Published for SISSA by Springer

Received: May 7, 2025
Accepted: June 10, 2025
Published: July 11, 2025

Integral of the double-emission eikonal function for a massive and a massless emitter at an arbitrary angle

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Abstract: We present an analytic calculation of the integrated double-emission eikonal function of a massive and a massless emitter whose momenta are at an arbitrary angle to each other. This quantity provides one of the required ingredients for extending the nested soft-collinear subtraction scheme to processes with massive final-state particles. To calculate it, we use the standard methodology involving reverse unitarity and its extension to cases with Heaviside functions, integration-by-parts technology and reduction to master integrals, and differential equations. In addition, we also describe a semi-numerical method based on the subtraction of infra-red and collinear singularities from the eikonal function, allowing us to extract divergences of the integrated eikonal function analytically, and to derive a simple integral representation for the finite remainder.

Keywords: Factorization, Renormalization Group, Higher-Order Perturbative Calculations, Jets and Jet Substructure, Resummation

ArXiv ePrint: 2504.20977

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Article funded by SCOAP^3.

https://doi.org/10.1007/JHEP07(2025)138

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1 Introduction

Perturbative predictions for various LHC processes have been extended to include next-to-next-to-leading order QCD corrections in recent years [1–23]. This progress was made possible by improvements in understanding how to compute two-loop amplitudes, both analytically [24–29] and numerically [30–34], and by continuous refinements of the NNLO subtraction and slicing schemes [35–53] with the aim of making them efficient and applicable to arbitrary processes.

For the subtraction schemes, one requires integrals of universal soft- and collinear limits of squared QCD amplitudes, over a phase space of unresolved partons. Making this statement precise is equivalent to specifying a particular subtraction scheme, but the need

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for such "integrated subtraction terms" is scheme-independent. In this paper we consider an integrated double-emission eikonal function in the context of the nested soft-collinear subtraction scheme [43] in a situation where one of the emitters is massive and the other one is massless, and their momenta are at an arbitrary angle to each other. We note that a similar computation for two massless emitters whose momenta are at an arbitrary angle to each other was performed in ref. [54], and for massive back-to-back emitters — in ref. [55]. To extend the nested soft-collinear subtraction scheme to processes with arbitrary number of massless and massive particles in the final state, one needs the integrated double-emission eikonal functions for the massless-massive and massive-massive emitters, which are at an arbitrary angle to each other. The goal of this paper is to present the required calculation for the massless-massive case.

We perform this calculation in two complementary ways. First, we show how to organize it using reverse unitarity [56] and its generalization for integrals with Heaviside functions [57]. Reverse unitarity enables the use of multi-loop technology, involving derivation of linear algebraic identities between different integrals with the help of the integration-by-parts (IBP) method [58, 59], reduction of many integrals contributing to the integrated eikonal to a few master integrals, and solving differential equations that these integrals satisfy. Although this methodology allows us to compute the integral of the double-emission eikonal function analytically, the calculation is quite challenging. Furthermore, a striking feature of the calculation is that results for integrals that appear at the intermediate stages are much more complex than the final result for the integrated double-emission eikonal function. For example, the system of differential equations for the master integrals that we require involves elliptic integrals which, however, cancel out in the final result for the integrated eikonal.

To shed light on this simplicity, we applied a different method to integrate the eikonal function which relies on constructing subtraction terms for the eikonal function itself. This approach allows us to extract all singular contributions from the integrated eikonal and devise a simple representation for the finite remainder that we then integrate numerically. It is based on the same idea that was used to compute the NNLO QCD contribution to the N-jettiness soft function in ref. [60], although there are important differences between the two cases at the practical level.

The paper is organized as follows. In section 2 we introduce the eikonal functions, as well as other quantities and notations that we need in the rest of the paper. In section 3 single-emission integrals and their iterations are discussed. In section 4 the analytic calculation of the integrated double-emission eikonal function is reported. In section 5 an alternative semi-numerical computation of the same quantity is discussed. We conclude in section 6. Various technical details, definitions of useful quantities, as well as instructions on how to use the supplementary material files provided with this paper are collected in appendices.

2 Notations and definitions

We study a generic partonic process

$$0 \to h_1(p_1) + \cdots + h_n(p_n) + H_{n+1}(p_{n+1}) + \cdots + H_N(p_N) + f_1(k_1) + f_2(k_2), \tag{2.1}$$

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where h_i and H_i are massless and massive partons, and $f_{1,2}$ are two massless, potentially unresolved partons which can be either two gluons or a $q\bar{q}$ pair. We then consider the double-soft limit, $k_1, k_2 \to 0$, with all other momenta in eq. (2.1) fixed. In this limit, the amplitude squared of the process in eq. (2.1) factorizes as follows [61]:

- if $f_{1,2}$ are gluons,

$$\lim_{k_1,k_2\to 0} |\mathcal{M}^{gg}(\{p\},k_1,k_2)|^2 \approx g_{s,b}^4 \Bigg\{ \frac{1}{2} \sum_{i,j,k,l}^{N} \mathcal{S}_{ij}(k_1)\mathcal{S}_{kl}(k_2)|\mathcal{M}^{\{(ij),(kl)\}}(\{p\})|^2 - C_A \sum_{i,j}^{N} \mathcal{S}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 \Bigg\}, \tag{2.2}$$

- if $f_1 = q$ and $f_2 = \bar{q}$,

$$\lim_{k_1,k_2\to 0} |\mathcal{M}^{q\bar{q}}(\{p\},k_1,k_2)|^2 \approx g_{s,b}^4 \, T_R \sum_{i,j}^{N} \mathcal{I}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 \,. \tag{2.3}$$

Quantities that appear in the above equations include two Casimir operators of the SU(3) group, $C_A = 3, T_R = 1/2$, the bare strong coupling constant $g_{s,b}$, as well as color-correlated matrix elements

$$|\mathcal{M}^{\{(ij),(kl)\}}(\{p\})|^2 = \langle \mathcal{M}(\{p\})|\{\boldsymbol{T}_i\cdot\boldsymbol{T}_j, \boldsymbol{T}_k\cdot\boldsymbol{T}_l\}|\mathcal{M}(\{p\})\rangle\,, \tag{2.4}$$

$$|\mathcal{M}^{\{(ij)\}}(\{p\})|^2 = \langle \mathcal{M}(\{p\})|\boldsymbol{T}_i\cdot\boldsymbol{T}_j|\mathcal{M}(\{p\})\rangle\,, \tag{2.5}$$

where $\boldsymbol{T}_i$ are operators of color charges [62] and $\{\cdot,\cdot\}$ denotes an anti-commutator. Sums in eqs. (2.2), (2.3) run over all pairs of hard color-charged emitters.

In eq. (2.2), the term containing the product of two single-eikonal factors

$$\mathcal{S}_{ij}(k) = \frac{(p_i\cdot p_j)}{(p_i\cdot k)(p_j\cdot k)}\,, \tag{2.6}$$

is the *Abelian* contribution. We note that $S_{ij}(k)$ also appears in the single-emission eikonal contribution relevant for computations at next-to-leading order.

The *non-Abelian* term, proportional to the color factor C_A, is more complicated. The eikonal function $\mathcal{S}_{ij}(k_1,k_2)$ reads

$$\mathcal{S}_{ij}(k_1,k_2) = \mathcal{S}_{ij}^0(k_1,k_2) + \left[m_i^2\mathcal{S}_{ij}^m(k_1,k_2) + m_j^2\mathcal{S}_{ji}^m(k_1,k_2)\right], \tag{2.7}$$

where quantities that appear in square brackets depend on the masses of the two emitters, $m_{i,j}$. Although the explicit dependence of the function S_{ij} on emitters' masses is shown in eq. (2.7), both functions $\mathcal{S}_{ij}^0(k_1,k_2)$ and $\mathcal{S}_{ij}^m(k_1,k_2)$ implicitly depend on them through their momenta $p_{i,j}$, $p_{i,j}^2 = m_{i,j}^2$.

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The first term in eq. (2.7), $\mathcal{S}_{ij}^0(k_1,k_2)$ is the same for massless and massive emitters [61]. It reads

$$\begin{aligned}\mathcal{S}_{ij}^0(k_1,k_2) = &\frac{(1-\varepsilon)}{(k_1\cdot k_2)^2}\frac{[(p_i\cdot k_1)(p_j\cdot k_2)+i\leftrightarrow j]}{(p_i\cdot k_{12})(p_j\cdot k_{12})}\\ &-\frac{(p_i\cdot p_j)^2}{2(p_i\cdot k_1)(p_j\cdot k_2)(p_i\cdot k_2)(p_j\cdot k_1)}\left[2-\frac{[(p_i\cdot k_1)(p_j\cdot k_2)+i\leftrightarrow j]}{(p_i\cdot k_{12})(p_j\cdot k_{12})}\right]\\ &+\frac{(p_i\cdot p_j)}{2(k_1\cdot k_2)}\left[\frac{2}{(p_i\cdot k_1)(p_j\cdot k_2)}+\frac{2}{(p_j\cdot k_1)(p_i\cdot k_2)}-\frac{1}{(p_i\cdot k_{12})(p_j\cdot k_{12})}\right.\\ &\quad\left.\times\left(4+\frac{[(p_i\cdot k_1)(p_j\cdot k_2)+i\leftrightarrow j]^2}{(p_i\cdot k_1)(p_j\cdot k_2)(p_i\cdot k_2)(p_j\cdot k_1)}\right)\right],\end{aligned} \tag{2.8}$$

where we have used the abbreviation $k_{12}=k_1+k_2$. The other two contributions in eq. (2.7) are only relevant for massive emitters. The function $\mathcal{S}_{ij}^m(k_1,k_2)$ is given by [63][1]

$$\begin{aligned}\mathcal{S}_{ij}^m(k_1,k_2) = &\frac{(p_i\cdot p_j)(p_j\cdot k_{12})}{2(p_i\cdot k_1)(p_j\cdot k_2)(p_i\cdot k_2)(p_j\cdot k_1)(p_i\cdot k_{12})}\\ &-\frac{1}{2(k_1\cdot k_2)(p_i\cdot k_{12})(p_j\cdot k_{12})}\left(\frac{(p_j\cdot k_1)^2}{(p_i\cdot k_1)(p_j\cdot k_2)}+\frac{(p_j\cdot k_2)^2}{(p_i\cdot k_2)(p_j\cdot k_1)}\right).\end{aligned} \tag{2.9}$$

In the quark-antiquark case, the eikonal function $\mathcal{I}_{ij}(k_1,k_2)$ reads

$$\mathcal{I}_{ij}(k_1,k_2) = \frac{[(p_i\cdot k_1)\,(p_j\cdot k_2)+i\leftrightarrow j]-(p_i\cdot p_j)\,(k_1\cdot k_2)}{(k_1\cdot k_2)^2\,(p_i\cdot k_{12})\,(p_j\cdot k_{12})}\,, \tag{2.10}$$

and there is no difference between massive and massless emitters.

It turns out to be convenient to make use of color conservation

$$\sum_{i=1}^{N}\boldsymbol{T}_i|\mathcal{M}(\{p\})\rangle = 0, \tag{2.11}$$

and the symmetry of functions $S_{ij}=S_{ji}$ and $I_{ij}=I_{ji}$ to write

$$\sum_{i,j}^{N}\mathcal{S}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 = \sum_{i<j}^{N}\widetilde{\mathcal{S}}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2, \tag{2.12}$$

$$\sum_{i,j}^{N}\mathcal{I}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 = \sum_{i<j}^{N}\widetilde{\mathcal{I}}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2, \tag{2.13}$$

where

$$\widetilde{\mathcal{S}}_{ij} = 2\mathcal{S}_{ij}-\mathcal{S}_{ii}-\mathcal{S}_{jj}, \tag{2.14}$$

$$\widetilde{\mathcal{I}}_{ij} = 2\mathcal{I}_{ij}-\mathcal{I}_{ii}-\mathcal{I}_{jj}. \tag{2.15}$$

Functions $\tilde{S}_{ij}, \tilde{I}_{ij}$ are more suitable for the analysis presented below, especially in section 5, because they only have physical singularities for each pair i,j.

[1]We note that a somewhat different expression for S_{ij}^m is given in ref. [41]. However, both expressions give the same result after summing over i,j in eqs. (2.2), (2.3) thanks to colour conservation.

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To compute the required double-soft contribution, we have to integrate the corresponding eikonal functions $\tilde{S}_{ij}$ and $\tilde{I}_{ij}$ over the phase space of two unresolved partons with momenta $k_{1,2}$. Working within the nested soft-collinear subtraction scheme, we have to fix the reference frame and restrict energies of unresolved partons by introducing an upper cut-off $E_{\max}$. Furthermore, energies of unresolved partons must be ordered. We call the parton with the larger (smaller) energy $\mathfrak{m}(\mathfrak{n})$, and refer to their momenta as $k_{\mathfrak{m},\mathfrak{n}}$, instead of $k_{1,2}$, which describe momenta without energy ordering.

We define the required double-emission phase-space integrals as [43]

$$\mathcal{S}\left[\mathcal{S}_{ij}\mathcal{S}_{kl}\right] = \int [\mathrm{d}k_\mathfrak{m}][\mathrm{d}k_\mathfrak{n}]\theta\left(E_{\max} - k_\mathfrak{m}^0\right)\theta\left(k_\mathfrak{m}^0 - k_\mathfrak{n}^0\right)\mathcal{S}_{ij}(k_\mathfrak{m})\mathcal{S}_{kl}(k_\mathfrak{n}), \tag{2.16}$$

$$\mathcal{S}\left[\Xi_{ij}\right] = \int [\mathrm{d}k_\mathfrak{m}][\mathrm{d}k_\mathfrak{n}]\theta\left(E_{\max} - k_\mathfrak{m}^0\right)\theta\left(k_\mathfrak{m}^0 - k_\mathfrak{n}^0\right)\Xi_{ij}\left(k_\mathfrak{m}, k_\mathfrak{n}\right), \tag{2.17}$$

where the eikonal function Ξ_{ij} is either $\tilde{\mathcal{S}}_{ij}$ (for gg emission) or $\tilde{\mathcal{I}}_{ij}$ (for $q\bar{q}$ emission),

$$[\mathrm{d}k] = \frac{\mathrm{d}^{d-1}k}{2k^0(2\pi)^{d-1}}, \tag{2.18}$$

is the phase-space element and d is the space-time dimension $d = 4 - 2\varepsilon$.[2]

The dependence of soft integrals on $E_{\max}$ can be made manifest. We use eq. (2.17) to demonstrate it. We use the integral representation for the first theta-function

$$\theta(b-a) = \int_0^1 \mathrm{d}z\ \delta(zb-a)\ b, \tag{2.19}$$

and obtain

$$\mathcal{S}\left[\Xi_{ij}\right] = \int_0^1 \frac{\mathrm{d}z}{z}\int [\mathrm{d}k_\mathfrak{m}][\mathrm{d}k_\mathfrak{n}]\delta\left(1 - \frac{k_\mathfrak{m}^0}{zE_{\max}}\right)\theta\left(k_\mathfrak{m}^0 - k_\mathfrak{n}^0\right)\Xi_{ij}\left(k_\mathfrak{m}, k_\mathfrak{n}\right). \tag{2.20}$$

Since the eikonal functions are homogeneous, i.e. $\Xi_{ij}(\lambda k_\mathfrak{m}, \lambda k_\mathfrak{n}) = \lambda^{-4}\Xi_{ij}(\{k_\mathfrak{m}, k_\mathfrak{n})$, and since the integration measure satisfies $[\mathrm{d}(\lambda k_i)] \to \lambda^{d-2}[\mathrm{d}k_i]$, we rescale both momenta $k_{\mathfrak{m},\mathfrak{n}}$ with $\lambda = zE_{\max}$ and find

$$\begin{aligned}\mathcal{S}\left[\Xi_{ij}\right] &= \int_0^1 \frac{\mathrm{d}z}{z}\frac{(zE_{\max})^{2(d-2)}}{(zE_{\max})^4}\int [\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]\delta\left(1 - l_\mathfrak{m}\cdot P\right)\theta\left(l_\mathfrak{m}\cdot P - l_\mathfrak{n}\cdot P\right)\Xi_{ij}\left(l_\mathfrak{m}, l_\mathfrak{n}\right) \\ &= -\frac{1}{4\varepsilon E_{\max}^{4\varepsilon}}\int [\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]\delta\left(1 - l_\mathfrak{m}\cdot P\right)\theta\left(l_\mathfrak{m}\cdot P - l_\mathfrak{n}\cdot P\right)\Xi_{ij}\left(l_\mathfrak{m}, l_\mathfrak{n}\right).\end{aligned} \tag{2.21}$$

We note that an auxiliary four-vector $P = (1, \vec{0})$ was introduced in eq. (2.21), to ensure that energies of unresolved partons are constrained. A similar formula is easily obtained for the integral of the product of two single eikonal functions $\mathcal{S}[S_{ij}S_{kl}]$.

Our goal is to compute the required soft integrals for a massive (i) and a massless (j) emitter. We label their momenta as $p_{i,j}$. If m is the mass parameter, then $p_i^2 = m^2$, and $p_j^2 = 0$. In the "laboratory frame" where $P = (1, \vec{0})$, we employ the following momenta parametrization

$$p_i = E_i(1, \beta\vec{n}_i), \qquad p_j = E_j(1, \vec{n}_j), \tag{2.22}$$

[2]We use dimensional regularization to regulate soft and collinear divergences throughout this paper.

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where the three-vectors $\vec{n}_{i,j}$ satisfy $\vec{n}_{i,j}^2 = 1$, and β is the velocity of the massive emitter

$$\beta = \sqrt{1 - \frac{m^2}{E_i^2}}. \tag{2.23}$$

In principle, the integrated eikonal function can depend on the following scalar products

$$p_i^2 = m^2, \quad P^2 = 1, \quad p_i \cdot p_j = E_i E_j (1 - \beta \cos\theta), \quad p_{i,j} \cdot P = E_{i,j}, \tag{2.24}$$

where θ is the angle between $\vec{n}_i$ and $\vec{n}_j$, i.e. $\vec{n}_i \cdot \vec{n}_j = \cos\theta$. Since the eikonal functions are homogeneous in $p_{i,j}$, the dependencies on $E_{i,j}$ cancel out, so that in practice integrals $\mathcal{S}$ are functions of β, $\cos\theta$, and the energy cut-off $E_{\max}$. The dependence on $E_{\max}$ has already been found, see eq. (2.21); thus, integrated eikonal functions have non-trivial functional dependence on β and $\cos\theta$ only.

3 Single emission integrals and their iterations

We start by considering the product of two single-soft eikonal functions,

$$\mathcal{S}\left[\mathcal{S}_{ij}\mathcal{S}_{kl}\right] = -\frac{1}{4\varepsilon E_{\max}^{4\varepsilon}} \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}]\delta\left(1 - l_{\mathfrak{m}} \cdot P\right) \theta\left(l_{\mathfrak{m}} \cdot P - l_{\mathfrak{n}} \cdot P\right) \mathcal{S}_{ij}(l_{\mathfrak{m}}) \mathcal{S}_{kl}(l_{\mathfrak{n}}). \tag{3.1}$$

We resolve the constraint imposed by the θ-function by writing

$$l_{\mathfrak{n}} \cdot P = t \; l_{\mathfrak{m}} \cdot P, \quad 0 < t < 1. \tag{3.2}$$

Integration over t and the energy component $P \cdot l_{\mathfrak{m}}$ of parton $\mathfrak{m}$ is elementary. We obtain

$$\mathcal{S}\left[\mathcal{S}_{ij}(k_{\mathfrak{m}})\mathcal{S}_{kl}(k_{\mathfrak{n}})\right] = \frac{\mathcal{G}_{ij}\,\mathcal{G}_{kl}}{8\varepsilon^2 E_{\max}^{4\varepsilon}}, \tag{3.3}$$

where $\mathcal{G}_{ij}, \mathcal{G}_{kl}$ are angular integrals which we write in the following way ($\mathcal{G}_{kl}$ is the same function up to $p_i \to p_k$, $p_j \to p_l$ reassignment),

$$\mathcal{G}_{ij} = \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}}{2\,(2\pi)^{d-1}} \frac{1 - \beta\cos\theta}{(1 - \beta\vec{n}_i \cdot \vec{n}_{\mathfrak{m}})(1 - \vec{n}_j \cdot \vec{n}_{\mathfrak{m}})}, \tag{3.4}$$

where integration over directions of the $(d-1)$-dimensional unit vector $\vec{n}_{\mathfrak{m}}$ is to be performed. We note that this integral can be expressed through an Appell function to all orders in ε [64], see appendix C. However, below we explain how to compute it using multi-loop methodology including integration-by-parts, reduction to master integrals and differential equations.

To compute the integral in eq. (3.4), we employ reverse unitarity [56] and other standard methods for multi-loop computations, such as integration-by-parts [58, 59] and differential equations [65–67]. To this end, we consider a family of integrals[3]

$$I(a_1, a_2) \equiv \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}}{2\,(2\pi)^{d-1}} \frac{(1 - \beta\cos\theta)}{(1 - \beta\vec{n}_i \cdot \vec{n}_{\mathfrak{m}})^{a_1}(1 - \vec{n}_j \cdot \vec{n}_{\mathfrak{m}})^{a_2}}\,, \quad a_i \in \mathbb{Z}, \tag{3.5}$$

[3]As we will see in section 5, we need to consider the case where the power of the "propagators" becomes ε-dependent. In general, such integrals can be expressed through Appell functions, see appendix C. In this section, we focus on integer powers of propagators only.

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that generalize $\mathcal{G}_{ij} = I(1,1)$. We would like to derive the differential equation for $I(1,1)$ with respect to β. To do this, we write the solid angle element as

$$d\Omega_{\mathfrak{m}}^{(d-1)} = 2\,d^d l_{\mathfrak{m}}\, \delta^+(l_{\mathfrak{m}}^2)\, \delta(1 - l_{\mathfrak{m}} \cdot P)\,, \tag{3.6}$$

and proceed with using reverse unitarity to rewrite δ-functions as "propagators". This leads to the differential equation

$$\frac{\partial I(1,1)}{\partial \beta} = \frac{1}{1-\beta\cos\theta}\left[\frac{1-2\varepsilon}{(1-\beta^2)\beta}\, I(0,0) + \frac{2\varepsilon\beta}{1-\beta^2}\, I(1,0) + \frac{2\varepsilon}{\beta}\, I(1,1)\right]. \tag{3.7}$$

Calculating integrals $I(0,0)$ and $I(1,0)$, and choosing the point $\beta = 0$ to compute the boundary condition,[4] we calculate $I(1,1)$ as a series expansion in ε. We find

$$I(1,1) = \frac{(1-2\varepsilon)}{\varepsilon}\mathcal{N}_\varepsilon \sum_{n=0} I_{11}^{(n)} \varepsilon^n, \tag{3.8}$$

where the normalization factor is

$$\mathcal{N}_\varepsilon = \frac{\pi^\varepsilon}{4\pi^2}\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}, \tag{3.9}$$

and the first three coefficients in eq. (3.8) read

$$\begin{aligned}
I_{11}^{(0)} &= -\frac{1}{2},\\
I_{11}^{(1)} &= \log\left(\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\right),\\
I_{11}^{(2)} &= -\left[\frac{1}{4}\log^2\left(\frac{1-\beta}{1+\beta}\right) + \log\left(\frac{1-\beta\cos\theta}{1+\beta}\right)\log\left(\frac{1-\beta\cos\theta}{1-\beta}\right)\right.\\
&\quad \left. + \mathrm{Li}_2\left(1-\frac{1-\beta\cos\theta}{1+\beta}\right) + \mathrm{Li}_2\left(1-\frac{1-\beta\cos\theta}{1-\beta}\right)\right].
\end{aligned} \tag{3.10}$$

We note that they agree with the results reported in refs. [68, 69]. With these expressions, it is straightforward to compute the soft integral in eq. (3.3). Judging from eq. (3.3), it may appear that terms up to $I_{11}^{(4)}$ will be required to obtain the finite contribution for the integrated double-soft subtraction term. This, however, may not be necessary since $\mathbb{S}[S_{ij}S_{kl}]$ will get combined with the iterated Catani's operator $\boldsymbol{I}_1$ [62, 70] which should ensure cancellation of $1/\varepsilon$ singularities in the iterated structure, reducing the depth of the ε-expansion in eq. (3.8) required to obtain $\mathcal{O}(\varepsilon^0)$ contributions to physical quantities. Nevertheless, for completeness, we also include higher order ε-expansion terms in eq. (3.8) in the supplementary material files provided with this paper.

4 Integrating double-emission eikonal function in the laboratory frame

We continue with the discussion of the non-Abelian contribution to the integrated double-emission eikonal function. Compared to the iterated single-emission case discussed in section 3,

[4]We note that at $\beta = 0$, the dependence of the integral on $\cos\theta$ disappears.

its calculation is much more involved. For this reason, it is important to design an efficient toolchain for calculating integrals appearing in eq. (2.21). The plan is to map these integrals on a small set of master integrals, and use differential equations to calculate them. We note that a similar approach was successfully used in calculations with two massless emitters at an arbitrary angle [54], and with two massive emitters in the back-to-back limit [55]. However, we will see that it remains challenging to utilize it in the massive-massless case.

4.1 IBP reduction and master integrals

The first step in making the computation of the integrated double-emission eikonal function defined in eq. (2.21) systematic is to classify contributing integrals. Since eikonal functions contain linear propagators, we perform partial fractioning and express all integrals through the following ones

$$\Theta^{\mathtt{def}}_{\mathtt{abc}}(g) = \int \frac{[\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}]\delta\left(1 - l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P - l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)^a(l_{\mathfrak{n}}\cdot p_i)^b(l_{\mathfrak{mn}}\cdot p_i)^c(l_{\mathfrak{m}}\cdot p_j)^d(l_{\mathfrak{n}}\cdot p_j)^e(l_{\mathfrak{mn}}\cdot p_j)^f(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})^g}, \tag{4.1}$$

where $l_{\mathfrak{mn}} = l_{\mathfrak{m}} + l_{\mathfrak{n}}$. The corresponding integrands contain quadratic and linear propagators, as well as constraints provided by delta- and theta-functions. We note that integrals without theta-functions, can be expressed through master integrals using publicly available codes [71, 72]. However, if integrands do contain theta-functions, a generalization of this approach is required.

There are several ways to achieve this. For example, one can write a theta-function as an integral of a delta-function, cf. eq. (2.19), and employ generalized unitarity to deal with integrals that only contain delta-function constraints afterwards. The drawback of this approach is that the integrals that we need to calculate become functions of three, rather than two variables. Although the reduction to master integrals and the derivation of differential equations are possible with this approach, we find that it leads to over-complicated intermediate results.

Thankfully, we do not need to follow this approach because there exists an alternative method for dealing with real-emission integrals that contain Heaviside functions. It was developed in ref. [57], and successfully applied in the calculation of N3LO zero-jettiness soft function [73–76].

This method is based on a simple observation that action of the differential operator on the integrand — a crucial step in establishing the IBP methodology [58, 59] — can be generalized to include integrands with theta-functions. The key to this generalization is the following equation

$$\frac{\partial}{\partial s}\theta(f(s)) = \delta(f(s))\frac{\partial}{\partial s}f(s), \tag{4.2}$$

where s is a scalar product constructed from internal and external momenta, and a typical function $f(s)$ is a linear combination of scalar products between various four-momenta.

There are two important consequences of eq. (4.2). First, each application of eq. (4.2) replaces a theta-function with a delta-function, and it is impossible to express such integrals through Θ-integrals defined in eq. (4.1). This makes the required reduction *inhomogeneous*,

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and we need to extend the set of considered integrals by including integrals with two delta-functions

$$\Delta^{\mathtt{def}}_{\mathtt{abc}}(g) = \int \frac{[\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}]\delta\left(1 - l_{\mathfrak{m}}\cdot P\right)\delta\left(1 - l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)^a(l_{\mathfrak{n}}\cdot p_i)^b(l_{\mathfrak{mn}}\cdot p_i)^c(l_{\mathfrak{m}}\cdot p_j)^d(l_{\mathfrak{n}}\cdot p_j)^e(l_{\mathfrak{mn}}\cdot p_j)^f(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})^g}. \tag{4.3}$$

Since such integrals do not contain Heaviside functions, they can be dealt with using any public code for IBP reduction.

Second, once a new delta-function appears in the integrand and is mapped on a new propagator as required by reverse unitarity, it is not guaranteed that this propagator is linearly-independent of the ones that are already present in an integral. To get rid of linearly-dependent propagators, we perform the partial fraction decomposition, each time eq. (4.2) is applied.

Although these additional reduction steps are relatively simple when taken separately, they require significant amount of work to be implemented in a working code. Luckily, many of these steps were designed and implemented for the calculation of the N3LO zero-jettiness soft function described in ref. [74]. Using the notion introduced in that paper to describe the present calculation, we need to consider two reduction levels. First, we express the original integrals from the eikonal function through the minimal set of Θ-integrals defined in eq. (4.1), and then proceed with the level-zero reduction of Δ-integrals shown in eq. (4.3), which is done with `Kira` [72]. After the reduction, we find that we require 52 master integrals including 37 of the Δ type and 15 of the Θ type. The full list of master integrals can be found in appendix A.

We note that the application of IBP identities has not resulted in a significant reduction in the number of integrals that need to be computed. Nevertheless, the reduction is very useful because it allows us to control the complexity of integrals that we choose to calculate. Furthermore, an opportunity to express any integral in eqs. (4.1) and (4.3) through master integrals is crucial for deriving differential equations that these integrals satisfy. Indeed, after differentiating an integrand of a particular integral with respect to one of the variables $x = \beta$ or $y = \cos\theta$, the obtained integrals can be expressed through master integrals again. This leads to differential equations

$$\partial_x \vec{J}(x,y) = M_x \vec{J}(x,y), \quad \partial_y \vec{J}(x,y) = M_y \vec{J}(x,y), \tag{4.4}$$

where $\vec{J}(x,y)$ is a vector of all master integrals that need to be computed and $M_{x,y}$ are matrices that satisfy the integrability condition

$$\partial_x M_y - \partial_y M_x = [M_x, M_y]\,. \tag{4.5}$$

Furthermore, linear IBP relations allow construction of the recurrence relations with respect to space-time dimension d. They read

$$\vec{J}(d) = L(d)\vec{J}(d+2), \tag{4.6}$$

thereby connecting integrals computed in d and $d+2$ dimensions. Eq. (4.6) is quite useful for numerical checks, since real-emission integrals that we are interested in become convergent for sufficiently large d, and can be calculated by a straightforward integration, see appendix D for details.

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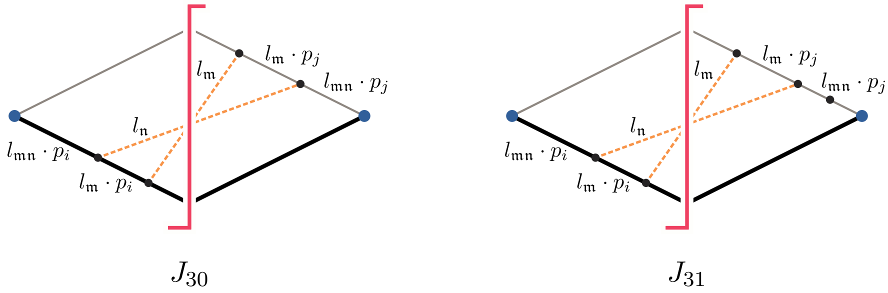

Figure 1. Integrals from the sector, which is not first-order factorizable. Bold (light) solid lines are massive (massless) eikonal propagators; vertical line represent a cut. Particles whose propagators are cut are considered on-shell.

4.2 Solution of the differential equations

In this section we explain how the differential equations in eq. (4.4) are solved. Following common practices, we aim at constructing solutions in terms of iterated integrals. The best way to do this is to bring the system into a global ε-form [67], by changing the basis of integrals if possible. The next-to-best option is to find a basis where matrices $M_{x,y}$ are block-triangular, which means that each matrix block is a lower-triangular matrix such that all of its diagonal entries are proportional to ε. For such matrices $M_{x,y}$, integrals $\vec{J}$ at a particular order in the ε-expansion depend on their lower-expansion orders, and simpler integrals.

Following methods described in ref. [77] it is possible to find a new set of master integrals such that all *but two* diagonal blocks of the system of differential equations are in the required form. We note that the transformation from the old to the new basis involves several complicated square roots.

Finding integrals that appear in the remaining two sectors is one of the major difficulties of this calculation, and we explain below how this challenge is addressed. The two irreducible blocks contain four integrals, that we refer to as $J_{30,31}$ and $J_{34,35}$; they can be found in appendix A. The homogeneous parts of the matrices $M_{x,y}$ for these blocks are identical; hence, we focus here on the first pair of integrals shown in figure 1.

For these integrals the second order differential operator (the Picard-Fuchs operator) constructed from the homogeneous part of the differential equation that $J_{30,31}$ satisfy, cannot be factorized into two first order differential operators. This implies that the solution of the system might involve elliptic integrals [77, 78]. It is useful to consider differential equations in a single variable x, and write them as follows

$$\partial_x \begin{pmatrix} J_{30} \\ J_{31} \end{pmatrix} = \sum_{i=0}^{2} \varepsilon^i A_i \begin{pmatrix} J_{30} \\ J_{31} \end{pmatrix} + B\vec{J}_s, \tag{4.7}$$

where A_i are ε-independent matrices of the diagonal block and the matrix B describes the inhomogeneous part of the differential equation. Integrals $\vec{J}_s$ are supposed to be known at this point.

Our goal is to construct a transformation to a new set of integrals, satisfying the system of equations shown in eq. (4.7), but without the ε-independent contribution. To accomplish

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this, we focus on the homogeneous part of eq. (4.7) and start by truncating it at $\mathcal{O}(\varepsilon^0)$. The matrix A_0 reads

$$A_0 = \begin{pmatrix} \frac{1}{2(1-x)} - \frac{1}{x} - \frac{1}{2(1+x)}, & -\frac{2}{x} - \frac{1-y}{1-x} + \frac{1+y}{1+x} \\ \frac{1}{4(1-x)(1-y)} - \frac{1}{4(1+x)(1+y)} - \frac{y}{2(1-y^2)(1-xy)}, & -\frac{1}{2(1-x)} + \frac{1}{2(1+x)} - \frac{1}{x-y} + \frac{2y}{1-xy} \end{pmatrix}. \tag{4.8}$$

We use it to turn a system of first-order differential equations into a single second order differential equation. It reads

$$\partial_x^2 h + p_1(x,y)\partial_x h + p_0(x,y)h = 0, \tag{4.9}$$

where $h = h(x,y)$ is a "homogeneous" version of the integral J_{30}, and the coefficients $p_{1,0}(x,y)$ are given by

$$\begin{aligned} p_1(x,y) &= \frac{y}{xy-1} + \frac{1}{x-y} + \frac{1}{x-1} + \frac{2}{x} + \frac{1}{x+1}, \\ p_0(x,y) &= \frac{y^2}{xy-1} - \frac{y}{x} + \frac{1}{x(x-y)} + \frac{1}{2(x-1)(x-y)} + \frac{1}{2(x+1)(x-y)} \\ &\quad + \frac{1}{2(x-1)(xy-1)} - \frac{1}{2(x+1)(xy-1)} + \frac{3}{2(x-1)} - \frac{3}{2(x+1)}. \end{aligned} \tag{4.10}$$

To simplify eq. (4.9), we define a new function

$$g(x,y) = x(1-xy)h(x,y), \tag{4.11}$$

and re-write the differential equation through a variable λ defined as

$$\lambda = \frac{(1-x^2)(1-y^2)}{(1-xy)^2}. \tag{4.12}$$

The transformed equation reads

$$4\lambda(1-\lambda)\,\partial_\lambda^2 g + 4(1-2\lambda)\,\partial_\lambda g - g = 0. \tag{4.13}$$

This equation is the second order differential equation that defines the complete elliptic integral of the first kind. It has two independent solutions $g(\lambda) = K(\lambda)$ and $g(\lambda) = K(1-\lambda)$ where $K(\lambda)$ is defined as follows

$$K(\lambda) = \int\limits_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\lambda t^2}}. \tag{4.14}$$

Interestingly, the parameter λ can be written as a sine squared of an angle σ which appears to be an angle between directions of vectors $\vec{P}$ and $\vec{p}_j$ *in the rest frame of the massive parton* p_i. We note that we use the rest frame of the parton i to set up a semi-numerical computation of the integrated double-emission eikonal function in section 5.

Having obtained the homogeneous solutions of the second-order equation, we follow ref. [78] and attempt to find a new set of master integrals J'_{30}, J'_{31} with the help of the following ansatz

$$J'_{30} = \frac{J_{30}}{\psi}\varepsilon^2, \quad J'_{31} = C_1(x,y)\frac{\psi^2}{\varepsilon}(\partial_x J'_{30}) + C_2(x,y)\psi^2 J'_{30}, \tag{4.15}$$

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where ψ is one of the solutions of eq. (4.9) that we take to be $\psi = K(\lambda)/x(1-xy)$. We note that this equation is a linear map

$$\begin{pmatrix} J'_{30} \\ J'_{31} \end{pmatrix} = \hat{T} \begin{pmatrix} J_{30} \\ J_{31} \end{pmatrix}, \tag{4.16}$$

where $\hat{T}$ is a two-by-two matrix with the zero entry in the upper right corner. In eq. (4.15), the integral J_{31} is hidden in the term $\partial_x J'_{30}$. Indeed, if the derivative of J'_{30} is computed explicitly, the term $\partial_x J_{30}$ appears; we replace this term using eq. (4.7) omitting the inhomogeneous term $B\vec{J}_s$ on the right-hand side. Since the two coefficients $C_{1,2}$ in eq. (4.15) are still arbitrary, no information is lost when using this map.

By requiring the ε-form of the new homogeneous block, the unknown coefficients $C_{1,2}$ can be fixed. We find

$$\begin{aligned} C_1(x,y) &= x^2(x-1)(x+1)(x-y)(xy-1), \\ C_2(x,y) &= x^4y^2 - 2x^2y^2 - 2x^3y + 2x^4 - 3x^2 + 4xy. \end{aligned} \tag{4.17}$$

The matrix T in eq. (4.16) evaluates to

$$\hat{T} = \begin{pmatrix} -\frac{x\varepsilon^2(xy-1)}{K(\lambda)}, & 0 \\ \frac{x\varepsilon^2 K(\lambda)\left(x^2y^2+2x^2-6xy+2y^2+1\right)}{1-xy} + x\varepsilon E(\lambda)(1-xy), & 2\varepsilon K(\lambda)(x-y)(xy-1) \end{pmatrix}, \tag{4.18}$$

where $E(\lambda)$ is the complete elliptic integral of the second type. The differential equation for the new integrals $J'_{30,31}$ becomes

$$\partial_x \begin{pmatrix} J'_{30} \\ J'_{31} \end{pmatrix} = \varepsilon A'_1 \begin{pmatrix} J'_{30} \\ J'_{31} \end{pmatrix} + B'\vec{J}_s, \tag{4.19}$$

where ε is factored out of the new homogeneous block, and the matrix A'_1 reads

$$A'_1 = \begin{pmatrix} -\frac{x^3y^2+2x^3-2x^2y-2xy^2-3x+4y}{(x-1)x(x+1)(x-y)(xy-1)}, & \frac{1}{\psi^2(x-1)x^2(x+1)(x-y)(xy-1)} \\ \frac{\psi^2x^2\left(x^2y^2-2x^2+2xy-2y^2+1\right)^2}{(x-1)(x+1)(x-y)(xy-1)}, & -\frac{x^3y^2+2x^3-2x^2y-2xy^2-3x+4y}{(x-1)x(x+1)(x-y)(xy-1)} \end{pmatrix}. \tag{4.20}$$

We note that A'_1 contains the function ψ and rational functions of x, y. The same transformation also applies to the second pair of master integrals, i.e., $J_{34,35}$, bringing their homogeneous block to ε-form.

Having diagonalized two elliptic blocks, we obtain the full set of differential equations

$$\begin{aligned} \partial_x \vec{J}'(x,y) &= \left(M'_{x,0} + \varepsilon M'_{x,1}\right) \vec{J}'(x,y), \\ \partial_y \vec{J}'(x,y) &= \left(M'_{y,0} + \varepsilon M'_{y,1}\right) \vec{J}'(x,y), \end{aligned} \tag{4.21}$$

where matrices $M'_{x,0}$ and $M'_{y,0}$ are lower-triangular. It is then possible to bring these systems of equations to an ε-form by rotating the integral basis with an ε-independent matrix T that satisfies

$$\partial_x T = M'_{x,0} T, \quad \partial_y T = M'_{y,0} T. \tag{4.22}$$

We use the Magnus series method [79] to find this matrix.

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After combining all transformations, the new system of differential equations is in ε-form

$$\partial_x \vec{\mathcal{J}}(x,y) = \varepsilon \widetilde{M}_x \vec{\mathcal{J}}(x,y), \quad \partial_y \vec{\mathcal{J}}(x,y) = \varepsilon \widetilde{M}_y \vec{\mathcal{J}}(x,y). \tag{4.23}$$

The matrices $\widetilde{M}_x$ and $\widetilde{M}_y$ contain

- rational functions of x and y;
- two square roots

$$R_1(y) = \sqrt{y^2-1}, \quad R_2(x,y) = \sqrt{1+8x^2-6xy+x^2y^2}, \tag{4.24}$$

  that do not couple to each other;
- the elliptic integral $K(\lambda)$, and a new function that is defined as an integral over $K(\lambda)$,

$$\mathcal{H}(x,y) = \int_0^x \mathrm{d}t \frac{(2t^2y^2+4t^2-ty^3-7ty+3y^2-1)\, K\left(\frac{(1-t^2)(1-y^2)}{(1-ty)^2}\right)}{(ty-1)R_2(t,y)^3}. \tag{4.25}$$

We solve the differential equations in eq. (4.23) in terms of iterated integrals [80]. We find that some of them involve complicated integration kernels containing square roots and elliptic integrals. However, it turns out that much of this complexity is superfluous since the results for individual integrals are much more complex than the final result for the integrated double-emission eikonal function.

Indeed, we find that all elliptic integrals disappear from the final result for the integrated massive-massless eikonal function. However, this does not happen naturally and, to reach this conclusion, it is extremely important to simplify intermediate and final expressions by carefully removing all linearly-dependent functions from the final result. Key to such simplifications is an observation that iterated integrals satisfy shuffle algebra relations, whose implementation in the `HarmonicSums` package [81] was very useful for the current calculation.

Another ingredient required for the final result are the boundary conditions, which cannot be determined from the differential equations. To fix them, it is best to compute required integrals at $\beta = 0$, which is the regular point for all integrals. A further benefit of this point is that the dependence on the angle θ disappears at $\beta = 0$, so that one computes constants and not functions of this angle. We present the needed boundary integrals in appendix B.

The final results for the integrated double-emission eikonal functions are expressed through iterated integrals defined as follows

$$I[w_1(t_1),\ldots,w_n(t_n)|x] = \int_0^x \mathrm{d}t_1 w_1(t_1) \int_0^{t_1} \mathrm{d}t_2 w_2(t_2) \cdots \int_0^{t_{n-1}} \mathrm{d}t_n w_n(t_n), \tag{4.26}$$

where

$$w(t) = \left\{ \frac{1}{t}, \frac{1}{t\pm 1}, \frac{1}{t-y}, \frac{y}{ty-1}, \frac{y\pm 1}{t(y\pm 1)-2}, \frac{1}{t-(y\pm\sqrt{y^2-1})}, \right. \\ \left. \frac{1}{tR_2(t,y)}, \frac{1}{(1\pm t)R_2(t,y)}, \frac{1}{(t-y)R_2(t,y)} \right\}. \tag{4.27}$$

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Iterated integrals without the square root R_2 can be expressed through standard Goncharov polylogarithms (GPLs) [82, 83] with the argument x and indices drawn from the following set

$$\left\{0, \pm 1, y \pm \sqrt{y^2-1}, \frac{2}{y \pm 1}, y, \frac{1}{y}\right\}. \tag{4.28}$$

When the square root R_2 is present in an integrand, the situation is more complex. However, we note that we can rationalize it by expressing x through a new variable η and y in the following way

$$x = \frac{2\eta}{\eta^2 + 6\eta y + 8(y^2-1)}. \tag{4.29}$$

The inverse transformation reads

$$\eta = \frac{1 - 3xy \pm R_2(x,y)}{x}. \tag{4.30}$$

As the result, the relevant iterated integrals with R_2 can also be expressed through GPLs with the argument η and indices drawn from the set

$$\left\{0, -2\,(y \pm 1)\,, -4\,(y \pm 1)\,, \frac{2(1-y^2)}{y}, -3y \pm \sqrt{y^2+8}, 2\left(-y \pm \sqrt{2-y^2}\right)\right\}. \tag{4.31}$$

4.3 Simplification of the final result

The result for the integrated double-emission eikonal function written in terms of GPLs with arguments $\{\eta, x\}$ and indices drawn from eqs. (4.28), (4.31) is not suitable for the high-precision, fast numerical evaluation. Because of this, we decided to construct an optimized form of the result, by expressing all GPLs up to weight four through normal logarithms, classical polylogarithms Li_n $(n = 2, 3, 4)$ and an additional function $\mathrm{Li}_{2,2}$, defined as

$$\mathrm{Li}_{2,2}(z_1, z_2) = \sum_{i>j>0}^{\infty} \frac{z_1^i z_2^j}{i^2 j^2} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{z_1^i}{(i+j)^2} \frac{(z_1 z_2)^j}{j^2}. \tag{4.32}$$

To do this, we employ the symbol technique [84, 85] since it provides a systematic way to derive non-trivial relations among GPLs.

We work with the integrated double-emission eikonal function as a whole, since significant simplifications can only be expected in the full result. We combine GPLs according to their weights and compute their symbols. From the list of symbols, we extract independent symbol letters and construct candidates for arguments of Li_n and $\mathrm{Li}_{2,2}$ functions, using the algorithms described in ref. [85].

Since there are too many functions that reproduce symbols of the identified sets of GPLs, we need to reduce their number by imposing additional conditions on their arguments. For example, we may try to require that, for physical values of β and $\cos\theta$, arguments of the polylogarithmic functions $\mathrm{Li}_n(z_1)$, $\mathrm{Li}_{2,2}(z_2, z_3)$ satisfy

$$\mathrm{Im} z_{1,2,3} = 0, \quad |z_1| \le 1, \quad |z_2| \le 1, \quad |z_2 z_3| \le 1. \tag{4.33}$$

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weight	1	2	3	4
number(s)	0	π^2	$\zeta_3, \pi^2\log(2)$	$\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\log^4(2)}{24}, \pi^4, \zeta_3\log(2), \pi^2\log^2(2)$

Table 1. Pure constants that are needed in the final step of the simplification procedure.

If this can be achieved, all functions become real-valued, and can be computed using convergent series expansion, leading to fast and efficient numerical evaluation. We note that when testing conditions in eq. (4.33), we choose a particular solution to eq. (4.29)

$$\eta = \frac{1 - 3xy - R(x,y)}{x}. \tag{4.34}$$

Unfortunately, it turns out that conditions in eq. (4.33) are too restrictive. In fact, we find that arguments of all $\mathrm{Li}_{2,2}$ functions can be chosen to satisfy eq. (4.33). However, for polylogarithms Li_n, it is only possible to choose argument z_1 that are smaller than one, i.e. $z_1 < 1$, as opposed to $|z_1| < 1$. Furthermore, we have to allow for the possibility that z_1 is *complex*. For such cases, we choose

$$\mathrm{Li}_n(z_1) + \mathrm{Li}_n(z_1^*), \quad \left[\mathrm{Li}_n(z_1) - \mathrm{Li}_n(z_1^*)\right]/i, \tag{4.35}$$

as suitable candidate functions.

Proceeding along these lines, we rewrite the original result for the integrated double-emission eikonal functions in terms of polylogarithmic functions with the chosen properties at the symbol level. The transformed and original expressions have identical symbols but are not yet the same, because of the missing polylogarithmic functions of lower weights multiplied by powers of π or ζ_3, and constant terms. Such terms can be restored by applying the so-called co-product formalism [86, 87]. By computing various co-products of the difference between the original and transformed results, one obtains symbols of a lower weight that can be reconstructed using the same functional basis. Finally, to fix the constants, we numerically evaluate the difference between the original and the reconstructed result, and use PSLQ algorithm [88] to express it through irrational constants of various weights, summarized in table 1. The motivation to consider these quantities comes from the expansion of the final result at small β, see eq. (4.41). We have checked that the old and new expressions agree with each other at multiple test points in the region $x \in [0,1], y \in [-1,1]$. We note that throughout the process of manipulating GPLs, computing symbols and co-products, we heavily relied on the package `PolyLogTools` [89–91] and used the library `GiNaC` [92, 93] to evaluate GPLs numerically.

The integrated double-emission eikonal functions are expressed in terms of conventional polylogarithms and $\mathrm{Li}_{2,2}$. This is extremely useful since, with the new representation, it takes about a *second* to obtain a numerical value for $\mathcal{S}[\tilde{S}_{ij}]$ and $\mathcal{S}[\tilde{I}_{ij}]$ for a generic phase space point in `Mathematica`, as opposed to *minutes* when using the original expressions in terms of GPLs. The final results for $\mathcal{S}[\tilde{S}_{ij}]$ and $\mathcal{S}[\tilde{I}_{ij}]$ can be found in the supplementary material files provided with this paper. There, a C code for fast and reliable evaluation of these quantities can be found, which uses algorithms described in ref. [94] to evaluate all the polylogarithmic functions. It significantly improves the efficiency of numerical evaluation, especially at the

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phase-space points where $\beta \sim 1$, and delivers results for a generic phase-space point in *milliseconds*.

We conclude this discussion by presenting the final results for the integrated double-emission eikonal functions up to order $\mathcal{O}\left(\varepsilon^0\right)$. We find

$$
\begin{aligned}
\mathcal{S}\left[\widetilde{\mathcal{S}}_{ij}\right] = \frac{-\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}}\Bigg\{ &-\frac{1}{4\varepsilon^3}+\frac{1}{\varepsilon^2}\left[\frac{1}{24}+\frac{1}{2}\log\left(\frac{(1-xy)^2}{1-x^2}\right)\right] \\
&+\frac{1}{\varepsilon}\left[\frac{119}{36}-\frac{13}{12}\log\left(\frac{(1-xy)^2}{1-x^2}\right)+\frac{\log\left(\frac{1-x}{x+1}\right)}{x}-\frac{11\log(2)}{6}\right. \\
&\left.-2\mathrm{Li}_2\left(\frac{x(y-1)}{1-x}\right)-2\mathrm{Li}_2\left(\frac{x(y+1)}{x+1}\right)-\frac{1}{2}\log^2\left(\frac{(1-xy)^2}{1-x^2}\right)-\frac{\pi^2}{24}\right] \\
&+\left[-\frac{589}{108}-\frac{31}{9}\log\left(\frac{(1-xy)^2}{1-x^2}\right)-\frac{19\log\left(\frac{1-x}{x+1}\right)}{6x}+\frac{269\log(2)}{36}\right. \\
&+\frac{f_1}{x}-\frac{f_2}{12}+\frac{11}{3}\log(2)\log\left(\frac{(1-xy)^2}{1-x^2}\right)+\frac{\pi^2}{12}+\frac{11\log^2(2)}{6} \\
&\left.-\frac{f_3}{3}+\frac{1}{12}\pi^2\log\left(\frac{(1-xy)^2}{1-x^2}\right)-\frac{11\zeta_3}{8}\right]+\mathcal{O}\left(\varepsilon\right)\Bigg\},
\end{aligned}
\tag{4.36}
$$

and

$$
\begin{aligned}
\mathcal{S}\left[\widetilde{\mathcal{I}}_{ij}\right] = \frac{-\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}}\Bigg\{ &-\frac{1}{12\varepsilon^2}+\frac{1}{\varepsilon}\left[\frac{25}{72}+\frac{1}{6}\log\left(\frac{(1-xy)^2}{1-x^2}\right)-\frac{\log(2)}{3}\right] \\
&+\left[-\frac{131}{216}-\frac{37}{36}\log\left(\frac{(1-xy)^2}{1-x^2}\right)+\frac{\log\left(\frac{1-x}{x+1}\right)}{3x}+\frac{59\log(2)}{36}\right. \\
&-\frac{2}{3}\mathrm{Li}_2\left(\frac{x(y-1)}{1-x}\right)-\frac{2}{3}\mathrm{Li}_2\left(\frac{x(y+1)}{x+1}\right)-\frac{1}{6}\log^2\left(\frac{(1-xy)^2}{1-x^2}\right) \\
&\left.+\frac{2}{3}\log(2)\log\left(\frac{(1-xy)^2}{1-x^2}\right)+\frac{\log^2(2)}{3}\right]+\mathcal{O}\left(\varepsilon\right)\Bigg\},
\end{aligned}
\tag{4.37}
$$

where we used the abbreviations

$$
\begin{aligned}
f_1 = &-\mathrm{Li}_2\left(\frac{x(y-1)}{1-x}\right)+\mathrm{Li}_2\left(\frac{x(y+1)}{x+1}\right)-4\mathrm{Li}_2\left(\frac{1-x}{2}\right)+4\mathrm{Li}_2(-x)-4\mathrm{Li}_2(x) \\
&+\log\left(\frac{1-x}{x+1}\right)\log\left(\frac{1-xy}{x+1}\right)-2\log^2\left(\frac{1-x}{2}\right)+\frac{1}{2}\log^2\left(\frac{1-x}{x+1}\right)+\frac{\pi^2}{3},
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
f_2 = &-40\mathrm{Li}_2\left(\frac{x(y-1)}{1-x}\right)-40\mathrm{Li}_2\left(\frac{x(y+1)}{x+1}\right)-20\log^2\left(\frac{1-xy}{1-x}\right) \\
&-20\log^2\left(\frac{1-xy}{x+1}\right)+13\log^2\left(\frac{1-x}{x+1}\right)
\end{aligned}
\tag{4.39}
$$

$$
\begin{aligned}
f_3 = &\,18\mathrm{Li}_3\left(\frac{x(y+1)}{x+1}\right)+12\mathrm{Li}_3\left(\frac{1-x}{1-xy}\right)-18\mathrm{Li}_3\left(\frac{x(1-y)}{1-xy}\right) \\
&+12\mathrm{Li}_3\left(\frac{(x+1)(1-y)}{2(1-xy)}\right)+12\mathrm{Li}_3\left(\frac{(1-x)(y+1)}{2(1-xy)}\right)+30\mathrm{Li}_3\left(\frac{1-xy}{x+1}\right)
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
&-12\mathrm{Li}_3\left(\frac{1-x}{x+1}\right)-12\mathrm{Li}_3\left(\frac{1-y}{2}\right)-12\mathrm{Li}_3\left(\frac{y+1}{2}\right)\\
&+\log(1-x)\Big[-27\mathrm{Li}_2\left(\frac{1-x}{2}\right)+33\mathrm{Li}_2(1-x)+27\mathrm{Li}_2(-x)+6\mathrm{Li}_2(x)\\
&+\frac{39}{2}\mathrm{Li}_2\left(\frac{1-y}{2}\right)+\frac{15}{2}\mathrm{Li}_2\left(\frac{2x(y+1)}{(x+1)^2}\right)+12\mathrm{Li}_2\left(\frac{x(y+1)}{x+1}\right)\\
&-15\mathrm{Li}_2\left(-\frac{1-x^2}{x^2-2yx+1}\right)-\frac{15}{2}\mathrm{Li}_2\left(\frac{2x(1-y)}{x^2-2yx+1}\right)+15\mathrm{Li}_2\left(\frac{x(x+1)(1-y)}{x^2-2yx+1}\right)\\
&+\frac{15}{2}\mathrm{Li}_2\left(\frac{(1-x)^2(y+1)}{2\,(x^2-2yx+1)}\right)-33\mathrm{Li}_2\left(\frac{1-x}{1-xy}\right)\Big]\\
&+\log(1-xy)\Big[15\mathrm{Li}_2\left(\frac{1-x}{2}\right)-42\mathrm{Li}_2(1-x)+12\mathrm{Li}_2(-x)-27\mathrm{Li}_2(x)\\
&-27\mathrm{Li}_2\left(\frac{1}{x+1}\right)-\frac{15}{2}\mathrm{Li}_2\left(\frac{1-y}{2}\right)-\frac{15}{2}\mathrm{Li}_2\left(\frac{2x(y+1)}{(x+1)^2}\right)+6\mathrm{Li}_2\left(\frac{x(y+1)}{x+1}\right)\\
&+15\mathrm{Li}_2\left(-\frac{1-x^2}{x^2-2yx+1}\right)+\frac{15}{2}\mathrm{Li}_2\left(\frac{2x(1-y)}{x^2-2yx+1}\right)-15\mathrm{Li}_2\left(\frac{x(x+1)(1-y)}{x^2-2yx+1}\right)\\
&-\frac{15}{2}\mathrm{Li}_2\left(\frac{(1-x)^2(y+1)}{2\,(x^2-2yx+1)}\right)+21\mathrm{Li}_2\left(\frac{1-x}{1-xy}\right)\Big]\\
&+\log(x+1)\Big[27\mathrm{Li}_2\left(\frac{1-x}{2}\right)-9\mathrm{Li}_2(1-x)-21\mathrm{Li}_2(-x)+3\mathrm{Li}_2(x)+9\mathrm{Li}_2\left(\frac{1}{x+1}\right)\\
&-12\mathrm{Li}_2\left(\frac{1-y}{2}\right)-18\mathrm{Li}_2\left(\frac{x(y+1)}{x+1}\right)-15\mathrm{Li}_2\left(\frac{2x(1-y)}{x^2-2yx+1}\right)+15\mathrm{Li}_2\left(\frac{x(x+1)(1-y)}{x^2-2yx+1}\right)\\
&-3\mathrm{Li}_2\left(\frac{1-x}{1-xy}\right)+15\mathrm{Li}_2\left(\frac{1-x^2}{2(1-xy)}\right)\Big]\\
&+\log(2)\Big[-\frac{45}{2}\mathrm{Li}_2\left(\frac{1-x}{2}\right)+\frac{45}{2}\mathrm{Li}_2\left(\frac{2x(1-y)}{x^2-2yx+1}\right)-\frac{45}{2}\mathrm{Li}_2\left(\frac{x(x+1)(1-y)}{x^2-2yx+1}\right)\\
&-\frac{45}{2}\mathrm{Li}_2\left(\frac{1-x^2}{2(1-xy)}\right)-\frac{45}{2}\mathrm{Li}_2\left(\frac{x(1-y)}{1-xy}\right)\Big]\\
&+\log(1-y)\Big[\frac{15}{2}\mathrm{Li}_2\left(\frac{1-x}{2}\right)-\frac{15}{2}\mathrm{Li}_2\left(\frac{2x(1-y)}{x^2-2yx+1}\right)+\frac{15}{2}\mathrm{Li}_2\left(\frac{x(x+1)(1-y)}{x^2-2yx+1}\right)\\
&+\frac{15}{2}\mathrm{Li}_2\left(\frac{1-x^2}{2(1-xy)}\right)+\frac{15}{2}\mathrm{Li}_2\left(\frac{x(1-y)}{1-xy}\right)\Big]\\
&-6\log^2(1-x)\log(1-y)+6\log^2(1-x)\log(1-xy)-\frac{15}{2}\log(1-x)\log^2(1-xy)\\
&-\frac{1}{2}\log^3(1-xy)+9\log(x+1)\log^2(1-xy)-\frac{3}{2}\log(1-y)\log^2(1-xy)\\
&+21\log(y+1)\log^2(1-xy)+21\log^2(x+1)\log(y+1)-\frac{9}{2}\log^2(x+1)\log(1-xy)\\
&+\frac{9}{2}\log(x+1)\log(1-x)\log(1-y)+\frac{39}{2}\log(1-x)\log(1-y)\log(y+1)\\
&-39\log(x+1)\log(1-x)\log(1-xy)+\frac{15}{2}\log(1-x)\log(1-y)\log(1-xy)\\
&+3\log(x+1)\log(1-y)\log(1-xy)-42\log(x+1)\log(y+1)\log(1-xy)
\end{aligned}
$$

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(x,y)	$\mathcal{S}\left[\tilde{S}_{ij}\right]$				$\mathcal{S}\left[\tilde{\mathcal{I}}_{ij}\right]$		
	ε^{-2}	ε^{-1}	ε^{0}	ε^{1}	ε^{-1}	ε^{0}	ε^{1}
$(0.1, 0.2)$	0.0264891	−0.40119472	0.066258771	2.1059932	0.11111398	0.020954787	0.083825324
$(0.9, 0.1)$	0.7777216	1.8762882	6.6664261	17.203106	0.36152480	0.47275868	1.0676022
$(0.8, 0.9)$	−0.7204734	−2.2382469	−4.0335936	−4.3260204	−0.13787352	−0.28578587	−0.35505627
$(0.8, -0.9)$	1.0948166	1.4353310	3.4641280	8.3534044	0.46722313	0.19603765	0.36091849

Table 2. Benchmark points for the integrated double-emission eikonal functions. The prefactor $-1/(4\varepsilon E_{\max}^{4\varepsilon})\mathcal{N}_\varepsilon^2$ is not included. The highest poles do not depend on x and y are not shown, cf. eqs. (4.36), (4.37).

$$
\begin{aligned}
&-\frac{39}{2}\log(1-y)\log(y+1)\log(1-xy)+\log^3(1-x)+12\log(x+1)\log^2(1-x)\\
&+39\log^2(x+1)\log(1-x)-\frac{19}{2}\log^3(x+1)\\
&+\log^2(2)\left[-\frac{69}{2}\log(1-xy)+51\log(1-x)+\frac{87}{2}\log(x+1)+\frac{15}{2}\log(1-y)\right]\\
&+\log(2)\Big[-\frac{69}{2}\log^2(1-xy)-\frac{69}{2}\log(1-x)\log(1-y)-\frac{39}{2}\log(1-x)\log(y+1)\\
&+\frac{99}{2}\log(1-x)\log(1-xy)-\frac{15}{2}\log(x+1)\log(1-y)+42\log(x+1)\log(1-xy)\\
&+27\log(1-y)\log(1-xy)+\frac{39}{2}\log(y+1)\log(1-xy)-15\log^2(1-x)\\
&-27\log^2(x+1)-\frac{135}{2}\log(x+1)\log(1-x)\Big]\\
&+\pi^2\left[\frac{33}{4}\log(1-xy)-\frac{9}{4}\log(1-x)-\frac{5}{4}\log(1-y)\right]\\
&-30\zeta_3-\frac{45\log^3(2)}{2}+\frac{15}{4}\pi^2\log(2).
\end{aligned}
$$

For reference, we present numerical results for a few benchmark points in table 2.

4.4 Checks of the result

We have performed several checks of the obtained results at various stages of the calculation. For example, we have computed master integrals numerically using dimensional shifts, see eq. (4.6), and direct integration as described in appendix D. To check complete results for $\mathcal{S}[\tilde{S}_{ij}]$ and $\mathcal{S}[\tilde{I}_{ij}]$, we recalculated them using the parameterization of the Heaviside function in eq. (2.19). Since the Heaviside function is mapped onto a delta-function, a conventional reduction to master integrals with the help of publicly-available codes becomes possible. However, since we only need to check the calculation described above, we decided to compute its small-β expansion, which is much easier to do. We then compare it to the results described in section 4.3 , which we also expand around $\beta = 0$. Perfect agreement up to order β^4 is found for the two results. It is also straightforward to extend the comparison to higher orders in the β-expansion. For completeness, we present the leading term in the β-expansion for

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the two integrated double-emission eikonal functions

$$
\begin{aligned}
\mathcal{S}\left[\widetilde{\mathcal{S}}_{ij}\right]\Bigg|_{\beta=0} &= \frac{-\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}}\Bigg[-\frac{1}{4\varepsilon^3}+\frac{1}{24\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{47}{36}-\frac{11\log(2)}{6}-\frac{\pi^2}{24}\right)\\
&+\left(-\frac{553}{108}+\frac{269\log(2)}{36}+\frac{\pi^2}{12}+\frac{11\log^2(2)}{6}-\frac{11\zeta_3}{8}\right)+\varepsilon\bigg(\frac{3887}{324}\\
&-\frac{1133\log(2)}{54}-\frac{47\pi^2}{144}-\frac{341\log^2(2)}{36}+\frac{99\zeta_3}{8}+\frac{11}{36}\pi^2\log(2)-\frac{11\log^3(2)}{9}\\
&-\frac{29\pi^4}{1440}-2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{\log^4(2)}{12}-\frac{7}{4}\zeta_3\log(2)+\frac{1}{12}\pi^2\log^2(2)\bigg)+O\left(\varepsilon^2\right)\Bigg],
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
\mathcal{S}\left[\widetilde{\mathcal{I}}_{ij}\right]\Bigg|_{\beta=0} &= \frac{-\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}}\Bigg[-\frac{1}{12\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{25}{72}-\frac{\log(2)}{3}\right)\\
&+\left(-\frac{275}{216}+\frac{59\log(2)}{36}+\frac{\log^2(2)}{3}\right)+\varepsilon\bigg(\frac{2737}{648}-\frac{577\log(2)}{108}-\frac{23\pi^2}{144}-\frac{95\log^2(2)}{36}\\
&+\frac{7\zeta_3}{4}+\frac{1}{18}\pi^2\log(2)-\frac{2\log^3(2)}{9}\bigg)+O\left(\varepsilon^2\right)\Bigg].
\end{aligned}
$$

Finally, in section 5 we describe an alternative computation of the integrated double-emission eikonal functions that does not involve IBP reduction, master integrals and differential equations. The agreement between calculations described in this section and in section 5 provides a very strong check on their correctness.

5 An alternative computation in the massive parton's rest frame

The final result for the integrated double-emission eikonal function in the two-gluon case, derived in the previous sections, is simpler than results for individual integrals. This is certainly true for the divergent contributions, but also for the finite part since elliptic integrals disappear from the final expression. It is interesting to understand the origin of this simplicity. A possible reason to expect it is Lorentz invariance, i.e. the possibility to choose a suitable reference frame (for example, the rest frame of the massive parton) to compute the integrated double-emission eikonal function.

Lorentz invariance would have been very useful for computing integrated double-emission eikonal functions, if not for the Heaviside functions that appear because of how subtraction terms in the nested soft-collinear subtraction scheme are defined. These Heaviside functions depend on the energies of unresolved partons in the laboratory frame and, therefore, transform in a non-trivial way under Lorentz boosts. Nevertheless, as we explain below it turns out to be very useful to compute the integrated eikonal in the rest frame of the massive parton, as it allows us to understand the simplicity of the result, at least partially.

The integrated double-emission eikonal function defined in eq. (2.21) is written in a Lorentz-invariant way. In the computation reported in the previous sections, we have taken $P=(1,\vec{0})$ defining the laboratory frame, but this is certainly not the only option. In fact, we find it useful to consider a frame where the massive parton is at rest $p_i=(m,\vec{0})$. In

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this frame, $P = \gamma_i(1, \beta\vec{n}_P)$, where β is the velocity of the massive parton in the lab frame, $\gamma_i = E_i/m$, where E_i is the energy of parton i in the lab frame, and $\vec{n}_P$ is $-\vec{n}_i$, where $\vec{n}_i$ is defined in the lab frame. The scalar products evaluate to

$$l_{\mathfrak{m}} \cdot P = l^0_{\mathfrak{m}}\, \gamma_i\, \rho_{\mathfrak{m}t}, \quad l_{\mathfrak{n}} \cdot P = l^0_{\mathfrak{n}}\, \gamma_i\, \rho_{\mathfrak{n}t}, \tag{5.1}$$

where $\rho_{\mathfrak{m}(\mathfrak{n})t} = 1 - \beta\vec{n}_P \cdot \vec{n}_{\mathfrak{m}(\mathfrak{n})}$. Furthermore, writing $l^0_{\mathfrak{n}} = \omega l^0_{\mathfrak{m}}$, using the homogeneity of Ξ_{ij}, and integrating over $l^0_{\mathfrak{m}}$, we obtain the following representation for the double-soft integral

$$\mathcal{S}\left[\Xi_{ij}\right] = \mathcal{N}_A \left\langle \int\limits_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\, \psi^{4\varepsilon}_{t,\mathfrak{m}j}\, \theta(\psi_{t,\mathfrak{m}j} - \omega\, \psi_{t,\mathfrak{n}j}) \left[\omega^2\, \Xi_{ij}(\mathfrak{m},\mathfrak{n})\right] \right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{5.2}$$

The brackets $\langle \ldots \rangle_{\mathfrak{m}\mathfrak{n}}$ denote integration over directions of partons $\mathfrak{m}$ and $\mathfrak{n}$. We note that in the above equation we have defined the function $\psi_{t,yz} = \rho_{yt}/\rho_{zt}$, and the normalization factor

$$\mathcal{N}_A = -\frac{1}{4\varepsilon}\left(\frac{E_{\max}}{\gamma_i \rho_{tj}}\right)^{-4\varepsilon} = -\frac{1}{4\varepsilon E^{4\varepsilon}_{\max}} \frac{(1-\beta^2)^{2\varepsilon}}{(1-\beta\cos\theta)^{4\varepsilon}}. \tag{5.3}$$

We also note that the double-emission eikonal function Ξ_{ij} in eq. (5.2) has to be evaluated in the rest frame of parton i. Furthermore, the four-momenta of partons $\mathfrak{m}, \mathfrak{n}$ in that equation should be taken to be $l_{\mathfrak{m}} = (1, \vec{n}_{\mathfrak{m}})$, $l_{\mathfrak{n}} = \omega(1, \vec{n}_{\mathfrak{n}})$. In the rest frame of i, scalar products $p_i \cdot l_{\mathfrak{m}}$, $p_i \cdot l_{\mathfrak{n}}$ become simple. Apart from the prefactor $\mathcal{N}_A$, the non-trivial dependence on the angle between $\vec{P}$ and $\vec{n}_j$ resides in functions $\psi_{t,\mathfrak{m}j}$ and $\psi_{t,\mathfrak{n}j}$, which appear in eq. (5.2) either in an ε-dependent power or inside the Heaviside function. Our goal is to exploit this fact and write the integral in eq. (5.2) as a sum of two terms, one that contains no divergences and can be calculated in *four* dimensions, and another one which is composed of simpler integrals, can be calculated analytically in a systematic way and contains all $1/\varepsilon$ poles. Such a separation, and in particular an analytic form of a divergent term, is very useful for establishing the explicit cancellation of infra-red poles in the context of NNLO calculations, see e.g. ref. [95].

We will focus on the case where soft partons are gluons, since it is more general than the case where a $q\bar{q}$ pair is soft. Denoting the integrated eikonal function for the two gluons as $G_{ij} = \mathcal{S}\left[\tilde{S}_{ij}\right]$, we proceed with iterative subtraction of singularities, starting with the soft one. We write

$$G_{ij} = S_\omega[G_{ij}] + \bar{S}_\omega[G_{ij}], \tag{5.4}$$

where $\bar{S}_\omega = 1 - S_\omega$ and the operator S_ω, acting on relevant quantities, retains the leading singular contribution in the $\omega \to 0$ limit. Computation of the strongly-ordered integral $S_\omega[G_{ij}]$ is discussed in section 5.1. The soft-subtracted integral[5]

$$\bar{S}_\omega\left[G_{ij}\right] = \mathcal{N}_A \left\langle \int\limits_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \psi^{4\varepsilon}_{t,\mathfrak{m}j} \theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) \bar{S}_\omega\left[\omega^2 \tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] \right\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{5.5}$$

requires further manipulations.

[5]Note that we do not act with the operator S_ω on the theta-function.

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For reasons that will become clear shortly, it is convenient to define a new quantity

$$\bar{S}_\omega[\Delta G_{ij}] = \bar{S}_\omega[G_{ij}] - \bar{S}_\omega[G_{ij}^{(0)}], \tag{5.6}$$

where

$$\bar{S}_\omega\left[G_{ij}^{(0)}\right] = \mathcal{N}_A \left\langle \int\limits_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \theta(1-\omega) \bar{S}_\omega \left[\omega^2 \tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] \right\rangle_{\mathfrak{mn}}. \tag{5.7}$$

Apart from the β-dependence in the normalization factor $\mathcal{N}_A$, the above quantity corresponds to the double-soft integral in the laboratory frame where the massive parton i is now at rest, i.e. $\beta = 0$. If the emitter i is at rest, the dependence on the emission angle disappears; therefore, the integral in eq. (5.7) is a number (as opposed to a function), and can be easily computed. Hence, we write

$$\bar{S}_\omega[G_{ij}] = \bar{S}_\omega[\Delta G_{ij}] + \bar{S}_\omega[G_{ij}^{(0)}], \tag{5.8}$$

and focus on the first term on the right-hand side in what follows.

Our goal is to remove all divergences from integrands in eqs. (5.5), (5.7). These divergences are collinear, and one can check that the soft-subtracted integrand is singular in only two cases, the triple-collinear one $\mathfrak{m}||\mathfrak{n}||j$, and the double-collinear one $\mathfrak{m}||\mathfrak{n}$. Note that the soft-subtracted double-emission eikonal is *not* singular in two other possible double-collinear limits $\mathfrak{m}||j$ and $\mathfrak{n}||j$, which simplifies the construction of the subtraction terms. To extract singularities, we write

$$\begin{aligned}\bar{S}_\omega\left[\Delta G_{ij}\right] = \mathcal{N}_A \Big\langle \int\limits_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \Big[C_{\mathfrak{mn}} + \bar{C}_{\mathfrak{mn}} \left(C_{j\mathfrak{mn}} + \bar{C}_{j\mathfrak{mn}}\right)\Big] \\ \times \left(\psi_{t,\mathfrak{m}j}^{4\varepsilon} \theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)\right) \bar{S}_\omega \left[\omega^2 \tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] \Big\rangle_{\mathfrak{mn}},\end{aligned} \tag{5.9}$$

where operators $C_{\mathfrak{mn}}$ and $C_{j\mathfrak{mn}}$ extract particular singular limits from all expressions that appear to the right of them, and $\bar{C}_{\mathfrak{mn}} = 1 - C_{\mathfrak{mn}}$, $\bar{C}_{j\mathfrak{mn}} = 1 - C_{j\mathfrak{mn}}$. The operator $C_{\mathfrak{mn}}$ extracts the double-collinear limit $\mathfrak{m}||\mathfrak{n}$ and the operator $C_{j\mathfrak{mn}}$ — the triple- collinear limit $\mathfrak{m}||\mathfrak{n}||j$.[6] An important simplification in eq. (5.9) occurs because

$$C_{j\mathfrak{mn}} \left(\psi_{t,\mathfrak{m}j}^{4\varepsilon} \theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)\right) \bar{S}_\omega \left[\omega^2 \tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] = 0, \tag{5.10}$$

which follows from the equations $C_{j\mathfrak{mn}}\psi_{t,\mathfrak{m}j} = 1$, $C_{j\mathfrak{mn}}\psi_{t,\mathfrak{n}j} = 1$. The consequence of eq. (5.10) is that the triple-collinear subtraction in eq. (5.9) is not needed; hence, we can replace $C_{j\mathfrak{mn}} + \bar{C}_{j\mathfrak{mn}}$ with the identity operator.

Taking into account this feature, we write

$$\bar{S}_\omega\left[\Delta G_{ij}\right] = \mathcal{N}_A \Big\langle \int\limits_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \left(C + \bar{C}\right) \bar{S}_\omega \left[\omega^2 \tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] \Big\rangle_{\mathfrak{mn}}, \tag{5.11}$$

[6]We note that we do not restrict action of $C_{\mathfrak{mn}}$ to parts of the unresolved phase space. As we will see later, it is not necessary to do this in the current case.

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where operators C and $\bar{C}$ read

$$\begin{aligned} C &= \left(\psi_{t,\mathfrak{m}j}^{4\varepsilon} - 1\right)\theta(1-\omega)\,C_{\mathfrak{mn}}, \\ \bar{C} &= \bar{C}_{\mathfrak{mn}}\left(\psi_{t,\mathfrak{m}j}^{4\varepsilon}\theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)\right), \end{aligned} \tag{5.12}$$

and we have used $C_{\mathfrak{mn}}\psi_{t,\mathfrak{m}j}^{4\varepsilon}\theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) = \psi_{t,\mathfrak{m}j}^{4\varepsilon}\theta(1-\omega)$. Since operator $\bar{C}$ removes all divergences from the integrand in eq. (5.11), we can expand it in powers of ε. A convenient first step is to write $\psi_{t,\mathfrak{m}j}^{4\varepsilon} = 1 + (\psi_{t,\mathfrak{m}j}^{4\varepsilon} - 1)$. Upon doing that, $\bar{C}$ splits into two terms $\bar{C} = \bar{C}_a + \bar{C}_b$, where

$$\begin{aligned} \bar{C}_a &= \bar{C}_{\mathfrak{mn}}\left[\theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)\right] = \left[\theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)\right], \\ \bar{C}_b &= \bar{C}_{\mathfrak{mn}}\left(\psi_{t,\mathfrak{m}j}^{4\varepsilon} - 1\right)\theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) \\ &= \left(\psi_{t,\mathfrak{m}j}^{4\varepsilon} - 1\right)\left[\theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)C_{\mathfrak{mn}}\right]. \end{aligned} \tag{5.13}$$

Since the normalization factor $\mathcal{N}_A$ is $\mathcal{O}(1/\varepsilon)$ (cf. eq. (5.3)), and the operator $\bar{C}_b$ is $\mathcal{O}(\varepsilon)$, as follows from the expansion of $\psi_{t,\mathfrak{m}j}^{4\varepsilon}$ in powers of ε, it only contributes to $\mathcal{O}(\varepsilon^0)$ part of the final result. Hence, it can be calculated numerically right away by integrating in four dimensions.

On the contrary, the integral that involves the operator $\bar{C}_a$ requires higher-order expansion in ε as it is multiplied with $\mathcal{N}_A \sim 1/\varepsilon$ and the corresponding integrand does not possess explicit ε-suppression factors. Hence, we need to simplify it further.

The key observation that allows us to do this is that this integral can be written in two complementary ways, as follows from the fact that the soft-subtracted double eikonal function is (nearly) invariant under a combined $\omega \to 1/\omega$ and $\mathfrak{m} \leftrightarrow \mathfrak{n}$ transformation. The relevant equation reads

$$\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Bigg|_{\substack{\omega\to 1/\omega \\ \mathfrak{m}\leftrightarrow\mathfrak{n}}} = \bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] + \frac{(\rho_{\mathfrak{mn}} - \rho_{\mathfrak{m}j} - \rho_{\mathfrak{n}j})(\rho_{\mathfrak{m}j} - \rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}, \tag{5.14}$$

and we note that the second term on the right-hand side of the above equation is much simpler than the complete double-emission eikonal function. To make use of eq. (5.14), we write

$$\begin{aligned} I_a &= \mathcal{N}_A\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\bar{C}_a\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Big\rangle_{\mathfrak{mn}} \\ &= \mathcal{N}_A\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\left[\theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)\right]\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Big\rangle_{\mathfrak{mn}} \\ &= \mathcal{N}_A\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1-2\varepsilon}}\left[\theta(\omega\psi_{t,\mathfrak{n}j} - \psi_{t,\mathfrak{m}j}) - \theta(\omega-1)\right]\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Bigg|_{\substack{\omega\to 1/\omega \\ \mathfrak{m}\leftrightarrow\mathfrak{n}}}\Big\rangle_{\mathfrak{mn}}. \end{aligned} \tag{5.15}$$

We then use eq. (5.15) to combine two different representations for the integral I_a and find

$$I_a = F_{a,1} + F_{a,2}, \tag{5.16}$$

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where the two integrals read

$$F_{a,1} = \frac{\mathcal{N}_A}{2}\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\left(\omega^{-2\varepsilon}-\omega^{2\varepsilon}\right) f_-(\omega,\psi_{t,\mathfrak{m}j},\psi_{t,\mathfrak{n}j})\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Big\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{5.17}$$

$$F_{a,2} = \frac{\mathcal{N}_A}{2}\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1-2\varepsilon}} f_+(\omega,\psi_{t,\mathfrak{m}j},\psi_{t,\mathfrak{n}j})\frac{(\rho_{\mathfrak{m}\mathfrak{n}}-\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})(\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\Big\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{5.18}$$

and the two functions $f_\mp$ are defined as

$$\begin{aligned} f_-(\omega,\psi_{t,\mathfrak{m}j},\psi_{t,\mathfrak{n}j}) &= \theta(\psi_{t,\mathfrak{m}j}-\omega\psi_{t,\mathfrak{n}j})-\theta(1-\omega),\\ f_+(\omega,\psi_{t,\mathfrak{m}j},\psi_{t,\mathfrak{n}j}) &= \theta(\omega\psi_{t,\mathfrak{n}j}-\psi_{t,\mathfrak{m}j})-\theta(\omega-1). \end{aligned} \tag{5.19}$$

The two integrals in eqs. (5.17), (5.18) are finite. Furthermore, since $(\omega^{-2\varepsilon}-\omega^{2\varepsilon}) = -4\varepsilon\log\omega+\mathcal{O}(\varepsilon^2)$, the integral $F_{a,1}$ only contributes to the finite part of eq. (5.2), and can be immediately calculated numerically in four dimensions. The second integral $F_{a,2}$ is sufficiently simple to be computed analytically; we discuss its computation in section 5.4.

The final expression for the integrated double-emission eikonal function therefore reads

$$G_{ij} = S_\omega\left[G_{ij}\right]+\bar{S}_\omega\left[G_{ij}^{(0)}\right]+\bar{S}_\omega\left[\Delta G_{ij}\right], \tag{5.20}$$

where the first and the second terms on the right-hand side are the strongly-ordered and the $\beta = 0$ contributions. The third term reads

$$\bar{S}_\omega[\Delta G_{ij}] = \bar{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}}+\bar{S}_\omega[\Delta G_{ij}]_a+\bar{S}_\omega[\Delta G_{ij}]_{\mathrm{num}}, \tag{5.21}$$

where

$$\bar{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} = \mathcal{N}_A\Big\langle \int_0^1 \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\left(\psi_{t,\mathfrak{m}j}^{4\varepsilon}-1\right)C_{\mathfrak{m}\mathfrak{n}}\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Big\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{5.22}$$

$$\bar{S}_\omega[\Delta G_{ij}]_a = \frac{\mathcal{N}_A}{2}\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1-2\varepsilon}} f_+(\omega,\psi_{t,\mathfrak{m}j},\psi_{t,\mathfrak{n}j})\frac{(\rho_{\mathfrak{m}\mathfrak{n}}-\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})(\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\Big\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{5.23}$$

$$\begin{aligned} \bar{S}_\omega\left[\Delta G_{ij}\right]_{\mathrm{num}} = {}& \frac{\mathcal{N}_A}{2}\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\left(\omega^{-2\varepsilon}-\omega^{2\varepsilon}\right) f_-(\omega,\psi_{t,\mathfrak{m}j},\psi_{t,\mathfrak{n}j})\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Big\rangle_{\mathfrak{m}\mathfrak{n}} \\ &+\mathcal{N}_A\Big\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\bar{C}_{\mathfrak{m}\mathfrak{n}}\left[\left(\psi_{t,\mathfrak{m}j}^{4\varepsilon}-1\right)\theta(\psi_{t,\mathfrak{m}j}-\omega\psi_{t,\mathfrak{n}j})\right]\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Big\rangle_{\mathfrak{m}\mathfrak{n}}. \end{aligned} \tag{5.24}$$

We can compute $\bar{S}_\omega[\Delta G_{ij}]_{\mathrm{num}}$ numerically in four dimensions, whereas $\bar{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}}$ and $\bar{S}_\omega[\Delta G_{ij}]_a$ need to be discussed further, and we do this in the follow-up sections.

5.1 The strongly-ordered term $S_\omega\left[G_{ij}\right]$

As explained in the previous section, we require the integral of the strongly-ordered double-emission eikonal function

$$S_\omega\left[G_{ij}\right] = \mathcal{N}_A\left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\psi_{t,\mathfrak{m}j}^{4\varepsilon}\theta(\psi_{t,\mathfrak{m}j}-\omega\psi_{t,\mathfrak{n}j})S_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{5.25}$$

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The strongly-ordered eikonal function evaluates to

$$S_\omega[\omega^2 \tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})] = \frac{2}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}} + \frac{2}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{n}j}} - \frac{2}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}} - \frac{1}{\rho_{\mathfrak{mn}}} + \frac{1}{\rho_{\mathfrak{n}j}} - \frac{\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{n}j}}. \tag{5.26}$$

Integrating over ω, we find

$$\int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \psi_{t,\mathfrak{m}j}^{4\varepsilon} \theta(\psi_{t,\mathfrak{m}j} - \omega\psi_{t,\mathfrak{n}j}) = -\frac{1}{2\varepsilon}\psi_{t,\mathfrak{m}j}^{2\varepsilon}\psi_{t,\mathfrak{n}j}^{2\varepsilon}. \tag{5.27}$$

Finally, using the symmetry of the integration measure with respect to the interchange of $\mathfrak{m}$ and $\mathfrak{n}$, we write the strongly-ordered contribution as

$$S_\omega\left[G_{ij}\right] = -\frac{\mathcal{N}_A}{2\varepsilon}\left\langle \psi_{t,\mathfrak{m}j}^{2\varepsilon}\psi_{t,\mathfrak{n}j}^{2\varepsilon}\left(\frac{1}{\rho_{\mathfrak{n}j}} - \frac{2}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}} - \frac{1}{\rho_{\mathfrak{mn}}} + \frac{4-\rho_{\mathfrak{n}j}}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}\right)\right\rangle_{\mathfrak{mn}}. \tag{5.28}$$

The first two terms in the above expression do not contain $1/\rho_{\mathfrak{mn}}$ and can be written using angular integrals defined in appendix C. We find

$$\left\langle \frac{\psi_{t,\mathfrak{m}j}^{2\varepsilon}\psi_{t,\mathfrak{n}j}^{2\varepsilon}}{\rho_{\mathfrak{n}j}}\right\rangle_{\mathfrak{mn}} = \rho_{tj}^{-4\varepsilon} I_{-2\varepsilon}^{(1)} I_{-2\varepsilon,1}^{(1)}, \tag{5.29}$$

$$\left\langle \frac{\psi_{t,\mathfrak{m}j}^{2\varepsilon}\psi_{t,\mathfrak{n}j}^{2\varepsilon}}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right\rangle_{\mathfrak{mn}} = \rho_{tj}^{-4\varepsilon}\left(I_{-2\varepsilon,1}^{(1)}\right)^2. \tag{5.30}$$

We note that we do not show arguments of the above integrals because they are always the same, i.e. $I_{a,b}^{(1)} = I_{a,b}^{(1)}\left[\rho_{tt},\rho_{tj}\right]$ and $I_a^{(1)} = I_a^{(1)}\left[\rho_{tt}\right]$, where $\rho_{tt} = 1-\beta^2$ and $\rho_{tj} = (1-\beta^2)/(1-\beta\cos\theta)$.

Using the $\mathfrak{m} \leftrightarrow \mathfrak{n}$ symmetry, we write the third term in eq. (5.28) as

$$\begin{aligned}\left\langle \frac{\psi_{t,\mathfrak{m}j}^{2\varepsilon}\psi_{t,\mathfrak{n}j}^{2\varepsilon}}{\rho_{\mathfrak{mn}}}\right\rangle_{\mathfrak{mn}} &= \left\langle \frac{\psi_{t,\mathfrak{m}j}^{4\varepsilon}}{\rho_{\mathfrak{mn}}}\right\rangle_{\mathfrak{mn}} - \frac{1}{2}\left\langle \left(\psi_{t,\mathfrak{m}j}^{2\varepsilon} - \psi_{t,\mathfrak{n}j}^{2\varepsilon}\right)^2 \frac{1}{\rho_{\mathfrak{mn}}}\right\rangle_{\mathfrak{mn}} \\ &= \rho_{tj}^{-4\varepsilon} I_{-4\varepsilon}^{(1)} - \frac{\varepsilon^2}{2}\left\langle \frac{\left(g_{t,\mathfrak{m}j}^{(2)} - g_{t,\mathfrak{n}j}^{(2)}\right)^2}{\rho_{\mathfrak{mn}}}\right\rangle_{\mathfrak{mn}},\end{aligned} \tag{5.31}$$

where in the last step we have used $\psi_{t,\mathfrak{m}j}^{2\varepsilon} = 1 + \varepsilon g_{t,\mathfrak{m}j}^{(2)}$. We emphasize that this equation is the definition of $g_{t,\mathfrak{m}j}^{(2)}$, and not the expansion of $\psi_{t,\mathfrak{m}j}^{2\varepsilon}$ to first power in ε. Finally, we note that the last term in eq. (5.31) gives a finite contribution to the final result for the integrated double-emission eikonal function, and can be numerically integrated in four dimensions.

The last term in eq. (5.28) is more complicated. To simplify it, we first substitute $\psi_{t,\mathfrak{m}j}^{2\varepsilon} = 1 + \varepsilon g_{t,\mathfrak{m}j}^{(2)}$, $\psi_{t,\mathfrak{n}j}^{2\varepsilon} = 1 + \varepsilon g_{t,\mathfrak{n}j}^{(2)}$ there, and make use of the fact that, unless two functions $g^{(2)}$ appear in the integrand, integration over $\mathfrak{m}$ or $\mathfrak{n}$ can be performed. Hence, we write

$$\left\langle \psi_{t,\mathfrak{m}j}^{2\varepsilon}\psi_{t,\mathfrak{n}j}^{2\varepsilon}\frac{4-\rho_{\mathfrak{n}j}}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}\right\rangle_{\mathfrak{mn}} = T_{4,0} + T_{4,1} + T_{4,2}, \tag{5.32}$$

where

$$T_{4,0} = \frac{1}{2\varepsilon}\rho_{tj}^{-2\varepsilon}\left(I_{-2\varepsilon}^{(1)} - 2(2-3\varepsilon)I_{-2\varepsilon,1}^{(1)}\right), \tag{5.33}$$

$$T_{4,1} = \varepsilon\left\langle g_{t,\mathfrak{m}j}^{(2)}(4-\rho_{\mathfrak{m}j})I_{1,1}^{(0)}\left[\rho_{\mathfrak{m}j}\right]\right\rangle_{\mathfrak{m}}, \tag{5.34}$$

$$T_{4,2} = \varepsilon^2\left\langle g_{t,\mathfrak{m}j}^{(2)} g_{t,\mathfrak{n}j}^{(2)}\frac{4-\rho_{\mathfrak{n}j}}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}\right\rangle_{\mathfrak{mn}}. \tag{5.35}$$

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Using the integral $I^{(0)}_{1,1}\left[\rho_{\mathfrak{m}j}\right]$ from eq. (C.6), and transformation rules for hypergeometric functions, the integral $T_{4,1}$ can be cast into the following form

$$T_{4,1} = 2^{\varepsilon}(1-2\varepsilon)\left\langle g^{(2)}_{t,\mathfrak{m}j}\frac{(\rho_{\mathfrak{m}j}-4)}{\rho_{\mathfrak{m}j}^{1+\varepsilon}}\,{}_2F_1\left(-\varepsilon,-\varepsilon;1-\varepsilon;1-\frac{\rho_{\mathfrak{m}j}}{2}\right)\right\rangle_{\mathfrak{m}}. \tag{5.36}$$

Expanding in ε to the required order, it is possible to replace the hypergeometric function with the following expression

$${}_2F_1\left(-\varepsilon,-\varepsilon;1-\varepsilon;1-\frac{\rho_{\mathfrak{m}j}}{2}\right) = 1+\varepsilon^2\mathrm{Li}_2\left(1-\frac{\rho_{\mathfrak{m}j}}{2}\right)+\mathcal{O}\left(\varepsilon^3\right). \tag{5.37}$$

The final result for the integral $T_{4,1}$ valid through $\mathcal{O}(\varepsilon^2)$ then easily follows. It reads

$$\begin{aligned} T_{4,1} = &-\frac{2^{\varepsilon}(1-2\varepsilon)^2}{\varepsilon^2}I^{(0)}_{\varepsilon}+\frac{2^{\varepsilon}(1-2\varepsilon)}{\varepsilon}\rho_{tj}^{-2\varepsilon}\left(I^{(1)}_{-2\varepsilon,\varepsilon}-4I^{(1)}_{-2\varepsilon,1+\varepsilon}\right)\\ &-2\varepsilon^2\left\langle\frac{4-\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{m}j}}\mathrm{Li}_2\left(1-\frac{\rho_{\mathfrak{m}j}}{2}\right)\log\frac{\rho_{t\mathfrak{m}}}{\rho_{tj}}\right\rangle_{\mathfrak{m}}. \end{aligned} \tag{5.38}$$

The integral $T_{4,2}$ diverges for $\mathfrak{m}||\mathfrak{n}$; hence, we subtract this limit by inserting $1 = C_{\mathfrak{mn}} + (1-C_{\mathfrak{mn}})$ into the integrand. Since

$$C_{\mathfrak{mn}}\,g^{(2)}_{t,\mathfrak{m}j}g^{(2)}_{t,\mathfrak{n}j}\frac{(4-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}} = \left(g^{(2)}_{t,\mathfrak{m}j}\right)^2\frac{(4-\rho_{\mathfrak{m}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}, \tag{5.39}$$

we find

$$\begin{aligned} \left\langle C_{\mathfrak{mn}}\,\varepsilon^2\,g^{(2)}_{t,\mathfrak{m}j}g^{(2)}_{t,\mathfrak{n}j}\frac{(4-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}\right\rangle_{\mathfrak{mn}} &= \frac{(1-2\varepsilon)(2-3\varepsilon)}{2\varepsilon^2}\\ +\frac{1-2\varepsilon}{2\varepsilon}\rho_{tj}^{-4\varepsilon}\left(I^{(1)}_{-4\varepsilon}-4I^{(1)}_{-4\varepsilon,1}\right)&-\frac{1-2\varepsilon}{\varepsilon}\rho_{tj}^{-2\varepsilon}\left(I^{(1)}_{-2\varepsilon}-4I^{(1)}_{-2\varepsilon,1}\right). \end{aligned} \tag{5.40}$$

The remaining integral with $\bar{C}_{\mathfrak{mn}} = 1-C_{\mathfrak{mn}}$ becomes finite, and can be calculated in four dimensions numerically. We obtain

$$\left\langle\overline{C}_{\mathfrak{mn}}\,\varepsilon^2\,g^{(2)}_{t,\mathfrak{m}j}g^{(2)}_{t,\mathfrak{n}j}\frac{(4-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}\right\rangle_{\mathfrak{mn}} = \varepsilon^2\left\langle g^{(2)}_{t,\mathfrak{m}j}\frac{g^{(2)}_{t,\mathfrak{n}j}(4-\rho_{\mathfrak{n}j})-g^{(2)}_{t,\mathfrak{m}j}(4-\rho_{\mathfrak{m}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}\right\rangle_{\mathfrak{mn}}. \tag{5.41}$$

The combined result for the strongly-ordered limit reads

$$\begin{aligned} \frac{S_\omega\left[G_{ij}\right]}{\mathcal{N}_A} = &-\frac{(1-2\varepsilon)(2-3\varepsilon)}{4\varepsilon^3}+\frac{(1-2\varepsilon)^2}{2^{1-\varepsilon}\varepsilon^3}I^{(0)}_{\varepsilon}+\frac{1}{2\varepsilon}\rho_{tj}^{-4\varepsilon}I^{(1)}_{-2\varepsilon,1}\left(2I^{(1)}_{-2\varepsilon,1}-I^{(1)}_{-2\varepsilon}\right)\\ &+\frac{(1-2\varepsilon)}{2\varepsilon^2}\rho_{tj}^{-4\varepsilon}\left(2I^{(1)}_{-4\varepsilon,1}-I^{(1)}_{-4\varepsilon}\right)+\frac{(1-2\varepsilon)}{2^{1-\varepsilon}\varepsilon^2}\rho_{tj}^{-2\varepsilon}\left(4I^{(1)}_{-2\varepsilon,1+\varepsilon}-I^{(1)}_{-2\varepsilon,\varepsilon}\right)\\ &+\frac{1}{4\varepsilon^2}\rho_{tj}^{-2\varepsilon}\left((1-4\varepsilon)I^{(1)}_{-2\varepsilon}-2(2-5\varepsilon)I^{(1)}_{-2\varepsilon,1}\right)+\varepsilon\left\langle\frac{4-\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{m}j}}\mathrm{Li}_2\left(1-\frac{\rho_{\mathfrak{m}j}}{2}\right)\log\frac{\rho_{t\mathfrak{m}}}{\rho_{tj}}\right\rangle_{\mathfrak{m}}\\ &-\frac{\varepsilon}{4}\left\langle\frac{(g^{(2)}_{t,\mathfrak{m}j}-g^{(2)}_{t,\mathfrak{n}j})^2}{\rho_{\mathfrak{mn}}}\right\rangle_{\mathfrak{mn}}-\frac{\varepsilon}{2}\left\langle g^{(2)}_{t,\mathfrak{m}j}\frac{g^{(2)}_{t,\mathfrak{m}j}(\rho_{\mathfrak{m}j}-4)-g^{(2)}_{t,\mathfrak{n}j}(\rho_{\mathfrak{n}j}-4)}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}\right\rangle_{\mathfrak{mn}}. \end{aligned} \tag{5.42}$$

5.2 Calculation of $\bar{S}_\omega\left[G_{ij}^{(0)}\right]$

It is straightforward to compute the term $\bar{S}_\omega\left[G_{ij}^{(0)}\right]$ in eq. (5.20), which does not depend on the velocity β and the angle between $\vec{n}_i$ and $\vec{n}_j$. It can be expressed in terms of the integrals discussed in section 4 if we choose $p_i = (m, \vec{0})$ in that section. We obtain

$$\begin{aligned}\frac{\bar{S}_\omega\left[G_{ij}^{(0)}\right]}{\mathcal{N}_A} &= -\frac{11}{24\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{65}{36} - \frac{11}{6}\log(2) - \frac{\pi^2}{24}\right) - \left(\frac{553}{108} - \frac{269}{36}\log(2) - \frac{\pi^2}{12}\right.\\ &\left. - \frac{11}{6}\log^2(2) + \frac{11}{8}\zeta_3\right) + \varepsilon\left(\frac{3887}{324} - \frac{1133}{54}\log(2) - \frac{47}{144}\pi^2 - \frac{341}{36}\log^2(2)\right.\\ &- \frac{11}{9}\log^3(2) + \frac{11}{36}\pi^2\log(2) + \frac{99}{8}\zeta_3 - \frac{29}{1440}\pi^4 + \frac{1}{12}\pi^2\log^2(2) - \frac{1}{12}\log^4(2)\\ &\left. - \frac{7}{4}\zeta_3\log(2) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)\right) + \mathcal{O}\left(\varepsilon^2\right).\end{aligned} \tag{5.43}$$

5.3 Integration of the double-collinear subtraction term

We proceed with the calculation of the integral of the double-collinear $\mathfrak{m}||\mathfrak{n}$ subtraction term. It reads

$$\bar{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} = \mathcal{N}_A\left\langle \int\limits_0^1 \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\left(\psi_{t,\mathfrak{m}j}^{4\varepsilon} - 1\right) C_{\mathfrak{mn}}\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{mn}}. \tag{5.44}$$

The soft-subtracted eikonal function $\bar{S}_\omega[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})]$ has a singularity in the limit $\mathfrak{m}||\mathfrak{n}$. Isolating this singularity requires care as it appears, naively, as the second-order pole in $\rho_{\mathfrak{mn}}$ whereas the pole is, in fact, first-order.

The relevant limit can be easily computed. We obtain

$$C_{\mathfrak{mn}}\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] = \frac{4(1-\varepsilon)\omega^2(\vec{n}_j\cdot\vec{\kappa})^2}{(1+\omega)^4\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}^2} - \frac{4\omega(2-\rho_{\mathfrak{m}j})}{(1+\omega)^2\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}}, \tag{5.45}$$

where $\vec{\kappa}$ is defined by the following equation

$$\vec{n}_\mathfrak{n} = \cos\theta_{\mathfrak{mn}}\vec{n}_\mathfrak{m} + \sin\theta_{\mathfrak{mn}}\,\vec{\kappa}, \quad \vec{\kappa}\cdot\vec{n}_\mathfrak{m} = 0, \quad \vec{\kappa}^2 = 1. \tag{5.46}$$

To integrate over directions of parton $\mathfrak{n}$ in eq. (5.44), we first average over directions of $\vec{\kappa}$. Using

$$\kappa^i\kappa^j \to \frac{\delta^{ij} - \vec{n}_\mathfrak{m}^i\vec{n}_\mathfrak{m}^j}{d-2}, \tag{5.47}$$

we obtain

$$\left\langle C_{\mathfrak{mn}}\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_\kappa = -2\frac{(2-\rho_{\mathfrak{m}j})}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{mn}}}\,\frac{\omega(2+3\omega+2\omega^2)}{(1+\omega)^4}. \tag{5.48}$$

Since integrations over energies and angles factorize, it is straightforward to complete the calculation. We find

$$\frac{S_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}}}{\mathcal{N}_A} = \frac{(1-\varepsilon)(1-2\varepsilon)}{\varepsilon^2}\gamma_\omega + \frac{1-2\varepsilon}{\varepsilon}\gamma_\omega\rho_{tj}^{-4\varepsilon}\left(2I_{-4\varepsilon,1}^{(1)} - I_{-4\varepsilon}^{(1)}\right), \tag{5.49}$$

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where

$$\gamma_\omega = \int_0^1 \frac{d\omega}{\omega^{1+2\varepsilon}} \frac{\omega(2+3\omega+2\omega^2)}{(1+\omega)^4} = \frac{11}{12} + \varepsilon\left(\frac{1}{12} + \frac{11}{3}\log(2)\right) + \varepsilon^2\left(\frac{1}{3} + \frac{11}{18}\pi^2\right) + \varepsilon^3\left(\frac{4}{3}\log(2) + 11\zeta_3\right) + \mathcal{O}\left(\varepsilon^4\right). \tag{5.50}$$

5.4 Calculation of $\bar{S}_\omega[\Delta G_{ij}]_a$

To compute $\bar{S}_\omega[\Delta G_{ij}]_a$ defined in eq. (5.23), we integrate over ω and combine the obtained result with the one where $\mathfrak{m}$ and $\mathfrak{n}$ are interchanged. We find

$$\bar{S}_\omega[\Delta G_{ij}]_a = \frac{\mathcal{N}_A}{8\varepsilon}\left\langle\left(\left(\frac{\psi_{t,\mathfrak{n}j}}{\psi_{t,\mathfrak{m}j}}\right)^{2\varepsilon} - \left(\frac{\psi_{t,\mathfrak{m}j}}{\psi_{t,\mathfrak{n}j}}\right)^{2\varepsilon}\right)\frac{(\rho_{\mathfrak{mn}}-\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})(\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right\rangle_{\mathfrak{mn}}. \tag{5.51}$$

To compute this integral, we expand in ε through the relevant order

$$\frac{1}{8\varepsilon}\left(\left(\frac{\psi_{\mathfrak{n}j}}{\psi_{\mathfrak{m}j}}\right)^{2\varepsilon} - \left(\frac{\psi_{\mathfrak{m}j}}{\psi_{\mathfrak{n}j}}\right)^{2\varepsilon}\right) = \frac{1}{4}\left(g^{(2)}_{t,\mathfrak{n}j} - g^{(2)}_{t,\mathfrak{m}j}\right) - \frac{\varepsilon}{8}\left(\left(g^{(2)}_{t,\mathfrak{n}j}\right)^2 - \left(g^{(2)}_{t,\mathfrak{m}j}\right)^2\right) + \mathcal{O}(\varepsilon^2), \tag{5.52}$$

and get rid of $g^{(2)}_{t,\mathfrak{m}j}$ terms by using the $\mathfrak{m} \leftrightarrow \mathfrak{n}$ permutation symmetry of the integrand in eq. (5.51). We find

$$\bar{S}_\omega[\Delta G_{ij}]_a = \frac{\mathcal{N}_A}{2}\left\langle\left(g^{(2)}_{t,\mathfrak{n}j} - \frac{\varepsilon}{2}\left(g^{(2)}_{t,\mathfrak{n}j}\right)^2\right)\frac{(\rho_{\mathfrak{mn}}-\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})(\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right\rangle_{\mathfrak{mn}}. \tag{5.53}$$

Integration over $\mathfrak{m}$ can be easily performed. We obtain

$$\left\langle\frac{(\rho_{\mathfrak{mn}}-\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})(\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right\rangle_{\mathfrak{m}} = \frac{1-\varepsilon}{\varepsilon} + \rho_{\mathfrak{n}j} I^{(0)}_{11}(\rho_{\mathfrak{n}j}). \tag{5.54}$$

Using

$$g^{(2)}_{t,\mathfrak{n}j} - \frac{\varepsilon}{2}\left(g^{(2)}_{t,\mathfrak{n}j}\right)^2 = \frac{1}{2\varepsilon}\left(\psi^{2\varepsilon}_{t,\mathfrak{n}j} - \psi^{-2\varepsilon}_{t,\mathfrak{n}j}\right) + \mathcal{O}(\varepsilon^2), \tag{5.55}$$

we find

$$\bar{S}_\omega[\Delta G_{ij}]_a = \frac{\mathcal{N}_A}{4\varepsilon}\left\langle\left(\psi^{2\varepsilon}_{t,\mathfrak{n}j} - \psi^{-2\varepsilon}_{t,\mathfrak{n}j}\right)\left(\frac{1-\varepsilon}{\varepsilon} + \rho_{\mathfrak{n}j} I^{(0)}_{11}(\rho_{\mathfrak{n}j})\right)\right\rangle_{\mathfrak{n}}. \tag{5.56}$$

Performing the remaining integrations, we obtain

$$\frac{\bar{S}_\omega[\Delta G_{ij}]_a}{\mathcal{N}_A} = \frac{1-\varepsilon}{4\varepsilon^2}\left(\rho_{tj}^{-2\varepsilon} I^{(1)}_{-2\varepsilon} - \rho_{tj}^{2\varepsilon} I^{(1)}_{2\varepsilon}\right) - \frac{1-2\varepsilon}{8\varepsilon^2}\left\langle\left(\rho_{tj}^{-2\varepsilon}\rho_{t,\mathfrak{n}}^{2\varepsilon} - \rho_{tj}^{2\varepsilon}\rho_{t,\mathfrak{n}}^{-2\varepsilon}\right)\rho_{\mathfrak{n}j}\ {}_2F_1(1,1,1-\varepsilon,\frac{\rho_{\mathfrak{n}j}}{2})\right\rangle_{\mathfrak{n}}. \tag{5.57}$$

We use transformation properties of the hypergeometric function to rewrite the above result as follows

$$\frac{\bar{S}_\omega[\Delta G_{ij}]_a}{\mathcal{N}_A} = \frac{1-\varepsilon}{4\varepsilon^2}\left(\rho_{tj}^{-2\varepsilon} I^{(1)}_{-2\varepsilon} - \rho_{tj}^{2\varepsilon} I^{(1)}_{2\varepsilon}\right) - \frac{1-2\varepsilon}{2^{2-\varepsilon}\varepsilon^2}\left(\rho_{tj}^{-2\varepsilon} I^{(1)}_{-2\varepsilon,\varepsilon} - \rho_{tj}^{2\varepsilon} I^{(1)}_{2\varepsilon,\varepsilon}\right) - \varepsilon\left\langle\log\frac{\rho_{t,\mathfrak{n}}}{\rho_{tj}}\ \mathrm{Li}_2\left(1-\frac{\rho_{\mathfrak{n}j}}{2}\right)\right\rangle_{\mathfrak{n}} + \mathcal{O}(\varepsilon^2). \tag{5.58}$$

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5.5 Combined expression for the double-soft integral

We are now in a position to present the result for the integrated double-emission eikonal function where the divergent part is extracted analytically, and the finite part is written as an integral that can be computed numerically in straightforward way. Our starting point is eq. (5.2) which we repeat here

$$\mathbb{S}[\tilde{S}_{ij}] = G_{ij} = S_\omega[G_{ij}] + \bar{S}_\omega[G_{ij}^{(0)}] + \bar{S}_\omega[\Delta G_{ij}]. \tag{5.59}$$

The result for $S_\omega[G_{ij}]$ can be found in eq. (5.42) and the result for $\bar{S}_\omega[G_{ij}^{(0)}]$ in eq. (5.43). The quantity $\bar{S}_\omega[\Delta G_{ij}]$ is defined in eq. (5.21). There are three terms there. The result for $\bar{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}}$ is given in eq. (5.49), the result for $\bar{S}_\omega[\Delta G_{ij}]_a$ can be found in eq. (5.57), and $\bar{S}_\omega[\Delta G_{ij}]_{\rm num}$ is finite and can be computed numerically.

To present the final result, we separate contributions that are known analytically (R) and the ones that are finite and can be integrated numerically (N). We write

$$\mathbb{S}\left[\tilde{S}_{ij}\right] = \mathcal{N}_A\left(R+N\right), \tag{5.60}$$

where

$$\begin{aligned} R = {}& \frac{\bar{S}_\omega\left[G_{ij}^{(0)}\right]}{\mathcal{N}_A} - \frac{(1-2\varepsilon)(2-3\varepsilon)}{4\varepsilon^3} + \frac{(1-\varepsilon)(1-2\varepsilon)\gamma_\omega}{\varepsilon^2} + \frac{(1-2\varepsilon)^2}{2^{1-\varepsilon}\varepsilon^3} I_\varepsilon^{(0)} \\ & - \frac{(1-2\varepsilon)}{2\varepsilon^2}\rho_{tj}^{-4\varepsilon}(1+2\varepsilon\gamma_\omega) I_{-4\varepsilon}^{(1)} - \frac{1-\varepsilon}{4\varepsilon^2}\rho_{tj}^{2\varepsilon} I_{2\varepsilon}^{(1)} + \frac{(1-2\varepsilon)}{\varepsilon^2}\rho_{tj}^{-4\varepsilon}(1+2\varepsilon\gamma_\omega) I_{-4\varepsilon,1}^{(1)} \\ & - \frac{2-5\varepsilon}{2\varepsilon^2}\rho_{tj}^{-2\varepsilon} I_{-2\varepsilon,1}^{(1)} + \frac{1}{\varepsilon}\rho_{tj}^{-4\varepsilon}\left(I_{-2\varepsilon,1}^{(1)}\right)^2 + \frac{(2-5\varepsilon)}{4\varepsilon^2}\rho_{tj}^{-2\varepsilon} I_{-2\varepsilon}^{(1)} - \frac{1}{2\varepsilon}\rho_{tj}^{-4\varepsilon} I_{-2\varepsilon}^{(1)} I_{-2\varepsilon,1}^{(1)} \\ & - \frac{3(1-2\varepsilon)}{2^{2-\varepsilon}\varepsilon^2}\rho_{tj}^{-2\varepsilon} I_{-2\varepsilon,\varepsilon}^{(1)} + \frac{(1-2\varepsilon)}{2^{2-\varepsilon}\varepsilon^2}\left(8\rho_{tj}^{-2\varepsilon} I_{-2\varepsilon,1+\varepsilon}^{(1)} + \rho_{tj}^{2\varepsilon} I_{2\varepsilon,\varepsilon}^{(1)}\right), \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} N = {}& -\varepsilon\left\langle \frac{\log^2 \rho_{t\mathfrak{m}}/\rho_{t\mathfrak{n}}}{\rho_{\mathfrak{m}\mathfrak{n}}} + \frac{2\log\psi_{\mathfrak{m}j}\left(\log\psi_{\mathfrak{m}j}(\rho_{\mathfrak{m}j}-4) - \log\psi_{\mathfrak{n}j}(\rho_{\mathfrak{n}j}-4)\right)}{\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{m}j}} \right\rangle_{\mathfrak{m}\mathfrak{n}} \\ & - 2\varepsilon\left\langle \int_0^\infty \frac{{\rm d}\omega\log\omega}{\omega}\left(\theta(\psi_{t,\mathfrak{m}j}-\omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)\right)\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}} \\ & + 4\varepsilon\left\langle \int_0^\infty \frac{{\rm d}\omega\log\psi_{t,\mathfrak{m}j}}{\omega}\left(\theta(\psi_{t,\mathfrak{m}j}-\omega\psi_{t,\mathfrak{n}j}) - \theta(1-\omega)C_{\mathfrak{m}\mathfrak{n}}\right)\bar{S}_\omega\left[\omega^2\tilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}} \\ & + \varepsilon\left\langle \frac{4-2\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{m}j}}{\rm Li}_2\left(1-\frac{\rho_{\mathfrak{m}j}}{2}\right)\log\frac{\rho_{t\mathfrak{m}}}{\rho_{tj}}\right\rangle_{\mathfrak{m}}. \end{aligned} \tag{5.62}$$

We have used the above representation to compute the integrated double-emission eikonal function $\mathbb{S}[\tilde{S}_{ij}]$ for various values of β and $\cos\theta$ and found excellent agreement with the results reported in section 4. We emphasize that all divergent contributions are known analytically[7] and that only integrals in eq. (5.62) need to be evaluated numerically. The quality and the speed of this numerical integration are acceptable; obviously, it depends rather strongly on values of β and $\cos\theta$, with values $\beta\sim 1$ being the most challenging.

[7] Explicit results for integrals $I_{\alpha,\beta}^{(1)}$ can be found in appendix C and in the supplementary material file.

6 Conclusion

We have described the computation of the integrated double-emission eikonal function with one massive and one massless emitter. This quantity is a required ingredient for extending the nested soft-collinear subtraction scheme [43] to processes with massive particles. We have shown that the extension of reverse unitarity to cases with Heaviside functions in the integrand [57] allows one to simplify the calculation significantly, and derive the differential equations for relevant master integrals in a straightforward way. Although the resulting differential equations contain two sectors with elliptic integrals, we were able to remove them from the final result and express it in terms of ordinary polylogarithms and $\mathrm{Li}_{2,2}$. As such, the result appears to be structurally similar to the one reported in ref. [54], but it is significantly more complex. Since, even after all the simplifications of the final result, it remains complicated, we have developed fast and efficient `C` code that can be used to compute the result for an arbitrary kinematic point with high precision.

Finally, we have studied the possibility to calculate the same quantity in an entirely different way, partially motivated by the calculation of the N-jettiness soft function in ref. [60]. The idea is to perform the computation in the rest frame of a massive parton, where the dependence on the original angle between two partons appears in kinematic constraints only. Identifying singularities of the double-emission eikonal function in this frame, allows us to compute the divergent contributions with relative ease, and design a simple representation for the finite remainder that can be integrated numerically right away. Apart from providing an important cross check for the calculation based on reverse unitarity and differential equations, this approach can open the way for computing integrated subtracted terms independently of the complexity of an "observable", used to define them. We look forward to further applications of this methodology in the future.

Acknowledgments

D.H. thanks Institute for Theoretical Particle Physics (TTP) at KIT for support during the early stages of this project. K.M. is grateful to CERN Theory Department for hospitality extended to him during work on this paper. This research was supported in part by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under grant no. 396021762 - TRR 257.

A Master integrals

In this appendix, the complete list of master integrals is given.

$$J_1 = \Delta^{0;0;0}_{0,0,0}(0) = \int [\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]\delta\left(1 - l_\mathfrak{m}\cdot P\right)\delta\left(1 - l_\mathfrak{n}\cdot P\right), \tag{A.1}$$

$$J_2 = \Delta^{0;0;1}_{0,0,0}(0) = \int [\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]\frac{\delta\left(1 - l_\mathfrak{m}\cdot P\right)\delta\left(1 - l_\mathfrak{n}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.2}$$

$$J_3 = \Delta^{0;0;0}_{0,0,1}(0) = \int [\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]\frac{\delta\left(1 - l_\mathfrak{m}\cdot P\right)\delta\left(1 - l_\mathfrak{n}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)}, \tag{A.3}$$

$$J_4 = \Delta^{0;0;0}_{0,0,2}(0) = \int [\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]\frac{\delta\left(1 - l_\mathfrak{m}\cdot P\right)\delta\left(1 - l_\mathfrak{n}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)^2}, \tag{A.4}$$

$$J_5 = \Delta^{0;0;0}_{1,0,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)}, \tag{A.5}$$

$$J_6 = \Delta^{0;1;0}_{1,0,0}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{n}\cdot p_j)(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.6}$$

$$J_7 = \Delta^{0;2;0}_{1,0,0}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{n}\cdot p_j)^2(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.7}$$

$$J_8 = \Delta^{0;1;0}_{2,0,0}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)^2(l_\mathtt{n}\cdot p_j)(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.8}$$

$$J_9 = \Delta^{1;0;0}_{1,0,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{m}\cdot p_j)}, \tag{A.9}$$

$$J_{10} = \Delta^{0;1;0}_{0,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{n}\cdot p_j)}, \tag{A.10}$$

$$J_{11} = \Delta^{0;1;-1}_{0,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)\,(l_\mathtt{mn}\cdot p_j)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{n}\cdot p_j)}, \tag{A.11}$$

$$J_{12} = \Delta^{0;1;0}_{0,0,1}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{n}\cdot p_j)(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.12}$$

$$J_{13} = \Delta^{0;2;0}_{0,0,1}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{n}\cdot p_j)^2(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.13}$$

$$J_{14} = \Delta^{0;1;0}_{0,0,2}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)^2(l_\mathtt{n}\cdot p_j)(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.14}$$

$$J_{15} = \Delta^{1;1;0}_{0,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{m}\cdot p_j)(l_\mathtt{n}\cdot p_j)}, \tag{A.15}$$

$$J_{16} = \Delta^{0;0;0}_{1,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{mn}\cdot p_i)}, \tag{A.16}$$

$$J_{17} = \Delta^{0;1;0}_{1,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{mn}\cdot p_i)(l_\mathtt{n}\cdot p_j)}, \tag{A.17}$$

$$J_{18} = \Delta^{0;0;0}_{1,1,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{n}\cdot p_i)}, \tag{A.18}$$

$$J_{19} = \Delta^{1;0;0}_{1,1,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{n}\cdot p_i)(l_\mathtt{m}\cdot p_j)}, \tag{A.19}$$

$$J_{20} = \Delta^{1;1;0}_{1,1,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{n}\cdot p_i)(l_\mathtt{m}\cdot p_j)(l_\mathtt{n}\cdot p_j)}, \tag{A.20}$$

$$J_{21} = \Delta^{0;0;1}_{0,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{mn}\cdot p_j)}, \tag{A.21}$$

$$J_{22} = \Delta^{0;0;2}_{0,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{mn}\cdot p_j)^2}, \tag{A.22}$$

$$J_{23} = \Delta^{1;0;1}_{0,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{mn}\cdot p_i)(l_\mathtt{m}\cdot p_j)(l_\mathtt{mn}\cdot p_j)}, \tag{A.23}$$

$$J_{24} = \Delta^{0;0;1}_{1,0,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{mn}\cdot p_j)}, \tag{A.24}$$

$$J_{25} = \Delta^{0;0;2}_{1,0,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta(1-l_\mathtt{m}\cdot P)\,\delta(1-l_\mathtt{n}\cdot P)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{mn}\cdot p_j)^2}, \tag{A.25}$$

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$$J_{26} = \Delta^{0,0;1}_{2,0,0}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)^2(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.26}$$

$$J_{27} = \Delta^{0,0;1}_{1,0,0}(1) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})}, \tag{A.27}$$

$$J_{28} = \Delta^{0,0;2}_{1,0,0}(1) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)^2(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})}, \tag{A.28}$$

$$J_{29} = \Delta^{0,0;1}_{2,0,0}(1) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)^2(l_{\mathfrak{mn}}\cdot p_j)(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})}, \tag{A.29}$$

$$J_{30} = \Delta^{0,0;1}_{1,0,1}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.30}$$

$$J_{31} = \Delta^{0,0;2}_{1,0,1}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)^2}, \tag{A.31}$$

$$J_{32} = \Delta^{0,0;1}_{1,0,2}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_i)^2(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.32}$$

$$J_{33} = \Delta^{1,0;1}_{1,0,1}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_i)(l_{\mathfrak{m}}\cdot p_j)(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.33}$$

$$J_{34} = \Delta^{0,0;1}_{1,1,0}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{n}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.34}$$

$$J_{35} = \Delta^{0,0;2}_{1,1,0}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{n}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)^2}, \tag{A.35}$$

$$J_{36} = \Delta^{0,0;1}_{1,2,0}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{n}}\cdot p_i)^2(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.36}$$

$$J_{37} = \Delta^{1,0;1}_{1,1,0}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\delta\left(1-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{m}}\cdot p_i)(l_{\mathfrak{n}}\cdot p_i)(l_{\mathfrak{m}}\cdot p_j)(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.37}$$

$$J_{38} = \Theta^{0,0;0}_{0,0,1}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)}, \tag{A.38}$$

$$J_{39} = \Theta^{0,0;0}_{0,0,2}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)^2}, \tag{A.39}$$

$$J_{40} = \Theta^{0,1;0}_{0,0,1}(1) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)(l_{\mathfrak{n}}\cdot p_j)(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})}, \tag{A.40}$$

$$J_{41} = \Theta^{0,2;0}_{0,0,1}(1) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)(l_{\mathfrak{n}}\cdot p_j)^2(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})}, \tag{A.41}$$

$$J_{42} = \Theta^{0,1;0}_{0,0,2}(1) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)^2(l_{\mathfrak{n}}\cdot p_j)(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})}, \tag{A.42}$$

$$J_{43} = \Theta^{0,0;1}_{0,0,1}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.43}$$

$$J_{44} = \Theta^{0,0;2}_{0,0,1}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{mn}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)^2}, \tag{A.44}$$

$$J_{45} = \Theta^{0,0;1}_{0,1,0}(0) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{n}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)}, \tag{A.45}$$

$$J_{46} = \Theta^{0,0;1}_{0,1,0}(1) \; = \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}] \frac{\delta\left(1-l_{\mathfrak{m}}\cdot P\right)\theta\left(l_{\mathfrak{m}}\cdot P-l_{\mathfrak{n}}\cdot P\right)}{(l_{\mathfrak{n}}\cdot p_i)(l_{\mathfrak{mn}}\cdot p_j)(l_{\mathfrak{m}}\cdot l_{\mathfrak{n}})}, \tag{A.46}$$

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$$J_{47} = \Theta^{0;0;2}_{0,1,0}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta\left(1 - l_\mathtt{m}\cdot P\right)\theta\left(l_\mathtt{m}\cdot P - l_\mathtt{n}\cdot P\right)}{(l_\mathtt{n}\cdot p_i)(l_\mathtt{mn}\cdot p_j)^2(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.47}$$

$$J_{48} = \Theta^{0;0;1}_{0,2,0}(1) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta\left(1 - l_\mathtt{m}\cdot P\right)\theta\left(l_\mathtt{m}\cdot P - l_\mathtt{n}\cdot P\right)}{(l_\mathtt{n}\cdot p_i)^2(l_\mathtt{mn}\cdot p_j)(l_\mathtt{m}\cdot l_\mathtt{n})}, \tag{A.48}$$

$$J_{49} = \Theta^{0;0;1}_{1,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta\left(1 - l_\mathtt{m}\cdot P\right)\theta\left(l_\mathtt{m}\cdot P - l_\mathtt{n}\cdot P\right)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{mn}\cdot p_i)(l_\mathtt{mn}\cdot p_j)}, \tag{A.49}$$

$$J_{50} = \Theta^{0;0;2}_{1,0,1}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta\left(1 - l_\mathtt{m}\cdot P\right)\theta\left(l_\mathtt{m}\cdot P - l_\mathtt{n}\cdot P\right)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{mn}\cdot p_i)(l_\mathtt{mn}\cdot p_j)^2}, \tag{A.50}$$

$$J_{51} = \Theta^{0;0;1}_{1,1,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta\left(1 - l_\mathtt{m}\cdot P\right)\theta\left(l_\mathtt{m}\cdot P - l_\mathtt{n}\cdot P\right)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{n}\cdot p_i)(l_\mathtt{mn}\cdot p_j)}, \tag{A.51}$$

$$J_{52} = \Theta^{0;0;2}_{1,1,0}(0) = \int [\mathrm{d}l_\mathtt{m}][\mathrm{d}l_\mathtt{n}] \frac{\delta\left(1 - l_\mathtt{m}\cdot P\right)\theta\left(l_\mathtt{m}\cdot P - l_\mathtt{n}\cdot P\right)}{(l_\mathtt{m}\cdot p_i)(l_\mathtt{n}\cdot p_i)(l_\mathtt{mn}\cdot p_j)^2}. \tag{A.52}$$

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B Boundary conditions

Here we list integrals used to compute the boundary conditions at $\beta = 0$.

$$J_1 = \mathcal{N}_\varepsilon{}^2, \tag{B.1}$$

$$\frac{J_2}{J_1} = \frac{2^{-1-2\varepsilon}(1-2\varepsilon)^2\Gamma^4(1-2\varepsilon)\Gamma(\varepsilon)}{(1-4\varepsilon)\Gamma(1-4\varepsilon)\Gamma^3(1-\varepsilon)} - \frac{(1-2\varepsilon)}{2\varepsilon}\,{}_3F_2(1,1-\varepsilon,2\varepsilon;2-2\varepsilon,1+\varepsilon;-1) \tag{B.2}$$

$$= \log(2) + \varepsilon\left(\frac{\pi^2}{12} - \log^2(2)\right) + \varepsilon^2\left(-\zeta_3 + \frac{2\log^3(2)}{3} - \frac{1}{6}\pi^2\log(2) + 4\log(2)\right)$$
$$+ \varepsilon^3\left(-12\zeta_3\log(2) - 16\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{5\pi^4}{144} + \frac{\pi^2}{3} - \log^4(2)\right.$$
$$\left. + \frac{5}{6}\pi^2\log^2(2) - 4\log^2(2) + 16\log(2)\right) + \mathcal{O}\left(\varepsilon^4\right),$$

$$\frac{J_{38}}{J_1} = \frac{1}{2}\left(\Psi\left(3/2-\varepsilon\right) - \Psi\left(1-\varepsilon\right)\right) \tag{B.3}$$
$$= (1-\log(2)) + \varepsilon\left(2 - \frac{\pi^2}{6}\right) + \varepsilon^2\left(4 - 3\zeta_3\right) + \varepsilon^3\left(8 - \frac{7\pi^4}{90}\right) + \mathcal{O}\left(\varepsilon^4\right),$$

$$\frac{J_{43}}{J_1} = \int_0^1 \frac{\mathrm{d}z}{1+z}\left[-z^{-2\varepsilon}\frac{(1-2\varepsilon)}{2\varepsilon}\,{}_3F_2\left(1,1-\varepsilon,2\varepsilon;2-2\varepsilon,1+\varepsilon;-\frac{1}{z}\right)\right. \tag{B.4}$$
$$\left. + z^{-\varepsilon}\frac{(1-2\varepsilon)^2\Gamma(1-2\varepsilon)^3\Gamma(\varepsilon)}{2(1-3\varepsilon)\Gamma(1-3\varepsilon)\Gamma^2(1-\varepsilon)}\,{}_2F_1\left(1-2\varepsilon,\varepsilon;2-3\varepsilon;-\frac{1}{z}\right)\right]$$
$$= \log(2) - \frac{1}{24}\pi^2 + \varepsilon\left(4\log(2) - \log^2(2) + \frac{1}{12}\pi^2 - \frac{17}{8}\zeta_3\right)$$
$$+ \varepsilon^2\left(+20\log(2) - 4\log^2(2) + \frac{1}{6}\pi^2 + \frac{2}{3}\log^3(2) - \frac{1}{6}\pi^2\log(2) - \zeta_3\right.$$
$$\left. - \frac{5}{12}\log^4(2) + \frac{1}{480}\pi^4 + \frac{5}{12}\pi^2\log^2(2) - \frac{35}{4}\zeta_3\log(2) - 10\mathrm{Li}_4\left(\frac{1}{2}\right)\right) + \mathcal{O}\left(\varepsilon^3\right),$$

$$\frac{J_{45}}{J_1} = -\frac{1}{2\varepsilon}\,{}_4F_3(1,1-2\varepsilon,1-\varepsilon,2\varepsilon;2-2\varepsilon,2-2\varepsilon,1+\varepsilon;-1) \tag{B.5}$$
$$+\frac{(1-2\varepsilon)^2\Gamma^3(1-2\varepsilon)\Gamma(\varepsilon)}{2(1-3\varepsilon)^2\Gamma(1-3\varepsilon)\Gamma^2(1-\varepsilon)}\,{}_3F_2(1-3\varepsilon,1-2\varepsilon,\varepsilon;2-3\varepsilon,2-3\varepsilon;-1)$$
$$=\left(\frac{\pi^2}{24}+\log(2)\right)+\varepsilon\left(\frac{5\zeta_3}{8}+\frac{\pi^2}{12}-\log^2(2)+4\log(2)\right)$$
$$+\varepsilon^2\Bigg(-\zeta_3-\frac{21}{4}\zeta_3\log(2)-6\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{61\pi^4}{1440}+\frac{\pi^2}{2}-\frac{\log^4(2)}{4}+\frac{2\log^3(2)}{3}$$
$$+\frac{1}{4}\pi^2\log^2(2)-4\log^2(2)-\frac{1}{6}\pi^2\log(2)+20\log(2)\Bigg)+\mathcal{O}\left(\varepsilon^3\right).$$

C Angular integrals

In this appendix, we define angular integrals which are used for calculations in section 5.

$$\Big\langle 1\Big\rangle_{\mathfrak{m}} = \Big\langle 1\Big\rangle_{\mathfrak{mn}} = 1, \tag{C.1}$$
$$\left\langle \frac{1}{\rho^n_{\mathfrak{m}x}}\right\rangle_{\mathfrak{m}} = I^{(1)}_n\left[\rho_{xx}\right], \qquad \rho_{xx}\neq 0, \tag{C.2}$$
$$\left\langle \frac{1}{\rho^a_{\mathfrak{m}x}\rho^b_{\mathfrak{m}y}}\right\rangle_{\mathfrak{m}} = I^{(0)}_{a,b}\left[\rho_{xy}\right], \qquad \rho_{xx}=\rho_{yy}=0, \tag{C.3}$$
$$\left\langle \frac{1}{\rho^a_{\mathfrak{m}x}\rho^b_{\mathfrak{m}y}}\right\rangle_{\mathfrak{m}} = I^{(1)}_{a,b}\left[\rho_{xx},\rho_{xy}\right], \quad \rho_{xx}\neq 0, \rho_{yy}=0. \tag{C.4}$$

These integrals can be computed in terms of hypergeometric functions and the Appell function

$$I^{(1)}_n\left[\rho_{11}\right] = \left(1+\sqrt{1-\rho_{11}}\right)^{-n}{}_2F_1\left(n,1-\varepsilon,2-2\varepsilon;\frac{2\sqrt{1-\rho_{11}}}{1+\sqrt{1-\rho_{11}}}\right), \tag{C.5}$$
$$I^{(0)}_{a,b}\left[\rho_{12}\right] = \frac{\Gamma(2-2\varepsilon)\,\Gamma(1-\varepsilon-a)\,\Gamma(1-\varepsilon-b)}{2^{a+b}\Gamma^2(1-\varepsilon)\,\Gamma(2-2\varepsilon-a-b)}\,{}_2F_1\left(a,b,1-\varepsilon;1-\frac{\rho_{12}}{2}\right), \tag{C.6}$$
$$I^{(1)}_{a,b}\left[\rho_{11},\rho_{12}\right] = \frac{2^{-b}}{\rho^a_{12}}\frac{\Gamma(2-2\varepsilon)\,\Gamma(1-\varepsilon-b)}{\Gamma(1-\varepsilon)\,\Gamma(2-2\varepsilon-b)} \tag{C.7}$$
$$\times F_1\left(a,1-\varepsilon-b,1-\varepsilon-b,2-2\varepsilon-b;1-\frac{1+\sqrt{1-\rho_{11}}}{\rho_{12}},1-\frac{1-\sqrt{1-\rho_{11}}}{\rho_{12}}\right).$$

Expansions of all integrals needed for computations in section 5 in ε to the required order can be found in the supplementary material files. Here, we present a few terms of such an expansion for illustration purposes. To present the result, we use variables $x=\beta$ and $y=\cos\theta$ that refer to the $P=(1,\vec{0})$ frame. In the massive-parton rest frame, we find

$$\rho_{tt}=1-x^2=1-\beta^2, \quad \rho_{tj}=\frac{1-\beta^2}{1-\beta\cos\theta}=\frac{1-x^2}{1-xy}. \tag{C.8}$$

The results for integrals read

$$I^{(1)}_{\alpha\varepsilon}\left[\rho_{tt}\right] = 1+\alpha\varepsilon\left(1+\frac{1-x}{2x}\log(1-x)-\frac{1+x}{2x}\log(1+x)\right) \tag{C.9}$$
$$+\varepsilon^2\Bigg(\alpha(\alpha+2)-\frac{\pi^2\alpha}{6x}+\frac{1}{2}\alpha^2\log^2(1-x)-\alpha^2\log(1-x)$$

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$$+\frac{\alpha}{x}\left(\frac{(\alpha(1+x)+2\log(2))}{2}-\frac{(\alpha(1+x)+2)\log(1-x)}{2}+\log(x)\right)\log\left(\frac{1-x}{1+x}\right)$$
$$+\frac{\alpha(\alpha(1+x)+x+3)}{4x}\log^2\left(\frac{1-x}{1+x}\right)+\frac{\alpha}{x}\mathrm{Li}_2\left(\frac{1-x}{1+x}\right)\Bigg)+\mathcal{O}\left(\varepsilon^3\right),$$

$$I^{(1)}_{\alpha\varepsilon,1}\left[\rho_{tt},\rho_{tj}\right]=-\frac{1}{2\varepsilon}+1+\frac{\alpha}{2}\log\frac{1-x^2}{1-xy}+\varepsilon\Bigg[\alpha\Bigg(-\frac{1}{2}\log^2\left(\frac{1-x^2}{1-xy}\right)\tag{C.10}$$
$$+\frac{1}{2}\log\left(1-x^2\right)\log\left(\frac{1-x^2}{1-xy}\right)-\log\left(\frac{1-x^2}{1-xy}\right)+\frac{1}{2}\log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{x(1-y)}{1-xy}\right)$$
$$+\frac{1}{2}\log\left(\frac{x+1}{1-xy}\right)\log\left(\frac{x(y+1)}{xy-1}\right)-\frac{1}{4}\log^2(1-x)-3\log^2(x+1)-2\pi^2$$
$$+\frac{1}{2}\mathrm{Li}_2\left(\frac{1-x}{1-xy}\right)+\frac{1}{2}\mathrm{Li}_2\left(\frac{1+x}{1-xy}\right)\Bigg)-\frac{1}{4}\alpha^2\log^2\left(\frac{1-x^2}{1-xy}\right)\Bigg]+\mathcal{O}\left(\varepsilon^2\right),$$

$$I^{(1)}_{\alpha\varepsilon,\varepsilon}\left[\rho_{tt},\rho_{tj}\right]=1+\varepsilon\left[1-\log(2)+\alpha\left(1+\frac{1}{2x}\left(+\log\frac{1-x}{1+x}-x\log\left(1-x^2\right)\right)\right)\right]\tag{C.11}$$
$$+\varepsilon^2\Bigg[-\frac{\pi^2}{6}+3+\frac{1}{2}(\log(2)-2)\log(2)+\alpha\Bigg(-\frac{1}{2x}\Bigg(-8x-\left((x-1)\log^2(1-x)\right)$$
$$-(x+1)\log^2(x+1)-2x\log^2(2)+(x-1)(1+\log(2))\log(1-x)$$
$$+(x\log(2)+1+\log(2))\log(x+1)+x\log(4(x+1))$$
$$+(1+x)\log\left(\frac{x-1}{xy-1}\right)\log\left(\frac{x(y-1)}{xy-1}\right)+(x-1)\log\left(\frac{x+1}{1-xy}\right)\log\left(\frac{x(y+1)}{xy-1}\right)\Bigg)$$
$$-\frac{(1+x)}{2x}\mathrm{Li}_2\left(\frac{1-x}{1-xy}\right)+\frac{(1-x)}{2x}\mathrm{Li}_2\left(\frac{1+x}{1-xy}\right)+\frac{2}{x}\mathrm{Li}_2(-x)-\frac{2}{x}\mathrm{Li}_2(x)$$
$$-\frac{(1-x)}{x}\mathrm{Li}_2\left(\frac{1-x}{2}\right)+\frac{(1+x)}{x}\mathrm{Li}_2\left(\frac{1+x}{2}\right)\Bigg)+\frac{\alpha^2}{4x}\Bigg(4x+(x-1)\log^2(1-x)$$
$$-2(x-1)\log(1-x)+(x+1)(\log(x+1)-2)\log(x+1)\Bigg)\Bigg]+\mathcal{O}\left(\varepsilon^3\right),$$

$$I^{(1)}_{\alpha\varepsilon,1+\varepsilon}\left[\rho_{tt},\rho_{tj}\right]=-\frac{1}{4\varepsilon}+\frac{1}{4}\left(2+\log(2)+\alpha\log\left(\frac{1-x^2}{1-xy}\right)\right)\tag{C.12}$$
$$+\varepsilon\Bigg[\frac{1}{24}\left(\pi^2-3\log(2)(4+\log(2))\right)+\alpha\Bigg(\frac{1}{2}\mathrm{Li}_2\left(\frac{1-x}{1-xy}\right)+\frac{1}{2}\mathrm{Li}_2\left(\frac{1+x}{1-xy}\right)$$
$$-\frac{1}{2}\log^2\left(\frac{1-x^2}{1-xy}\right)+\frac{1}{2}\log\left(1-x^2\right)\log\left(\frac{1-x^2}{1-xy}\right)-\frac{(2+\log(2))}{4}\log\left(\frac{1-x^2}{1-xy}\right)$$
$$+\frac{1}{2}\log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{x(1-y)}{1-xy}\right)+\frac{1}{2}\log\left(\frac{1+x}{1-xy}\right)\log\left(\frac{x(1+y)}{xy-1}\right)$$
$$-\frac{1}{4}\log^2(1-x)-\frac{1}{4}\log^2(x+1)-\frac{1}{6}\pi^2\Bigg)-\frac{1}{8}\alpha^2\log^2\left(\frac{x^2-1}{xy-1}\right)\Bigg]+\mathcal{O}\left(\varepsilon^2\right).$$

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D Direct integration

To check master integrals, we compute them numerically in higher-dimensional space-time where they are finite. We use the following parameterization of vectors

$$\begin{aligned} p_i &= (1,\ \vec{0}_{d-5},\ 0,\ 0,\ \beta\sin\theta,\ \beta\cos\theta),\\ p_j &= (1,\ \vec{0}_{d-5},\ 0,\ 0,\ 0,\ 1),\\ l_1 &= (1,\ \vec{0}_{d-5},\ 0,\ \sin\theta_{11}\sin\theta_{12},\ \sin\theta_{11}\cos\theta_{12},\ \cos\theta_{11}),\\ l_2 &= z(1,\ \vec{0}_{d-5},\ \sin\theta_{21}\sin\theta_{22}\sin\theta_{23},\ \sin\theta_{21}\sin\theta_{22}\cos\theta_{23},\ \sin\theta_{21}\cos\theta_{22},\ \cos\theta_{21}). \end{aligned} \tag{D.1}$$

The integration measure reads

$$\int \mathrm{d}^d l\ \delta(l^2)\theta(E_l)f(l) = \int \frac{\mathrm{d}^{d-1}\vec{l}}{2E_l} f(l) = \int \frac{\mathrm{d}E_l}{2E_l^{3-d}} \int \mathrm{d}\Omega_l^{(d-1)} f(l). \tag{D.2}$$

Parameterizing the relevant angles as $\cos\theta_{ij} = 1 - 2\lambda_{ij}$, $\sin\theta_{ij} = 2\sqrt{\lambda_{ij}\bar{\lambda}_{ij}}$, we write integration over solid angles as follows

$$\begin{aligned} \int \mathrm{d}\Omega_1^{(d-1)}\mathrm{d}\Omega_2^{(d-1)} &= 32\Omega^{(d-4)}\Omega^{(d-3)} \int_0^1 \mathrm{d}\lambda_{11}\mathrm{d}\lambda_{12}\mathrm{d}\lambda_{21}\mathrm{d}\lambda_{22}\mathrm{d}\lambda_{23}\\ &\times \Lambda_{11}^{d_0-4}\Lambda_{12}^{d_0-5}\Lambda_{21}^{d_0-4}\Lambda_{22}^{d_0-5}\Lambda_{23}^{d_0-6} \sum_{k=0}^{\infty} \varepsilon^k \frac{\left[-\log\left(\Lambda_{11}^2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{22}^2\Lambda_{23}^2\right)\right]^k}{k!}, \end{aligned} \tag{D.3}$$

where $\Lambda_{ij} = 2\sqrt{\lambda_{ij}(1-\lambda_{ij})}$ and $d = d_0 - 2\varepsilon$.

E Supplementary material files

The following supplementary material files, in `Mathematica`-readable format, are provided with this paper:

- `I11` contains the expansion of the single-emission soft integral, defined in eq. (3.8), up to $\mathcal{O}(\varepsilon^3)$ in terms of conventional polylogarithms and $\mathrm{Li}_{2,2}$;
- `Im0exp` contains expansion of angle integrals $I_{a,b}^{(1)}[\rho_{xx}, \rho_{xy}]$ to higher orders in ε, see appendix C;
- `SSm0_tldI` contains the integrated double-emission eikonal function $\mathcal{S}\left[\widetilde{\mathcal{I}}_{ij}\right]$, with the normalization factor $-\mathcal{N}_\varepsilon^2/(4\varepsilon E_{\max}^{4\varepsilon})$ omitted, in terms of polylogarithms and $\mathrm{Li}_{2,2}$;
- `SSm0_tldS` contains the integrated double-emission eikonal function $\mathcal{S}\left[\widetilde{\mathcal{S}}_{ij}\right]$, with the normalization factor $-\mathcal{N}_\varepsilon^2/(4\varepsilon E_{\max}^{4\varepsilon})$ omitted, in terms of polylogarithms and $\mathrm{Li}_{2,2}$.

To retrieve the results for integrated eikonal functions, files `SSm0_tldI` and `SSm0_tldS` have to loaded into a `Mathematica` session, together with the package `PolylogTools` [89], since it provides an interface to `GiNaC` [93] which is needed for $\mathrm{Li}_{2,2}$ evaluation. An example of a `Mathematica` session is shown below.

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```
(* Numerical values specify beta (xNum) and cos(theta) (yNum) values *)
With[{xNum = 0.2, yNum = 0.3},
(* Gluons, Quarks *)
   { Get["SSm0_tldS"],Get["SSm0_tldI"]}
    /. {eta -> (1 - 3*x*y - Sqrt[1 + 8*x^2 - 6*x*y + x^2*y^2])/x}
    /. {Li22[a_, b_] :> Ginsh[Li[{2, 2}, {b, a}], {x -> xNum, y -> yNum}]}
    /. {x -> xNum, y -> yNum}]
(* Produced output:
{
-(1/(4 ep^3)) + 0.00020226/ep^2 - 0.440986/ep + 0.0147285 + 2.08178 ep + O[ep]^2,
-(1/(12 ep^2)) + 0.102352/ep + 0.018446 + 0.0847638 ep + O[ep]^2
} *)
```

Here the output refers to $\mathscr{S}\left[\widetilde{S}_{ij}\right]$ and $\mathscr{S}\left[\widetilde{\mathcal{I}}_{ij}\right]$, divided by the normalization factor $-\mathcal{N}_\varepsilon^2/(4\varepsilon E_{\max}^{4\varepsilon})$, computed at the kinematic point $\beta = 0.2$ and $\cos\theta = 0.3$.

We also provide an implementation of $\mathscr{S}\left[\widetilde{S}_{ij}\right]$ and $\mathscr{S}\left[\widetilde{\mathcal{I}}_{ij}\right]$ in a `C` code. The code is available from the `Github` repository, and can be obtained using the following command

```
git clone https://github.com/apik/SSm0.git
```

Our implementation supports fast and accurate numerical evaluation of the relevant $\mathrm{Li}_{2,3,4}(x)$ and $\mathrm{Li}_{2,2}(x, y)$ functions with machine precision. We use algorithms described in ref. [94] with modifications suitable for our case, where only real-valued functions are involved and their arguments satisfy certain constraints. After downloading the code from GitHub, one can build the library, create an executable, and run it for the same point $\beta = 0.2, \cos\theta = 0.3$ using the following commands

```
$ make
$ ./ex_SSm0 0.2 0.3
```

The following output should then appear on the screen

```
   beta       =   0.20000000
   cos(theta) =   0.30000000
                      \tilde{I}(quarks)             \tilde{S}(gluons)
   ep^-3                 0.0000000000                  -0.2500000000
   ep^-2                -0.0833333333                   0.0002022602
   ep^-1                 0.1023516932                  -0.4409857886
   ep^ 0                 0.0184460303                   0.0147284599
   ep^ 1                 0.0847637908                   2.0817758455
```

To obtain $\mathscr{S}\left[\widetilde{S}_{ij}\right]$ and $\mathscr{S}\left[\widetilde{\mathcal{I}}_{ij}\right]$, one has to multiply these numbers by the normalization factor $-\mathcal{N}_\varepsilon^2/(4\varepsilon E_{\max}^{4\varepsilon})$.

Data Availability Statement. This article has no associated data or the data will not be deposited.

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Code Availability Statement. This article has code included as electronic supplementary material.

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