H}$, or equivalently

$$R(P^+ - P^-)u = \iota f(x,u)$$

in $\mathcal{H}'$ where $R\colon \mathcal{H} \to \mathcal{H}'$ denotes the Riesz isomorphism and ι is the adjoint of the embedding $\mathcal{H} \hookrightarrow L^{p_\star} \cap L^{p^\star}$. Here the operator $R(P^+ - P^-)$ generalizes the the weak formulation of $\mathcal{L}$.

- We do not distinguish between $u \in \mathcal{H}$ and its image under the embedding of ($\mathcal{A}$A.2).

The second part of ($\mathcal{A}$A.2) is fulfilled for example if Ω is a separable locally compact space and the embedding $\mathcal{H} \hookrightarrow L^p(\Omega)$ is locally compact for some $p \in [p_\star, p^\star]$.

A.2. Proof of main result

We work under the assumptions of Theorem A.1.1. Let us first investigate the nonlinearity

$$I(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x.$$

Lemma A.2.1. *We have $I \in C^1(\mathcal{H}; [0,\infty))$ with $I(0) = 0$, $I'(u) = o(\|u\|)$ as $u \to 0$. Moreover, I is weakly lower semicontinuous and satisfies the weak Ambrosetti-Rabinowitz condition $0 < 2I(u) < I'(u)[u]$ for $u \in \mathcal{H} \setminus \{0\}$.*

Proof. Assumption ($\mathcal{A}$A.3) allows us to estimate $|F(x,s)| \lesssim |s|^{p_\star} + |s|^{p^\star}$, so the embedding ($\mathcal{A}$A.2) shows that I is well-defined. By standard arguments (cf. e.g. [86]) one can show that I is continuously differentiable with derivative given by

$$I'(u)[h] = \int_\Omega f(x, u(x))h(x)\,\mathrm{d}x$$

for $u, h \in \mathcal{H}$. Using ($\mathcal{A}$A.3) once more, we estimate

$$\begin{aligned}\|I'(u)\| &= \|f(x,u)\|_{(L^{p_\star}\cap L^{p^\star})'} \lesssim \|f(x,u)\|_{L^{(p_\star)'}+L^{(p^\star)'}} \lesssim \left\||u|^{p_\star-1}\right\|_{(p_\star)'} + \left\||u|^{p^\star-1}\right\|_{(p^\star)'} \\ &= \|u\|_{p_\star}^{p_\star-1} + \|u\|_{p^\star}^{p^\star-1} \lesssim \|u\|^{p_\star-1} + \|u\|^{p^\star-1} = \mathrm{o}(\|u\|)\end{aligned}$$

as $u \to 0$.

Next, for $s \neq 0$ we have

$$\frac{F(x,s)}{s^2} = \int_0^1 \frac{f(x,\theta s)}{\theta s}\theta\,\mathrm{d}\theta \tag{A.4}$$

and hence $0 < F(x,s) < s^2 \int_0^1 \frac{f(x,s)}{s}\theta\,\mathrm{d}\theta = \frac{1}{2}sf(x,s)$. The Ambrosetti-Rabinowitz condition follows immediately from this. $I(0) = 0$ is clear by definition. Lastly, $f(x,\cdot)$ is increasing by ($\mathcal{A}$A.3) (iii), so $F(x,\cdot)$ is convex and therefore I is weakly lower semicontinuous. □

Lemma A.2.2. *Let $A \subseteq \mathcal{H}\setminus\{0\}$ be weakly compact. Then $I(su)/s^2 \to \infty$ as $s \to \infty$ uniformly in $u \in A$.*

Proof. Assume for a contradiction that there exist $s_n > 0$, $u_n \in \mathcal{H} \setminus \{0\}$ with $s_n \to \infty$ and $u_n \rightharpoonup u \neq 0$ such that $I(s_n u_n)/s_n^2$ is bounded. By ($\mathcal{A}$A.2) we may assume that $u_n \to u$ pointwise a.e. Using equation (A.4) and assumption ($\mathcal{A}$A.3), for $x \in \Omega$ we see that $\frac{F(x,s)}{s^2}$ is increasing in s on $(0,\infty)$ and decreasing on $(-\infty, 0)$ with limit ∞ at $s \to \pm\infty$. Fatou's Lemma yields

$$\liminf_{n\to\infty} \frac{I(s_n u_n)}{s_n^2} \geq \int_\Omega \liminf_{n\to\infty} \frac{F(x, s_n u_n)}{s_n^2}\,\mathrm{d}x \geq \int_\Omega \infty \mathbb{1}_{\{u\neq 0\}}\,\mathrm{d}x = \infty,$$

a contradiction. □

Let us now focus on J, where the following partition of $\mathcal{H}$ will be fundamental for the description of $\mathcal{M}$.

Definition A.2.3. *For $u \in \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-)$, set $\mathcal{H}(u) = (0,\infty)u \oplus \mathcal{H}^0 \oplus \mathcal{H}^-$. Note that $\mathcal{H}(u) = \mathcal{H}(\frac{u^+}{\|u^+\|})$ and we have the partition $\mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-) = \cup_{v \in S^+}\mathcal{H}(v)$ where $S^+ := \{v \in \mathcal{H}^+ \colon \|v\| = 1\}$ is the unit sphere in $\mathcal{H}^+$.*

In the following four lemmas, we show that there exists an isomorphism $m\colon S^+ \to \mathcal{M}$ satisfying the identity $\mathcal{M} \cap \mathcal{H}(v) = \{m(v)\}$.

Lemma A.2.4. *There exists $\delta > 0$ such that $\sup_{u\in\mathcal{H}(v)} J(u) \geq \delta$ for all $v \in S^+$.*

Proof. We have

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \mathrm{o}(\|u\|^2)$$

as $u \to 0$ in $\mathcal{H}$ by Lemma A.2.1. Thus for small $r > 0$ and arbitrary $v \in S^+$, we have $\sup_{u\in\mathcal{H}(v)} J(u) \geq J(rv) \geq \frac{1}{4}\|rv\|^2 = \frac{r^2}{4}$. □

Lemma A.2.5. *Let $(u_n) \subseteq \mathcal{H}$ be an unbounded sequence on which $J(u_n)$ is bounded from below. Then $\left(\frac{u_n^+}{\|u_n\|}\right)$ has no accumulation points.*

Proof. Set $v_n := \frac{u_n}{\|u_n\|}$ and $s_n := \|u_n\|$. We first show $v_n \rightharpoonup 0$ in $\mathcal{H}$. If this were false, then up to a subsequence we have $v_n \rightharpoonup v \neq 0$. Now

$$\frac{\inf_{n\in\mathbb{N}} J(u_n)}{s_n^2} \leq \frac{J(u_n)}{s_n^2} = \frac{1}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{I(s_n v_n)}{s_n^2} \to -\infty$$

as $n \to \infty$ due to Lemma A.2.2 applied to the weakly compact set $\{v_n, v\colon n \in \mathbb{N}\}$, a contradiction. So indeed $v_n \rightharpoonup 0$, and $v_n^0 \to 0$ follows since $\mathcal{H}^0$ is finite-dimensional. From the above inequality we further have

$$\|v_n^+\|^2 = \|v_n^-\|^2 + \frac{2I(s_n v_n)}{s_n^2} + \frac{2J(u_n)}{s_n^2} \geq \|v_n^-\|^2 + 0 + \mathrm{o}(1)$$

which together with $1 = \|v_n\|^2 = \|v_n^+\|^2 + \|v_n^-\|^2 + \mathrm{o}(1)$ shows $1 \geq \|v_n^+\| \geq \frac{1}{\sqrt{2}} + \mathrm{o}(1)$. Combined with $v_n^+ \rightharpoonup 0$ it follows that (v_n^+) has no accumulation points. □

Lemma A.2.6. *For any $v \in S^+$, $J|_{\mathcal{H}(v)}$ attains its maximum.*

Proof. Let (u_n) be a maximizing sequence for $J|_{\mathcal{H}(v)}$. Then the bounded sequence $\left(\frac{u_n^+}{\|u_n\|}\right)$ lies in the one-dimensional space $\mathbb{R}v$, and therefore it has accumulation points. By Lemma A.2.5 the sequence (u_n) is bounded.

Therefore we have $u_n \rightharpoonup u$ up to a subsequence. From lower semicontinuity of I (see Lemma A.2.1) and of the norm we obtain

$$J(u) \geq \lim_{n\to\infty} J(u_n) \geq \delta > 0,$$

where the last estimate is due to Lemma A.2.4. Clearly $u \in \overline{\mathcal{H}(v)}$ holds, and since $J \leq 0$ on $\mathcal{H}^0 \oplus \mathcal{H}^- = \overline{\mathcal{H}(v)} \setminus \mathcal{H}(v)$ we necessarily have $u \in \mathcal{H}(v)$, hence u is a maximizer. □

Lemma A.2.7. *For $v \in S^+$, any nonzero critical point of $J|_{\mathcal{H}(v)}$ is a strict maximizer.*

Proof. Denote the critical point by $u \in \mathcal{H}(v) \setminus \{0\}$, and let $(1+s)u + w \in \mathcal{H}(v)$ with $s > -1$ and $w \in \mathcal{H}^0 \oplus \mathcal{H}^-$. We calculate

$$\begin{aligned}
&J((1+s)u+w)\\
&\quad = \tfrac{1}{2}\left\|(1+s)^2u^+\right\|^2 - \tfrac{1}{2}\left\|(1+s)u^- + w^-\right\|^2 - I((1+s)u+w)\\
&\quad = \tfrac{1}{2}\left\|u^+\right\|^2 + \langle u^+, (s+\tfrac{s^2}{2})u^+\rangle - \tfrac{1}{2}\left\|u^-\right\|^2 - \langle u^-, (s+\tfrac{s^2}{2})u^- + (1+s)w^-\rangle - \tfrac{1}{2}\left\|w^-\right\|^2\\
&\qquad - I((1+s)u+w)\\
&\quad = J(u) + J'(u)[(s+\tfrac{s^2}{2})u + (1+s)w] - \tfrac{1}{2}\left\|w\right\|^2\\
&\qquad + I(u) + I'(u)[(s+\tfrac{s^2}{2})u + (1+s)w] - I((1+s)u+w).
\end{aligned}$$

So it suffices to show the inequality

$$I((1+s)u+w) \geq I(u) + I'(u)[(s+\tfrac{s^2}{2})u + (1+s)w],$$

with strict inequality for $w = 0$ and $s, u \neq 0$. This follows from the pointwise estimate

$$F(x,(1+s)u+w) \geq F(x,u) + f(x,u)[(s+\tfrac{s^2}{2})u + (1+s)w] \tag{A.5}$$

for $x \in \Omega$, $s \in (-1,\infty)$, $u, w \in \mathbb{R}$, again with strict inequality for $w = 0$ and $s, u \neq 0$. We fix u and w. If $u = 0$ we have $F(x,w) \geq 0$ so that (A.5) holds. Thus we may assume $u \neq 0$. For large s, (A.5) is true since F grows superquadratically in s. Inserting $s = -1$, (A.5) also holds since $F(x,w) \geq 0 > F(x,u) - \frac{1}{2}f(x,u)u$ by Lemma A.2.1 and its proof. If (A.5) were false for some s, then the minimum of the difference

$$F(x,(1+s)u+w) - F(x,u) - f(x,u)[(s+\tfrac{s^2}{2})u + (1+s)w]$$

is attained at some $s \in (-1,\infty)$. Taking the derivative w.r.t s we find

$$f(x,(1+s)u+w)u = f(x,u)((1+s)u+w).$$

By strict monotonicity of $\frac{f(x,u)}{|u|}$ either (1) $z := (1+s)u+w = 0$ or (2) $z = u$ hold or (3) u and z have opposite signs.

If $z = u \neq 0$, using $w = -su$ we calculate

$$f(x,u)[(s+\tfrac{s^2}{2})u + (1+s)w] = -\tfrac{s^2}{2}f(x,u)u \leq 0,$$

where the last inequality is strict for $s \neq 0$. If $z = 0$ or $uz < 0$ (and $u \neq 0$), we use $F(x,u) - \frac{1}{2}f(x,u)u < 0$ and $f(x,u)u > 0$ to estimate

$$\begin{aligned}
F(x,z) \geq 0 &> \Big(F(x,u) - \tfrac{1}{2}f(x,u)u\Big) - \tfrac{1+(1+s)^2}{2}f(x,u)u + (1+s)f(x,u)z\\
&= F(x,u) + f(x,u)[(s+\tfrac{s^2}{2})u + (1+s)w].
\end{aligned}$$

□

Definition A.2.8. *For $u \in \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-)$, let $m(u) \in \mathcal{H}(u)$ denote the unique maximizer of $J|_{\mathcal{H}(u)}$, cf. Lemmas A.2.6 and A.2.7.*

Remark A.2.9. Note that $m(u) \in \mathcal{M}$. Since $\mathcal{H}(u) = \mathcal{H}(\frac{u^+}{\|u^+\|})$, m is uniquely determined by its restriction to the sphere S^+. Moreover, $m|_{S^+} : S^+ \to \mathcal{M}$ is a bijection by Lemma A.2.7 since we have $\mathcal{M} = \Big\{u \in \mathcal{H} \setminus (\mathcal{H}^0 \oplus \mathcal{H}^-) \colon u \text{ is CP of } J|_{\mathcal{H}(u)}\Big\}$.

Lemma A.2.10. *$m\colon \mathcal{H}\setminus(\mathcal{H}^0\oplus\mathcal{H}^-)\to\mathcal{M}$ is continuous.*

Proof. Let $u_n\to u$ be a convergent sequence in $\mathcal{H}\setminus(\mathcal{H}^0\oplus\mathcal{H}^-)$. We write $v_n:=\frac{u_n^+}{\|u_n^+\|}$, $v:=\frac{u^+}{\|u^+\|}$ and $m(u_n)=s_nv_n+w_n$, $m(u)=sv+w$ with $s_n,s>0$ and $w_n,w\in\mathcal{H}^0\oplus\mathcal{H}^-$.

In particular, $\frac{m(u_n)^+}{\|m(u_n)\|}=\frac{s_n}{\|m(u_n)\|}\frac{u_n^+}{\|u_n^+\|}$ is bounded and converges along a subsequence since $u_n^+\to u^+\neq 0$. By Lemmas A.2.4 and A.2.5 the sequence $(m(u_n))$ is bounded.

Up to a subsequence, which we again denote by index n, we therefore have $s_n\to s_\star$, $w_n\rightharpoonup w_\star$ for some $s_\star\geq 0$, $w_\star\in\mathcal{H}^0\oplus\mathcal{H}^-$. Using Definition A.2.8 and weak lower semicontinuity of the norm and I, we obtain

$$\begin{aligned}
J(sv+w)&\geq J(s_\star v+w_\star)\\
&=\frac12 s_\star^2-\frac12\|w_\star^-\|^2-I(s_\star v+w_\star)\\
&\geq\frac12 s_\star^2-\frac12\limsup_{n\to\infty}\|w_n^-\|^2-\limsup_{n\to\infty}I(s_nv_n+w_n)\\
&\geq\liminf_{n\to\infty}J(s_nv_n+w_n)\\
&\geq\liminf_{n\to\infty}J(sv_n+w)\\
&=J(sv+w).
\end{aligned}$$

We have equality everywhere, and from the first (in)equality and uniqueness of the maximizer we get $s_\star=s, w_\star=w$. The second (in)equality implies $\|w_\star^-\|=\limsup_{n\to\infty}\|w_n^-\|$ which combined with $w_n^-\rightharpoonup w_\star^-$ shows $w_n^-\to w_\star^-$. As $\mathcal{H}^0$ is finite-dimensional, we also have $w_n^0\to w_\star^0$. In total we have shown $m(u_n)\to m(u)$ up to a subsequence. By considering subsequences of subsequences, convergence of the full sequence follows. □

To conclude the proof, we apply Ekeland's variational principle to $E:=J\circ m$. We first investigate E.

Lemma A.2.11. *The map $E=J\circ m\colon\mathcal{H}\setminus(\mathcal{H}^0\oplus\mathcal{H}^-)\to\mathbb{R}$ is continuously differentiable with*

$$E'(u)[h]=\frac{\|m(u)^+\|}{\|u^+\|}J'(m(u))[h] \tag{A.6}$$

for $u\in\mathcal{H}\setminus(\mathcal{H}^0\oplus\mathcal{H}^-)$ and $h\in\mathcal{H}$.

Proof. By Lemmas A.2.1 and A.2.10, the right-hand side of (A.6) is continuous, so it suffices to show that E is differentiable with derivative given by (A.6). So we fix $u\in\mathcal{H}\setminus(\mathcal{H}^0\oplus\mathcal{H}^-)$ and for small $h\in\mathcal{H}$ we write $m(u+h)=s_h(u+h)+w_h$ with $s_h>0$, $w_h\in\mathcal{H}^0\oplus\mathcal{H}^-$. By Lemma A.2.10 we have $m(u+h)\to m(u)$, i.e., $s_h\to s_0$ and $w_h\to w_0$ as $h\to 0$ in $\mathcal{H}$. Then, using the definition of m we estimate

$$\begin{aligned}
E(u+h)-E(u)&=J(s_h(u+h)+w_h)-J(s_0u+w_0)\\
&\leq J(s_h(u+h)+w_h)-J(s_hu+w_h)\\
&=J'(s_h(u+\theta_hh)+w_h)[s_hh]\\
&=s_0J'(s_0u+w_0)[h]+\mathrm{o}(\|h\|)
\end{aligned}$$

as $h\to 0$ with some $\theta_h\in[0,1]$, and similarly we get

$$E(u+h)-E(u)\geq J(s_0(u+h)+w_0)-J(s_0u+w_0)$$

$$
\begin{aligned}
&= J'(s_0(u + \theta_h h) + w_0)[s_0 h] \\
&= s_0 J'(s_0 u + w_0)[h] + \mathrm{o}(\|h\|).
\end{aligned}
$$

The claim follows from this since

$$\left\|m(u)^+\right\| = \left\|(s_0 u + w_0)^+\right\| = s_0\left\|u^+\right\|. \qquad \square$$

Proof of Theorem A.1.1. By Lemmas A.2.4 and A.2.11, Ekeland's variational principle (cf. [86]) shows there exists a minimizing Palais-Smale sequence $v_n \in S^+$ for $E|_{S^+}$ with $(E|_{S^+})'(v_n) \to 0$. Since $E'(v_n)[v_n] = 0$, we have $E'(v_n) \to 0$. As

$$\frac{1}{2}\|m(v_n)^+\|^2 \geq J(m(v_n)) \geq \delta > 0$$

by Lemma A.2.4, $\|m(v_n)^+\|$ is bounded from below by $\sqrt{2\delta}$, and hence (A.6) shows that $u_n := m(v_n)$ is a Cerami sequence for J with $J(u_n) \to \inf_{S^+} E = \inf_{\mathcal{M}} J = \mathfrak{c}$. $\square$

Appendix B

Two Banach spaces and their properties

B.1. Vector-valued L^2-spaces

This section consists of [41, Appendix A].

Following [35], we give a definition of the Hilbert space $L^2(\mu)$ appearing in Section 2.3 where μ is an increasing matrix-valued function.

Definition B.1.1. *We call a function $\mu\colon \mathbb{R} \to \mathbb{C}^{d\times d}$* increasing *if $\mu(y) - \mu(x)$ is Hermitian and positive semidefinite for all $y \geq x$.*

Lemma B.1.2. *Let $\mu\colon \mathbb{R} \to \mathbb{C}^{d\times d}$ be increasing. Then the coefficients μ_{ij} are locally of bounded variation, and hence there exist $\mathbb{C}$-valued measures ν_{ij}, defined on the bounded Borel subsets of $\mathbb{R}$, which are σ-additive for sets with bounded union, such that*

$$\int_{\mathbb{R}} \varphi \,\mathrm{d}\mu_{ij} = \int_{\mathbb{R}} \varphi \,\mathrm{d}\nu_{ij}$$

for all $\varphi \in C_c(\mathbb{R})$. Here, the left-hand side is a Riemann-Stieltjes integral.

Proof. By positive semidefiniteness, for $y \geq x$ we have

$$\begin{aligned} |\mu_{ij}(y) - \mu_{ij}(x)| &\leq \sqrt{(\mu_{ii}(y) - \mu_{ii}(x))(\mu_{jj}(y) - \mu_{jj}(x))} \\ &\leq \tfrac{1}{2}(\mu_{ii}(y) - \mu_{ii}(x) + \mu_{jj}(y) - \mu_{jj}(x)). \end{aligned}$$

Since μ_{ii} and μ_{jj} are increasing, it follows that

$$\|\mu_{ij}\|_{BV[a,b]} \leq \tfrac{1}{2}(\mu_{ii}(b) - \mu_{ii}(a) + \mu_{jj}(b) - \mu_{jj}(a)) < \infty.$$

Existence of the measure ν_{ij} then follows from the Riesz-Markov-Kakutani representation theorem, see [79, Theorem 6.19]. □

Definition B.1.3. *Let $\mu\colon \mathbb{R} \to \mathbb{C}^{d\times d}$ be increasing, ν_{ij} be the measures from Lemma B.1.2, and $f\colon \mathbb{R} \to \mathbb{C}^d$ be Borel measurable. Let ν be a σ-finite Borel measure on $\mathbb{R}$ such that all ν_{ij} are absolutely continuous with respect to ν. Then define*

$$\|f\|^2_{L^2(\mu)} := \int_{\mathbb{R}} f_i \overline{f_j} \,\mathrm{d}\mu_{ij}(\lambda) := \int_{\mathbb{R}} \left(f_i \overline{f_j} \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\nu} \right) \mathrm{d}\nu,$$

where we used Einstein summation convention and $\frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\nu}$ is the Radon-Nikodým derivative.

Remark B.1.4. By [35, Lemma XIII.5.7], the matrix $\left(\frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\nu}\right)_{i,j}$ is positive semidefinite ν-almost everywhere. Hence, the last integrand above is nonnegative and therefore the integral exists in $[0, \infty]$. Note that such ν always exists and that the $L^2(\mu)$-norm does not depend on the choice of ν. One can for example take $\nu(E) := \sup_{n\in\mathbb{N}} \sum_{i,j=1}^d |\nu_{ij}|(E \cap [-n, n])$.

Definition B.1.5. *Define $L^2(\mu)$ as the quotient space of $\{f\colon \mathbb{R}\to\mathbb{C}^d \text{ meas.} \mid \|f\|_{L^2(\mu)}<\infty\}$ modulo $\{f\colon \mathbb{R}\to\mathbb{C}^d \text{ meas.} \mid \|f\|_{L^2(\mu)}=0\}$. By [35, Theorem XIII.5.10], $L^2(\mu)$ is a Hilbert space with inner product*

$$\langle f,g\rangle_{L^2(\mu)} := \int_{\mathbb{R}} f_i\overline{g_j}\,\mathrm{d}\mu_{ij}(\lambda) := \int_{\mathbb{R}}\left(f_i\overline{g_j}\frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\nu}\right)\mathrm{d}\nu.$$

Remark B.1.6. Multiplication with matrix-valued functions need not be well-defined on $L^2(\mu)$. Consider for example

$$\mu(\lambda)=\begin{pmatrix}\lambda & -\lambda\\ -\lambda & \lambda\end{pmatrix}\in\mathbb{C}^{2\times 2},\qquad M=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

so that $\|f\|^2_{L^2(\mu)}=\int_{\mathbb{R}}|f_1-f_2|^2\,\mathrm{d}\lambda$. In particular, $(g,g)^\top=0$ in $L^2(\mu)$ for arbitrary measurable g, whereas $\left\|M(g,g)^\top\right\|_{L^2(\mu)}=2\|g\|_{L^2(\mathbb{R})}$ need not be zero, nor finite. On the other hand, multiplication with scalar-valued measurable functions $m\colon\mathbb{R}\to\mathbb{C}$ is well-defined, and in addition $\|mf\|_{L^2(\mu)}\le\sup_{x\in\mathbb{R}}|m(x)|\cdot\|f\|_{L^2(\mu)}$ holds.

B.2. Fractional Sobolev spaces on the torus

In this section we collected the results of [73, Appendix B] and [71, Appendix A] on the fractional Sobolev-Slobodeckij space $W^{s,p}(\mathbb{T})$ of differentiability order $s\in[0,\infty)$ and integrability order $p\in[1,\infty]$. We discuss their embedding properties as well as relations to the fractional Laplacian when $p=2$. These results are presented in a form that is sufficient for our applications.

Definition B.2.1. *For $s\in(0,1)$ and $p\in[1,\infty)$, we set*

$$\begin{aligned}
[f]^p_{W^{s,p}(\mathbb{T})} &:= \int_{\mathbb{T}}\int_{\mathbb{R}}\frac{|f(t)-f(t+h)|^p}{|h|^{1+sp}}\,\mathrm{d}h\mathrm{d}t\\
\|f\|^p_{W^{s,p}(\mathbb{T})} &:= \|f\|^p_{L^p(\mathbb{T})}+[f]^p_{W^{s,p}(\mathbb{T})},\\
W^{s,p}(\mathbb{T}) &:= \Big\{f\in L^p(\mathbb{T})\colon [f]_{W^{s,p}(\mathbb{T})}<\infty\Big\}.
\end{aligned}$$

We also set $W^{0,p}(\mathbb{T}):=L^p(\mathbb{T})$ and $W^{s,\infty}(\mathbb{T}):=C^s(\mathbb{T})$. For $s\ge 1$ and $p\in[1,\infty]$ we set $W^{s,p}(\mathbb{T}):=\Big\{u\in W^{\lfloor s\rfloor,p}(\mathbb{T})\colon u^{\lfloor s\rfloor}\in W^{s-\lfloor s\rfloor,p}(\mathbb{T})\Big\}$ where $W^{\lfloor s\rfloor,p}(\mathbb{T})$ is the classical Sobolev space. Lastly, we write $H^s(\mathbb{T})=W^{s,2}(\mathbb{T})$.

Remark B.2.2. For $\sigma>0$ and $h\in\mathbb{T}$ we set

$$\tilde{K}_\sigma(h):=T\sum_{k\in\mathbb{Z}}|h+kT|^{-1-\sigma}.$$

This allows us to write

$$[f]^p_{W^{s,p}(\mathbb{T})}=\int_{\mathbb{T}}\int_{\mathbb{T}}\tilde{K}_{sp}(h)|f(t)-f(t+h)|^p\,\mathrm{d}h\mathrm{d}t.$$

We denote by d the metric on $\mathbb{T}$, i.e., $d(t,\tau)=\min_{\hat{t}\in t,\hat{\tau}\in\tau}|\hat{t}-\hat{\tau}|$. Note that $\tilde{K}_\sigma(h)\eqsim d(0,h)^{-1-\sigma}$ holds uniformly in h.

Remark B.2.3. Let us show that piecewise constant functions lie in $H^s(\mathbb{T})$ precisely for $s < \frac{1}{2}$. For simplicity, we consider the function $f = \mathbb{1}_{[0,1]}$ on $\mathbb{T} = \mathbb{R}/_{2\mathbb{Z}}$. We calculate the $W^{s,p}(\mathbb{T})$-seminorm of f for $s > 0$, $p \in [1, \infty)$ as

$$
\begin{aligned}
[f]^p_{W^{s,p}(\mathbb{T})} \eqsim \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(x) - f(y)|^p}{d(x,y)^{1+sp}} \,\mathrm{d}y\mathrm{d}x &= \frac{1}{2} \int_0^1 \int_1^2 \frac{1}{d(x,y)^{1+sp}} \,\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2} \int_0^1 \int_1^2 \frac{\mathbb{1}_{\{y<x+1\}}}{(y-x)^{1+sp}} + \frac{\mathbb{1}_{\{y>x+1\}}}{(2+x-y)^{1+sp}} \,\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2sp} \int_0^1 \left[\frac{-1}{(y-x)^{sp}} \right]_1^{x+1} + \left[\frac{1}{(2+x-y)^{sp}} \right]_{x+1}^2 \mathrm{d}x \\
&= \frac{1}{2sp} \int_0^1 \frac{1}{(1-x)^{sp}} + \frac{1}{x^{sp}} - 2 \,\mathrm{d}x \\
&= \begin{cases} \infty, & sp \geq 1, \\ \frac{1}{2sp}\left(\frac{2}{1-sp} - 2\right), & sp < 1. \end{cases}
\end{aligned}
$$

Setting $p = 2$, we verify the claim.

B.2.1. Embeddings between fractional Sobolev spaces

Let us note that the fractional Gagliardo-Nirenberg inequality of Lemma B.2.4 as well as the Sobolev embeddings of Lemmas B.2.5 and B.2.6 hold on the torus.

Lemma B.2.4. *Let $s_1, s_2 \in [0,1)$, $\theta \in (0,1)$, $p_1, p_2 \in [1,\infty]$ and $s = \theta s_1 + (1-\theta)s_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then $\|f\|_{W^{s,p}(\mathbb{T})} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\mathbb{T})} \|f\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{T})}$ holds.*

Lemma B.2.5. *Let $s_1, s_2 \in (0,1), p_1, p_2 \in [1,\infty]$ with $s_2 < s_1$ and $\frac{1}{p_2} - s_2 \geq \frac{1}{p_1} - s_1$ with strict inequality for $s_1 p_1 = 1$. Then $\|f\|_{W^{s_2,p_2}(\mathbb{T})} \lesssim \|f\|_{W^{s_1,p_1}(\mathbb{T})}$ holds.*

Lemma B.2.6. $W^{1+s,p}(\mathbb{T}) \hookrightarrow C^{1,s-\frac{1}{p}}(\mathbb{T})$ *for $s \in (0,1), p \in (1,\infty)$ with $sp > 1$.*

Proof of Lemmas B.2.4 to B.2.6. We first remark that all results hold if $W^{s,p}(\mathbb{T})$ is replaced by $W^{s,p}(I)$ where I is a bounded interval. This space is defined by

$$
W^{s,p}(I) = \left\{ f \in L^p(I) \colon [f]_{W^{s,p}(I)} \coloneqq \left(\int_I \int_I \frac{|f(x) - f(y)|^p}{|x-y|^{1+sp}} \,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}} < \infty \right\}
$$

for $s \in (0,1)$, $1 \leq p < \infty$, and $W^{s,\infty}(I) = C^s(I)$, $W^{0,p}(I) = L^p(I)$.

Indeed, on intervals the Gagliardo-Nirenberg inequality holds by [16]. Also on intervals we have from [69] that $W^{s_1,p_1}(I) \hookrightarrow L^q(I)$ for $\frac{1}{q} = \frac{1}{p_1} - s_1$ if $s_1 p_1 < 1$, $W^{s_1,p_1}(I) \hookrightarrow L^q(I)$ for $1 \leq q < \infty$ if $s_1 p_1 = 1$, and that $W^{s_1,p_1}(I) \hookrightarrow C^{\alpha}(I)$ for $-\alpha = \frac{1}{p_1} - s_1$ if $s_1 p_1 > 1$. From these properties we obtain Lemma B.2.6 (on intervals), and we can deduce the Sobolev embedding of Lemma B.2.5 (on intervals) by applying the Gagliardo-Nirenberg inequality.

Then, the statements of Lemmas B.2.4 to B.2.6 follow from the results on intervals since the norms $\|f\|_{W^{s,p}(\mathbb{T})}$ and $\|f\|_{W^{s,p}([0,2T])}$ are equivalent for periodic f. □

Lastly, we consider the composition with a Hölder continuous function as a map between fractional Sobolev spaces.

Lemma B.2.7. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be r-Hölder continuous. Then the map*

$$W^{s,p}(\mathbb{T}) \to W^{rs,p/r}(\mathbb{T}), u \mapsto g \circ u$$

is well-defined for $s \in [0,1)$ and $p \in [1,\infty)$.

Proof. By assumption, there exists $C > 0$ such that $|g(x) - g(y)| \leq C|x-y|^r$ holds for all $x, y \in \mathbb{R}$. First, let $u \in L^p(\mathbb{T})$. Then

$$\begin{aligned}\|g(u)\|_{L^{p/r}(\mathbb{T})}^{p/r} &= \int_{\mathbb{T}} |g(u(t))|^{p/r}\,\mathrm{d}t \leq 2^{p/r-1}\int_{\mathbb{T}}\Big(|g(u(t)) - g(0)|^{p/r} + |g(0)|^{p/r}\Big)\,\mathrm{d}t \\ &\leq 2^{p/r-1}\int_{\mathbb{T}}\Big(C^{p/r}|u(t)|^p + |g(0)|^{p/r}\Big)\,\mathrm{d}t = 2^{p/r-1}\Big(C^{p/r}\|u\|_{L^p(\mathbb{T})}^p + T|g(0)|^{p/r}\Big),\end{aligned}$$

so $g(u) \in L^{p/r}(\mathbb{T})$. Now let $u \in W^{s,p}(\mathbb{T})$ with $s \in (0,1)$. Then

$$\begin{aligned}[g(u)]_{W^{rs,p/r}(\mathbb{T})}^{p/r} &\eqsim \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|g(u(t_1)) - g(u(t_2))|^{p/r}}{d(t_1,t_2)^{1+sp}}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &\leq \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{C^{p/r}|u(t_1) - u(t_2)|^p}{d(t_1,t_2)^{1+sp}}\,\mathrm{d}t_1\,\mathrm{d}t_2 \eqsim C^{p/r}[u]_{W^{s,p}(\mathbb{T})}^p.\end{aligned}$$

□

B.2.2. The fractional Laplacian

We first show that the fractional Laplacian $|\partial_t|^s f = \mathcal{F}^{-1}[|\omega k|^s \hat{f}_k]$ can be expressed using a singular integral.

Lemma B.2.8. *Let $s \in (0,2)$ and $f \in C^{1,1}(\mathbb{T})$. Then*

$$\begin{aligned}|\partial_t|^s f(t) &= C_s \int_{\mathbb{R}} \frac{2f(t) - f(t+h) - f(t-h)}{|h|^{1+s}}\,\mathrm{d}h \\ &= \int_{\mathbb{T}} K_s(h)(2f(t) - f(t+h) - f(t-h))\,\mathrm{d}h\end{aligned}$$

holds where

$$C_s := \left(2\int_{\mathbb{R}} \frac{1-\cos(\eta)}{|\eta|^{1+s}}\,\mathrm{d}\eta\right)^{-1}, \qquad \textit{and} \qquad K_s(h) := C_s T \sum_{k\in\mathbb{Z}} |h + kT|^{-1-\sigma}.$$

For a proof see [78, Theorem 2.5] where the case $T = 2\pi$ is discussed[1]. In [78] a principal value formulation is used which can be avoided by using the above symmetric representation, as discussed in [69, Lemma 3.2].

Related to Lemma B.2.8 is the fact that the seminorm $[f]_{W^{s,2}(\mathbb{T})}$ coincides with $\||\partial_t|^s f\|_2$ up to a constant, and in particular that $W^{s,2}(\mathbb{T}) = H^s(\mathbb{T})$.

Lemma B.2.9. *Let $s \in (0,1)$ and $u \in H^s(\mathbb{T})$. Then*

$$\||\partial_t|^s u\|_{L^2(\mathbb{T})}^2 = C_{2s}[u]_{W^{s,2}(\mathbb{T})}^2 = \int_{\mathbb{T}}\int_{\mathbb{T}} K_{2s}(h)|f(t) - f(t+h)|^2\,\mathrm{d}h\mathrm{d}t$$

holds.

[1]The constant in [78, Theorem 2.5] has a typo: σ needs to be replaced by 2σ. Then the constant in [78] coincides with C_s up to a factor of 2.

This can be shown in the same way as [69, Proposition 3.4]. Formally, it follows from Lemma B.2.8 (with $2s$ instead of s) by multiplying the identity with f and then integrating.

Additionally, the following version of the Kenig-Ponce-Vega inequality (cf. [48]) holds on the torus.

Lemma B.2.10. *Let $f, g \in C^\infty(\mathbb{T})$, $s \in (0,2)$, $s_1, s_2 \in (0,1)$ with $s < s_1 + s_2$, and $p, p_1, p_2 \in [1,\infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\||\partial_t|^s(fg) - f|\partial_t|^s g - g|\partial_t|^s f\|_{L^p(\mathbb{T})} \lesssim [f]_{W^{s_1,p_1}(\mathbb{T})}[g]_{W^{s_2,p_2}(\mathbb{T})}$$

holds.

Proof. We only consider the case $p_1, p_2 < \infty$. Using Lemma B.2.8, we write

$$\begin{aligned}
&|\partial_t|^s(fg)(t) - f(t)|\partial_t|^s g(t) - g(t)|\partial_t|^s f(t) \\
&\quad = \int_{\mathbb{T}} K_s(h)(2f(t)g(t) - f(t+h)g(t+h) - f(t-h)g(t-h))\,\mathrm{d}h \\
&\qquad - f(t)\int_{\mathbb{T}} K_s(h)(2g(t) - g(t+h) - g(t-h))\,\mathrm{d}h \\
&\qquad - g(t)\int_{\mathbb{T}} K_s(h)(2f(t) - f(t+h) - f(t-h))\,\mathrm{d}h \\
&\quad = -\int_{\mathbb{T}} K_s(h)\big((f(t) - f(t+h))(g(t) - g(t+h)) + (f(t) - f(t-h))(g(t) - g(t-h))\big)\,\mathrm{d}h \\
&\quad = -2\int_{\mathbb{T}} K_s(h)(f(t) - f(t+h))(g(t) - g(t+h))\,\mathrm{d}h
\end{aligned}$$

Now let $r := 1 - \frac{1}{p} + s - s_1 - s_2 < \frac{1}{p'}$. Using Hölder's inequality twice, we estimate

$$\begin{aligned}
&\left\|\int_{\mathbb{T}} K_s(h)(f(t) - f(t+h))(g(t) - g(t+h))\,\mathrm{d}h\right\|_{L^p(\mathbb{T})} \\
&\quad \leq \int_{\mathbb{T}} \|K_s(h)|f(t) - f(t+h)||g(t) - g(t+h)|\|_{L^p(\mathbb{T})}\,\mathrm{d}h \\
&\quad \leq \|d(0,h)^{-r}\|_{L^{p'}(\mathbb{T})}\|K_s(h)d(0,h)^r|f(t) - f(t+h)||g(t) - g(t+h)|\|_{L^p(\mathbb{T}\times\mathbb{T})} \\
&\quad \lesssim \left\|\sqrt[p_1]{\tilde{K}_{s_1p_1}(h)}|f(t) - f(t+h)|\right\|_{L^{p_1}(\mathbb{T}\times\mathbb{T})}\left\|\sqrt[p_2]{\tilde{K}_{s_2p_2}(h)}|f(t) - f(t+h)|\right\|_{L^{p_2}(\mathbb{T}\times\mathbb{T})} \\
&\quad = [f]_{W^{s_1,p_1}(\mathbb{T})}[g]_{W^{s_2,p_2}(\mathbb{T})}
\end{aligned}$$

where we also used that

$$K_s(h)d(0,h)^r \eqsim d(0,h)^{-1-s+r} = d(0,h)^{-\frac{1}{p_1}-s_1-\frac{1}{p_2}-s_2} \eqsim \sqrt[p_1]{\tilde{K}_{s_1p_1}(h)}\,\sqrt[p_2]{\tilde{K}_{s_2p_2}(h)}. \qquad \square$$

Lastly, we make the following observation on derivatives of time-antiperiodic functions. The proof, which follows via the Fourier transform from the fact that the zero Fourier mode vanishes, is omitted.

Lemma B.2.11. *Let the function $v \in L^1(\mathbb{T})$ be $\frac{T}{2}$-antiperiodic in time. Then for any $s > 0$ and $\sigma \in \mathbb{R}$ we have*

$$\|v\|_{L^2(\mathbb{T})} \leq \frac{1}{\omega^s}\||\partial_t|^s v\|_{L^2(\mathbb{T})} \quad \textit{and thus} \quad \||\partial_t|^\sigma v\|_{L^2(\mathbb{T})} \leq \frac{1}{\omega^s}\left\||\partial_t|^{\sigma+s} v\right\|_{L^2(\mathbb{T})}.$$

If furthermore $\mathcal{F}_k[v] = 0$ for $|k| < K$, then these estimates can be improved to

$$\|v\|_{L^2(\mathbb{T})} \leq \frac{1}{(K\omega)^s}\||\partial_t|^s v\|_{L^2(\mathbb{T})} \quad \textit{and} \quad \||\partial_t|^\sigma v\|_{L^2(\mathbb{T})} \leq \frac{1}{(K\omega)^s}\left\||\partial_t|^{\sigma+s} v\right\|_{L^2(\mathbb{T})}.$$

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Eidesstattliche Erklärung

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