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Finitely generated infinite torsion groups that are residually finite simple

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ABSTRACT

We show that every countable residually finite torsion group G embeds in a finitely generated torsion group Γ that is residually *finite simple*. In particular we show the existence of finitely generated infinite torsion groups that are residually finite simple, which answers a question of Olshanskii and Osin.

1. Introduction

Let $\mathcal{C}$ be a class of groups. A group G is said to be *residually* $\mathcal{C}$ if the intersection of all normal subgroups N of G with $G/N \in \mathcal{C}$ is the trivial group. It is a classical problem in group theory to determine the classes of groups $\mathcal{C}$ for which a given group is residually $\mathcal{C}$ and a lot of research has been done in this direction, see e.g. [13]. A special instance of the latter was formulated in 1987 by Gromov [4] and became a notorious open problem in geometric group theory: Is every hyperbolic group residually finite, i.e. residually $\mathcal{F}$, where $\mathcal{F}$ denotes the class of finite groups? In 2008 it was shown by Olshanskii and

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Osin [12] that an affirmative answer to Gromov's question would imply the existence of finitely generated infinite torsion groups that are residually $\mathcal{FS}$, where $\mathcal{FS}$ denotes the class of finite simple groups. It was therefore natural to ask the following, see [12, Problem 3.4].

Problem 1.1. Does there exist an infinite finitely generated torsion group that is residually $\mathcal{FS}$?

Despite of a variety of techniques that are known to produce infinite finitely generated residually finite torsion groups that range from amenable [3,5] and non-amenable branch groups [14,6] to groups with Kazhdan's property (T) [1,2], and groups with positive first ℓ^2-Betti number [11,8], there was no construction known so far that produces infinite finitely generated torsion groups that are residually $\mathcal{FS}$. In fact there is a big obstruction for infinite finitely generated torsion groups to be residually $\mathcal{FS}$. To make this more precise, let us write $\mathcal{FS}_k$ to denote the subclass of $\mathcal{FS}$ that consists of groups that do not contain a subgroup isomorphic to $\mathrm{Alt}(k)$. The following observation is an easy consequence of a result of Lubotzky and Segal [10, Theorem 16.4.2(i), page 394] and certainly well-known to the experts. However, we take the opportunity to state it here. A proof of it will be given below Proposition 3.9.

Proposition 1.2. *Every finitely generated group G that is residually $\mathcal{FS}_k$ for some k can be realized as a subdirect product of finitely many linear groups.*

In particular, if such a group G is infinite, it admits an infinite finitely generated linear quotient, which is virtually torsion free. It therefore follows that the class of finitely generated groups that are residually $\mathcal{FS}_k$ for some k does not contain an infinite torsion group. In view of this, it can be easily seen that an affirmative answer to Problem 1.1 implies the existence of a torsion group Γ that is a subdirect product in $\prod_{i=1}^{\infty} S_i$, where $S_i \in \mathcal{FS}$ contains an isomorphic copy of $\mathrm{Alt}(i)$. We will show that such a group Γ indeed exists and thereby answer Problem 1.1 affirmatively. In fact we will see that every countable residually finite torsion group embeds in a group Γ as above.

Theorem 1.3. *Every countable residually finite torsion group embeds into a finitely generated torsion group that is residually $\mathcal{FS}$.*

The proof of Theorem 1.3 is based on the following idea. Consider a group G and a sequence of G-sets $(\Omega_i)_{i\in\mathbb{N}}$ that are represented by homomorphisms $\alpha_i\colon G \to \mathrm{Sym}(\Omega_i)$. For each i let $\tau_i \in \mathrm{Sym}(\Omega_i)$ be a permutation of Ω_i. Then, under suitable assumptions on τ_i and α_i, the subgroup Γ of $\prod_{i\in\mathbb{N}} \mathrm{Sym}(\Omega_i)$ that is generated by $(\tau_i)_{i\in\mathbb{N}}$ and the image of

$$\alpha\colon G \to \prod_{i\in\mathbb{N}} \mathrm{Sym}(\Omega_i),\ g \mapsto (\alpha_i(g))_{i\in\mathbb{N}}$$

will keep some of the properties of G, e.g. being torsion, while gaining some extra properties, e.g. being residually $\mathcal{FS}$. A related idea was recently applied in a work of Kionke and the author [7] in order to produce new examples of infinite finitely generated amenable simple groups.

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2. Extending actions of torsion groups

For the rest of this section we fix a torsion group G that acts on a set Ω. Let $\alpha\colon G \to \mathrm{Sym}(\Omega)$ denote the corresponding homomorphism. Let us moreover fix an element $p \in \Omega$ and let $\Omega^+ := \Omega \cup \{q\}$ for some $q \notin \Omega$. We are interested in the subgroup Γ of $\mathrm{Sym}(\Omega^+)$ that is generated by $\alpha(G)$ and the transposition $\tau = (p, q)$.

Notation 2.1. Let $F(X)$ denote the free group over a set X and let $w = x_{i_1} \dots x_{i_\ell} \in F(X)$ be a reduced word of length $\ell \in \mathbb{N}_0$. For each $0 \leq k \leq \ell$ we write $w_{[k]} := x_{i_{\ell-k+1}} \dots x_{i_\ell}$ to denote the terminal subword of length k in w.

Let us now consider the free group $F := F(G \cup \{\tau\})$. To simplify the notation we will often interpret a word $w \in F$ as an element of Γ, respectively G if $w \in F(G)$, as long as no ambiguity is possible.

Definition 2.2. For each word $w \in F$ of length $\ell \in \mathbb{N}_0$ and each point $\xi \in \Omega^+$, we define the *w-trace of ξ* as the sequence

$$\mathrm{Tr}_w(\xi) := (w_{[i]} \cdot \xi)_{i=1}^{\ell}.$$

Note that the w-trace of an element ξ does not necessarily contain ξ. Let us now fix a finite sequence $g_1, \dots, g_k$ of elements in G. In what follows we will study traces for the words

$$v_{n,i} = (g_1 \dots g_k)^n g_1 \dots g_i$$

and

$$w_{n,i} = (\tau g_1 \dots \tau g_k)^n \tau g_1 \dots \tau g_i$$

in F, where $n \in \mathbb{N}_0$ and $0 \leq i < k$.

Notation 2.3. Given a group H and an element $h \in H$, we write $\mathrm{o}_H(h) \in \mathbb{N} \cup \{\infty\}$ to denote the order of h in H.

Let us consider the element $g := g_1 \dots g_k \in G$ and let $N = \mathrm{o}_G(g)$.

Lemma 2.4. *Let $\xi \in \Omega^+$ and let $0 \leq i < k$. Suppose that p is not contained in $\mathrm{Tr}_{w_{N,i}}(\xi)$. Then p is not contained in $\mathrm{Tr}_{w_{n,i}}(\xi)$ for every $n \in \mathbb{N}_0$.*

Proof. If $\xi = q$, then p is clearly contained in $\mathrm{Tr}_{w_{N,i}}(\xi)$ so that there is nothing to show. Let us therefore assume that $\xi \in \Omega$ and that $\mathrm{Tr}_{w_{N,i}}(\xi)$ does not contain p. Since τ fixes every point in $\Omega \setminus \{p\}$, it follows that $\mathrm{Tr}_{v_{N,i}}(\xi)$ does not contain p. Thus there is no non-trivial terminal subword u of $(g_1 \dots g_k)^N g_1 \dots g_i$ that satisfies $u(\xi) = p$. Since

$$(g_1 \dots g_k)^N g_1 \dots g_i \cdot \xi = g_1 \dots g_i \cdot \xi,$$

it follows that $\mathrm{Tr}_{v_{aN+r,i}}(\xi)$ does not contain p for every $a \in \mathbb{N}_0$ and every $r < k$. Thus the same is true for $w_{aN+r,i}$, which proves the lemma. □

Lemma 2.5. *The element p is contained in $\mathrm{Tr}_{w_{N,i}}(p)$ for every $0 \leq i < k$.*

Proof. Suppose that p is not contained in $\mathrm{Tr}_{w_{N,i}}(p)$. Since τ fixes every point in $\Omega \setminus \{p\}$, it follows that p is not contained in $\mathrm{Tr}_{v_{N,i}}(p)$. However, this is not possible since the word $(g_{i+1} \dots g_k g_1 \dots g_i)^N$, which represents the trivial element in G, is a non-trivial terminal subword of $v_{N,i}$. □

Lemma 2.6. *Let $\xi \in \Omega^+$ and let $n \in \mathbb{N}_0$. Suppose that $\mathrm{Tr}_{w_{n,0}}(\xi)$ contains p. Then there are natural numbers m_1, m_2, j with $0 \leq m_1 < m_2 < N(k+1)$ and $0 \leq j < k$ such that*

$$w_{N(k+1),0} \cdot \xi = w_{m_1,j} \cdot p = w_{m_2,j} \cdot p.$$

Proof. From Lemma 2.4 we know that $\mathrm{Tr}_{w_{N,0}}(\xi)$ contains p. Thus there are integers $n_1 < N$ and $i_1 < k$ with

$$w_{N,0} \cdot \xi = w_{n_1,i_1} \cdot p$$

and therefore

$$w_{(k+1)N,0} \cdot \xi = w_{kN+n_1,i_1} \cdot p.$$

Now an inductive application of Lemma 2.5 provides us with integers $n_2, n_3, \dots, n_{k+1} < N$ and $i_2, i_3, \dots, i_{k+1} < k$ such that

$$\begin{aligned}w_{kN+n_1,i_1}\cdot p &= w_{(k-1)N+n_1+n_2,i_2}\cdot p\\ &= w_{(k-2)N+n_1+n_2+n_3,i_3}\cdot p\\ &\vdots\\ &= w_{n_1+\ldots+n_{k+1},i_{k+1}}\cdot p.\end{aligned}$$

Regarding this, the lemma follows from the pigeonhole principle applied to the sequence of indices $i_1,\ldots,i_{k+1}$. □

Lemma 2.7. *For every $\xi\in\Omega^+$ there is a natural number $m\le N(k+1)$ such that $w_{m,0}(\xi)=\xi$.*

Proof. Suppose first that p is not contained in $\mathrm{Tr}_{w_{N,0}}(\xi)$. Then p is not contained in $\mathrm{Tr}_{v_{N,0}}(\xi)$ and we obtain

$$w_{N,0}\cdot\xi=(g_1\ldots g_k)^N\cdot\xi=\xi.$$

Suppose next that p is contained in $\mathrm{Tr}_{w_{N,0}}(\xi)$. From Lemma 2.6 we know that there are natural numbers m_1,m_2,j with $0\le m_1<m_2<N(k+1)$ and $0\le j<k$ such that

$$w_{N(k+1),0}\cdot\xi=w_{m_1,j}\cdot p=w_{m_2,j}\cdot p.$$

In view of this, we see that $w_{m_2-m_1,0}\cdot\xi=\xi$, where $m_2-m_1\le(k+1)N$. □

3. Embedding torsion groups

In this section we will apply Lemma 2.7 in the case where the involved groups are finitely generated and residually finite. This will enable us to prove Theorem 1.3 from the introduction.

3.1. The finitely generated case

Let G, Γ, and Ω^+ be as above. Suppose now that G is finitely generated and let X be a finite generating set of G. In this case we can define the *torsion growth function of G with respect to X* as the function

$$T_G^X:\mathbb{N}\to\mathbb{N},\ \ell\mapsto\max\{\,\mathrm{o}_G(g)\mid g\in B_G^X(\ell)\,\},$$

where $B_G^X(\ell)$ denotes the set of elements of G whose word length with respect to X is bounded above by ℓ. We consider the generating set $X^+:=\alpha(X)\cup\{\tau\}$ of Γ.

Lemma 3.1. *Let $\ell\in\mathbb{N}$, let $\gamma\in B_\Gamma^{X^+}(\ell)$, and let $\xi\in\Omega^+$. The size of the orbit $\langle\gamma\rangle\cdot\xi$ is bounded above by $T_G^X(\ell)\cdot(\ell+1)$.*

Proof. Since the claim is trivial otherwise, we may assume that γ does not lie in $B_{\alpha(G)}^{\alpha(X)}(\ell)$. Thus, up to conjugation, we may assume that γ is represented by a word of the form

$$w = \tau g_1 \tau \dots \tau g_r,$$

where $\sum_{i=1}^{r} |g_i|_{\alpha(X)} \leq \ell$ and therefore $|g_1 \dots g_r|_X \leq \ell$. In this case we know from Lemma 2.7 that there is a natural number

$$m \leq T_G(\ell)(r+1) \leq T_G(\ell)(\ell+1)$$

such that $\gamma^m(\xi) = \xi$. □

Note that Lemma 3.1 has the following immediate consequence.

Corollary 3.2. *Every element $\gamma \in \Gamma$ satisfies*

$$\gamma^{(T_G^X(|\gamma|_{X^+}) \cdot (|\gamma|_{X^+}+1))!} = 1,$$

where $|\gamma|_{X^+}$ denotes the word length of γ with respect to X^+. In particular, Γ is a torsion group and $T_\Gamma^{X^+}$ is bounded above by the function $n \mapsto (T_G^X(n) \cdot (n+1))!$.

3.2. Families of actions

The crucial point of Corollary 3.2 is that the function

$$n \mapsto (T_G^X(n) \cdot (n+1))!$$

depends neither on the action of Γ on Ω^+ nor on the choice of the point $p \in \Omega$. This allows us to apply Corollary 3.2 simultaneously to a family of G-actions. To do so, we consider a family $(\Omega_i)_{i \in I}$ of G-sets Ω_i. Let $\alpha_i \colon G \to \mathrm{Sym}(\Omega_i)$ denote the homomorphism corresponding to the action of G on Ω_i. For each $i \in I$ we fix an element $p_i \in \Omega_i$ and let $(q_i)_{i \in \Omega}$ be a family of pairwise different elements that do not lie in $\cup_{i \in \mathbb{N}} \Omega_i$. Let $\Omega_i^+ := \Omega_i \cup \{q_i\}$ and let $\tau_i = (p_i, q_i) \in \mathrm{Sym}(\Omega_i^+)$. We consider the homomorphism

$$\alpha_I \colon G \to \prod_{i \in I} \mathrm{Sym}(\Omega_i^+), \ g \to (\alpha_i(g))_{i \in I}$$

and the sequence $\tau_I := (\tau_i)_{i \in I} \in \prod_{i \in I} \mathrm{Sym}(\Omega_i^+)$. Let Γ_I denote the subgroup of $\prod\limits_{i \in I} \mathrm{Sym}(\Omega_i^+)$ that is generated by $\alpha_I(G)$ and τ_I and let $X_I := \alpha_I(X) \cup \{\tau_I\}$, which is a finite generating set of Γ_I.

Proposition 3.3. *The torsion function $T_{\Gamma_I}^{X_I}$ of Γ_I with respect to X_I satisfies*

$$T_{\Gamma_I}^{X_I}(n) \leq (T_G^X(n) \cdot (n+1))!$$

for every $n \in \mathbb{N}$. In particular, Γ_I is a torsion group.

Proof. The claim directly follows by applying Corollary 3.2 simultaneously to the actions of Γ_I on Ω_i^+, which are given by the canonical homomorphisms $\Gamma_I \to \mathrm{Sym}(\Omega_i^+)$ for every $i \in I$. □

3.3. The residually finite case

Let us now assume that G is an infinite finitely generated residually finite torsion group. In this case we can choose a properly decreasing chain $(N_i)_{i\in\mathbb{N}}$ of finite index normal subgroups of G that satisfies $\cap_{i\in\mathbb{N}} N_i = 1$. Let $\Omega_i := G/N_i$ and let $\alpha_i\colon G \to \mathrm{Sym}(\Omega_i)$ denote the action of G that is given by left translation. Then, using the assumption that $(N_i)_{i\in\mathbb{N}}$ is properly decreasing, we see that the homomorphism

$$\alpha_{\geq n}\colon G \to \prod_{i\geq n}^{\infty} \mathrm{Sym}(\Omega_i),\ \ g \to (\alpha_i(g))_{i\geq n}$$

is injective for every $n \in \mathbb{N}$. As before, we fix an element $p_i \in \Omega_i$ for each $i \in \mathbb{N}$ and a family $(q_i)_{i\in\mathbb{N}}$ of pairwise different elements that do not lie in $\cup_{i\in\mathbb{N}}\Omega_i$. We write $\Omega_i^+ := \Omega_i \cup \{q_i\}$ and consider the elements $\tau_i = (p_i, q_i) \in \mathrm{Sym}(\Omega_i^+)$ and $\tau := (\tau_i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} \mathrm{Sym}(\Omega_i^+)$. Let $\Gamma \leq \prod\limits_{i\geq n}^{\infty} \mathrm{Sym}(\Omega_i^+)$ denote the subgroup that is generated by $\alpha_{\geq 1}(G)$ and τ.

Lemma 3.4. *Let Y be a finite set and let $G \leq \mathrm{Sym}(Y)$ be a subgroup that acts transitively on Y. Let $Y^+ = Y \cup \{z\}$, where $z \notin Y$. For every $y \in Y$ the group $\mathrm{Sym}(Y^+)$ is generated by G and the transposition (y, z).*

Proof. Let H denote the subgroup of $\mathrm{Sym}(Y^+)$ that is generated by G and (y, z). Since G acts transitively on Y it follows that every transposition of the form (x, z) with $x \in Y$ is a conjugate of (y, z) in H and therefore lies in H. By conjugating such a transposition (x, z) with a transposition (x', z), where $x' \notin \{x, z\}$, we obtain $(x, z)^{(x',z)} = (x, x') \in H$. Now the proof follows from the well-known fact that $\mathrm{Sym}(Y^+)$ is generated by all transpositions in $\mathrm{Sym}(Y^+)$. □

Recall that a subgroup H of a product of groups $P = \prod_{i\in I} K_i$ is called *subdirect product* if the canonical map $H \to K_i$ is surjective for every $i \in I$.

Corollary 3.5. *The subgroup Γ of $\prod_{i=1}^{\infty} \mathrm{Sym}(\Omega_i^+)$ is a subdirect product.*

Proof. This is a direct consequence of Lemma 3.4 and the definition of Γ. □

Lemma 3.6. *Let $(n_i)_{i\in\mathbb{N}}$ be a sequence of pairwise different natural numbers, let $P := \prod_{i=1}^{\infty} \mathrm{Sym}(n_i)$, and let $H \leq P$ be a finitely generated subdirect product. Let $\iota\colon H \to P$*

denote the inclusion map. For each $k \in \mathbb{N}$ *let* $\mathrm{pr}_{\geq k} \colon P \to \prod_{i=k}^{\infty} \mathrm{Sym}(n_i)$ *denote the canonical projection. There is a natural number* m *such that the image of the group* $K := H \cap \prod_{i=1}^{\infty} \mathrm{Alt}(n_i)$ *under* $\mathrm{pr}_{\geq m} \circ \iota$ *is a subdirect product in* $\prod_{i=m}^{\infty} \mathrm{Alt}(n_i)$*.*

Proof. Let $\pi \colon \prod_{i=1}^{\infty} \mathrm{Sym}(n_i) \to \prod_{i=1}^{\infty} \mathrm{Sym}(n_i)^{\mathrm{ab}} \cong \prod_{i=1}^{\infty} \mathbb{F}_2$ denote the abelianization. Note that K is the kernel of $\pi \circ \iota$. Since H is finitely generated, its image in $\prod_{i=1}^{\infty} \mathbb{F}_2$ is finite and thus K has finite index, say k, in H. Since there are only finitely many alternating groups that admit proper subgroups of index at most k, it follows that the canonical map $K \to \mathrm{Alt}(n_i)$ is surjective for almost every i. Thus the lemma follows if m is chosen big enough. □

We are now ready to prove the main result.

Theorem 3.7. *Every countable residually finite torsion group* G *embeds into a finitely generated torsion group that is residually in the class* $\mathcal{FS}$*.*

Proof. From [15, Theorem B] we know that every countable residually finite torsion group embeds into a finitely generated residually finite torsion group. On the other hand, it is shown in [9, Theorem 1.1] that every finitely generated, residually finite torsion group embeds into a finitely generated, residually finite perfect torsion group. Regarding this, we can assume that G is perfect. Let $(N_i)_{i \in \mathbb{N}}$ be a strictly decreasing sequence of finite index normal subgroups of G with $\cap_{i \in \mathbb{N}} N_i = \{1\}$ and let $\Omega_i = G/N_i$. From Corollary 3.5 we know that G embeds in a finitely generated subdirect product Γ in $\prod_{i=1}^{\infty} \mathrm{Sym}(n_i)$, where $n_i = |\Omega_i| + 1$. In this case Lemma 3.6 provides us with a number $m \in \mathbb{N}$ such that the projection image of $K := \Gamma \cap \prod_{i=1}^{\infty} \mathrm{Alt}(n_i)$ in $\prod_{i=m}^{\infty} \mathrm{Sym}(n_i)$ is a subdirect product of $\prod_{i=m}^{\infty} \mathrm{Alt}(n_i)$. Since G is perfect we have $G \leq K$. Moreover the restriction of the projection $\prod_{i=1}^{\infty} \mathrm{Sym}(n_i) \to \prod_{i=m}^{\infty} \mathrm{Sym}(n_i)$ to G is injective since the sequence $(N_i)_{i \in \mathbb{N}}$ was chosen to be decreasing. Thus G embeds into the image of K in $\prod_{i=m}^{\infty} \mathrm{Alt}(n_i)$. Since the latter is a finitely generated torsion group and a subdirect product of $\prod_{i=m}^{\infty} \mathrm{Alt}(n_i)$, this completes the proof. □

We conclude this article with a proof of Proposition 1.2, which we deduce from the following result of Lubotzky and Segal, see [10, Theorem 16.4.2(i), page 394]. To formulate the latter, we write $\mathcal{L}_d$ to denote the class of all groups that admit a d-dimensional faithful representation, i.e. embed into $\mathrm{GL}_d(K)$ for some field K.

Theorem 3.8. *Every finitely generated group* G *that is residually in* $\mathcal{L}_d$ *for some* $d \in \mathbb{N}$ *can be realized as a subdirect product of finitely many linear groups.*

Recall that we write $\mathcal{FS}_k$ to denote the class of finite simple groups that do not contain a subgroup isomorphic to $\mathrm{Alt}(k)$.

Proposition 3.9. *Every finitely generated group G that is residually in $\mathcal{FS}_k$ for some k can be realized as a subdirect product of finitely many linear groups.*

Proof. Let $k \geq 5$ be a natural number. According to [10, Proposition 16.4.4, page 346], there is a number $\ell \in \mathbb{N}$ such that every finite simple group of classical Lie type ${}^*X_\ell$ contains a subgroup isomorphic to $\mathrm{SL}_k(\mathbb{F}_q)/N$, where N is a subgroup of the center of $\mathrm{SL}_k(\mathbb{F}_q)$. Let $\iota\colon\ \mathrm{Alt}(k) \to \mathrm{SL}_k(\mathbb{F}_q)$ denote the standard embedding and let $\pi\colon\ \mathrm{SL}_k(\mathbb{F}_q) \to \mathrm{SL}_k(\mathbb{F}_q)/N$ be the quotient map. Since $\mathrm{Alt}(k)$ is a non-abelian simple group, it follows that $\iota(\mathrm{Alt}(k)) \cap N = \{1\}$ and hence that $\pi \circ \iota(\mathrm{Alt}(k))$ is a subgroup of $\mathrm{SL}_k(\mathbb{F}_q)/N$ that is isomorphic to $\mathrm{Alt}(k)$. In particular we see that every finite simple group of classical Lie type ${}^*X_\ell$ contains a subgroup that is isomorphic to $\mathrm{Alt}(k)$. On the other hand, we know from [10, Proposition 16.4.6, page 347] that there is some $n \in \mathbb{N}$ such that every simple group Q of (not necessarily classical) Lie type ${}^*X_\ell$ is contained in $\mathrm{SL}_n(\mathbb{F}_q)$ for some field $\mathbb{F}_q$. As a consequence, we see that Q lies in $\mathcal{L}_n$. By combining the latter with the fact that the Lie rank of the exceptional Lie types is bounded above by 8, we deduce that $\mathcal{FS}_k \subseteq \mathcal{L}_{\max(n,8)}$. Thus we can apply Theorem 3.8 to deduce that G is a subdirect product of finitely many linear groups. □

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