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A new T-compatibility condition and its application to the discretization of the damped time-harmonic Galbrun's equation

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We consider the approximation of weakly T-coercive operators. The main property to ensure the convergence thereof is the regularity of the approximation (in the vocabulary of discrete approximation schemes). In a previous work the existence of discrete operators T_n, which converge to T in a discrete norm, was shown to be sufficient to obtain regularity. Although this framework proved useful for many applications, for some instances the former assumption is too strong. Thus, in this article we report a weaker criterion for which the discrete operators T_n only have to converge point-wise, but in addition a weak T-coercivity condition has to be satisfied on the discrete level. We apply the new framework to prove the convergence of certain H^1-conforming finite element discretizations of the damped time-harmonic Galbrun's equation, which is used to model the oscillations of stars. A main ingredient in the latter analysis is the uniformly stable invertibility of the divergence operator on certain spaces, which is related to the topic of stable discretizations of the Stokes equation.

Keywords: discrete approximation schemes; weak T-coercivity; Galbrun's equation.

1. Introduction

An origin of the T-coercivity technique to analyse equations of nonweakly coercive form can be found in the theory of Maxwell's equation and goes back at least to Buffa *et al.* (2002) and Buffa (2005). The idea to use a discrete variant to prove the stability of approximations can be found, e.g., in Hohage & Nannen (2015) and Bonnet-BenDhia *et al.* (2018). In Halla (2021c) this approach was formalized to a framework to prove the convergence of Galerkin approximations of holomorphic eigenvalue problems, and was successfully applied for perfectly matched layer methods to scalar isotropic (Halla, 2021a) and anisotropic (Halla, 2022b) materials, Maxwell problems in conductive media (Halla, 2021c), modified Maxwell Steklov problems (Halla, 2021b) and Maxwell transmission problems for dispersive media (Unger, 2021; Halla, 2023b). In particular, Halla (2021c) is build upon the much broader framework of *discrete approximation schemes* (Stummel, 1970; Vainikko, 1976), which originated in the 1970s and the best results for eigenvalue problems in this context are Karma (1996a,b). The main contribution of Halla (2021c) was to provide a practical criterion to prove the regularity of approximations, which allows to apply the results achieved for discrete approximation schemes, although for some applications it turns out that the T-compatibility criterion of Halla (2021c) is too strong, and hence we present in this article a weaker variant. Some similarity can be drawn to the analysis of p-finite element methods for Maxwell problems (Boffi *et al.*, 2011), for which (opposed to h-finite element methods) the cochain projections are not uniformly L^2 bounded, and hence the discrete compactness property is obtained in Boffi *et al.* (2011) by an alternative technique.

Primarily the T-coercivity approach serves a technique for the analysis of partial differential equations and the numerical analysis of respective discretizations. However, T-coercivity techniques can also be used to construct new numerical schemes. Indeed, if feasible, the operator T can be included in the discretized variational formulation as, e.g., done in Ciarlet & Jamelot (2024) and Halla *et al.* (2024). Having now the discretization of a weakly coercive problem at hand, the stability of the approximations follows in a straightforward manner.

This article is motivated by the study of approximations to the damped time-harmonic Galbrun's equation. The Galbrun's equation (Galbrun, 1931) is a linearization of the nonlinear Euler equations with the Lagrangian perturbation of displacement as unknown, and is used in aeroacoustics (Maeder *et al.*, 2020), as well as in an extended version in asteroseismology (Lynden-Bell & Ostriker, 1967). We refer to Hägg & Berggren (2021) for a well-posedness analysis in the time domain. In the time-harmonic domain an approach in aeroacoustics is to use a stabilized formulation, which is justified by the introduction of an additional transport equation for the vorticity, and we refer to the well-posedness analysis in Bonnet-BenDhia *et al.* (2012). Different to aeroacoustics in asteroseismology there exists a significant damping of waves that allows the equation to be analysed in a more direct way, see the well-posedness results Halla & Hohage (2021) and Halla (2022a). In the second part of this article we apply our new framework to the approximation of the damped time-harmonic Galbrun's equation as considered in Halla & Hohage (2021):

$$
\begin{aligned}
&-\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)^2\mathbf{u} - \nabla\left(\rho c_s^2 \operatorname{div}\mathbf{u}\right) + (\operatorname{div}\mathbf{u})\nabla p \\
&-\nabla(\nabla p\cdot\mathbf{u}) + (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\mathbf{u} + \gamma\rho(-i\omega)\mathbf{u} = \mathbf{f} \quad \text{in } \mathscr{O},
\end{aligned} \tag{1.1}
$$

where $\rho, p, \phi, c_s, \mathbf{b}, \Omega$ and $\mathbf{f}$ denote density, pressure, gravitational potential, sound speed, background velocity, angular velocity of the frame and sources; $\partial_{\mathbf{b}} := \sum_{l=1}^{3} \mathbf{b}_l \partial_{x_l}$ denotes the directional derivative in direction $\mathbf{b}$, $\operatorname{Hess}(p)$ the Hessian of p and $\mathscr{O} \subset \mathbb{R}^3$ a bounded domain; and damping is modelled by the term $-i\omega\gamma\rho\mathbf{u}$ with damping coefficient γ. The main challenge to tackle this equation can already be observed in the case $p, \phi = 0, \Omega = 0$. We discretize (1.1) with conforming $\mathbf{H}^1$ finite elements. To guarantee the stability of the approximation we use vectorial finite element spaces that admit a suitable uniformly stable inversion of the divergence operator. In particular, let $X_h \subset \mathbf{H}^1$ be a Lagrangian vectorial finite element space of order k and $Q_h \subset L^2$ be a scalar finite element space and $L^2_0 := \{u \in L^2 \colon \int_\Omega u \,\mathrm{d}x = 0\}$. Then, we require that there exists a uniformly bounded inverse of the (discrete) divergence operator acting on the spaces $Q_h \cap L^2_0 \to X_h \cap \mathbf{H}^1_0$. Such methods have been developed in the field of computational fluid dynamics for the stable discretization of incompressible Stokes and Navier–Stokes equations, cf. e.g., John (2016).

Especially convenient for the analysis are the so-called *divergence free finite elements*, meaning that the approximative solutions to the Stokes equations are exactly divergence-free. However, note that there exist sophisticated techniques to construct such elements and not all *divergence-free finite elements* fit our needs. The pioneering work for *divergence-free finite elements* was set by Scott and Vogelius (Scott & Vogelius, 1985), who established respective results (suitable for our purpose) in two dimensions for triangular quasi-uniform meshes with *finite degree of degeneracy* and polynomial degree $k \geq 4$ (the quasi-uniformity is actually not necessary due to Falk & Neilan (2013)). In three dimensions, Zhang (Zhang, 2011) reported a generalization to uniform tetrahedral grids for $k \geq 6$, and his results in Zhang (2009) indicate that for general tetrahedral grids suitable orders are $k \geq 8$. The application of convenient finite element spaces on specialized meshes generated by barycentric refinements (suitable for our purpose) received extensive attention and we refer, e.g., to Arnold & Qin (1992), Zhang (2005) and Guzmán & Neilan (2018). In general, such schemes are related to respective discretizations of suitable deRahm complexes with high regularity (Christiansen & Hu, 2018; Guzmán *et al.*, 2020; Neilan, 2020). There exist also several results for elements on quadrilateral grids for which we refer to the bibliographies of John *et al.* (2017) and Neilan (2020). Other approaches to construct *divergence-free finite elements* include enriched finite elements, nonconforming elements, discontinuous Galerkin methods and isogeometric methods.

Although we will make use of the advantages of *divergence-free finite elements* in the analysis, we note that the more important property is the stable Stokes approximation. A comparison and analysis of different robust finite element discretizations for a simplified Galbrun's equation is presented in Alemán *et al.* (2022). Approximations with $H(\mathrm{div})$-conforming finite elements and DGFEMs are analysed in van Beeck (2023), Halla (2023a) and Halla *et al.* (2025), employing the framework of the current article.

The remainder of this article is structured as follows. In Section 2 we report a multipurpose framework based on a weak T-compatibility condition (weaker than in Halla (2021c)) to obtain the regularity, and hence the stability of approximations. Although in this article we consider only conforming discretizations to (1.1), we formulate the framework in a general way to include also nonconforming approximations (Halla, 2023a). In Section 3 we apply the former framework to discretizations of (1.1). In particular, in Section 3.5 we consider a simplified case of (1.1) to present the main ideas and in Section 3.6 we treat the general case. In Section 4 we present computational examples to accompany our theoretical results and we conclude in Section 5.

2. Abstract framework

This section discusses a multipurpose framework for the analysis of approximations of linear operators. In Section 2.1 we review the framework and important definitions, as well as sufficient conditions for the convergence of the approximative solution. We aim to apply this framework to operators that are Fredholm with index zero; however, have the structure of 'coercive+compact' only up to a bijection. Such operators are called weakly T-coercive; a precise definition is given in Section 2.2. Note that this property is equivalent to an operator being Fredholm with index zero, and the construction of a suitable T operator is the tool to prove this property. Here, we study a way how this property can be mimiced on the discrete level to ensure convergent approximations.

2.1 *Discrete approximation schemes*

We consider discrete approximation schemes in Hilbert spaces. Note that the forthcoming setting is a bit more restrictive than the schemes considered in Stummel (1970), Vainikko (1976) and Karma (1996a), but more convenient for our purposes. For two Hilbert spaces $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ let $L(X, Y)$ be the space of bounded linear operators from X to Y, and set $L(X) := L(X, X)$.

Definition 1. We call $\{X_n, A_n, p_n\}_{n\in\mathbb{N}}$ a *discrete approximation scheme* of $A \in L(X)$ if the following properties hold: Let $(X_n, \langle\cdot,\cdot\rangle_{X_n})_{n\in\mathbb{N}}$ be a sequence of finite-dimensional Hilbert spaces and $A_n \in L(X_n)$.

And let $p_n \in L(X, X_n)$ such that $\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X$ for each $u \in X$. We then define the following properties of a discrete approximation scheme:

(i) A sequence $(u_n)_{n\in\mathbb{N}}, u_n \in X_n$ is said to *converge* to $u \in X$, if $\lim_{n\to\infty} \|p_n u - u_n\|_{X_n} = 0$.

(ii) A sequence $(u_n)_{n\in\mathbb{N}}, u_n \in X_n$ is said to be *compact*, if for every subsequence $\mathbb{N}' \subset \mathbb{N}$ there exists a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $(u_n)_{n\in\mathbb{N}''}$ converges (to $u \in X$).

(iii) A sequence of operators $(A_n)_{n\in\mathbb{N}}, A_n \in L(X_n)$ is said to *approximate* $A \in L(X)$, if $\lim_{n\to\infty} \|A_n p_n u - p_n A u\|_{X_n} = 0$. In a finite element vocabulary it might be more convenient to denote this property as *asymptotic consistency*.

(iv) A sequence of operators $(A_n)_{n\in\mathbb{N}}, A_n \in L(X_n)$ is said to be *compact*, if for every bounded sequence $(u_n)_{n\in\mathbb{N}}, u_n \in X_n$, $\|u_n\|_{X_n} \le C$ the sequence $(A_n u_n)_{n\in\mathbb{N}}$ is compact.

(v) A sequence of operators $(A_n)_{n\in\mathbb{N}}, A_n \in L(X_n)$ is said to be *stable*, if there exist constants $C, n_0 > 0$ such that A_n is invertible and $\|A_n^{-1}\|_{L(X_n)} \le C$ for all $n > n_0$.

(vi) A sequence of operators $(A_n)_{n\in\mathbb{N}}, A_n \in L(X_n)$ is said to be *regular*, if $\|u_n\|_{X_n} \le C$ and the compactness of $(A_n u_n)_{n\in\mathbb{N}}$ implies the compactness of $(u_n)_{n\in\mathbb{N}}$.

Note that we do not demand that the spaces X_n are subspaces of X. Instead, we demand the existence of the projection operators p_n. The vocabulary introduced in definition 1 and used throughout the manuscript may not be familiar to every reader. We hence refer to the corresponding properties of a discrete approximation scheme with an upper index linking to the corresponding property in definition 1. The central properties we are looking for in a discrete approximation scheme are regularity[(vi)] and asymptotic consistency[(iii)], which are sufficient for the convergence[(i)] of discrete solutions. To emphasize this we recall in the following some well known results.

LEMMA 1. Let $A \in L(X)$ be bijective and $(A_n)_{n\in\mathbb{N}}, A_n \in L(X_n)$ be a discrete approximation scheme that is regular[(vi)] and approximates[(iii)] A. Then, $(A_n)_{n\in\mathbb{N}}$ is stable[(iv)].

Proof. Follows from statement (3) of (Karma, 1996a, theorem 2). We give a proof for the sake of completeness. Assume the contrary, i.e., there exists a normalized sequence $u_n \in X_n$ with $A_n u_n = 0$. Since A_n is regular, there exists $u \in X$ and subsequence that we do not rename, such that $\lim \|u_n - p_n u\| = 0$. As A_n approximates A we have $\lim \|A_n p_n u - p_n A u\|_{X_n} = 0$. Therefore, $\lim \|p_n A u\|_{X_n} = 0$. Since A is injective, it follows $u = 0$. This contradicts $\|u_n\|_{X_n} = 1$ and $\lim_{n\to\infty} \|u_n - p_n u\|_{X_n} = 0$, and hence the claim is proven. □

LEMMA 2. Let $A \in L(X)$ be bijective and $(A_n)_{n\in\mathbb{N}}, A_n \in L(X_n)$ be a discrete approximation scheme that is stable[(v)] and approximates[(iii)] A. Let $n_0 > 0$ be such that A_n is bijective for all $n > n_0$, u, u_n be the solutions to $Au = f$ and $A_n u_n = f_n \in X_n$, and assume that $\lim_{n\to\infty} \|p_n f - f_n\|_{X_n} = 0$. Then, $\lim_{n\to\infty} \|p_n u - u_n\|_{X_n} = 0$. If the approximation is a conforming Galerkin scheme, i.e., $X_n \subset X$ and p_n is the orthogonal projection onto $X_n, f_n = p_n f$, then there exists a constant $C > 0$ such that $\|u - u_n\|_X \le C \inf_{u_n' \in X_n} \|u - u_n'\|_X$ for all $n > n_0$.

Proof. Using that A_n is stable, i.e., has a bounded inverse, followed by the triangle inequality, we estimate

$$\begin{aligned}
\|p_n u - u_n\|_{X_n} &\le \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \|A_n p_n u - A_n u_n\|_{X_n} \\
&\le \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \left(\|A_n p_n u - p_n A u\|_{X_n} + \|p_n A u - A_n u_n\|_{X_n}\right) \\
&= \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \left(\|A_n p_n u - p_n A u\|_{X_n} + \|p_n f - f_n\|_{X_n}\right).
\end{aligned}$$

It holds that $\lim_{n\to\infty} \|A_n p_n u - p_n A u\|_{X_n} = 0$, because $(A_n)_{n\in\mathbb{N}}$ approximates(iii) A, and that $\lim_{n\to\infty} \|p_n f - f_n\|_{X_n} = 0$ by assumption. Hence, the first claim is proven.

For the second claim we recall that we are in the setting of a conforming Galerkin scheme. We estimate, using the triangle inequality, the stability of A_n and the definition of the projection p_n, to obtain

$$\begin{aligned}
\|u - u_n\|_X &\le \|u - p_n u\|_X + \|p_n u - u_n\|_X \\
&\le \|u - p_n u\|_X + \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \|p_n A p_n u - p_n A u\|_X \\
&\le \|u - p_n u\|_X + \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \|A\|_{L(X)} \|u - p_n u\|_X \\
&= \left(1 + \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \|A\|_{L(X)}\right) \|u - p_n u\|_X.
\end{aligned}$$

□

2.2 *The new T-compatibility condition*

Definition 2. We define the following properties for an operator A.

(i) An operator $A \in L(X)$ is called coercive, if there exists a constant $C > 0$ such that $|\langle Au, u\rangle_X| \ge C\|u\|_X^2$ for all $u \in X$.

(ii) An operator $A \in L(X)$ is called weakly coercive, if there exists a compact operator $K \in L(X)$ such that $A + K$ is coercive.

(iii) An operator A is called (weakly) right T-coercive, if $T \in L(X)$ is bijective and AT is (weakly) coercive.

Our definition of weak T-coercivity is in spirit equivalent to the generalized Gårding inequality in (Buffa *et al.*, 2002, prop. 3). The generalized Gårding inequality in [11] follows from our definition of weak T-coercivity by applying the triangle inequality. However, the reverse direction seems to require an additional argument.

The next theorem provides a sufficient setting for a discrete approximation of a (weakly) right T-coercive operator. This theorem is key for the discretization and its analysis in Section 3.

Theorem 3. Let sequences $(A_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ and $B, T \in L(X)$ satisfy the following: There exists a constant $C > 0$ such that for each $n \in \mathbb{N}$ it holds $A_n, T_n, B_n, K_n \in L(X_n)$, $\|T_n\|_{L(X_n)}, \|T_n^{-1}\|_{L(X_n)}, \|B_n\|_{L(X_n)}, \|B_n^{-1}\|_{L(X_n)} \le C$, B is bijective, $(K_n)_{n\in\mathbb{N}}$ is compact(iv) and

$$\lim_{n\to\infty} \|T_n p_n u - p_n T u\|_{X_n} = 0, \qquad \lim_{n\to\infty} \|B_n p_n u - p_n B u\|_{X_n} = 0 \qquad \forall u \in X, \tag{2.1a}$$

$$A_n T_n = B_n + K_n. \tag{2.1b}$$

Then, $(A_n)_{n\in\mathbb{N}}$ is regular(vi).

Proof. Let $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$ be a uniformly bounded sequence $\|u_n\|_{X_n} \le C$, $(f_n)_{n\in\mathbb{N}}$ with $f_n := A_n u_n$ be compact and $\mathbb{N}' \subset \mathbb{N}$ be an arbitrary subsequence. Consider a converging subsequence $(f_n)_{n\in\mathbb{N}''}$ with $\mathbb{N}'' \subset \mathbb{N}'$ and denote the limit as $f \in X$ such that $\lim_{n\in\mathbb{N}''} \|A_n u_n - p_n f\|_{X_n} = 0$. We then obtain from (2.1b) that $B_n T_n^{-1} u_n + K_n T_n^{-1} u_n = A_n u_n = f_n \to f$ for $n \in \mathbb{N}'', n \to \infty$. Since T_n^{-1} is bounded and

$(K_n)_{n\in\mathbb{N}}$ is compact[(iv)], $(T_n^{-1}u_n)_{n\in\mathbb{N}''}$ is bounded and we can choose a converging subsequence $(g_n)_{n\in\mathbb{N}'''}$ with $g_n = K_nT_n^{-1}u_n$, $\mathbb{N}''' \subset \mathbb{N}''$ and limit $g \in X$ such that $\lim_{n\in\mathbb{N}'''} \|g_n - p_ng\|_{X_n} = 0$. We observe that there holds $u_n = T_nB_n^{-1}(f_n - g_n)$. Finally, we want to exploit the properties in (2.1a) on T_n and B_n to show $\lim_{n\in\mathbb{N}'''} \|u_n - p_nTB^{-1}(f-g)\|_{X_h} = 0$, which implies the compactness[(ii)] of $(u_n)_{n\in\mathbb{N}}$. We start with a triangle inequality

$$\|u_n - p_nTB^{-1}(f-g)\|_{X_n} \leq \underbrace{\|u_n - T_nB_n^{-1}p_n(f-g)\|_{X_n}}_{I} + \underbrace{\|p_nTB^{-1}(f-g) - T_nB_n^{-1}p_n(f-g)\|_{X_n}}_{II}$$

and bound the two contributions I and II one after another:

$$\begin{aligned}
I &\leq \|T_n\|_{L(X_n)}\|B_n^{-1}\|_{L(X_n)}\|(f_n - g_n) - p_n(f-g)\|_{X_n} \leq C^2\left(\|f_n - p_nf\|_{X_n} + \|p_n - p_ng\|_{X_n}\right),\\
II &\leq \|p_nTB^{-1}(f-g) - T_np_nB^{-1}(f-g)\|_{X_n} + \|T_np_nB^{-1}(f-g) - T_nB_n^{-1}p_n(f-g)\|_{X_n}\\
&\leq \|p_nTB^{-1}(f-g) - T_np_nB^{-1}(f-g)\|_{X_n} + C^2\|B_np_nB^{-1}(f-g) - p_n(f-g)\|_{X_n},
\end{aligned}$$

where the latter right-hand side terms converge to zero for $n\to\infty$ by the assumptions in (2.1a). Hence, $(u_n)_{n\in\mathbb{N}'''}$ converges (to $TB^{-1}(f-g)$), and thus A_n is regular[(vi)]. □

We call a sesquilinear form $a(\cdot,\cdot)$ compact or (weakly) (right T-)coercive, if its Riesz representation $A \in L(X)$ (defined by $\langle Au, u'\rangle_X = a(u,u')$ for all $u, u' \in X$) admits the respective property.

3. Discrete approximations of the damped time-harmonic Galbrun's equation

In this section we analyse approximations to (1.1). After introducing the weak formulation of the problem in Section 3.1 we discuss a Helmholtz-type decomposition and a density result in Section 3.2 and Section 3.3, respectively. The discrete approximation is then introduced in Section 3.4 and analysed in two steps in Section 3.5 and Section 3.6, where in Section 3.5 we treat the case of homogeneous pressure and gravity and treat the general case in Section 3.6.

3.1 *Preliminaries, notation and weak formulation*

To this end, we first set our notation, and specify our assumptions on the parameters and the domain. Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedron. We consider $\mathcal{O}$ to be the default domain for all functions spaces, i.e., $L^2 := L^2(\mathcal{O})$, etc. Let $L^2_0 := \{u \in L^2 :\ \mathrm{mean}(u) = 0\}$. Further, for a scalar function space X we use the boldface notation for its vectorial variant, i.e., $\mathbf{X} := (X)^3$. If not specified otherwise all function spaces are considered over $\mathbb{C}$. We introduce the following subspaces of $\mathbf{H}^1$ with zero (normal) trace:

$$\mathbf{H}^1_{\boldsymbol{\nu}0} := \{\mathbf{u} \in \mathbf{H}^1 : \boldsymbol{\nu}\cdot\mathbf{u} = 0 \text{ on } \partial\mathcal{O}\} \quad \text{and} \quad \mathbf{H}^1_0 := (H^1_0)^3,$$

where $H^1_0 := \{\mathbf{u} \in H^1 : \mathbf{u} = 0 \text{ on } \partial\mathcal{O}\}$ is the subspace of H^1 with zero trace. By $C_{PS} > 0$ we denote the Poincaré–Steklov constant of $\mathcal{O}$, which satisfies

$$C_{PS}\|u\|_{H^1} \leq \|\nabla u\|_{\mathbf{L}^2} \quad \text{for all } u \in H^1_0. \tag{3.1}$$

We denote scalar products as $\langle\cdot,\cdot\rangle_X$, whereas a scalar product without index always means the L^2-scalar product for scalar and vectorial functions. We employ the notation $A \lesssim B$, if there exists a constant $C > 0$ such that $A \le CB$. The constant $C > 0$ may be different at each occurrence and can depend on the domain $\mathscr{O}$, the physical parameters $\rho, c_s, p, , \phi, \gamma, \mathbf{b}, \omega, \Omega$ and on the sequence of Galerkin spaces $(X_n)_{n\in\mathbb{N}}$. However, it will always be independent of the index n and any involved functions that may appear in the terms A and B. Let the frequency $\omega \in \mathbb{R}\setminus\{0\}$ and the angular velocity of the frame $\Omega \in \mathbb{R}^3$. Let the sound speed, density and damping parameter $c_s, \rho, \gamma : \mathscr{O} \to \mathbb{R}$ be measurable such that

$$\underline{c_s} \le c_s \le \overline{c_s}, \qquad \underline{\rho} \le \rho \le \overline{\rho}, \qquad \underline{\gamma} \le \gamma \le \overline{\gamma}, \tag{3.2}$$

with constants $0 < \underline{c_s}, \overline{c_s}, \underline{\rho}, \overline{\rho}, \underline{\gamma}, \overline{\gamma}$. Let the pressure and gravitational potential $p, \phi \in W^{2,\infty}$. Let the source term $\mathbf{f} \in \mathbf{L}^2$. Further, let the flow $\mathbf{b} \in W^{1,\infty}(\mathscr{O}, \mathbb{R}^3)$ such that $\operatorname{div}(\rho\mathbf{b}) \in L^2$ and $\boldsymbol{\nu}\cdot\mathbf{b} = 0$ on $\partial\mathscr{O}$ and $\mathbf{b}$ be compactly supported in $\mathscr{O}$. This ensures that the distributional streamline derivative operator $\partial_{\mathbf{b}}\mathbf{u} := \mathbf{b}\cdot\nabla\mathbf{u}$ is well defined w.r.t. the inner product $\langle\rho\cdot,\cdot\rangle$ for $\mathbf{u} \in \mathbf{L}^2$ (Halla & Hohage, 2021), and we define

$$\mathbb{X} := \{\mathbf{u} \in \mathbf{L}^2 : \ \operatorname{div}\mathbf{u} \in L^2, \ \partial_{\mathbf{b}}\mathbf{u} \in \mathbf{L}^2, \boldsymbol{\nu}\cdot\mathbf{u} = 0 \text{ on } \partial\mathscr{O}\}$$

with inner product

$$\langle\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}} := \langle\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle + \langle\partial_{\mathbf{b}}\mathbf{u}, \partial_{\mathbf{b}}\mathbf{u}'\rangle + \langle\mathbf{u}, \mathbf{u}'\rangle$$

and the associated norm $\|\mathbf{u}\|_{\mathbb{X}}^2 = \langle\mathbf{u}, \mathbf{u}\rangle_{\mathbb{X}}$. Note that the smoothness $\mathbf{b} \in W^{1,\infty}(\mathscr{O}, \mathbb{R}^3)$ of the flow will be required to obtain density results for the space $\mathbb{X}$. There exists a constant $C_{XH} > 0$ such that

$$C_{XH}\|\mathbf{u}\|_{\mathbb{X}} \le \|\mathbf{u}\|_{\mathbf{H}^1} \quad \text{for all } \mathbf{u} \in \mathbf{H}^1. \tag{3.3}$$

We further assume the conservation of mass $\operatorname{div}(\rho\mathbf{b}) = 0$, which allows us to reformulate (1.1) in the weak form as in Halla & Hohage (2021): find $\mathbf{u} \in \mathbb{X}$ such that

$$a(\mathbf{u}, \mathbf{u}') = \langle\mathbf{f}, \mathbf{u}'\rangle \quad \forall\mathbf{u}' \in \mathbb{X} \tag{3.4}$$

with the sesquilinear form

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}') := {} & \langle c_s^2\rho\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle - \langle\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}'\rangle \\ & + \langle\operatorname{div}\mathbf{u}, \operatorname{grad}p\cdot\mathbf{u}'\rangle + \langle\operatorname{grad}p\cdot\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle + \langle(\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\mathbf{u}, \mathbf{u}'\rangle \\ & - i\omega\langle\gamma\rho\mathbf{u}, \mathbf{u}'\rangle. \end{aligned} \tag{3.5}$$

3.2 *Topological decomposition*

A crucial tool to analyse (3.5) and to construct a proper operator T in Halla & Hohage (2021) is a Helmholtz-type decomposition of vector fields in $\mathbb{X}$. To this end, let us recall that a vector space Y is called the direct algebraic sum of subspaces $Y_1, \ldots, Y_N \subset Y$, denoted by $Y = \bigoplus_{n=1,\ldots,N} Y_n$, if each element $y \in Y$ has a unique representation of the form $y = \sum_{n=1}^N y_n$ with $y_n \in Y_n$. We refer to $Y = \bigoplus_{n=1,\dots,N} Y_n$ as the algebraic decomposition of Y. Note that there exist associated projection operators $P_{Y_n}\colon Y \to Y_n\colon y \mapsto y_n$ with $\operatorname{ran} P_{Y_n} = Y_n$ and $\ker P_{Y_n} = \bigoplus_{m=1,\dots,N, m\neq n} Y_m$. An algebraic decomposition of a Hilbert space is called a topological decomposition, if all associated projection operators P_{Y_n} are continuous. We set

$$\begin{aligned} V &:= \{\mathbf{u} \in \mathbf{H}_0^1\colon \langle \nabla \mathbf{u}, \nabla \mathbf{u}'\rangle = 0 \text{ for all } \mathbf{u}' \in \mathbf{H}_0^1 \text{ with } \operatorname{div} \mathbf{u}' = 0\}, \\ W &:= \{\mathbf{u} \in \mathbb{X}\colon \ \operatorname{div} \mathbf{u} = 0\}. \end{aligned} \tag{3.6}$$

From (Acosta *et al.*, 2006, theorem 4.1) we know that $D\mathbf{v} := \operatorname{div} \mathbf{v}$, $D \in L(V, L_0^2)$ is bijective. We make use of the notation D for the divergence operator to emphasize that it has a bounded inverse on V, and we will always consider it in the space $D^{-1} \in L(L_0^2, V)$. Note that $D = \operatorname{div}$ is also bounded and well defined on $\mathbb{X}$. While the choice of D is deceptively simple in the case of homogeneous pressure and gravity, it is not trivial in the case of heterogeneous pressure and gravity, as we will see in Section 3.6. On V the sesquilinear form $\langle \operatorname{div} \cdot, \operatorname{div} \cdot\rangle$ defines an inner product equivalent to the $\mathbf{H}_0^1$ inner product.

The projections onto V and W are given by

$$P_V\mathbf{u} := D^{-1} \operatorname{div} \mathbf{u} \quad \text{and} \quad P_W\mathbf{u} := \mathbf{u} - P_V\mathbf{u}.$$

Thus, $V \oplus W$ is a topological decomposition of $\mathbb{X}$. If there is no conflict of notation we use the abbreviations $\mathbf{v} := P_V\mathbf{u}$, $\mathbf{w} := P_w\mathbf{u}$ for $\mathbf{u} \in \mathbb{X}$.

3.3 *Density results*

PROPOSITION 4 (Variation of prop. 3.5 of Bredies (2008)). Let $l \in \mathbb{N}$ and $\Lambda\colon \mathscr{D}(\Lambda) \subset L^2(\mathscr{O}, \mathbb{R}^3) \to L^2(\mathscr{O}, \mathbb{R}^l)$ with $C_0^\infty(\mathscr{O}, \mathbb{R}^3) \subset \mathscr{D}(\Lambda)$ be a closed linear operator with the property that

1. $u \in \mathscr{D}(\Lambda)$ if and only if for each $\zeta \in C_0^\infty(\mathscr{O})$ follows $\zeta u \in \mathscr{D}(\Lambda)$,
2. for each $u \in \mathscr{D}(\Lambda)$ and $\zeta \in C_0^\infty(\mathscr{O})$ follows $\operatorname{supp} \Lambda(\zeta u) \subset \operatorname{supp} \zeta$,
3. for each $u \in \mathscr{D}(\Lambda)$ with compact support in $\mathscr{O}$ there exists a $\delta_0 > 0$ such that the sequence of mollified $u_\delta := u * G_\delta$ satisfies $\|\Lambda u_\delta\|_{L^2(\mathscr{O},\mathbb{R}^l)} \le C$ for every $\delta \in (0, \delta_0)$ and some $C > 0$.

Then, for each $\varepsilon > 0$ and $u \in L^2(\mathscr{O}, \mathbb{R}^3)$ with $\Lambda u \in L^2(\mathscr{O}, \mathbb{R}^l)$ there exists a $\tilde{u} \in C^\infty(\mathscr{O}, \mathbb{R}^3)$ such that

$$\|u - \tilde{u}\|^2_{L^2(\mathscr{O},\mathbb{R}^3)} + \|\Lambda u - \Lambda\tilde{u}\|^2_{L^2(\mathscr{O},\mathbb{R}^l)} < \varepsilon. \tag{3.7}$$

THEOREM 5. Let $\mathbf{b} \in W^{1,\infty}(\mathscr{O}, \mathbb{R}^3)$ and $\mathbf{b}$ be compactly supported in $\mathscr{O}$. Then, $\mathbf{H}_0^1$ is dense in $\mathbb{X}$.

Proof. Let $\chi \in C_0^\infty(\mathscr{O})$ be a cut-off function with values in $[0, 1]$, $\mathscr{O} \supset G_2 \supset G_1 \supset \operatorname{supp} \mathbf{b}$, $\chi = 1$ on G_1 and $\chi = 0$ on $\mathscr{O} \setminus G_2$, where $\operatorname{dist}(\partial G_2, \partial\mathscr{O}) > 0$. Let $\mathbf{u} \in \mathbb{X}$ and $\varepsilon > 0$. Since $\|(1-\chi)\mathbf{u}\|_{\mathbb{X}} = \|(1-\chi)\mathbf{u}\|_{H(\mathrm{div};\mathscr{O})}$, we can find $\tilde{\mathbf{u}}_1 \in \mathbf{C}_0^\infty(\mathscr{O})$ such that $\|(1-\chi)\mathbf{u} - \tilde{\mathbf{u}}_1\|_{\mathbb{X}} < \varepsilon/2$, see, e.g., Ern & Guermond (2020). To find a suitable smooth approximation of $\chi\mathbf{u}$ we apply Proposition 4 to $\Lambda\mathbf{u} := (\operatorname{div} \mathbf{u}, \partial_{\mathbf{b}}\mathbf{u})^\top$, i.e., $l = 4$. The first assumption of Proposition 4 follows from the product rule (see, e.g., (Bredies, 2008, lemma 3.7) for details on $\partial_{\mathbf{b}}$). The second assumption of Proposition 4 holds, because Λ is a differential operator. The third assumption of Proposition 4 follows from (Bredies, 2008, lemma 3.8) and convenient manipulations for the smoothing in $H(\mathrm{div})$, i.e., $\operatorname{div}(\mathbf{u} * G_\delta)(\mathbf{x}) = \int_{\mathscr{O}} \mathbf{u}(\mathbf{y}) \cdot \nabla_{\mathbf{x}} G_\delta(\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{y} =$

$-\int_{\mathcal{O}} \mathbf{u}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G_\delta(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathcal{O}} G_\delta(\mathbf{x}-\mathbf{y})\,\mathrm{div}_{\mathbf{y}}\,\mathbf{u}(\mathbf{y})\,\mathrm{d}\mathbf{y}$. The claimed bound follows now from the properties of G_δ. Thus, there exists $\tilde{\mathbf{u}}_2 \in \mathbf{C}^\infty(\mathcal{O})$ such that $\|\chi\mathbf{u} - \tilde{\mathbf{u}}_2\|_{\mathbb{X}} < \varepsilon/2$. Since the support of $\chi\mathbf{u}$ is compact in $\mathcal{O}$, $\tilde{\mathbf{u}}_2$ can be chosen with compact support too, and hence satisfies the necessary boundary condition. Thus, the proof is finished. □

THEOREM 6. Let $\mathbf{b} \in W^{1,\infty}(\mathcal{O}, \mathbb{R}^3)$ and supp $\mathbf{b}$ be compact in $\mathcal{O}$. Then, $\mathbf{C}_0^\infty$ is dense in $\mathbb{X}$.

Proof. Since $\mathbf{C}_0^\infty$ is dense in $\mathbf{H}_0^1$, the claim follows from Theorem 5 and (3.3). □

3.4 $\mathbf{H}^1$-*conforming discretization*

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of shape-regular simplical meshes of $\mathcal{O}$ with maximal element diameter $h_n \to 0$ for $n \to \infty$. For $k \in \mathbb{N}$ we denote by P_k the space of scalar polynomials with maximal degree k. We consider finite element spaces

$$\mathbb{X}_n := \{\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\nu}0} : \mathbf{u}|_T \in (P_k(T))^3\ \forall T \in \mathcal{T}_n\},$$

with fixed uniform polynomial degree $k \in \mathbb{N}$. It readily follows $\mathbb{X}_n \subset \mathbf{H}^1_{\boldsymbol{\nu}0} \subset \mathbb{X}$.

Let us note that the previous assumption that $\mathcal{O}$ is polygonal is crucial for $\mathbb{X}_n$ to be a proper finite element space with the usual approximation quality. We discuss the construction of such a finite element space in Appendix A. For curved boundaries, especially in the case of curved boundaries that are approximated with only C^0-continuous discrete boundaries, the construction of $\mathbb{X}_n$ is hardly possible or computationally unfeasible. In these cases one typically resorts to Lagrange multiplier-based or Nitsche-like techniques in order to weakly impose the boundary condition $\mathbf{u}\cdot\boldsymbol{\nu} = 0$ through the variational formulation that is then posed on $\{\mathbf{u} \in \mathbf{H}^1 : \mathbf{u}|_T \in (P_k(T))^3\ \forall T \in \mathcal{T}_n\}$. In the numerical examples below, we will use a Nitsche-based (weak) imposition of the boundary conditions, while in the analysis we assume $\mathbf{u}\cdot\boldsymbol{\nu} = 0$ to be imposed as *essential* boundary conditions in $\mathbb{X}_n$.

$\mathbb{X}_n$ allows for proper approximation of $\mathbf{u} \in \mathbb{X}$:

LEMMA 7. It holds

$$\lim_{n\to\infty} \inf_{\mathbf{u}'_n \in \mathbb{X}_n} \|\mathbf{u} - \mathbf{u}'_n\|_{\mathbb{X}} = 0 \quad \text{for each} \quad \mathbf{u} \in \mathbb{X}.$$

Proof. Let $\mathbf{u} \in \mathbb{X}$ be given. Since $\mathbf{C}_0^\infty$ is dense in $\mathbb{X}$ (see theorem 6), we can find for each $\varepsilon > 0$ a function $\mathbf{u}_\varepsilon \in \mathbf{C}_0^\infty$ such that $\|\mathbf{u} - \mathbf{u}_\varepsilon\|_{\mathbb{X}} < \varepsilon$. Further, the canonical interpolation operator I_{h_n} is well defined for $\mathbf{u}_\varepsilon$ and yields the estimate $\|\mathbf{u}_\varepsilon - I_{h_n}\mathbf{u}_\varepsilon\|_{\mathbf{H}^1} \le Ch_n\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^2}$ with a constant $C > 0$ independent of h_n. Since $\mathbf{u}_\varepsilon$ has compact support, it also follows that $I_{h_n}\mathbf{u}_\varepsilon \in \mathbf{H}^1_0$, and thus $I_{h_n}\mathbf{u}_\varepsilon \in \mathbb{X}_n$. Hence, we estimate

$$\lim_{n\to\infty} \inf_{\mathbf{u}'_n \in \mathbb{X}_n} \|\mathbf{u} - \mathbf{u}'_n\|_{\mathbb{X}} \le \lim_{n\to\infty} \inf_{\mathbf{u}'_n \in \mathbb{X}_n} (\|\mathbf{u} - \mathbf{u}_\varepsilon\|_{\mathbb{X}} + \|\mathbf{u}_\varepsilon - \mathbf{u}'_n\|_{\mathbb{X}}) \le \varepsilon + \lim_{n\to\infty} \|\mathbf{u}_\varepsilon - I_{h_n}\mathbf{u}_\varepsilon\|_{\mathbb{X}}$$

$$\lesssim \varepsilon + \lim_{n\to\infty} \|\mathbf{u}_\varepsilon - I_{h_n}\mathbf{u}_\varepsilon\|_{\mathbf{H}^1} \lesssim \varepsilon + \lim_{n\to\infty} h_n\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^2} = \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, it follows $\lim_{n\to\infty} \inf_{\mathbf{u}'_n \in \mathbb{X}_n} \|\mathbf{u} - \mathbf{u}'_n\|_{\mathbb{X}} = 0$. □

Let $P_{\mathbb{X}_n} \in L(\mathbb{X}, \mathbb{X}_n)$ be the $\mathbb{X}$-orthogonal projection onto $\mathbb{X}_n$. Lemma 7 implies that $\lim_{n\to\infty} \|\mathbf{u} - P_{\mathbb{X}_n}\mathbf{u}\|_{\mathbb{X}} = 0$ for each $\mathbf{u} \in \mathbb{X}$.

Based on $\mathbb{X}_n$ we can formulate the discrete problem as

$$\text{find } \mathbf{u}_n \in \mathbb{X}_n \text{ s.t.} \quad a(\mathbf{u}_n, \mathbf{u}_n') = \langle \mathbf{f}, \mathbf{u}_n' \rangle \quad \forall \mathbf{u}_n' \in \mathbb{X}_n. \tag{3.8}$$

Let $A \in L(\mathbb{X})$ be the operator associated with $a(\cdot,\cdot)$ and $A_n := P_{\mathbb{X}_n} A|_{\mathbb{X}_n} \in L(\mathbb{X}_n)$. Then, the introduced Galerkin approximation constitutes a discrete approximation scheme as described in Section 2.1, whereat $p_n = P_{\mathbb{X}_n}$. To guarantee the stability of the approximations we impose the following assumption. Let

$$Q_n := \{f \in L^2_0 \colon f|_T \in P_{k-1}(T) \quad \forall T \in \mathscr{T}_n\} \tag{3.9}$$

and let $P_{Q_n} \in L(L^2_0, Q_n)$ be the associated orthogonal projection.

A key observation of the following analysis is that a discrete inf-sup-stability for the discrete divergence operator and the spaces $\mathbb{X}_n$ and Q_n allows to obtain a discrete counterpart of the Helmholtz-type decomposition that is required for the discrete T_n operator in the T-coercivity analysis.

Assumption 8. There exists a constant $\beta_{\mathrm{disc}} > 0$ such that

$$\inf_{f_n \in Q_n \setminus \{0\}} \sup_{\mathbf{u}_n \in \mathbb{X}_n \setminus \{0\}} \frac{|\langle \operatorname{div} \mathbf{u}_n, f_n \rangle|}{\|\nabla \mathbf{u}_n\|_{(L^2)^{3\times 3}} \|f_n\|_{L^2}} > \beta_{\mathrm{disc}}$$

for all $n \in \mathbb{N}$.

The choice of Q_n in (3.9) relates to Scott–Vogelius elements in the discretization of the Stokes problem. In order to ensure its stability, and hence to make sure that assumption 8 is satisfied, it is usually necessary to apply special meshes (barycentric refinement) and/or sufficiently large polynomial degree k, see, e.g., Scott & Vogelius (1985), Arnold & Qin (1992), Zhang (2005, 2011), Guzmán & Neilan (2018) and Neilan (2020).

While the Scott–Vogelius element satisfies the stronger condition $\operatorname{div}(\mathbb{X}_n) \subset Q_n$, this property is not essential for the validity of the analysis, as will also be clarified in remark 2. Assumption 8 can often be relaxed if Q_n is replaced by another finite element space and in the discrete formulation div is replaced by $\operatorname{div}_h := P_{Q_n} \operatorname{div}$. We will comment on this type of discretizations and the necessary adjustments in the analysis in more detail in remark 2 after the first *a priori* error bounds, below.

3.5 *Homogeneous pressure and gravity*

In this section we consider a simplified case of (1.1) in which the pressure and gravitational potential are assumed to be constant before we consider the general case in the subsequent section. Equation (3.5) reduces to

$$a(\mathbf{u}, \mathbf{u}') = \langle c_s^2 \rho \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}' \rangle - \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega \times)\mathbf{u}, (\omega + i\partial_{\mathbf{b}} + i\Omega \times)\mathbf{u}' \rangle - i\omega \langle \gamma \rho \mathbf{u}, \mathbf{u}' \rangle.$$

We aim to establish the stability of $(A_n)_{n\in\mathbb{N}}$ by means of Theorem 3 and Lemma 1. To this end, we need to construct operators T_n with respective properties. Of course the natural approach is to mimic the analysis from the continuous level (Halla & Hohage, 2021). However, for the analysis in this article we will rely on a slightly different construction than used in Halla & Hohage (2021). The reason thereof is that this new variant can be mimicked more easily on the discrete level. While the analysis presented here is an important set-up for the discrete problem, compared with the results in Halla & Hohage (2021) it is suboptimal, as the assumption on the Mach number $\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}$ is more restrictive.

LEMMA 9. Let $\beta > 0$ be the inf-sup constant of the divergence on $\mathscr{O}$. Let $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} < \beta^2 \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}$. Let $T := P_V - P_W$. Then, $T \in L(\mathbb{X})$ is bijective with inverse $T^{-1} = T$ and A is weakly right T-coercive.

Proof. Since P_V and P_W are the projections of a topological decomposition, it holds that $T \in L(\mathbb{X})$ and $TT = (P_V - P_W)(P_V - P_W) = P_V P_V + P_W P_W = P_V + P_W = I$. Using $T\mathbf{u} = \mathbf{v} - \mathbf{w}$ we have that $\langle AT\mathbf{u}, \mathbf{u}\rangle_{\mathbb{X}} = a(\mathbf{v} - \mathbf{w}, \mathbf{v} + \mathbf{w})$. It then holds $AT = B + K$ for B and K defined by

$$\begin{aligned}\langle B\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}} :=& \langle c_s^2\rho \operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}'\rangle - \langle \rho i\partial_{\mathbf{b}}\mathbf{v}, i\partial_{\mathbf{b}}\mathbf{v}'\rangle \\ &- \langle \rho i\partial_{\mathbf{b}}\mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}'\rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, i\partial_{\mathbf{b}}\mathbf{v}'\rangle \\ &+ \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}'\rangle + i\omega\langle\gamma\rho\mathbf{w}, \mathbf{w}'\rangle \end{aligned} \tag{3.10}$$

$$\begin{aligned}\langle K\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}} :=& - \langle \rho(\omega + i\Omega\times)\mathbf{v}, (\omega + i\Omega\times)\mathbf{v}'\rangle - \langle \rho(\omega + i\Omega\times)\mathbf{v}, i\partial_{\mathbf{b}}\mathbf{v}'\rangle - \langle \rho i\partial_{\mathbf{b}}\mathbf{v}, (\omega + i\Omega\times)\mathbf{v}'\rangle \\ &- \langle \rho(\omega + i\Omega\times)\mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}'\rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\Omega\times)\mathbf{v}'\rangle \\ &- i\omega\langle\rho\gamma\mathbf{v}, \mathbf{v}'\rangle + i\omega\langle\rho\gamma\mathbf{w}, \mathbf{v}'\rangle - i\omega\langle\rho\gamma\mathbf{v}, \mathbf{w}'\rangle, \quad \text{for all } \mathbf{u}, \mathbf{u}' \in \mathbb{X}. \end{aligned} \tag{3.11}$$

The terms appearing in definition of K can be represented, e.g., as

$$\langle \rho(\omega + i\Omega\times)\mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}'\rangle = \langle P_W^* B^*_{(\omega+i\partial_{\mathbf{b}}+i\Omega\times)} M_{\rho(\omega+i\Omega\times)} E_{V,\mathbf{L}^2} P_V \mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}}$$

with the embedding $E_{V,\mathbf{L}^2} \in L(V, \mathbf{L}^2)$, the multiplication operator $M_{\rho(\omega+i\Omega\times)} \in L(\mathbf{L}^2)$ and $B_{(\omega+i\partial_{\mathbf{b}}+i\Omega\times)} \in L(W, \mathbf{L}^2)$, $B_{(\omega+i\partial_{\mathbf{b}}+i\Omega\times)}\mathbf{w} := (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}$. Since the embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^2$ is compact, V embeds continuously into $\mathbf{H}^1$ and each term in (3.11) contains at least one operator $E_{V,\mathbf{L}^2}$ or $E^*_{V,\mathbf{L}^2}$ it follows that $K \in L(\mathbb{X})$ is compact. We now show that B is coercive, and hence that A is bijective. Let $\tau \in (0, \pi/2)$. We compute

$$\begin{aligned}\frac{1}{\cos\tau}\operatorname{Re}\left(e^{-i\tau\operatorname{sgn}\omega}\langle B\mathbf{u}, \mathbf{u}\rangle_{\mathbb{X}}\right) =& \langle c_s^2\rho\operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}\rangle - \langle\rho i\partial_{\mathbf{b}}\mathbf{v}, i\partial_{\mathbf{b}}\mathbf{v}\rangle + \tan\tau|\omega|\langle\gamma\rho\mathbf{w}, \mathbf{w}\rangle \\ &+ \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}\rangle \\ &- 2\tan\tau\operatorname{sgn}\omega\operatorname{Im}\left(\langle\rho i\partial_{\mathbf{b}}\mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}\rangle\right).\end{aligned}$$

We estimate the last term by the Cauchy–Schwarz inequality and the weighted Young inequality $|2ab| \leq (1-\varepsilon)^{-1}a^2 + (1-\varepsilon)b^2$ with an additional parameter $\varepsilon \in (0,1)$, $a = \tan\tau\|\sqrt{\rho}\partial_{\mathbf{b}}\mathbf{v}\|_{L^2}$, and $b = \|\sqrt{\rho}(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}\|_{L^2}$ and obtain

$$\begin{aligned}\frac{1}{\cos\tau}\operatorname{Re}\left(e^{-i\tau\operatorname{sgn}\omega}\langle B\mathbf{u}, \mathbf{u}\rangle_{\mathbb{X}}\right) \geq& \langle c_s^2\rho\operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}\rangle - \left(1 + (1-\varepsilon)^{-1}\tan^2\tau\right)\langle\rho i\partial_{\mathbf{b}}\mathbf{v}, i\partial_{\mathbf{b}}\mathbf{v}\rangle \\ &+ \varepsilon\langle\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}\rangle + \tan\tau|\omega|\langle\gamma\rho\mathbf{w}, \mathbf{w}\rangle.\end{aligned}$$

We estimate further

$$\langle c_s^2\rho \operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}\rangle - \left(1+(1-\varepsilon)^{-1}\tan^2\tau\right)\langle \rho i\partial_{\mathbf{b}}\mathbf{v}, i\partial_{\mathbf{b}}\mathbf{v}\rangle$$
$$\geq \left(\beta^2\underline{c_s}^2\underline{\rho} - \overline{c_s}^2\overline{\rho}\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}\left(1+(1-\varepsilon)^{-1}\tan^2\tau\right)\right)\langle\nabla\mathbf{v},\nabla\mathbf{v}\rangle.$$

Due to the assumption of this lemma we can choose small enough $\tau\in(0,\pi/2)$ and $\varepsilon>0$ such that the constant in the right-hand side is positive. Since $\|\mathbf{v}\|_{\mathbb{X}} \lesssim \|\nabla\mathbf{v}\|_{L^2}$ for $\mathbf{v}\in\mathbf{H}^1_0$, this yields coercivity in $\mathbf{v}$. As in Halla & Hohage (2021) a weighted Young's inequality shows that

$$\varepsilon\langle\rho(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w},(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w}\rangle+\tan\tau|\omega|\langle\gamma\rho\mathbf{w},\mathbf{w}\rangle \gtrsim \left\|\partial_{\mathbf{b}}\mathbf{w}\right\|^2_{L^2}+\|\mathbf{w}\|^2_{L^2}=\|\mathbf{w}\|^2_{\mathbb{X}}.$$

Thus, $|\langle B\mathbf{u},\mathbf{u}\rangle_{\mathbb{X}}| \gtrsim \|\mathbf{v}\|^2_{\mathbb{X}}+\|\mathbf{w}\|^2_{\mathbb{X}} \gtrsim \|\mathbf{u}\|^2_{\mathbb{X}}$ and the claim follows. □

3.5.1 *Regular approximation.*

LEMMA 10. Let Assumption 8 be satisfied. Then, the spaces

$$\begin{aligned} V_n &:= \left\{\mathbf{u}_n\in\mathbb{X}_n\cap\mathbf{H}^1_0\colon \langle\nabla\mathbf{u}_n,\nabla\mathbf{u}'_n\rangle=0 \quad \forall\mathbf{u}'_n\in\mathbf{H}^1_0\cap W_n\right\},\\ W_n &:= \{\mathbf{u}_n\in\mathbb{X}_n\colon \operatorname{div}\mathbf{u}_n=0\}, \end{aligned} \tag{3.12}$$

form a topological decomposition of $\mathbb{X}_n$ with projections

$$P_{V_n}\mathbf{u}_n := D_n^{-1}\operatorname{div}\mathbf{u}_n \quad\text{and}\quad P_{W_n}\mathbf{u}_n := \mathbf{u}_n - P_{V_n}\mathbf{u}_n,$$

being uniformly bounded in $n\in\mathbb{N}$, where for $q_n\in Q_n$ the function $D_n^{-1}q_n\in V_n$ is the unique solution to

$$\text{find } \mathbf{v}_n\in V_n \text{ such that } \operatorname{div}\mathbf{v}_n=q_n, \tag{3.13}$$

i.e., $D_n^{-1}\in L(Q_n,V_n)$.

Proof. Assumption 8 ensures that (3.13) admits a unique solution $\mathbf{v}_n$ that satisfies $\beta_{\text{disc}}\|\nabla\mathbf{v}_n\|_{(L^2)^{3\times3}} \leq \|q_n\|_{L^2}$. Since $P_{V_n}P_{V_n}\mathbf{u}_n = D_n^{-1}\operatorname{div}P_{V_n}\mathbf{u}_n = D_n^{-1}\operatorname{div}\mathbf{u}_n = P_{V_n}\mathbf{u}_n$, P_{V_n} is indeed a projection. Assumption 8 ensures that P_{V_n} is uniformly bounded. Due $\ker P_{V_n}=W_n$ the spaces V_n, W_n form indeed a topological decomposition of $\mathbb{X}_n$. □

We abbreviate $\mathbf{v}_n := P_{V_n}\mathbf{u}_n$, $\mathbf{w}_n := P_{W_n}\mathbf{u}_n$ for $\mathbf{u}_n\in\mathbb{X}_n$.

LEMMA 11. Let Assumption 8 be satisfied. Then,

$$T_n := P_{V_n} - P_{W_n} = T_n^{-1}\in L(\mathbb{X}_n)$$

is uniformly bounded in $n\in\mathbb{N}$.

Proof. Since the spaces V_n and W_n form a topological decomposition of $\mathbb{X}_n$, it follows that $T_n = T_n^{-1}$. The uniform boundedness of T_n and T_n^{-1} follows from the uniform boundedness of P_{V_n}. □

LEMMA 12. For each $\mathbf{v} \in V$ it holds that $\lim_{n\to\infty} \left\| \mathbf{v} - D_n^{-1} P_{Q_n} \operatorname{div} \mathbf{v} \right\|_{\mathbf{H}^1} = 0$.

Proof. $(\mathbf{v}, 0)$ solves the problem to find $(\mathbf{u}, \mathbf{z}) \in \mathbf{H}_0^1 \times (W \cap \mathbf{H}_0^1)$ such that

$$\langle \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}' \rangle + \langle \nabla \mathbf{u}, \nabla \mathbf{z}' \rangle + \langle \nabla \mathbf{z}, \nabla \mathbf{u}' \rangle = \langle \operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{u}' \rangle,$$

for all $(\mathbf{u}', \mathbf{z}') \in \mathbf{H}_0^1 \times (W \cap \mathbf{H}_0^1)$, and $(D_n^{-1} P_{Q_n} \operatorname{div} \mathbf{v}, 0)$ solves the problem to find $(\mathbf{u}_n, \mathbf{z}_n) \in \mathbb{X}_n \cap \mathbf{H}_0^1 \times (W_n \cap \mathbf{H}_0^1)$ such that

$$\langle \operatorname{div} \mathbf{u}_n, \operatorname{div} \mathbf{u}_n' \rangle + \langle \nabla \mathbf{u}_n, \nabla \mathbf{z}_n' \rangle + \langle \nabla \mathbf{z}_n, \nabla \mathbf{u}_n' \rangle = \langle \operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{u}_n' \rangle,$$

for all $(\mathbf{u}_n', \mathbf{z}_n') \in \mathbb{X}_n \cap \mathbf{H}_0^1 \times (W_n \cap \mathbf{H}_0^1)$. The latter is a conforming Galerkin approximation of the former. It can be seen that both equations are uniformly stable by testing with $(\mathbf{v} + \mathbf{z}, \mathbf{w})$ and $(\mathbf{v}_n + \mathbf{z}_n, \mathbf{w}_n)$, respectively. With a Céa lemma it only remains to show that $\lim_{n\to\infty} \inf_{\mathbf{u}_n \in \mathbb{X}_n \cap \mathbf{H}_0^1} \left\| \mathbf{u} - \mathbf{u}_n \right\|_{\mathbf{H}^1} = 0$ and $\lim_{n\to\infty} \inf_{\mathbf{z}_n \in W_n \cap \mathbf{H}_0^1} \left\| \mathbf{z} - \mathbf{z}_n \right\|_{\mathbf{H}^1} = 0$ for $\mathbf{u} = \mathbf{v} \in V \subset \mathbf{H}_0^1$ and $\mathbf{z} = 0$, respectively. The first result is standard, while the second is trivial as $\mathbf{z} = 0 \in W_n$. □

Next, we shall establish the point-wise limit of T_n.

LEMMA 13. For each $\mathbf{u} \in \mathbb{X}$ it holds $\lim_{n\to\infty} \|T_n P_{\mathbb{X}_n} \mathbf{u} - P_{\mathbb{X}_n} T \mathbf{u}\|_{\mathbb{X}} = 0$.

Proof. It suffices to prove $\lim_{n\to\infty} \|P_{V_n} P_{\mathbb{X}_n} \mathbf{u} - P_{\mathbb{X}_n} P_V \mathbf{u}\|_{\mathbb{X}} = 0$. We use $P_{V_n} = D_n^{-1} P_{Q_n} \operatorname{div}$ and estimate

$$\begin{aligned}
&\|P_{V_n} P_{\mathbb{X}_n} \mathbf{u} - P_{\mathbb{X}_n} P_V \mathbf{u}\|_{\mathbb{X}} \leq \|D_n^{-1} P_{Q_n} \operatorname{div} P_{\mathbb{X}_n} \mathbf{u} - P_V \mathbf{u}\|_{\mathbb{X}} + \|P_V \mathbf{u} - P_{\mathbb{X}_n} P_V \mathbf{u}\|_{\mathbb{X}} \\
&\leq \|D_n^{-1} P_{Q_n} \operatorname{div} \mathbf{u} - P_V \mathbf{u}\|_{\mathbb{X}}\| + \|D_n^{-1} P_{Q_n} \operatorname{div}\|_{L(\mathbb{X})} \|\mathbf{u} - P_{\mathbb{X}_n} \mathbf{u}\|_{\mathbb{X}} + \|P_V \mathbf{u} - P_{\mathbb{X}_n} P_V \mathbf{u}\|_{\mathbb{X}}.
\end{aligned}$$

The claim follows now from $\operatorname{div} \mathbf{u} = \operatorname{div} P_V \mathbf{u}$, Lemma 12, (3.3) and the point-wise convergence of $P_{\mathbb{X}_n}$ (see Lemma 7). □

LEMMA 14. If $\|c_s^{-1} \mathbf{b}\|_{\mathbf{L}^\infty}^2 < \beta_{\text{disc}}^2 \frac{\underline{c_s^2 \rho}}{\overline{c_s^2 \rho}}$ then $(A_n)_{n\in\mathbb{N}}$ is regular[(vi)], in the sense of definition 1.

Proof. We apply Theorem 3. In the previous part of Section 3.5.1 we already constructed T_n and showed that $T_n \in L(\mathbb{X}_n)$ and $T_n^{-1} = T_n \in L(\mathbb{X}_n)$ are uniformly bounded. Further, Lemma 13 shows that T_n converges pointwise. Next, we need to split $A_n T_n = B_n + K_n$ into a stable part $B_n \in L(\mathbb{X}_n)$ and a compact part $K_n \in L(\mathbb{X}_n)$. To do so we stick very closely to the lines of Halla & Hohage (2021). Recall that $\langle A_n T_n \mathbf{u}_n, \mathbf{u}_n \rangle_{\mathbb{X}} = a(\mathbf{v}_n - \mathbf{w}_n, \mathbf{v}_n + \mathbf{w}_n)$. Hence, it holds $A_n T_n = B_n + K_n$ with B_n and K_n defined by

$$\begin{aligned}
\langle B_n \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}} := &\langle c_s^2 \rho \operatorname{div} \mathbf{v}_n, \operatorname{div} \mathbf{v}_n' \rangle - \langle \rho i\partial_{\mathbf{b}} \mathbf{v}_n, i\partial_{\mathbf{b}} \mathbf{v}_n' \rangle \\
&- \langle \rho i\partial_{\mathbf{b}} \mathbf{v}_n, (\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}_n' \rangle + \langle \rho (\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}_n, i\partial_{\mathbf{b}} \mathbf{v}_n' \rangle \\
&+ \langle \rho (\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}_n, (\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}_n' \rangle + i\omega \langle \gamma \rho \mathbf{w}_n, \mathbf{w}_n' \rangle
\end{aligned}$$

and

$$\begin{aligned}\langle K_n \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}} := &- \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle - \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, i\partial_{\mathbf{b}}\mathbf{v}_n' \rangle - \langle \rho i\partial_{\mathbf{b}}\mathbf{v}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle \\ &- \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n' \rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle \\ &- i\omega\langle \rho\gamma\mathbf{v}_n, \mathbf{v}_n' \rangle + i\omega\langle \rho\gamma\mathbf{w}_n, \mathbf{v}_n' \rangle - i\omega\langle \rho\gamma\mathbf{v}_n, \mathbf{w}_n' \rangle\end{aligned}$$

for all $\mathbf{u}_n, \mathbf{u}_n' \in \mathbb{X}_n$. The operator K_n is compact due to the compact Sobolev embedding from $V_n \subset \mathbf{H}^1$ to $\mathbf{L}^2$. It is straightforward to see that B_n is uniformly bounded and that B_n converges pointwise to the operator $B \in L(\mathbb{X})$ defined in (3.10). The uniform coercivity of B_n follows along the lines of the proof of Lemma 9, with the constant β replaced by β_{disc}. Hence, the claim is proven. □

3.5.2 *Convergence.*

THEOREM 15. Let p and ϕ be constant. Let $\mathbf{u}$ be the solution to (1.1). Let Assumption 8 be satisfied and $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} < \beta^2_{\text{disc}} \frac{\underline{c_s}^2 \underline{\rho}}{\overline{c_s}^2 \overline{\rho}}$. Then, there exists an index $n_0 > 0$ such that for all $n > n_0$ the solution $\mathbf{u}_n$ to (3.8) exists and $\mathbf{u}_n$ converges to $\mathbf{u}$ in the $\mathbb{X}$-norm with the best approximation estimate $\|\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}} \lesssim \inf_{\mathbf{u}_n' \in \mathbb{X}_n} \|\mathbf{u} - \mathbf{u}_n'\|_{\mathbb{X}}$.

Proof. Due to Lemma 14 the approximation scheme $(A_n)_{n\in\mathbb{N}}$ is regular. Since A is bijective (Halla & Hohage, 2021), the claim follows from Lemmas 1 and 2. □

REMARK 1. Note that for smooth solutions $\mathbf{u} \in \mathbf{H}^{1+s}$, $s > 0$ we can obtain convergence rates by convenient techniques:

$$\inf_{\mathbf{u}_n' \in \mathbb{X}_n} \|\mathbf{u} - \mathbf{u}_n'\|_{\mathbb{X}} \lesssim \inf_{\mathbf{u}_n' \in \mathbb{X}_n} \|\mathbf{u} - \mathbf{u}_n'\|_{\mathbf{H}^1} \lesssim h^{\min(s,k)} \|\mathbf{u}\|_{\mathbf{H}^{1+s}}.$$

REMARK 2. The considered discrete setting can be generalized by replacing the divergence operator div in the discrete formulation by a discrete version $\operatorname{div}_h : \mathbb{X}_n \to Q_n$ with a space Q_n that is potentially different to the one in (3.9). In this case also assumption 8 would be relaxed w.r.t. div and Q_n. One important case, which is known as the *Taylor–Hood* discretization in fluid dynamics, is obtained from $Q_n = \{f \in L^2_0 : f|_T \in P_{k-1}(T) \quad \forall T \in \mathcal{T}_n\} \cap H^1$ and $\operatorname{div}_h = P_{Q_n} \operatorname{div}$. For the implementation of div_h one typically introduces an auxiliary variable, the so-called pseudo-pressure so that $\langle \operatorname{div}_h \mathbf{u}_n, \operatorname{div}_h \mathbf{u}_n' \rangle$ becomes $\langle q_n, \operatorname{div} \mathbf{u}_n' \rangle + \langle \operatorname{div} \mathbf{u}_n, q_n' \rangle - \langle q_n, q_n' \rangle$, where $\mathbf{u}_n$ and q_n and $\mathbf{u}_n'$ and q_n' are the trial and the test functions in $\mathbb{X}_n$ and Q_n, respectively. Let us briefly sketch the changes in the analysis that would be necessary to account for this change in the discrete formulation. First, note that replacing div with div_h in (3.8) would lead to a *nonconforming* discretization. Hence, we would need to prove asymptotic consistency[(iii)], i.e., the corresponding sequence of discrete operators A_n approximates[(iii)] A, which has been trivial for the Galerkin approximation. In the discrete subspace splitting W_n would need to be defined w.r.t. to div_h (instead of div), as well as the corresponding projection onto V_n in (3.13). With only minor changes also the proof of lemma 12 would carry over to this setting so that finally convergence of the corresponding discrete solution $\mathbf{u}_n$ to the continuous solution $\mathbf{u}$ would follow. Alternatively, an equivalent conforming discretization could be analysed by introducing the pseudo-pressure formulation already on the continuous level. In the remainder of the analysis in this manuscript we will continue to focus on to the case of the divergence operator div and the space Q_n as in (3.9). However, in the numerical examples below we will also consider a Taylor–Hood-type discretization and compare it with the chosen setting of Scott–Vogelius-type elements.

3.6 *Heterogeneous pressure and gravity*

In this section we expand the analysis from the previous section and consider heterogeneous pressure p and gravitational potential ϕ.

3.6.1 *Analysis on the continuous level.* As in Halla & Hohage (2021) we introduce $\mathbf{q} := c_s^{-2}\rho^{-1}\nabla p$ and express

$$a(\mathbf{u},\mathbf{u}') = \langle c_s^2\rho(\operatorname{div}+\mathbf{q}\cdot)\mathbf{u}, (\operatorname{div}+\mathbf{q}\cdot)\mathbf{u}'\rangle - \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}'\rangle$$
$$- i\omega\langle\rho\gamma\mathbf{u},\mathbf{u}'\rangle + \langle(\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi) - c_s^2\rho\,\mathbf{q}\otimes\mathbf{q})\mathbf{u},\mathbf{u}'\rangle. \tag{3.14}$$

However, in the forthcoming analysis we will deviate from Halla & Hohage (2021) and avoid the introduction of an additional third space Z in the topological decomposition of $\mathbb{X}$. Consider now the divergence operator $D \in L(V, L_0^2), D\mathbf{v} := \operatorname{div}\mathbf{v}$. We know that $D^{-1} \in L(L_0^2, V)$. For heterogeneous pressure our analysis leads us to consider $D\mathbf{v} + \mathbf{q}\cdot\mathbf{v}$ instead of $D\mathbf{v}$. A necessary ingredient for our analysis is that the new operator $D + \mathbf{q}\cdot$ is invertible on suitable spaces. Since we cannot ensure this property for $D + \mathbf{q}\cdot$, we work instead with a slight modification.

LEMMA 16. There exist operators $M \in L(\mathbb{X}, L^2)$ and $F \in L(\mathbb{X}, L_0^2)$ with finite-dimensional range such that $\tilde{D} \in L(V, L_0^2)$ defined by $\tilde{D}\mathbf{v} := D\mathbf{v} + \mathbf{q}\cdot\mathbf{v} + M\mathbf{v} + F\mathbf{v}$ is bijective.

Proof. First let $M\mathbf{v} := -\operatorname{mean}(\mathbf{q}\cdot\mathbf{v})$ for which it follows that $D + \mathbf{q}\cdot + M \in L(V, L_0^2)$. The new operator acts now on the same spaces as D and we can perform a perturbation analysis. Indeed, D is bijective and $\mathbf{q}\cdot + M \in L(V, L_0^2)$ is compact from V to L_0^2 due to the continuous embedding $V \hookrightarrow \mathbf{H}^1$ (and because the range of M is one-dimensional). Thus, $D + \mathbf{q}\cdot + M$ is a Fredholm operator with index zero, i.e., the range of $D + \mathbf{q}\cdot + M$ is closed and $N := \dim\ker(D + \mathbf{q}\cdot + M) = \operatorname{codim}\operatorname{ran}(D + \mathbf{q}\cdot + M) < +\infty$. However, we have no tool at our disposal to ensure that $N = 0$ (which would imply the bijectivity of $D + \mathbf{q}\cdot + M$). Thus, we perform an additional modification as follows, where we note that the case $N = 0$ is included. We use that $\langle\operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}'\rangle$ is an equivalent scalar product to $\langle\mathbf{v},\mathbf{v}'\rangle_{\mathbf{H}^1}$ on V. Let $\psi_n, n = 1, \ldots, N$ be an orthonormal basis with respect to $\langle\operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}'\rangle$ of $\ker(D + \mathbf{q}\cdot + M)$, $\phi_n, n = 1, \ldots, N$ be an orthonormal basis of $\operatorname{ran}(D + \mathbf{q}\cdot + M)^\perp$ and set $F\mathbf{v} := \sum_{n=1}^N \phi_n\langle\operatorname{div}\mathbf{v}, \operatorname{div}\psi_n\rangle$. Then,

$$\tilde{D} \in L(V, L_0^2),\ \tilde{D}\mathbf{v} := D\mathbf{v} + \mathbf{q}\cdot\mathbf{v} + M\mathbf{v} + F\mathbf{v}$$

is bijective. □

Note that $\tilde{D}$ is also bounded and well defined on $\mathbb{X}$, i.e., $\tilde{D} \in L(\mathbb{X}, L_0^2)$. Although the inverse $\tilde{D}^{-1}$ will always be considered in the space $L(L_0^2, V)$, for $\mathbf{u} \in \mathbb{X}$ we construct a topological decomposition mirroring the one in the homogeneous case, in (3.6). As $\tilde{D}$ is bijective on $L(V, L_0^2)$, we keep V as in (3.6) and define

$$\widetilde{W} := \{\mathbf{u} \in \mathbb{X} : \tilde{D}\mathbf{u} = 0\}, \tag{3.15}$$

where we use the tilde to indicate the difference to the homogeneous case. The projections onto V and $\widetilde{W}$ are now given by

$$\widetilde{P}_V\mathbf{u} := \tilde{D}^{-1}\tilde{D}\mathbf{u} \quad \text{and} \quad P_{\widetilde{W}}\mathbf{u} := \mathbf{u} - \widetilde{P}_V\mathbf{u};$$

note that, while V is the same as in the homogeneous case, the projection $\widetilde{P}_V$ is different, now defined with respect to $\tilde{D}$. Now, $V \oplus \widetilde{W}$ is again a topological decomposition of $\mathbb{X}$. We keep using the abbreviations $\mathbf{v} := \widetilde{P_V}\mathbf{u}$, $\mathbf{w} := P_{\widetilde{W}}\mathbf{u}$ for $\mathbf{u} \in \mathbb{X}$.

Since $\mathbf{v} \in V$, it holds

$$\|\operatorname{div} \mathbf{v}\|_{L^2} \geq \beta \|\nabla \mathbf{v}\|_{(L^2)^{3\times 3}}.$$

Further, it follows that

$$\begin{aligned}
(\operatorname{div} + \mathbf{q}\cdot)\mathbf{w} &= (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u} - (\operatorname{div} + \mathbf{q}\cdot)\mathbf{v} \\
&= (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u} - (\operatorname{div} + \mathbf{q}\cdot + M + F)\mathbf{v} + (M+F)\mathbf{v} \\
&= (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u} - (\operatorname{div} + \mathbf{q}\cdot + M + F)\mathbf{u} + (M+F)\mathbf{v} \\
&= -(M+F)\mathbf{u} + (M+F)\mathbf{v} \\
&= -(M+F)(\mathbf{u} - \mathbf{v}) \\
&= -(M+F)\mathbf{w}
\end{aligned} \tag{3.16}$$

is a compact operator, which is almost as good as being zero. Hence, the decomposition $\mathbf{u} = \mathbf{v} + \mathbf{w}$ satisfies our wishes. Thus, we build

$$T\mathbf{u} := \mathbf{v} - \mathbf{w}. \tag{3.17}$$

Let $\lambda_-(\underline{\underline{m}}) \in L^\infty$ be the smallest eigenvalue of the symmetric matrix

$$\underline{\underline{m}} := -\rho^{-1}\operatorname{Hess}(p) + \operatorname{Hess}(\phi). \tag{3.18}$$

Further, let

$$C_M := \max\left\{0, \sup_{x\in\mathcal{O}} \frac{-\lambda_-(\underline{\underline{m}}(x))}{\gamma(x)}\right\} \quad \text{and} \quad \theta := \arctan(C_M/|\omega|) \in [0, \pi/2) \tag{3.19}$$

for $\omega \neq 0$.

Corollary 17. Let $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} < \beta^2 \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}} \frac{1}{1+\tan^2\theta}$. Then, A is weakly right T-coercive.

Proof. Using T as defined in Equation (3.17) we can split $AT = B + K$ with $B, K \in L(\mathbb{X})$ given by

$$\begin{aligned}
\langle B\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}} :=& -\langle \rho i\partial_{\mathbf{b}}\mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}'\rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, i\partial_{\mathbf{b}}\mathbf{v}'\rangle \\
&+ \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}'\rangle - \langle \rho i\partial_{\mathbf{b}}\mathbf{v}, i\partial_{\mathbf{b}}\mathbf{v}'\rangle \\
&+ i\omega\langle \gamma\rho\mathbf{w}, \mathbf{w}'\rangle + \langle \rho\underline{\underline{m}}\mathbf{w}, \mathbf{w}'\rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}'\rangle \\
&+ \langle c_s^2\rho(\mathbf{q}\cdot\mathbf{w}), (\mathbf{q}\cdot\mathbf{w}')\rangle + \langle \rho F\mathbf{w}, F\mathbf{w}'\rangle + \langle \rho M\mathbf{w}, M\mathbf{w}'\rangle
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\langle K\mathbf{u},\mathbf{u}'\rangle_{\mathbb{X}} :=& -\langle\rho(\omega+i\Omega\times)\mathbf{v},(\omega+i\Omega\times)\mathbf{v}'\rangle-\langle\rho(\omega+i\Omega\times)\mathbf{v},i\partial_{\mathbf{b}}\mathbf{v}'\rangle-\langle\rho i\partial_{\mathbf{b}}\mathbf{v},(\omega+i\Omega\times)\mathbf{v}'\rangle\\ &-\langle\rho(\omega+i\Omega\times)\mathbf{v},(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w}'\rangle+\langle\rho(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w},(\omega+i\Omega\times)\mathbf{v}'\rangle\\ &-i\omega\langle\gamma\rho\mathbf{v},\mathbf{w}'\rangle+i\omega\langle\gamma\rho\mathbf{w},\mathbf{v}'\rangle-i\omega\langle\gamma\rho\mathbf{v},\mathbf{v}'\rangle+\langle\rho\underline{\underline{m}}\mathbf{w},\mathbf{v}'\rangle-\langle\rho\underline{\underline{m}}\mathbf{v},\mathbf{w}'\rangle-\langle\rho\underline{\underline{m}}\mathbf{v},\mathbf{v}'\rangle\\ &+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v},\operatorname{div}\mathbf{v}'\rangle+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v},\operatorname{div}\mathbf{v}'\rangle-\langle c_s^2\rho\operatorname{div}\mathbf{w},\mathbf{q}\cdot\mathbf{v}'\rangle+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v},\operatorname{div}\mathbf{w}'\rangle\\ &-\langle c_s^2\rho(\operatorname{div}+\mathbf{q}\cdot)\mathbf{w},(\operatorname{div}+\mathbf{q}\cdot)\mathbf{w}'\rangle-\langle c_s^2\rho(\operatorname{div}+\mathbf{q}\cdot)\mathbf{w},\operatorname{div}\mathbf{v}'\rangle+\langle c_s^2\rho\operatorname{div}\mathbf{v},(\operatorname{div}+\mathbf{q}\cdot)\mathbf{w}'\rangle\\ &-\langle\rho F\mathbf{w},F\mathbf{w}'\rangle-\langle\rho M\mathbf{w},M\mathbf{w}'\rangle\end{aligned}$$

for all $\mathbf{u},\mathbf{u}'\in\mathbb{X}$. The operator K is compact due to the compact Sobolev embedding from $V\subset\mathbf{H}^1$ to $\mathbf{L}^2$, due to the compactness of F,M and Equation (3.16). Next, we show that B is coercive. Let $\tau\in(0,\pi/2-\theta)$. First, we note that

$$\begin{aligned}&\frac{1}{\cos(\theta+\tau)}\operatorname{Re}\Big(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\Big(\langle\rho(i\omega\gamma+\underline{\underline{m}})\mathbf{w},\mathbf{w}\rangle\Big)\Big)\\ &\quad=\operatorname{Re}\Big(\langle\rho(i\omega\gamma+\underline{\underline{m}})\mathbf{w},\mathbf{w}\rangle\Big)+\operatorname{sgn}\omega\tan(\theta+\tau)\operatorname{Im}\Big(\langle\rho(i\omega\gamma+\underline{\underline{m}})\mathbf{w},\mathbf{w}\rangle\Big)\\ &\quad=\langle\rho\underline{\underline{m}}\mathbf{w},\mathbf{w}\rangle+|\omega|\tan(\theta+\tau)\langle\rho\gamma\mathbf{w},\mathbf{w}\rangle\geq\langle\rho\lambda_-(\underline{\underline{m}})\mathbf{w},\mathbf{w}\rangle+|\omega|\tan(\theta+\tau)\langle\rho\gamma\mathbf{w},\mathbf{w}\rangle\\ &\quad\geq|\omega|(\tan(\theta+\tau)-\tan\theta)\langle\rho\gamma\mathbf{w},\mathbf{w}\rangle,\end{aligned}$$

whereat the last estimate is due to the definition of θ (3.19). We compute

$$\begin{aligned}\frac{1}{\cos(\theta+\tau)}\operatorname{Re}\Big(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle B\mathbf{u},\mathbf{u}\rangle_{\mathbb{X}}\Big)\geq&\ \langle c_s^2\rho\operatorname{div}\mathbf{v},\operatorname{div}\mathbf{v}\rangle-\langle\rho i\partial_{\mathbf{b}}\mathbf{v},i\partial_{\mathbf{b}}\mathbf{v}\rangle\\ &+\langle\rho(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w},(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w}\rangle\\ &+(\tan(\theta+\tau)-\tan\theta)|\omega|\langle\gamma\rho\mathbf{w},\mathbf{w}\rangle\\ &+\langle\rho F\mathbf{w},F\mathbf{w}\rangle+\langle\rho M\mathbf{w},M\mathbf{w}\rangle+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{w},\mathbf{q}\cdot\mathbf{w}\rangle\\ &+2\tan(\theta+\tau)\operatorname{sgn}\omega\operatorname{Im}\big(\langle\rho i\partial_{\mathbf{b}}\mathbf{v},(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w}\rangle\big).\end{aligned}$$

We proceed now as in the proof of Lemma 9 and estimate

$$\begin{aligned}&\frac{1}{\cos(\theta+\tau)}\operatorname{Re}\Big(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle B\mathbf{u},\mathbf{u}\rangle_{\mathbb{X}}\Big)\\ &\quad\geq\langle c_s^2\rho\operatorname{div}\mathbf{v},\operatorname{div}\mathbf{v}\rangle-\Big(1+(1-\varepsilon)^{-1}\tan^2(\theta+\tau)\Big)\langle\rho i\partial_{\mathbf{b}}\mathbf{v},i\partial_{\mathbf{b}}\mathbf{v}\rangle\\ &\quad+\varepsilon\langle\rho(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w},(\omega+i\partial_{\mathbf{b}}+i\Omega\times)\mathbf{w}\rangle\\ &\quad+(\tan(\theta+\tau)-\tan\theta)|\omega|\langle\gamma\rho\mathbf{w},\mathbf{w}\rangle\\ &\quad+\langle\rho F\mathbf{w},F\mathbf{w}\rangle+\langle\rho M\mathbf{w},M\mathbf{w}\rangle+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{w},\mathbf{q}\cdot\mathbf{w}'\rangle.\end{aligned}$$

The same reasoning as in the proof of Lemma 9 yields

$$\langle c_s^2\rho \operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{v}\rangle - \Big(1 + (1-\varepsilon)^{-1}\tan^2(\theta+\tau)\Big)\langle \rho i\partial_{\mathbf{b}}\mathbf{v}, i\partial_{\mathbf{b}}\mathbf{v}\rangle \gtrsim \|\mathbf{v}\|_{\mathbb{X}}^2$$

and

$$\varepsilon\langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}\rangle + (\tan(\theta+\tau) - \tan\theta)|\omega|\langle \gamma\rho\mathbf{w}, \mathbf{w}\rangle \gtrsim \|\partial_{\mathbf{b}}\mathbf{w}\|_{\mathbf{L}^2}^2 + \|\mathbf{w}\|_{\mathbf{L}^2}^2.$$

Using Equation (3.16) we know that $\operatorname{div} \mathbf{w} = -\mathbf{q}\cdot\mathbf{w} - M\mathbf{w} - F\mathbf{w}$ and we obtain further that

$$\begin{aligned}
&\varepsilon\langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}\rangle \\
&\quad + (\tan(\theta+\tau) - \tan\theta)|\omega|\langle \gamma\rho\mathbf{w}, \mathbf{w}\rangle \\
&\quad + \langle \rho F\mathbf{w}, F\mathbf{w}\rangle + \langle \rho M\mathbf{w}, M\mathbf{w}\rangle + \langle c_s^2\rho\mathbf{q}\cdot\mathbf{w}, \mathbf{q}\cdot\mathbf{w}\rangle \\
&\gtrsim \|\operatorname{div}\mathbf{w}\|_{L^2}^2 + \|\partial_{\mathbf{b}}\mathbf{w}\|_{\mathbf{L}^2}^2 + \|\mathbf{w}\|_{\mathbf{L}^2}^2 = \|\mathbf{w}\|_{\mathbb{X}}^2.
\end{aligned}$$

Thus, B is uniformly coercive and the proof is finished. □

3.6.2 *The discrete topological decomposition.* Now, we mimic this construction on the discrete level. Let P_{Q_n} be the orthogonal projection onto Q_n. Consider the discrete operator

$$\tilde{D}_n := P_{Q_n}\tilde{D}|_{V_n}.$$

Note that $V_n \not\subset V$ and hence $\tilde{D}_n$ is a nonconforming approximation of $\tilde{D}$. Compared with the homogeneous case, we must first ensure that the discrete operator is a suitable approximation of $\tilde{D}$.

LEMMA 18. Let Assumption 8 be satisfied. $\tilde{D}_n \in L(V_n, Q_n)$ with $p_n = D_n^{-1}P_{Q_n}\operatorname{div} \in L(V, V_n), P_{Q_n} \in L(L_0^2, Q_n)$ forms a discrete approximation scheme of $\tilde{D} \in L(V, L_0^2)$, which approximates $\tilde{D}$ and is stable. In particular, it holds $\lim_{n\to\infty}\|\tilde{D}_n^{-1}P_{Q_n}\tilde{D}\mathbf{v} - \mathbf{v}\|_{\mathbf{H}^1} = 0$ for each $\mathbf{v}\in V$.

Proof. Due to Lemma 12 and, since P_{Q_n} is an orthogonal projection, it easily follows that the approximation is a discrete approximation scheme. For the approximation property we compute for $\mathbf{v}\in V$

$$\begin{aligned}
\|\tilde{D}_n p_n\mathbf{v} - P_{Q_n}\tilde{D}\mathbf{v}\|_{L^2} &= \|\tilde{D}_n D_n^{-1}P_{Q_n}\operatorname{div}\mathbf{v} - P_{Q_n}\tilde{D}\mathbf{v}\|_{L^2} \\
&= \|P_{Q_n}\tilde{D}D_n^{-1}P_{Q_n}\operatorname{div}\mathbf{v} - P_{Q_n}\tilde{D}\mathbf{v}\|_{L^2} \\
&\le \|P_{Q_n}\tilde{D}\|_{L(\mathbf{H}^1, L^2)}\|D_n^{-1}P_{Q_n}\operatorname{div}\mathbf{v} - \mathbf{v}\|_{\mathbf{H}^1},
\end{aligned}$$

whereat the right-hand side tends to zero for $n\to\infty$ due to Lemma 12. By construction $\tilde{D}\in L(V, L_0^2)$ is bijective, and hence the regularity of $\tilde{D}_n$ implies its stability. Since we can split $\tilde{D}_n = D_n + P_{Q_n}(q\cdot +M+F)|_{V_n}$ into a stable part D_n and a compact part $P_{Q_n}(q\cdot +M+F)|_{V_n}$, the regularity of $\tilde{D}_n$ follows similarly as in the proof of Theorem 3. The last claim follows from

$$\|\tilde{D}_n^{-1}P_{Q_n}\tilde{D}\mathbf{v} - \mathbf{v}\|_{\mathbf{H}^1} \le \|\tilde{D}_n^{-1}P_{Q_n}\tilde{D}\mathbf{v} - D_n^{-1}P_{Q_n}\operatorname{div}\mathbf{v}\|_{\mathbf{H}^1} + \|D_n^{-1}P_{Q_n}\operatorname{div}\mathbf{v} - \mathbf{v}\|_{\mathbf{H}^1}.$$

Here, the first terms tends to zero, because the discrete approximation scheme of $\tilde{D}$ is stable ($p_n = D_n^{-1} P_{Q_n} \operatorname{div}, \tilde{D}^{-1}\tilde{D}\mathbf{v} = \mathbf{v}$), and the second term tends to zero due to Lemma 12. □

Now that we have shown that $\tilde{D}_n$ is a suitable approximation of $\tilde{D}$, we can proceed similarly as in the homogeneous case, by defining a topological decomposition of $\mathbb{X}_n$.

LEMMA 19. Let Assumption 8 be satisfied. Then, the space V_n as in (3.12) together with the space

$$\widetilde{W}_n := \{\mathbf{u}_n \in \mathbb{X}_n : \tilde{D}\mathbf{u}_n = 0\},$$

form a topological decomposition of $\mathbb{X}_n$ with projections

$$\widetilde{P}_{V_n}\mathbf{u}_n := \tilde{D}_n^{-1} P_{Q_n} \tilde{D}\mathbf{u}_n \quad \text{and} \quad P_{\widetilde{W}_n}\mathbf{u}_n := \mathbf{u}_n - \widetilde{P}_{V_n}\mathbf{u}_n,$$

being uniformly bounded in $n \in \mathbb{N}$, where for $q_n \in Q_n$ the function $\tilde{D}_n^{-1} q_n \in V_n$ is the unique solution to

$$\text{find } \mathbf{v}_n \in V_n \text{ such that } \tilde{D}\mathbf{v}_n = q_n. \tag{3.21}$$

Proof. Lemma 18 enures that (3.21) admits a unique solution, and that $\widetilde{P}_{V_n}$ is uniformly bounded. Since $\mathbf{v}_n \in V_n$, it also holds

$$\|\operatorname{div} \mathbf{v}_n\|_{L^2} \geq \beta_{\text{disc}} \|\nabla \mathbf{v}_n\|_{(L^2)^{3\times 3}}.$$

Since $\widetilde{P}_{V_n}\widetilde{P}_{V_n}\mathbf{u}_n = \tilde{D}_n^{-1} P_{Q_n} \operatorname{div} \widetilde{P}_{V_n}\mathbf{u}_n = D_n^{-1} P_{Q_n} P_{Q_n} \operatorname{div} \mathbf{u}_n = \widetilde{P}_{V_n}\mathbf{u}_n$, $\widetilde{P}_{V_n}$ is indeed a projection. Since $\ker \widetilde{P}_{V_n} = \widetilde{W}_n$, the spaces V_n and $\widetilde{W}_n$ form indeed a topological decomposition of $\mathbb{X}_n$. □

We abbreviate $\mathbf{v}_n := \widetilde{P}_{V_n}\mathbf{u}_n$, $\mathbf{w}_n := P_{\widetilde{W}_n}\mathbf{u}_n$ for $\mathbf{u}_n \in \mathbb{X}_n$. For $\mathbf{w}_n$ we compute

$$\begin{aligned}
P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot)\mathbf{w}_n &= P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}_n - P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot)\mathbf{v}_n \\
&= P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}_n - P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot + M + F)\mathbf{v}_n + P_{Q_n}(M+F)\mathbf{v}_n \\
&= P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}_n - P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot + M + F)\mathbf{u}_n + P_{Q_n}(M+F)\mathbf{v}_n \\
&= -P_{Q_n}(M+F)\mathbf{u}_n + P_{Q_n}(M+F)\mathbf{v}_n \\
&= -P_{Q_n}(M+F)(\mathbf{u}_n - \mathbf{v}_n) \\
&= -P_{Q_n}(M+F)\mathbf{w}_n,
\end{aligned} \tag{3.22}$$

which shows that $P_{Q_n}(\operatorname{div} + \mathbf{q}\cdot)\mathbf{w}_n$ defines a compact sequence of operators. This sets up the discrete counterpart of the T operator, and allows us to proceed just like in the homogeneous case, with the following lemma.

LEMMA 20. Let Assumption 8 be satisfied. Then, $T_n := \widetilde{P}_{V_n} - P_{\widetilde{W}_n} = T_n^{-1} \in L(\mathbb{X}_n)$ is uniformly bounded in $n \in \mathbb{N}$.

Proof. Since the spaces $V_n, \widetilde{W}_n$ form a topological decomposition of $\mathbb{X}_n$, it follows that $T_n = T_n^{-1}$. The uniform boundedness of T_n, T_n^{-1} follows from the uniform boundedness of $\widetilde{P}_{V_n}$, i.e., lemma 18. □

LEMMA 21. For each $\mathbf{u} \in \mathbb{X}$ it holds $\lim_{n\to\infty} \|T_n P_{\mathbb{X}_n}\mathbf{u} - P_{\mathbb{X}_n} T\mathbf{u}\|_{\mathbb{X}} = 0$.

Proof. We can proceed similarly as in the proof of Lemma 13. It suffices to prove $\lim_{n\to\infty} \|\widetilde{P}_{V_n} P_{\mathbb{X}_n}\mathbf{u} - P_{\mathbb{X}_n}\widetilde{P}_V \mathbf{u}\|_{\mathbb{X}} = 0$. We use $\widetilde{P}_{V_n} = \tilde{D}_n^{-1} P_{Q_n} \tilde{D}$ and estimate

$$\begin{aligned}
&\|\widetilde{P}_{V_n} P_{\mathbb{X}_n}\mathbf{u} - P_{\mathbb{X}_n}\widetilde{P}_V \mathbf{u}\|_{\mathbb{X}}\\
&\leq \|\tilde{D}_n^{-1} P_{Q_n} \tilde{D} P_{\mathbb{X}_n}\mathbf{u} - \widetilde{P}_V\mathbf{u}\|_{\mathbb{X}} + \|\widetilde{P}_V\mathbf{u} - P_{\mathbb{X}_n}\widetilde{P}_V\mathbf{u}\|_{\mathbb{X}}\\
&\leq \|\tilde{D}_n^{-1} P_{Q_n} \tilde{D}\mathbf{u} - \widetilde{P}_V\mathbf{u}\|_{\mathbf{H}^1} + \|\tilde{D}_n^{-1} P_{Q_n} \tilde{D}\|_{L(\mathbb{X},V_n)} \|\mathbf{u} - P_{\mathbb{X}_n}\mathbf{u}\|_{\mathbb{X}} + \|\widetilde{P}_V\mathbf{u} - P_{\mathbb{X}_n}\widetilde{P}_V\mathbf{u}\|_{\mathbb{X}}.
\end{aligned}$$

The second two summands vanish in the limit $n \to \infty$ due to the point-wise convergence of $P_{\mathbb{X}_n}$. For the first term we apply $\tilde{D}\mathbf{u} = \tilde{D}\widetilde{P}_V\mathbf{u}$ and Lemma 18. □

3.6.3 *Regularity.* Let $Q_n^+ := Q_n \oplus \operatorname{span}\{1\}$ and $P_{Q_n^+}$ be the L^2 orthogonal projection onto Q_n^+ given by

$$P_{Q_n^+} := P_{Q_n} + M.$$

LEMMA 22. If $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} < \beta^2_{\text{disc}} \frac{\underline{c_s}^2 \underline{\rho}}{\overline{c_s}^2 \overline{\rho}} \frac{1}{1+\tan^2\theta}$ then $(A_n)_{n\in\mathbb{N}}$ is regular.

Proof. We proceed similarly to the proof of Lemma 14 and apply Theorem 3. In the previous part of this Section 3.5.1, we already constructed T_n and showed that $T_n \in L(\mathbb{X}_n)$ and $T_n^{-1} = T_n \in L(\mathbb{X}_n)$ are uniformly bounded. Further, Lemma 21 shows that T_n converges pointwise. Next, we split $A_nT_n = B_n + K_n$ into a stable part $B_n \in L(\mathbb{X}_n)$ and a compact part $K_n \in L(\mathbb{X}_n)$. Recall that $\langle A_nT_n\mathbf{u}_n, \mathbf{u}_n'\rangle_{\mathbb{X}} = a(\mathbf{v}_n - \mathbf{w}_n, \mathbf{v}_n' + \mathbf{w}_n')$. We start by considering the terms involving $(\operatorname{div} + \mathbf{q}\cdot)$. Note that

$$\begin{aligned}
&\langle c_s^2\rho \operatorname{div} T_n\mathbf{u}_n, \operatorname{div}\mathbf{u}_n'\rangle\\
&= \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \operatorname{div}\mathbf{v}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \operatorname{div}\mathbf{v}_n'\rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \operatorname{div}\mathbf{w}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \operatorname{div}\mathbf{w}_n'\rangle,
\end{aligned}$$

$$\langle c_s^2\rho\mathbf{q}\cdot T_n\mathbf{u}_n, \operatorname{div}\mathbf{u}_n'\rangle = \langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_n, \operatorname{div}\mathbf{v}_n'\rangle - \langle c_s^2\rho\mathbf{q}\cdot\mathbf{w}_n, \operatorname{div}\mathbf{v}_n'\rangle + \langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_n, \operatorname{div}\mathbf{w}_n'\rangle - \langle c_s^2\rho\mathbf{q}\cdot\mathbf{w}_n, \operatorname{div}\mathbf{w}_n'\rangle$$

and

$$\langle c_s^2\rho \operatorname{div} T_n\mathbf{u}_n, \mathbf{q}\cdot\mathbf{u}_n'\rangle = \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \mathbf{q}\cdot\mathbf{v}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \mathbf{q}\cdot\mathbf{v}_n'\rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \mathbf{q}\cdot\mathbf{w}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \mathbf{q}\cdot\mathbf{w}_n'\rangle.$$

Hence,

$$\begin{aligned}
&\langle c_s^2\rho \operatorname{div} T_n\mathbf{u}_n, \operatorname{div}\mathbf{u}_n'\rangle + \langle c_s^2\rho\mathbf{q}\cdot T_n\mathbf{u}_n, \operatorname{div}\mathbf{u}_n'\rangle + \langle c_s^2\rho \operatorname{div} T_n\mathbf{u}_n, \mathbf{q}\cdot\mathbf{u}_n'\rangle\\
&= \langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_n, \operatorname{div}\mathbf{v}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \mathbf{q}\cdot\mathbf{v}_n'\rangle + \langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_n, \operatorname{div}\mathbf{w}_n'\rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \mathbf{q}\cdot\mathbf{v}_n'\rangle &(3.23)\\
&\quad - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \operatorname{div}\mathbf{w}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \mathbf{q}\cdot\mathbf{w}_n'\rangle - \langle c_s^2\rho\mathbf{q}\cdot\mathbf{w}_n, \operatorname{div}\mathbf{w}_n'\rangle &(3.24)\\
&\quad + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, (\operatorname{div} + \mathbf{q}\cdot)\mathbf{w}_n'\rangle - \langle c_s^2\rho(\operatorname{div} + \mathbf{q}\cdot)\mathbf{w}_n, \operatorname{div}\mathbf{v}_n'\rangle &(3.25)\\
&\quad + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \operatorname{div}\mathbf{v}_n'\rangle. &(3.26)
\end{aligned}$$

Line (3.23) can be moved to the compact operator K_n due to the compact Sobolev embedding from $V_n \subset \mathbf{H}^1$ to $\mathbf{L}^2$. To treat line (3.24) we note that

$$\mathbf{q}\cdot\mathbf{w}_n = P_{Q_n}(\mathbf{q}\cdot\mathbf{w}_n) + M(\mathbf{q}\cdot\mathbf{w}_n) + (1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n)$$

and express

$$\begin{aligned}
(3.24) &= -\langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \operatorname{div}\mathbf{w}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, P_{Q_n^+}(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho P_{Q_n^+}(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n'\rangle \\
&\quad - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, M(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho M(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n'\rangle \\
&\quad - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, (1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho (1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n'\rangle \\
&= \langle c_s^2\rho P_{Q_n^+}(\mathbf{q}\cdot\mathbf{w}_n), P_{Q_n^+}(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho(\operatorname{div} + P_{Q_n^+}\mathbf{q}\cdot)\mathbf{w}_n, (\operatorname{div} + P_{Q_n^+}\mathbf{q}\cdot)\mathbf{w}_n'\rangle \\
&\quad - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, M(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho M(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n'\rangle \\
&\quad - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, (1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho (1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n'\rangle \\
&= \langle c_s^2\rho P_{Q_n^+}(\mathbf{q}\cdot\mathbf{w}_n), P_{Q_n^+}(\mathbf{q}\cdot\mathbf{w}_n')\rangle \\
&\quad - \langle c_s^2\rho P_{Q_n^+}F\mathbf{w}_n, P_{Q_n^+}F\mathbf{w}_n'\rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, M(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho M(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n'\rangle \\
&\quad - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, (1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n')\rangle - \langle c_s^2\rho (1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n'\rangle
\end{aligned}$$

by means of (3.22). The first line in the former right-hand side is put into B_n. Since F and M are compact, the second line is put into K_n. The third line tends to zero and is also put into K_n. Indeed, we compute, e.g.,

$$\begin{aligned}
|\langle c_s^2\rho(1-P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{v}_n'\rangle| &= |\langle \mathbf{q}\cdot\mathbf{w}_n, (1-P_{Q_n^+})(c_s^2\rho \operatorname{div}\mathbf{v}_n')\rangle| \\
&\leq \|\mathbf{q}\cdot\mathbf{w}_n\|_{L^2} \left\|(1-P_{Q_n^+})(c_s^2\rho \operatorname{div}\mathbf{v}_n')\right\|_{L^2},
\end{aligned}$$

and by means of the discrete commutator property (Bertoluzza, 1999) we estimate

$$\begin{aligned}
\left\|(1-P_{Q_n^+})(c_s^2\rho \operatorname{div}\mathbf{v}_n')\right\|_{L^2}^2 &= \sum_{\tau\in\mathcal{T}_n} \left\|(1-P_{Q_n^+})(c_s^2\rho \operatorname{div}\mathbf{v}_n')\right\|_{L^2(\tau)}^2 \\
&= \sum_{\tau\in\mathcal{T}_n} \left\|(1-P_{Q_n^+})((c_s^2\rho - c_\tau) \operatorname{div}\mathbf{v}_n')\right\|_{L^2(\tau)}^2 \\
&\leq \sum_{\tau\in\mathcal{T}_n} \left\|c_s^2\rho - c_\tau\right\|_{L^\infty(\tau)}^2 \|\operatorname{div}\mathbf{v}_n'\|_{L^2(\tau)}^2 \\
&\lesssim h_n^2 \sum_{\tau\in\mathcal{T}_n} \left\|c_s^2\rho\right\|_{W^{1,\infty}(\tau)}^2 \|\operatorname{div}\mathbf{v}_n'\|_{L^2(\tau)}^2 \qquad (3.27)
\end{aligned}$$

with suitably chosen constants $c_\tau, \tau \in \mathscr{T}_n$. Line (3.25) is treated similarly to line (3.24). Finally, the line (3.26) is moved to the operator B_n. Hence, it holds $A_n T_n = B_n + K_n$ with B_n and K_n defined by

$$
\begin{aligned}
\langle K_n \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}} := &- \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle - \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, i\partial_{\mathbf{b}}\mathbf{v}_n' \rangle - \langle \rho i\partial_{\mathbf{b}}\mathbf{v}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle \\
&- \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n' \rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle \\
&- i\omega\langle \gamma\rho\mathbf{v}_n, \mathbf{w}_n' \rangle + i\omega\langle \gamma\rho\mathbf{w}_n, \mathbf{v}_n' \rangle - i\omega\langle \gamma\rho\mathbf{v}_n, \mathbf{v}_n' \rangle \\
&+ \langle \rho\underline{\underline{m}}\mathbf{w}_n, \mathbf{v}_n' \rangle - \langle \rho\underline{\underline{m}}\mathbf{v}_n, \mathbf{w}_n' \rangle - \langle \rho\underline{\underline{m}}\mathbf{v}_n, \mathbf{v}_n' \rangle \\
&- \langle \rho F\mathbf{w}_n, F\mathbf{w}_n' \rangle - \langle \rho M\mathbf{w}_n, M\mathbf{w}_n' \rangle \\
(3.23) \Rightarrow &\left\{ +\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_n, \operatorname{div}\mathbf{v}_n' \rangle - \langle c_s^2\rho \operatorname{div}\mathbf{w}_n, \mathbf{q}\cdot\mathbf{v}_n' \rangle + \langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_n, \operatorname{div}\mathbf{w}_n' \rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \mathbf{q}\cdot\mathbf{v}_n' \rangle \right. \\
(3.24) \Rightarrow &\begin{cases} -\langle c_s^2\rho P_{Q_n}F\mathbf{w}_n, P_{Q_n}F\mathbf{w}_n' \rangle \\ -\langle c_s^2\rho \operatorname{div}\mathbf{w}_n, (1 - P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n') \rangle - \langle c_s^2\rho(1 - P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n' \rangle \\ -\langle c_s^2\rho \operatorname{div}\mathbf{w}_n, M(\mathbf{q}\cdot\mathbf{w}_n') \rangle - \langle c_s^2\rho M(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{w}_n' \rangle \end{cases} \\
(3.25) \Rightarrow &\begin{cases} -\langle c_s^2\rho P_{Q_n}F\mathbf{w}_n, \operatorname{div}\mathbf{v}_n' \rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, P_{Q_n}F\mathbf{w}_n' \rangle \\ -\langle c_s^2\rho(1 - P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{v}_n' \rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, (1 - P_{Q_n^+})(\mathbf{q}\cdot\mathbf{w}_n') \rangle \\ -\langle c_s^2\rho \operatorname{div}\mathbf{v}_n, M(\mathbf{q}\cdot\mathbf{w}_n') \rangle - \langle c_s^2\rho M(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{v}_n' \rangle \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle B_n \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}} := &- \langle \rho i\partial_{\mathbf{b}}\mathbf{v}_n, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n' \rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n, i\partial_{\mathbf{b}}\mathbf{v}_n' \rangle \\
&+ \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n' \rangle - \langle \rho i\partial_{\mathbf{b}}\mathbf{v}_n, i\partial_{\mathbf{b}}\mathbf{v}_n' \rangle \\
&+ i\omega\langle \gamma\rho\mathbf{w}_n, \mathbf{w}_n' \rangle + \langle \rho\underline{\underline{m}}\mathbf{w}_n, \mathbf{w}_n' \rangle + \langle \rho F\mathbf{w}_n, F\mathbf{w}_n' \rangle + \langle \rho M\mathbf{w}_n, M\mathbf{w}_n' \rangle \\
&+ \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \operatorname{div}\mathbf{v}_n' \rangle + \langle c_s^2\rho P_{Q_n}(\mathbf{q}\cdot\mathbf{w}_n), P_{Q_n}(\mathbf{q}\cdot\mathbf{w}_n') \rangle
\end{aligned}
\tag{3.28}
$$

for all $\mathbf{u}_n, \mathbf{u}_n' \in \mathbb{X}_n$, where $\underline{\underline{m}}$ is as defined in (3.18). The operator $(K_n)_{n\in\mathbb{N}}$ is indeed compact due to the compact Sobolev embedding from $V_n \subset \mathbf{H}^1$ to $\mathbf{L}^2$, because M and F have a finite-dimensional range and because terms involving $1 - P_{Q_n}$ tend to zero due to (3.27). It is straightforward to see that B_n is uniformly bounded and that B_n converges pointwise to the operator $B \in L(\mathbb{X})$ defined in Equation (3.20). It remains to show that B_n is uniformly coercive. This follows along the lines of the proof of Corollary 17, whereat β is replaced by β_{disc} and we use that $\langle c_s^2\rho\mathbf{q}\cdot\mathbf{w}_n, \mathbf{q}\cdot\mathbf{w}_n \rangle \lesssim \|\mathbf{w}_n\|_{\mathbf{L}^2}^2$. □

3.6.4 *Convergence.*

THEOREM 23. Let Assumption 8 be satisfied and $\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 < \beta_{\text{disc}}^2 \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}} \frac{1}{1+\tan^2\theta}$. Let $\mathbf{u}$ be the solution to (1.1). Then, there exists an index $n_0 > 0$ such that for all $n > n_0$ the solution $\mathbf{u}_n$ to (3.8) exists and $\mathbf{u}_n$ converges to $\mathbf{u}$ in the $\mathbb{X}$-norm with the best approximation estimate $\|\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}} \lesssim \inf_{\mathbf{u}_n' \in \mathbb{X}_n} \|\mathbf{u} - \mathbf{u}_n'\|_{\mathbb{X}}$.

Proof. Due to Lemma 22 the approximation scheme $(A_n)_{n\in\mathbb{N}}$ is regular. Since A is bijective (Halla & Hohage, 2021), the claim follows from Lemmas 1 and 2. □

Note that Remark 1 concerning convergence rates still applies.

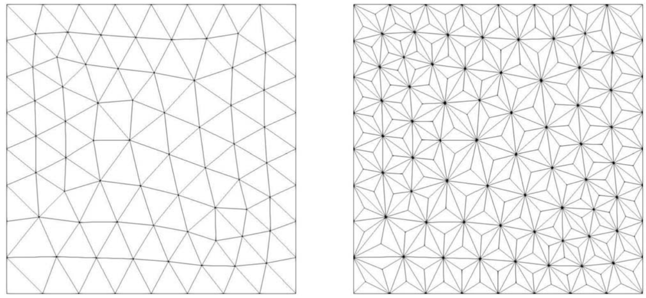

FIG. 1. The coarsest mesh of the sequence of unstructured simplicial meshes is shown on the left, with mesh size $h = 0.5$. To construct the second sequence of meshes we apply barycentric mesh refinement, resulting in the mesh on the right.

REMARK 3. In remark 2 we already discussed the possibility for different choices of the space Q_n from the one in (3.9). The choice of Taylor–Hood-type discretization using $Q_n = \{f \in L^2_0 : f|_T \in P_{k-1}(T) \quad \forall T \in \mathscr{T}_n\} \cap H^1$ is also possible in the case of heterogeneous pressure and gravity. In this case we modify the terms

$$\langle c_s^2 \rho (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}, (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}' \rangle - \langle c_s^2 \rho\, \mathbf{q} \otimes \mathbf{q}\mathbf{u}, \mathbf{u}' \rangle,$$

in (3.14) by inserting the projection P_{Q_n} onto the H^1 conforming space of polynomials and obtain

$$\langle c_s^2 \rho P_{Q_n} (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}, P_{Q_n} (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}' \rangle - \langle c_s^2 \rho\, P_{Q_n}(\mathbf{q} \cdot \mathbf{u}), P_{Q_n}(\mathbf{q} \cdot \mathbf{u}') \rangle.$$

As already discussed in remark 2, this can be implemented using auxiliary variables.

4. Numerical examples

The method has been implemented using `NGSolve` (Schöberl, 2014) and reproduction material is available in Halla *et al.* (2023). In this section we present numerical examples in the two-dimensional case. We work with the sesquilinear form given in Equation (3.5) and finite element spaces

$$\mathbb{X}_n := \{\mathbf{u} \in \mathbf{H}^1, \mathbf{u}|_\tau \in (P_k(\tau))^2 \quad \forall \tau \in \mathscr{T}_n\}, \tag{4.1}$$

with fixed uniform polynomial degree $k \in \mathbb{N}$. The error will be measured in the $\|\cdot\|_{\mathbb{X}}$-norm. We focus on testing the restrictions posed by Assumption 8 and the smallness assumption on the Mach number $\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}$. In two dimensions Assumption 8 requires either barycentric refined meshes and polynomial degree $k \geq 2$ or $k \geq 4$, provided that the meshes have finite degree of degeneracy (Scott & Vogelius, 1985). To put these conditions to the test we will consider two sequences of meshes of the domain $\mathscr{O} = (-4, 4)^2$. First, shape-regular unstructured simplicial meshes, which include some (nearly)singular vertices. We will refer to this mesh sequence as unstructured meshes. These meshes are used to construct the second sequence of meshes. For each mesh in the first sequence we apply barycentric mesh refinement once, constructing the second sequence of meshes. We will refer to those as the barycentric refined meshes. A mesh of each type is presented in Fig. 1.

First, we aim to recreate the results obtained in Chabassier & Duruflé (2018), which use periodic boundary conditions. Then, we consider the case of the boundary condition used in this work, given by $\boldsymbol{\nu} \cdot \mathbf{u} = 0$.

4.1 *Periodic boundary*

We aim to recreate the setting of numerical examples presented in Chabassier & Duruflé (2018). While we will use the same setting of parameters, there are a few differences. The major difference is that our formulation uses a slightly different damping term (Halla & Hohage, 2021) than the one considered in Chabassier & Duruflé (2018). Furthermore, in Chabassier & Duruflé (2018) quadrilateral meshes were considered, whereas we will use the simplicial meshes described in Fig. 1. The setting is as follows: we consider as computational domain the square $(-4,4)^2$ with periodic boundary conditions, and a source term given by

$$\mathbf{f} = (-i\omega + \partial_{\mathbf{b}}) \begin{pmatrix} g \\ 0 \end{pmatrix}, \tag{4.2}$$

where $g(x,y)$ is the Gaussian given by $g(x,y) = \sqrt{a/\pi}\, \exp(-a(x^2+y^2))$. Here, $a = \log(10^6)$ so that g is equal to 10^{-6} on the unit circle. The parameters are chosen as

$$\begin{aligned} &\rho = 1.5 + 0.2\cos(\pi x/4)\sin(\pi y/2), && c_s^2 = 1.44 + 0.16\rho, && \omega = 0.78 \times 2\pi, \\ &\gamma = 0.1, && \varOmega = (0,0), && p = 1.44\rho + 0.08\rho^2, \end{aligned} \tag{4.3}$$

and finally, the background flow is given by

$$\mathbf{b} = \frac{\alpha}{\rho} \begin{pmatrix} 0.3 + 0.1\cos(\pi y/4) \\ 0.2 + 0.08\sin(\pi x/4) \end{pmatrix}. \tag{4.4}$$

The error in the $\|\cdot\|_{\mathbb{X}}$-norm is considered against a reference solution computed with polynomial degree $k = 5$ and mesh size $h = 1.5 \times 2^{-6}$. Plots of the reference solution are shown in Fig. 2 (compare with (Chabassier & Duruflé, 2018, figs 8 and 12)).

In Fig. 3 we compare convergence rates for $\alpha = 0.1$, putting us safely into the regime of sub-sonic flow, thus satisfying the assumption on the Mach number in Theorem 23. If the additional inf-sup stability assumption 8 is satisfied, then, from theorem 23 together with Remark 1, we expect convergence rate of order $\mathcal{O}(h^k)$. In Fig. 3 we compare different approaches to satisfy inf-sup stability. We consider the two different types of mesh sequences for polynomial degrees $k = 1,2,3,4$. On unstructured meshes the error is given in Fig. 3 on the left. There, we observe good convergence rates for $k = 4$, after a pre-asymptotic phase, which might be caused by nearly singular vertices. For the meshes using barycentric refinement we observe convergence rates of order $\mathcal{O}(h^k)$ for $k \geq 2$, shown in Fig. 3, in the centre. These observations align with the requirements for stability of the Scott–Vogelius element, showing that Assumption 8 is necessary. We also show the error for the Taylor–Hood variant, which we discussed in remarks 2 and 3, in Fig. 3, on the right. The method used an H^1 conforming choice for the space Q_n, and we use unstructured meshes. The method suffers from a long pre-asymptotic phase and shows a worse approximation error compared with the other two methods. The rates agree with Remark 1.

Now that we observed the importance of using an inf-sup stable method, we set out to numerically test the smallness requirement on the Mach number in Theorem 23. To do so we use an inf-sup stable

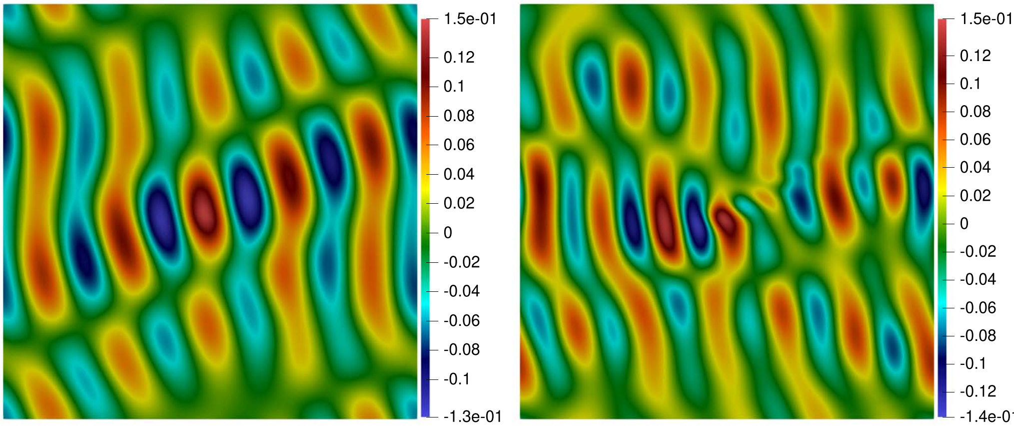

FIG. 2. The real part of the first entry of the reference solution computed with $k = 5$ and $h = 1.5 \times 2^{-6}$ for two different values of the coefficient of the flow field $\mathbf{b}$, $\alpha = 0.2$ on the left and $\alpha = 1.5$ on the right.

method and we compare different values of the coefficient for the background flow, $\alpha = 0.2, 0.5, 1.5, 3$, in Fig. 4. We fix $k = 4$ and use unstructured meshes. We compare against a reference solution computed with $k = 5$ and $h = 1.5 \times 2^{-6}$. The reference solution for $\alpha = 0.2, 1.5$ is presented in Fig. 2. As the reference solution changes with α, and can give an unreliable comparison, we additionally consider the consistency error, as in Chabassier & Duruflé (2018). Let us denote

$$S_1 = -\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)^2\mathbf{u}_n - \gamma\rho i\omega\mathbf{u}_n, \;\; S_2 = \nabla\left(\rho c_s^2 \operatorname{div}\mathbf{u}_n\right) - (\operatorname{div}\mathbf{u}_n)\nabla p + \nabla(\nabla p\cdot\mathbf{u}_n) - \operatorname{Hess}(p)\mathbf{u}_n,$$

where the differential operators are applied elementwise. The consistency error measures the difference between the two terms, which should be (virtually) zero outside the ball $B_{1.5}$ where the source term is located. Following (Chabassier & Duruflé, 2018, section 3.2) we define the consistency error by

$$\text{consistency error} = \frac{\|S_1 - S_2\|_{\mathbf{L}^2(\mathscr{O}\setminus B_{1.5})}}{\|S_1\|_{\mathbf{L}^2(\mathscr{O}\setminus B_{1.5})}},$$

where the error is measured on the domain without the disc with radius 1.5 centred at the origin, denoted by $\mathscr{O}\setminus B_{1.5}$, thus removing the effects of the source term $\mathbf{f}$.

Estimating the inf-sup constant numerically, see remark 4, we have $\beta_{\text{disc}} \approx 0.17$ for the considered meshes. Our assumption on the Mach number in theorem 23 then corresponds to

$$\beta_{\text{disc}}^2 \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}} \frac{1}{1+\tan^2\theta} \approx 0.008. \tag{4.5}$$

The Mach number is approximately $\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 \approx 0.002, 0.012, 0.115, 0.463$ for $\alpha = 0.2, 0.5, 1.5, 3$. With the choice of $\alpha = 1.5$ and $\alpha = 3$ we exceed the upper bound notably. We observe in Fig. 4 that the error and the consistency error worsen considerably for $\alpha = 1.5, 3$ and an optimal rate of convergence is not visible for the considered mesh widths. On the other hand, for the choice $\alpha = 0.5$ we still observe optimal convergence, showing that the bound is not sharp.

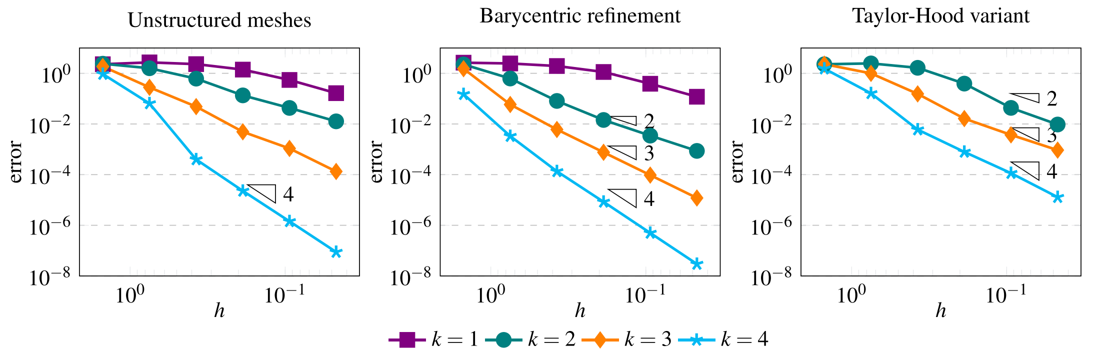

FIG. 3. Convergence against a reference solution computed with polynomial degree $k = 5$ and mesh size $h = 1.5 \times 2^{-6}$. We consider the setting described in (4.2) and (4.3) with periodic boundary conditions and fixed $\alpha = 0.2$ for the background flow $\mathbf{b}$ given in (4.4), and different polynomial degree $k = 1, 2, 3, 4$ and varying mesh size. From left to right we consider unstructured meshes, barycentric refined meshes and the Taylor–Hood variant, i.e., unstructured meshes with $Q_n \subset H^1$. The error is measured in the $\|\cdot\|_{\mathbb{X}}$-norm.

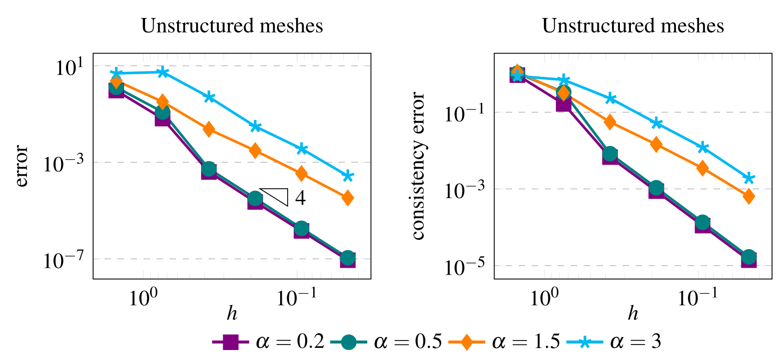

FIG. 4. On unstructured meshes with periodic boundary conditions we consider the error in the $\|\cdot\|_{\mathbb{X}}$-norm (left) and consistency-error (right) against a reference solution for different values of the coefficient in the background flow $\mathbf{b}$, given in (4.4), and fixed polynomial order $k = 4$.

REMARK 4. We have estimated the inf-sup constant β_{disc} in Assumption 8 numerically by computing smallest singular value of the matrix $\mathbf{M} = \mathbf{S}_q^{-1/2}\mathbf{B}\mathbf{S}_{\mathbf{u}}^{-1/2}$, where

$$(\mathbf{S}_q)_{i,j} = \langle q_j, q_i \rangle, \ (\mathbf{B})_{i,j} = \langle \operatorname{div} \mathbf{u}_j, q_i \rangle \text{ and } (\mathbf{S}_{\mathbf{u}})_{i,j} = \langle \nabla \mathbf{u}_j, \nabla \mathbf{u}_i \rangle + \langle \mathbf{u}_j, \mathbf{u}_i \rangle,$$

for a basis $(\mathbf{u}_i)_{i=1}^N$ for $\mathbb{X}_n$, as chosen in (4.1), of polynomial degree k and a basis $(q_i)_{i=1}^M$ for $P_{k-1}(\mathscr{T})$. We recall that the finite element space $\mathbb{X}_n$ does not include any boundary conditions, which is why we chose a stronger norm for $\mathbf{u}$ in the denominator.

4.2 *Normal boundary condition*

In this section we are considering the boundary condition $\boldsymbol{\nu} \cdot \mathbf{u} = 0$. We do not introduce a new finite element, instead we continue to use the finite element space defined in (4.1), and we incorporate the

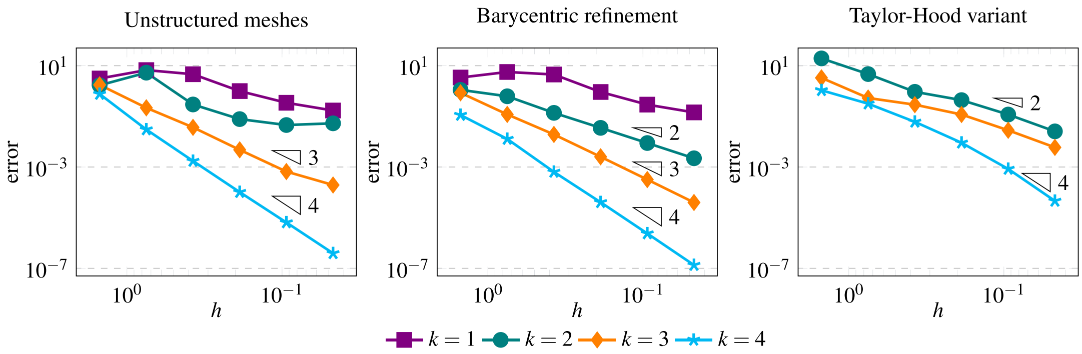

FIG. 5. Convergence towards an exact solution given in (4.7) for different polynomial orders $k = 1, 2, 3, 4$ for varying mesh sizes h. We consider homogeneous normal boundary condition with fixed $\alpha = 0.1$ in the background flow $\mathbf{b}$, given in Equation (4.6). From left to right we consider unstructured meshes, barycentric refined meshes and unstructured meshes with $Q_n \subset H^1$. The error is measured in the $\|\cdot\|_{\mathbb{X}}$-norm.

boundary condition using Nitsche's method. Therefore, we add the following terms to (3.5):

$$-\langle c_s^2 \rho \mathbf{u} \cdot \boldsymbol{\nu}, \operatorname{div} \mathbf{u}' \rangle_{\partial\mathcal{O}} - \langle c_s^2 \rho \operatorname{div} \mathbf{u}, \mathbf{u}' \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{O}} + \frac{\lambda k^2}{h} \langle c_s^2 \rho \mathbf{u} \cdot \boldsymbol{\nu}, \mathbf{u}' \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{O}},$$

where we choose $\lambda = 2^{15}$. We again consider the domain $\mathcal{O} = (-4, 4)^2$ and the parameters as in (4.3). Only the background flow is changed to satisfy $\mathbf{b} \cdot \boldsymbol{\nu} = 0$ on $\partial\mathcal{O}$, and will now be given by

$$\mathbf{b} = \frac{\alpha}{\rho} \begin{pmatrix} \sin(\pi x) \cos(\pi y) \\ -\cos(\pi x) \sin(\pi y) \end{pmatrix}. \tag{4.6}$$

The flow additionally fulfils $\operatorname{div}(\rho \mathbf{b}) = 0$ in $\mathcal{O}$. In the following text we will consider two examples with different source terms.

As in the periodic study we start again with a low Mach number flow, which satisfies the assumption in Theorem 23, and compare different approaches to satisfy inf-sup stability. To this end, we consider convergence against a manufactured solution, by choosing the source term $\mathbf{f}$ such that the solution will be given by

$$\frac{1}{\rho} \begin{pmatrix} (1+i)g \\ -(1+i)g \end{pmatrix}, \tag{4.7}$$

where g is again the Gaussian with $a = \log(10^6)$. As g equals 10^{-6} on the unit circle, we can consider the boundary conditions fulfilled numerically to a reasonable degree. Results for fixed $\alpha = 0.1$ are shown in Fig. 5. We consider unstructured meshes and barycentric refined meshes. Further, we include the Taylor–Hood variant outlined in remarks 2 and 3 using unstructured meshes and an H^1-conforming choice for the space Q_n. For unstructured and barycentric refined meshes, we observe the expected convergence rates for $k \geq 3$ and $k \geq 2$, respectively. The method with $Q_n \subset H^1$ shows again a long pre-asymptotic phase and a worse approximation error, compared with the other two methods. Furthermore, for $k = 3$ we observe only a long pre-asymptotic phase; optimal convergence rate is never reached.

Second, we consider again the source term given in (4.2), this time including the boundary condition, and compare against a reference solution, computed using $k = 5$ and $h = 1.5 \times 2^{-6}$, in Fig. 6. Before we

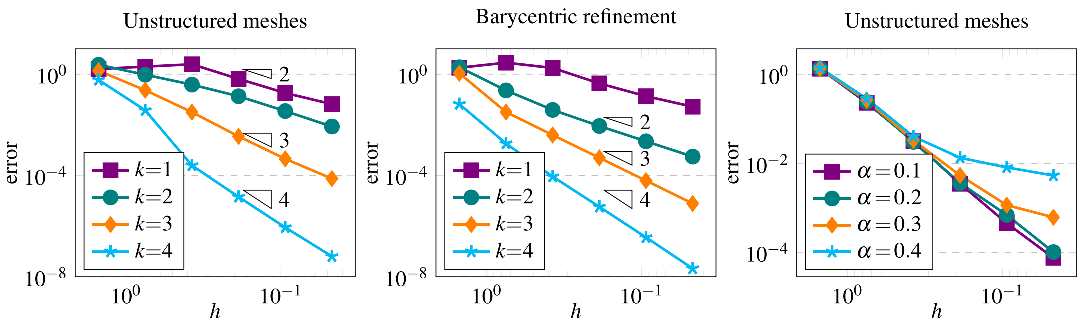

FIG. 6. Convergence against a reference solution on the unstructured meshes on the left and barycentric refined meshes in the middle, considering polynomial degree $k = 1, 2, 3, 4$. Here, $\alpha = 0.1$ and the background flow is as in (4.6). On the right we consider the unstructured meshes with $k = 4$ and different background flows.

investigate the behaviour for larger Mach numbers, we test convergence against the reference solution for different mesh types and polynomial orders. In the first two plots in Fig. 6 we fix $\alpha = 0.1$ and consider two different mesh types. For both methods we observe good convergence rates of order $\mathcal{O}(h^k)$; however, barycentric refinement shows more stable rates and a better error overall.

Next, we put the assumption on the Mach number in Theorem 23 to the test. In Fig. 6, on the right, we consider unstructured meshes, and different values of the coefficient α for the flow given in (4.6). We choose $\alpha = 0.1, 0.2, 0.3, 0.4$, resulting in the corresponding Mach numbers $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} \approx$ 2e−3, 0.01, 0.02, 0.04. From (4.5) we recall that the bound on the Mach number is approximately 0.008. True to Theorem 23 with the assumptions fulfilled in the case $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} \approx 0.01$ we observe the rates given in Remark 1. Similar to the periodic case, we still observe convergence for $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} \approx 0.01$, even though it is larger than our estimated bound. Nonetheless, for $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} \approx 0.02, 0.04$ we observe a loss of optimal convergence.

5. Conclusion

In this article we reported in Theorem 3 a new T-compatibility criterion to obtain the regularity of approximations. As an example of application we considered the damped time-harmonic Galbrun's equation (which is used in asteroseismology), and we proved in Theorem 23 convergence for discretizations with divergence stable (Assumption 8) $\mathbf{H}^1$ finite elements. Although the results of this article constitute only a first step in the numerical analysis for the oscillations of stars. The subsonic Mach number assumption

$$\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} < \beta^2_{\text{disc}} \frac{\underline{c_s}^2 \underline{\rho}}{\overline{c_s}^2 \overline{\rho}} \tag{5.1}$$

is far from being optimal. In stars the density decays with increasing radius, and hence the ratio $\frac{\underline{c_s}^2 \underline{\rho}}{\overline{c_s}^2 \overline{\rho}}$ becomes very small. Thus, a goal is to get rid of this factor in (5.1) by a more refined analysis or possibly by more sophisticated discretization methods. In addition, it is desired to replace in (5.1) the discrete inf-sup constant of the divergence β_{disc} with a better constant closer to 1. The reported computational examples serve only to illustrate the convergence of the finite element method and computational experiments with realistic parameters for stars are eligible. In particular, a numerical realization of a transparent boundary condition is necessary (Barucq *et al.*, 2021; Hohage *et al.*, 2021; Halla, 2022a). Finally, we aim to apply the new T-compatibility technique to a number of equations/discretizations for which Halla (2021c) is too rigid.

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References

Acosta, G., Durán, R. G. & Muschietti, M. A. (2006) Solutions of the divergence operator on John domains. *Adv. Math.*, **206**, 373–401.

Alemán, T., Halla, M., Lehrenfeld, C. & Stocker, P. (2022) Robust finite element discretizations for a simplified Galbrun's equation. *eccomas2022*. Available at: https://doi.org/10.23967/eccomas.2022.206.

Arnold, D. N. & Qin, J. (1992) Quadratic velocity/linear pressure Stokes elements. *Advances in Computer Methods for Partial Differential Equations-VII* (Vichnevetsky R., Knight D. & Richter G. eds). New Brunswick, NJ, USA: IMACS, pp. 28–34. Available at: https://umn.edu/~arnold/papers/imacs92.pdf.

Barucq, H., Faucher, F., Fournier, D., Gizon, L. & Pham, H. (2021) Outgoing modal solutions for Galbrun's equation in helioseismology. *J. Differ. Equations*, **286**, 494–530.

Bertoluzza, S. (1999) The discrete commutator property of approximation spaces. *C. R. Acad. Sci. Paris Sér. I Math.*, **329**, 1097–1102. Available at: https://doi.org/10.1016/S0764-4442(00)88481-5.

Boffi, D., Costabel, M., Dauge, M., Demkowicz, L. & Hiptmair, R. (2011) Discrete compactness for the p-version of discrete differential forms. *SIAM J. Numer. Anal.*, **49**, 135–158.

Bonnet-BenDhia, A.-S., Carvalho, C. & Ciarlet, P. (2018) Mesh requirements for the finite element approximation of problems with sign-changing coefficients. *Numer. Math.*, **138**, 801–838.

Bonnet-BenDhia, A.-S., Mercier, J.-F., Millot, F., Pernet, S. & Peynaud, E. (2012) Time-harmonic acoustic scattering in a complex flow: a full coupling between acoustics and hydrodynamics. *Commun. Comput. Phys.*, **11**, 555–572.

Bredies, K. (2008) *A variational weak weighted derivative: Sobolev spaces and degenerate elliptic equations. Report, Institute of Mathematics and Scientific Computing*. Austria: University of Graz. Available at: https://imsc.uni-graz.at/bredies/papers/weighted_weak_derivative_elliptic.pdf.

Buffa, A. (2005) Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations. *SIAM J. Numer. Anal.*, **43**, 1–18.

Buffa, A., Costabel, M. & Schwab, C. (2002) Boundary element methods for Maxwell's equations on non-smooth domains. *Numer. Math.*, **92**, 679–710.

Chabassier, J. & Duruflé, M. (2018) *Solving time-harmonic Galbrun's equation with an arbitrary flow. Application to helioseismology. Research Report RR-9192*. Bordeaux: INRIA. Available at: https://hal.inria.fr/hal-01833043.

Christiansen, S. H. & Hu, K. (2018) Generalized finite element systems for smooth differential forms and Stokes' problem. *Numer. Math.*, **140**, 327–371.

Ciarlet, P. & Jamelot, E. (2025) Explicit T-coercivity for the Stokes problem: A coercive finite element discretization. *Comput. Math. Appl.*, **188**, 137–159. Hal-04414789. Available at: https://inria.hal.science/hal-04414789.

Ern, A. & Guermond, J.-L. (2020) *Finite Elements I. Approximation and Interpolation, Volume 72 of Texts Appl. Math*. Cham: Springer. Available at: https://doi.org/10.1007/978-3-030-56341-7.

Falk, R. S. & Neilan, M. (2013) Stokes complexes and the construction of stable finite elements with pointwise mass conservation. *SIAM J. Numer. Anal.*, **51**, 1308–1326.

Galbrun, H. (1931) *Propagation d'une Onde Sonore Dans L'atmosphre et théorie Des Zones de Silence*. Paris: Gauthier-Villars.

Guzmán, J., Lischke, A. & Neilan, M. (2020 Id/No 13) Exact sequences on Powell–Sabin splits. *Calcolo*, **57**, 25.

Guzmán, J. & Neilan, M. (2018) Inf-sup stable finite elements on barycentric refinements producing divergence-free approximations in arbitrary dimensions. *SIAM J. Numer. Anal.*, **56**, 2826–2844.

Hägg, L. & Berggren, M. (2021) On the well-posedness of Galbrun's equation. *J. Math. Pures Appl.*, **150**, 112–133.

Halla, M. (2021a) Analysis of radial complex scaling methods: scalar resonance problems. *SIAM J. Numer. Anal.*, **59**, 2054–2074.

Halla, M. (2021b) Electromagnetic Steklov eigenvalues: approximation analysis. *ESAIM: Math. Model. Numer. Anal.*, **55**, 57–76. Available at: https://doi.org/10.1051/m2an/2020075.

Halla, M. (2021c) Galerkin approximation of holomorphic eigenvalue problems: weak T-coercivity and T-compatibility. *Numer. Math.*, **148**, 387–407.

Halla, M. (2022a) On the treatment of exterior domains for the time-harmonic equations of stellar oscillations. *SIAM J. Math. Anal.*, **54**, 5268–5290.

Halla, M. (2022b) Radial complex scaling for anisotropic scalar resonance problems. *SIAM J. Numer. Anal.*, **60**, 2713–2730.

Halla, M. (2023a) Convergence analysis of nonconform $H(\mathrm{div})$-finite elements for the damped time-harmonic Galbrun's equation. *arXiv:2306.03496*. 10.48550/arxiv.2306.03496.

Halla, M. (2023b) On the approximation of dispersive electromagnetic eigenvalue problems in two dimensions. *IMA J. Numer. Anal.*, **43**, 535–559.

Halla, M. & Hohage, T. (2021) On the well-posedness of the damped time-harmonic Galbrun equation and the equations of stellar oscillations. *SIAM J. Math. Anal.*, **53**, 4068–4095.

Halla, M., Hohage, T. & Oberender, F. (2024) A new numerical method for scalar eigenvalue problems in heterogeneous, dispersive, sign-changing materials. *arXiv:2401.16368*. Available at: https://doi.org/10.48550/arXiv.2401.16368.

Halla, M., Lehrenfeld, C. & Stocker, P. (2023) A new T-compatibility condition and its application to the discretization of the damped time-harmonic Galbrun's equation. *GRO.data*. Available at: https://doi.org/10.25625/BISNQ9.

Halla, M., Lehrenfeld, C. & van Beeck, T. (2025) Hybrid discontinuous Galerkin discretizations for the damped time-harmonic Galbrun's equation arXiv:2504.09547. Available at: https://doi.org/10.48550/arXiv.2504.09547.

Hohage, T., Lehrenfeld, C. & Preuss, J. (2021) Learned infinite elements. *SIAM J. Sci. Comput.*, **43**, a3552–a3579.

Hohage, T. & Nannen, L. (2015) Convergence of infinite element methods for scalar waveguide problems. *BIT*, **55**, 215–254.

John, V. (2016) *Finite element methods for incompressible flow problems*. Cham: Springer. Available at: https://doi.org/10.1007/978-3-319-45750-5.

John, V., Linke, A., Merdon, C., Neilan, M. & Rebholz, L. G. (2017) On the divergence constraint in mixed finite element methods for incompressible flows. *SIAM Rev.*, **59**, 492–544.

Karma, O. (1996a) Approximation in eigenvalue problems for holomorphic Fredholm operator functions. *I. Numer. Funct. Anal. Optim.*, **17**, 365–387.

Karma, O. (1996b) Approximation in eigenvalue problems for holomorphic Fredholm operator functions. II. (convergence rate). *Numer. Funct. Anal. Optim.*, **17**, 389–408. Available at: https://doi.org/10.1080/01630569608816700.

Lynden-Bell, D. & Ostriker, J. P. (1967) On the stability of differentially rotating bodies. *Mon. Not. R. Astron. Soc.*, **136**, 293–310.

Maeder, M., Gabard, G. & Marburg, S. (2020) 90 years of Galbrun's equation: an unusual formulation for aeroacoustics and hydroacoustics in terms of the Lagrangian displacement. *J. Theor. Comput. Acoust.*, **28**, 2050017. Available at: https://doi.org/10.1142/S2591728520500176.

Neilan, M. (2020) The Stokes complex: a review of exactly divergence-free finite element pairs for incompressible flows. In *75 years of mathematics of computation. Symposium celebrating 75 years of mathematics of computation, Institute for Computational and Experimental Research in Mathematics, ICERM, Providence,* *RI, USA, November 1–3, 2018*, (Brenner S., Shparlinski I., Shu C.-W. & Szyld D. eds). 141–158. Providence, RI: American Mathematical Society (AMS). Available at: https://doi.org/10.1090/conm/754/15142.

Schöberl, J. (2014) *C++ 11 Implementation of Finite Elements in NGSolve. Institute for Analysis and Scientific Computing*. Vienna University of Technology.

Scott, L. R. & Vogelius, M. (1985) Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *ESAIM. Math. Model. Numer. Anal.*, **19**, 111–143. Available at: https://doi.org/10.1051/m2an/1985190101111.

Stummel, F. (1970) Diskrete Konvergenz linearer Operatoren. I. *Math. Ann.*, **190**, 45–92.

Unger, G. (2021) Convergence analysis of a Galerkin boundary element method for electromagnetic resonance problems. *Partial Differ. Equ. Appl.*, **2**, 39. Available at: https://doi.org/10.1007/s42985-020-00049-5.

Vainikko, G. (1976) *Funktionalanalysis der Diskretisierungsmethoden*. Leipzig: B. G. Teubner Verlag Mit Englischen und Russischen Zusammenfassungen, *Teubner-Texte Math*.

Van Beeck, T. (2023) *On Stable Discontinuous Galerkin Discretizations for Galbrun's Equation*. Master's thesis, University of Göttingen, GRO.data, Göttingen, Germany. Available at: https://doi.org/10.25625/KGHQWV.

Zhang, S. (2005) A new family of stable mixed finite elements for the 3D Stokes equations. *Math. Comp.*, **74**, 543–554.

Zhang, S. (2009) A family of 3D continuously differentiable finite elements on tetrahedral grids. *Appl. Numer. Math.*, **59**, 219–233.

Zhang, S. (2011) Divergence-free finite elements on tetrahedral grids for $k \geq 6$. *Math. Comp.*, **80**, 669–695.

Appendix A. Construction of $\mathbf{H}^1_{\boldsymbol{\nu}0}$-conforming finite element space

The convenient way to obtain a vectorial $\mathbf{H}^1$ finite element space is to use a scalar H^1 finite element space Y_n and to use $(Y_n)^3$. Hence, if u_j and $\mathrm{dof}_j(u)$ are the basis functions and degrees of freedom of Y_n then $u_j\mathbf{e}_l$ and $\mathrm{dof}_j(\mathbf{e}_l \cdot \mathbf{u})$, $l = 1, 2, 3$ with Cartesian unit vectors $\mathbf{e}_j$ are the basis functions and degrees of freedom of $(Y_n)^3$. However, with this construction it is not clear how to handle the boundary condition $\boldsymbol{\nu} \cdot \mathbf{u} = 0$, and hence the question how to construct finite element spaces of $\mathbf{H}^1_{\boldsymbol{\nu}0}$ remains. To solve this issue for each j we reorganize $u_j\mathbf{e}_l$, $\mathrm{dof}_j(\mathbf{e}_l \cdot \mathbf{u})$, $l = 1, 2, 3$ into tangential basis functions and DoFs $\mathbf{u}_j^{\tan,l}$, $\mathrm{dof}_j^{\tan}$, $l = 1, \ldots, L_j$ and nontangential ones $\mathbf{u}_j^{\mathrm{nontan},l}$, $\mathrm{dof}_j^{\mathrm{nontan},l}$, $l = 1, \ldots, 3 - L_j$. Here, $L_j = 0$ if dof_j is a vertex DoF associated with a vertex of $\partial\mathcal{O}$, $L_j = 1$ if dof_j is a vertex or edge DoF associated with an edge of $\partial\mathcal{O}$, and $L_j = 2$ if dof_j is a vertex, edge of face DoF associated with a face of $\partial\mathcal{O}$. Note that the tangential basis functions $\mathbf{u}_j^{\tan,l}$ will satisfy $\boldsymbol{\nu} \cdot \mathbf{u}_j^{\tan,l} = 0$, whereas the nontangential basis functions $\mathbf{u}_j^{\mathrm{nontan},l}$ will in general satisfy neither $\boldsymbol{\nu} \times \mathbf{u}_j^{\mathrm{nontan},l} = 0$ nor $\boldsymbol{\nu} \cdot \mathbf{u}_j^{\mathrm{nontan},l} = 0$. However, $\boldsymbol{\nu} \cdot \mathbf{u} = 0$ will imply $\mathrm{dof}_j^{\mathrm{nontan},l}(\mathbf{u}) = 0$. For $L_j = 2$ we simply choose tangential vectors $\mathbf{t}_1, \mathbf{t}_2$ and the normal vector $\boldsymbol{\nu}$ and set $\mathbf{u}_j^{\tan,l} = u_j\mathbf{t}_l$, $\mathrm{dof}_j^{\tan,l}(\mathbf{u}) := \mathrm{dof}_j(\mathbf{t}_l \cdot \mathbf{u})$, $l = 1, 2$ and $\mathbf{u}_j^{\mathrm{nontan},1} = u_j\boldsymbol{\nu}$, $\mathrm{dof}_j^{\mathrm{nontan},1}(\mathbf{u}) := \mathrm{dof}_j(\boldsymbol{\nu} \cdot \mathbf{u})$. For $L_j = 1$ we choose the tangential vector $\mathbf{t}$ associated with the edge of $\partial\mathcal{O}$ and $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ as the normal vectors of the two adjacent faces. Thence, we set $\mathbf{u}_j^{\tan,1} = u_j\mathbf{t}$, $\mathrm{dof}_j^{\tan,1}(\mathbf{u}) := \mathrm{dof}_j(\mathbf{t} \cdot \mathbf{u})$, $l = 1, 2$ and $\mathbf{u}_j^{\mathrm{nontan},l} = u_j\boldsymbol{\nu}_l$, $\mathrm{dof}_j^{\mathrm{nontan},l}(\mathbf{u}) := \mathrm{dof}_j(\boldsymbol{\nu}_l \cdot \mathbf{u})$, $l = 1, 2$. For $L_j = 0$ there exist no tangential DoFs. Let $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3$ be the normal vectors of the three adjacent faces. Thence, we set $\mathbf{u}_j^{\mathrm{nontan},l} = u_j\boldsymbol{\nu}_l$, $\mathrm{dof}_j^{\mathrm{nontan},l}(\mathbf{u}) := \mathrm{dof}_j(\boldsymbol{\nu}_l \cdot \mathbf{u})$, $l = 1, 2, 3$. Thus, to obtain an $\mathbf{H}^1_{\boldsymbol{\nu}0}$ conforming finite element space we simply set the DoFs associated with the nontangential DoFs to zero. Hence, for $\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\nu}0} \cap \mathbf{H}^s$ the obtained finite element space has the same approximation properties as $(Y_n)^3$.