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Sobolev stability for the 2D MHD equations in the non-resistive limit

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Abstract

In this article, we consider the stability of the 2D magnetohydrodynamics equations close to a combination of Couette flow and a constant magnetic field. We consider the ideal conductor limit for the case when viscosity ν is larger than resistivity κ, $\nu \geqslant \kappa > 0$. For this regime, we establish a bound on the Sobolev stability threshold. Furthermore, for $\kappa \leqslant \nu^3$ this system exhibits instability, which leads to norm inflation of size $\nu\kappa^{-\frac{1}{3}}$.

Keywords: magnetohydrodynamics, stability threshold, enhanced dissipation

Mathematics subject classification: 76E25, 76E30, 76E05

Contents

1. Introduction

The equations of magnetohydrodynamics (MHDs)

$$\begin{aligned} \partial_t V + (V\cdot\nabla)V + \nabla\Pi &= \nu\Delta V + (B\cdot\nabla)B, \\ \partial_t B + (V\cdot\nabla)B &= \kappa\Delta B + (B\cdot\nabla)V, \\ \nabla\cdot V = \nabla\cdot B &= 0, \\ (t,x,y) &\in \mathbb{R}^+\times\mathbb{T}\times\mathbb{R} =: \Omega, \end{aligned} \tag{1}$$

model the evolution of a magnetic field $B:\Omega\to\mathbb{R}^2$ interacting with the velocity $V:\Omega\to\mathbb{R}^2$ of a conducting fluid. The MHD equations are a common model used in astrophysics, planetary magnetism and controlled nuclear fusion [Dav16]. The quantities $\nu,\kappa\geqslant 0$ correspond to fluid viscosity and magnetic resistivity. The pressure $\Pi:\Omega\to\mathbb{R}$ ensures that the velocity remains divergence-free. A fundamental problem of fluid dynamics and plasma physics is the stability and long-time behavior of solutions to equation (1) and in particular stability of specific solutions. We consider the combination of an affine shear flow, called Couette flow, and a constant magnetic field:

$$\begin{aligned} V_s &= ye_1, \\ B_s &= \alpha e_1. \end{aligned}$$

In particular, the solution combines the effects of mixing due to shear and coupling by the magnetic field. The Couette flow mixes any perturbation, which leads to increased dissipation rates, called enhanced dissipation, and stabilizes the equation. The coupling with a constant magnetic field propagates this mixing to magnetic perturbations. However, the magnetic field weakens the mixing, especially if viscosity is larger than resistivity, inviscid damping gets counteracted by algebraic growth for specific time regimes.

In the related case of the Navier–Stokes equation, that is when no magnetic field is present, one observes turbulent solutions as viscosity reaches small values. In contrast, the linearized problem around Couette flow is stable for all values of the viscosity. These phenomena are known as the Sommerfeld paradox [LL11] and highlight instability due to nonlinear effects. In [BM15, DM18, DZ21, BM14, IJ13] various authors show sharp stability in Gevrey 2 spaces (spaces between C^∞ and analytic) for the inviscid case $\nu = 0$. The nonlinear instability can be suppressed by the viscosity for initial data sufficiently small in Sobolev spaces, ensuring stability [BVW18, MZ22, BGM17].

When considering the MHD equations without Couette flow, the constant magnetic field stabilizes the equation. The dynamics of small initial perturbations of the ideal MHD equation around a strong enough magnetic field is close to the linearized system [BSS88]. For stability in several dissipation regimes we refer to [WZ17, RWXZ14, HXY18, Sch88, CF23, Koz89] and references therein. However, global in time wellposedness for the non-resistive case is still open (see the discussion in [CF23]). Furthermore, a shear flow leads to qualitatively different behavior and instabilities [HT01, HHKL18].

Recently, the MHD equation around Couette flow has gathered significant interest [Lis20, KZ23, ZZ24, Dol23, KZ25]. Already on a linear level, the behavior of the MHD changes for different values of ν and κ. In [Lis20] Liss proved the first stability threshold for the MHD equations. He considered the full dissipative regime of $\kappa=\nu>0$ and proved the stability of the three-dimensional MHD equation for initial data which is sufficiently small in Sobolev spaces. For the analogous two-dimensional problem, Dolce [Dol23] proved stability in the more general setting of $0<\kappa^3\lesssim\nu\leqslant\kappa$. In [KZ25] Zillinger and the author considered the case of only horizontal resistivity and full viscosity and established stability for small data in Sobolev spaces. For the regime of vanishing viscosity $\nu=0$ and non-vanishing resistivity $\kappa>0$, in [KZ23] we constructed a linear stability and instability mechanism around nearby traveling waves in Gevrey 2 spaces. In a corresponding nonlinear stability result, Zhao and Zi [ZZ24] proved the almost matching nonlinear result of Gevrey σ stability for $1\leqslant\sigma<2$ and for sufficiently small perturbations.

The results mentioned above on stability around Couette flow focus on the setting when resistivity is larger than viscosity $\nu\leqslant\kappa$. Indeed in the setting $\nu>0$ and $\kappa=0$, the magnetic effects dominate leading to a linear instability mechanism and thus a growth of the magnetic field by νt for specific initial data [KZ25].

In this paper, we consider the setting $0<\kappa\leqslant\nu$. In particular, this also includes the non-resistive limit $\kappa\downarrow 0$ independent of ν. To the author's knowledge the stability of the regime $\kappa<\nu$ has not previously been studied for the MHD equation around Couette flow. To state the main result, we define the perturbative unknowns

$$\begin{aligned} v(x,y,t) &= V(x+yt,y,t)-V_s,\\ b(x,y,t) &= B(x+yt,y,t)-B_s, \end{aligned}$$

where the change of variables $x\mapsto x+yt$ follows the characteristics of the Couette flow. For these unknowns, equation (1) becomes

$$\begin{aligned} \partial_t v+v_2e_1-2\partial_x\Delta_t^{-1}\nabla_t v_2 &= \nu\Delta_t v+\alpha\partial_x b+(b\cdot\nabla_t)b-(v\cdot\nabla_t)v-\nabla_t\pi,\\ \partial_t b-b_2e_1 &= \kappa\Delta_t b+\alpha\partial_x v+(b\cdot\nabla_t)v-(v\cdot\nabla_t)b,\\ \nabla_t\cdot v=\nabla_t\cdot b &= 0. \end{aligned}\tag{2}$$

Due to the change of variables the spatial derivatives become time-dependent, i.e. $\partial_y^t=\partial_y-t\partial_x$, $\nabla_t=(\partial_x,\partial_y^t)^{\mathrm{T}}$, $\Delta_t=\partial_x^2+(\partial_y^t)^2$ and $\Lambda_t=(-\Delta_t)^{\frac{1}{2}}$.

For equation (2) we establish Lipschitz stability for initial data which is sufficiently small in Sobolev spaces, in the sense that there exists a bound on the initial data $\varepsilon_0=\varepsilon_0(\nu,\kappa)$ and a Lipschitz constant $L=L(\nu,\kappa)$ such that for initial data which satisfies

$$\|(v,b)_{\mathrm{in}}\|_{H^N}=\varepsilon\leqslant\varepsilon_0,$$

the corresponding solution is globally bounded in time by

$$\|(v,b)(t)\|_{H^N}\leqslant L\varepsilon.$$

For the non-resistive case, $\kappa = 0$, global wellposedness is an open problem and so Lipschitz stability in Sobolev spaces is unclear. Thus, naturally the question arises, which ε_0 and L are optimal and how they behave in the limit $\nu, \kappa \downarrow 0$.

We denote a Sobolev stability threshold as $\gamma_1, \gamma_2 \in \mathbb{R}$, such that for $\varepsilon_0 = c_0 \nu^{\gamma_1} \kappa^{\gamma_2}$ with small $c_0 > 0$ we obtain

$$\|(v,b)_{\text{in}}\|_{H^N} \leqslant c_0 \nu^{\gamma_1} \kappa^{\gamma_2} \to \text{stability},$$
$$\|(v,b)_{\text{in}}\|_{H^N} \gg c_0 \nu^{\gamma_1} \kappa^{\gamma_2} \to \text{possible instability}.$$

This extends the common convention in the field (e.g. see [BVW18]) to allow for two independent parameters ν and κ. In particular, it agrees with the common convention when restricting to the case $\nu \approx \kappa$. It allows us to discuss cases where κ tends to zero much quicker than ν. Establishing a possible instability is highly nontrivial since for the nonlinear setting it is difficult to construct solutions that exhibit norm inflation. To the author's knowledge, there does not exist any nonlinear instability result for the MHD equation around Couette flow in Sobolev spaces.

For accessibility and simplicity of notation, we state our main result as the following theorem (see theorem 3.1 for a detailed description).

Theorem 1.1. *Consider $\alpha > \frac{1}{2}$, $N \geqslant 5$ and a small enough constant $c_0 = c_0(\alpha) > 0$. Let $0 < \kappa \leqslant \nu \leqslant \frac{1}{40}(1 - \frac{1}{2\alpha})^{\frac{6}{5}}$, then we obtain Sobolev stability for initial data which is sufficiently small in Sobolev spaces, where the estimates qualitatively differ for the regimes $\kappa \gtrsim \nu^3$ and $\kappa \lesssim \nu^3$. More precisely:*

- *In the regime of $\nu^3 \lesssim \kappa$, for all initial data which satisfy*

$$\|(v,b)_{\text{in}}\|_{H^N} = \varepsilon \leqslant c_0 \nu^{\frac{1}{12}} \kappa^{\frac{1}{2}},$$

  the global in time solution (v, b) of (2) satisfies the Lipschitz bound

$$\sup_{t>0} \|(v,b)(t)\|_{H^N} \lesssim \varepsilon.$$

- *In the regime of $\nu^3 \gtrsim \kappa$, for all initial data which satisfy*

$$\|(v,b)_{\text{in}}\|_{H^N} = \varepsilon \leqslant c_0 \nu^{-\frac{11}{12}} \kappa^{\frac{5}{6}},$$

  the global in time solution (v, b) of (2) satisfies the Lipschitz bound

$$\sup_{t>0} \|(v,b)(t)\|_{H^N} \lesssim \nu \kappa^{-\frac{1}{3}} \varepsilon.$$

In particular, we obtain Lipschitz stability for the Lipschitz constant $L \approx \max(1, \nu\kappa^{-\frac{1}{3}})$ for the smallness parameter $\varepsilon_0 \approx \min(\nu^{\frac{1}{12}}\kappa^{\frac{1}{2}}, \nu^{-\frac{11}{12}}\kappa^{\frac{5}{6}})$.

In the proof, we employ an energy method similar to [BBZD23, MZZ23, Zil21, Dol23, KZ25]. In the following, we outline the main challenges and novelties of the proof:

- The imbalance of resistivity κ and viscosity ν yields two cases $\nu^3 \lesssim \kappa$ and $\nu^3 \gtrsim \kappa$ (or equivalently $1 \lesssim \nu\kappa^{-\frac{1}{3}}$ or $1 \gtrsim \nu\kappa^{-\frac{1}{3}}$). These cases give different values for L, namely 1 and $\nu\kappa^{-\frac{1}{3}}$. By proposition 2, the norm inflation of $\nu\kappa^{-\frac{1}{3}}$ appears in the linear dynamics and thus is sharp.

- We consider the case $\nu^3 \gtrsim \kappa$. On certain time scales the viscosity is so strong that fluid effects get suppressed while the effects of the magnetic field dominate. Thus, the term $\partial_t b = e_1 b_2$ in (1) generates algebraic growth in specific regimes (see section 2.2). Estimating this linear effect yields the norm inflation by $L = \nu\kappa^{-\frac{1}{3}}$. The algebraic growth appears on different time scales depending on the frequency, a precise estimate of the nonlinear terms is necessary.
- For the case $\nu^3 \lesssim \kappa$ the algebraic growth is bounded by a finite constant. In the subcase $\nu = \kappa$ the sum of the threshold parameters is $\gamma_1 + \gamma_2 = \frac{7}{12}$ which is a slight improvement over $\frac{2}{3}$ in [Dol23]. This improvement is attained by the choice of our adapted unknowns which changes the structure of the nonlinearity.
- In the proof of theorem 1.1 we perform a low and high frequency decomposition $a = a_{\mathrm{hi}} + a_{\mathrm{low}}$. For high frequencies, the nonlinear term consist of $a_{\mathrm{low}}\nabla_t a_{\mathrm{hi}}$, called transport term and $a_{\mathrm{hi}}\nabla_t a$, called reaction term (including hi–hi interactions). Compared to the Navier–Stokes equation, in the case of the MHD equation, it is vital to bound the transport term precisely. In particular, for $\kappa \lesssim \nu^3$ the previously mentioned algebraic growth affects the estimate of the transport term strongly.
- The threshold is determined by the nonlinear term $v\nabla_t b = \Lambda_t^{-1}\nabla^\perp p_1 \nabla b$ acting on b in (2), for the natural unknown $p_1 = \Lambda_t^{-1}\nabla_t^\perp \cdot v$ (which we discuss later in more detail). In our estimates we rely on two stabilizing effects, the strong viscosity of v and the Λ_t^{-1} in front of p_1. For the nonlinear term $v\nabla_t b$ both effects fall onto v. Due to the weaker integrability of the b this term determines the threshold after integrating in time.

With the main challenges in mind, let us comment on the results:

- The size of the constant magnetic field $\alpha > \frac{1}{2}$ results in a strong interaction between v and b. Due to this interaction, the decay in v and growth in b are in balance (see lemma 1). Constants may depend on α and degenerate as $\alpha \downarrow \frac{1}{2}$. For example we obtain $\lim_{\alpha\downarrow\frac{1}{2}} c_0(\alpha) = 0$.
- Figure 1 shows which areas stability has been proven. The graphic shows only qualitative behavior and after rescaling we obtain the same graphic. The resistivity κ is on the vertical axis and the viscosity ν is on the horizontal axis. We prove stability for the regime $0 < \kappa \leqslant \nu$, which we divide into two segments: $\nu^3 \lesssim \kappa$ in orange and $\nu^3 \gtrsim \kappa$ in red. In [Dol23] Dolce considered the regime of $0 < (\frac{16}{\alpha}\kappa)^3 \leqslant \nu \leqslant \kappa$, which is in blue. The authors of [ZZ24] considered the line $\nu = 0$ which is in purple. The black line corresponds to $\nu = \kappa > 0$ of [Lis20].

  Stability for the regimes $0 < \nu \leqslant (\frac{16}{\alpha}\kappa)^3$, $\kappa = 0 < \nu$ and $\kappa = \nu = 0$ remain open. For the set $0 < \nu \ll \kappa^3$, we expect that an adjusted application of the methods used in this article yield stability. We expect stability for the case $\kappa = 0$ and $0 < \nu$ to be very difficult since we obtain linear growth for the $p_2 = \Lambda_t^{-1}\nabla_t^\perp \cdot b$ unknown. For $\Lambda_t^{-1} p_2$ we obtain linear stability but then there is no time decay in the magnetic field and so we lack an important stabilizing effect. In the inviscid case, $\kappa = \nu = 0$ the linearized system is stable in the p unknowns. However, due to the lack of dissipation, it is very challenging to bound the nonlinear terms.
- Our threshold consists of parameters γ_1 and γ_2. An alternative notation is to impose the relation $\nu \approx \kappa^\delta$ for some $0 \leqslant \delta \leqslant 1$. With that convention we obtain stability if $\varepsilon \leqslant c_0\kappa^{\gamma(\delta)}$ for

$$\gamma = \begin{cases} \frac{1}{2} + \frac{\delta}{12} & \delta \geqslant \frac{1}{3} \\ \frac{5}{6} - \frac{11}{12}\delta & \text{otherwise.} \end{cases}$$

The remainder of this article is structured as follows:

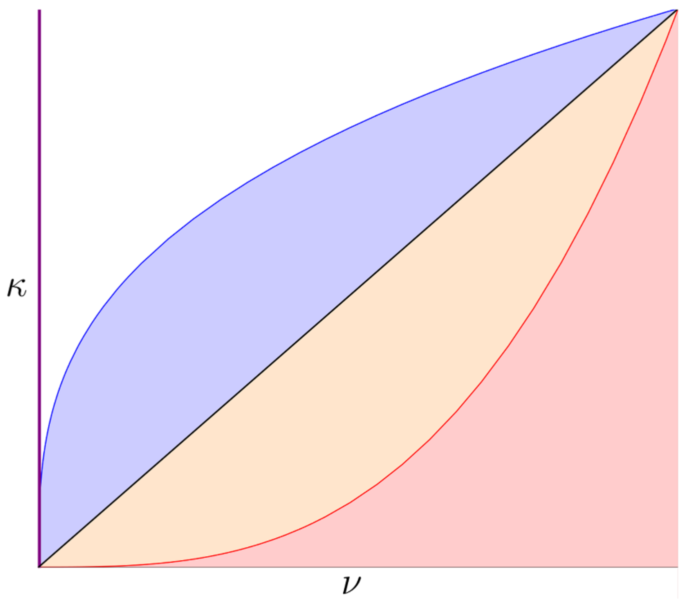

Figure 1. Sketch of areas with results for stability.

- In section 2 we discuss the linearized system. We identify two different time regions where 'circular movement' or 'strong viscosity' determine the linearized behavior. We estimate both effects separately and then establish the estimates for the linearized system.
- In section 3 we prove the main theorem. We employ a bootstrap approach, where we control errors in proposition 3.1. The main difficulty is to bound the linear growth and the nonlinear effect of $v\nabla_t b$ acting on b.

1.1. Notations and conventions

For $a, b \in \mathbb{R}$ we denote their minimum and maximum as

$$\min(a,b) = a \wedge b,$$
$$\max(a,b) = a \vee b.$$

We write $f \lesssim g$ if $f \leqslant Cg$ for a constant C independent of ν and κ. Furthermore, we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. We denote the Lebesgue spaces $L^p = L^p(\mathbb{T}\times\mathbb{R})$ and the Sobolev spaces $H^N = H^N(\mathbb{T}\times\mathbb{R})$ for some $N \in \mathbb{N}$. For time-dependent functions, we denote $L^pH^s = L^p_tH^s$ as the space with the norm

$$\|f\|_{L^pH^s} = \left\| \|f\|_{H^s(\mathbb{T}\times\mathbb{R})} \right\|_{L^p(0,T)}, \tag{3}$$

where omit writing the T. We write the time-dependent spatial derivatives

$$\partial_y^t = \partial_y - t\partial_x,$$
$$\nabla_t = \left(\partial_x, \partial_y^t\right)^{\mathrm{T}},$$
$$\Delta_t = \partial_x^2 + \left(\partial_y^t\right)^2,$$

and the half Laplacians as

$$\Lambda = (-\Delta)^{\frac{1}{2}},$$
$$\Lambda_t = (-\Delta_t)^{\frac{1}{2}}.$$

The function $f \in H^N$ is decomposed into its x average and the orthogonal complement

$$f_= (y) = \int f(x,y)\,\mathrm{d}x,$$
$$f_{\neq} = f - f_=.$$

1.2. The adapted unknowns

For the following, it is useful to change to the unknowns $p_{1,\neq} = \Lambda_t^{-1}\nabla_t^{\perp} \cdot v_{\neq}$ and $p_{2,\neq} = \Lambda_t^{-1}\nabla_t^{\perp} \cdot b_{\neq}$. However, since $\Lambda_t^{-1}\nabla_t^{\perp}$ is not a bounded operator on the x average, we define

$$\begin{aligned}
p_{1,\neq} &= \Lambda_t^{-1}\nabla_t^{\perp} \cdot v_{\neq},\\
p_{1,=} &= v_{1,=},\\
p_{2,\neq} &= \Lambda_t^{-1}\nabla_t^{\perp} \cdot b_{\neq},\\
p_{2,=} &= b_{1,=},\\
p &= (p_1, p_2).
\end{aligned}$$

The unknowns p correspond to the unknowns of [KZ25] and a adaption of the unknowns (z,q) of [Dol23], i.e. $p_{\neq}$ corresponds to $\partial_x^{-1}(z_{\neq}, q_{\neq})$. Thus (2) can be equivalently expressed as

$$\begin{aligned}
\partial_t p_1 - \partial_x \partial_y^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1 + \Lambda_t^{-1}\nabla_t^{\perp}\left((b\cdot\nabla_t)\,b - (v\cdot\nabla_t)\,v\right),\\
\partial_t p_2 + \partial_x \partial_y^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \kappa \Delta_t p_2 + \Lambda_t^{-1}\nabla_t^{\perp}\left((b\cdot\nabla_t)\,v - (v\cdot\nabla_t)\,b\right),\\
p|_{t=0} &= p_{\mathrm{in}}.
\end{aligned} \tag{4}$$

These unknowns are particularly useful since

$$\|Ap_1\|_{L^2} = \|Av\|_{L^2},$$
$$\|Ap_2\|_{L^2} = \|Ab\|_{L^2},$$

for all Fourier multipliers A such that one side is finite. We note that we can recover $v_{\neq}$ and $b_{\neq}$ from $p_{\neq}$ by

$$v_{\neq} = -\Lambda_t^{-1}\nabla_t^{\perp} p_{1,\neq}$$
$$b_{\neq} = -\Lambda_t^{-1}\nabla_t^{\perp} p_{2,\neq}.$$

The operator $\Lambda_t^{-1}\nabla_t^{\perp}$ can be seen as the perpendicular Riesz transform shifted in time on frequency space applied to either a vector or a scalar. It satisfies $(\Lambda_t^{-1}\nabla_t^{\perp}) \circ (\Lambda_t^{-1}\nabla_t^{\perp}) = -Id$ in H^N.

2. Linear stability and norm inflation

In this section, we consider the behavior of the linearized version of (4):

$$\begin{aligned}\partial_t p_1 - \partial_x \partial_y^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1,\\ \partial_t p_2 + \partial_x \partial_y^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \kappa \Delta_t p_2.\end{aligned} \tag{5}$$

For this equation, we establish the following proposition:

Proposition 1 (Linear energy estimate). *Consider $\alpha > \frac{1}{2}$, $N \geqslant 0$ and $0 < \kappa \leqslant \nu$. Let $p_{\mathrm{in}} \in H^N$ with $p_{\mathrm{in},=} = \int p_{\mathrm{in}} \mathrm{d}x = 0$, then the solution p of (5) satisfies the bound*

$$\|p(t)\|_{H^N} \lesssim \mathrm{e}^{-c\kappa^{\frac{1}{3}} t} \left(1 + \nu \kappa^{-\frac{1}{3}}\right) \|p_{\mathrm{in}}\|_{H^N}. \tag{6}$$

Furthermore, we obtain for specific initial data norm inflation of $\nu\kappa^{-\frac{1}{3}}$. The proof uses lemma 3 from [KZ25].

Proposition 2 (Linear norm inflation). *Consider $\alpha > \frac{1}{2}$, $N \geqslant 1$ and $\nu \geqslant \max(2\kappa, \kappa^{\frac{1}{3}})$, then there exist initial data p_{in} and such that at the time $T = \kappa^{-\frac{1}{3}}$ the solution p of (5) satisfies*

$$\|p(T)\|_{H^N} \gtrsim \nu \kappa^{-\frac{1}{3}} \|p_{\mathrm{in}}\|_{H^N}. \tag{7}$$

For the proof of the propositions, we perform a Fourier transform $(x,y) \mapsto (k,\xi)$ and replace p_1 by ip_1. Hence the system (5) can be equivalently (for $k \neq 0$) written as

$$\begin{aligned}\partial_t p_1 &= -\frac{t - \frac{\xi}{k}}{1 + \left(t - \frac{\xi}{k}\right)^2} p_1 - \alpha k p_2 - \nu k^2 \left(1 + \left(t - \frac{\xi}{k}\right)^2\right) p_1,\\ \partial_t p_2 &= \frac{t - \frac{\xi}{k}}{1 + \left(t - \frac{\xi}{k}\right)^2} p_2 + \alpha k p_1 - \kappa k^2 \left(1 + \left(t - \frac{\xi}{k}\right)^2\right) p_2.\end{aligned} \tag{8}$$

Here, with slight abuse of notation, we omit writing the Fourier transformation. This equation has several effects that appear in different regimes of $t - \frac{\xi}{k}$, which we discuss in the following. The effect of circular movement appears for $|t - \frac{\xi}{k}| \lesssim \nu^{-1}$ and the effect of strong viscosity for $\nu^{-1} \lesssim |t - \frac{\xi}{k}| \lesssim \kappa^{-\frac{1}{3}}$. Then before proceeding to the proof of the propositions, we briefly sketch these effects.

2.1. Circular movement

To highlight the effect of the constant magnetic field α in (5) we consider the toy model

$$\begin{aligned}\partial_t p_1 &= -\alpha k p_2,\\ \partial_t p_2 &= \alpha k p_1.\end{aligned} \tag{9}$$

This is solved by

$$p(t) = \begin{pmatrix} \cos(\alpha k t) & -\sin(\alpha k t) \\ \sin(\alpha k t) & \cos(\alpha k t) \end{pmatrix} p_{\mathrm{in}}.$$

We call this effect of the constant magnetic field circular movement, which leads to a transfer between p_1 and p_2. This circular movement is counteracted by viscosity for times away from $\frac{\xi}{k}$.

2.2. Effect of strong viscosity

Let us consider the case when $0<\kappa \ll \nu$ and for simplicity of notation let $k=1$ and $\xi=0$. Due to the viscosity, we obtain $p_1\approx 0$ for large times $t\geqslant t_0\gg 1$. Then from (8) we deduce the toy model

$$\partial_t p_2 = \left(\tfrac{t}{1+t^2} - \kappa\left(1+t^2\right)\right)p_2. \tag{10}$$

The first term in (10) leads to linear growth until the resistivity is strong enough for the second term to take over. This is seen in the explicit solution of (10)

$$p_2(t) = \tfrac{\langle t\rangle}{\langle t_0\rangle}\exp\left(-\kappa\int_{t_0}^{t} 1+\tau^2\,\mathrm{d}\tau\right)p_2(t_0).$$

This is estimated by

$$p_2(t) \lesssim t_0^{-1}\kappa^{-\frac{1}{3}}\mathrm{e}^{-c\kappa^{\frac{1}{3}}(t-t_0)}p_2(t_0),$$

which corresponds to the maximal growth which we obtain. In the following, we will see that $t_0\approx \nu^{-1}$ is the time after which viscosity dominates. The reader may expect that the enhanced dissipation timescale $\nu^{-\frac{1}{3}}$ would be the relevant timescale, but the combination of circular movement and the viscosity gives enough decay for p_2 such that the linear growth gets suppressed until the time ν^{-1}.

2.3. Proof of proposition 1

Proof. For simplicity of notation, we introduce the new variable $s=t-\frac{\xi}{k}$ and initial time $s_{\mathrm{in}}=-\frac{\xi}{k}$. Then equation (8) reads

$$\begin{aligned}
\partial_s p_1 &= -\tfrac{s}{1+s^2}p_1 - \alpha k p_2 - \nu k^2\left(1+s^2\right)p_1,\\
\partial_s p_2 &= \tfrac{s}{1+s^2}p_2 + \alpha k p_1 - \kappa k^2\left(1+s^2\right)p_2.
\end{aligned}$$

Further we change the unknown to $\tilde p = \exp(\frac{\kappa}{2}k^2(s-s_{\mathrm{in}}+\frac{1}{3}(s^3-s_{\mathrm{in}}^3)))p$. For $\tilde\kappa=\frac{\kappa}{2}$ and $\tilde\nu = \nu-\frac{\kappa}{2}$, this yield the equation

$$\begin{aligned}
\partial_s \tilde p_1 &= -\tfrac{s}{1+s^2}\tilde p_1 - \alpha k \tilde p_2 - \tilde\nu k^2\left(1+s^2\right)\tilde p_1,\\
\partial_s \tilde p_2 &= \tfrac{s}{1+s^2}\tilde p_2 + \alpha k \tilde p_1 - \tilde\kappa k^2\left(1+s^2\right)\tilde p_1.
\end{aligned}$$

Let us denote $s_0:=\nu^{-1}$ and in the following, we distinguish between times $|s|\leqslant s_0$ and $|s|\geqslant s_0$. We first consider the case $s_{\mathrm{in}}\leqslant -s_0$. For $|s|\leqslant s_0$, the circular movement is not suppressed by the viscosity.

We define the energy $E=|\tilde p|^2+\frac{1}{\alpha k}\frac{2s}{1+s^2}\tilde p_1\tilde p_2$, then E is a positive quadratic form due to our assumption $\alpha>\frac{1}{2}$ and satisfies

$$\left(1-\tfrac{1}{2\alpha k}\right)|\tilde p|^2 \leqslant E \leqslant \left(1+\tfrac{1}{2\alpha k}\right)|\tilde p|^2.$$

We calculate the time derivative

$$\begin{aligned}&\partial_s E+\tilde{\nu}k^2\left(1+s^2\right)\tilde{p}_1^2+\tilde{\kappa}k^2\left(1+s^2\right)\tilde{p}_2^2\\&\quad=\tfrac{1}{\alpha k}\partial_s\left(\tfrac{2s}{1+s^2}\right)\tilde{p}_1\tilde{p}_2-2s\tfrac{(\tilde{\nu}-\tilde{\kappa})k}{\alpha}\tilde{p}_1\tilde{p}_2\\&\quad\leqslant\tfrac{1}{\alpha k}\partial_s\left(\tfrac{2s}{1+s^2}\right)\tilde{p}_1\tilde{p}_2+\tfrac{1}{2}\tilde{\nu}k^2\left(1+s^2\right)\tilde{p}_1^2+\tfrac{2\tilde{\nu}}{\alpha^2}\tilde{p}_2^2\end{aligned}$$

and so with $|\tilde{p}|^2\leqslant\frac{2\alpha}{2\alpha-1}E$ we infer

$$|\partial_s E|\leqslant\tfrac{\alpha}{\alpha-\frac{1}{2}}\left(\tfrac{1}{1+s^2}+2\tfrac{\tilde{\nu}}{\alpha^2}\right)E.$$

Gronwall's lemma implies

$$E(s_0)\leqslant\exp\left(\tfrac{\alpha}{\alpha-\frac{1}{2}}\left(\pi+2\tfrac{\tilde{\nu}}{\alpha^2}|s_0|\right)\right)E(-s_0).$$

Since $\nu s_0=1$, we deduce

$$E(s_0)\lesssim E(-s_0)$$

and thus

$$|\tilde{p}(s_0)|\lesssim|\tilde{p}(-s_0)|. \tag{11}$$

Consider the case $|s|\geqslant s_0$, we calculate

$$\begin{aligned}\tfrac{1}{2}\partial_s|\tilde{p}|^2\leqslant&\left(-\tfrac{s}{1+s^2}-\tilde{\nu}k^2\left(1+s^2\right)\right)\tilde{p}_1^2\\&+\left(\tfrac{s}{1+s^2}-\tilde{\kappa}k^2\left(1+s^2\right)\right)\tilde{p}_2^2,\end{aligned}$$

and since $(-\frac{s}{1+s^2}-\tilde{\nu}k^2(1+s^2))\leqslant 0$ for all $|s|\geqslant s_0$ we conclude

$$\partial_s|\tilde{p}|^2\leqslant\left(\tfrac{2s}{1+s^2}-\tilde{\kappa}k^2\left(1+s^2\right)\right)_+\tilde{p}_2^2.$$

Thus we obtain the estimate

$$|\tilde{p}(s)|^2\leqslant\begin{cases}|\tilde{p}(s_{\text{in}})|^2 & s\leqslant -s_0,\\ \frac{1+s^2}{1+s_0^2}|\tilde{p}(s_0)|^2 & s_0\leqslant s\leqslant 2\left(\kappa k^2\right)^{-\frac{1}{3}},\\ \left(1+4\nu^2\kappa^{-\frac{2}{3}}k^{-\frac{4}{3}}\right)|\tilde{p}(s_0)|^2 & s_0\vee 2\left(\kappa k^2\right)^{-\frac{1}{3}}\leqslant s.\end{cases}$$

Combining this with (11) we infer

$$|\tilde{p}(s)|\lesssim\left(1+\nu\kappa^{-\frac{1}{3}}k^{-\frac{2}{3}}\right)|p(s_{\text{in}})|.$$

The case $s_{\text{in}}\geqslant -s_0$ is established similarly since we only bound the growth. With

$$\exp\left(-\tfrac{\kappa}{2}k^2\left(s-s_{\text{in}}+\tfrac{1}{3}\left(s^3-s_{\text{in}}^3\right)\right)\right)\lesssim \mathrm{e}^{-c\kappa^{\frac{1}{3}}t}$$

we deduce

$$
\begin{aligned}
|p(s)| &\lesssim e^{-c\kappa^{\frac{1}{3}}t}|\tilde{p}|(s) \\
&\lesssim \left(1+\nu\kappa^{-\frac{1}{3}}k^{-\frac{2}{3}}\right)e^{-c\kappa^{\frac{1}{3}}t}|p|(s_{\text{in}}).
\end{aligned}
$$

Equation (8) decouples in ξ and k, so we infer the proposition with this estimate. □

2.4. *Proof of proposition 2*

Proof. We introduce the notations $\tilde{p}(t,k,\xi) = \exp(\kappa k^2 \int_0^t (1+(\tau-\frac{\xi}{k}k)^2)\mathrm{d}\tau)p(t,k,\xi)$ and $\tilde{\nu} := \nu-\kappa$. Then the equations (8) read

$$
\begin{aligned}
\partial_t\tilde{p}_1 &= -\frac{t-\frac{\xi}{k}k}{1+\left(t-\frac{\xi}{k}k\right)^2}\tilde{p}_1 - \alpha k\tilde{p}_2 - \tilde{\nu}k^2\left(1+\left(t-\frac{\xi}{k}k\right)^2\right)\tilde{p}_1, \\
\partial_t\tilde{p}_2 &= \frac{t-\frac{\xi}{k}k}{1+\left(t-\frac{\xi}{k}k\right)^2}\tilde{p}_2 + \alpha k\tilde{p}_1, \\
\tilde{p}_{\text{in}} &= p_{\text{in}}.
\end{aligned} \tag{12}
$$

We point out that this exactly agrees with the linearized equation with the non-resistive case. Specifically in lemma 3 of [KZ25] it was shown that there exists frequency localized initial data such that for all $t>0$ and frequencies $\xi \in [2\frac{\alpha^2}{\tilde{\nu}}, 4\frac{\alpha^2}{\tilde{\nu}}]$ it holds that

$$
\tilde{p}_2(t,-1,\xi) \geqslant \frac{t}{2\xi}p_{2,\text{in}}(-1,\xi) \gtrsim \tilde{\nu}tp_{2,\text{in}}(-1,\xi). \tag{13}
$$

From this lower bound, we deduce the norm inflation for the non-resistive limit. For times $\tau \in [0,\kappa^{-\frac{1}{3}}]$ and frequencies $\xi \in [2\frac{\alpha^2}{\tilde{\nu}}, 4\frac{\alpha^2}{\tilde{\nu}}]$ we obtain

$$
|\tau+\xi| \lesssim \nu^{-1}+\kappa^{-\frac{1}{3}}.
$$

Then there exist $C = C(\alpha)$, such that

$$
1 \geqslant \exp\left(-\kappa\int_0^{\mathrm{T}}\left(1+(t+\xi)^2\right)\mathrm{d}t\right) \gtrsim \exp\left(-C\kappa\left(\nu^{-1}+\kappa^{-\frac{1}{3}}\right)^2\kappa^{-\frac{1}{3}}\right) \geqslant \exp(-2C) \gtrsim 1,
$$

where we used in the last estimate, that $\kappa \leqslant \nu^3$ yields $\kappa(\nu^{-1}+\kappa^{-\frac{1}{3}})^2\kappa^{-\frac{1}{3}} \leqslant 2$. From (13) and $\tilde{\nu} = \nu-\kappa \geqslant \frac{1}{2}\nu$, since $\nu \geqslant 2\kappa$, we deduce, that

$$
p_2(T,-1,\xi) = \exp\left(-\kappa\int_0^{\mathrm{T}}\left(1+(t+\xi)^2\right)\mathrm{d}t\right)p_2(T,-1,\xi) \gtrsim \nu\kappa^{-\frac{1}{3}}p_{2,\text{in}}(-1,\xi).
$$

From this, we infer the norm inflation

$$
\|p(T)\|_{H^N} \gtrsim \nu\kappa^{-\frac{1}{3}}\|p_{\text{in}}\|_{H^N}.
$$

□

3. Sobolev stability for the nonlinear system

The following theorem is a more general statement of theorem 1.1. We dedicate the remainder of the section to the proof.

Theorem 3.1. *Let $\alpha > \frac{1}{2}$ and $N \geqslant 5$, then there exist $c_0, c > 0$, such that for all $0 < \kappa \leqslant \nu \leqslant \frac{1}{40}(1 - \frac{1}{2\alpha})^{\frac{6}{5}}$ there exist $L = \max(1, \nu\kappa^{-\frac{1}{3}})$, such that for all initial data, which satisfy*

$$\begin{aligned} \|(v,b)_{\text{in},\neq}\|_{H^N} &= \varepsilon \leqslant c_0 L^{-1} \nu^{\frac{1}{12}} \kappa^{\frac{1}{2}}, \\ \|(v,b)_{\text{in},=}\|_{H^N} &\leqslant \tilde{\varepsilon}, \qquad \text{with } \varepsilon \leqslant \tilde{\varepsilon} \leqslant \nu^{-\frac{1}{12}} \varepsilon \end{aligned} \tag{14}$$

the corresponding solution of (2) satisfies the bound

$$\begin{aligned} \|(v,b)_{\neq}(t)\|_{L^\infty H^N} + \|\nabla_t (\sqrt{\nu} v, \sqrt{\kappa} b)_{\neq}\|_{L^2 H^N} &\lesssim L \mathrm{e}^{-c\kappa^{\frac{1}{3}} t} \varepsilon, \\ \|(v,b)_{=}(t)\|_{L^\infty H^N} + \|\partial_y (\sqrt{\nu} v, \sqrt{\kappa} b)_{=}\|_{L^2 H^N} &\lesssim \tilde{\varepsilon}. \end{aligned} \tag{15}$$

Furthermore, we obtain the following enhanced dissipation estimates

$$\begin{aligned} \|v_{\neq}\|_{L^2 H^N} &\lesssim L \nu^{-\frac{1}{6}} \mathrm{e}^{-c\kappa^{\frac{1}{3}} t} \varepsilon, \\ \|b_{\neq}\|_{L^2 H^N} &\lesssim L \kappa^{-\frac{1}{6}} \mathrm{e}^{-c\kappa^{\frac{1}{3}} t} \varepsilon. \end{aligned}$$

This theorem implies theorem 1.1. With slight abuse of notation, we write L as the ν and κ dependent part of the Lipschitz constant. We prove this theorem by using a bootstrap method. Let A be the Fourier weight

$$A := M |\nabla|^N \exp\left(c\kappa^{\frac{1}{3}} t \mathbf{1}_{\neq}\right),$$

where $M = M_L M_1 M_\kappa M_\nu M_{\nu^3}$ are defined as

$$\begin{aligned} \frac{-\dot{M}_L}{M_L} &= \frac{t - \frac{\xi}{k}}{1 + \left(\frac{\xi}{k} - t\right)^2} \mathbf{1}_{\left\{\nu^{-1} \leqslant t - \frac{\xi}{k} \leqslant (c_1 \kappa k^2)^{-\frac{1}{3}}\right\}} && k \neq 0, \\ \frac{-\dot{M}_1}{M_1} &= C_\alpha \frac{|k| + \nu^{\frac{1}{12}} |k|^2}{k^2 + (\xi - kt)^2} && k \neq 0, \\ \frac{-\dot{M}_\nu}{M_\nu} &= \frac{\nu^{\frac{1}{3}}}{1 + \nu^{\frac{2}{3}} \left(t - \frac{\xi}{k}\right)^2} && k \neq 0, \\ \frac{-\dot{M}_\kappa}{M_\kappa} &= \frac{\kappa^{\frac{1}{3}}}{1 + \kappa^{\frac{2}{3}} \left(t - \frac{\xi}{k}\right)^2} && k \neq 0, \\ \frac{-\dot{M}_{\nu^3}}{M_{\nu^3}} &= \frac{C_\alpha \nu}{1 + \nu^2 \left(t - \frac{\xi}{k}\right)^2} && k \neq 0, \\ M_\cdot(t=0) &= M_\cdot(k=0) = 1. \end{aligned}$$

The weight M_L is an adaption of the weight $m^{\frac{1}{2}}$ in [Lis20] to our setting. The weight M_1 corresponds to the inviscid damping part of the weight M of [BMV16] to the power of $1 + |k|\nu^{\frac{1}{12}}$. We use the $|k|$ to bound terms involving $\partial_x \Lambda_t^{-1}$ and the $\nu^{\frac{1}{12}}$ to obtain the commutator estimates. The method of using time-dependent Fourier weights is common when working with solutions around Couette flow and the other weights are modifications of previously used weights (see [BVW18, MZ22, Lis20, ZZ24]). For simplicity, here we only state their main properties and refer to appendix A for a detailed description. The constants $C_\alpha = \frac{2}{\min(1,\alpha-\frac{1}{2})}$, $c = \frac{1}{200}(1-\frac{1}{2\alpha})^2$ and $c_1 = \frac{1}{20}(1-\frac{1}{2\alpha})$ are determined through the linear estimates. For the weights we obtain

$$\begin{aligned} L^{-1} \leqslant \min\left(1, \nu^{-1}\kappa^{\frac{1}{3}}k^{\frac{2}{3}}\right) \lesssim M_L \leqslant 1, \\ M_1 \approx M_\kappa \approx M_\nu \approx M_{\nu^3} \approx 1. \end{aligned} \tag{16}$$

We note that the weight M_L is distinct from the others due to its lower bound L^{-1}, which depends on ν and κ. The weight M_L is necessary to bound the linear growth in the region $\nu^{-1} \lesssim t - \frac{\xi}{k} \lesssim (\kappa k^2)^{-\frac{1}{3}}$. Controlling the effects of M_L is one of the main challenges in the proof of theorem 3.1. We recall the unknowns p and equation (4)

$$\begin{aligned} \partial_t p_1 - \partial_x \partial_y^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1 + \Lambda_t^{-1} \nabla_t^\perp \left((b\cdot\nabla_t) b - (v\cdot\nabla_t) v \right), \\ \partial_t p_2 + \partial_x \partial_y^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \kappa \Delta_t p_2 + \Lambda_t^{-1} \nabla_t^\perp \left((b\cdot\nabla_t) v - (v\cdot\nabla_t) b \right), \\ p|_{t=0} &= p_{\text{in}}. \end{aligned} \tag{17}$$

Let $\chi \in C^\infty(\mathbb{R}_+ \times \mathbb{Z} \times \mathbb{R})$ be a Fourier multiplier defined by

$$\chi = \chi(k,\xi) = \begin{cases} 1 & |t - \frac{\xi}{k}| \leqslant \nu^{-1} \\ 0 & |t - \frac{\xi}{k}| \geqslant 2\nu^{-1} \end{cases} \tag{18}$$

with a smooth extension on $\nu^{-1} \leqslant |t - \frac{\xi}{k}| \leqslant 2\nu^{-1}$ satisfying

$$\partial_t \chi \leqslant 2\nu. \tag{19}$$

We define the main energy

$$E := \|Ap_{\neq}\|_{L^2}^2 + \tfrac{2}{\alpha} \Re \langle \partial_y^t \Delta_t^{-1} \chi A p_{1,\neq}, A p_{2,\neq} \rangle.$$

Here $\Re$ denotes the real part, in the following, we omit writing the symbol $\Re$ since we derive an upper bound. As $\alpha > \frac{1}{2}$, this energy is positive definite and satisfies

$$\left(1 - \tfrac{1}{2\alpha}\right) \|Ap\|_{L^2} \leqslant E \leqslant \left(1 + \tfrac{1}{2\alpha}\right) \|Ap\|_{L^2}. \tag{20}$$

In the following, we assume initial data as in theorem 3.1, i.e. (14). We use a bootstrap approach to prove the following two estimates globally in time:

The **energy estimate without *x*-average**

$$\begin{aligned} &\|E\|_{L^\infty} + \|A\nabla_t \otimes \left(\sqrt{\nu} p_{1,\neq}, \sqrt{\kappa} p_{2,\neq}\right)\|_{L^2L^2}^2 \\ &\quad + \sum_{j=1,\nu,\kappa,\nu^3} \|\sqrt{\tfrac{-\dot{M}_j}{M_j}} A p_{\neq}\|_{L^2L^2}^2 \leqslant (C\varepsilon)^2. \end{aligned} \tag{21}$$

The **energy estimate with *x*-average**

$$\|p_=\|_{L^\infty H^N}^2 + \|\partial_y \left(\sqrt{\nu} p_{1,=}, \sqrt{\kappa} p_{2,=}\right)\|_{L^2 H^N}^2 \leqslant (C\tilde{\varepsilon})^2. \tag{22}$$

We then prove that the equality in the estimates is not attained at time T. By local wellposedness, the estimates thus remain valid at least for a short additional time. This contradicts the maximality and thus T has to be infinite. We note that we suppress in our notation the T in the estimates (see (3)). With $1 \leqslant \kappa^{-\frac{1}{3}}(\frac{-\dot{M}_\kappa}{M_\kappa} + \kappa k^2(1+(t-\frac{\xi}{k})^2)$ and $1 \leqslant \nu^{-\frac{1}{3}}(\frac{-\dot{M}_\nu}{M_\nu} + \nu k^2(1+(t-\frac{\xi}{k})^2))$ we infer from (20) and (21) the enhanced dissipation estimates

$$\|Ap_{1,\neq}\|_{L^2L^2} \leqslant 2\left(1-\tfrac{1}{2\alpha}\right)^{-1}\nu^{-\frac{1}{6}}C\varepsilon, \tag{23}$$

$$\|Ap_{2,\neq}\|_{L^2L^2} \leqslant 2\left(1-\tfrac{1}{2\alpha}\right)^{-1}\kappa^{-\frac{1}{6}}C\varepsilon. \tag{24}$$

By the construction of M_1 we obtain

$$\|\partial_x\Lambda_t^{-1}Ap\|_{L^2} \lesssim \nu^{-\frac{1}{12}}\varepsilon.$$

We obtain the energy estimate by deriving the energy E

$$\begin{aligned}&\partial_t E + 2\|A\nabla_t \otimes (\nu p_{1,\neq}, \kappa p_{2,\neq})\|_{L^2}^2 + 2\|\sqrt{\tfrac{-\dot{M}}{M}}Ap_{\neq}\|_{L^2}^2\\ &=L_1 + L_{NR} + L_R + NL_{\neq} + ONL,\end{aligned} \tag{25}$$

where we define

$$\begin{aligned}
L_1 &= 2c\kappa^{\frac{1}{3}}\|Ap_{\neq}\|_{L^2}^2\\
L_{NR} &= -2\Re\langle A(1-\chi)p_{1,\neq}, \partial_x\partial_x^t\Delta_t^{-1}Ap_{1,\neq}\rangle\\
&\quad + 2\Re\langle A(1-\chi)p_{2,\neq}, \partial_x\partial_x^t\Delta_t^{-1}Ap_{2,\neq}\rangle\\
L_R &= \tfrac{4}{\alpha}\Re\langle\chi\,\partial_y^t\Delta_t^{-1}Ap_{1,\neq}, \dot{A}p_{2,\neq}\rangle\\
&\quad + \tfrac{2}{\alpha}\Re\langle\chi\,\partial_t\left(\partial_y^t\Delta_t^{-1}\right)Ap_{1,\neq}, Ap_{2,\neq}\rangle\\
&\quad + \tfrac{2}{\alpha}\Re\langle\partial_t(\chi)\,\partial_y^t\Delta_t^{-1}Ap_{1,\neq}, Ap_{2,\neq}\rangle\\
&\quad + \tfrac{2|\nu+\kappa|}{\alpha}\Re\langle\chi\,\partial_y^t Ap_{1,\neq}, Ap_{2,\neq}\rangle\\
NL_{\neq} &= 2\langle Av_{\neq}, A((b\cdot\nabla_t)b - (v\cdot\nabla_t)v)\rangle\\
&\quad + 2\langle Ab_{\neq}, A((b\cdot\nabla_t)v - (v\cdot\nabla_t)b)\rangle\\
ONL &= \tfrac{2}{\alpha}\Re\langle\chi A\partial_y^t\Delta_t^{-1}b_{\neq}, A((b\cdot\nabla_t)b - (v\cdot\nabla_t)v)\rangle\\
&\quad + \tfrac{2}{\alpha}\Re\langle\chi A\partial_y^t\Delta_t^{-1}v_{\neq}, A((b\cdot\nabla_t)v - (v\cdot\nabla_t)b)\rangle.
\end{aligned}$$

Furthermore, for the energy of x-averages, we obtain

$$\begin{aligned}
&\partial_t\|p_=\|_{H^N} + \|\partial_y(\nu p_{1,=}, \kappa p_{2,=})\|_{H^N}\\
&\quad\leqslant \langle\langle\partial_y\rangle^N v_{1,=}, \langle\partial_y\rangle^N((b\cdot\nabla_t)b - (v\cdot\nabla_t)v)_=\rangle\\
&\qquad + \langle\langle\partial_y\rangle^N b_{1,=}, \langle\partial_y\rangle^N((b\cdot\nabla_t)v - (b\cdot\nabla_t)v)_=\rangle\\
&\quad= NL_=.
\end{aligned} \tag{26}$$

In the following subsections, we establish the following proposition:

Proposition 3.1 (Control of errors). *Under the assumptions of theorem 3.1, there exists a constant $C = C(\alpha) > 0$ such that if (21) and (22) are satisfied for $T > 0$, then the following estimate holds*

$$\begin{aligned}\int_0^T L_1 + L_R + L_{NR}\,\mathrm{d}t &\leqslant \tfrac{17+\frac{3}{2}\alpha}{10}(C\varepsilon)^2 + 2\Big\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_2\Big\|^2_{L^2L^2},\\ \int_0^T NL_{\neq} + ONL\,\mathrm{d}t &\lesssim L\nu^{-\frac{1}{12}}\kappa^{-\frac{1}{2}}\varepsilon^3 + \left(L\kappa^{-\frac{1}{3}} + \kappa^{-\frac{1}{2}}\right)\tilde{\varepsilon}\varepsilon^2,\\ \int_0^T NL_{=}\,\mathrm{d}t &\lesssim L\nu^{-\frac{1}{12}}\kappa^{-\frac{1}{2}}\tilde{\varepsilon}\varepsilon^2.\end{aligned} \tag{27}$$

With this proposition we deduce theorem 3.1:

Proof of theorem 3.1. By a standard application of the Banach fixed-point theorem we obtain local well-posedness, see appendix B. Thus for all initial data, there exists a time interval $[0,T]$ such that (21) and (22) hold. Let T^* be the maximal time such that (21) and (22) hold. Let c_0 be a given, small constant and suppose for the sake of contradiction that $T^* < \infty$. With the estimates (25)–(27) and since c_0 is small we obtain that the estimates (21) and (22) do not attain equality. Thus by local existence, T^* is not the maximal time and thus we obtain a contradiction. Therefore, for small enough c_0, (21) and (22) hold global in time and so we infer theorem 3.1. □

The remainder of the section is dedicated to the proof of proposition 3.1. We rearrange and use partial integration to infer that

$$\begin{aligned}&\langle Av_{\neq}, (b\cdot\nabla_t)Ab_{\neq} - (v\cdot\nabla_t)Av_{\neq}\rangle + \langle Ab_{\neq}, (b\cdot\nabla_t)Av_{\neq} - (v\cdot\nabla_t)Ab_{\neq}\rangle\\ &\quad= \langle b, \nabla_t(Av_{\neq}Ab_{\neq})\rangle - \tfrac{1}{2}\langle v, \nabla_t(Av_{\neq}Av_{\neq}) + \nabla_t(Ab_{\neq}Ab_{\neq})\rangle\\ &\quad= 0.\end{aligned} \tag{28}$$

The NL term consists of trilinear products with the unknowns

$$a^1a^2a^3 \in \{vvv, vbb, bbv, bvb\}. \tag{29}$$

Thus, we denote the nonlinear terms

$$\begin{aligned}NL_{\neq}\left[a^1a^2a^3\right] &= \langle Aa^1_{\neq}, A\left(\left(a^2_{\neq}\cdot\nabla_t\right)a^3_{\neq}\right)_{\neq} - \left(a^2_{\neq}\cdot\nabla_t\right)Aa^3_{\neq}\rangle,\\ &\quad+ \langle Aa^1_{\neq}, A\left(\left(a^2_{=}\cdot\nabla_t\right)a^3_{\neq}\right) - \left(a^2_{=}\cdot\nabla_t\right)a^3_{\neq}\rangle,\\ &\quad+ \langle Aa^1_{\neq}, A\left(\left(a^2_{\neq}\cdot\nabla_t\right)a^3_{=}\right)\rangle,\\ NL_{=}\left[a^1a^2a^3\right] &= \langle\langle\partial_y\rangle^N a^1_{=}, \langle\partial_y\rangle^N\left(\left(a^2_{\neq}\cdot\nabla_t\right)a^3_{\neq}\right)_{=}\rangle.\end{aligned}$$

If we do not use specific choices for $a^1a^2a^3$ we write just NL. Similarly, we use $a^1a^2a^3 \in \{bvv, bbb, vbv, vvb\}$ for ONL. Furthermore, we always use i such that $p_i = \Lambda_t^{-1}\nabla_t^{\perp}a^2$ in the sense that $i=1$ if $a^2 = v$ and $i=2$ if $a^2 = b$. We perform the energy estimates in the next subsections:

- In section 3.1 we estimate the linear error terms. In this subsection, the split with χ into resonant and non-resonant regions depending on ν is vital.
- In section 3.2 we conclude the energy estimate for the nonlinear term without x average. Here it is necessary to perform a low and high frequency decomposition. This gives us a reaction and a transport term. In particular, for $\kappa\downarrow 0$ bounding the transport term is very challenging due to the linear growth.
- In sections 3.3–3.5 we estimate nonlinear terms with an x-average component.

- In section 3.6 we estimate nonlinear term which arise due χ in the resonant regions. For these terms, we obtain an additional Λ_t^{-1}, which has a stabilizing effect. This stabilizing effect is necessary due to a nonlinear term consisting of only magnetic components.

3.1. Linear estimates

In this section, we establish estimates of the linear errors L_1, L_R and L_{NR} of (25). In order to estimate L_1, we use (23) and (24) to deduce

$$\int L_1 \mathrm{d}\tau = 2c\kappa^{\frac{1}{3}} \|Ap_{\neq}\|^2_{L^2L^2} \leqslant 8\left(1-\tfrac{1}{2\alpha}\right)^{-1} c\,(C\varepsilon)^2.$$

For the L_{NR} terms in (25), we infer

$$\begin{aligned}\Re\langle A(1-\chi)p_{1,\neq}, A\partial_x\partial_x^t\Delta_t^{-1}p_{1,\neq}\rangle &= \Re\sum_{k\neq 0}\int \mathrm{d}\xi\,(1-\chi)\,\frac{t-\frac{\xi}{k}}{1+\left(t-\frac{\xi}{k}\right)^2}|Ap_1|^2\\ &\leqslant \nu^3\|A\nabla_t p_{1,\neq}\|^2_{L^2},\end{aligned}$$

since $(1-\chi)\frac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} \leqslant (1-\chi)(\nu)^3(1+(t-\frac{\xi}{k})^2)$ due to $\chi=1$ for $|t-\frac{\xi}{k}|\leqslant \nu^{-1}$. Furthermore, using (34) we estimate

$$\begin{aligned}&\Re\langle A(1-\chi)p_{2,\neq}, A\partial_x\partial_x^t\Delta_t^{-1}p_2\rangle\\ &\quad= \Re\sum_{k\neq 0}\int \mathrm{d}\xi\,(1-\chi)\,\frac{t-\frac{\xi}{k}}{1+\left(t-\frac{\xi}{k}\right)^2}|Ap_2|^2\\ &\quad\leqslant \Re\sum_{k\neq 0}\int \mathrm{d}\xi\,(1-\chi)\left(\frac{-\dot{M}_L}{M_L}+\kappa c_1\left(1+\left(t-\frac{\xi}{k}\right)^2\right)\right)|Ap_2|^2\\ &\quad\leqslant \|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_{2,\neq}\|^2_{L^2}+\kappa c_1\|\nabla_t Ap_{2,\neq}\|^2_{L^2}.\end{aligned}$$

Thus with (21) we deduce

$$\int L_{NR}\,\mathrm{d}\tau \leqslant \left(2\nu^2+2c_1\right)(C\varepsilon)^2 + 2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_{2,\neq}\|^2_{L^2L^2}.$$

For L_R, we estimate in frequency space

$$\begin{aligned}|\mathcal{F}\left(\mathbf{1}_{\neq}\partial_y^t\Delta_t^{-1}\right)| &= |\left(\frac{\xi-kt}{k^2+(\xi-kt)^2}\right)_{k\neq 0}| \leqslant \tfrac{1}{2},\\ |\mathcal{F}\left(\left(\tfrac{-\dot{M}_L}{M_L}\right)\partial_y^t\Delta_t^{-1}\right)| &\leqslant \left|\left(\frac{\left(t-\frac{\xi}{k}\right)^2}{k\left(1+\left(t-\frac{\xi}{k}\right)^2\right)^2}\right)_{k\neq 0}\right| \leqslant C_\alpha^{-1}\frac{-\dot{M}_1}{M_1},\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. So it follows that

$$\begin{aligned}&\tfrac{4}{\alpha}\Re\langle\chi\,\partial_y^t\Delta_t^{-1}Ap_{1,\neq}, \dot{A}p_{2,\neq}\rangle\\ &\quad= \tfrac{4}{\alpha}\Re\langle\chi Ap_{1,\neq}, \left(c\kappa^{\frac{1}{3}}+\frac{\dot{M}_1}{M_1}+\frac{\dot{M}_L}{M_L}+\frac{\dot{M}_\kappa}{M_\kappa}+\frac{\dot{M}_\nu}{M_\nu}+\frac{\dot{M}_{\nu^3}}{M_{\nu^3}}\right)\partial_y^t\Delta_t^{-1}Ap_{2,\neq}\rangle\end{aligned}$$

$$\leqslant \tfrac{2c}{\alpha}\kappa^{\frac{1}{3}}\|Ap_{\neq}\|_{L^2}^2 + \left(1 + C_\alpha^{-1}\right)\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{\neq}\|_{L^2}^2$$
$$+ \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\kappa}{M_\kappa}}Ap_{\neq}\|_{L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\nu}{M_\nu}}Ap_{\neq}\|_{L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_{\nu^3}}{M_{\nu^3}}}Ap_{\neq}\|_{L^2}^2.$$

We use the estimate in frequency space

$$|\mathcal{F}\left(\left(\left(\partial_x^2 - \left(\partial_y^t\right)^2\right)\Delta_t^{-2}\right)_{\neq}\right)| = \left|\left(\frac{1-\left(t-\frac{\xi}{k}k\right)^2}{k^2\left(1+\left(t-\frac{\xi}{k}k\right)^2\right)^2}\right)_{k\neq 0}\right| \leqslant C_\alpha^{-1}\tfrac{-\dot{M}_1}{M_1},$$

to infer that

$$\tfrac{1}{\alpha}\Re\langle\chi Ap_{1,\neq}, A\partial_x^{-1}\left(\partial_x^2 - \left(\partial_y^t\right)^2\right)\Delta_t^{-2}p_{2,\neq}\rangle \leqslant C_\alpha^{-1}\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{1,\neq}\|_{L^2}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{2,\neq}\|_{L^2}$$
$$\leqslant C_\alpha^{-1}\tfrac{1}{2\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{\neq}\|_{L^2}^2.$$

With (19) we deduce

$$\Re\langle\partial_y^t\Delta_t^{-1}\partial_t(\chi)Ap_{1,\neq}, Ap_{2,\neq}\rangle \leqslant \nu\|Ap_{1,\neq}\|_{L^2}\|A\Lambda_t^{-1}p_{2,\neq}\|_{L^2}.$$

By the Fourier support of χ (see (18)) and the definition of M_{ν^3} we obtain $\chi \leqslant 2C_\alpha^{-1}\nu^{-1}\frac{-\dot{M}_{\nu^3}}{M_{\nu^3}}\chi$, which yields

$$\tfrac{|\nu+\kappa|}{\alpha}\Re\langle\chi A\partial_y^t p_{1,\neq}, Ap_{2,\neq}\rangle \leqslant 2C_\alpha^{-1}\tfrac{\nu^{\frac{1}{2}}}{\alpha}\|A\partial_y^t p_{1,\neq}\|_{L^2}\|\sqrt{\tfrac{-M_{\nu^3}}{M_{\nu^3}}}Ap_{2,\neq}\|_{L^2}.$$

Thus for the linear error L_R we infer

$$\int L_R \mathrm{d}\tau \leqslant \left(\tfrac{2c}{\alpha} + \nu^{\frac{5}{6}}\right)(C\varepsilon)^2$$
$$+ \tfrac{1+C_\alpha^{-1}}{\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}p_{\neq}\|_{L^2L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\kappa}{M_\kappa}}p_{\neq}\|_{L^2L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\nu}{M_\nu}}p_{\neq}\|_{L^2L^2}^2$$
$$+ \tfrac{1+C_\alpha^{-1}}{2\alpha}\|\sqrt{\tfrac{-\dot{M}_{\nu^3}}{M_{\nu^3}}}Ap_{2,\neq}\|_{L^2L^2} + C_\alpha^{-1}\tfrac{1}{2\alpha}\nu\|A\partial_y^t p_{1,\neq}\|_{L^2L^2}^2.$$

Combining the estimates for all linear terms, we obtain

$$\int L + L_R + L_{NR}\,\mathrm{d}\tau$$
$$\leqslant \left(\left(8 + \tfrac{2}{\alpha}\right)\left(1 - \tfrac{1}{2\alpha}\right)^{-1}c + 2\nu^2 + 2c_1 + \nu^{\frac{5}{6}}\right)(C\varepsilon)^2$$
$$+ 2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_2\|_{L^2L^2}^2$$
$$+ \left(1 + \tfrac{3}{2}C_\alpha^{-1}\right)\tfrac{1}{\alpha}\|\chi\sqrt{\tfrac{-\dot{M}_1}{M_1}}p_{\neq}\|_{L^2L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\kappa}{M_\kappa}}p_{\neq}\|_{L^2L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\nu}{M_\nu}}p_{\neq}\|_{L^2L^2}^2$$
$$+ \tfrac{1+C_\alpha^{-1}}{2\alpha}\|\sqrt{\tfrac{-M_{\nu^3}}{M_{\nu^3}}}Ap_{2,\neq}\|_{L^2L^2} + C_\alpha^{-1}\tfrac{1}{2\alpha}\nu\|A\partial_y^t p_{1,\neq}\|_{L^2L^2}^2$$
$$\leqslant \left(12\left(1 - \tfrac{1}{2\alpha}\right)^{-1}c + 2\nu^2 + 2c_1 + \nu^{\frac{5}{6}} + \tfrac{1+2C_\alpha^{-1}}{2\alpha}\right)(C\varepsilon)^2$$
$$+ 2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_{2,\neq}\|_{L^2L^2}^2.$$

Since $\alpha > \frac{1}{2}$ we deduce $\frac{1+2C_\alpha^{-1}}{\alpha} < 1 + \frac{1}{2\alpha}$. Choosing the constants such that

$$\begin{aligned} c &= \tfrac{1}{200}\left(1 - \tfrac{1}{2\alpha}\right)^2, \\ c_1 &= \tfrac{1}{20}\left(1 - \tfrac{1}{2\alpha}\right), \end{aligned}$$

and recalling that

$$\nu \leqslant \tfrac{1}{40}\left(1 - \tfrac{1}{2\alpha}\right)^{\frac{6}{5}},$$

we conclude that $(12(1-\frac{1}{2\alpha})^{-1}c + 2\nu^2 + 2c_1 + \nu^{\frac{5}{6}} + \frac{1+2C_\alpha^{-1}}{2\alpha} < \frac{17+\frac{3}{2\alpha}}{20}$. Thus we obtain the estimate

$$\int L_1 + L_R + L_{NR} \mathrm{d}\tau \leqslant \tfrac{17+\frac{3}{2\alpha}}{20}\left(C\varepsilon\right)^2 + 2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}} A p_{2,\neq}\|^2_{L^2L^2}.$$

This yields the first estimate of proposition 3.1.

3.2. Nonlinear terms without an x-average

We apply the notation of (29) and aim to estimate terms of the form

$$\begin{aligned} &\langle A a^1_{\neq}, A\left(\left(a^2_{\neq}\cdot\nabla_t\right)a^3_{\neq}\right) - \left(a^2_{\neq}\cdot\nabla_t\right)A a^3_{\neq}\rangle \\ &\quad\leqslant \sum_{k,l,k-l\neq 0}\iint \mathrm{d}\left(\xi,\eta\right)\tfrac{|A(k,\xi)-A(l,\eta)|}{A(k-l,\xi-\eta)A(l,\eta)}\tfrac{|\xi l - k\eta|}{\left((k-l)^2+(\xi-\eta-(k-l)t)^2\right)^{\frac{1}{2}}} \\ &\qquad\qquad |Aa^1|\left(k,\xi\right)|Ap_i|\left(k-l,\xi-\eta\right)|Aa^3|\left(l,\eta\right) \\ &\quad= T + R. \end{aligned}$$

Here, we split the integral into the *reaction R* and the *transport T* terms which correspond to the sets

$$\begin{aligned} \Omega_R &= \left\{|k-l,\xi-\eta| \geqslant \tfrac{1}{8}|l,\eta|\right\}, \\ \Omega_T &= \left\{|k-l,\xi-\eta| < \tfrac{1}{8}|l,\eta|\right\}, \end{aligned}$$

in Fourier space. We split the weights

$$\begin{aligned} A\left(k,\xi\right) - A\left(l,\eta\right) = {} & \mathrm{e}^{ct\kappa^{\frac{1}{3}}}\left(M_L\left(k,\xi\right) - M_L\left(l,\eta\right)\right)M_1\left(k,\xi\right)M_\kappa\left(k,\xi\right)M_\nu\left(k,\xi\right)M_{\nu^3}\left(k,\xi\right)|k,\xi|^N \\ & + \mathrm{e}^{ct\kappa^{\frac{1}{3}}}\left(|k,\xi|^N - |l,\eta|^N\right)M_L\left(l,\eta\right)M_1\left(k,\xi\right)M_\kappa\left(k,\xi\right)M_\nu\left(k,\xi\right)M_{\nu^3}\left(k,\xi\right) \\ & + \mathrm{e}^{ct\kappa^{\frac{1}{3}}}\left(M_1\left(k,\xi\right) - M_1\left(l,\eta\right)\right)M_L\left(l,\eta\right)M_\kappa\left(k,\xi\right)M_\nu\left(k,\xi\right)M_{\nu^3}\left(k,\xi\right)|l,\eta|^N \\ & + \mathrm{e}^{ct\kappa^{\frac{1}{3}}}\left(M_\kappa\left(k,\xi\right) - M_\kappa(l,\eta)\right)M_1(l,\eta)M_L(l,\eta)M_\nu(k,\xi)M_{\nu^3}(k,\xi)|l,\eta|^N \\ & + \mathrm{e}^{ct\kappa^{\frac{1}{3}}}\left(M_\nu(k,\xi) - M_\nu(l,\eta)\right)M_1(l,\eta)M_L(l,\eta)M_\kappa(l,\eta)M_{\nu^3}(k,\xi)|l,\eta|^N \\ & + \mathrm{e}^{ct\kappa^{\frac{1}{3}}}\left(M_{\nu^3}(k,\xi) - M_{\nu^3}(l,\eta)\right)M_1(l,\eta)M_L(l,\eta)M_\kappa(l,\eta)M_\nu(l,\eta)|l,\eta|^N \end{aligned}$$

and thus by (16) we estimate

$$\frac{|A(k,\xi)-A(l,\eta)|}{A(k-l,\xi-\eta)A(l,\eta)} \lesssim \mathrm{e}^{-ct\kappa^{\frac{1}{3}}} \frac{|M_L(k,\xi)-M_L(l,\eta)|}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \frac{|\xi,\eta|^N}{|l,\eta|^N|k-l,\xi-\eta|^N}$$
$$+ \mathrm{e}^{-ct\kappa^{\frac{1}{3}}} \frac{\left||k,\xi|^N-|l,\eta|^N\right|}{|l,\eta|^N|k-l,\xi-\eta|^N} \frac{1}{M_L(k-l,\xi-\eta)}$$
$$+ \mathrm{e}^{-ct\kappa^{\frac{1}{3}}} \sum_{j=1,\kappa,\nu,\nu^3} |M_j(k,\xi)-M_j(l,\eta)| \frac{1}{|k-l,\xi-\eta|^N} \frac{1}{M_L(k-l,\xi-\eta)}. \tag{30}$$

Reaction term: On the set Ω_R it holds that $|k-l,\xi-\eta| \geqslant \frac{1}{8}|l,\eta|$, thus $|k,\xi|,|l,\eta| \lesssim |k-l,\xi-\eta|$. From (16) and (30) and $\frac{|k,\xi|^N}{|l,\eta|^N|k-l,\xi-\eta|^N}, \frac{||k,\xi|^N-|l,\eta|^N|}{|l,\eta|^N|k-l,\xi-\eta|^N} \lesssim \frac{1}{|l,\eta|^N}$ we infer

$$\frac{|A(k,\xi)-A(l,\eta)|}{A(k-l,\xi-\eta)A(l,\eta)} \lesssim \frac{1}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \frac{1}{|l,\eta|^N}.$$

With $\xi l - k\eta = (\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)$ and Hölder's inequality we deduce

$$R = \mathrm{e}^{-ct\kappa^{\frac{1}{3}}} \sum_{k,l,k-l\neq 0} \int \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_R} \frac{|A(k,\xi)-A(l,\eta)|}{A(k-l,\xi-\eta)A(l,\eta)} \frac{|\xi l-k\eta|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}}$$
$$|Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)$$
$$\lesssim \mathrm{e}^{-ct\kappa^{\frac{1}{3}}} \sum_{k,l,k-l\neq 0} \int \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_R} \frac{1}{|l,\eta|^N} \frac{1}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \frac{|(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}}$$
$$|Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)|Aa^3|(l,\eta)$$
$$\lesssim \|Aa^1_{\neq}\|_{L^2} \|\tfrac{1}{M_L}Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2}$$
$$+ \|Aa^1_{\neq}\|_{L^2} \|\partial_x \tfrac{1}{M_L}\Lambda_t^{-1}Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L}A\partial_y^t a^3_{\neq}\|_{L^2}.$$

We use (36) and (37) to infer

$$R \lesssim \|Aa^1_{\neq}\|_{L^2} \left(\|Ap_{i,\neq}\|_{L^2} + \nu\|\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right)Ap_{i,\neq}\|_{L^2}\right) \left(\|Aa^3_{\neq}\|_{L^2} + \nu\|A\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right)a^3_{\neq}\|_{L^2}\right)$$
$$+ L\|Aa^1_{\neq}\|_{L^2} \left(\|A\partial_x\Lambda_t^{-1}p_{i,\neq}\|_{L^2} + \nu\|Ap_{i,\neq}\|_{L^2}\right) \|\partial_y^t Aa^3_{\neq}\|_{L^2}.$$

By (29), $a^j = v$ for at least one j, thus

$$\|Aa^1_{\neq}\|_{L^2} \left(\|Ap_{i,\neq}\|_{L^2} + \nu\|\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right)Ap_{i,\neq}\|_{L^2}\right) \left(\|Aa^3_{\neq}\|_{L^2} + \nu\|A\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right)a^3_{\neq}\|_{L^2}\right)$$
$$\leqslant \left(1+\nu^2\kappa^{-\frac{2}{3}}\right) \|Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|Aa^3_{\neq}\|_{L^2}$$
$$\leqslant L\left(1+\nu\kappa^{-\frac{1}{3}}\right) \|Av_{\neq}\|_{L^2} \|A(v_{\neq},b_{\neq})\|^2_{L^2}$$

and

$$L\|Aa^1_{\neq}\|_{L^2}\nu\|Ap_{i,\neq}\|_{L^2}\|\partial_y^t Aa^3_{\neq}\|_{L^2} \leqslant \nu L\|Av_{\neq}\|_{L^2}\|A(v_{\neq},b_{\neq})\|_{L^2}\|\partial_y^t A(v_{\neq},b_{\neq})\|_{L^2}$$
$$+ \nu L\|Av_{\neq}\|_{L^2}\|A(v_{\neq},b_{\neq})\|_{L^2}\|\partial_y^t Av_{\neq}\|_{L^2}.$$

We use the definition of M_1 to infer

$$L\|Aa^1_{\neq}\|_{L^2}\|A\partial_x\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|\partial_y^t Aa^3_{\neq}\|_{L^2} \leqslant L\nu^{-\frac{1}{12}}\|Aa^1_{\neq}\|_{L^2}\|A\sqrt{-\tfrac{\partial_t M_1}{M_1}}p_{i,\neq}\|_{L^2}\|\partial_y^t Aa^3_{\neq}\|_{L^2}.$$

Therefore, combining these three estimates we obtain

$$\begin{aligned}R \lesssim\, & L\left(1+\nu\kappa^{-\frac{1}{3}}\right)\|Av_{\neq}\|_{L^2}\|A(v_{\neq},b_{\neq})\|_{L^2}^2\\&+\nu L\|Av_{\neq}\|_{L^2}\|A(v_{\neq},b_{\neq})\|_{L^2}\|\partial_y^t A(v_{\neq},b_{\neq})\|_{L^2}\\&+\nu L\|A(v_{\neq},b_{\neq})\|_{L^2}^2\|\partial_y^t Av_{\neq}\|_{L^2}\\&+L\nu^{-\frac{1}{12}}\|Aa^1_{\neq}\|_{L^2}\|A\sqrt{-\tfrac{\partial_t M_1}{M_1}}p_{i,\neq}\|_{L^2}\|\partial_y^t Aa^3_{\neq}\|_{L^2}.\end{aligned}$$

Integrating in time and using the Bootstrap assumption yields

$$\int R\,\mathrm{d}\tau \lesssim L\nu^{-\frac{1}{12}}\kappa^{-\frac{1}{2}}\varepsilon^3.$$

Transport term: On the set Ω_T it holds that $|k-l,\xi-\eta|<\frac{1}{8}|l,\eta|$ and thus it follows that $|k,\xi|\approx|l,\eta|$. By the mean value theorem, there exists $\theta\in[0,1]$ such that

$$\begin{aligned}\left||k,\xi|^N-|l,\eta|^N\right| &\leqslant N|k-l,\xi-\eta||k-\theta l,\xi-\theta\eta|^{N-1}\\&\lesssim |k-l,\xi-\eta||l,\eta|^{N-1}.\end{aligned}$$

Thus with (30) and lemma 5 we conclude, that

$$\frac{|A(k,\xi)-A(l,\eta)|}{A(k-l,\xi-\eta)A(l,\eta)} \lesssim \frac{1}{M_L(l,\eta)}\left(\frac{1}{|l|}+\nu^{\frac{1}{12}}\right)\frac{1}{|k-l,\xi-\eta|^{N-1}} \tag{31}$$

$$+\sum_{j=\kappa,\nu,\nu^3}|M_j(k,\xi)-M_j(l,\eta)|\frac{1}{M_L(l,\eta)}\frac{1}{|k-l,\xi-\eta|^N} \tag{32}$$

$$+\frac{M_L(k,\xi)-M_L(l,\eta)}{M_L(k-l,\xi-\eta)M_L(l,\eta)}\frac{1}{|k-l,\xi-\eta|^N}. \tag{33}$$

Based on this estimate, in the following we distinguish between different regimes in frequency,

$$\begin{aligned}T=&\sum_{k,l,k-l\neq 0}\int \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_T}\frac{|A(k,\xi)-A(l,\eta)|}{A(k-l,\xi-\eta)A(l,\eta)}\frac{|\xi l-k\eta|}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}}\\&\qquad|Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)\\ \lesssim&\sum_{k,l,k-l\neq 0}\int \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_T}\frac{1}{M_L(l,\eta)}\left(\frac{1}{|l|}+\nu^{\frac{1}{12}}\right)\frac{1}{|k-l,\xi-\eta|^{N-1}}\frac{|\xi l-k\eta|}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}}\\&\qquad|Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)\\&+\sum_{k,l,k-l\neq 0}\int \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_T}\mathbf{1}_{|\frac{\eta}{l}-t|\geqslant|\frac{\xi-\eta}{k-l}-t|}\frac{\sum_{j=\kappa,\nu,\nu^3}|M_j(k,\xi)-M_j(l,\eta)|}{M_L(l,\eta)}\frac{1}{|k-l,\xi-\eta|^N}\frac{|\xi l-k\eta|}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}}\\&\qquad|Aa^1|(k,\xi)|Ap_i|(k-l,\xi-\eta)|Aa^3|(l,\eta)\\&+\sum_{k,l,k-l\neq 0}\int \mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{|\frac{\eta}{l}-t|\leqslant|\frac{\xi-\eta}{k-l}-t|}\frac{\sum_{j=\kappa,\nu,\nu^3}|M_j(k,\xi)-M_j(l,\eta)|}{M_L(l,\eta)}\frac{1}{|k-l,\xi-\eta|^N}\frac{|\xi l-k\eta|}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}}\\&\qquad|Aa^1|(k,\xi)|Ap_i|(k-l,\xi-\eta)|Aa^3|(l,\eta)\\&+\sum_{k,l,k-l\neq 0}\int \mathrm{d}(\xi,\eta)\mathbf{1}_{\Omega_T}\frac{|M_L(k,\xi)-M_L(l,\eta)|}{M_L(k-l,\xi-\eta)M_L(l,\eta)}\frac{1}{|k-l,\xi-\eta|^N}\frac{|\xi l-k\eta|}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}}\\&\qquad|Aa^1|(k,\xi)|Ap_i|(k-l,\xi-\eta)|Aa^3|(l,\eta)\\=&\,T_{1,1}+T_{1,2}+T_{1,3}+T_2.\end{aligned}$$

Here, the $T_{1,1}$ term is due to estimate (31). For (32) we distinguish between the frequencies $|\frac{\eta}{l}-t| \geqslant |\frac{\xi-\eta}{k-l}-t|$ in $T_{1,2}$ and $|\frac{\eta}{l}-t| \leqslant |\frac{\xi-\eta}{k-l}-t|$ in $T_{1,3}$. The M_L commutator (33) is T_2, which requires further splitting. For $T_{1,1}$ we use $\xi l - k\eta = (\xi-\eta-(k-l)t)l - (k-l)(\eta-lt)$, (16) and (37) to estimate

$$\begin{aligned}
T_{1,1} = & \sum_{k,l,k-l\neq 0} \iint \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_T} \tfrac{1}{M_L(l,\eta)} \left(\tfrac{1}{|l|} + \nu^{\frac{1}{12}}\right) \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{|\xi l - k\eta|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}} \\
& \quad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta) \\
& \lesssim \|Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} Aa^3_{\neq}\|_{L^2} + \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} A\partial_y^t a^3_{\neq}\|_{L^2} \\
& \quad + \nu^{\frac{1}{12}} \|Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} A\partial_x a^3_{\neq}\|_{L^2} \\
& \leqslant L \|Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|Aa^3_{\neq}\|_{L^2} + L \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2} \\
& \quad + \nu^{\frac{1}{12}} \|Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|\Lambda_t a^3_{\neq}\|_{L^2}.
\end{aligned}$$

After integrating in time we deduce that

$$\int T_{1,1} \mathrm{d}\tau \lesssim \left(L + \nu^{-\frac{1}{12}}\right) \kappa^{-\frac{1}{2}} \varepsilon^3.$$

For $T_{1,2}$ we use $|\frac{\eta}{l}-t| \geqslant |\frac{\xi-\eta}{k-l}-t|$ to infer that $|\xi l - k\eta| = |(\xi-\eta-(k-l)t)l - (k-l)(\eta - lt)| \leqslant 2|(k-l)(\eta-lt)|$. Furthermore, with $\sum_{i=\nu,\kappa,\nu^3} |M_i(k,\xi) - M_i(l,\eta)| \approx 1$ and (35) we conclude that

$$\begin{aligned}
T_{1,2} \lesssim & \sum_{k,l,k-l\neq 0} \int \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_T} \mathbf{1}_{|\frac{\eta}{l}-t| \geqslant |\frac{\xi-\eta}{k-l}-t|} \tfrac{1}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{|(k-l)(\eta-lt)|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}} \\
& \quad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta) \\
& \lesssim \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} A\partial_y^t a^3_{\neq}\|_{L^2} \\
& \lesssim L \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2}.
\end{aligned}$$

So after integrating in time, we obtain

$$\int T_{1,2} \mathrm{d}\tau \lesssim L \kappa^{-\frac{1}{2}} \varepsilon^3.$$

For $T_{1,3}$, we use $|\frac{\eta}{l}-t| \leqslant |\frac{\xi-\eta}{k-l}-t|$ to infer $\xi l - k\eta \leqslant 2(\xi-\eta-(k-l)t)l$. Furthermore, with (43) we deduce

$$\sum_{j=\kappa,\nu,\nu^3} |M_j(k,\xi) - M_j(l,\eta)| \lesssim \nu^{\frac{1}{3}} \tfrac{|\xi l - k\eta|}{|kl|}.$$

Combining these two estimates by Hölder's inequality and (35) it follows, that

$$\begin{aligned}
T_{1,3} \lesssim & \sum_{k,l,k-l\neq 0} \int \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_T} \mathbf{1}_{|\frac{\eta}{l}-t| \leqslant |\frac{\xi-\eta}{k-l}-t|} \tfrac{1}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{\nu^{\frac{1}{3}} (\xi l - k\eta)^2}{kl\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}} \\
& \quad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta) \\
& \lesssim \sum_{k,l,k-l\neq 0} \int \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_T} \mathbf{1}_{|\frac{\eta}{l}-t| \leqslant |\frac{\xi-\eta}{k-l}-t|} \tfrac{1}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{\nu^{\frac{1}{3}} (\xi-\eta-(k-l)t)^2 l^2}{kl\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}}
\end{aligned}$$

$$
\begin{aligned}
&\qquad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)\\
&\lesssim \nu^{\frac13}\sum_{k,l,k-l\neq 0}\int \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_T}\mathbf{1}_{|\frac{\eta}{l}-t|\leqslant|\frac{\xi-\eta}{k-l}-t|}\frac{1}{M_L(l,\eta)}\frac{|\xi-\eta-(k-l)t|}{|k-l,\xi-\eta|^{N-1}}\\
&\qquad |Aa^1|(k,\xi)|Ap_i|(k-l,\xi-\eta)|Aa^3|(l,\eta)\\
&\lesssim \nu^{\frac13}\|Aa^1_{\neq}\|_{L^2}\|A\Lambda_t p_{i,\neq}\|_{L^2}\|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2}\\
&\lesssim L\nu^{\frac13}\|Aa^1_{\neq}\|_{L^2}\|A\Lambda_t p_{i,\neq}\|_{L^2}\|Aa^3_{\neq}\|_{L^2}.
\end{aligned}
$$

Thus integrating in time yields

$$\int T_{1,3}\mathrm{d}\tau \leqslant L\nu^{\frac16}\kappa^{-\frac12}\varepsilon^3.$$

To estimate the T_2 term, we split the integral into the sets

$$
\begin{aligned}
\Omega_1 &= \left\{\min\left(t-\tfrac{\eta}{l},t-\tfrac{\xi-\eta}{k-l}\right)\geqslant \nu^{-1}\right\},\\
\Omega_2 &= \left\{t-\tfrac{\eta}{l}\geqslant \nu^{-1}\geqslant t-\tfrac{\xi-\eta}{k-l}\right\},\\
\Omega_3 &= \left\{t-\tfrac{\xi-\eta}{k-l}\geqslant \nu^{-1}\geqslant t-\tfrac{\eta}{l}\right\},\\
\Omega_4 &= \left\{t-\tfrac{\xi}{}k\geqslant \nu^{-1}\geqslant \max\left(t-\tfrac{\eta}{l},t-\tfrac{\xi-\eta}{k-l}\right)\right\}.
\end{aligned}
$$

For frequencies such that $\nu^{-1}\geqslant \max(t-\frac{\eta}{l},t-\frac{\xi}{}k)$, then $M_L(k,\xi)-M_L(l,\eta)=0$ and hence the commutator vanishes. Thus the sets Ω_j covers all regions of the support. The sets Ω_1,Ω_2 and Ω_3 are chosen to distinguish between $\frac{1}{M_L}=1$ and $\frac{1}{M_L}>1$ for different frequencies and on set Ω_4 we use strong dissipation in the first component. We split the set T_2 into

$$
\begin{aligned}
T_2 &= \sum_{k,l,k-l\neq 0}\int \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_T}\frac{|M_L(k,\xi)-M_L(l,\eta)|}{M_L(k-l,\xi-\eta)M_L(l,\eta)}\frac{|\xi l-k\eta|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac12}}\frac{1}{|k-l,\xi-\eta|^N}\\
&\qquad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)\,(\mathbf{1}_{\Omega_1}+\mathbf{1}_{\Omega_2}+\mathbf{1}_{\Omega_3}+\mathbf{1}_{\Omega_4})\\
&= T_{2,1}+T_{2,2}+T_{2,3}+T_{2,4}.
\end{aligned}
$$

For $T_{2,1}$ we use (35) and the definition of Ω_1 to infer

$$
\begin{aligned}
\mathbf{1}_{\Omega_1}\frac{1}{M_L(k-l,\xi-\eta)M_L(l,\eta)} &\leqslant \mathbf{1}_{\Omega_1}\left(1+\nu\langle t-\tfrac{\eta}{l}\wedge\kappa^{-\frac13}\rangle\right)\left(1+\nu\langle t-\tfrac{\xi-\eta}{k-l}\wedge\kappa^{-\frac13}\rangle\right)\\
&\leqslant 2\nu^2\mathbf{1}_{\Omega_1}\langle t-\tfrac{\eta}{l}\wedge\kappa^{-\frac13}\rangle\langle t-\tfrac{\xi-\eta}{k-l}\wedge\kappa^{-\frac13}\rangle.
\end{aligned}
$$

This yields the estimate

$$
\begin{aligned}
T_{2,1} &\leqslant 2\nu^2\sum_{k,l,k-l\neq 0}\iint \mathrm{d}(\xi,\eta)\,1_{\Omega_1}\langle t-\tfrac{\eta}{l}\wedge\kappa^{-\frac13}\rangle\langle t-\tfrac{\xi-\eta}{k-l}\wedge\kappa^{-\frac13}\rangle\frac{1}{|k-l,\xi-\eta|^{N-1}}\\
&\qquad \frac{|(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac12}}|Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)\\
&\lesssim \nu^2\|Aa^1_{\neq}\|_{L^2}\|\left(\Lambda_t\wedge\kappa^{-\frac13}\right)Ap_{i,\neq}\|_{L^2}\|A\Lambda_t a^3_{\neq}\|_{L^2}\\
&\quad + L\nu\|Aa^1_{\neq}\|_{L^2}\|\left(\Lambda_t\wedge\kappa^{-\frac13}\right)\Lambda_t^{-1}\Lambda^{-1}Ap_{i,\neq}\|_{L^2}\|A\partial_y^t a^3_{\neq}\|_{L^2}
\end{aligned}
$$

and so

$$\int T_{2,1} \mathrm{d}\tau \lesssim L\nu^{\frac{5}{6}}\kappa^{-\frac{1}{2}}\varepsilon^3.$$

Now we consider $T_{2,2}$. As before by (35) and the definition of Ω_2 we infer

$$\begin{aligned} T_{2,2} \leqslant & \sum_{k,l,k-l\neq 0} \iint \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_2} \nu \langle t - \tfrac{\eta}{l} \wedge \kappa^{-\frac{1}{3}} \rangle \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{|(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}} \\ & \qquad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta) \\ & \lesssim \nu \|Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|A\Lambda_t a^3_{\neq}\|_{L^2} \\ & \quad + L\|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2}. \end{aligned}$$

Integrating in time yields

$$\int T_{2,2} \mathrm{d}\tau \lesssim L\kappa^{-\frac{1}{2}}\varepsilon^3.$$

To estimate $T_{2,3}$, we need to distinguish between different choices of a. Using (35) and the definition of Ω_3 we estimate

$$\begin{aligned} T_{2,3} = & \sum_{k,l,k-l\neq 0} \iint \mathrm{d}(\xi,\eta)\, \mathbf{1}_{\Omega_3} \nu \langle t - \tfrac{\xi-\eta}{k-l} \wedge \kappa^{-\frac{1}{3}} \rangle \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{|(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}} \\ & \qquad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta) \\ & \lesssim \nu \|Aa^1_{\neq}\|_{L^2} \|\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) Ap_{i,\neq}\|_{L^2} \|A\partial_x a^3_{\neq}\|_{L^2} \\ & \quad + \nu \|Aa^1_{\neq}\|_{L^2} \|A\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) \Lambda^{-1}\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2} \end{aligned}$$

and thus after integrating in time

$$\begin{aligned} \int T_{2,3}[vvv]\, \mathrm{d}\tau &\lesssim \nu^{-\frac{1}{2}}\varepsilon^3, \\ \int T_{2,3}[bvb]\, \mathrm{d}\tau &\lesssim \kappa^{-\frac{1}{2}}\varepsilon^3, \\ \int T_{2,3}[bbv]\, \mathrm{d}\tau &\lesssim \kappa^{-\frac{1}{2}}\varepsilon^3. \end{aligned}$$

In the case of vbb, we use (35) and the definition of Ω_3 to estimate

$$\begin{aligned} T_{2,3}[vbb] = & \sum_{k,l,k-l\neq 0} \iint \mathrm{d}(\xi,\eta)\, \nu \langle t - \tfrac{\xi-\eta}{k-l} \rangle \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{|(\xi-\eta-(k-l)t)k-(k-l)(\xi-kt)|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}} \\ & \qquad |Av|(k,\xi)\,|Ap_2|(k-l,\xi-\eta)\,|Ab|(l,\eta) \\ & \lesssim \nu \|\partial_x Av_{\neq}\|_{L^2} \|\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) Ap_{2,\neq}\|_{L^2} \|Ab_{\neq}\|_{L^2} \\ & \quad + \nu \|\partial_y^t Av_{\neq}\|_{L^2} \|\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) \Lambda_t^{-1} Ap_{2,\neq}\|_{L^2} \|Ab_{\neq}\|_{L^2} \\ & \leqslant \nu \|Ab_{\neq}\|_{L^2} \|A\nabla_t v_{\neq}\|_{L^2} \|A\Lambda_t p_{2,\neq}\|_{L^2}. \end{aligned}$$

Thus after integrating in time, we obtain

$$\int T_{2,3}[vbb]\,\mathrm{d}\tau \lesssim \nu^{\frac{1}{2}}\kappa^{-\frac{1}{2}}\varepsilon^3.$$

For $T_{2,4}$ we obtain that $M(l,\eta) = M(k-l,\xi-\eta) = 1$. We use $t - \frac{\xi}{k} \geqslant \nu \geqslant \max(t - \frac{\eta}{l}, t - \frac{\xi-\eta}{k-l})$ to deduce that

$$1 = \frac{kt-\xi}{kt-\xi} = \frac{k}{k}\frac{t-\frac{\xi}{k}}{t-\frac{\xi}{k}} \leqslant \nu\frac{|\xi-kt|}{|k|}.$$

With $\xi l - k\eta = (\xi - \eta - (k-l)t)l - (k-l)(\eta - lt)$ we infer that

$$\begin{aligned}
T_{2,4} &= \sum_{k,l,k-l\neq 0}\iint \mathrm{d}(\xi,\eta)\,\frac{1}{|k-l,\xi-\eta|^N}\frac{|\xi l-k\eta|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}}\\
&\qquad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)\\
&= \nu\sum_{k,l,k-l\neq 0}\iint \mathrm{d}(\xi,\eta)\,|\xi-kt|\frac{1}{|k-l,\xi-\eta|^{N-1}}\frac{|(\xi-\eta-(k-l)t)l|}{|l|\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}}\\
&\qquad |Aa^1|(k,\xi)\,|Ap_i|(k-l,\xi-\eta)\,|Aa^3|(l,\eta)\\
&\quad + \sum_{k,l,k-l\neq 0}\iint \mathrm{d}(\xi,\eta)\,\frac{1}{|k-l,\xi-\eta|^{N-1}}\frac{|(k-l)(\eta-lt)|}{\left((k-l)^2+(\xi-\eta-(k-l)\tau)^2\right)^{\frac{1}{2}}}\\
&\qquad |Aa^1|(k,\xi)|Ap_i|(k-l,\xi-\eta)|Aa^3|(l,\eta)\\
&\leqslant \nu\|A\partial_y^t a^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|Aa^3_{\neq}\|_{L^2}\\
&\quad + \|Aa^1_{\neq}\|_{L^2}\|\Lambda_t^{-1}Ap_{i,\neq}\|_{L^2}\|A\partial_y^t a^3_{\neq}\|_{L^2}.
\end{aligned}$$

Thus integrating in time yields

$$\int T_{2,4}\mathrm{d}\tau \lesssim \kappa^{-\frac{1}{2}}\varepsilon^3.$$

3.3. Nonlinear terms with an x-average in the second component

We apply the notation of (29)

$$\begin{aligned}
&\langle Aa^1_{\neq}, A\left(a^2_{1,=}\partial_x a^3_{\neq}\right) - a^2_{1,=}\partial_x Aa^3_{\neq}\rangle\\
&\quad\leqslant \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\,|Aa^1|(k,\xi)\,|A(k,\xi)-A(k,\eta)|\,|k||a^2_1|(0,\xi-\eta)\,|a^3|(k,\eta)\\
&\quad= R+T.
\end{aligned}$$

Here we split into reaction and transport terms according to the sets

$$\begin{aligned}
\Omega_R &= \left\{|\xi-\eta| \geqslant \tfrac{1}{8}|k,\eta|\right\},\\
\Omega_T &= \left\{|\xi-\eta| < \tfrac{1}{8}|k,\eta|\right\}.
\end{aligned}$$

Reaction term On the set Ω_R it holds that $|\xi-\eta| \geqslant \frac{1}{8}|k,\eta|$, then we obtain $|A(k,\xi)-A(k,\eta)| \lesssim |\xi-\eta|^N$ and thus with (35), it follows that

$$\begin{aligned}
&\sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)|Aa^1|(k,\xi)\,|A(k,\xi)-A(k,\eta)|\,|k||a_1^2|(0,\xi-\eta)\,|a^3|(k,\eta)\\
&\qquad\lesssim \|Aa^1_{\neq}\|_{L^2}\|a^2_{=}\|_{H^N}\|\partial_x a^3_{\neq}\|_{L^\infty}\\
&\qquad\lesssim \|Aa^1_{\neq}\|_{L^2}\|a^2_{=}\|_{H^N}\|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2}\\
&\qquad\lesssim L\|Aa^1_{\neq}\|_{L^2}\|a^2_{=}\|_{H^N}\|Aa^3_{\neq}\|_{L^2}.
\end{aligned}$$

Integrating in time yields a bound

$$\int R\,\mathrm{d}\tau \lesssim L\kappa^{-\frac{1}{3}}\varepsilon^2\tilde{\varepsilon}.$$

Transport term On the set Ω_L it holds that $|k,\eta| \geqslant \frac{1}{8}|\xi-\eta|$. By the mean value theorem there exists a $\theta\in[0,1]$

$$\left||k,\eta|^N-|k,\xi|^N\right| \lesssim |\xi-\eta||k,\eta-\theta\xi|^{N-1} \lesssim |\xi-\eta||k,\eta|^{N-1}.$$

Thus, we can estimate the difference in A by

$$\begin{aligned}
|A(k,\xi)-A(k,\eta)| &\lesssim (M_L(k,\xi)-M_L(k,\eta))\,|k,\xi|^N\\
&\quad+M_L(k,\eta)\,|k,\xi|^N\sum_{j=1,\kappa,\nu,\nu^3}|M_j(k,\xi)-M_j(k,\eta)|\\
&\quad+M_L(k,\eta)\left(|k,\eta|^N-|k,\xi|^N\right)\\
&\lesssim \tfrac{1}{k}|\xi-\eta||k,\xi|^N
\end{aligned}$$

where we used (40), (42) and (43) to estimate the differences in M_j. So we infer, that

$$T\leqslant \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_T}|Aa^1|(k,\xi)\,|a_1^2|(0,\xi-\eta)\,\tfrac{1}{M_L(k,\eta)}|Aa^3|(k,\eta),$$

and thus integrating in time yields

$$\int T\mathrm{d}\tau \lesssim L\|Aa^1_{\neq}\|_{L^2L^2}\|a^2_{=}\|_{L^\infty H^N}\|Aa^3_{\neq}\|_{L^2L^2} \lesssim L\kappa^{-\frac{1}{3}}\varepsilon^2\tilde{\varepsilon}.$$

3.4. Nonlinear terms with an x-average in the third component

We aim to estimate

$$\begin{aligned}
&\langle Aa^1_{1,\neq}, A\left(a^2_{2,\neq}\partial_y a^3_{1,=}\right)\rangle\\
&\quad=\sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\,|Aa_1^1|(k,\xi)A(k,\xi)\,\tfrac{|k\eta|}{\sqrt{k^2+(\xi-\eta-kt)^2}}|p_i|(k,\xi-\eta)\,|a_1^3|(0,\eta)\\
&\quad=R+T
\end{aligned}$$

where we split into the reaction and transport terms according to the sets

$$\Omega_R = \left\{|k,\xi-\eta| \geqslant \tfrac{1}{8}|\eta|\right\},$$
$$\Omega_T = \left\{|k,\xi-\eta| < \tfrac{1}{8}|\eta|\right\}.$$

Reaction term On the set Ω_R it holds that $|k,\xi-\eta| \geqslant \frac{1}{8}|\eta|$. With (35) we infer

$$\begin{aligned} R &= \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_R}|Aa_1^1|(k,\xi)A(k,\xi)\,\tfrac{|k\eta|}{\sqrt{k^2+(\xi-\eta-kt)^2}}|p_i|(k,\xi-\eta)\,|a_1^3|(0,\eta) \\ &\lesssim \|Aa^1_{1,\neq}\|_{L^2}\|A\tfrac{1}{M_L}\partial_x\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|\partial_y a^3_{1,=}\|_{L^\infty} \\ &\lesssim \|Aa^1_{1,\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|a^3_{1,=}\|_{H^N}. \end{aligned}$$

Integrating in time then yields

$$\int R\mathrm{d}\tau \lesssim \kappa^{-\frac{1}{3}}\varepsilon^2\tilde{\varepsilon}.$$

Transport term On the set Ω_T it holds that $|k,\xi-\eta| \leqslant \frac{1}{8}|\eta|$, then with (35) we estimate

$$\begin{aligned} T &= \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\,\mathbf{1}_{\Omega_T}|Aa_1^1|(k,\xi)A(k,\xi)\,\tfrac{|k\eta|}{\sqrt{k^2+(\xi-\eta-kt)^2}}|p_i|(k,\xi-\eta)\,|a_1^3|(0,\eta) \\ &\lesssim \|Aa^1_{\neq}\|_{L^2}\|\partial_x\Lambda_t^{-1}p_{i,\neq}\|_{L^\infty}\|\partial_y a^3_{=}\|_{H^N} \\ &\lesssim \|Aa^1_{\neq}\|_{L^2}\|\tfrac{1}{M_L}A\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|\partial_y a^3_{=}\|_{H^N} \\ &\lesssim \|Aa^1_{\neq}\|_{L^2}\left(\|\Lambda_t^{-1}Ap_i\|_{L^2}+\nu\|Ap_{i,\neq}\|_{L^2}\right)\|\partial_y a^3_{=}\|_{H^N}. \end{aligned}$$

Integrating in time yields

$$\int T\mathrm{d}\tau \lesssim \kappa^{-\frac{1}{2}}\varepsilon^2\tilde{\varepsilon}.$$

3.5. *Nonlinear terms with an x-average in first component*

Now we turn to

$$\begin{aligned} &\langle\langle\partial_y\rangle^N a^1_{=,1}, \langle\partial_y\rangle^N\left(\left(a^2_{\neq}\cdot\nabla_t\right)a^3_{\neq,1}\right)_=\rangle \\ &\quad\leqslant \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\,\langle\xi\rangle^{2N}|a_1^1|(0,\xi)\,\tfrac{|k\xi|}{\sqrt{k^2+(\xi-\eta+kt)^2}}|p_i|(-k,\xi-\eta)\,|a_1^3|(k,\eta). \end{aligned}$$

Applying Hölder's inequality, the Sobolev embedding and the definition of A yields

$$\begin{aligned} &\langle\langle\partial_y\rangle^N a^1_{1,=}, \langle\partial_y\rangle^N(a^2_{\neq}\nabla_t a^3_{\neq,1})_=\rangle \\ &\quad\leqslant \|\partial_y\langle\partial_y\rangle^N a^1_{=}\|_{L^2}(\|\partial_x\Lambda_t^{-1}p_{i,\neq}\|_{L^\infty}\|\langle\partial_y\rangle^N a^3_{\neq}\|_{L^2}+\|\langle\partial_y\rangle^N\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|\partial_x a^3_{\neq}\|_{L^\infty} \\ &\quad\leqslant \|\partial_y a^1_{=}\|_{H^N}\|A\tfrac{1}{M_L}\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|A\tfrac{1}{M_L}a^3_{\neq}\|_{L^2}. \end{aligned}$$

With (35) we infer

$$\begin{aligned}
&\langle \langle \partial_y\rangle^N a^1_{1,=}, \langle \partial_y\rangle^N \left(a^2_{\neq} \nabla_t a^3_{\neq,1}\right)_{=}\rangle \\
&\quad \lesssim \|\partial_y a^1_{=}\|_{H^N} \left(\|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} + \nu \|Ap_{i,\neq}\|_{L^2}\right) \left(\|Aa^3_{\neq}\|_{L^2} + \nu \|A\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) a^3_{\neq}\|_{L^2}\right) \\
&\quad \lesssim \|\partial_y a^1_{=}\|_{H^N} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \left(\|Aa^3_{\neq}\|_{L^2} + \nu \|\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) a^3_{\neq}\|_{L^2}\right) \\
&\qquad + \nu \|\partial_y a^1_{=}\|_{H^N} \|Ap_{i,\neq}\|_{L^2} \left(\|Aa^3_{\neq}\|_{L^2} + \nu \|\left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) a^3_{\neq}\|_{L^2}\right).
\end{aligned}$$

Integrating in time yields

$$\int \langle \langle \partial_y\rangle^N a^1_{=}, \langle \partial_y\rangle^N \left(a^2_{\neq} \nabla_t a^3_{\neq,1}\right)_{=}\rangle \mathrm{d}\tau \lesssim L\kappa^{-\frac{1}{2}} \varepsilon^2 \tilde{\varepsilon}.$$

3.6. Other nonlinear terms

In this subsection, we aim to estimate

$$\begin{aligned}
\Re\langle \chi A\partial_y^t \Delta_t^{-1} a^1_{\neq}, A\left(\left(a^2 \cdot \nabla_t\right) a^3\right)\rangle &= \Re\langle \chi A\partial_y^t \Delta_t^{-1} a^1_{\neq}, A\left(\left(a^2_{\neq} \cdot \nabla_t\right) a^3_{\neq}\right)\rangle \\
&\quad + \Re\langle \chi A\partial_y^t \Delta_t^{-1} a^1_{\neq}, A\left(\left(a^2_{\neq} \cdot \nabla_t\right) a^3_{=}\right)\rangle \\
&\quad + \Re\langle \chi A\partial_y^t \Delta_t^{-1} a^1_{\neq}, A\left(\left(a^2_{=} \cdot \nabla_t\right) a^3_{\neq}\right)\rangle.
\end{aligned}$$

with the choices $a^1a^2a^3 \in \{bvv, bbb, vbv, vvb\}$. We start with the case of no x-averages and use

$$\xi l - k\eta = (\xi - kt)(l-k) + k(\xi - \eta - (k-l)t)$$

and (35) to infer

$$\begin{aligned}
&\Re\langle \chi A\partial_y^t \Delta_t^{-1} a^1_{\neq}, A\left(\left(a^2_{\neq} \cdot \nabla_t\right) a^3_{\neq}\right)\rangle \\
&\quad \leqslant \sum_{k,l,k-l\neq 0} \iint \mathrm{d}(\xi,\eta) \frac{|\xi - kt|}{k^2 + (\xi - kt)^2} \frac{|\xi l - k\eta|}{\sqrt{(k-l)^2 + (\xi - \eta - (k-l)t)^2}} A^2 \\
&\qquad \times (k,\xi) |a^1| (k,\xi) |p_i| (k-l, \xi - \eta) |a^3| (l,\eta) \\
&\quad \lesssim \|Aa^1_{\neq}\|_{L^2} \|\tfrac{1}{M_L} \partial_x \Lambda_t^{-1} Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} Aa^3_{\neq}\|_{L^2} \\
&\qquad + \|\partial_x \Lambda_t^{-1} Aa^1_{\neq}\|_{L^2} \|\tfrac{1}{M_L} Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} Aa^3_{\neq}\|_{L^2} \\
&\quad \lesssim L\|Aa^1_{\neq}\|_{L^2} \left(\|\partial_x \Lambda_t^{-1} Ap_{i,\neq}\|_{L^2} + \nu \|Ap_{i,\neq}\|_{L^2}\right) \|Aa^3_{\neq}\|_{L^2} \\
&\qquad + L\left(1 + \nu\kappa^{-\frac{1}{3}}\right) \|\partial_x \Lambda_t^{-1} Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|Aa^3_{\neq}\|_{L^2}.
\end{aligned}$$

Thus integrating in time yields

$$\int \Re\langle \chi A\partial_y^t \Delta_t^{-1} a^1_{\neq}, A\left(a^2_{\neq} \nabla_t a^3_{\neq}\right)\rangle \mathrm{d}\tau \lesssim L\kappa^{-\frac{1}{2}} \varepsilon^3.$$

For the case, when the average is in the second component, we use partial integration to estimate

$$\begin{aligned}
&\Re\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A\left(a^2_{1,=}\partial_x a^3_{\neq}\right)\rangle \\
&\quad = -\Re\langle \chi A\partial_x\partial_y^t\Delta_t^{-1}a^1_{\neq}, A\left(a^2_{1,=}a^3_{\neq}\right)\rangle \\
&\quad \lesssim \|\partial_x\Lambda_t^{-1}a^1_{\neq}\|_{L^2}\|a^2_{=}\|_{H^N}\|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2} \\
&\quad \lesssim L\|\partial_x\Lambda_t^{-1}a^1_{\neq}\|_{L^2}\|a^2_{=}\|_{H^N}\|Aa^2_{\neq}\|_{L^2}
\end{aligned}$$

and thus integrating in time and using $L=\max(1,\nu\kappa^{-\frac{1}{3}})$ yields

$$\int \Re\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A\left(a^2_{1,=}\partial_x a^3_{\neq}\right)\rangle \mathrm{d}\tau \lesssim \kappa^{-\frac{1}{2}}\varepsilon^2\tilde{\varepsilon}.$$

For the case when the average is in the third component, we obtain

$$\begin{aligned}
&\Re\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A\left(a^2_{2,\neq}\partial_y a^3_{=}\right)\rangle \\
&\quad = \sum_{k\neq 0}\iint \mathrm{d}(\xi,\eta)\,\chi\,\frac{|\xi-kt|}{k^2+(\xi-kt)^2}\frac{|k\eta|}{\sqrt{k^2+(\xi-\eta-t)^2}}\frac{A(\xi,k)}{A(\xi-\eta,k)} \\
&\qquad \times |Aa^1|(k,\xi)\,|Ap_i|(k,\xi-\eta)\,|a^3|(0,\eta).
\end{aligned}$$

Thus by $\eta=\xi-kt-(\xi-\eta-kt)$ we estimate

$$\begin{aligned}
&\Re\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A\left(a^2_{2,\neq}\partial_y a^3_{=}\right)\rangle \\
&\quad \leqslant \|\left(\partial_y^t\right)^2\Delta_t^{-1}Aa^1_{\neq}\|_{L^2}\|\partial_x\Lambda_t^{-1}\tfrac{1}{M_L}Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N} \\
&\qquad + \|\partial_x\partial_y^t\Delta_t^{-1}Aa^1_{\neq}\|_{L^2}\|\tfrac{1}{M_L}Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N} \\
&\quad \leqslant L\|Aa^1_{\neq}\|_{L^2}\|\partial_x\Lambda_t^{-1}Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N} \\
&\qquad + L\|\partial_x\Lambda_t^{-1}Aa^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N}.
\end{aligned}$$

Integrating in time and using $L=\max(1,\nu\kappa^{-\frac{1}{3}})$ yields

$$\int \Re\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A\left(a^2_{2,\neq}\partial_y a^3_{=}\right)\rangle \mathrm{d}\tau \lesssim \kappa^{-\frac{1}{2}}\varepsilon^2\tilde{\varepsilon}.$$

Which concludes the estimate

$$\int ONL\mathrm{d}\tau \lesssim L\kappa^{-\frac{1}{2}}\tilde{\varepsilon}\varepsilon^2.$$

Combining the estimates of sections 3.2–3.6 completes the proof of proposition 3.1 and thus theorem 3.1. □

In this article, we have shown that the MHD equations around Couette flow with magnetic resistivity smaller than fluid viscosity $\nu\geqslant\kappa>0$ are stable for initial data which is small enough in Sobolev spaces. If the resistivity is much smaller than the viscosity, $\nu\kappa^{-\frac{1}{3}}>1$, large viscosity destabilizes the equation, leading to norm inflation of size $\nu\kappa^{-\frac{1}{3}}$. Controlling this norm inflation is a major new challenge compared to other dissipation regimes.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Construction of the weights

Let A be the Fourier weight

$$A := M\langle\nabla\rangle \mathrm{e}^{c\kappa^{\frac{1}{3}}t\mathbf{1}_{\neq}},$$

with $M = M_1 M_L M_\kappa M_\nu M_{\nu^3}$ defined as

$$\begin{aligned}
\tfrac{-\dot M_L}{M_L} &= \tfrac{t-\frac{\xi}{k}}{1+\left(\frac{\xi}{k}-t\right)^2}\mathbf{1}_{\left\{\nu^{-1}\leqslant t-\frac{\xi}{k}\leqslant (c_1\kappa k^2)^{-\frac{1}{3}}\right\}} && k\neq 0,\\
\tfrac{-\dot M_1}{M_1} &= C_\alpha \tfrac{|k|+\nu^{\frac{1}{12}}|k|^2}{k^2+(\xi-kt)^2} && k\neq 0,\\
\tfrac{-\dot M_\nu}{M_\nu} &= \tfrac{\nu^{\frac{1}{3}}}{1+\nu^{\frac{2}{3}}\left(t-\frac{\xi}{k}\right)^2} && k\neq 0,\\
\tfrac{-\dot M_\kappa}{M_\kappa} &= \tfrac{\kappa^{\frac{1}{3}}}{1+\kappa^{\frac{2}{3}}\left(t-\frac{\xi}{k}\right)^2} && k\neq 0,\\
\tfrac{-\dot M_{\nu^3}}{M_{\nu^3}} &= \tfrac{C_\alpha \nu}{1+\nu^2\left(t-\frac{\xi}{k}\right)^2} && k\neq 0,\\
M.\,(t=0) &= M.\,(k=0) = 1.
\end{aligned}$$

The weight M_L is an adaption of the weight $m^{\frac{1}{2}}$ in [Lis20] to the present setting and M_{ν^3} we use to differentiate between resonant and non-resonant regions. The method of using time-dependent Fourier weights is common when working at solutions around Couette flow and the other weights are modifications of previously used weights (see [BVW18, MZ22, Lis20, ZZ24] for shear related systems such as Navier–Stokes). The constants $C_\alpha = \frac{2}{\min(1,\alpha-\frac{1}{2})}$, $c = \frac{1}{20}(1-\frac{1}{2\alpha})^2$ and $c_1 = \frac{1}{20}(1-\frac{1}{2\alpha})$ are determined through the linear estimates. For the weights we obtain that for all times $t>0$, it holds that

$$\begin{aligned}
M_1 \approx M_\kappa \approx M_\nu \approx M_{\nu^3} \approx 1,\\
L^{-1} \leqslant \min\left(1, \nu^{-1}\kappa^{\frac{1}{3}}k^{\frac{2}{3}}\right) \lesssim M_L \leqslant 1.
\end{aligned}$$

Lemma 3 (M_L properties). *The weight M_L satisfies the following bounds*

$$\mathbf{1}_{|t-\frac{\xi}{k}|\geqslant \nu^{-1}} \tfrac{t-\frac{\xi}{k}}{1+\left(t-\frac{\xi}{k}\right)^2} \leqslant \tfrac{-\dot M_L}{M_L} + \kappa k^2 c_1\left(1+\left(t-\tfrac{\xi}{k}\right)^2\right), \tag{34}$$

$$\tfrac{1}{M_L(k,\xi)} \lesssim 1+\nu\left(\langle t-\tfrac{\xi}{k}\rangle \wedge \kappa^{-\frac{1}{3}}\right). \tag{35}$$

Furthermore, it follows for $a \in H^1$, that

$$\|\tfrac{1}{M_L} a_{\neq}\|_{L^2} \lesssim \|a_{\neq}\|_{L^2} + \nu \| \left(\Lambda_t \wedge \kappa^{-\frac{1}{3}}\right) a_{\neq}\|_{L^2}, \tag{36}$$

$$\|\tfrac{1}{M_L} \partial_x a_{\neq}\|_{L^2} \lesssim \|\Lambda_t a_{\neq}\|_{L^2}. \tag{37}$$

Proof. This (34) is a consequence of the definition of M_L. For estimate (35), we use that for $s_0 \in \mathbb{R}$ the ODE $f'(s) = \frac{s}{1+s^2} f(s), f(s_0) = 1$ is solved by $f(s) = \sqrt{\frac{1+s^2}{1+s_0^2}}$ for all $s \in \mathbb{R}$. Therefore, we obtain that

$$M_L = \begin{cases} 1, & t - \frac{\xi}{k} \leqslant \nu^{-1}, \\ \sqrt{\frac{1+\left(t-\frac{\xi}{k}\right)^2}{1+\nu^{-2}}}, & \nu^{-1} \leqslant t - \frac{\xi}{k} \leqslant \left(c_1 \kappa k^2\right)^{-\frac{1}{3}}, \\ \sqrt{\frac{1+\left(c_1 \kappa k^2\right)^{-\frac{2}{3}}}{1+\nu^{-2}}}, & \left(c_1 \kappa k^2\right)^{-\frac{1}{3}} \leqslant t - \frac{\xi}{k}. \end{cases}$$

By the estimate $\sqrt{\frac{1+(t-\frac{\xi}{k})^2}{1+\nu^{-2}}} \leqslant \nu \langle t - \frac{\xi}{k} \rangle$ and $\sqrt{\frac{1+(c_1 \kappa k^2)^{-\frac{2}{3}}}{1+\nu^{-2}}} \leqslant \langle (c_1 \kappa k^2)^{-\frac{1}{3}} \rangle \nu \lesssim \nu \kappa^{-\frac{1}{3}}$ estimate (35) follows. Estimates (36) and (37) are a direct consequence of (35). □

Lemma 4 (Enhanced dissipation estimates). *The weights M_ν and M_κ satisfy the following bounds*

$$\tfrac{1}{2}\nu^{\frac{1}{3}} \leqslant \tfrac{-\dot{M}_\kappa}{M_\kappa} + \nu \left(k^2 + (\xi - kt)^2\right), \tag{38}$$

$$\tfrac{1}{2}\kappa^{\frac{1}{3}} \leqslant \tfrac{-\dot{M}_\nu}{M_\nu} + \kappa \left(k^2 + (\xi - kt)^2\right). \tag{39}$$

Proof. This follows immediately from the definition of M_ν and M_κ. □

Lemma 5 (Difference estimates). *Let $k, l \in \mathbb{Z} \setminus \{0\}$ and $\xi, \eta \in \mathbb{R}$, then there hold the following bounds on differences*

$$1 - \tfrac{M_1(k,\xi)}{M_1(k,\eta)} \lesssim \tfrac{|\xi - \eta|}{|k|}, \tag{40}$$

$$1 - \tfrac{M_1(k,\xi)}{M_1(l,\eta)} \lesssim \tfrac{|k-l|}{|l|} + \nu^{\frac{1}{12}}, \tag{41}$$

$$M_L(k,\eta) - M_L(k,\xi) \leqslant 2\tfrac{|\xi-\eta|}{k}, \tag{42}$$

$$1 - \tfrac{M_j(k,\xi)}{M_j(l,\eta)} \leqslant 2 j^{\frac{1}{3}} \tfrac{|\xi l - k\eta|}{|kl|}, \qquad j \in \left\{\kappa, \nu, \nu^3\right\} \tag{43}$$

Proof. We start with the M_1 estimate (40) and consider $M_1(k,\xi) \leqslant M_1(k,\eta)$

$$\begin{aligned} 1 - \tfrac{M_1(k,\xi)}{M_1(k,\eta)} &= 1 - \exp\left(-\left|\int_0^t \tfrac{|k| + |k|^2 \nu^{\frac{1}{12}}}{k^2 + (\xi - k\tau)^2} - \tfrac{|k| + |k|^2 \nu^{\frac{1}{12}}}{k^2 + (\eta - kt)^2} \mathrm{d}\tau\right|\right), \\ &\leqslant 1 - \exp\left(-\int_{\left[\frac{\xi}{k}, \frac{\eta}{k}\right] \cup t - \left[\frac{\xi}{k}, \frac{\eta}{k}\right]} 1 \, \mathrm{d}\tau\right) \lesssim \tfrac{|\xi - \eta|}{|k|}. \end{aligned}$$

The case $M_1(k,\xi) \geqslant M_1(k,\eta)$ follows by the same argument and $M_1(k,\xi) \approx M_1(k,\eta) \approx 1$. For (41) we consider the case $M_1(k,\xi) \leqslant M_1(l,\eta)$ and infer that

$$1 - \tfrac{M_1(k,\xi)}{M_1(l,\eta)} = 1 - \exp\left(-\left|\int_0^t \tfrac{|k| + |k|^2 \nu^{\frac{1}{12}}}{k^2 + (\xi - kt)^2} - \tfrac{|l| + |l|^2 \nu^{\frac{1}{12}}}{l^2 + (\eta - lt)^2}\right|\right).$$

Then we use that for $x \geqslant 0$ that $1 - e^{-x} \leqslant x$ and thus

$$1 - \frac{M_1(k,\xi)}{M_1(l,\eta)} \leqslant \left| \int_0^t \frac{|k|+|k|^2\nu^{\frac{1}{12}}}{k^2+(\xi-kt)^2} - \frac{|l|+|l|^2\nu^{\frac{1}{12}}}{l^2+(\eta-lt)^2} \right| \leqslant 2\pi \left(\frac{1}{l\wedge k} + \nu^{\frac{1}{12}} \right) \lesssim \frac{|k-l|}{|l|} + \nu^{\frac{1}{12}}.$$

The case $M_1(k,\xi) \geqslant M_1(l,\eta)$ follows by the same argument and $M_1(k,\xi) \approx M_1(l,\eta) \approx 1$. For (42) we consider the case $M_L(k,\xi) \leqslant M_L(k,\eta)$ and thus

$$M_L(k,\eta) - M_L(k,\xi) = M_L(k,\eta)\left(1 - \frac{M_L(k,\xi)}{M_L(k,\eta)}\right) \lesssim 1 - \frac{M_L(k,\xi)}{M_L(k,\eta)}.$$

We infer

$$\begin{aligned} 1 - \frac{M_L(k,\xi)}{M_L(k,\eta)} &= 1 - \exp\left(- \left| \int_0^t \frac{\tau - \frac{\xi}{k}}{1+\left(\tau - \frac{\xi}{k}\right)^2} \mathrm{d}\tau - \int_0^t \frac{\tau - \frac{\eta}{k}}{1+\left(\tau-\frac{\eta}{k}\right)^2} \mathrm{d}\tau \right| \right), \\ &= 1 - \exp\left(- \int_{-\left[\frac{\xi}{k},\frac{\eta}{k}\right] \cup t - \left[\frac{\xi}{k},\frac{\eta}{k}\right]} 1 \mathrm{d}\tau \right), \\ &\leqslant 2\frac{|\xi-\eta|}{k}. \end{aligned}$$

The case $M_L(k,\xi) \geqslant M_L(k,\eta)$ follows by the same argument.

For (43) we estimate the M_κ difference, since the M_ν and M_{ν^3} differences are done similar. Let $M_\kappa(k,\xi) \geqslant M_\kappa(l,\eta)$, then it follows

$$\begin{aligned} 1 - \frac{M_\kappa(k,\xi)}{M_\kappa(l,\eta)} &= 1 - \exp\left(-\kappa^{\frac{1}{3}} \left| \int_0^t \frac{1}{1+\kappa^{\frac{2}{3}}\left(t-\frac{\xi}{k}\right)^2} - \frac{1}{1+\kappa^{\frac{2}{3}}\left(t-\frac{\eta}{l}\right)^2} \right| \right), \\ &\leqslant 1 - \exp\left(-\kappa^{\frac{1}{3}} \left| \int_0^t \mathbf{1}_{-\left[\frac{\xi}{k},\frac{\eta}{l}\right] \cup t - \left[\frac{\xi}{k},\frac{\eta}{l}\right]}(\tau) \mathrm{d}\tau \right| \right), \\ &\leqslant 1 - \exp\left(-2\kappa^{\frac{1}{3}} \left| \frac{\xi}{k} - \frac{\eta}{l} \right| \right), \\ &\lesssim \kappa^{\frac{1}{3}} \frac{|\xi l - k\eta|}{|kl|}. \end{aligned}$$

The case $M_\kappa(k,\xi) \leqslant M_\kappa(l,\eta)$ follows from the same steps and $M_\kappa(k,\xi) \approx M_\kappa(l,\eta)$.

□

Appendix B. Local wellposedness

We expect the local wellposedness result to be well-known, but were not able to find it stated in the literature. In the following, we prove the local wellposedness by a standard application of the Banach fixed-point theorem.

Proposition 6. *Consider equation (17) with initial data $p_{\mathrm{in}} \in H^N$ for $N \geqslant 5$. Then there exists a time T such that there exists a unique solution $p(t) \in H^N$ to (17) for all $t \in [0,T]$.*

Proof. We prove existence with the Banach fixed-point theorem. Let $T = 1 + 2\|p_{\mathrm{in}}\|_{H^N}(1+\frac{8}{\kappa})$ and let X be the space

$$X = \left\{ p \in L^\infty H^N \cap CH^{N-2} : \ p(t=0) = p_{\mathrm{in}}, \ \|p\|^2_{L^\infty H^N} + \frac{\kappa}{2}\|\nabla_t p\|^2_{L^\infty H^N} \leqslant 2\|p_{\mathrm{in}}\|^2_{H^N} \right\}$$

with the norm

$$\|p\|^2_X := \|p\|^2_{L^\infty H^N} + \frac{\kappa}{2}\|\nabla_t p\|^2_{L^\infty H^N}.$$

We define $F: X \mapsto X$ as a mapping $q \mapsto p = F(q)$ such that p solves

$$\begin{aligned}
&\partial_t p_1 - \partial_x \partial_y^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 = \nu \Delta_t p_1 + \Lambda_t^{-1} \nabla_t^{\perp} \left(\nabla_t^{\perp} \Lambda_t^{-1} q_2 \nabla_t b - \nabla_t^{\perp} \Lambda_t^{-1} q_1 \nabla_t v \right), \\
&\partial_t p_2 + \partial_x \partial_y^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 = \kappa \Delta_t p_2 + \Lambda_t^{-1} \nabla_t^{\perp} \left(\nabla_t^{\perp} \Lambda_t^{-1} q_2 \nabla_t v - \nabla_t^{\perp} \Lambda_t^{-1} q_1 \nabla_t b \right), \\
&p|_{t=0} = p_{\text{in}}.
\end{aligned}$$

Then the mapping F satisfies:

(1) The mapping $F: X \to X$ is well defined on X.
(2) The mapping F is a contraction, i.e. $\|f(p) - F(\tilde{p})\|_{L^\infty H^N} \leqslant \frac{1}{2} \|p - \tilde{p}\|_{L^\infty H^N}$.

Since X is a complete metric space, if we prove (1) and (2), then it follows that F has a unique fixpoint by the Banach fixed-point theorem.

(1) Let $q \in X$, then we obtain for $p = F(q)$

$$\begin{aligned}
\partial_t \|p\|_{H^N}^2 + \kappa \|\nabla_t p\|_{H^N}^2 &\leqslant \|p\|_{H^N}^2 + \langle \Lambda^N v, \Lambda^N \left(\nabla_t^{\perp} \Lambda_t^{-1} q_2 \nabla_t b - \nabla_t^{\perp} \Lambda_t^{-1} q_1 \nabla_t v \right) \rangle \\
&\quad + \langle \Lambda^N b, \Lambda^N \left(\nabla_t^{\perp} \Lambda_t^{-1} q_2 \nabla_t v - \nabla_t^{\perp} \Lambda_t^{-1} q_1 \nabla_t b \right) \rangle \\
&\leqslant \|p\|_{H^N}^2 + \|p\|_{H^N} \|q\|_{H^N} \|\nabla_t p^{n+1}\|_{H^N} \\
&\leqslant \|p\|_{H^N}^2 + \tfrac{2}{\kappa} \|p\|_{H^N}^2 \|q\|_{H^N}^2 + \tfrac{\kappa}{2} \|\nabla_t p\|_{H^N}^2.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\|p\|_X^2 &\leqslant \|p_{\text{in}}\|_{H^N}^2 + T \left(1 + \tfrac{1}{\kappa} \|q\|_{L^\infty H^N}^2 \right) \|p\|_{L^\infty H^N}^2 \\
&\leqslant \|p_{\text{in}}\|_{H^N}^2 + T \left(1 + \tfrac{2}{\kappa} \|p_{\text{in}}\|_{H^N}^2 \right) \|p\|_{L^\infty H^N}^2.
\end{aligned}$$

Since

$$T \left(1 + \tfrac{4}{\kappa} \|p_{\text{in}}\|_{H^N}^2 \right) < \tfrac{1}{2}$$

we infer the bound

$$\|p\|_X^2 \leqslant 2 \|p_{\text{in}}\|_{H^N}^2.$$

As $\partial_t p \in H^{N-2}$, it follows that $p \in CH^{N-2}$ and thus $p \in X$.

(2) We show that F is a contraction. Let $q, \tilde{q} \in X$ we denote $p = F(q)$ and $\tilde{p} = F(\tilde{q})$. We need to show that

$$\|p - \tilde{p}\|_X < \tfrac{1}{2} \|q - \tilde{q}\|_X,$$

by time estimate we obtain

$$\begin{aligned}
\partial_t \|p - \tilde{p}\|_{H^N}^2 + \kappa \|\nabla_t (p - \tilde{p})\|_{H^N}^2 &\leqslant \|p - \tilde{p}\|_{H^N}^2 \\
&\quad + \langle \Lambda^N (v - \tilde{v}), \Lambda^N \left(\nabla_t^{\perp} \Lambda_t^{-1} q_2 \nabla_t b - \nabla_t^{\perp} \Lambda_t^{-1} q_1 \nabla_t v \right) \rangle \\
&\quad + \langle \Lambda^N \left(b - \tilde{b} \right), \Lambda^N \left(\nabla_t^{\perp} \Lambda_t^{-1} q_2 \nabla_t v - \nabla_t^{\perp} \Lambda_t^{-1} q_1 \nabla_t b \right) \rangle
\end{aligned}$$

$$
\begin{aligned}
&-\langle\Lambda^N(v-\tilde v),\Lambda^N\left(\nabla_t^\perp\Lambda_t^{-1}\tilde q_2\nabla_t\tilde b-\nabla_t^\perp\Lambda_t^{-1}\tilde q_1\nabla_t v\right)\rangle\\
&-\langle\Lambda^N\left(b-\tilde b\right),\Lambda^N\left(\nabla_t^\perp\Lambda_t^{-1}\tilde q_2\nabla_t\tilde v-\nabla_t^\perp\Lambda_t^{-1}\tilde q_1\nabla_t\tilde b\right)\rangle\\
\leqslant{}& \|p-\tilde p\|_{H^N}^2+\|p-\tilde p\|_{H^N}\left(\|q-\tilde q\|_{H^N}\|\nabla_t\tilde p\|_{H^N}+\|q\|_{H^N}\|\nabla_t(p-\tilde p)\|_{H^N}\right)\\
\leqslant{}& \|p-\tilde p\|_{H^N}^2\left(1+\tfrac{2}{\kappa}\|q\|_{H^N}^2\right)\\
&+\|p-\tilde p\|_{H^N}\|q-\tilde q\|_{H^N}\|\nabla_t\tilde p\|_{H^N}+\tfrac{\kappa}{2}\|\nabla_t(p-\tilde p)\|_{H^N}^2.
\end{aligned}
$$

Integrating in time yields

$$
\begin{aligned}
\|p-\tilde p\|_{L^\infty H^N}^2&+\tfrac{\kappa}{2}\|\nabla_t(p-\tilde p)\|_{L^2H^N}^2\\
&\leqslant \|p-\tilde p\|_{L^\infty H^N}^2 T\left(1+\tfrac{2}{\kappa}\|q\|_{L^\infty H^N}^2\right)\\
&\quad+\sqrt{T}\|p-\tilde p\|_{L^\infty H^N}\|q-\tilde q\|_{L^\infty H^N}\|\nabla_t\tilde p\|_{L^2H^N}\\
&\leqslant \|p-\tilde p\|_{L^\infty H^N}^2 T\left(1+\tfrac{2}{\kappa}\|q\|_{L^\infty H^N}^2+4\|\nabla_t\tilde p\|_{L^2H^N}^2\right)\\
&\quad+\tfrac{T}{4}\|q-\tilde q\|_{L^\infty H^N}^2.
\end{aligned}
$$

Choosing T such that

$$
T\left(1+\tfrac{2}{\kappa}\|q\|_{H^N}^2+\|\nabla_t\tilde p\|_{L^2H^N}^2\right)\leqslant T+2T\|p_{\mathrm{in}}\|_{H^N}\left(1+\tfrac{8}{\kappa}\right)<\tfrac{1}{2},
$$

it follows, that

$$
\|p-\tilde p\|_X^2\leqslant\tfrac{1}{2}\|p-\tilde p\|_{H^N}+\tfrac{T}{4}\|q-\tilde q\|_{L^\infty H^N}^2.
$$

We hence conclude, that

$$
\|p-\tilde p\|_X^2\leqslant\tfrac{1}{2}\|q-\tilde q\|_X^2.
$$

□

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