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Power corrections to the production of a color-singlet final state in hadron collisions in the N-jettiness slicing scheme at NLO QCD

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Abstract: We compute next-to-leading power corrections in the zero-jettiness variable for the production of colorless final states at hadron colliders at next-to-leading order in QCD. To assess if the process-independence of leading power contributions can be extended, we attempt to construct generic expansions of phase spaces and matrix elements squared through next-to-leading power in the zero-jettiness. We highlight challenges associated with the collinear limit, where universality no longer holds at the subleading power, making the result process-dependent. We show that quantities that need to be calculated in the collinear limit can be obtained using Berends-Giele currents, enabling computation of power corrections to high-multiplicity final states. As a concrete example, we apply our method to compute power corrections in the zero-jettiness for lepton pair as well as multi-photon production in $q\bar{q}$ collisions.

Contents

1 Introduction

Early perturbative computations in QED (see, e.g. Ref. [1]) were performed using methods that, currently, would be classified as "slicing". The idea of such a method is to split the real emission contribution into singular and regular parts by introducing a parameter that distinguishes between unresolved (soft and collinear) and resolved (hard) radiation. Although recognizing the difference between two types of emissions is helpful, in practice slicing methods suffer from large cancellations when resolved and unresolved parts of the calculation are combined. This drawback led to a rejection of the slicing methods as a suitable tool for computing higher-order perturbative (mostly QCD) corrections to complex collider processes. Since worthy computational alternatives in the form of subtraction schemes [2, 3] appeared, development of slicing methods was put on a back burner for a while.

Slicing methods made a remarkable comeback with the advent of next-to-next-to-leading order (NNLO) computations, starting with the proposal to use the transverse momentum of a color-singlet final state as a slicing parameter [4]. Since this method was only suitable for processes without final-state jets at leading order, it was later suggested to use the so-called N-jettiness as the slicing parameter for generic NNLO computations [5–7]. We note in passing that other slicing variables have recently been proposed for processes with final-state jets [8, 9].

Nevertheless, the use of slicing methods is still hindered by very large numerical cancellations between the different contributions to physical cross sections. These cancellations are caused by the need to take the slicing parameter to be very small, to ensure the independence of the final result on its value. Thus, efficient numerical implementations remain a challenge for modern slicing schemes, especially when applied to complex processes. To overcome this challenge, one needs to compute the *unresolved* contribution more accurately; to achieve this, a description of real-emission amplitudes and cross sections beyond leading soft and collinear limits is required.

Such power-suppressed terms were studied in a number of publications in recent years, focusing mostly on computations at next-to-leading order [10–30]. However, these calculations typically address relatively simple processes and it is unclear how to generalize them to *arbitrary* collider processes and higher perturbative orders.

The goal of this paper is to make a step in this direction and to explore power corrections that arise when a process, where an *arbitrary* color-singlet final state is produced in the collision of a $q\bar{q}$ pair, is studied in the context of the N-jettiness slicing scheme at NLO QCD. Calculation of power corrections in the N-jettiness variable for such processes requires us to understand the expansion of two building blocks – the phase space and the matrix element squared – around the limit of the vanishing N-jettiness for the radiative process $q\bar{q} \to X + g$.

Since these building blocks appear to be process-dependent, it is crucial to investigate to what extent a process-independent calculation of the first subleading N-jettiness power correction is possible. In this respect, the so-called Low-Burnett-Kroll (LBK) theorem [31–34], that allows one to compute the next-to-soft corrections by calculating derivatives of the Born process with respect to momenta of external particles, serves as an inspiration. Similarly, next-to-collinear terms in the expansion of a generic matrix element can be related to matrix elements of simpler processes [16, 35] although this case is more complex than the next-to-soft one.

Another source of these power-suppressed terms is kinematic dependences of various observables or dependence of fiducial cross sections on selection cuts. For simplicity, in this paper we will only consider observables that have smooth dependence on kinematics although it is known that this is not always the case [36–38]. As this paper focuses on providing general framework for computing power corrections to arbitrary collider processes, we do not examine such observables in what follows.

In what follows, we discuss the next-to-leading power corrections in the N-jettiness variable by considering the process $q\bar{q} \to X + g$ where X is an arbitrary colorless state. In Section 2 we begin with the discussion of a toy example – the power corrections to vector boson production. In Section 3 we turn to the study of an arbitrary process, we explain how to use momenta transformations to enable the expansion of the phase space and the matrix elements in the limit of small N-jettiness beyond leading power. In Section 4 we combine the soft and collinear contributions obtained in the previous section, and derive the formula for power corrections. In Section 5 we explain how various quantities that appear in the final formula can be computed by relating them to generalizations of Berends-Giele currents [39]. In Section 6 we first apply the general formula to the processes $q\bar{q} \to \gamma^* \to e^+e^-$ and $q\bar{q} \to \gamma\gamma$, for which we derive expressions for power corrections analytically, and then we compute the power correction to $q\bar{q} \to 4\gamma$ numerically, further showcasing the general nature of the derived formula. We conclude in Section 7. Some technical aspects of the calculation are discussed in appendices.

2 A simple example: vector boson production

We start by considering the production of a vector boson in $q\bar{q}$ collisions, studying power corrections that arise from both the phase space and the matrix element of this process. We note that studying this process has advantages and disadvantages. On the one hand, since this process is very simple, many computational steps can be performed explicitly. On the other hand, because of its $2 \to 1$ nature, the phase space of this process is singular which precludes us from using the same computational methods that we develop in the rest of the paper to address power corrections to a general partonic process.

We consider the tree-level process

$$q(p_1) + \bar{q}(p_2) \to V, \tag{2.1}$$

and define the cross section convoluted with a smooth function of the partonic center-of-mass energy squared

$$\Sigma_0 = \int \mathrm{d}s \; \mathcal{L}(s)\sigma_0(s). \tag{2.2}$$

We can think about this function as a product of parton distribution function integrated over the ratio of two Bjorken variables keeping their product fixed. We note that the need to introduce this function into a computation is related to the fact that the phase space of process in Eq. (2.1) is overconstrained. Precisely for this reason, this step is not necessary when power corrections for more complex processes are studied.

The cross section σ_0 for the process in Eq. (2.1) evaluates to

$$\sigma_0 = \frac{2\pi\delta(s - m_V^2)}{8sN_c^2} \, |\mathcal{M}_0(p_1, p_2)|^2. \tag{2.3}$$

Under the assumption that the vector boson only has vector coupling to quarks, $-ig_V\gamma^\mu$, the matrix element squared reads

$$|\mathcal{M}_0(p_1, p_2)|^2 = g_V^2 N_c \, 4s(1 - \epsilon). \tag{2.4}$$

We then find the following result for the convoluted cross section

$$\Sigma_0 = \frac{\pi g_V^2}{N_c}(1 - \epsilon) \, \mathcal{L}(m_V^2). \tag{2.5}$$

We are interested in developing a method for a systematic computation of power corrections in the zero-jettiness variable. To find them, we have to consider the real emission process

$$q(p_1) + \bar{q}(p_2) \to V + g(k), \tag{2.6}$$

and work at fixed zero-jettiness of the final state. We therefore write

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma}{\mathrm{d}\tau} = \frac{1}{\Sigma_0}\int \mathrm{d}s \, \mathcal{L}(s)\frac{\mathrm{d}\sigma(s)}{\mathrm{d}\tau}. \tag{2.7}$$

To compute the partonic cross section, we work in the center-of-mass frame. We write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{1}{8sN_c^2}\int [\mathrm{d}k][\mathrm{d}p_V](2\pi)^d\delta(p_1 + p_2 - p_V - k)\delta(\tau - T_0)|M(p_1, p_2, k)|^2, \tag{2.8}$$

where T_0 is the zero-jettiness function defined as

$$T_0 = \min\left(\frac{s_{1k}}{Q_1}, \frac{s_{2k}}{Q_2}\right), \tag{2.9}$$

with $s_{ik} = 2p_i \cdot k$. The two quantities $Q_{1,2}$ are arbitrary normalization constants. We will use $Q_1 = Q_2 = Q$, throughout the paper, unless stated otherwise. Finally, the matrix element squared for the process in Eq. (2.6) reads

$$|M(p_1, p_2, k)|^2 = 8C_F g_V^2 g_s^2 N_c \, (1 - \epsilon)\left[\frac{2 \; s \; m_V^2}{s_{1k}s_{2k}} + (1 - \epsilon)\left(\frac{s_{1k}}{s_{2k}} + \frac{s_{2k}}{s_{1k}}\right) - 2\epsilon\right], \tag{2.10}$$

with $s = 2p_1 \cdot p_2$.

To proceed further, we integrate over the momentum of the vector boson p_V; the phase space becomes

$$
\begin{aligned}
[\mathrm{d}p_V](2\pi)^d\delta(p_1+p_2-p_V-k) &= 2\pi\ \mathrm{d}^d p_V\ \delta(p_V^2-m_V^2)\delta(p_1+p_2-p_V-k) \\
&= 2\pi\ \delta(s-m_V^2-2\sqrt{s}\omega_k).
\end{aligned}
\tag{2.11}
$$

We remove the zero-jettiness δ-function by integrating over gluon energy ω_k, finding that the energy of the gluon is given by

$$
\omega_k^* = \frac{\tau Q}{\sqrt{s}\psi_k}, \tag{2.12}
$$

with

$$
\psi_k = \min\left(\rho_{1k},\rho_{2k}\right), \quad \rho_{ik} = 1-\cos\theta_{ik}. \tag{2.13}
$$

We use this result in the expression for the cross section and obtain

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{2\pi[\alpha_s]}{8sN_c^2}\int[\mathrm{d}\Omega_k]\delta\left(s-m_V^2-\frac{2\tau Q}{\psi_k}\right)\left(\frac{\tau Q}{\sqrt{s}\psi_k}\right)^{1-2\epsilon}\frac{Q}{\sqrt{s}\psi_k}\frac{|\mathcal{M}(p_1,p_2,k^*)|^2}{g_s^2}, \tag{2.14}
$$

where we introduced the following notations

$$
[\alpha_s] = \frac{g_s^2\Omega^{(d-2)}}{2(2\pi)^{d-1}}, \quad [\mathrm{d}\Omega_k] = \frac{\mathrm{d}\Omega_k^{(d-1)}}{\Omega^{(d-2)}}. \tag{2.15}
$$

Integrating over s, we find

$$
\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma}{\mathrm{d}\tau} = \frac{C_F[\alpha_s]}{\tau^{1+2\epsilon}}\int[\mathrm{d}\Omega_k]\,\frac{\mathcal{L}\left(s^*\right)}{\mathcal{L}(m_V^2)}\left(\frac{Q}{\sqrt{s^*}\psi_k}\right)^{2-2\epsilon}\frac{\tau^2|\mathcal{M}(p_1,p_2,k^*)|^2}{C_F N_c g_s^2 g_V^2(1-\epsilon)4s^*}, \tag{2.16}
$$

where

$$
s^* = m_V^2+\frac{2\tau Q}{\psi_k}, \tag{2.17}
$$

and we should use this value to evaluate energies associated with momenta $p_{1,2}$.

We have to construct an expansion of $\mathrm{d}\Sigma/\mathrm{d}\tau$ in Eq. (2.16) in powers of τ. The expansion is controlled by the value of the emission angle θ_k of the gluon which is parametrized by Ω_k. If θ_k is $\mathcal{O}(1)$, the energy of the emitted gluon is $\mathcal{O}(\tau)$ and the required expansion is the soft expansion. If, on the other hand, θ_k or $\pi-\theta_k$ is $\mathcal{O}(\tau/m_V)$, we have to expand in θ_k or $\pi-\theta_k$. This case corresponds to two collinear contributions, since the gluon can be emitted either along the direction of p_1 or p_2. Two distinct integration regions – the soft one and the collinear one – are associated with two "branches" of the cross section with respect to τ: the soft contribution is proportional to $\tau^{-1-2\epsilon}$, whereas the collinear ones are proportional to $\tau^{-1-\epsilon}$. Hence, schematically, we can write

$$
\frac{d\Sigma}{\mathrm{d}\tau} \sim \tau^{-1-2\epsilon}f_s(\tau)+\tau^{-1-\epsilon}f_c(\tau), \tag{2.18}
$$

and we compute each of the two contributions by performing an appropriate Taylor expansion of the *integrand* and then integrating the resulting expression over the angle θ_k.

We compute the *soft* contribution by Taylor-expanding in τ. We obtain

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(s)}}{\mathrm{d}\tau} = \frac{4C_F[\alpha_s]}{\tau^{1+2\epsilon}}\left(\frac{Q}{m_V}\right)^{-2\epsilon}\int [\mathrm{d}\Omega_k]\frac{\psi_k^{2\epsilon}}{\rho_{1k}\rho_{2k}} \times\left[1-\frac{Q\tau}{m_V^2\psi_k}\left(2-2\epsilon-2m_V^2\mathcal{L}_1(m_V^2)\right)\right], \tag{2.19}$$

where $\mathcal{L}_1(m_V^2) = \mathcal{L}^{-1}(m_V^2)\ \mathrm{d}\mathcal{L}(m_V^2)/\mathrm{d}m_V^2$. We note that $\mathcal{O}(\tau)$ correction contains $1/\psi_k$ factor which makes the integration over angle θ_k *power-divergent.* This is a feature that will also appear when zero-jettiness corrections to a general process are discussed in the next section. Due to the simplicity of the process and the knowledge of the matrix element, explicit integration over θ_k is straightforward. It yields

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(s)}}{\mathrm{d}\tau} = \frac{4C_F[\alpha_s]}{\epsilon\tau^{1+2\epsilon}}\left(\frac{Q}{m_V}\right)^{-2\epsilon}\left[1-\frac{Q\tau}{m_V^2}\frac{(1-2\epsilon)}{(1-\epsilon)}\left(1-\epsilon-m_V^2\mathcal{L}_1(m_V^2)\right)\right]. \tag{2.20}$$

To construct a Laurent expansion in ϵ, we write

$$\tau^{-1-n\epsilon} = -\frac{\delta(\tau)}{n\epsilon}+L_0(\tau)-n\epsilon\ L_1(\tau)+\mathcal{O}(\epsilon^2), \tag{2.21}$$

with

$$L_n(x) = \left[\frac{\theta(x)\log^n(x)}{x}\right]_+. \tag{2.22}$$

Using it, we obtain the leading- and subleading-power soft contributions

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(s),LP}}{\mathrm{d}\tau} = 4C_F[\alpha_s]\Bigg[-\frac{\delta(\tau)}{2\epsilon^2}+\frac{L_0(\tau)+\delta(\tau)\ln\frac{Q}{m_V}}{\epsilon}-2L_1(\tau) -2L_0(\tau)\ln\frac{Q}{m_V}-\delta(\tau)\ln^2\frac{Q}{m_V}\Bigg], \tag{2.23}$$

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(s),NLP}}{\mathrm{d}\tau} = \frac{4C_F[\alpha_s]Q}{m_V^2}\Bigg\{\frac{m_V^2\mathcal{L}_1(m_V^2)-1}{\epsilon} -\left[m_V^2\mathcal{L}_1(m_V^2)-1\right]\left(1+2\ln\frac{Q\tau}{m_V}\right)+1\Bigg\}. \tag{2.24}$$

We continue with the construction of the *collinear* expansion. We consider the case where the gluon is emitted along the direction of the incoming quark with the momentum p_1; the other case, where the gluon is emitted along the direction of the anti-quark with momentum p_2 is analogous. To construct the collinear expansion, we go back to Eq. (2.16) and consider the case of the small emission angle , $\theta_{1k}\ll 1$. It follows that the function ψ_k in this case is given by ρ_{1k}. We need to perform the expansion of the integrand assuming that the emission angle is small, and extend the integration region over the angle (or related variable) to *infinity.* We write

$$s^* \to m_V^2+\frac{2\tau Q}{\rho}, \tag{2.25}$$

where $\rho = 1 - \cos\theta_{1k}$ and assume that $\rho \sim \tau Q/m_V^2$. The integration region over θ_k extends from $\rho = 0$ to $\rho = \infty$. We then change the integration variable $\rho \to z$,

$$\rho = \frac{2\tau Q}{m_V^2}\frac{z}{1-z}, \tag{2.26}$$

with $0 < z < 1$, expand the resulting formula in τ and obtain

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(c1)}}{\mathrm{d}\tau} = \frac{C_F[\alpha_s]}{\tau^{1+2\epsilon}}\left(\frac{\tau}{Q}\right)^{\epsilon}\int_0^1 \frac{\mathrm{d}z}{z}\frac{\mathcal{L}\left(m_V^2/z\right)}{\mathcal{L}(m_V^2)}\,(1-z)^{-\epsilon}\left[\tilde{P}_{qq}(z) + \frac{\tau Q}{m_V^2}z\tilde{P}^{(1)}_{qq}(z)\right], \tag{2.27}$$

where

$$\tilde{P}_{qq}(z) = \frac{1+z^2}{1-z} - \epsilon(1-z), \qquad \tilde{P}^{(1)}_{qq}(z) = \frac{(4-z)z - 1 + 2\epsilon z - \epsilon^2(1-z)^2}{(1-z)^2}. \tag{2.28}$$

We note that the function $\tilde{P}^{(1)}_{qq}(z)$ has a power singularity at $z = 1$. Again, we will see the appearance of these power singularities in the general case discussed in the next section. In the current case, since both the matrix element and the phase space are simple, we can integrate by parts obtaining expressions that can be expanded in ϵ

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}z}{z}\,\mathcal{L}\left(\frac{m_V^2}{z}\right)(1-z)^{-\epsilon}z\tilde{P}^{(1)}_{qq}(z) &= \int_0^1 \mathrm{d}z\,(1-z)^{-1-\epsilon} \\ \times\Bigg\{2\left[1+\frac{(1+\epsilon^2)(1-z)}{(1+\epsilon)}\right]\mathcal{L}\left(\frac{m_V^2}{z}\right) &- \frac{2m_V^2}{z^2}\,\mathcal{L}'\left(\frac{m_V^2}{z}\right)\left[z - \frac{(1+\epsilon^2)(1-z)^2}{2(1+\epsilon)}\right]\Bigg\}.\end{aligned} \tag{2.29}$$

It is straightforward to extract the $1/\epsilon$ divergences from the above expression by applying Eq. (2.21) to construct the ϵ-expansion of $\tau^{-1-\epsilon}$ and $(1-z)^{-1-\epsilon}$. This leads to the leading-power contribution

$$\begin{aligned}\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(c1),LP}}{\mathrm{d}\tau} =& C_F[\alpha_s]\int_0^1 \frac{\mathrm{d}z}{z}\frac{\mathcal{L}\left(m_V^2/z\right)}{\mathcal{L}(m_V^2)}\Bigg\{\frac{2\delta(\tau)\delta(1-z)}{\epsilon^2} - \frac{\delta(\tau)(1+z^2)L_0(1-z)}{\epsilon} \\ &- 2\frac{\delta(1-z)\left[L_0(\tau)+\delta(\tau)\ln Q\right]}{\epsilon} + \delta(\tau)\delta(1-z)\ln^2 Q \\ &+ \delta(\tau)\left[1-z+(1+z^2)L_1(1-z)+(1+z^2)L_0(1-z)\ln Q\right] \\ &+ 2\delta(1-z)\left[L_1(\tau)+L_0(\tau)\ln Q\right] + (1+z^2)L_0(\tau)L_0(1-z)\Bigg\},\end{aligned} \tag{2.30}$$

and the next-to-leading power contribution

$$\begin{aligned}\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(c1),NLP}}{\mathrm{d}\tau} =& -C_F[\alpha_s]\frac{Q}{m_V^2}\int_0^1 \mathrm{d}z\frac{\mathcal{L}\left(m_V^2/z\right)}{\mathcal{L}(m_V^2)}\Bigg\{\frac{2\delta(1-z)\left[m_V^2\,\mathcal{L}_1\left(m_V^2\right)-1\right]}{\epsilon} \\ &+ L_0(1-z)\left[4-2z+\frac{m_V^2\,\mathcal{L}_1\left(m_V^2/z\right)(1+(z-4)z)}{z^2}\right] \\ &+ 2\delta(1-z)\ln(Q\tau)\left[1-m_V^2\,\mathcal{L}_1\left(m_V^2\right)\right]\Bigg\}.\end{aligned} \tag{2.31}$$

As we already mentioned, the above results describe the collinear emissions off the quark with momentum p_1. The contribution where an anti-quark with momentum p_2 emits collinear gluons, doubles the above result.

By comparing Eqs. (2.24) and (2.31), we observe that at next-to-leading power all ϵ-divergences cancel between the soft and collinear contributions. This is the main result of our analysis. However, for completeness, we also comment on the leading-power contribution which is well-understood.

Indeed, to get the cancellation of ϵ-poles at leading power, the virtual contributions and the collinear renormalization contributions need to be added to the real emission one. For $q\bar{q} \to V$ annihilation, they are given by the following expressions

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(v)}}{\mathrm{d}\tau} = C_F[\alpha_s](m_V)^{2\epsilon}\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \frac{2\pi^2}{3}\right)\delta(\tau), \tag{2.32}$$

and

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{(cs1)}}{\mathrm{d}\tau} = \frac{C_F[\alpha_s](m_V)^{2\epsilon}}{\epsilon}\int\limits_0^1 \frac{\mathrm{d}z}{z}\frac{\mathcal{L}\left(m_V^2/z\right)}{\mathcal{L}(m_V^2)}\left[(1+z^2)L_0(1-z) + \frac{3}{2}\delta(1-z)\right]\delta(\tau). \tag{2.33}$$

Upon combining Eqs. (2.32, 2.33, 2.30, 2.23) and accounting for the second collinear region, we observe the cancellation of all $1/\epsilon$ poles in the leading-power contribution.

The next-to-leading power correction to the zero-jettiness regulated cross section for $q\bar{q} \to V$ is obtained upon combining results from Eqs. (2.24, 2.31). It reads

$$\begin{aligned}\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma^{NLP}}{\mathrm{d}\tau} = \frac{4C_F[\alpha_s]Q}{m_V^2}&\Bigg\{1 + \left[1 - m_V^2\ \mathcal{L}_1\left(m_V^2\right)\right]\left(1 + \ln\frac{Q\tau}{m_V^2}\right) \\ &- \int\limits_0^1 \mathrm{d}z\frac{\mathcal{L}\left(m_V^2/z\right)}{\mathcal{L}(m_V^2)}L_0(1-z)\left[2 - z + \frac{m_V^2\mathcal{L}_1\left(m_V^2/z\right)(1+(z-4)z)}{2\ z^2}\right]\Bigg\},\end{aligned} \tag{2.34}$$

where $\mathcal{L}_1$ is defined right after Eq. (2.19) and L_0 in Eq. (2.22).

To check this result, we make use of the fact that, when working at finite τ, the real emission contribution in (2.16) is ϵ-finite and can be computed by numerical integration over the gluon emission angle θ_k. We then write a small-τ expansion

$$\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma}{\mathrm{d}\tau} = \tau^{-1}\left(A_1\ln\tau + A_2\right) + A_3\ln\tau + A_4 + \tau\left(A_5\ln\tau + A_6\right) + A_7\tau^2 + \dots. \tag{2.35}$$

The coefficients $A_{1,..,7}$ are functions of m_V, Q and the function $\mathcal{L}$, for which we take a Gaussian distribution

$$\mathcal{L}(s) = \frac{2}{\sqrt{2\pi}\sigma_s}\exp\left(-\frac{s^2}{2\sigma_s^2}\right). \tag{2.36}$$

For numerical calculations, we choose $\sigma_s^2 = 2.47\ m_V^2$, and $m_V = 90$ GeV.

Both the leading-power coefficients A_1, A_2, and the next-to-leading power corrections A_3, A_4 are obtained from Eqs. (2.23, 2.24, 2.30, 2.31). We then write

$$X(\tau) = \frac{1}{A_4}\left[\frac{1}{\Sigma_0}\frac{\mathrm{d}\Sigma}{\mathrm{d}\tau} - A_1\frac{\ln\tau}{\tau} - \frac{A_2}{\tau} - A_3\ln\tau\right], \tag{2.37}$$

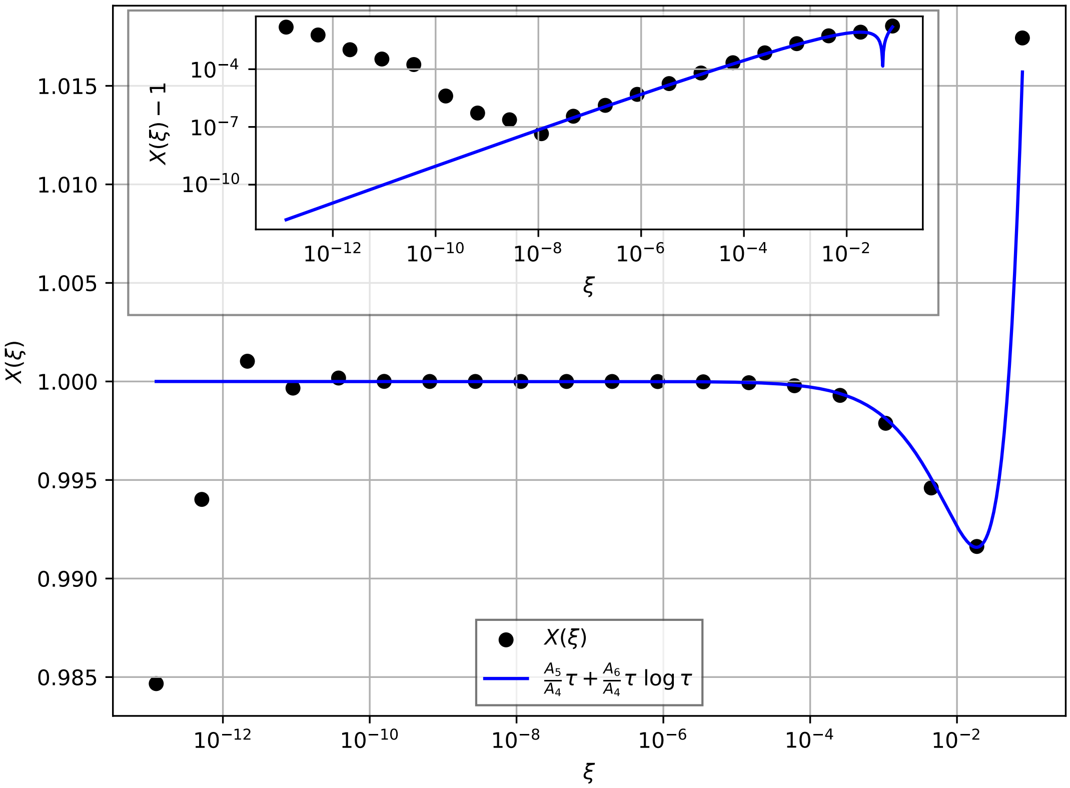

Figure 1. The parameter $X(\tau)$ expressed as a function of ξ, which relates the small-jettiness expansions of the vector boson production cross section and a numerical integration of (2.16) at $Q = 100$ GeV and $m_V = 90$ GeV and with the Gaussian luminosity distribution given in Eq. (2.36). The parameter approaches unity for lower values of ξ before the numerical integration becomes unreliable. A fit of the missing linear terms in τ is presented in blue.

and note that at small τ this quantity should approach one,

$$\lim_{\tau \to 0} X(\tau) = 1. \tag{2.38}$$

Since τ is a dimensionful variable, the expansion proceeds in powers of the parameter $\xi = Q\tau/m_V^2$. Hence, choosing $\tau < 0.1\, m_V$ and $Q = 100$ GeV, we expect that the deviation of $X(\tau)$ from unity is about a percent at $\xi \sim 0.1$, and that it decreases at smaller values of τ. Numerical results for $X(\tau)$ are shown in Figure 1, where X is plotted as a function of ξ. We observe that $X(\tau)$ approaches the limiting value $X(\tau) = 1$ at $\xi > 10^{-8}$ and that for smaller values of ξ, it starts to deviate from its limiting value due to the loss of accuracy in the numerical integration. A fit of the next-to-next-leading-power terms is also shown, demonstrating they can explain the (tiny) difference between $X(\tau)$ and its asymptotic value.

The example presented in this section showcases the main aspects of the calculation of the zero-jettiness power corrections to hadron collisions processes. In the next section, we will discuss how to generalize this computation to arbitrary color-singlet final state. However, before doing that, we would like to reiterate the following points that emphasize the differences between vector boson production and a general case.

- In vector boson production, the final-state phase space is simple, so that one can easily parametrize it and perform all integrations explicitly. However, for processes with higher final-state multiplicities this is not possible. Thus, new approaches are needed to enable a systematic way to expand an arbitrary phase space in the zero-jettiness variable.
- The matrix element that describes vector boson production is simple and can be written down explicitly. Hence, expanding it through subleading terms in relevant limits is straightforward. However, for high-multiplicity final states it may not be possible to compute the relevant matrix element analytically. Hence, one needs to understand how to compute subleading terms in the soft and collinear expansions in a process-independent manner.
- In the case of vector boson production the phase space is overconstrained, i.e. it is proportional to $\delta(s - M_V^2)$ at leading order. This feature is particular to $2 \to 1$ processes, and it makes the zero-jettiness expansion of *partonic* processes in such cases more complex, than for processes with more final-state particles. To ameliorate this problem, the zero-jettiness expansion of *hadronic* cross sections for $2 \to 1$ processes is usually studied. In the toy example discussed in this section, we introduced a luminosity function and computed an analog of a hadronic cross section. In the result for subleading-power corrections shown in Eq. (2.34), the derivative of the luminosity function $\mathcal{L}$ appears, which is analogous to the derivatives of parton distribution functions found in the previous studies of the Drell-Yan and Higgs production processes [11, 13–16].

3 Power corrections: general considerations

We consider the following leading order process

$$f_a(p_a) + f_b(p_b) \to X(P_X), \tag{3.1}$$

where f_a and f_b are the initial-state partons, which we take to be a quark and an anti-quark, and X denotes a generic colorless final state with the momentum P_X composed of m massless particles. To discuss the next-to-leading power corrections in the zero-jettiness variable, we add a gluon with the momentum k to the process in Eq. (3.1) and write the differential cross section as

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \mathcal{N} \int [\mathrm{d}\tilde{P}_X]_m [\mathrm{d}k] (2\pi)^d \delta(p_a + p_b - \tilde{P}_X - k) \\
\times\, \delta(\tau - T_0(p_a, p_b, k))\; \mathcal{O}(\tilde{P}_X) \sum_{\mathrm{col,pol}} |\mathcal{M}|^2(p_b, p_a, k, \tilde{P}_X).
\end{aligned}
\tag{3.2}
$$

In Eq. (3.2) we used $[\mathrm{d}k] = \mathrm{d}^{d-1}\vec{k}/(2\,(2\pi)^{d-1}k^0)$ and $[\mathrm{d}\tilde{P}_X]_m = \prod_{i=1}^{m} [\mathrm{d}\tilde{p}_i]$. The zero-jettiness function is defined as follows

$$
T_0(p_a, p_b, k) = \min\left[\frac{2p_a \cdot k}{Q}, \frac{2p_b \cdot k}{Q}\right],
\tag{3.3}
$$

with Q being an arbitrary normalization factor of mass dimension one. Furthermore, $\mathcal{O}$ is an observable that depends on the momenta of colorless particles comprising the final state X, and $\mathcal{N}$ is the cross-section normalization that contains the flux factor, color- and spin-averaging terms, etc. We note that we have used a new notation for the momentum of the colorless final state by writing it with a tilde, $P_X \to \tilde{P}_X$. The reason for doing this will become clear later.

Using a reference frame where the collision axis is the z-axis, and writing the zero-jettiness variable in terms of the energy and the polar angle of the emitted gluon, it is easy to see that the constraint $\tau = T_0(p_a, p_b, k)$ implies that either the gluon energy or its transverse momentum squared is $\mathcal{O}(\tau)$. The expansions around these distinct limits can be performed independently of each other, as we show below. We will start with the construction of the soft expansion.

3.1 The soft contribution

A gluon with momentum k is considered to be soft if $k \sim \tau$. Since we are interested in the relative $\mathcal{O}(\tau/\sqrt{s})$ correction, where $s = 2p_a \cdot p_b$, we only need to expand the integrand in Eq. (3.2) to the *first* subleading order in k. To facilitate this expansion, we use the momentum mapping that absorbs k into the momentum of the colorless final state [40], and write

$$
P_{ab}^{\mu} = \lambda^{-1} \, [\Lambda_s]^{\mu}{}_{\nu} \, (P_{ab}^{\nu} - k^{\nu}).
\tag{3.4}
$$

In Eq. (3.4) $P_{ab} = p_a + p_b$, $\Lambda_s^{\mu\nu}$ is the matrix of a Lorentz boost that we specify below, and λ is a constant that is defined from the condition

$$
\lambda^2 P_{ab}^2 = (P_{ab} - k)^2.
\tag{3.5}
$$

It follows from the above equation that

$$
\lambda = \sqrt{1 - \frac{2P_{ab} \cdot k}{P_{ab}^2}} \approx 1 - \frac{P_{ab} \cdot k}{P_{ab}^2} + \mathcal{O}(k^2).
\tag{3.6}
$$

We then write

$$
\begin{aligned}
\mathrm{d}\Phi_m(p_a, p_b, \tilde{P}_X, k) &= [\mathrm{d}\tilde{P}_X]_m [\mathrm{d}k] (2\pi)^d \delta^{(d)}(p_a + p_b - \tilde{P}_X - k) \\
&= [\mathrm{d}\tilde{P}_X]_m [\mathrm{d}k] (2\pi)^d \delta^{(d)} \left(\lambda \Lambda_s^{-1} P_{ab} - \tilde{P}_X\right) \\
&= [\mathrm{d}\tilde{P}_X]_m [\mathrm{d}k] (2\pi)^d \delta^{(d)} \left(\lambda \Lambda_s^{-1} \left(P_{ab} - \lambda^{-1} \Lambda_s \tilde{P}_X\right)\right) \\
&= [\mathrm{d}\tilde{P}_X]_m [\mathrm{d}k]\ \lambda^{-d}\ (2\pi)^d \delta^{(d)} \left(P_{ab} - \lambda^{-1} \Lambda_s \tilde{P}_X\right).
\end{aligned}
\tag{3.7}
$$

To further simplify this expression, we use the fact that $\tilde{P}_X = \sum\limits_{i=1}^{m} \tilde{p}_i$, so that

$$
[\mathrm{d}\tilde{P}_X]_m = \prod_{i=1}^{m} \frac{\mathrm{d}^d \tilde{p}_i}{(2\pi)^{d-1}}\ \delta_+(\tilde{p}_i^2). \tag{3.8}
$$

We then write

$$
\tilde{p}_i = \lambda \Lambda_s^{-1} p_i, \tag{3.9}
$$

and since Λ_s is a Lorentz transformation, we find

$$
[\mathrm{d}\tilde{P}_X]_m = \lambda^{m(d-2)} \prod_{i=1}^{m} \frac{\mathrm{d}^d p_i}{(2\pi)^{d-1}} \delta(p_i^2) = \lambda^{m(d-2)} [\mathrm{d}P_X]_m. \tag{3.10}
$$

Hence, we obtain

$$
\begin{aligned}
\mathrm{d}\Phi_m(p_a, p_b, \tilde{P}_X, k) &= \mathrm{d}\Phi_m(p_a, p_b, P_X)\ [\mathrm{d}k]\ \lambda^{m(d-2)-d} \\
&\approx \mathrm{d}\Phi_m(p_a, p_b, P_X)\ [\mathrm{d}k]\ \left(1 - \kappa_m\ \frac{P_{ab} \cdot k}{P_{ab}^2}\right),
\end{aligned}
\tag{3.11}
$$

where

$$
\mathrm{d}\Phi_m(p_a, p_b, P_X) = [\mathrm{d}P_X]_m\ (2\pi)^d \delta(p_a + p_b - P_X), \tag{3.12}
$$

is the phase space of the Born process $q\bar{q} \to X$, and

$$
\kappa_m = m(d-2) - d. \tag{3.13}
$$

Putting everything together, we find

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \mathcal{N} \int \mathrm{d}\Phi_m(p_a, p_b, P_X)\ &\int [\mathrm{d}k] \left(1 - \kappa_m \frac{P_{ab} \cdot k}{P_{ab}^2}\right) \delta(\tau - T_0(p_a, p_b, k)) \\
\times\ \mathcal{O}(\lambda \Lambda_s^{-1} P_X) &\sum_{\mathrm{col,pol}} |\mathcal{M}|^2(p_b, p_a, k, \lambda \Lambda_s^{-1} P_X).
\end{aligned}
\tag{3.14}
$$

Since we are interested in $\mathcal{O}(\tau)$ corrections, we need the matrix element squared to the first subleading order in the expansion in k. The matrix element itself scales as $1/k$, so that we need to find $\mathcal{O}(1)$ terms in the expansion. The required terms can be obtained from the Low-Burnett-Kroll theorem [31, 32], as we explain shortly.

Before discussing the expansion of the matrix element, we derive the formula for the Lorentz boost Λ_s^{-1}. We start with a general formula for the boost $\Lambda_{\mathrm{gen}}(Q_f, Q_i)$, that transforms a vector Q_i to a vector Q_f; this formula can be found in Eq. (A.2). The Lorentz transformation that we need (c.f. Eq. (3.4)) reads

$$\Lambda_s^{-1} = \Lambda_{\text{gen}}(P_{ab} - k, \lambda P_{ab}). \tag{3.15}$$

Since in the soft limit $k \sim \tau$, Λ_s^{-1} is nearly the identity matrix; to determine $\mathcal{O}(\tau)$ corrections to the cross section, we need the Lorentz boost to the first order in k. Simplifying the expression for Λ_s^{-1}, we find

$$\left[\Lambda_s^{-1}\right]_{\mu\nu} = g_{\mu\nu} - B_{\mu\nu} + \mathcal{O}(k^2), \tag{3.16}$$

where

$$B^{\mu\nu} = \frac{k^\mu P_{ab}^\nu - P_{ab}^\mu k^\nu}{P_{ab}^2}. \tag{3.17}$$

We turn to the discussion of the expansion of the matrix element in the soft limit. We ignore the color charges since for the process we consider it is trivial to restore them at the end of the calculation. Separating emissions off the external legs and the structure-dependent radiation, we write

$$\begin{aligned}\mathcal{M}(p_b, p_a, k, \tilde{P}_X) = -g_s \epsilon_\mu^* \bar{v}_b \Bigg[& N(p_b, p_a - k, \tilde{P}_X)\,(J_a^\mu + S_a^\mu) \\ & + \left(-J_b^\mu + S_b^\mu\right) N(p_b - k, p_a, \tilde{P}_X) + N_{\text{str}}^\mu(p_b, p_a, k, \tilde{P}_X) \Bigg] u_a,\end{aligned} \tag{3.18}$$

where

$$J_a^\mu = \frac{2p_a^\mu - k^\mu}{d_a}, \quad J_b^\mu = \frac{2p_b^\mu - k^\mu}{d_b}, \quad S_a^\mu = \frac{\sigma^{\mu\nu} k_\nu}{d_a}, \quad S_b^\mu = \frac{\sigma^{\mu\nu} k_\nu}{d_b}, \tag{3.19}$$

with $d_a = (p_a - k)^2$, $d_b = (p_b - k)^2$ and $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/2$.

The structure-dependent contribution to the amplitude can be restored by requiring that the Ward identity is fulfilled, namely that the amplitude vanishes if the gluon polarization vector ϵ^μ is replaced with its momentum k^μ. This implies that in the soft limit

$$\begin{aligned}\mathcal{M} = -g_s \epsilon_\mu^* \bar{v}_b \Bigg[& N(p_a - k)\,(J_a^\mu + S_a^\mu) + \left(-J_b^\mu + S_b^\mu\right) N(p_b - k) \\ & - \left[\frac{\partial}{\partial p_{a,\mu}} - \frac{\partial}{\partial p_{b,\mu}}\right] N \Bigg] u_a,\end{aligned} \tag{3.20}$$

where we only show the k-dependent momenta in the arguments of the function N. Since the currents $J_{a,b}^\mu$ scale as $1/k$, we need to expand the k-dependent functions N in powers of the gluon momentum. We then obtain

$$\mathcal{M} = -g_s \epsilon_\mu^* \bar{v}_b \left[J^\mu N - (L^\mu N) + \left(N S_a^\mu + S_b^\mu N\right) \right] u_a, \tag{3.21}$$

where the function N is now k-independent and

$$J^\mu = J_a^\mu - J_b^\mu, \qquad L^\mu = L_a^\mu - L_b^\mu, \tag{3.22}$$

with

$$L_a^\mu = J_a^\mu k^\nu \frac{\partial}{\partial p_a^\nu} + \frac{\partial}{\partial p_{a,\mu}}, \qquad L_b^\mu = J_b^\mu k^\nu \frac{\partial}{\partial p_b^\nu} + \frac{\partial}{\partial p_{b,\mu}}. \tag{3.23}$$

Upon squaring the soft amplitude and summing over polarizations of all external particles, we find

$$g_s^{-2}|\mathcal{M}(p_b, p_a, k, \tilde{P}_X)|^2 \approx -J_\mu J^\mu |\mathcal{M}|^2(p_b, p_a, \tilde{P}_X) + J^\mu L_\mu |\mathcal{M}|^2(p_b, p_a, \tilde{P}_X) + \dots, \tag{3.24}$$

where the ellipses denote terms that are finite in the $k \to 0$ limit. We note that the first term on the right-hand side in Eq. (3.24) provides the leading contribution that scales as $1/k^2$. Hence, we need to account for the momenta redefinitions in that term. Momenta redefinitions impact particles that comprise the color-singlet system X. Working through first subleading order in k, we obtain

$$\begin{aligned} g_s^{-2}|\mathcal{M}(p_b, p_a, k, \tilde{P}_X)|^2 \approx \Bigg[- J_\mu J^\mu \left(1 - \sum_{i=1}^{m} \left[\frac{P_{ab}\cdot k}{P_{ab}^2} p_i^\rho + B^{\rho\sigma} p_{i,\sigma} \right] \frac{\partial}{\partial p_i^\rho} \right) \\ + J^\mu L_\mu \Bigg] |\mathcal{M}|^2(p_b, p_a, P_X). \end{aligned} \tag{3.25}$$

Combining this expression with Eq. (3.14), we observe that the integration over k can be performed in a process-independent way and that for computing the soft contribution to the zero-jettiness cross section through next-to-leading power, we need to calculate two distinct integrals

$$\begin{aligned} I_1 &= g_s^2 \int [\mathrm{d}k]\ \delta(\tau - T_0(p_a, p_b, k))\ \frac{2 p_a \cdot p_b}{(p_a\cdot k)(p_b \cdot k)}, \\ I_2^\mu &= g_s^2 \int [\mathrm{d}k]\ \delta(\tau - T_0(p_a, p_b, k))\ \frac{k^\mu}{(p_a\cdot k)(p_b \cdot k)}. \end{aligned} \tag{3.26}$$

The second integral can be written as

$$I_2^\mu = I_2 \frac{P_{ab}^\mu}{s}, \tag{3.27}$$

so that

$$I_2 = 2 I_2^\mu p_{b,\mu} = g_s^2 \int [\mathrm{d}k] \delta(\tau - T_0(p_a, p_b, k))\ \frac{2}{(p_a \cdot k)}. \tag{3.28}$$

Using these definitions, we find the following results for the integrals that are needed to compute $\mathrm{d}\sigma/\mathrm{d}\tau$ in the soft limit

$$\begin{aligned} & g_s^2 \int [\mathrm{d}k]\ \delta\left(\tau - T_0(p_a, p_b, k)\right) \left(1 - \kappa_m \frac{P_{ab}\cdot k}{P_{ab}^2}\right)\ (-J_\mu J^\mu) = I_1 - \kappa_m I_2, \\ & g_s^2 \int [\mathrm{d}k]\ \delta\left(\tau - T_0(p_a, p_b, k)\right)\ (J_\mu J^\mu) \frac{P_{ab}\cdot k}{P_{ab}^2} p_i^\rho = -I_2 p_i^\rho, \\ & g_s^2 \int [\mathrm{d}k]\ \delta\left(\tau - T_0(p_a, p_b, k)\right)\ (J_\mu J^\mu) B^{\rho\sigma} = 0, \\ & g_s^2 \int [\mathrm{d}k]\ \delta\left(\tau - T_0(p_a, p_b, k)\right)\ J^\mu L_\mu = -I_2 \left(p_a^\mu \frac{\partial}{\partial p_a^\mu} + p_b^\mu \frac{\partial}{\partial p_b^\mu} \right). \end{aligned} \tag{3.29}$$

Putting everything together, and accounting for the fact that the observable $\mathcal{O}$ also depends on the boosted momenta, we obtain

$$\frac{\mathrm{d}\sigma^{(s)}}{\mathrm{d}\tau} = \mathcal{N} \int [\mathrm{d}\Phi_m(p_a, p_b, P_X)] \Bigg\{ \mathcal{O}(P_X) \Bigg[I_1 - \kappa_m I_2 - I_2 \sum_{i \in L_f} p_i^\mu \frac{\partial}{\partial p_i^\mu} \Bigg] |\mathcal{M}|^2(p_b, p_a, P_X) - I_2 \, |\mathcal{M}|^2(p_b, p_a, P_X) \sum_{i=1}^{m} p_i^\mu \frac{\partial}{\partial p_i^\mu} \mathcal{O}(P_X) \Bigg\}, \tag{3.30}$$

where L_f is the list that includes all particles in the Born process Eq. (3.1). We note that in the last term in Eq. (3.30) the sum can be extended to include initial partons if an observable does not depend on them and the corresponding derivatives vanish.

To finalize the calculation, we need to compute integrals $I_{1,2}$. To do this, we integrate over the energy of the gluon with momentum k, removing the zero-jettiness δ-function. Then, using the following expressions for the angular integrals

$$\int \left[\mathrm{d}\Omega_{\vec{k}}^{(d-1)}\right] \frac{\psi_k^{2\epsilon}}{\rho_{ak}\rho_{bk}} = \frac{1}{\epsilon}, \qquad \int \left[\mathrm{d}\Omega_{\vec{k}}^{(d-1)}\right] \frac{\psi_k^{2\epsilon-1}}{\rho_{ik}} = \frac{1}{2\epsilon} - \frac{1}{2} - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2), \quad i = a, b, \tag{3.31}$$

where $\left[\mathrm{d}\Omega_{\vec{k}}^{(d-1)}\right] = \mathrm{d}\Omega_{\vec{k}}^{(d-1)}/\Omega^{(d-2)}$, $\psi_k = \min(\rho_{ak}, \rho_{bk})$ and $\rho_{ik} = 1 - \cos\theta_{ik}$, $i = a, b$, we find

$$I_1 = [\alpha_s] \left(\frac{Q}{\sqrt{s}}\right)^{-2\epsilon} \frac{4}{\epsilon\, \tau^{1+2\epsilon}}, \quad I_2 = [\alpha_s] \left(\frac{Q\tau}{\sqrt{s}}\right)^{-2\epsilon} \frac{4Q}{s} \left(\frac{1}{2\epsilon} - \frac{1}{2} - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)\right), \tag{3.32}$$

where $[\alpha_s]$ is defined in Eq. (2.15). It is straightforward to use Eq. (3.30) together with the results for the two integrals $I_{1,2}$ to determine both leading and subleading zero-jettiness contributions to the cross section of a process in Eq. (3.1), that originate from the emission of a soft gluon. Since the physical result requires including the contributions of the collinear emissions, we refrain from presenting the expansion of Eq. (3.30) in powers of ϵ. Nevertheless, for illustration purposes, we show the $1/\epsilon$-divergence of the subleading soft contribution which can be easily obtained from Eq. (3.30). This contribution comes entirely from the divergent part of the integral I_2. After restoring the appropriate color factor, we obtain

$$\frac{\mathrm{d}\sigma^{s,\mathrm{div}}}{\mathrm{d}\tau} = -\frac{2C_F[\alpha_s]\,\mathcal{N}}{\epsilon} \frac{(Q\tau)^{-2\epsilon}}{s^{-\epsilon}} \frac{Q}{s} \left(\kappa_m + \sum_{i \in L_f} p_i^\mu \frac{\partial}{\partial p_i^\mu} \right) |\mathcal{M}|^2 \, \mathcal{O}(P_X). \tag{3.33}$$

3.2 The first collinear contribution: $\vec{k} || \vec{p}_a$

As the next step, we need to construct expansions in the zero-jettiness variable around the collinear limits. We will start with the case where the gluon is emitted along the direction of the incoming quark with momentum p_a. The case where the gluon is emitted along the direction of the incoming anti-quark is completely analogous; we discuss it in the next subsection.

Similarly to the case of the soft emission considered earlier, we perform a momenta mapping [40] that allows us to construct the collinear expansion. To do this, we start by re-writing the gluon momentum k as follows

$$k = \frac{k \cdot P_{ab}}{p_a \cdot p_b} p_a + \tilde{k}_a = (1-x) p_a + \tilde{k}_a. \tag{3.34}$$

The momentum conservation condition[1]

$$p_a + p_b = k + \tilde{Q}_X, \tag{3.35}$$

becomes

$$x p_a + p_b - Q_X = 0, \tag{3.36}$$

where

$$Q_X = \tilde{Q}_X + \tilde{k}_a. \tag{3.37}$$

It is easy to show that Q_X^2 and $\tilde{Q}_X^2$ are the same

$$\begin{aligned} Q_X^2 &= (\tilde{Q}_X + \tilde{k}_a)^2 = (p_a + p_b - k + \tilde{k}_a)^2 = (P_{ab} - \frac{k \cdot P_{ab}}{p_a \cdot p_b} p_a)^2 \\ &= P_{ab}^2 - 2k \cdot P_{ab} = (P_{ab} - k)^2 = \tilde{Q}_X^2. \end{aligned} \tag{3.38}$$

Since $\tilde{Q}_X^2 = Q_X^2$, we can obtain one of these momenta by Lorentz-boosting the other. We therefore write

$$\begin{aligned} &[\mathrm{d}\tilde{Q}_X]_m [\mathrm{d}k] (2\pi)^d \delta(P_{ab} - \tilde{Q}_X - k) = [\mathrm{d}\tilde{Q}_X]_m [\mathrm{d}k] (2\pi)^d \delta(x p_a + p_b - Q_X) \\ &= [\mathrm{d}\tilde{Q}_X]_m [\mathrm{d}k] (2\pi)^d \delta(x p_a + p_b - \Lambda_a(Q_X, \tilde{Q}_X) \tilde{Q}_X), \end{aligned} \tag{3.39}$$

where the Lorentz boost $\Lambda_a(Q_X, \tilde{Q}_X)$ is defined as follows

$$Q_X = \Lambda_a(Q_X, \tilde{Q}_X)\, \tilde{Q}_X. \tag{3.40}$$

Since $\tilde{Q}_X = \sum\limits_{i=1}^{m} \tilde{p}_i$, we perform the required boost for each final-state particle $\tilde{p}_i = \Lambda_a^{-1}(Q_X, \tilde{Q}_X) p_i$ and obtain

$$\begin{aligned} &\prod_{i=1}^{m} [\mathrm{d}p_i][\mathrm{d}k](2\pi)^d \delta\left(x p_a + p_b - \sum_{i=1}^{m} p_i \right) \\ &= \int \mathrm{d}\xi \, \prod_{i=1}^{m} [\mathrm{d}p_i](2\pi)^d \delta\left(\xi p_a + p_b - \sum_{i=1}^{m} p_i \right) \, [\mathrm{d}k] \, \delta(x - \xi). \end{aligned} \tag{3.41}$$

The Lorentz transformation $\Lambda_a(Q_X, \tilde{Q}_X)$ can be found in Eq. (A.2), where one should identify $Q_i = \tilde{Q}_X$, $Q_f = Q_X$.

[1] At variance with the previous section, here we denote the momentum of the colorless final state X as $\tilde{Q}_X$. We do this because we need several redefinitions of this momentum, before we reach the final formula in Sec. 4. There, we will return to the notation P_X for the momentum of the final state X.

Since we will have to apply this transformation to all final-state particles and then expand the result around the collinear limit, we need to simplify Λ_a. To do this, we introduce the notation

$$Q_a = xp_a + p_b, \tag{3.42}$$

so that

$$Q_f = Q_a, \quad Q_i = Q_a - \tilde{k}_a. \tag{3.43}$$

We use Eq. (3.34) to write $\tilde{k}_a$ as

$$\tilde{k}_a^\mu = k^\mu - \frac{k \cdot P_{ab}}{p_a \cdot p_b}\, p_a^\mu. \tag{3.44}$$

To simplify the expression for $\tilde{k}_a$ further, we perform the Sudakov decomposition of the vector k and write

$$k^\mu = \alpha p_a^\mu + \beta p_b^\mu + k_\perp^\mu, \tag{3.45}$$

where the transverse momentum $k_\perp$ satisfies $p_{a,b} \cdot k_\perp = 0$. We then compute the coefficients α and β, and find

$$\tilde{k}_a^\mu = \frac{\omega_k}{\sqrt{s}}(2 - \rho_{ak})p_a^\mu + \frac{\omega_k}{\sqrt{s}}\rho_{ak}p_b^\mu + k_\perp^\mu, \tag{3.46}$$

where ω_k is the gluon's energy and $\rho_{ak} = 1 - \cos\theta_{ak}$ was introduced earlier. Furthermore, we also made use of the fact that vectors $p_{a,b}$ are back-to-back, and that $2p_a \cdot p_b = s$.

The absolute value of the vector $k_\perp$ is determined from the on-shell condition $k^2 = 0$. We derive

$$k_\perp^2 = -\omega_k^2 \rho_{ak}(2 - \rho_{ak}). \tag{3.47}$$

Using Eq. (3.46), we write $\tilde{k}_a^\mu$ in Eq. (3.44) as follows

$$\tilde{k}_a^\mu = \frac{\omega_k}{\sqrt{s}}\rho_{ak}(p_b - p_a)^\mu + \omega_k\sqrt{\rho_{ak}(2 - \rho_{ak})}\,n_\perp^\mu = \frac{2kp_a}{s}(p_b - p_a)^\mu + k_\perp^\mu. \tag{3.48}$$

The important point is that $\tilde{k}_a^\mu$ vanishes in the soft $\omega_k \to 0$ and in the collinear $\rho_{ak} \to 0$ limits, which allows us to construct the expansion of the Lorentz-boost matrix Λ_a, which becomes the identity matrix in both of these limits.

The boost operator depends on Q_a and $\tilde{k}_a$; the collinear expansion is the expansion in small $\tilde{k}_a$. Since $\tilde{k}_a \sim \sqrt{\rho_{ak}}$ and we need to account for $\mathcal{O}(\rho_{ak})$ terms, we must expand Λ_a to *second order* in $\tilde{k}_a$. The expansion can be simplified if we notice that

$$2\tilde{k}_a \cdot Q_a = \tilde{k}_a^2. \tag{3.49}$$

The above equation follows from the equality $Q_f^2 = Q_a^2 = Q_i^2 = (Q_a - \tilde{k}_a)^2$. Hence, in this case

$$(Q_f + Q_i)^2 = 4Q_a^2 - \tilde{k}_a^2. \tag{3.50}$$

Using this result, we easily arrive at the expressions for the boost operator Λ_a and its inverse, shown in Eqs (A.4, A.5).

We continue with the simplification of the starting expression for the cross section in Eq. (3.2) in the collinear $\vec{k}||\vec{p}_a$ limit. The first point is that the jettiness constraint is simplified in this limit, since $\psi_k = \rho_{ak}$. It is important to emphasize that the above formula is valid not only in the strict $\vec{k}||\vec{p}_a$ limit, but also in its neighborhood. Because of this, we do not expand the zero-jettiness function around the collinear limit below. Hence, in the first step we write

$$\begin{aligned} \frac{\mathrm{d}\sigma^{ca}}{\mathrm{d}\tau} &= \mathcal{N} \int [\mathrm{d}\tilde{Q}_X]_m [\mathrm{d}k] (2\pi)^d \delta(p_a + p_b - k - \tilde{Q}_X) \\ &\times \delta\left(\tau - \frac{\sqrt{s}\omega_k \rho_{ak}}{Q}\right) \; \mathcal{O}(\tilde{Q}_X) \sum_{\mathrm{col,pol}} |\mathcal{M}|^2(p_b, p_a, k, \tilde{Q}_X). \end{aligned} \tag{3.51}$$

Following the above discussion, we perform the momenta transformation and obtain

$$\begin{aligned} \frac{\mathrm{d}\sigma^{ca}}{\mathrm{d}\tau} &= \mathcal{N} \int_0^1 \mathrm{d}x \; [\mathrm{d}Q_X]_m (2\pi)^d \delta(x p_a + p_b - Q_X) \; \int [\mathrm{d}k] \delta\left(1 - \frac{2\omega_k}{\sqrt{s}} - x\right) \\ &\times \delta\left(\tau - \frac{\sqrt{s}\omega_k \rho_{ak}}{Q}\right) \mathcal{O}(\Lambda_a^{-1} Q_X) \sum_{\mathrm{pol,col}} |\mathcal{M}(p_b, p_a, k, \Lambda_a^{-1} Q_X)|^2. \end{aligned} \tag{3.52}$$

The matrix of the inverse Lorentz transformation Λ_a^{-1} is given in Eq. (A.5).

The product of the gluon phase-space element $[\mathrm{d}k]$ and two δ-functions in Eq. (3.52) can be simplified, since these delta-functions fix the gluon energy and its emission angle relative to the direction of the quark with momentum p_a. We find

$$\begin{aligned} &[\mathrm{d}k] \; \delta\left(1 - \frac{2\omega_k}{\sqrt{s}} - x\right) \delta\left(\tau - \frac{\sqrt{s}\omega_k \rho_{ak}}{Q}\right) = \frac{\Omega^{(d-2)}}{2(2\pi)^{d-1}} \, \mathrm{d}\omega_k \; \omega_k^{1-2\epsilon} \\ &\times \mathrm{d}\rho_{ak} \; \rho_{ak}^{-\epsilon} \; (2 - \rho_{ak})^{-\epsilon} \frac{\mathrm{d}\Omega^{(d-2)}}{\Omega^{(d-2)}} \delta\left(1 - \frac{2\omega_k}{\sqrt{s}} - x\right) \delta\left(\tau - \frac{\sqrt{s}\omega_k \rho_{ak}}{Q}\right) \\ &= \frac{\Omega^{(d-2)}}{2(2\pi)^{d-1}} \; \frac{Q^{1-\epsilon}\tau^{-\epsilon}}{2} (1-x)^{-\epsilon} \left(1 + \frac{\epsilon \rho_{ak}^*}{2}\right) [\mathrm{d}\Omega^{(d-2)}], \end{aligned} \tag{3.53}$$

where

$$\omega_k = \frac{\sqrt{s}}{2}(1-x), \qquad \rho_{ak}^* = \frac{2Q\tau}{s(1-x)}, \qquad [\mathrm{d}\Omega^{(d-2)}] = \frac{\mathrm{d}\Omega^{(d-2)}}{\Omega^{(d-2)}}. \tag{3.54}$$

We now put everything together and write the collinear contribution as follows

$$\begin{aligned} \frac{\mathrm{d}\sigma^{ca}}{\mathrm{d}\tau} &= \frac{C_F [\alpha_s] Q^{1-\epsilon}}{2\tau^{1+\epsilon}} \mathcal{N} \int_0^1 \mathrm{d}x \; \mathrm{d}\Phi_m^{xa} \; \left[\mathrm{d}\Omega_k^{(d-2)}\right] \; (1-x)^{-\epsilon} \left(1 + \frac{\epsilon \rho_{ak}^*}{2}\right) \\ &\times \; \mathcal{O}(\Lambda_a^{-1} Q_X) \sum_{\mathrm{pol,col}} C_F^{-1} g_s^{-2} \tau \; |\mathcal{M}(p_b, p_a, k, \Lambda_a^{-1} Q_X)|^2, \end{aligned} \tag{3.55}$$

where

$$\mathrm{d}\Phi_m^{xa} = [\mathrm{d}Q_X]_m (2\pi)^d \delta(x p_a + p_b - Q_X). \tag{3.56}$$

To determine the subleading contributions to the cross section, we use Eq. (3.55) as a starting point and expand the matrix element squared and the observable around the collinear limit using explicit expressions for the Lorentz boost Λ_a. We integrate the result of the expansion over the azimuthal angle of the emitted gluon, leading to the final expression which depends on the scalar products of $p_{a,b}$ and p_i, as well as on the derivatives of the *observable* with respect to the momenta p_i. This expression will have to be combined with the contribution of the other collinear limit ($\vec{k}||\vec{p}_b$) and the contribution of the soft limit, to arrive at the next-to-leading power correction to the differential cross section subject to the zero-jettiness constraint.

To proceed further, we need to construct the collinear expansion of the matrix element squared. We write

$$g_s^2 C_F F_a = \sum_{\text{pol}} |\mathcal{M}|^2(p_b, p_a, k, \Lambda_a^{-1} Q_X) = \sum_{\text{pol}} |\mathcal{M}|^2(\Lambda_a p_b, \Lambda_a p_a, \Lambda_a k, Q_X), \tag{3.57}$$

where we have used the Lorentz invariance of the matrix element squared to move the action of the Lorentz boost to p_a, p_b and k. The advantage of the above formula is that the action of the boost is now "localized"; we need to consider changes in the momenta of the initial partons and the momentum of the gluon k, but the momenta of the final-state color-neutral particles do not need to be changed.

We require the expression of the matrix element that is suitable for the study of the collinear limit. Furthermore, the collinear limit should be written in such a way that the soft singularity in the corresponding expressions can be isolated. Finally, it is convenient to first expand the matrix element squared around the collinear limit, and apply the boost later; this approach keeps the expressions more compact since the boost will have to be applied to fewer terms.

We write the matrix element as

$$\mathcal{M} = -g_s T^a \epsilon_\nu^* \bar{v}_b \left[N(p_b, p_a - k, Q_X) \frac{(\hat{p}_a - \hat{k})\gamma^\nu}{(-2p_a \cdot k)} + N^\nu_{\text{fin},a} \right] u_a. \tag{3.58}$$

The collinear $\vec{k}||\vec{p}_a$ singularity is present in the first term of the above expression, whereas the second term is subleading in the collinear limit. However, that term still has a soft singularity. For this reason, it is convenient to write it as follows

$$N^\nu_{\text{fin},a} = R^\nu_{\text{fin}}(p_b, p_a, k, Q_X) + \gamma^\nu \frac{(\hat{p}_b - \hat{k})}{2p_b \cdot k} N(p_b - k, p_a, Q_X). \tag{3.59}$$

We note that R^ν_{fin} arises from diagrams where the gluon is emitted from off-shell lines and, therefore, it contains neither collinear nor soft singularities. In what follows we will denote $N(p_b, p_a - k, Q_X)$ as $N_a(p_b, p_a - k, Q_X)$ and $N(p_b - k, p_a, Q_X)$ as $N_b(p_b - k, p_a, Q_X)$, and we will not write their arguments explicitly unless it is needed.

In the collinear limit $\vec{k}||\vec{p}_a$, we write the matrix element squared as a sum of three terms

$$F_a = F_{aa} + F_{ar} + F_{rr,a}, \tag{3.60}$$

where the first term on the right-hand side refers to the square of the diagram where the gluon is emitted off the parton a, the second term refers to the interference of this diagram with the remaining ones, and the third one to the contribution of the remaining diagrams squared.

We begin by considering the first term on the right-hand side of Eq. (3.60) and write it as

$$F_{aa} = \frac{1}{(2p_a k)^2}\mathrm{Tr}\left[N_a \,\left(\hat{p}_a - \hat{k}\right)\gamma^\mu \hat{p}_a \gamma^\nu \left(\hat{p}_a - \hat{k}\right)\, N_a^+ \hat{p}_b\right]\,\rho^{(a)}_{\mu\nu}, \tag{3.61}$$

where

$$\rho^{(a)}_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu p_{b,\nu} + p_{b,\mu} k_\nu}{k\cdot p_b}, \tag{3.62}$$

and the arguments of N_a are not shown. The superscript a in the gluon density matrix in Eq. (3.62) indicates that a gauge choice for the gluon polarization vector is made to simplify the expansion around the $\vec{k}||\vec{p}_a$ collinear limit. We will need to expand F_{aa} through terms that scale as $\mathcal{O}((kp_a)^0)$ and, once the expansion is constructed, apply the Lorentz boost Λ_a to momenta p_a, p_b and k in the resulting formula.

We begin with the simplification of F_{aa}. A straightforward algebra gives

$$\begin{aligned}(\hat{p}_a - \hat{k})\gamma^\mu \hat{p}_a \gamma^\nu (\hat{p}_a - \hat{k})\, \rho^{(a)}_{\mu\nu} = (2k\cdot p_a)\Big[&(d-2)\hat{k} \\ &+ \frac{1}{k\cdot p_b}\left(\hat{p}_a\hat{p}_b(\hat{p}_a - k) + \left(\hat{p}_a - \hat{k}\right)\hat{p}_b\hat{p}_a\right)\Big].\end{aligned} \tag{3.63}$$

Since

$$\hat{p}_a\hat{p}_b(\hat{p}_a - k) + (\hat{p}_a - \hat{k})\hat{p}_b\hat{p}_a = 2(p_a - k)\cdot p_b\,\hat{p}_a + 2p_a\cdot k\,\hat{p}_b + 2p_a\cdot p_b\,(\hat{p}_a - \hat{k}), \tag{3.64}$$

we obtain

$$F_{aa} = \frac{1}{2p_a\cdot k}\mathrm{Tr}\left[N_a\left(-2(\hat{p}_a - \hat{k}) - 2\epsilon\hat{k} + \frac{2p_a\cdot p_b}{p_b\cdot k}\left(2\hat{p}_a - \hat{k}\right) + \frac{2p_a\cdot k}{p_b\cdot k}\hat{p}_b\right)N_a^+\hat{p}_b\right]. \tag{3.65}$$

We can further simplify the above equation by substituting $k = (1-x)p_a + \tilde{k}_a$ and neglecting all terms that contribute to the expansion around the collinear limit beyond the next-to-leading power. We then find

$$\begin{aligned}F_{aa} = \frac{1}{2p_a\cdot k}\mathrm{Tr}\Big[N_a\Big(&2\hat{p}_a P_{qq}(x) - 2\left(\frac{x}{1-x} + \epsilon\right)\hat{\tilde{k}}_a \\ &+ \frac{2p_a\cdot k}{(1-x)p_a\cdot p_b}\left[\hat{p}_b + \frac{1+x}{1-x}\hat{p}_a\right]\Big)N_a^+\hat{p}_b\Big],\end{aligned} \tag{3.66}$$

where

$$P_{qq}(x) = \frac{1+x^2}{1-x} - \epsilon(1-x), \tag{3.67}$$

is the standard collinear splitting function. We note that the last term in Eq. (3.66) is already of the right order in the collinear expansion. For this reason it does not require further manipulations, i.e. the Lorentz boost does not need to be applied to it. Furthermore, it is convenient to express $\tilde{k}_a$ through $k_\perp$ using Eq. (3.48). We obtain

$$
\begin{aligned}
F_{aa} = {} & \frac{2P_{qq}(x)}{2p_a \cdot k} \mathrm{Tr}\left[N_a\, \hat{p}_a\, N_a^+ \hat{p}_b\right] - \frac{2}{2p_a \cdot k}\left(\frac{x}{1-x} + \epsilon\right) \mathrm{Tr}\left[N_a \hat{k}_\perp N_a^+ \hat{p}_b\right] \\
& + \frac{2}{s}\left(\frac{(1+2x-x^2)}{(1-x)^2} + \epsilon\right) \mathrm{Tr}\left[N_a \hat{p}_a N_a^+ \hat{p}_b\right] + \frac{2(1-\epsilon)}{s} \mathrm{Tr}\left[N_a \hat{p}_b N_a^+ \hat{p}_b\right].
\end{aligned}
\tag{3.68}
$$

The last two terms do not require further manipulations, they are already in the right form. The first term is the leading collinear contribution; it must be expanded to account for subleading collinear terms.

The second term on the right-hand side in Eq. (3.68) requires discussion. As we mentioned earlier, we will have to boost the momenta p_a, p_b, k to compute the matrix element squared. Since the second term in Eq. (3.68) is proportional to $k_\perp$, anything that arises from $N_{a,b}$ or $\hat{p}_{a,b}$ after the boost can only contain $k_\perp$ since all other terms contribute beyond next-to-leading order in the zero-jettiness expansion. However, boosting $k_\perp$ will generate a term that is proportional to $2p_a k$, which will then contribute to F_{aa} at the right order. Since

$$
\Lambda_a{}^{\mu}{}_{\nu}\, k_\perp^\nu = k_\perp^\mu + \frac{Q_a^\mu}{Q_a^2}(1-x)(2k \cdot p_a), \tag{3.69}
$$

with $Q_a = x p_a + p_b$, it is convenient to introduce a new vector

$$
\kappa_a^\mu = k_\perp^\mu - \frac{\tilde{Q}_a^\mu}{\tilde{Q}_a^2}(1-x)(2k \cdot p_a), \tag{3.70}
$$

which after the boost becomes $k_\perp$,

$$
\Lambda_{a,\nu}^{\mu}\, \kappa_a^\nu = k_\perp^\mu + \mathcal{O}(k_\perp^3). \tag{3.71}
$$

Thus, if we express the before-the-boost result through κ_a, computing the boost for κ_a-dependent terms becomes straightforward.

Hence, we rewrite the vector $k_\perp$ through κ_a using Eq. (3.68), and find

$$
\mathrm{Tr}\left[N_a \hat{k}_\perp N_a^+ \hat{p}_b\right] = \mathrm{Tr}\left[N_a \hat{\kappa}_a N_a^+ \hat{p}_b\right] + 2k \cdot p_a \frac{(1-x)}{sx} \mathrm{Tr}\left[N_a \left(x\hat{p}_a + \hat{p}_b\right) N_a^+ \hat{p}_b\right]. \tag{3.72}
$$

Combining the last term in Eq. (3.72) with the third and the fourth term in Eq. (3.68), we obtain

$$
\begin{aligned}
F_{aa} = {} & \frac{2P_{qq}(x)}{2p_a \cdot k} \mathrm{Tr}\left[N_a\, \hat{p}_a\, N_a^+ \hat{p}_b\right] - \frac{2}{2p_a \cdot k}\left(\frac{x}{1-x} + \epsilon\right) \mathrm{Tr}\left[N_a \hat{\kappa}_a N_a^+ \hat{p}_b\right] \\
& + \frac{2}{s}\left(\frac{(1+x+x^2-x^3)}{(1-x)^2} + \epsilon x\right) \mathrm{Tr}\left[N_a \hat{p}_a N_a^+ \hat{p}_b\right] - \frac{2\epsilon}{sx} \mathrm{Tr}\left[N_a \hat{p}_b N_a^+ \hat{p}_b\right].
\end{aligned}
\tag{3.73}
$$

As we already mentioned this form is convenient because after the boost, κ_a will become $k_\perp$; this implies that only $k_\perp$ terms from other momenta will be needed. Since, after averaging

$$
k_\perp^\mu k_\perp^\nu \to \frac{k_\perp^2}{2(1-\epsilon)} g_\perp^{\mu\nu} = -\frac{2p_a k(1-x)}{2(1-\epsilon)} g_\perp^{\mu\nu}, \tag{3.74}
$$

where

$$g_{\perp}^{\mu\nu} = g^{\mu\nu} - \frac{p_a^\mu p_b^\nu + p_b^\mu p_a^\nu}{p_a \cdot p_b}, \tag{3.75}$$

such terms do not lead to soft and collinear singularities. Hence, the soft singularities are only present in the *first* and *third* terms on the r.h.s. of Eq. (3.73).

We continue with the discussion of the interference contribution in Eq. (3.60). We write it as

$$F_{ar} = \mathrm{Tr}\left[\frac{N_a(\hat{p}_a - \hat{k})\gamma^\mu \hat{p}_a N_{\mathrm{fin},a}^{+,\nu}\hat{p}_b}{(-2p_a \cdot k)}\right] \rho_{\mu\nu}^{(a)} + \mathrm{c.c.} \tag{3.76}$$

We now split $\rho_{\mu\nu}^{(a)}$ into two terms

$$\rho_{\mu\nu}^{(a)} = \rho_{\mu\nu}^{(a,1)} + \rho_{\mu\nu}^{(a,2)}, \tag{3.77}$$

where

$$\rho_{\mu\nu}^{(a,1)} = -g_{\mu\nu} + \frac{p_{b\mu}k_\nu}{k \cdot p_b}, \quad \rho_{\mu\nu}^{(a,2)} = \frac{k_\mu p_{b\nu}}{k \cdot p_b}. \tag{3.78}$$

We will first compute $F_{ar}^{(2)}$ which we obtain by replacing $\rho_{\mu\nu}^{(a)}$ in Eq. (3.76) with $\rho_{\mu\nu}^{(a,2)}$. Using

$$(\hat{p}_a - k)\gamma^\mu \hat{p}_a \rho_{\mu\nu}^{(a,2)} = \hat{p}_a \gamma^\mu \hat{p}_a \rho_{\mu\nu}^{(a,2)} = \frac{2kp_a}{k \cdot p_b}\hat{p}_a \, p_{b\nu}, \tag{3.79}$$

we obtain

$$F_{ar}^{(2)} = -\frac{2p_{b,\nu}}{s(1-x)}\mathrm{Tr}\left[N_a \hat{p}_a N_{\mathrm{fin},a}^{+,\nu}\hat{p}_b\right] + \mathrm{c.c.}, \tag{3.80}$$

where we already took the collinear limit.

It is important to understand how the soft limit can be extracted from this expression especially since there is a term in $N_{\mathrm{fin},a}^{\nu,+}$ that contains the soft singularity. We use Eq. (3.59) and write

$$N_{\mathrm{fin},a}^{\nu,+} = R_{\mathrm{fin}}^{\nu,+} + N_b^+ \frac{(\hat{p}_b - \hat{k})}{2p_b \cdot k}\gamma^\nu. \tag{3.81}$$

Using it in Eq. (3.80), we find

$$F_{ar}^{(2)} = -\frac{2p_{b,\nu}}{s(1-x)}\mathrm{Tr}\left[N_a \hat{p}_a R_{\mathrm{fin}}^{+,\nu}\hat{p}_b\right] + \mathrm{c.c.}, \tag{3.82}$$

and the soft singularity is now explicit.

The other contribution $F_{ar}^{(1)}$ is obtained by replacing $\rho_{\mu\nu}^{(a)}$ in Eq. (3.76) with $\rho_{\mu\nu}^{(a,1)}$. To simplify the result in this case, we write

$$\left(\hat{p}_a - \hat{k}\right)\gamma^\mu \hat{p}_a \rho_{\mu\nu}^{(a,1)} = \left(\hat{p}_a - \hat{k}\right) 2p_a^\mu \rho_{\mu\nu}^{(a,1)} + \hat{k}\hat{p}_a \gamma^\mu \rho_{\mu\nu}^{(a,1)}. \tag{3.83}$$

Replacing $\hat{k}\hat{p}_a$ in the second term with $\hat{\tilde{k}}_a \hat{p}_a$ and writing there $\hat{p}_a \gamma^\mu = 2p_a^\mu - \gamma^\mu \hat{p}_a$, we find

$$\left(\hat{p}_a - \hat{k}\right)\gamma^\mu \hat{p}_a \rho_{\mu\nu}^{(a,1)} = \left(\hat{p}_a - \hat{k} + \hat{\tilde{k}}_a\right) 2p_a^\mu \rho_{\mu\nu}^{(a,1)} - \hat{\tilde{k}}_a \gamma^\mu \hat{p}_a \rho_{\mu\nu}^{(a,1)}. \tag{3.84}$$

It is easy to show that, through the right order in the zero-jettiness expansion, the following equation holds

$$p_a^\mu \rho_{\mu\nu}^{(a,1)} = \frac{\tilde{k}_{a,\nu}}{1-x} + \frac{2k\cdot p_a}{(1-x)s}\, p_{a,\nu}. \tag{3.85}$$

We also find

$$\hat{\tilde{k}}_a \gamma^\mu \hat{p}_a \rho_{\mu\nu}^{(a,1)} = -\hat{\tilde{k}}_a \gamma_\nu \hat{p}_a + \frac{1}{k\cdot p_b}\hat{\tilde{k}}_a \hat{p}_b \hat{p}_a k_\nu. \tag{3.86}$$

The Ward identity implies

$$\hat{p}_a N_{\mathrm{fin},a}^{+,\nu} \hat{p}_b\; k_\nu = \hat{p}_a N_a^+ \hat{p}_b. \tag{3.87}$$

Putting the above results together, we obtain

$$\begin{aligned} F_{ar}^{(a,1)} &= 2\mathrm{Tr}\left[\frac{N_a x \hat{p}_a N^+_{\mathrm{fin},a,\nu}\hat{p}_b}{(-2p_a\cdot k)}\right]\left(\frac{\tilde{k}_a^\nu}{1-x} + \frac{2k\cdot p_a}{(1-x)s}p_a^\nu\right) + \mathrm{Tr}\left[\frac{N_a \hat{\tilde{k}}_a \gamma_\nu \hat{p}_a N_{\mathrm{fin},a}^{+,\nu}\hat{p}_b}{(-2p_a\cdot k)}\right] \\ &\quad - \frac{1}{k\cdot p_b}\mathrm{Tr}\left[\frac{N_a \hat{\tilde{k}}_a \hat{p}_b \hat{p}_a N_a^+ \hat{p}_b}{(-2p_a\cdot k)}\right] + \mathrm{c.c.}. \end{aligned} \tag{3.88}$$

We observe that the interference terms are proportional to $\tilde{k}_a/(-2p_a\cdot k)$. Hence, to obtain the final result, the functions N and $N_{\mathrm{fin}}^{+,\nu}$ have to be expanded to first order in $\tilde{k}_a$. We also note that in the last term in Eq. (3.88) we can replace the $1/(k\cdot p_b)$ factor with $2/(s(1-x))$, without compromising the accuracy of the collinear expansion.

We continue with the analysis of the individual terms in Eq. (3.88), aiming at isolating those that have a soft singularity. We begin with the first term in Eq. (3.88) and write

$$\begin{aligned} &2\mathrm{Tr}\left[\frac{N_a x \hat{p}_a N^+_{\mathrm{fin},a,\nu}\hat{p}_b}{(-2p_a\cdot k)}\right]\left(\frac{\tilde{k}_a^\nu}{1-x} + \frac{2k\cdot p_a}{(1-x)s}\, p_a^\nu\right) \\ &= -\frac{2}{s(1-x)}\mathrm{Tr}\left[N_a x \hat{p}_a \left(R_{\mathrm{fin}}^{\nu,+} p_{b,\nu} + N_b^+\right)\hat{p}_b\right] \\ &+ \frac{2}{(1-x)(-2p_a\cdot k)}\mathrm{Tr}\left[N_a x\hat{p}_a\left(R_{\mathrm{fin}}^{\nu,+} k_{\perp,\nu}\hat{p}_b - N_b^+ \frac{\hat{p}_a \hat{k}_\perp}{s}\hat{p}_b\right)\right]. \end{aligned} \tag{3.89}$$

It is convenient to express the last term in Eq. (3.89) through a vector $\tilde{\kappa}_a$, following the discussion above. We find

$$\begin{aligned} &2\mathrm{Tr}\left[\frac{N_a x \hat{p}_a N^+_{\mathrm{fin},a,\nu}\hat{p}_b}{(-2p_a\cdot k)}\right]\left(\frac{\tilde{k}_a^\nu}{1-x} + \frac{2k\cdot p_a}{(1-x)s}p_a^\nu\right) = \\ &-\frac{2}{s(1-x)}\mathrm{Tr}\left[N_a x \hat{p}_a\left(R^+_{\mathrm{fin},\nu}\frac{p_b^\nu}{x} + N_b^+\right)\hat{p}_b\right] - \frac{2}{s}\mathrm{Tr}\left[N_a x \hat{p}_a R_{\mathrm{fin}}^{\nu,+} p_{a,\nu}\hat{p}_b\right] \\ &+ \frac{2\kappa_{a,\nu}}{(1-x)(-2p_a\cdot k)}\mathrm{Tr}\left[N_a x \hat{p}_a\left(R_{\mathrm{fin}}^{\nu,+} - N_b^+\frac{\hat{p}_a\gamma^\nu}{s}\right)\hat{p}_b\right], \end{aligned} \tag{3.90}$$

and the soft singularity is only present in the first term on the right-hand side of the above equation, thanks to the argument mentioned below Eq. (3.73).

Next, we need to consider the second term in Eq. (3.88). We write

$$\mathrm{Tr}\left[\frac{N_a\hat{\tilde{k}}_a\gamma_\nu\hat{p}_aN^{+,\nu}_{\mathrm{fin},a}\hat{p}_b}{(-2p_a\cdot k)}\right] = \mathrm{Tr}\left[\frac{N_a\hat{k}_\perp\gamma_\nu\hat{p}_aN^{+,\nu}_{\mathrm{fin},a}\hat{p}_b}{(-2p_a\cdot k)}\right] - \frac{1}{s}\mathrm{Tr}\left[N_a(\hat{p}_b-\hat{p}_a)\gamma_\nu\hat{p}_aN^{+,\nu}_{\mathrm{fin}}\hat{p}_b\right], \tag{3.91}$$

and then replace $k_\perp$ with κ_a in the first term. After simplifications, we find

$$\begin{aligned}\mathrm{Tr}\left[\frac{N_a\hat{\tilde{k}}_a\gamma_\nu\hat{p}_aN^{+,\nu}_{\mathrm{fin},a}\hat{p}_b}{(-2p_a\cdot k)}\right] &= \mathrm{Tr}\left[\frac{N_a\hat{\kappa}_a\gamma_\nu\hat{p}_aN^{+,\nu}_{\mathrm{fin},a}\hat{p}_b}{(-2p_a\cdot k)}\right] + \frac{1}{sx}\mathrm{Tr}\left[N_a\hat{p}_b\gamma^\nu\hat{p}_aN_b^+\frac{\hat{p}_a\gamma_\nu\hat{p}_b}{s}\right]\\ &+ \frac{1}{sx}\mathrm{Tr}\left[N_a(x^2\hat{p}_a-\hat{p}_b)\gamma_\nu\hat{p}_aR^{\nu,+}_{\mathrm{fin}}\hat{p}_b\right] + \frac{2x}{s(1-x)}\mathrm{Tr}\left[N_a\hat{p}_aN_b^+\hat{p}_b\right].\end{aligned} \tag{3.92}$$

The soft divergence resides in the last term on the right-hand side of the above equation.

Finally, we need to analyze the last term in Eq. (3.88). It reads

$$-\frac{1}{kp_b}\mathrm{Tr}\left[\frac{N_a\hat{\tilde{k}}_a\hat{p}_b\hat{p}_aN_a^+\hat{p}_b}{(-2p_a\cdot k)}\right] = -\frac{2}{s(1-x)}\mathrm{Tr}\left[\frac{N_a\hat{\kappa}_a\hat{p}_b\hat{p}_aN_a^+\hat{p}_b}{(-2p_a\cdot k)}\right] - \frac{2x}{s(1-x)}\mathrm{Tr}\left[N_a\hat{p}_aN_a^+\hat{p}_b\right]. \tag{3.93}$$

The soft singularity is in the second term on the right-hand side of Eq. (3.93).

The last contribution we need to compute is $F_{rr,a}$; it is finite in the collinear $\vec{k}||\vec{p}_a$ limit. It is also finite in the soft limit thanks to our choice of the gluon density matrix. The result reads

$$\begin{aligned}F_{rr,a} &= \frac{1}{s}\mathrm{Tr}\left[R^\mu_{\mathrm{fin}}\hat{p}_aN_b^+\hat{p}_a\gamma_\mu\hat{p}_b\right] + \mathrm{c.c.}\\ &+ \frac{2}{s}\mathrm{Tr}\left[N_b\hat{p}_aN_b^+\hat{p}_a\right] - \mathrm{Tr}\left[R^\nu_{\mathrm{fin}}\hat{p}_aR^{\mu,+}_{\mathrm{fin}}\hat{p}_b\right]g_{\perp,\mu\nu}.\end{aligned} \tag{3.94}$$

We now collect all the terms that contribute to the function F_a defined in Eq. (3.60) through the required order in the collinear expansion. We pay particular attention to separating terms that exhibit soft and collinear singularities[2] from the ones that do not. We then write (discarding $\mathcal{O}(\epsilon)$ contributions in terms that are neither soft- nor collinear-

[2] Such singular terms come from Eqs (3.73, 3.82) and Eqs (3.90, 3.92, 3.93).

divergent)

$$
\begin{aligned}
F_a = {} & \frac{2P_{qq}(x)}{2p_a\cdot k}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] + \frac{2}{s}\frac{(1+x+x^2-x^3)}{(1-x)^2}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] \\
& - \frac{4p_b^\nu}{s(1-x)}\mathrm{Tr}\left[N_a\hat{p}_a R^+_{\mathrm{fin},\nu}\hat{p}_b\right] + \mathrm{c.c.} - \frac{2x}{s(1-x)}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] + \mathrm{c.c.} \\
& + \frac{1}{sx}\mathrm{Tr}\left[N_a\hat{p}_b\gamma^\nu\hat{p}_a N_b^+\frac{\hat{p}_a\gamma_\nu\hat{p}_b}{s}\right] + \mathrm{c.c.} - \frac{1}{sx}\mathrm{Tr}\left[N_a\hat{p}_b\gamma_\nu\hat{p}_a R^{\nu,+}_{\mathrm{fin}}\hat{p}_b\right] + \mathrm{c.c.} \\
& - \frac{2}{2p_a\cdot k}\frac{x}{1-x}\mathrm{Tr}\left[N_a\hat{\kappa}_a N_a^+\hat{p}_b\right] + \mathrm{Tr}\left[\frac{N_a\hat{\kappa}_a\gamma_\nu\hat{p}_a N^{+,\nu}_{\mathrm{fin},a}\hat{p}_b}{(-2p_a\cdot k)}\right] + \mathrm{c.c.} \\
& + \frac{2\kappa_{a,\nu}}{(1-x)(-2p_a\cdot k)}\mathrm{Tr}\left[N_a x\hat{p}_a\left(R^{\nu,+}_{\mathrm{fin}} - N_b^+\frac{\hat{p}_a\gamma^\nu}{s}\right)\hat{p}_b\right] + \mathrm{c.c.} \\
& - \frac{2}{s(1-x)}\mathrm{Tr}\left[\frac{N_a\hat{\kappa}_a\hat{p}_b\hat{p}_a N_a^+\hat{p}_b}{(-2p_a\cdot k)}\right] + \mathrm{c.c.} + F_{rr,a}.
\end{aligned}
\tag{3.95}
$$

We note that the complex conjugation, indicated by c.c. in the above formula, always refers to the term that appears immediately to the left of it.

The next-to-last term in Eq. (3.95) can be simplified if we combine it with its conjugate. Then

$$
\mathrm{Tr}\left[\frac{N_a\hat{\kappa}_a\hat{p}_b\hat{p}_a N_a^+\hat{p}_b}{(-2p_a\cdot k)}\right] + \mathrm{c.c.} = \mathrm{Tr}\left[\frac{N_a\left[\hat{\kappa}_a\hat{p}_b\hat{p}_a + \hat{p}_a\hat{p}_b\hat{\kappa}_a\right]N_a^+\hat{p}_b}{(-2p_a\cdot k)}\right]. \tag{3.96}
$$

Since

$$
\hat{\kappa}_a\hat{p}_b\hat{p}_a + \hat{p}_a\hat{p}_b\hat{\kappa}_a = 2(\kappa_a\cdot p_b)\hat{p}_a - 2(\kappa_a\cdot p_a)\hat{p}_b + s\hat{\kappa}_a, \tag{3.97}
$$

we obtain

$$
\begin{aligned}
F_a = {} & \frac{2P_{qq}(x)}{2p_a\cdot k}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] + \frac{2}{s}\frac{(1+x+x^2-x^3)}{(1-x)^2}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] \\
& - \frac{4p_b^\nu}{s(1-x)}\mathrm{Tr}\left[N_a\hat{p}_a R^+_{\mathrm{fin},\nu}\hat{p}_b\right] + \mathrm{c.c.} - \frac{2x}{s(1-x)}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] + \mathrm{c.c.} \\
& + \frac{1}{sx}\mathrm{Tr}\left[N_a\hat{p}_b\gamma^\nu\hat{p}_a N_b^+\frac{\hat{p}_a\gamma_\nu\hat{p}_b}{s}\right] + \mathrm{c.c.} - \frac{1}{sx}\mathrm{Tr}\left[N_a\hat{p}_b\gamma_\nu\hat{p}_a R^{\nu,+}_{\mathrm{fin}}\hat{p}_b\right] + \mathrm{c.c.} \\
& - \frac{2}{s}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] + \frac{2}{sx}\mathrm{Tr}\left[N_a\hat{p}_b N_a^+\hat{p}_b\right] + F_{rr,a} \\
& + \frac{2}{2p_a\cdot k}\mathrm{Tr}\left[N_a\hat{\kappa}_a N_a^+\hat{p}_b\right] + \mathrm{Tr}\left[\frac{N_a\hat{\kappa}_a\gamma_\nu\hat{p}_a N^{+,\nu}_{\mathrm{fin},a}\hat{p}_b}{(-2p_a\cdot k)}\right] + \mathrm{c.c.} \\
& + \frac{2\kappa_{a,\nu}}{(1-x)(-2p_a\cdot k)}\mathrm{Tr}\left[N_a x\hat{p}_a\left(R^{\nu,+}_{\mathrm{fin}} - N_b^+\frac{\hat{p}_a\gamma^\nu}{s}\right)\hat{p}_b\right] + \mathrm{c.c.}.
\end{aligned}
\tag{3.98}
$$

All terms that appear in the above formula should be evaluated for the boosted momenta p_1, p_2, k. In practice, this concerns the first term and the last three terms on the right-hand side in Eq. (3.98) which, after the boost, will have to be expanded to the required order in $k_\perp$. The result of this expansion for the last three terms in Eq. (3.98) is free of both collinear and soft singularities but it is somewhat messy; we present the corresponding formulas in Appendix B.

3.3 The second collinear contribution: $\vec{k}||\vec{p}_b$

We need to consider the second collinear contribution which arises when the gluon is emitted along the direction of an anti-quark with momentum p_b. The construction of the Lorentz transformation and the parametrization of the gluon momentum is identical to what has been discussed in the previous section except that the replacement $p_a \leftrightarrow p_b$ should be applied.

The simplification of the matrix element proceeds as in the previous subsection. It is easy to see that in addition to the $p_a \leftrightarrow p_b$ transformation, we also need to perform the replacement $N_a \leftrightarrow -N_b^+$. We find

$$
\begin{aligned}
F_b &= \frac{2P_{qq}(x)}{2p_b \cdot k}\mathrm{Tr}\left[N_b^+\hat{p}_b N_b \hat{p}_a\right] + \frac{2}{s}\frac{(1+x+x^2-x^3)}{(1-x)^2}\mathrm{Tr}\left[N_b^+\hat{p}_b N_b \hat{p}_a\right] \\
&+ \frac{4p_a^\nu}{s(1-x)}\mathrm{Tr}\left[N_b^+\hat{p}_b R_{\mathrm{fin},\nu}\hat{p}_a\right] + \mathrm{c.c.} - \frac{2x}{s(1-x)}\mathrm{Tr}\left[N_b^+\hat{p}_b N_b \hat{p}_a\right] + \mathrm{c.c.} \\
&+ \frac{1}{sx}\mathrm{Tr}\left[N_b^+\hat{p}_a\gamma^\nu\hat{p}_b N_a \frac{\hat{p}_b\gamma_\nu\hat{p}_a}{s}\right] + \mathrm{c.c.} + \frac{1}{sx}\mathrm{Tr}\left[N_b^+\hat{p}_a\gamma_\nu\hat{p}_b R_{\mathrm{fin}}^\nu\hat{p}_a\right] + \mathrm{c.c.} \\
&- \frac{2}{s}\mathrm{Tr}\left[N_b^+\hat{p}_b N_b \hat{p}_a\right] + \frac{2}{sx}\mathrm{Tr}\left[N_b^+\hat{p}_a N_b \hat{p}_a\right] + F_{rr,b} \\
&+ \frac{2}{2p_b \cdot k}\mathrm{Tr}\left[N_b^+\hat{\kappa}_b N_b \hat{p}_a\right] + \mathrm{Tr}\left[\frac{N_b^+\hat{\kappa}_b\gamma_\nu\hat{p}_b N_{\mathrm{fin},b}^\nu\,\hat{p}_a}{(2p_b \cdot k)}\right] + \mathrm{c.c.} \\
&+ \frac{2\kappa_{b,\nu}}{(1-x)(2p_b \cdot k)}\mathrm{Tr}\left[N_b^+ x\hat{p}_b\left(R_{\mathrm{fin}}^\nu + N_a\frac{\hat{p}_b\gamma^\nu}{s}\right)\hat{p}_a\right] + \mathrm{c.c.},
\end{aligned}
\tag{3.99}
$$

where

$$
N_{\mathrm{fin},b}^\nu = R_{\mathrm{fin}}^\nu - \frac{N_a(\hat{p}_a - \hat{k})\gamma^\nu}{2p_a \cdot k}. \tag{3.100}
$$

Similar to the collinear case $\vec{k}||\vec{p}_a$ discussed in the preceding section, the first two lines contain divergent terms and the remaining terms are finite in the collinear and soft limits. The boost that needs to be applied here differs from the boost in the case $\vec{k}||\vec{p}_a$. We denote the required Lorentz boost as Λ_b; it is given in Appendix A.

4 Combining soft and collinear contributions

In this section, we extract collinear and soft singularities from the different contributions to the differential cross section, and derive the finite result for the next-to-leading term in the zero-jettiness expansion.

4.1 The first collinear region: $\vec{k}||\vec{p}_a$

We begin with the contribution of the first collinear region where the gluon is emitted along the direction of the incoming quark with momentum p_a. The differential cross section reads

$$\begin{aligned}\frac{\mathrm{d}\sigma^{ca}}{\mathrm{d}\tau} = \frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^{1+\epsilon}}\mathcal{N}\int\limits_0^1 \mathrm{d}x\ \mathrm{d}\Phi_m^{xa}\left[\mathrm{d}\Omega_k^{(d-2)}\right](1-x)^{-\epsilon} \\ \times\left(1+\frac{\epsilon Q\tau}{s(1-x)}\right)\mathcal{O}(\tilde{Q}_X)\ \tau\ F_a,\end{aligned} \tag{4.1}$$

where the Born phase space $\mathrm{d}\Phi_m^{xa}$ can be found in Eq. (3.56), $\tilde{Q}_X = \Lambda_a^{-1}Q_X$, F_a is given in Eq. (3.98), and the momenta p_a, p_b, k which appear in that equation should be boosted. It is convenient to write, with the required accuracy,

$$\begin{aligned}\left(1+\frac{\epsilon Q\tau}{s(1-x)}\right)F_a =& \frac{2P_{qq}(x)}{2p_a\cdot k}\mathrm{Tr}\left[N_a\hat{p}_a N_a^+\hat{p}_b\right] + \frac{4(1+\epsilon)}{s(1-x)^2}\mathrm{Tr}\left[N_a\hat{x}p_a N_a^+\hat{p}_b\right] \\ &- \frac{4p_b^\nu}{s(1-x)}\mathrm{Tr}\left[N_a\hat{p}_a R^+_{\mathrm{fin},\nu}\hat{p}_b\right] + \mathrm{c.c.} + F_{a,\mathrm{reg}},\end{aligned} \tag{4.2}$$

where the function $F_{a,\mathrm{reg}}$ does not have soft or collinear singularities. Among the four terms that appear on the right-hand side of Eq. (4.2), the first one requires the expansion of the reduced matrix element in $k_\perp$, the second term has a *power* divergence at $x=1$, and the third one has a regular soft singularity.

We begin with the discussion of the first term on the right-hand side of Eq. (4.2). We note that momenta that appear in that term still have to be boosted. Hence, we write

$$\begin{aligned}\frac{\mathrm{d}\sigma^{ca,1}}{\mathrm{d}\tau} = \frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^{1+\epsilon}}\mathcal{N}\int\limits_0^1 \mathrm{d}x\ \mathrm{d}\Phi_m^{xa}\left[\mathrm{d}\Omega_k^{(d-2)}\right](1-x)^{-\epsilon}\mathcal{O}(\Lambda_a^{-1}Q_X) \\ \times\ \tau\ \frac{2P_{qq}(x)}{(2p_a\cdot k)\ x}\ \mathrm{Tr}\left[N_a(p_b,p_a-k,Q_X)\ x\hat{p}_a N_a^+(p_b,p_a-k,Q_X)\hat{p}_b\right]_{\Lambda_a}.\end{aligned} \tag{4.3}$$

The subscript of the trace function indicates that momenta p_a, p_b and k should be boosted with the matrix Λ_a.

To proceed further, we need to expand the trace function and the observable in Eq. (4.3) around the collinear limit, and extract the soft singularity that is present in $P_{qq}(x)$ from all terms in such an expansion. As the first step, we discuss the (standard) leading collinear contribution which is obtained by setting $\Lambda_a \to 1$ and neglecting the transverse momentum of the gluon k. We find

$$\frac{\mathrm{d}\sigma^{ca,1,\mathrm{LP}}}{\mathrm{d}\tau} = \frac{[\alpha_s]C_F Q^{-\epsilon}}{\tau^{1+\epsilon}}\mathcal{N}\int\limits_0^1 \mathrm{d}x\ \mathrm{d}\Phi_m^{xa}\ \mathcal{O}(Q_X)\ \frac{P_{qq}(x)}{x(1-x)^\epsilon}\ |\mathcal{M}(p_b,xp_a,Q_X)|^2. \tag{4.4}$$

In deriving Eq. (4.4) we have used the equality $2p_a k = \tau Q$, and the fact that in the collinear limit

$$\mathrm{Tr}\left[N_a(p_b,p_a-k,Q_X)\ x\hat{p}_a N_a^+(p_b,p_a-k,Q_X)\hat{p}_b\right]_{\Lambda_a} \to |\mathcal{M}(p_b,xp_a,Q_X)|^2, \tag{4.5}$$

where $|\mathcal{M}(p_b, xp_a, Q_X)|^2$ is the spin-summed matrix element for the elastic process $q\bar{q} \to X$ where the quark q and the anti-quark $\bar{q}$ have momenta xp_a and p_b, respectively. There is a soft singularity present in the splitting function, but it is straightforward to extract it and we do not discuss this point further.

The *subleading* terms require more effort. We start with the discussion of the trace function and write

$$\begin{aligned}
&\mathrm{Tr}\left[N_a(p_b, p_a - k, Q_X)\; x\hat{p}_a\; N_a^+(p_b, p_a - k, Q_X)\; \hat{p}_b\right]_{\Lambda_a} \\
&= \mathrm{Tr}\left[N_a(\Lambda_a p_b, \Lambda_a(p_a - k), Q_X)\; (\Lambda_a x\hat{p}_a)\; N_a^+(\Lambda_a p_b, \Lambda_a(p_a - k), Q_X)\; (\Lambda_a p_b)\right] \\
&= \mathrm{Tr}\big[N_a(p_b + \delta p_{a1}, xp_a - \delta p_{a1}, Q_X)\; (x\hat{p}_a + \delta\hat{p}_{a2}) \\
&\qquad \times N_a^+(p_b + \delta p_{a1}, xp_a - \delta p_{a1}, Q_X)\; (\hat{p}_b + \delta\hat{p}_{a1})\big].
\end{aligned} \tag{4.6}$$

The momenta shifts shown in Eq. (4.6) are easily obtained using the explicit form of the boost operator Λ_a, c.f. Appendix A. We find

$$\begin{aligned}
\delta p_{a1} &= \frac{k_\perp}{2} + \frac{2p_a k}{s}\left(p_b + (1-x)\pi_{a1}\right), \\
\delta p_{a2} &= \frac{k_\perp}{2} - \frac{2p_a k}{s}\left(p_a - (1-x)\pi_{a2}\right),
\end{aligned} \tag{4.7}$$

with

$$\pi_{a1} = \frac{p_a}{4} + \frac{3p_b}{4x}, \quad \pi_{a2} = \frac{3p_a}{4} + \frac{p_b}{4x}. \tag{4.8}$$

The non-trivial step, required to move forward, is the expansion of Eq. (4.6) in powers of $k_\perp$. Since the shifts in Eq. (4.7) are linear in $k_\perp$, the trace in Eq. (4.6) needs to be expanded through *second* order in $k_\perp$. To organize this expansion efficiently, it is convenient to rewrite Eq. (4.6) by introducing the following momenta

$$\mathcal{P}_a = xp_a - \delta p, \quad \mathcal{P}_b = p_b + \delta p, \quad \delta p = \frac{k_\perp}{2} + \frac{k_\perp^2}{4sx}(p_b - xp_a). \tag{4.9}$$

These momenta are constructed in such a way that $\mathcal{P}_a^2 = \mathcal{P}_b^2 = 0$ with $\mathcal{O}(k_\perp^2)$ accuracy, and $\mathcal{P}_a + \mathcal{P}_b = xp_a + p_b$. Using these momenta, we find

$$xp_a - \delta p_{a1} = \mathcal{P}_a - \frac{2k\cdot p_a}{sx}p_b, \quad p_b + \delta p_{a1} = \mathcal{P}_b + \frac{2k\cdot p_a}{sx}p_b. \tag{4.10}$$

Furthermore, we find

$$xp_a + \delta p_{a2} = \mathcal{P}_a + k_\perp - \frac{2k\cdot p_a}{s}xp_a. \tag{4.11}$$

We are now in a position to rewrite Eq. (4.6) in the following way

$$\begin{aligned}
&\mathrm{Tr}\left[N_a(p_b, p_a - k, Q_X)\; x\hat{p}_a\; N_a^+(p_b, p_a - k, Q_X)\; \hat{p}_b\right]_{\Lambda_a} \\
&= \mathrm{Tr}\Bigg[N_a\left(\mathcal{P}_b + \frac{2k\cdot p_a}{sx}p_b, \mathcal{P}_a - \frac{2k\cdot p_a}{sx}p_b, Q_X\right)\left(\hat{\mathcal{P}}_a + \hat{k}_\perp - \frac{2k\cdot p_a}{s}x\hat{p}_a\right) \\
&\qquad \times N_a^+\left(\mathcal{P}_b + \frac{2k\cdot p_a}{sx}p_b, \mathcal{P}_a - \frac{2k\cdot p_a}{sx}p_b, Q_X\right)\left(\hat{\mathcal{P}}_b + \frac{2k\cdot p_a}{sx}\hat{p}_b\right)\Bigg]. \\
&= \left(1 + \frac{2k\cdot p_a}{s}\frac{1-x}{x}\right)|\mathcal{M}|^2(\mathcal{P}_b, \mathcal{P}_a, Q_X) + k_{\perp,\mu}\mathrm{Tr}\left[N_a\gamma^\mu N_a^+\hat{p}_b\right] \\
&- \frac{2k\cdot p_a}{sx}p_{b,\mu}\left(\mathrm{Tr}[N_a^{(1),\mu}xp_a N_a^+ p_b] + \text{c.c.}\right) - \frac{k_{\perp,\mu}k_\perp^\nu}{2}D_\nu^{ax,b}\mathrm{Tr}\left[N_a\gamma^\mu N_a^+\hat{p}_b\right],
\end{aligned} \tag{4.12}$$

where $D^{xa,b}_\mu = x^{-1}\partial/\partial p^\mu_a - \partial/\partial p^\mu_b$, and the quantity $N_a^{(1),\mu}$ is defined through the expansion of the function N_a as follows

$$N_a(q_b - \delta q, q_a + \delta q, Q_X) = N_a(q_b, q_a, Q_x) + \delta q_\mu N_a^{(1),\mu}(q_b, q_a, Q_X). \tag{4.13}$$

Note that in Eq. (4.12) we have written the matrix element as a function of two momenta $\mathcal{P}_a$, $\mathcal{P}_b$. This is possible because both of these momenta are on-shell and the momentum conservation $\mathcal{P}_a + \mathcal{P}_b = Q_X$ is assured. It remains to expand the matrix element squared through the right order in $k_\perp$. The result reads

$$\begin{aligned}|\mathcal{M}|^2(\mathcal{P}_b, \mathcal{P}_a, Q_X) = &\left[1 - \frac{k_\perp^\mu}{2} D^{xa,b}_\mu - \frac{k_\perp^2}{4sx}(p_b^\mu - x p_a^\mu) D^{xa,b}_\mu \right.\\ &\left. + \frac{k_\perp^\mu k_\perp^\nu}{8} D^{xa,b}_\mu D^{xa,b}_\nu \right] |\mathcal{M}|^2(p_b, x p_a, Q_X).\end{aligned} \tag{4.14}$$

As the last step, we need to combine Eqs (4.12, 4.14) and average over the directions of the vector $k_\perp$ in $\mathcal{O}(k_\perp^2)$ terms.[3] It is convenient to write the result as follows

$$\begin{aligned}&\mathrm{Tr}\left[N_a(p_b, p_a - k, Q_X)\, x\hat{p}_a\, N_a^+(p_b, p_a - k, Q_X)\, \hat{p}_b)\right]_{\Lambda_a} = |\mathcal{M}|^2(p_b, x p_a, Q_X)\\ &- \frac{k_\perp^\mu}{2}\left(D^{xa,b}_\mu |\mathcal{M}|^2(p_b, x p_a, Q_X) - 2\mathrm{Tr}\left[N_a \gamma_\mu N_a^+ \hat{p}_b\right]\right) + \frac{2kp_a}{s} W_a(x).\end{aligned} \tag{4.15}$$

In the above equation the function $W_a(x)$ is defined as

$$W_a(x) = -p_{b,\mu} W^\mu_{a1}(x) + (1-x) W_{a2}(x), \tag{4.16}$$

where

$$\begin{aligned}W^\mu_{a1}(x) =& \mathrm{Tr}\left[N_a^{(1),\mu}\, x\hat{p}_a\, N_a^+ \hat{p}_b\right] + \mathrm{c.c.},\\ W_{a2}(x) =& -\frac{1}{x} p_{b,\mu} W^\mu_{a1} + \frac{1}{4x}\left(4 + (p_b^\mu - x p_a^\mu) D^{xa,b}_\mu\right) |\mathcal{M}|^2(p_b, x p_a, Q_X)\\ &- \frac{s}{16} g_\perp^{\mu\nu}\left(D^{xa,b}_\nu D^{xa,b}_\mu\, |\mathcal{M}|^2(p_b, x p_a, Q_X) - 4 D^{xa,b}_\mu\, \mathrm{Tr}\left[N_a \gamma_\nu N_a^+ \hat{p}_b\right]\right).\end{aligned} \tag{4.17}$$

Another quantity that we need to expand is the observable $\mathcal{O}$ because it depends on the transformed momentum of the final-state colorless particles

$$\tilde{Q}_X = \Lambda_a^{-1} Q_X. \tag{4.18}$$

We write

$$\left[\Lambda_a^{-1}\right]^{\mu\nu}(x) = g^{\mu\nu} + b_a^{\mu\nu,\alpha} k_{\perp,\alpha} + \frac{2k\cdot p_a}{s} l_a^{\mu\nu}, \tag{4.19}$$

[3] Note that we do not discard terms linear in $k_\perp$ because such terms may get combined with the collinear expansion of an observable $\mathcal{O}$ producing a non-vanishing result. However, since $\mathcal{O}(k_\perp^2)$ terms that appear from the expansion of the matrix element squared will always be multiplied by $\mathcal{O}(k_\perp^0)$ terms in the observable, we can average over directions of $k_\perp$ in such terms right away.

where

$$b_a{}^{\mu\nu}_{\alpha} = \frac{Q_a^\mu g_\alpha^\nu - Q_a^\nu g_\alpha^\mu}{sx}, \quad l_a^{\mu\nu}(x) = \omega_{ab}^{\mu\nu} + (1-x)\left[\frac{1}{2x}\omega_{ab}^{\mu\nu} + \frac{Q_a^\mu Q_a^\nu}{2sx^2} + \frac{g_\perp^{\mu\nu}}{4x}\right], \tag{4.20}$$

with

$$\omega_{ab}^{\mu\nu} = \frac{p_a^\mu p_b^\nu - p_b^\mu p_a^\nu}{p_a \cdot p_b}. \tag{4.21}$$

We note that in Eq. (4.20), we have replaced $k_\perp^\mu k_\perp^\nu$ with its average value, since there will be no further dependencies on $k_\perp$ when this term is combined with an amplitude squared. Furthermore, we took the four-dimensional limit because such terms do not present soft or collinear singularities. Finally, we note that

$$\lim_{x\to 1} l_a^{\mu\nu}(x) = \omega_{ab}^{\mu\nu}. \tag{4.22}$$

Using the above results, we easily expand the observable $\mathcal{O}$ around the collinear limit

$$\begin{aligned} \mathcal{O}(\Lambda_a^{-1}Q_X) = \Bigg[& 1 + \left(k_\perp^\alpha b_a{}^{\mu\nu}_{\alpha} + \frac{2k\cdot p_a}{s} l_a^{\mu\nu}(x)\right) L_{\mu\nu} \\ & - \frac{1}{2}(1-x)k\cdot p_a \, t_a^{\mu\mu_1,\nu\nu_1} L_{\mu\mu_1} L_{\nu\nu_1}\Bigg]\mathcal{O}(Q_X). \end{aligned} \tag{4.23}$$

In Eq. (4.23), the differential operator $L^{\mu\nu}$ reads

$$L^{\mu\nu} = \sum_{i=1}^m p_i^\nu \frac{\partial}{\partial p_{i,\mu}}, \tag{4.24}$$

and

$$t_a^{\mu\mu_1,\nu\nu_1} = g_\perp^{\alpha\beta} \, b_a{}^{\mu\mu_1}_{\alpha} b_a{}^{\nu\nu_1}_{\beta}, \tag{4.25}$$

is a rank-four tensor.

Having computed all the different terms in Eq. (4.3) to the required order in the collinear expansion, we are in the position to write the different contributions to the cross section as an expansion in ϵ. In this respect, we note that the only divergence in the subleading term comes from the soft singularity of the splitting function, so that it is straightforward to extract it. We write

$$P_{qq}(x)\,(1-x)^{-\epsilon} = -\frac{2}{\epsilon}\delta(1-x) + \bar{P}_{qq}(x) + \mathcal{O}(\epsilon), \tag{4.26}$$

where

$$\bar{P}_{qq}(x) = \frac{2}{(1-x)_+} - (1+x). \tag{4.27}$$

Using this representation, we derive the following result for the next-to-leading power contribution that originates from the first term in Eq. (4.2)

$$
\begin{aligned}
\frac{\mathrm{d}\sigma^{ca,1,\mathrm{NLP}}}{\mathrm{d}\tau} = &-\frac{2[\alpha_s]C_F Q^{1-\epsilon}\mathcal{N}}{s\tau^\epsilon \epsilon}\mathrm{d}\Phi_m(p_a,p_b,P_X)\Big[-p_{b,\mu}W^{\mu}_{a1}(1) \\
&+|\mathcal{M}|^2(p_b,p_a,..)\;\omega^{\mu\nu}_{ab}L_{\mu\nu}\Big]\mathcal{O}(P_X) \\
&+\frac{[\alpha_s]C_F Q\mathcal{N}}{s}\int\limits_0^1 \mathrm{d}x\;\mathrm{d}\Phi_m(xp_a,p_b,P_X)\frac{\bar{P}_{qq}(x)}{x}\Big\{W_a(x) \\
&+\frac{s}{4}(1-x)g^{\rho\alpha}_{\perp}\Big[D^{xa,b}_{\rho}\;|\mathcal{M}|^2(p_b,xp_a,...)-2\mathrm{Tr}\left[N_a\gamma_\rho N^+_a\hat{p}_b\right]\Big]b_{a\alpha}^{\mu\nu}L_{\mu\nu} \\
&+|\mathcal{M}|^2(p_b,xp_a,...)\;l^{\mu\nu}_a(x)L_{\mu\nu} \\
&-\frac{s\;(1-x)}{4}|\mathcal{M}|^2(p_b,xp_a,...)t^{\mu\mu_1,\nu\nu_1}_a L_{\mu\mu_1}L_{\nu\nu_1}\Big\}\;\mathcal{O}(P_X).
\end{aligned}
\tag{4.28}
$$

Next we consider the second term in Eq. (4.2). That term is already subleading in the collinear limit which means that no Lorentz boost needs to be applied to it. Consequently, we can replace $\tilde{Q}_X$ with Q_X everywhere. We also note that the trace in that term gives the squared matrix element of the leading order process with x-dependent kinematics

$$
\mathrm{Tr}\left[N_a(p_b,xp_a,Q_X)x\hat{p}_a N^+_a(p_b,xp_a,Q_X)\hat{p}_b\right] = |\mathcal{M}|^2(p_b,xp_a,Q_X). \tag{4.29}
$$

However, the complication arises because this term is *linearly divergent* in the soft limit; for this reason, it requires additional manipulations. We begin by writing this term explicitly

$$
\frac{\mathrm{d}\sigma^{ca,2}}{\mathrm{d}\tau} = \frac{2(1+\epsilon)[\alpha_s]C_F}{s\tau^\epsilon Q^{-1+\epsilon}}\mathcal{N}\int\limits_0^1 \mathrm{d}x\;\mathrm{d}\Phi^{xa}_m\;\frac{\mathcal{O}(Q_X)}{(1-x)^{2+\epsilon}}\;|\mathcal{M}|^2(p_b,xp_a,Q_X), \tag{4.30}
$$

where $\mathrm{d}\Phi^{xa}_m$ is given in Eq. (3.56).

To extract singularities from this expression, we would like to remove the x-dependence from the phase space. We do this by performing a boost, along the lines of what was done for the discussion of the soft contribution in Sec. 3.1. To this end, we write

$$
0 = xp_a + p_b - Q_X = p_a + p_b - (1-x)p_a - Q_X, \tag{4.31}
$$

and treat $(1-x)p_a$ as the "soft gluon momentum". Similarly to the discussion in Sec. 3.1, we remove it using a Lorentz boost along with the rescaling

$$
p_a + p_b = \lambda^{-1}\Lambda_{ax}\;(xp_a + p_b). \tag{4.32}
$$

Using the fact that boosts do not change squares of four-momenta, it is easy to see that $\lambda = \sqrt{x}$. Following the steps discussed in the section dedicated to soft emissions, we find

$$
\mathrm{d}\Phi^{xa}_m = \mathrm{d}\Phi_m(xp_a,p_b,Q_X) = \mathrm{d}\Phi_m(p_a,p_b,P_X)\;\lambda^{\kappa_m}, \tag{4.33}
$$

where κ_m is defined in Eq. (3.13) and the relation between Q_X and P_X reads

$$Q_X = \lambda \Lambda_{ax}^{-1} P_X. \tag{4.34}$$

We then find

$$\begin{aligned}\frac{\mathrm{d}\sigma^{ca,2}}{\mathrm{d}\tau} = \frac{2(1+\epsilon)[\alpha_s] C_F Q^{1-\epsilon}}{s\tau^{\epsilon}} \mathcal{N} \int_0^1 \mathrm{d}x \; \mathrm{d}\Phi_m^{ab} \; \lambda^{\kappa_m} (1-x)^{-\epsilon-2} \\ \times \mathcal{O}(\lambda \Lambda_{ax}^{-1} P_X) \; |\mathcal{M}|^2 (p_b, x p_a, \lambda \Lambda_{ax}^{-1} P_X).\end{aligned} \tag{4.35}$$

To extract singularities from this expression and to regulate them, we require the expansion of the matrix element squared, the λ^{κ_m} factor and the observable $\mathcal{O}$ through first order in $(1-x)$. The boost operator, as well as its expansion in $(1-x)$ is given in Eqs (A.12, A.13). Using those equations and the expansion of λ around $x = 1$, $\lambda = 1-(1-x)/2+\mathcal{O}((1-x)^2)$, we obtain

$$\lambda (\Lambda_{ax}^{-1})^{\mu\nu} = g^{\mu\nu} - \frac{(1-x)}{2} \left(g^{\mu\nu} + \omega_{ab}^{\mu\nu} \right) + \mathcal{O}((1-x)^2). \tag{4.36}$$

We then find

$$\begin{aligned}&|\mathcal{M}|^2 \left(p_b, x p_a, \lambda \Lambda_{ax}^{-1} P_X \right) = \\ &\left[1 - \frac{(1-x)}{2} \left(2 p_a^{\mu} \frac{\partial}{\partial p_{a,\mu}} + \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma} \right) L_{\rho\sigma} \right) \right] |\mathcal{M}|^2 \left(p_b, p_a, P_X \right) + \mathcal{O}((1-x)^2),\end{aligned} \tag{4.37}$$

and

$$\mathcal{O}(\lambda \Lambda_{ax}^{-1} P_X) = \left[1 - \frac{(1-x)}{2} \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma} \right) L_{\rho\sigma} \right] \mathcal{O}(P_X) + \mathcal{O}((1-x)^2). \tag{4.38}$$

It is now convenient to define a new function to represent the subtracted expression

$$\begin{aligned}&W_3^{(a)}(x, p_b, p_a, P_X, \mathcal{O}(P_X)) = \lambda^{\kappa_m} \mathcal{O}(\lambda \Lambda_{ax}^{-1} P_X) |\mathcal{M}|^2 (p_b, x p_a, \lambda \Lambda_{ax}^{-1} P_X) - \\ &\left[1 - \frac{(1-x)}{2} \left(\kappa_m + 2 p_a^{\mu} \frac{\partial}{\partial p_a^{\mu}} + \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma} \right) L_{\rho\sigma} \right) \right] |\mathcal{M}|^2 \left(p_b, p_a, P_X \right) \mathcal{O}(P_X),\end{aligned} \tag{4.39}$$

where we have assumed that the observable $\mathcal{O}$ is independent of the momentum p_a. It follows from Eqs (4.37, 4.38) that in the soft limit $W_3^{(a)}$ vanishes as $\mathcal{O}((1-x)^2)$ and, therefore, can be integrated with the $1/(1-x)^2$ factor which appears in the cross section computation, c.f. Eq. (4.30).

We can now write the complete result that originates from the second term in Eq. (4.2) in the following way[4]

$$
\begin{aligned}
\frac{\mathrm{d}\sigma^{ca,2,\mathrm{NLP}}}{\mathrm{d}\tau} &= \frac{(1+\epsilon)[\alpha_s]C_F Q^{1-\epsilon}}{s\tau^\epsilon\epsilon}\mathcal{N}\,\mathrm{d}\Phi_m(p_a,p_b,P_X)\Bigg[-\frac{2\epsilon}{1+\epsilon}+\kappa_m \\
&+\left(2p_a^\mu\frac{\partial}{\partial p_a^\mu}+\left(g^{\rho\sigma}+\omega_{ab}^{\rho\sigma}\right)L_{\rho\sigma}\right)\Bigg]\mathcal{O}(P_X)\,|\mathcal{M}|^2(p_b,p_a,P_X) \\
&+\frac{2[\alpha_s]C_F Q}{s}\mathcal{N}\int_0^1 \mathrm{d}x\,\mathrm{d}\Phi_m(p_a,p_b,P_X)\frac{W_3^{(a)}(x,p_b,p_a,P_X,\mathcal{O}(P_X))}{(1-x)^2} \\
&= \frac{[\alpha_s]C_F Q^{1-\epsilon}}{s\tau^\epsilon\epsilon}\mathcal{N}\,\mathrm{d}\Phi_m(p_a,p_b,P_X)\Bigg[-2\epsilon+\kappa_m \\
&+\left(2p_a^\mu\frac{\partial}{\partial p_a^\mu}+\left(g^{\rho\sigma}+\omega_{ab}^{\rho\sigma}\right)L_{\rho\sigma}\right)\Bigg]\mathcal{O}(P_X)\,|\mathcal{M}|^2(p_b,p_a,P_X) \\
&-\frac{[\alpha_s]C_F Q}{s}\mathcal{N}\int_0^1 \mathrm{d}x\,\mathrm{d}\Phi_m(xp_a,p_b,P_X)\,\frac{1}{(1-x)_+} \\
&\times\left(\kappa_m+2p_a^\mu\frac{\partial}{\partial p_a^\mu}+\left(g^{\rho\sigma}+\omega_{ab}^{\rho\sigma}\right)L_{\rho\sigma}\right)\mathcal{O}(P_X)\,|\mathcal{M}|^2(p_b,xp_a,P_X),
\end{aligned}
\tag{4.40}
$$

where in the first term integration over x has been performed and the $W_3^{(a)}$ term was rewritten in terms of the plus distribution.

Finally, we need to consider the third term in Eq. (4.2). This term is also subleading in the collinear expansion which means that no boost is required. Its contribution to the cross section reads

$$
\begin{aligned}
\frac{\mathrm{d}\sigma^{ca,3}}{\mathrm{d}\tau} &= -\frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^\epsilon}\mathcal{N}\int_0^1 \mathrm{d}x\,\frac{\mathrm{d}\Phi_m(xp_a,p_b,P_X)\,\mathcal{O}(P_X)}{(1-x)^{1+\epsilon}} \\
&\times\frac{4p_b^\nu}{s}\left(\mathrm{Tr}\left[N_a\hat{p}_a R^+_{\mathrm{fin},\nu}\hat{p}_b\right]+\mathrm{c.c.}\right).
\end{aligned}
\tag{4.41}
$$

We then replace $(1-x)^{-1-\epsilon}$ with the plus distribution in the standard way and find

$$
\begin{aligned}
\frac{\mathrm{d}\sigma^{ca,3}}{\mathrm{d}\tau} &= -\frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^\epsilon}\mathcal{N}\int_0^1 \mathrm{d}x\,\mathrm{d}\Phi_m(xp_a,p_b,P_X)\left[-\frac{1}{\epsilon}\delta(1-x)+\frac{1}{(1-x)_+}\right] \\
&\times\,\mathcal{O}(P_X)\,\frac{4p_b^\nu}{s}\left(\mathrm{Tr}\left[N_a\hat{p}_a R^+_{\mathrm{fin},\nu}\hat{p}_b\right]+\mathrm{c.c.}\right).
\end{aligned}
\tag{4.42}
$$

The $1/\epsilon$ divergent term requires $R_{\mathrm{fin},\nu}(p_b,p_a,k,P_X)$, with $k=(1-x)p_a$ at $x=1$ (i.e., the soft limit). We can obtain it using gauge invariance. From the transversality of the gluon emission amplitude it follows that

$$
\bar{v}_b\left(N(p_b-k,p_a,P_X)-N(p_b,p_a-k,P_X)+k^\mu R_{\mathrm{fin},\mu}(p_b,p_a,k,P_X)\right)u_a=0. \tag{4.43}
$$

[4]We provide a detailed derivation of this formula in Appendix C.

We are interested in the soft $k \to 0$ limit. Since $R_{\text{fin},\nu}(p_b, p_a, k, P_X)$ in that equation is multiplied with k_ν, we can replace it with $R_{\text{fin},\nu}(p_b, p_a, 0, P_X)$. The difference of the two Green's functions can be computed using the discussion in Sec. 5, where it is explained how such an expansion should be constructed. In particular, if we compute it starting from the incoming quark momentum and do not let momentum k flow into the colorless final state, then we have to replace p_b with $P_X + k - p_a$ in the functions N in Eq. (4.43). It follows that

$$N(p_b - k, p_a, P_X) - N(p_b, p_a - k, P_X) = k^\mu N^{(1),\mu}(p_b, p_a, P_X). \tag{4.44}$$

Employing this result in Eq. (4.43) and making use of the fact that it is valid for small, but otherwise arbitrary vectors k^μ, we find

$$p_b^\nu \bar{v}_b R_{\text{fin},\nu} u_a \Big|_{k\to 0} = -p_{b,\nu} \bar{v}_b N_a^{(1),\nu} u_a \Big|_{k\to 0}. \tag{4.45}$$

Hence, we obtain

$$\begin{aligned}\frac{\mathrm{d}\sigma^{ca,3,\mathrm{NLP}}}{\mathrm{d}\tau} &= -\frac{[\alpha_s] C_F Q^{1-\epsilon}}{2\tau^\epsilon \epsilon} \mathcal{N}\, \mathrm{d}\Phi_m(p_a, p_b, P_X)\, \mathcal{O}(P_X) \\ &\times \frac{4p_b^\nu}{s} \left(\mathrm{Tr}\left[N_a \hat{p}_a N_\nu^{(1),+} \hat{p}_b \right]_{x=1} + \text{c.c.} \right) \\ &- \frac{[\alpha_s] C_F Q}{2} \mathcal{N} \int\limits_0^1 \frac{\mathrm{d}x}{(1-x)_+}\, \mathrm{d}\Phi_m(x p_a, p_b, P_X)\, \mathcal{O}(P_X) \\ &\times \frac{4p_b^\nu}{s} \left(\mathrm{Tr}\left[N_a \hat{p}_a R^+_{\text{fin},\nu} \hat{p}_b \right] + \text{c.c.} \right).\end{aligned} \tag{4.46}$$

We note that the $\mathcal{O}(1/\epsilon)$ term in Eq. (4.46) is exactly canceled by the first term in Eq. (4.28).

4.2 The second collinear region: $\vec{k}||\vec{p}_b$

We continue with the contribution of the collinear region where the gluon is emitted along the direction of the incoming anti-quark. The differential cross section in this case reads

$$\begin{aligned}\frac{\mathrm{d}\sigma^{cb}}{\mathrm{d}\tau} &= \frac{[\alpha_s] C_F Q^{1-\epsilon}}{2\tau^{1+\epsilon}} \mathcal{N} \int\limits_0^1 \mathrm{d}x \mathrm{d}\Phi_m(p_a, x p_b, \tilde{Q}_X) \left[\mathrm{d}\Omega_k^{(d-2)} \right] (1-x)^{-\epsilon} \\ &\times \left(1 + \frac{\epsilon Q \tau}{s(1-x)} \right) \mathcal{O}(\tilde{Q}_X)\, \tau\, F_b,\end{aligned} \tag{4.47}$$

where $\tilde{Q}_X = \Lambda_b^{-1} Q_X$, F_b is given in Eq. (3.99) and the momenta p_a, p_b, k that appear there should be boosted. Similar to the collinear case where $\vec{k}||\vec{p}_a$, it is convenient to write, with the required accuracy,

$$\begin{aligned}\left(1 + \frac{\epsilon Q\tau}{s(1-x)} \right) F_b &= \frac{2P_{qq}(x)}{2p_b \cdot k} \mathrm{Tr}\left[N_b^+ \hat{p}_b N_b \hat{p}_a \right] + \frac{4(1+\epsilon)}{s(1-x)^2} \mathrm{Tr}\left[N_b^+ \hat{x} p_b N_b \hat{p}_a \right] \\ &+ \frac{4p_a^\nu}{s(1-x)} \mathrm{Tr}\left[N_b^+ \hat{p}_b R_{\text{fin},\nu} \hat{p}_a \right] + \text{c.c.} + F_{b,\text{reg}},\end{aligned} \tag{4.48}$$

where the function $F_{b,\text{reg}}$ is free from both soft and collinear singularities.

In principle, the discussion of the $\vec{k}||\vec{p}_b$ collinear limit follows very closely the discussion in the previous section. Nevertheless, we decided to repeat it one more time for the sake of clarity.

We begin with the first term on the right-hand side of Eq. (4.48). We note that momenta that appear in that term still have to be boosted with the matrix Λ_b. Hence, we write

$$\begin{aligned}\frac{\mathrm{d}\sigma^{cb,1}}{\mathrm{d}\tau} &= \frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^{1+\epsilon}}\mathcal{N}\int\limits_0^1 \mathrm{d}x\ \mathrm{d}\Phi_m(p_a, xp_b, Q_X)[\mathrm{d}\Omega_k^{(d-2)}](1-x)^{-\epsilon}\mathcal{O}(\Lambda_b^{-1}Q_X) \\ &\times\ \tau\ \frac{2P_{qq}(x)}{2p_b\cdot k\ x}\ \mathrm{Tr}\left[N_b^{+}(p_b-k, p_a, Q_X)\ x\hat{p}_b\ N_b(p_b-k, p_a, Q_X)\ \hat{p}_a\right]_{\Lambda_b},\end{aligned} \tag{4.49}$$

where the subscript of the trace function indicates that momenta p_a, p_b and k should be boosted with the matrix Λ_b.

Following the discussion of the $\vec{k}||\vec{p}_a$ case, we first show the result for the leading power contribution that is obtained by setting $\Lambda_b \to 1$. We find

$$\begin{aligned}\frac{\mathrm{d}\sigma^{cb,1,\mathrm{LP}}}{\mathrm{d}\tau} &= \frac{[\alpha_s]C_F Q^{-\epsilon}}{\tau^{1+\epsilon}}\mathcal{N}\int\limits_0^1 \mathrm{d}x\ \mathrm{d}\Phi_m(p_a, xp_b, Q_X)\ \frac{P_{qq}(x)}{x(1-x)^{\epsilon}} \\ &\times \mathcal{O}(Q_X)\ |\mathcal{M}(xp_b, p_a, P_X)|^2,\end{aligned} \tag{4.50}$$

where we have used $2p_b k = \tau Q$, and the fact that in the collinear limit

$$\mathrm{Tr}\left[N_b^{+}(p_b-k, p_a, Q_X)\ x\hat{p}_b\ N_b(p_b-k, p_a, Q_X)\ \hat{p}_a\right]_{\Lambda_b} \to |\mathcal{M}(xp_b, p_a p_b, Q_X)|^2. \tag{4.51}$$

There is a soft singularity present in the splitting function P_{qq}, but it is straightforward to extract it.

Computing the subleading terms in the τ-expansion requires more effort. We start by showing formulas for the trace

$$\begin{aligned}&\mathrm{Tr}\left[N_b^{+}(p_b-k, p_a, Q_X)\ x\hat{p}_b\ N_b(p_b-k, p_a, Q_X)\ \hat{p}_a\right]_{\Lambda_b} \\ &= \mathrm{Tr}\big[N_b^{+}(xp_b-\delta p_{b1}, p_a+\delta p_{b1}, Q_X)(x\hat{p}_b+\delta\hat{p}_{b2}) \\ &\qquad \times N_b(xp_b-\delta p_{b1}, p_a+\delta p_{b1}, Q_X)(\hat{p}_a+\hat{\delta}p_{b1})\big],\end{aligned} \tag{4.52}$$

where

$$\begin{aligned}\delta p_{b1} &= \frac{k_\perp}{2} + \frac{2k\cdot p_b}{s}\left(p_a+(1-x)\pi_{b1}\right), \\ \delta p_{b2} &= \frac{k_\perp}{2} - \frac{2p_b\cdot k}{s}\left(p_b-(1-x)\pi_{b2}\right),\end{aligned} \tag{4.53}$$

and

$$\pi_{b1} = \frac{p_b}{4} + \frac{3p_a}{4x}, \quad \pi_{b2} = \frac{3p_b}{4} + \frac{p_a}{4x}. \tag{4.54}$$

We note that, thanks to the explicit $(1-x)$ factors in front of vectors $\pi_{b1,b2}$, they do not contribute to soft singularities. Proceeding as in the previous section, we find

$$\begin{aligned}&\mathrm{Tr}\left[N_b^{+}(p_b-k, p_a, Q_X)\ x\hat{p}_b\ N_b(p_b-k, p_a, Q_X)\ \hat{p}_a)\right]_{\Lambda_b} = |\mathcal{M}|^2(xp_b, p_a, Q_X) \\ &- \frac{k_\perp^\mu}{2}\left(D_\mu^{xb,a}\left[|\mathcal{M}|^2(xp_b, p_a, Q_X)\right] - 2\mathrm{Tr}\left[N_b^{+}\gamma_\mu N_b\hat{p}_a\right]\right) + \frac{2k\cdot p_b}{s}W_b(x),\end{aligned} \tag{4.55}$$

where $D_\mu^{xb,a} = x^{-1}\partial/\partial p_b^\mu - \partial/\partial p_a{}^\mu$ and we defined

$$W_b(x) = -p_{a,\mu}\, W_{b1}^\mu(x) + (1-x)W_{b2}(x), \tag{4.56}$$

with

$$\begin{aligned} W_{b1}^\mu &= \mathrm{Tr}\left[N_b^{(1),\mu,+}\, x\hat{p}_b\, N_b\, \hat{p}_a\right] + \text{c.c.}, \\ W_{b2} &= -\frac{1}{x}p_{a,\mu}W_{b1}^\mu + \frac{1}{4x}\left(4 + (p_a^\mu - xp_b^\mu)D_\mu^{xb,a}\right)|\mathcal{M}|^2(xp_b, p_a, Q_X) \\ &\quad - \frac{s}{16}g_\perp^{\mu\nu}\left(D_\nu^{xb,a}D_\mu^{xb,a}\,|\mathcal{M}|^2(xp_b, p_a, Q_X) - 4D_\mu^{xb,a}\,\mathrm{Tr}\left[N_b^+\gamma_\nu N_b\hat{p}_a\right]\right). \end{aligned} \tag{4.57}$$

The function $N_b^{(1),\mu}$ in the above equation is defined in Appendix B.

We also need to expand the observable $\mathcal{O}$ since it depends on the transformed momenta $\tilde{Q}_X = \Lambda_b^{-1}Q_X$. We write

$$\Lambda_b^{-1\mu\nu}(x) = g^{\mu\nu} + b_b^{\mu\nu,\alpha}k_{\perp,\alpha} + \frac{2k\cdot p_b}{s}l_b^{\mu\nu}, \tag{4.58}$$

where

$$b_{b\alpha}^{\mu\nu} = \frac{Q_b^\mu g_\alpha^\nu - Q_b^\nu g_\alpha^\mu}{sx}, \quad l_b^{\mu\nu}(x) = \omega_{ba}^{\mu\nu} + (1-x)\left[\frac{1}{2x}\omega_{ba}^{\mu\nu} + \frac{Q_b^\mu Q_b^\nu}{2sx^2} + \frac{1}{4x}g_\perp^{\mu\nu}\right], \tag{4.59}$$

and we have replaced $k_\perp^\mu k_\perp^\nu$ with its average value, since there will be no further dependencies on $k_\perp$ when the contribution of this term is taken into account. We also took the four-dimensional limit because such terms do not contribute to soft and collinear singularities. Finally, we note that

$$\lim_{x\to 1} l_b^{\mu\nu}(x) = \omega_{ba}^{\mu\nu}. \tag{4.60}$$

Using the above results, we find

$$\begin{aligned} \mathcal{O}(\Lambda_b^{-1}Q_X) = \Big[1 &+ \left(k_\perp^\alpha b_{b\alpha}^{\mu\nu} + \frac{2k\cdot p_b}{s}l_b^{\mu\nu}(x)\right)L_{\mu\nu} \\ &- \frac{1}{2}(1-x)k\cdot p_b\, t_b^{\mu\mu_1,\nu\nu_1}L_{\mu\mu_1}L_{\nu\nu_1}\Big]\mathcal{O}(Q_X), \end{aligned} \tag{4.61}$$

where the differential operator $L^{\mu\nu}$ is given in Eq. (4.24) and the tensor $t_b^{\mu\mu_1,\nu\nu_1}$ reads

$$t_b^{\mu\mu_1,\nu\nu_1} = g_\perp^{\alpha\beta}\, b_{b\alpha}^{\mu\mu_1}b_{b\beta}^{\nu\nu_1}. \tag{4.62}$$

Using these results and following the discussion of the $\vec{k}||\vec{p}_a$ case, we obtain

$$\begin{aligned} \frac{\mathrm{d}\sigma^{cb,1,\mathrm{NLP}}}{\mathrm{d}\tau} = &\frac{2[\alpha_s]C_F Q^{1-\epsilon}\mathcal{N}}{s\tau^\epsilon\epsilon}\mathrm{d}\Phi_m(p_a,p_b,P_X)\Big[p_{a,\mu}W_{b1}^\mu(1) \\ &- |\mathcal{M}|^2(p_b,p_a,...)\,\omega_{ba}^{\mu\nu}L_{\mu\nu}\Big]\mathcal{O}(P_X) \\ &+ \frac{[\alpha_s]C_F Q\mathcal{N}}{s}\int_0^1 \mathrm{d}x\,\mathrm{d}\Phi_m(p_a,xp_b,P_X)\frac{\bar{P}_{qq}(x)}{x}\Big\{W_b(x) \\ &+ \frac{s}{4}(1-x)g_\perp^{\rho\alpha}\Big[D_\rho^{xb,a}|\mathcal{M}|^2(xp_b,p_a,..) \\ &- 2\mathrm{Tr}\left[N_b^+\gamma_\rho N_b\hat{p}_a\right]\Big]b_{b\alpha}^{\mu\nu}L_{\mu\nu} + |\mathcal{M}|^2(xp_b,p_a,..)\,l_b^{\mu\nu}(x)L_{\mu\nu} \\ &- \frac{s(1-x)}{4}|\mathcal{M}|^2(xp_b,p_a,...)t_b^{\mu\mu_1,\nu\nu_1}L_{\mu\mu_1}L_{\nu\nu_1}\Big\}\mathcal{O}(P_X). \end{aligned} \tag{4.63}$$

Next we consider the second term in Eq. (4.48). The trace evaluates to

$$\mathrm{Tr}\left[N_b^+(xp_b,p_a,Q_X)\; x\hat{p}_b\; N_b^+(xp_b,p_a,Q_X)\;\hat{p}_a\right] = |\mathcal{M}|^2(xp_b,p_a,Q_X), \tag{4.64}$$

and, following steps described in the previous section, we find

$$\frac{\mathrm{d}\sigma^{cb,2}}{\mathrm{d}\tau} = \frac{2(1+\epsilon)[\alpha_s]C_F}{s\tau^\epsilon Q^{-1+\epsilon}}\mathcal{N}\int_0^1 \mathrm{d}x\;\mathrm{d}\Phi_m(p_a,xp_b,Q_X)\frac{\mathcal{O}(Q_X)}{(1-x)^{2+\epsilon}}\;|\mathcal{M}|^2(xp_b,p_a,Q_X). \tag{4.65}$$

Extracting the singularity at $x=1$, as discussed in the previous section, we obtain

$$\begin{aligned}\frac{\mathrm{d}\sigma^{cb,2,\mathrm{NLP}}}{\mathrm{d}\tau} &= \frac{[\alpha_s]C_F Q^{1-\epsilon}}{s\tau^\epsilon\epsilon}\mathcal{N}\;\mathrm{d}\Phi_m(p_a,p_b,P_X)\Big[-2\epsilon+\kappa_m \\ &+\left(2p_b^\mu\frac{\partial}{\partial p_b^\mu}+\left(g^{\rho\sigma}+\omega_{ba}^{\rho\sigma}\right)L_{\rho\sigma}\right)\Big]\mathcal{O}(P_X)\;|\mathcal{M}|^2(p_b,p_a,P_X) \\ &-\frac{[\alpha_s]C_F Q}{s}\mathcal{N}\int \mathrm{d}x\;\mathrm{d}\Phi_m(p_a,xp_b,P_X)\;\frac{1}{(1-x)_+} \\ &\times\left(\kappa_m+2p_b^\mu\frac{\partial}{\partial p_b^\mu}+\left(g^{\rho\sigma}+\omega_{ba}^{\rho\sigma}\right)L_{\rho\sigma}\right)\mathcal{O}(P_X)\;|\mathcal{M}|^2(xp_b,p_a,P_X).\end{aligned} \tag{4.66}$$

Finally, we need to consider the third term in Eq. (4.48). This term is also subleading in the collinear expansion which means that no boost is required. The contribution to the cross section reads

$$\begin{aligned}\frac{\mathrm{d}\sigma^{cb,3}}{\mathrm{d}\tau} &= \frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^\epsilon}\mathcal{N}\int_0^1 \mathrm{d}x\;\frac{\mathrm{d}\Phi_m(p_a,xp_b,P_X)}{(1-x)^{1+\epsilon}}\mathcal{O}(P_X) \\ &\times\frac{4p_a^\nu}{s}\left(\mathrm{Tr}\left[N_b^+\hat{p}_b R_{\mathrm{fin},\nu}\hat{p}_a\right]+\mathrm{c.c.}\right).\end{aligned} \tag{4.67}$$

We then replace $(1-x)^{-\epsilon-1}$ with the plus distribution in the standard way and find

$$\begin{aligned}\frac{\mathrm{d}\sigma^{cb,3}}{\mathrm{d}\tau} &= \frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^\epsilon}\mathcal{N}\int_0^1 \mathrm{d}x\;\mathrm{d}\Phi_m(p_a,xp_b,P_X)\left[-\frac{1}{\epsilon}\delta(1-x)+\frac{1}{(1-x)_+}\right] \\ &\times\;\mathcal{O}(P_X)\;\frac{4p_a^\nu}{s}\left(\mathrm{Tr}\left[N_b^+\hat{p}_b R_{\mathrm{fin},\nu}\hat{p}_a\right]+\mathrm{c.c.}\right).\end{aligned} \tag{4.68}$$

We compute the $x=1$ contribution following the discussion in the previous section and find

$$\begin{aligned}\frac{\mathrm{d}\sigma^{cb,3,\mathrm{NLP}}}{\mathrm{d}\tau} &= \frac{[\alpha_s]C_F Q^{1-\epsilon}}{2\tau^\epsilon\epsilon}\mathcal{N}\;\mathrm{d}\Phi_m(p_a,p_b,P_X)\;\mathcal{O}(P_X) \\ &\times\frac{4p_a^\nu}{s}\left(\mathrm{Tr}\left[N^+\hat{p}_b N_\nu^{(1)}\hat{p}_a\right]_{x=1}+\mathrm{c.c.}\right) \\ &+\frac{[\alpha_s]C_F Q}{2}\mathcal{N}\int_0^1\frac{\mathrm{d}x}{(1-x)_+}\;\mathrm{d}\Phi_m(p_a,xp_b,P_X)\;\mathcal{O}(P_X) \\ &\times\frac{4p_a^\nu}{s}\left(\mathrm{Tr}\left[N_b^+\hat{p}_b R_{\mathrm{fin},\nu}\hat{p}_a\right]+\mathrm{c.c.}\right).\end{aligned} \tag{4.69}$$

The $1/\epsilon$ pole in the first term of the above equation is canceled by the first term in Eq. (4.63).

4.3 The final result for the next-to-leading power correction

In this section, we combine all the different contributions, and derive the final formula for the production of an arbitrary colorless final state X in the $q\bar{q} \to X$ process at next-to-leading power in the zero-jettiness expansion at NLO QCD. We need to account for the soft and two collinear contributions, presented in Eqs (3.30, 3.98, 3.99), using further simplifications of the last two equations (the collinear contributions) discussed in Secs. 3.2 and 3.3, and in Appendix B.

Using the above results, it is straightforward to check that, at next-to-leading power, all $1/\epsilon$ poles cancel after summing soft and collinear contributions. However, the result contains a $\ln\tau$-enhanced term, which appears as a consequence of the mismatch of the ϵ-dependent exponents of τ in the soft and collinear contributions. We note that the $1/\epsilon$ poles proportional to the tensor $\omega_{ab}^{\mu\nu}$ cancel when taking the sum of both collinear regions.

We write the next-to-leading power contribution in the expansion of the $q\bar{q} \to X$ cross section in the zero-jettiness as the sum of three terms

$$\frac{\mathrm{d}\sigma^{\mathrm{NLP}}}{\mathrm{d}\tau} = \frac{[\alpha_s] C_F Q}{s} \mathcal{N} \left\{ 2 \left[\ln\left(\frac{Q\tau}{s}\right) + 1 \right] \mathrm{C}^{\mathrm{NLP},s} + \mathrm{C}^{\mathrm{NLP},a} + \mathrm{C}^{\mathrm{NLP},b} \right\}. \tag{4.70}$$

The finite remnant of the soft and soft-collinear contributions read

$$\mathrm{C}^{\mathrm{NLP},s} = \int \mathrm{d}\Phi_m(p_a, p_b, P_X) \left(\kappa_m + \sum_{i \in L_f} p_i^\mu \frac{\partial}{\partial p_i^\mu} \right) |\mathcal{M}(p_b, p_a, P_X)|^2 \, \mathcal{O}(P_X), \tag{4.71}$$

where the sum extends over all particles in the process.[5] We note that for amplitudes with *massless* particles *only*, the following equation holds

$$\left(\kappa_m + \sum_{i \in L_f} p_i^\mu \frac{\partial}{\partial p_i^\mu} \right) |\mathcal{M}(p_b, p_a, P_X)|^2 = 0. \tag{4.72}$$

This result follows from the fact that the mass dimension of the amplitude squared with two initial-state and m final-state particles is $(-\kappa_m)$ and that the derivative operator in the above equation probes the mass dimension of the amplitude squared in the massless case.

The expressions for the two collinear remnants $C^{\mathrm{NLP},a(b)}$ are more complex. The $\vec{k}||\vec{p}_a$ contribution reads

$$\mathrm{C}^{\mathrm{NLP},a} = -2 \int \mathrm{d}\Phi_m \, |\mathcal{M}(p_b, p_a, P_X)|^2 \, \mathcal{O}(P_X) + \int \mathrm{d}x \, \mathrm{d}\Phi_m^{xa} \left\{ \frac{\bar{P}_{qq}(x)}{x} \left[W_a(x) \right.\right.$$
$$\left. + \frac{s}{4}(1-x) g_\perp^{\rho\alpha} \left(D_\rho^{xa,b} \, |\mathcal{M}|^2(p_b, xp_a, ...) - 2\mathrm{Tr}\left[N_a \gamma_\rho N_a^+ \hat{p}_b \right] \right) b_{a\alpha}^{\mu\nu} L_{\mu\nu} \right.$$

[5] We remind the reader that the validity of Eq. (4.71) requires that the observable $\mathcal{O}$ is independent of momenta p_a, p_b.

$$
\begin{aligned}
&+ |\mathcal{M}|^2(p_b, xp_a, ...)\, l_a^{\mu\nu}(x) L_{\mu\nu} - \frac{s\,(1-x)}{4} |\mathcal{M}|^2(p_b, xp_a, ...) t_a^{\mu\mu_1,\nu\nu_1} L_{\mu\mu_1} L_{\nu\nu_1} \Bigg] \\
&- \frac{1}{(1-x)_+} \left(\kappa_m + 2p_a^\mu \frac{\partial}{\partial p_a^\mu} + \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma}\right) L_{\rho\sigma} \right) |\mathcal{M}|^2(p_b, xp_a, ...) \\
&- \frac{2p_b^\nu}{(1-x)_+} \left(\mathrm{Tr}\left[N_a \hat{p}_a R^+_{\mathrm{fin},\nu} \hat{p}_b \right] + \mathrm{c.c.} \right) + F_{\mathrm{fin},a} \\
&+ \frac{s}{4}(1-x) g_\perp^{\alpha\beta} \Bigg[-2\mathrm{Tr}\left[N_a \gamma_\beta N_a^+ \hat{p}_b \right] \\
&+ \mathrm{Tr}\left[N_a \gamma_\beta \gamma_\rho \hat{p}_a \left(R_{\mathrm{fin}}^{\rho,+} + \frac{N_b^+ (\hat{p}_b - (1-x)\hat{p}_a)\gamma^\rho}{(1-x)s} \right) \hat{p}_b \right] + \mathrm{c.c.} \\
&+ \frac{2x}{1-x} \mathrm{Tr}\left[N_a \hat{p}_a \left(R^+_{\mathrm{fin},\beta} - \frac{N_b^+ \hat{p}_a \gamma_\beta}{s} \right) \hat{p}_b \right] + \mathrm{c.c.} \Bigg] b_{a\,\alpha}^{\ \mu\nu} L_{\mu\nu} \Bigg\} \mathcal{O}(P_X),
\end{aligned}
\tag{4.73}
$$

where

$$
\mathrm{d}\Phi_m = \mathrm{d}\Phi(p_a, p_b, P_X), \qquad \mathrm{d}\Phi_m^{xa} = \mathrm{d}\Phi(xp_a, p_b, P_X), \tag{4.74}
$$

$W_a(x)$ is defined in Eq. (4.16), $F_{\mathrm{fin},a}$ can be found in Appendix B, and the functions N_a, N_b and R^μ_{fin} appearing in the above expression should be evaluated with the following arguments

$$
\begin{aligned}
N_a &= N_a(p_b, xp_a, P_X), \\
N_b &= N_b(p_b - (1-x)p_a, p_a, P_X), \\
R^\mu_{\mathrm{fin}} &= R^\mu_{\mathrm{fin}}(p_b, p_a, (1-x)p_a, P_X).
\end{aligned}
\tag{4.75}
$$

We note that many terms in Eq. (4.73) involve derivatives of the observable $\mathcal{O}$; these terms are written for a generic case and may simplify significantly if a definite observable is considered. We will see examples of this in what follows.

The second collinear contribution with $\vec{k}||\vec{p}_b$, that we referred to as $\mathrm{C}^{\mathrm{NLP},b}$ above, can be obtained from Eq. (4.73) by making the following replacements

$$
p_a \leftrightarrow p_b, \qquad N_a \leftrightarrow -N_b^+, \tag{4.76}
$$

which also implies replacing the following quantities

$$
\begin{gathered}
W_a(x) \to W_b(x), \quad D^{xa,b} \to D^{xb,a}, \quad b_a \to b_b, \quad l_a \to l_b, \quad t_a \to t_b, \\
\omega_{ab}^{\rho\sigma} \to \omega_{ba}^{\rho\sigma}, \quad F_{\mathrm{fin},a} \to F_{\mathrm{fin},b}, \quad \mathrm{d}\Phi_m^{xa} \to \mathrm{d}\Phi_m^{xb}.
\end{gathered}
\tag{4.77}
$$

Furthermore, the Green's functions that would appear in $C^{\mathrm{NLP,b}}$ will have to be evaluated for the following arguments

$$
\begin{aligned}
N_a &= N_a(p_b, p_a - (1-x)p_b, P_X), \\
N_b &= N_b(xp_b, p_a, P_X), \\
R^\mu_{\mathrm{fin}} &= R^\mu_{\mathrm{fin}}(p_b, p_a, (1-x)p_b, P_X).
\end{aligned}
\tag{4.78}
$$

Several terms in the collinear contributions $C^{\text{NLP},a(b)}$ can be simplified further although we do not try to do this systematically. As an example, consider the term

$$\omega_{ab}^{\mu\nu} L_{\mu\nu} |M|^2(p_b, p_a, P_X). \tag{4.79}$$

in Eq. (4.73). Since $\omega_{ab}^{\mu\nu}$ is an antisymmetric tensor, we can think of it as part of an infinitesimal Lorentz transformation

$$[\Lambda_\delta]^{\mu\nu} = g^{\mu\nu} + \delta\omega_{ab}^{\mu\nu} + \mathcal{O}(\delta^2). \tag{4.80}$$

Because the matrix element squared is invariant under Lorentz transformations, we can write

$$|\mathcal{M}|^2(p_b, p_a, \Lambda_\delta P_X) = |\mathcal{M}|^2(\Lambda_\delta^{-1} p_b, \Lambda_\delta^{-1} p_a, P_X). \tag{4.81}$$

The inverse infinitesimal transformation is obtained by replacing $\delta \to -\delta$ in Eq. (4.80). Finally, expanding Eq. (4.81) in δ, we find

$$\omega_{ab}^{\mu\nu} L_{\mu\nu} |\mathcal{M}|^2(p_b, p_a, P_X) = -\left(p_a^\mu \frac{\partial}{\partial p_a^\mu} - p_b^\mu \frac{\partial}{\partial p_b^\mu} \right) |\mathcal{M}|^2(p_b, p_a, P_X), \tag{4.82}$$

which might be helpful for calculating this quantity for complex physical processes.

5 How to compute Green's functions that appear in the formula for power corrections

The general formula for subleading zero-jettiness corrections, derived in the previous section, is complicated because it involves Green's functions whose relation to amplitudes is obscure. Thus, for such a formula to be useful, one has to understand how the relevant Green's functions can be calculated. It turns out that methods developed for computing high-multiplicity QCD amplitudes more than thirty years ago [39] are suitable for this purpose.[6]

Although we are certain that the discussion in this section can be made fully general, for the sake of definiteness, we consider the case when the state X consists of N photons. The observable function $\mathcal{O}(P_X)$ is chosen in such a way that photons are hard and not collinear to the incoming quark and anti-quark; hence, we treat them as hard particles throughout the calculation.

We need to understand how to compute the Green's functions $N_{a,b}$, $N_{a,b}^{(1),\mu}$ etc., as well as R_{fin}^ν and its expansion to first order in $k_\perp$. We will start with the discussion of the two simplest Green's functions $N_{a,b}$. To calculate them, we introduce the quark current $\hat{J}$ (c.f. Fig. 2) which depends on the momentum of the incoming quark (that we denote as q_a) and the momenta and polarization vectors of N photons. The momentum of the anti-quark is obtained from the momentum conservation. The current reads $\hat{J}(q_a, \psi_N)$, where the set ψ_N is given by $\psi_N = \{(p_1, \epsilon_1), (p_2, \epsilon_2), .., (p_N,\ \epsilon_N)\}$, and (p_i, ϵ_i) denote the momentum and

[6] The extension of these methods beyond QCD is discussed in Ref. [41].

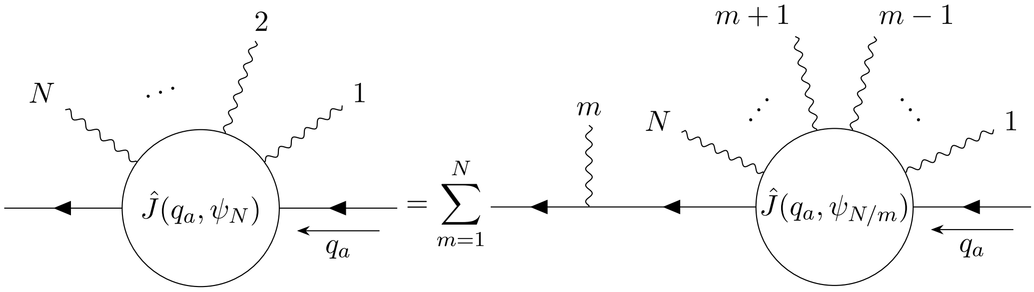

Figure 2. Pictorial representation of Eq. (5.1).

the polarization vector of the photon i. The current is a four-by-four matrix that satisfies the following recurrence relation

$$\hat{J}(q_a, \psi_N) = \frac{i}{\hat{q}_a - \hat{Q}_N} \sum_{m=1}^{N} (ie_q \hat{\epsilon}_m)\, \hat{J}(q_a, \psi_{N/m}), \tag{5.1}$$

where e_q is the quark electric charge,

$$Q_N = \sum_{m=1}^{N} p_m, \tag{5.2}$$

$\psi_{N/m}$ denotes the original set ψ_N from which the photon m is removed, and the recursion starts by identifying $\hat{J}(q_a, \{\})$ with the identity matrix. A schematic representation of Eq. (5.1) is shown in Fig. 2.

Eq. (5.1) is general; it allows us to compute the current $\hat{J}$ and obtain the Green's functions $N_{a,b}$ from it. This is achieved by simply removing the propagator $i/(\hat{q}_a - \hat{Q}_N)$ from Eq. (5.1). We then find

$$N_{a,b} = \sum_{m=1}^{N} (ie_q \hat{\epsilon}_m)\, \hat{J}\left(q_{a,b}, \psi_{N/m}\right), \tag{5.3}$$

where the two vectors $q_{a,b}$ are different for the two cases. For example, many terms in the final formula involve functions N_a and N_b in the collinear $\vec{k}||\vec{p}_a$ limit. In that case

$$q_a = x p_a, \quad q_b = p_a. \tag{5.4}$$

For the $\vec{k}||\vec{p}_b$ case,

$$q_a = p_a - (1-x) p_b, \quad q_b = p_a. \tag{5.5}$$

In addition, we require the expansion of these Green's functions for certain deformations of the quark momentum q; we will denote such deformations by δq. The important feature of these deformations is that they do not affect momenta and polarizations of colorless particles of the final state X. Thanks to this feature, it becomes straightforward to compute the expansion of the functions $N_{a,b}$ with respect to such deformations. Writing

$$\hat{J}(q + \delta q, \psi_N) = \hat{J}^{(0)}(q, \psi_N) + \delta q_\mu \hat{J}^{(1),\mu}(q, \psi_N) + \frac{\delta q_\nu \delta q_\mu}{2} \hat{J}^{(2),\mu\nu}(q, \psi_N) + \mathcal{O}(\delta q^3), \tag{5.6}$$

we can derive equations that currents $\hat{J}^{(0)}$, $\hat{J}^{(1),\mu}$ and $\hat{J}^{(2),\mu\nu}$ satisfy. In fact, the equation for $J^{(0)}$ is identical to Eq. (5.1). The equations for $\hat{J}^{(1),\mu}$ and $\hat{J}^{(2),\mu\nu}$ read

$$
\begin{aligned}
\hat{J}^{(1),\mu}(q_a,\psi_N) &= -\frac{1}{\hat{q}_a-\hat{Q}_N}\gamma^\mu J^{(0)}(q_a,\psi_N) + \frac{i}{\hat{q}_a-\hat{Q}_N}\sum_{m=1}^{N}\left(ie_q\hat{\epsilon}_m\right)\hat{J}^{(1),\mu}(q_a,\psi_{N/m}),\\
\hat{J}^{(2),\mu\nu}(q_a,\psi_N) &= -\frac{1}{\hat{q}_a-\hat{Q}_N}\left[\gamma^\mu J^{(1),\nu}(q_a,\psi_N)+\gamma^\nu J^{(1),\mu}(q_a,\psi_N)\right]\\
&\quad + \frac{i}{\hat{q}_a-\hat{Q}_N}\sum_{m=1}^{N}\left(ie_q\hat{\epsilon}_m\right)\hat{J}^{(2),\mu\nu}(q_a,\psi_{N/m}).
\end{aligned}
\tag{5.7}
$$

To start the recursion, we use $\hat{J}^{(0)}(q_a)=\hat{1}$, $\hat{J}^{(1),\mu}(q_a)=0$ and $\hat{J}^{(2),\mu\nu}(q_a)=0$. To compute relevant Green's functions, we write their expansions as

$$
N_{a,b}(q+\delta q,P_X) = N_{a,b}(q,P_X) + \delta q_\mu N_{a,b}^{(1),\mu}(q,P_X) + \frac{\delta q_\mu \delta q_\nu}{2}N_{a,b}^{(2),\mu\nu}(q,P_X)+\dots, \tag{5.8}
$$

where q is the quark momentum and ellipses stand for terms with higher powers of δq. Then, using Eqs (5.3, 5.7), we find

$$
\begin{aligned}
N_{a,b}^{(1),\mu} &= \sum_{m=1}^{N}\left(ie_q\hat{\epsilon}_m\right)\hat{J}^{(1),\mu}\left(q_{a,b},\psi_{N/m}\right),\\
N_{a,b}^{(2),\mu\nu} &= \sum_{m=1}^{N}\left(ie_q\hat{\epsilon}_m\right)\hat{J}^{(2),\mu\nu}\left(q_{a,b},\psi_{N/m}\right).
\end{aligned}
\tag{5.9}
$$

The two vectors $q_{a,b}$ are given in Eqs (5.4, 5.5) and we identify $Q_N = P_X$.

The formula in Eq. (4.73) requires us to compute derivatives of the Born matrix element squared. Although one can calculate these derivatives for simple processes, where matrix elements squared are known, it becomes difficult to do so in complicated cases with a large number of particles. To facilitate computing derivatives also in such cases, we relate them to the Green's functions that we have already introduced. In particular, we find

$$
\begin{aligned}
D_\mu^{xa,b}N_a(p_b,xp_a,P_X) &= N_a^{(1),\mu}(p_b,xp_a,P_X),\\
D_\nu^{xa,b}D_\mu^{xa,b}N_a(p_b,xp_a,P_X) &= N_a^{(2),\mu\nu}(p_b,xp_a,P_X).
\end{aligned}
\tag{5.10}
$$

Given these relations, we can replace

$$
\begin{aligned}
D_\mu^{xa,b}\,|\mathcal{M}|^2(xp_a,p_b,Q_X) \to &\,\mathrm{Tr}\left[N_{a,\mu}^{(1)}x\hat{p}_aN_a^+\hat{p}_b\right]+\mathrm{Tr}\left[N_ax\hat{p}_aN_{a,\mu}^{(1),+}\hat{p}_b\right]\\
&+\mathrm{Tr}\left[N_a\gamma_\mu N_a^+\hat{p}_b\right]-\mathrm{Tr}\left[N_ax\hat{p}_aN_a^+\gamma_\mu\right].
\end{aligned}
\tag{5.11}
$$

For the term with the second-order derivative in the function $W_a(x)$ we find

$$
\begin{aligned}
g_\perp^{\mu\nu}\,D_\nu^{xa,b}D_\mu^{xa,b}\,|\mathcal{M}|^2(p_b,xp_a,Q_X) \to g_\perp^{\mu\nu}\Big\{&\mathrm{Tr}\left[N_{a,\mu\nu}^{(2)}x\hat{p}_aN_a^+\hat{p}_b\right]+\text{c.c.}\\
&+2\mathrm{Tr}\left[N_{a,\mu}^{(1)}x\hat{p}_aN_{a,\nu}^{(1),+}\hat{p}_b\right]+2\mathrm{Tr}\left[N_{a,\mu}^{(1)}\gamma_\nu N_a^+\hat{p}_b\right]+\text{c.c.}\\
&-2\mathrm{Tr}\left[N_{a,\mu}^{(1)}x\hat{p}_aN_a^+\gamma_\nu\right]+\text{c.c.}-2\mathrm{Tr}\left[N_a\gamma_\mu N_a^+\gamma_\nu\right]\Big\}.
\end{aligned}
\tag{5.12}
$$

We note that the above replacements are only valid if they are done simultaneously in *all* relevant terms.

The last ingredients required for the final formula for subleading power corrections involve the Green's function $R^{\nu}_{\rm fin}$ and its expansion to first order in the momentum $k_\perp$. To compute these quantities, we introduce the current $\hat{G}^{\nu}$ that depends on the quark momentum q, the gluon momentum k and the photon momenta and polarization vectors. This current satisfies the following equation

$$\hat{G}^{\nu}(q,k;\psi_N) = \frac{i}{\hat{q}-\hat{k}-\hat{Q}_N}\left[i\gamma^{\nu}\hat{J}(q,\psi_N) + \sum_{m=1}^{N}(ie_q\hat{\epsilon}_m)\,\hat{G}^{\nu}\left(q,k;\psi_{N/m}\right)\right], \tag{5.13}$$

where the first term on the right-hand side describes the gluon emission off the anti-quark leg, and the second term refers to a situation where the emission of one of the N photons happens last, see Fig. 3. The boundary condition for the recursion is

$$\hat{G}^{\nu}(q,k,\{\}) = 0, \tag{5.14}$$

because gluon emissions off the external quark line should not be considered. For the same reason, the expression for $R^{\nu}_{\rm fin}$ reads

$$R^{\nu}_{\rm fin}(q,k,\psi_N) = \sum_{m=1}^{N}(ie_q\hat{\epsilon}_m)\,\hat{G}^{\nu}\left(q,k;\psi_{N/m}\right). \tag{5.15}$$

For the case $\vec{k}||\vec{p}_a$, we require $R^{\nu}_{\rm fin}$ in the strict collinear limit, in which case $q = p_a$ and $k = (1-x)p_a$. For the case $\vec{k}||\vec{p}_b$, $R^{\nu}_{\rm fin}$ should be evaluated for $q = p_a$ and $k = (1-x)p_b$.

We also require the expansion of $R^{\nu}_{\rm fin}$ to first order in $k_\perp$. Since the dependencies on $k_\perp$ arise after one of the two collinear boosts is applied to momenta p_a, k and p_b, we will define the expansion of the current G^{ν} for *particular momentum deformations only.* We begin with the $\vec{k}||\vec{p}_a$ case. Applying the Λ_a-boost, to p_a and k and expanding in $k_\perp$, we write

$$\begin{aligned}\hat{G}^{\nu}\left(p_a+\frac{k_\perp}{2x},(1-x)p_a+\frac{k_\perp(1+x)}{2x},\psi_N\right) &= \hat{G}^{(0)\nu}\left(p_a,(1-x)p_a,\psi_N\right)\\ &+\hat{G}^{(1)\nu,\mu}\left(p_a,(1-x)p_a,\psi_N\right)k_{\perp,\mu}+\cdots,\end{aligned} \tag{5.16}$$

where $\hat{G}^{(0)\nu}$ is computed with the help of Eq. (5.13). The recurrence relation for $\hat{G}^{(1)\nu,\mu}$ reads

$$\begin{aligned}\hat{G}^{(1),\nu\mu}(p_a,(1-x)p_a,\psi_N) &= \frac{1}{2}\frac{1}{x\hat{p}_a-\hat{Q}_N}\gamma^{\mu}\hat{G}^{(0),\nu}(p_a,(1-x)p_a,\psi_N)\\ -\frac{1}{x\hat{p}_a-\hat{Q}_N}&\left[\frac{1}{2x}\gamma^{\nu}\hat{J}^{(1),\mu}(p_a,\psi_N)+\sum_{m=1}^{N}(e_q\hat{\epsilon}_m)\hat{G}^{(1),\nu\mu}(p_a,(1-x)p_a,\psi_{N/m})\right],\end{aligned} \tag{5.17}$$

and the recursion starts with $G^{(1),\nu\mu} = 0$. Defining the expansion of $R^{\nu}_{\rm fin}$ as

$$\begin{aligned}&R^{\nu}_{\rm fin}\left(p_b+\frac{k_\perp}{2},p_a+\frac{k_\perp}{2x},(1-x)p_a+\frac{k_\perp(1+x)}{2x},P_X\right)\\ &\approx R^{(0),\nu}_{\rm fin}(p_a,p_b,(1-x)p_a,P_X)+k_{\perp,\mu}R^{(1),\nu\mu}_{\rm fin}(p_a,p_b,(1-x)p_a,P_X)+\mathcal{O}(k_\perp^2),\end{aligned} \tag{5.18}$$

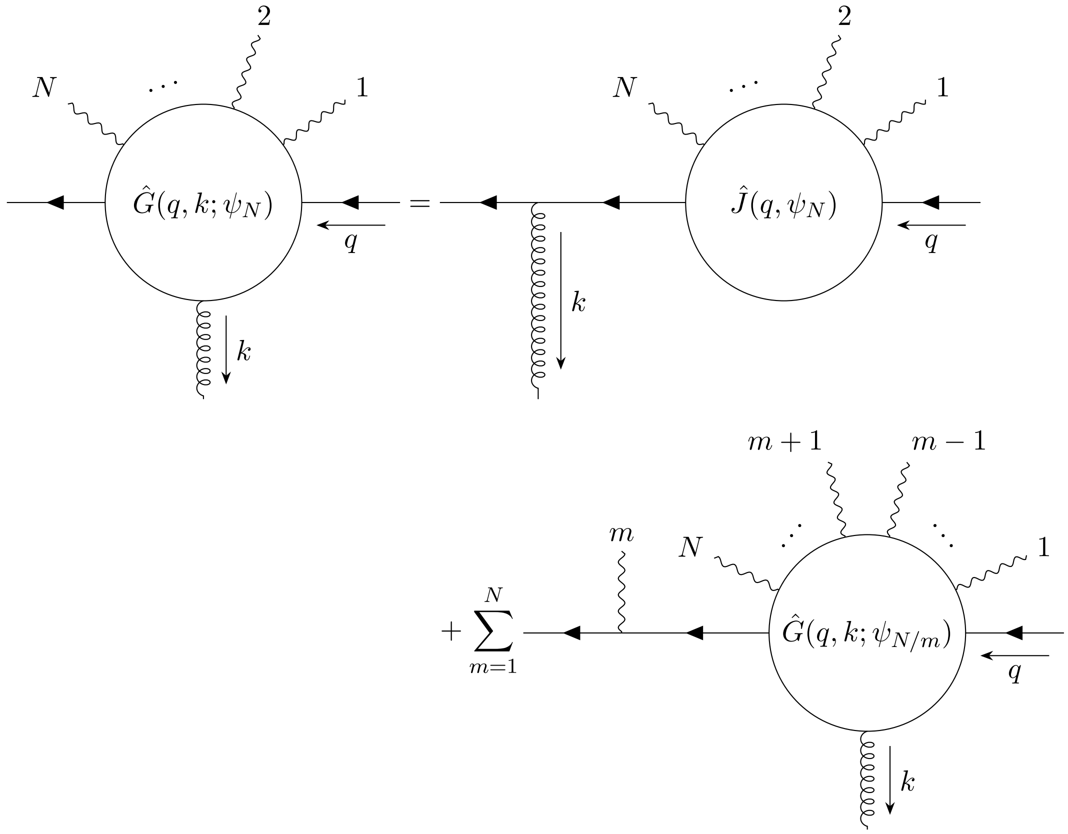

Figure 3. Pictorial representation of Eq. (5.13).

we find

$$R^{(1),\nu\mu}_{\rm fin}(p_b,p_a,(1-x)p_a,P_X)=\sum_{m=1}^{N}(ie_q\hat{\epsilon}_m)\,\hat{G}^{(1),\nu\mu}\left(p_a,(1-x)p_a,\psi_{N/m}\right). \tag{5.19}$$

For the $\vec{k}||\vec{p}_b$ case, we apply the Λ_b boost and write

$$\begin{aligned}\hat{G}^{\nu}\left(p_a+\frac{k_\perp}{2},(1-x)p_b+\frac{k_\perp(1+x)}{2x},\psi_N\right)&=\hat{G}^{(0)\nu}\left(p_a,(1-x)p_b,\psi_N\right)\\&+\hat{G}^{(1),\nu\mu}\left(p_a,(1-x)p_b,\psi_N\right)k_{\perp,\mu}+\cdots.\end{aligned} \tag{5.20}$$

We then derive an equation for $\hat{G}^{(1),\nu\mu}$. It reads

$$\begin{aligned}\hat{G}^{(1),\nu\mu}(p_a,(1-x)p_b,\psi_N)&=\frac{1}{2x}\frac{1}{\hat{P}_{abN}}\gamma^\mu\hat{G}^{(0),\nu}(p_a,(1-x)p_b,\psi_N)\\-\frac{1}{\hat{P}_{abN}}&\left[\frac{1}{2}\gamma^\nu\hat{J}^{(1),\mu}(p_a,\psi_N)+\sum_{m=1}^{N}(e_q\hat{\epsilon}_m)\hat{G}^{(1),\nu\mu}(p_a,(1-x)p_b,\psi_{N/m})\right],\end{aligned} \tag{5.21}$$

where $P_{abN}=p_a-p_b(1-x)-Q_N$. Defining the expansion of $R^{\nu}_{\rm fin}$ as

$$\begin{aligned}&R^{\nu}_{\rm fin}\left(p_b+\frac{k_\perp}{2x},p_a+\frac{k_\perp}{2},(1-x)p_b+\frac{k_\perp(1+x)}{2x},P_X\right)\\&\approx R^{(0),\nu}_{\rm fin}(p_a,p_b,(1-x)p_b,P_X)+k_{\perp,\mu}R^{(1),\nu\mu}_{\rm fin}(p_a,p_b,(1-x)p_b,P_X)+\mathcal{O}(k_\perp^2),\end{aligned} \tag{5.22}$$

we find for the $\vec{k}||\vec{p}_b$ case

$$R^{(1),\nu\mu}_{\rm fin}(p_a,p_b,(1-x)p_b,P_X) = \sum_{m=1}^{N} (ie_q\hat{\epsilon}_m)\,\hat{G}^{(1),\nu\mu}\left(p_a,(1-x)p_b,\psi_{N/m}\right). \tag{5.23}$$

6 Examples of application

In this section, we apply the master formula in Eq. (4.73) to compute the next-to-leading power correction in the zero-jettiness variable to various processes. We start with the Drell-Yan process $q\bar{q} \to l^+l^-$ and the two-photon production $q\bar{q} \to \gamma\gamma$. These processes are sufficiently simple to allow an analytic computation of the subleading contribution in the zero-jettiness expansion. Then, we turn to the process $q\bar{q} \to 4\gamma$. In this case, the matrix element and the required Green's functions are complicated, so that we employ the generalized currents introduced in the previous section to perform the calculation.

6.1 The Drell-Yan process

We consider the photon-mediated production of a pair of leptons in the annihilation of a quark and an anti-quark

$$q(p_a) + \bar{q}(p_b) \to \gamma^* \to l(p_1) + \bar{l}(p_2). \tag{6.1}$$

The calculation of the next-to-leading power corrections involves several quantities that we need to specify. They include the leading order matrix element and the phase space appearing in Eq. (4.73). Since the $1/\epsilon$ singularities have already been canceled, we can compute the relevant quantities in four space-time dimensions.

The analysis of the collinear $\vec{k}||\vec{p}_a$ contribution requires the (boosted) phase space $\mathrm{d}\Phi_2^{xa}$ that corresponds to the process in Eq. (6.1) where the quark momentum p_a is replaced with xp_a. The phase space reads

$$\mathrm{d}\Phi_2^{xa} = \frac{1}{8\pi}\mathrm{d}\beta\frac{\mathrm{d}\varphi}{(2\pi)}. \tag{6.2}$$

In Eq. (6.2) φ is the azimuthal angle of the outgoing lepton in the reference frame where the z-axis is aligned with the collision axis, and the parameter $\beta \in [0,1]$ is related to the polar angle of the lepton. With this parametrization, the momenta $p_{1,2}$ read

$$\begin{aligned} p_1 &= x(1-\beta)p_a + \beta p_b + \sqrt{xs\beta(1-\beta)}n_\perp, \\ p_2 &= x\beta p_a + (1-\beta)p_b - \sqrt{xs\beta(1-\beta)}n_\perp, \end{aligned} \tag{6.3}$$

where $s = 2p_a\cdot p_b$, $p_{a,b}\cdot n_\perp = 0$ and $n_\perp^2 = -1$. We note that the phase space parametrization in Eq. (6.2) does not depend on the parameter x, so that if we set $x=1$ also in Eq. (6.3), we obtain both the $x=1$ Born phase space and the momenta parametrization.

For the collinear region $\vec{k}||\vec{p}_b$, we require the phase space $\mathrm{d}\Phi_2^{xb}$. We can use Eq. (6.2) to describe it provided that we use the following parametrization of the momenta $p_{1,2}$

$$\begin{aligned} p_1 &= (1-\beta)p_a + x\beta p_b + \sqrt{xs\beta(1-\beta)}n_\perp, \\ p_2 &= \beta p_a + x(1-\beta)p_b - \sqrt{xs\beta(1-\beta)}n_\perp. \end{aligned} \tag{6.4}$$

This parametrization ensures that in the soft $x = 1$ limit Eqs (6.3, 6.4) coincide.

The appropriately normalized Born matrix element squared summed over polarizations and colors reads

$$\sum_{\text{pol,col}} \frac{|\mathcal{M}|^2(p_b, p_a; p_1, p_2)}{4N_c(Q_q e^2)^2} = 2\frac{s_{a1}^2 + s_{b1}^2}{s^2} = 2(1 - 2\beta + 2\beta^2), \tag{6.5}$$

where $s_{a1} = 2p_a \cdot p_1$, $s_{b1} = 2p_b \cdot p_1$, N_c is the number of colors, e is the positron electric charge and Q_q is the electric charge of the quark in units of e. The leading order cross section evaluates to

$$\mathrm{d}\sigma_0 = 16\pi\, \bar{\sigma}_0\, \mathrm{d}\Phi_2^{ab}\, (1 - 2\beta + 2\beta^2), \tag{6.6}$$

where

$$\bar{\sigma}_0 = \frac{\pi Q_q^2 \alpha_{\text{QED}}^2}{s N_c}, \tag{6.7}$$

and $\mathrm{d}\Phi_2^{ab}$ is given by Eq. (6.2).

To compute the next-to-leading power corrections from the master formula in Eq. (4.73), we have to calculate a significant number of terms. We will perform the computation setting $N_c \to 1, Q_q \to 1$ and $e \to 1$ and restore the relevant factors at the end. Then, we have to use

$$|\mathcal{M}|^2(p_b, p_a; p_1, p_2) \to 8\frac{s_{a1}^2 + s_{b1}^2}{s^2} = 8(1 - 2\beta + 2\beta^2), \tag{6.8}$$

With this normalization, the Green's functions $N_{a,b}$ read

$$N_a = N_b = \frac{i}{s_{12}}\, \gamma_\mu\, \left(\bar{u}(p_1)\gamma^\mu v(p_2)\right). \tag{6.9}$$

It follows from Eq. (6.9) that neither N_a nor N_b depends on p_a and p_b, which means that it is not affected by the boost and that it does not depend on the gluon momentum k. Hence, we find

$$N_{a,b}^{(1),\mu} = 0. \tag{6.10}$$

Furthermore, in case of the Drell-Yan process, no gluon emissions from the internal lines can occur, which implies that

$$R_{\text{fin}}^{\mu} = 0. \tag{6.11}$$

Another simplification is that for a $2 \to 2$ process $\kappa_2 = 0$, which follows from the fact that Born amplitudes for such processes have vanishing mass dimension.

With these preliminary remarks out of the way, we proceed with the calculation of the subleading power corrections, using the general formula in Eq. (4.70). We will start with the discussion of the collinear $\vec{k}||\vec{p}_a$ contribution which means that we employ the parametrization of momenta $p_{1,2}$ given in Eq. (6.3) to write the corresponding expressions. Several ingredients need to be discussed.

- *Traces that involve p_a, p_b and some combinations of N_a and N_b.* These are straightforward to compute given the expressions for these Green's functions. We find e.g.

$$\begin{aligned} F_{\rm fin,a} =& -8\left(1-2\beta+2\beta^2\right) + 32\,\frac{\beta(1-\beta)}{x^2} + 16\,\frac{\beta(1-\beta)}{x} + 16\,\frac{\beta(1-\beta)}{x^2} \\ & - 16\,\frac{(1-x)\beta(1-\beta)}{x} - 8\,\frac{1+2\beta-2\beta^2}{x^2} - 8\,\frac{1-2\beta+2\beta^2}{x}. \end{aligned} \tag{6.12}$$

- *Terms that involve derivatives of the various quantities w.r.t. momenta of the incoming partons.* Such derivatives appear in several terms in Eq. (4.70) and also in the definition of the function $W_a(x)$, c.f. Eq. (4.16). We start by discussing derivatives of the matrix element squared. In principle, these derivatives may not be uniquely defined given the need to account for the momentum conservation, etc. However, in our formulas, the potentially ambiguous derivatives, always involve contractions that make them unique. For example, we find

$$g_\perp^{\mu\nu} D_\nu^{ab} |M^2(p_b,p_a,p_1,p_2)|^2 = \frac{16 p_{1,\perp}^\mu}{s_{12}^2}\left(s_{a1}-s_{b1}\right), \tag{6.13}$$

where $D_\nu^{ab} = \partial/\partial p_a^\nu - \partial/\partial p_b^\nu$. Furthermore, using Eq. (4.82) and Eq. (4.72), it is easy to see that

$$\left[\kappa_2 + 2p_a^\mu \frac{\partial}{\partial p_a^\mu} + (g^{\mu\nu}+\omega_{ab}^{\mu\nu})L_{\mu\nu}\right] |M^2(p_b,p_a,p_1,p_2)| = 0. \tag{6.14}$$

- Terms that involve derivatives and traces can be computed in a straightforward way using the above results. For example, we find a compact expression for the function W_a,

$$W_a(x) = 4\frac{1-x}{x}(1-2\beta+2\beta^2). \tag{6.15}$$

We also find that in the Drell-Yan case

$$g_\perp^{\rho\alpha}\left(D_\rho^{xa,b}\,|\mathcal{M}|^2(p_b,xp_a,...) - 2{\rm Tr}\left[N_a\gamma_\rho N_a^+\hat{p}_b\right]\right) b_{a\alpha}^{\mu\nu} L_{\mu\nu} = 0. \tag{6.16}$$

Another contribution with derivative operators and traces evaluates to

$$\begin{aligned} & s(1-x)\frac{g_\perp^{\alpha\beta}}{4}\Bigg\{ -2{\rm Tr}\left[N_a\gamma_\beta N_a^+\hat{p}_b\right] \\ & + {\rm Tr}\left[N_a\gamma_\beta\gamma_\rho\hat{p}_a\left(R_{\rm fin}^{\rho,+} + \frac{N_b^+(\hat{p}_b-(1-x)\hat{p}_a)\gamma^\rho}{(1-x)s}\right)\hat{p}_b\right] + {\rm c.c.} \\ & + \frac{2x}{1-x}{\rm Tr}\left[N_a\hat{p}_a\left(R_\beta^{{\rm fin},+} - \frac{N_b^+\hat{p}_a\gamma_\beta}{s}\right)\hat{p}_b\right] + {\rm c.c.}\Bigg\} b_{a\alpha}^{\mu\nu}L_{\mu\nu} \\ & = -4(1-x^2)\frac{(1-2\beta)}{x^2}\Bigg\{\left[p_1^\mu - (1-2\beta)(1-\beta)\,xp_a^\mu + (1-2\beta)\beta\,p_b^\mu\right]\partial_{1\mu} \\ & - \left[p_2^\mu + (1-2\beta)\beta\,xp_a^\mu - (1-2\beta)(1-\beta)\,p_b^\mu\right]\partial_{2\mu}\Bigg\}, \end{aligned} \tag{6.17}$$

where $\partial_{1(2),\mu}$ are derivatives $\partial/\partial p_{1,2}^\mu$.

Expressions for the case $\vec{k}||\vec{p}_b$ can be obtained from the formulas for $\vec{k}||\vec{p}_a$ by replacing

$$\beta \to 1-\beta, \qquad p_a \leftrightarrow p_b. \tag{6.18}$$

The total subleading contribution is obtained from Eq. (4.70), using the partial results described above. We find

$$\begin{aligned}
\frac{d\sigma^{\rm DY,NLP}}{d\tau} &= \frac{4[\alpha_s]C_F Q}{s} d\sigma_0 \left[\left(-1+\frac{1}{2}\mathcal{D}\right) + \frac{1}{2}\log\left(\frac{\tau Q}{s}\right)\mathcal{D}\right]\mathcal{O}(p_1,p_2) \\
&+ \frac{2[\alpha_s]C_F Q}{s}\int_0^1 d\sigma_0\, dx \Bigg[-\frac{1}{2(1-x)_+}\left(\mathcal{D}\Big|_{ca} + \mathcal{D}\Big|_{cb}\right) \\
&+ \left(\frac{\left[\bar\beta\, p_a^\mu - \beta\, p_b^\mu\right]\partial_{1\mu} + \left[\beta\, p_a^\mu - \bar\beta\, p_b^\mu\right]\partial_{2\mu}}{2\,(1-x)_+}\right)\Bigg|_{ca} \\
&+ \left(\frac{\bar\beta\, p_a^\mu \partial_{1\mu} + \beta\, p_a^\mu \partial_{2\mu}}{2} + \frac{\mathcal{P}(\beta,x,p_a,p_b;p_1,p_2,\partial_1,\partial_2)}{8(1-2\beta+2\beta^2)}\right)\Bigg|_{ca} \\
&- \left(\frac{\left[\bar\beta\, p_a^\mu - \beta\, p_b^\mu\right]\partial_{1\mu} + \left[\beta\, p_a^\mu - \bar\beta\, p_b^\mu\right]\partial_{2\mu}}{2\,(1-x)_+}\right)\Bigg|_{cb} \\
&+ \left(\frac{\beta\, p_b^\mu \partial_{1\mu} + \bar\beta\, p_b^\mu \partial_{2\mu}}{2} + \frac{\mathcal{P}(\bar\beta,x,p_b,p_a;p_1,p_2,\partial_1,\partial_2)}{8(1-2\beta+2\beta^2)}\right)\Bigg|_{cb}\Bigg]\mathcal{O}(p_1,p_2),
\end{aligned} \tag{6.19}$$

where $\bar\beta = 1-\beta$ and $d\sigma_0$ is given in Eq. (6.6). We note that bars with a subscript ca or cb indicate that after applying derivatives to the observable $\mathcal{O}(p_1,p_2)$, the ensuing scalar products must be evaluated in a particular collinear kinematics given in Eqs (6.3, 6.4) for the ca and cb cases, respectively. The differential operator $\mathcal{D}$ reads

$$\mathcal{D} = p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu}. \tag{6.20}$$

The other differential operator $\mathcal{P}(\beta,x,p_a,p_b;p_1,p_2,\partial_1,\partial_2)$ appearing in Eq. (6.19) also acts on the observable $\mathcal{O}(p_1,p_2)$. It is given by the following expression

$$\begin{aligned}
&\mathcal{P}(\beta,x,p_a,p_b;p_1,p_2,\partial_1,\partial_2) = -2\left(\frac{1+x^2}{x^2}\left(1-6\beta+6\beta^2\right) + \frac{2f_0(\beta)}{x}\right) \\
&+ g_2(x,\beta)\, p_1^\mu\partial_{1\mu} + g_2(x,1-\beta)\, p_2^\mu\partial_{2\mu} + g_1(x,\beta)\, p_a^\mu\partial_{1\mu} + \frac{g_1(x_1,\beta)}{x}\, p_b^\mu\partial_{2\mu} \\
&+ g_1(x,1-\beta) p_a^\mu\partial_{2\mu} + \frac{g_1(x_1,1-\beta)}{x} p_b^\mu\partial_{1\mu} \\
&+ \frac{(1+x^2)f_0(\beta)}{2x^2}\Bigg\{\Big[-2(1-\beta)^2x^2\, p_a^\mu p_a^\nu - 2\beta^2\, p_b^\mu p_b^\nu \\
&- x\,(p_a p_b)\, g^{\mu\nu} + 2\left(x\, p_a^\mu p_1^\nu + p_b^\mu p_1^\nu\right) + 4x\beta(1-\beta)\, p_a^\mu p_b^\nu\Big]\partial_{1\nu}\partial_{1\mu} \\
&+ \Big[\left(f_0(\beta)-2\right)\left(x^2\, p_a^\mu p_a^\nu + p_b^\mu p_b^\nu\right) + \left(x\, p_1^\mu p_a^\nu + p_1^\mu p_b^\nu\right) + \left(x\, p_a^\mu p_2^\nu + p_b^\mu p_2^\nu\right) \\
&- x\,(p_a p_b)\, g^{\mu\nu} - x\left(1-2\beta^2\right) p_a^\mu p_b^\nu + x\left(1-4\beta+2\beta^2\right) p_b^\mu p_a^\nu\Big]\partial_{2\nu}\partial_{1\mu} \\
&+ (p_1 \leftrightarrow p_2,\ \beta \leftrightarrow 1-\beta)\Bigg\},
\end{aligned} \tag{6.21}$$

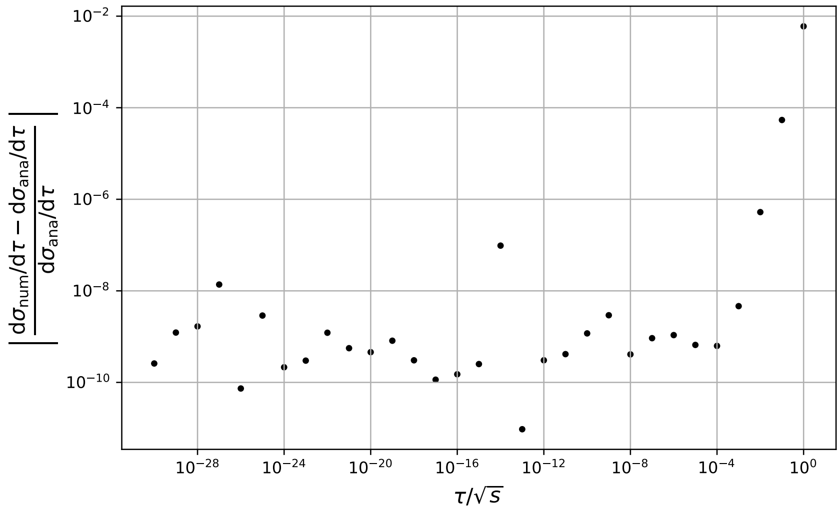

Figure 4. Relative difference between the small-jettiness expansions of the Drell-Yan cross section with an observable and its numerical integration.

where $x_1 = 1/x$,

$$\begin{aligned} g_1(x,\beta) &= -4(1-\beta)f_0(\beta) + f_1(1-\beta)x + \frac{f_2(1-\beta)}{x}, \\ g_2(x,\beta) &= f_3(1-\beta) + \frac{f_3(\beta)}{x^2}, \end{aligned} \tag{6.22}$$

and

$$\begin{aligned} &f_0(\beta) = 1 - 2\beta + 2\beta^2, \quad f_1(\beta) = -14\beta^3 + 16\beta^2 - 7\beta + 1, \\ &f_2(\beta) = 1 + \beta - 8\beta^2 + 10\beta^3, \quad f_3(\beta) = f_0(-\beta) - 2. \end{aligned} \tag{6.23}$$

We note that the complexity of the above formula is related to the fact that the observable $\mathcal{O}(p_1, p_2)$ is considered to be generic. If e.g. all derivatives applied to an observable, that appear in the final formula, are dropped, the expression for next-to-leading power corrections for the Drell-Yan process simplifies dramatically.

To ensure the correctness of the master formula, we perform the following checks. First, we repeat the calculation of the zero-jettiness power corrections to the Drell-Yan process employing explicit expression for the matrix element squared and using explicit parametrization of the phase space. We find a complete analytic agreement between the result of such "explicit" calculation with the result that one obtains when using the master formula to derive the power-suppressed term.

We can also use the master formula to rederive results for the vector boson production discussed in Sec. 2. To do so, we choose the observable that constrains the invariant mass of two leptons

$$\mathcal{O}(p_1, p_2) = \delta(2p_1 \cdot p_2 - m_V^2). \tag{6.24}$$

Since in Sec. 2 we work with hadronic, rather than partonic cross sections, we need to do the same here. Hence, we employ Eq. (2.7) to calculate power corrections to the hadronic cross section using the master formula for $\mathrm{d}\sigma(s)/\mathrm{d}\tau$ adapted to the Drell-Yan case, and integrate over the leptonic final states to compare with Eq. (2.34). The derivatives of the observable will be related (via integration by parts) to derivatives of the luminosity function and other terms in the formula. We find that $\mathcal{O}(\tau \log \tau)$ terms agree immediately, while other $\mathcal{O}(\tau)$ terms agree after integrating by parts over z, to transform terms with $\mathcal{L}_1\left(m_V^2/z\right)$ into ones with $\mathcal{L}\left(m_V^2/z\right)$. The relevant equation reads

$$\int\limits_0^1 \mathrm{d}z\, \mathcal{L}_1\left(m_V^2/z\right) f(z) = -\frac{f(1)}{m_V^2} + \frac{1}{\mathcal{L}\left(m_V^2\right)} \int\limits_0^1 \mathrm{d}z\, \mathcal{L}\left(m_V^2/z\right) \left(\frac{z^2}{m_V^2} f(z)\right)' . \tag{6.25}$$

We may use Eq. (6.19) to recover results for power corrections, obtained for the Drell-Yan process $q\bar{q} \to V$ in Ref. [16]. This calculation is more complex because in addition to the invariant mass, the rapidity of the vector boson (or, equivalently, the dilepton rapidity) is constrained. Furthermore, the definition of the zero-jettiness variable used in Ref. [16] differs from what we employ here since Born-projected momenta for the incoming partons are used there to compute zero-jettiness. Nevertheless, choosing the observable

$$\mathcal{O}(p_1, p_2) = \delta\left(2p_1 \cdot p_2 - M^2\right)\ \delta\left(\frac{1}{2}\ln\frac{P_b \cdot (p_1+p_2)}{P_a \cdot (p_1+p_2)} - Y\right), \tag{6.26}$$

where $P_{a,b}$ are the hadronic initial-state momenta, and adding the appropriate convolution with parton distribution functions, it is possible to use our master formula to rederive results of Ref. [16]. More information about this comparison can be found in Appendix D.

Another check of the master formula originates from an opportunity to perform another specialized computation for Drell-Yan process $q\bar{q} \to l^+l^-$. Indeed, as shown in Appendix E, it is straightforward to derive power corrections to the rapidity of *one* of the charged leptons, using dedicated phase-space parametrisation. Alternatively, we can derive the same result from the master formula. To this end, we choose the observable

$$\mathcal{O}(p_1, p_2) = \delta(\tilde{y} - y_0) = \delta\left(\frac{1}{2}\ln\frac{p_b \cdot p_1}{p_a \cdot p_1} - y_0\right), \tag{6.27}$$

that fixes the lepton rapidity. For the comparison it is important to realize that the variable x in this section and the variable z in the appendix are, in fact, the same, and a both given by the following formula.

$$x = z = \frac{1}{s}(p_1+p_2)^2. \tag{6.28}$$

Furthermore, the rapidity is expressed through variables x, β differently in two collinear sectors, i.e.

$$y_{ca} = \frac{1}{2}\ln\frac{x(1-\beta)}{\beta}, \qquad y_{cb} = \frac{1}{2}\ln\frac{(1-\beta)}{x\beta}. \tag{6.29}$$

It follows

$$\beta = \begin{cases} \left(1+\frac{1}{x}e^{2y}\right)^{-1}, & \text{sector } ca, \\ \left(1+xe^{2y}\right)^{-1}, & \text{sector } cb. \end{cases} \tag{6.30}$$

After expressing everything in terms of the rapidity and evaluating the derivatives of the observable (that lead to derivatives of the rapidity-constraining delta function that we need to get rid of by integrating by parts), we find perfect agreement in the distributions in y_0 obtained from both approaches.

In addition, we have performed a numerical check, as we now describe. To keep things simple, we take an observable that constrains the invariant mass of the two leptons

$$\mathcal{O}(p_1, p_2) = \theta\left((p_1+p_2)^2 - s_0\right), \tag{6.31}$$

and compute the fiducial cross section by performing the phase-space integration in Eq. (3.2) at fixed values of τ. To this end, we remove the δ-function responsible for the overall energy-momentum conservation by integrating over the three-momentum of one of the leptons and the energy of another lepton, following the discussion in Appendix E. The zero-jettiness δ-function is removed by integrating over the energy of the emitted gluon. Integrations over the emission angles of one of the leptons and the gluon are performed numerically. For the numerical integration itself, we take $s_0 = 0.1$ GeV^2, $Q = 0.1$ GeV, $s = 1$ GeV^2 and set all couplings and charges to 1. We perform the calculation of the differential cross section at finite τ for $\tau \in [\tau_{\min}, 1]$, where $\tau_{\min}$ is 10^{-30}.

We wish to compare the results of the numerical and analytic computations. The latter (for the subleading power) is given in Eq. (6.19); they have to be supplemented with leading power results that are well known. An important feature of the observable in Eq. (6.31) is that it is invariant under Lorentz boosts applied to leptons. This leads to significant simplifications in the final formula for next-to-leading power corrections to the Drell-Yan process. We find

$$\begin{aligned}\frac{\mathrm{d}\sigma^{\mathrm{DY,NLP}}}{\mathrm{d}\tau} = \frac{4[\alpha_s]C_F Q}{s}\mathrm{d}\sigma_0\Bigg[-1 - \frac{x_0}{1-x_0} - \frac{1}{4(1-2\beta+2\beta^2)} \\ \times \int\limits_0^1 \mathrm{d}x \left(\frac{1+x^2}{x^2}\left(1-6\beta+6\beta^2\right) + \frac{2f_0(\beta)}{x}\right)\theta(sx-s_0)\Bigg],\end{aligned} \tag{6.32}$$

where $x_0 = s_0/s$. We note that there is no $\log\tau$ term in the subleading power corrections for the observable in Eq. (6.31), but this feature is certainly observable-dependent.

In Fig. 4 we plot the relative difference between the numerical and analytic results, normalized to the analytic result. At very small values of τ, the precision of the numerical calculation is insufficient to constrain subleading power corrections, but it is good enough to check the leading power contributions. However, for values $\tau \in [10^{-8}, 10^{-4}]$, the precision becomes sufficient to enable the check of the subleading power correction.

We performed a numerical fit for the τ-independent coefficients of the fiducial cross section defined with the observable in Eq. (6.31). Making an ansatz

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \tau^{-1}\left(\log\tau C_{\mathrm{LP,\,LL}} + C_{\mathrm{LP,\,NLL}}\right) + \log\tau C_{\mathrm{NLP,\,LL}} + C_{\mathrm{NLP,\,NLL}} + \tau C_{\mathrm{NNLP}} + \tau^2 C_{\mathrm{N3LP}} + \tau^3 C_{\mathrm{N4LP}}\,,$$

we compute $\mathrm{d}\sigma/\mathrm{d}\tau$ for different values of τ and perform a standard χ^2 fit to determine the coefficients. The fit is done using the values within the range $\tau \in [10^{-30}, 10^0]$. The result

Table 1. Comparison of the expansion coefficients of the fiducial cross section of a Drell-Yan process in the zero-jettiness variable through next-to-leading power, obtained through a numerical fit and an analytic computation, for the observable in Eq. (6.31). We take $s_0 = 0.1~\mathrm{GeV}^2, Q = 0.1~\mathrm{GeV}, s = 1~\mathrm{GeV}^2$ and set all couplings and charges to 1.

coefficient	fit	analytic
$C_{\mathrm{LP,LL}}$	−4.740 740 718	−4.740 740 741
$C_{\mathrm{LP,NLL}}$	13.741 118 266	13.741 118 217
$C_{\mathrm{NLP,LL}}$	0.000 17	0.000 00
$C_{\mathrm{NLP,NLL}}$	−1.0710	−1.0725

of the fit for the relevant terms is shown in Table 1 together with the results obtained from the analytic computation. Excellent agreement among the τ-independent coefficients is observed.

6.2 Production of two photons in $q\bar{q}$ collisions

Next we consider the production of two photons in the annihilation of a quark and an anti-quark

$$q(p_a) + \bar{q}(p_b) \to \gamma(p_1) + \gamma(p_2). \tag{6.33}$$

Since this is also a $2 \to 2$ process, we can use the same phase space and momenta parametrization as in the Drell-Yan case. However, the main difference between the two cases is that in the di-photon production the quantities R^{μ}_{fin} and $N^{(1),\mu}_{a,b}$ do not vanish. Because of this, we can check all the entries in the master formula for subleading power corrections given in Eq. (4.73).

For the di-photon production Eq. (6.33), the leading order cross section reads

$$\mathrm{d}\sigma_0^{2\gamma} = 16\pi\, \bar{\sigma}_0^{2\gamma}\, \mathrm{d}\Phi_2\, \frac{(1-2\beta+2\beta^2)}{\beta(1-\beta)}, \tag{6.34}$$

where the phase space and the momenta parametrization can be found in Eqs (6.2, 6.3), and

$$\bar{\sigma}_0^{2\gamma} = \frac{\pi Q_q^4 \alpha_{\mathrm{QED}}^2}{s N_c}. \tag{6.35}$$

Similar to the Drell-Yan case, we perform the computation setting $N_c \to 1, Q_q \to 1$ and $e \to 1$, and restore the relevant factors at the end. With this normalization, the required Green's functions can be computed either using formulas provided in Sec. 5 or simply collecting relevant Feynman diagrams which, for the process in Eq. (6.33) is quite straightforward. We find

$$N_{a,b}(p_b, p_a, P_X) = -i\left[\frac{\gamma_\nu(\hat{p}_a - \hat{p}_1)\gamma_\mu}{s_{a1}} + \frac{\gamma_\mu(\hat{p}_a - \hat{p}_2)\gamma_\nu}{s_{a2}}\right]\epsilon_1^{\mu *}\epsilon_2^{\nu *}, \tag{6.36}$$

where ϵ_i are the polarization vectors of the photons and $s_{ai} = 2p_a p_i$, $i = 1, 2$. We note that, when constructing $N_{a,b}$, we obtain the momentum of an anti-quark (p_b) using momentum conservation. Similarly, the function $R_{\rm fin}$ is easy to construct; it reads

$$R^{\rho}_{\rm fin}(p_b,p_a,k,P_X) = i\Bigg[\frac{\gamma_\nu(\hat{p}_a-\hat{k}-\hat{p}_1)\gamma^\rho(\hat{p}_a-\hat{p}_1)\gamma_\mu}{(s_{a1}+s_{ak}-s_{k1})\;s_{a1}} + \frac{\gamma_\mu(\hat{p}_a-\hat{p}_2)\gamma^\rho(\hat{p}_a-\hat{k}-\hat{p}_2)\gamma_\nu}{(s_{a2}+s_{ak}-s_{k2})\;s_{a2}}\Bigg]\epsilon_1^{\mu*}\epsilon_2^{\nu*}. \tag{6.37}$$

We note that the above expressions can be used for both $\vec{k}||\vec{p}_a$ and $\vec{k}||\vec{p}_b$ cases. In the first case, in Eq. (6.37) we have to take $p_a \to x p_a$, $k \to (1-x)p_a$ in the strict collinear limit, and in the second case $p_a \to p_a$ and $k \to (1-x)p_b$.

Given the above expressions, it is clear that the Green's functions $N^{(1),\rho}_{a,b}$ and $R^{(1),\rho\sigma}_{\rm fin}$ do not vanish. They can be obtained by expanding the above formulas in the relevant small parameters, routing the momentum perturbation in a particular way. We find

$$N^{(1),\rho}_{a,b}(p_b,p_a,P_X) = -i\Bigg[\frac{\gamma_\nu\gamma^\rho\gamma_\mu}{s_{a1}} + 2(p_a^\rho-p_1^\rho)\,\frac{\gamma_\nu(\hat{p}_a-\hat{p}_1)\gamma_\mu}{s_{a1}^2} + \frac{\gamma_\mu\gamma^\rho\gamma_\nu}{s_{a2}} + 2(p_a^\rho-p_2^\rho)\,\frac{\gamma_\mu(\hat{p}_a-\hat{p}_2)\gamma_\nu}{s_{a2}^2}\Bigg]\epsilon_1^{\mu*}\epsilon_2^{\nu*}, \tag{6.38}$$

and

$$R^{(1),\rho\sigma}_{\rm fin}(p_b,p_a,k,P_X) = \frac{i}{2}\epsilon_1^{\mu*}\epsilon_2^{\nu*}\Bigg[\frac{\gamma_\nu(\hat{p}_a-\hat{k}-\hat{p}_1)\gamma^\rho\gamma^\sigma\gamma_\mu}{x\;(s_{a1}+s_{ak}-s_{k1})\;s_{a1}} - \frac{\gamma_\nu\gamma^\sigma\gamma^\rho(\hat{p}_a-\hat{p}_1)\gamma_\mu}{(s_{a1}+s_{ak}-s_{k1})\;s_{a1}} + 2\left(\frac{p_a^\sigma-p_1^\sigma}{x s_{a1}} - \frac{p_a^\sigma-k^\sigma-p_1^\sigma}{(s_{a1}+s_{ak}-s_{k1})}\right)\frac{\gamma_\nu(\hat{p}_a-\hat{k}-\hat{p}_1)\gamma^\rho(\hat{p}_a-\hat{p}_1)\gamma_\mu}{(s_{a1}+s_{ak}-s_{k1})\;s_{a1}}\Bigg] + \{1\leftrightarrow 2\}\,. \tag{6.39}$$

The comment about momenta assignments for the two collinear cases below Eq. (6.37) also applies for $N^{(1),\mu}$ and $R^{(1),\rho\sigma}_{\rm fin}$ in the above formulas.

We proceed with the calculation of the subleading power corrections, and discuss various contributions that appear in the general formula Eq. (4.70).

- For the finite reminder (c.f. Appendix B) in the $\vec{k}||\vec{p}_a$ case, we find

$$F^{2\gamma}_{\rm fin,a} = 8\left(\frac{1-x}{\beta(1-\beta)x}\right)^2 - 8\,\frac{1+2\beta-2\beta^2}{\beta(1-\beta)x} + 32. \tag{6.40}$$

- As we already mentioned, since we consider massless particles, the following equation holds

$$\left[\kappa_m + 2p_a^\mu\frac{\partial}{\partial p_a^\mu} + (g^{\mu\nu}+\omega_{ab}^{\mu\nu})L_{\mu\nu}\right]|\mathcal{M}^2(p_b,p_a,p_1,p_2)| = 0. \tag{6.41}$$

- The function W_a evaluates to

$$W_a^{2\gamma}(x) = 2\,\frac{1-x}{x}\,\frac{(1-2\beta+2\beta^2)}{(1-\beta)^2\beta^2} + \frac{32}{x}. \tag{6.42}$$

  It contains one term that does not vanish in the soft $x\to 1$ limit.

- We also find

$$g_\perp^{\rho\alpha}\left(D_\rho^{xa,b}\,|\mathcal{M}|^2(p_b,xp_a,...)-2\mathrm{Tr}\left[N_a\gamma_\rho N_a^+\hat{p}_b\right]\right)b_{a\,\alpha}^{\;\mu\nu}L_{\mu\nu}$$
$$=8\frac{(1-2\beta+2\beta^2)}{sx(1-\beta)\beta}\Bigg\{\left[\frac{(1-2\beta)}{(1-\beta)\beta}\,p_1^\mu-\frac{(1-2\beta)^2}{\beta}\,xp_a^\mu+\frac{(1-2\beta)^2}{(1-\beta)}\,p_b^\mu\right]\partial_{1\mu} \tag{6.43}$$
$$-\left[\frac{(1-2\beta)}{(1-\beta)\beta}\,p_2^\mu+\frac{(1-2\beta)^2}{(1-\beta)}\,xp_a^\mu-\frac{(1-2\beta)^2}{\beta}\,p_b^\mu\right]\partial_{2\mu}\Bigg\}.$$

- Another contribution with derivatives and traces that involves multiple Green's functions evaluates to

$$s(1-x)\frac{g_\perp^{\alpha\beta}}{4}\Bigg\{-2\mathrm{Tr}\left[N_a\gamma_\beta N_a^+\hat{p}_b\right]$$
$$+\mathrm{Tr}\left[N_a\gamma_\beta\gamma_\rho\hat{p}_a\left(R_{\mathrm{fin}}^{\rho,+}+\frac{N_b^+(\hat{p}_b-(1-x)\hat{p}_a)\gamma^\rho}{(1-x)s}\right)\hat{p}_b\right]+\mathrm{c.c.}$$
$$+\frac{2x}{1-x}\mathrm{Tr}\left[N_a\hat{p}_a\left(R_\beta^{\mathrm{fin},+}-\frac{N_b^+\hat{p}_a\gamma_\beta}{s}\right)\hat{p}_b\right]+\mathrm{c.c.}\Bigg\}b_{a\,\alpha}^{\;\mu\nu}L_{\mu\nu} \tag{6.44}$$
$$=-\frac{4(1-2\beta)(1-2x^2\beta+2x^2\beta^2)}{x^2(1-\beta)^2\beta^2}$$
$$\times\Bigg\{\left[p_1^\mu-(1-2\beta)(1-\beta)\,xp_a^\mu+(1-2\beta)\beta\,p_b^\mu\right]\partial_{1\mu}$$
$$-\left[p_2^\mu+(1-2\beta)\beta\,xp_a^\mu-(1-2\beta)(1-\beta)\,p_b^\mu\right]\partial_{2\mu}\Bigg\}.$$

As in the previous section, to get $\mathrm{C}^{\mathrm{NLP},b}$ we should replace

$$\beta\to 1-\beta,\qquad p_a\leftrightarrow p_b, \tag{6.45}$$

in the above formulas.

With all the necessary ingredients, the total subleading contribution can be obtained using Eq. (4.70). In this case we get

$$\frac{\mathrm{d}\sigma^{2\gamma,\mathrm{NLP}}}{\mathrm{d}\tau}=\frac{4[\alpha_s]C_FQ}{s}\mathrm{d}\sigma_0^{2\gamma}\left[\left(-1+\frac{1}{2}\mathcal{D}\right)+\frac{1}{2}\log\left(\frac{\tau Q}{s}\right)\mathcal{D}\right]\mathcal{O}(p_1,p_2)$$
$$+\frac{2[\alpha_s]C_FQ}{s}\mathrm{d}\sigma_0^{2\gamma}\int_0^1\mathrm{d}x\,\Bigg\{-\frac{1}{2(1-x)_+}\left(\mathcal{D}\Big|_{ca}+\mathcal{D}\Big|_{cb}\right)$$
$$+\left(\frac{\left[\bar{\beta}\,p_a^\mu-\beta\,p_b^\mu\right]\partial_{1\mu}+\left[\beta\,p_a^\mu-\bar{\beta}\,p_b^\mu\right]\partial_{2\mu}}{2\,(1-x)_+}\right)\Bigg|_{ca} \tag{6.46}$$
$$+\left(\frac{\bar{\beta}\,p_a^\mu\partial_{1\mu}+\beta\,p_a^\mu\partial_{2\mu}}{2}+\frac{\mathcal{P}_{2\gamma}(\beta,x,p_a,p_b;p_1,p_2,\partial_1,\partial_2)}{8(1-2\beta+2\beta^2)}\right)\Bigg|_{ca}$$
$$-\left(\frac{\left[\bar{\beta}\,p_a^\mu-\beta\,p_b^\mu\right]\partial_{1\mu}+\left[\beta\,p_a^\mu-\bar{\beta}\,p_b^\mu\right]\partial_{2\mu}}{2\,(1-x)_+}\right)\Bigg|_{cb}$$

$$+\left(\frac{\beta\, p_b^\mu \partial_{1\mu} + \bar\beta\, p_b^\mu \partial_{2\mu}}{2} + \frac{\mathcal{P}_{2\gamma}(\bar\beta, x, p_b, p_a; p_1, p_2, \partial_1, \partial_2)}{8(1-2\beta+2\beta^2)}\right)\Bigg|_{cb} \Bigg\} \mathcal{O}(p_1,p_2),$$

where $\bar\beta = 1 - \beta$ and vertical bars indicate that terms have to be evaluated in the appropriate collinear kinematics. The differential operator $\mathcal{D}$ is defined in Eq. (6.20) and $\mathcal{P}_{2\gamma}(\beta, x, p_a, p_b; p_1, p_2, \partial_1, \partial_2)$ is given by

$$\begin{aligned}
&\mathcal{P}_{2\gamma}(\beta, x, p_a, p_b; p_1, p_2, \partial_1, \partial_2) = \mathcal{P}(\beta, x, p_a, p_b; p_1, p_2, \partial_1, \partial_2) \\
&+ \left(\frac{1+x^2}{x^2}\,\frac{5+4(\bar\beta\beta)^2}{\bar\beta\,\beta} - \frac{8}{x\,\bar\beta\,\beta}\right) + g_\gamma(x,\beta)\, p_a^\mu \partial_{1\mu} + \frac{g_\gamma\,(x_1,\beta)}{x}\, p_b^\mu \partial_{2\mu} \\
&+ g_\gamma(x,\bar\beta) p_a^\mu \partial_{2\mu} + \frac{g_\gamma\,(x_1,\bar\beta)}{x} p_b^\mu \partial_{1\mu} - \frac{g_\gamma(x,\beta)}{x\bar\beta(1-2\beta)} p_1^\mu \partial_{1\mu} + \frac{g_\gamma(x,\bar\beta)}{x\beta(1-2\beta)} p_2^\mu \partial_{2\mu},
\end{aligned} \tag{6.47}$$

where

$$g_\gamma(x,\beta) = \frac{(1-2\beta)^2}{\beta}\,\frac{(1-x^2)}{x}. \tag{6.48}$$

Similar to the Drell-Yan case, the results shown above were checked against the direct expansion of the NLO matrix element squared of the process $q(p_a) + \bar q(p_b) \to \gamma(p_1) + \gamma(p_2) + g(k)$ through next-to-leading power in the gluon momentum in the soft and collinear limits and then integrating over the unresolved phase space. Full agreement with the above formulas has been found. This completes the check of all the entries present in the master formula given in Eq. (4.73).

6.3 Production of four photons in $q\bar q$ collisions

In this subsection, we apply the master formula presented at the end of Sec. 4, to calculate the subleading power corrections in the zero-jettiness to the production of a high-multiplicity colorless final state. For this purpose, we developed a `FORTRAN` code capable of computing the subleading power corrections to the production of an *arbitrary* number of photons in $q\bar q$ collisions.

The central element of the code is the computation of the generalized currents described in Sec. 5 which can be done using recursive functions in `FORTRAN 90` for an arbitrary number of final-state particles N. The use of such functions makes coding straightforward. However, it also requires careful optimization since the calculation of matrix currents is, in fact, quite expensive. In addition, phase space routines for an arbitrary number of final-state particles are available (see e.g. [42] and [43]), making it straightforward to write a program to compute the subleading power correction in the zero-jettiness variable to a process $q\bar q \to N\gamma$.[7]

An important limitation of the current code is that it works for one observable at a time. This observable should be such that it keeps all photons hard (i.e., not collinear to the incoming quarks and not soft) or, at the very least, it should regulate the cross section

[7] In practice, we have employed the multi-particle phase-space generator written by K. Asteriadis.

in potentially singular regions of the phase space. A possible choice is the product of the squared transverse momenta of all photons, i.e.

$$\mathcal{O}(P_{N\gamma}) = \frac{p_{1,\perp}^2\, p_{2,\perp}^2\, \cdots\, p_{N,\perp}^2}{s^N}, \tag{6.49}$$

where $s = 2p_a \cdot p_b$. The transverse momentum squared of the i-th photon is given by

$$p_{i,\perp}^2 = 2\frac{(p_a \cdot p_i)(p_b \cdot p_i)}{p_a \cdot p_b}. \tag{6.50}$$

We have checked the numerical code by using it to calculate the subleading power corrections for the production of two photons (using the observable given in Eq. (6.49)), and comparing the result with the integration of the analytic expression for subleading corrections to the $q\bar{q} \to \gamma\gamma$ process presented in the previous subsection. We found excellent agreement between the results of the two calculations.

We then used the numerical code to compute the subleading power zero-jettiness correction to $q\bar{q} \to 4\gamma$ for the observable in Eq. (6.49). We found that computation of the subleading power correction for four-photon production with a percent precision required $\mathcal{O}(10\ 000)$ CPU hours. This is to be contrasted with $\mathcal{O}(5)$ CPU hours needed to compute the fiducial leading order cross section for the four-photon production. This increase is related to the complexity and the number of the many different currents that are required at subleading power but, probably, with further optimization, significant improvements in efficiency can be achieved.

In order to validate our numerical results for the zero-jettiness power correction to $q\bar{q} \to 4\gamma$, we used the same `FORTRAN` code to compute bin-integrated cross section for $q\bar{q} \to 4\gamma + g$ with the observable in Eq. (6.49), i.e.

$$\int\limits_{\tau_{\min}}^{\tau_{\max}} \mathrm{d}\tau \frac{\mathrm{d}\sigma_{4\gamma}}{\mathrm{d}\tau}\, \mathcal{O}(P_{4\gamma}), \tag{6.51}$$

for several bins $[\tau_{\min}, \tau_{\max}]$ drawn from the interval $\tau \in [10^{-4}, 1]$. We then write

$$\frac{\mathrm{d}\sigma_{4\gamma}}{\mathrm{d}\tau} = \frac{\mathrm{d}\sigma_{4\gamma}^{\mathrm{LP}}}{\mathrm{d}\tau} + \frac{\mathrm{d}\sigma_{4\gamma}^{\mathrm{NLP}}}{\mathrm{d}\tau}, \tag{6.52}$$

and use the well-known result for the leading-power cross section $\mathrm{d}\sigma_{4\gamma}^{\mathrm{LP}}/\mathrm{d}\tau$, and the following ansatz

$$\frac{\mathrm{d}\sigma_{4\gamma}^{\mathrm{NLP}}}{\mathrm{d}\tau} = \log\tau(C_{0,\mathrm{NLP}} + \tau C_{0,\mathrm{NNLP}} + \tau^2 C_{0,\mathrm{N3LP}}) + C_{1,\mathrm{NLP}} + \tau C_{1,\mathrm{NNLP}} + \tau^2 C_{1,\mathrm{N3LP}} + \tau^3 C_{1,\mathrm{N4LP}}\,,$$

for the subleading one. We then perform a standard χ^2 fit to determine coefficients in the above equation by integrating the ansatz for each τ-bin. The fitted coefficients $C_{0,\mathrm{NLP}}$ and $C_{1,\mathrm{NLP}}$ are then used to determine the subleading soft and collinear coefficients using Eq. (4.70).

The results of the numerical evaluation of the subleading power corrections to four-photon production and their comparison with the fitted results is shown in Table 2. We

Table 2. Next-to-leading power coefficients as defined in Eq. (4.70) for 4γ production at $\sqrt{s} = 200$ GeV with the observable defined in Eq. (6.49). In this case, the collinear coefficient $\mathrm{C}^{\mathrm{NLP},b}$ is equal to $\mathrm{C}^{\mathrm{NLP},a}$. To compute these coefficients, we have set quark electric charges to one, $ee_q \to 1$. Fitted results for the next-to-leading power coefficients are compared with the results obtained by a numerical integration of the derived analytic formula.

coefficient	numeric	fitted
$\mathrm{C}^{\mathrm{NLP},s}$	$2.61598(7) \times 10^{-7}$	$2.5(1) \times 10^{-7}$
$\mathrm{C}^{\mathrm{NLP},a}$	$8.61(8) \times 10^{-7}$	$8.9(5) \times 10^{-7}$

find an agreement between numerically-calculated and fitted coefficients within the error of the fit. We note that further reduction of the fit error is possible, but would require a more significant computational effort.

7 Conclusions

We discussed the computation of next-to-leading power corrections in the zero-jettiness variable to the production of *arbitrary* colorless final states at hadron colliders at next-to-leading order in perturbative QCD. Our goal was to investigate whether a similar degree of universality that exists for leading power corrections can be achieved for the subleading ones. We have relied on the powerful tools developed to study infra-red and collinear limits of QCD which employ momenta redefinition and Lorentz boosts, and we have shown how to use these methods to construct an expansion of the generic phase space and matrix elements squared at next-to-leading power, restricted to the production of colorless finals states.

The most challenging aspect of these expansions comes from the collinear limit where the universality of the limit is lost at next-to-leading power in the sense that the result depends on the radiative process albeit in the simplified kinematics. We have argued that complicated Green's functions that arise from these expansions can be calculated recursively using analogs of Berends-Giele currents [39] which should enable applications of the derived formulas to processes with high multiplicity final states. We have provided an example by computing the next-to-leading power correction in the zero-jettiness variable to the fiducial cross section for the production of four hard photons in $q\bar{q}$ collisions, and we have constructed a numerical code which can be used to compute such power corrections to $q\bar{q} \to N\gamma$ process for any N.

We note that we only considered the $q\bar{q}$ annihilation channel in this paper whereas also $qg \to X + q$ and similar channels are needed for a complete next-to-leading order computation. The most important difference between $qg \to X + q$ and $q\bar{q} \to X + g$ channels is that in the former, a soft final-state quark only contributes to subleading power so that the analysis of the soft limit is significantly simpler than in $q\bar{q} \to X + g$ case. On the other hand, we do not anticipate any significant differences between qg and $q\bar{q}$ channels in the collinear limits. Thus, we believe that the methodology developed in this paper can be applied to all partonic channels in a straightforward way.

An important shortcoming for the numerical implementation of our method is its explicit dependence on observables. This is in strong contrast to calculations at leading power where one obtains all observables in a single Monte-Carlo run by computing many of them for each generated kinematic point, and storing them in histogram bins. It is important to find a way to do this for power corrections as well, since it will make such computations observable-independent and significantly more efficient.

Eventually, one would like to extend the current understanding of the next-to-leading power corrections in the context of existing slicing schemes to arbitrary collider processes, similar to what has been achieved at leading power. This is a highly non-trivial task, and there are lessons that one can take from the computation described in this paper. For example, at next-to-leading order, next-to-leading power contributions to arbitrary processes originate exclusively from soft and collinear limits that can be treated independently. Similar to the leading power case, at next-to-leading power the soft contributions can be treated universally and the collinear contributions – which appear to be the major bottleneck – are localized on the external legs. At the same time, extension to QCD final states will require understanding of jet algorithms and their interplay with power corrections, and, as we already see, observables introduce a significant degree of complexity into the analysis of subleading power corrections even for colorless final states. All in all, it remains to be seen to what extent the approach introduced in this paper can be used to extend slicing schemes to next-to-leading power for *arbitrary* processes at NLO QCD and beyond.

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A Explicit formulas for boosts

For the analysis of collinear contributions, four boosts are required. In the main text, they are denoted as Λ_a, Λ_b, Λ_{ax}, Λ_{bx}. In this appendix we present these quantities explicitly.

A general formula that describes a Lorentz boost that transforms a four-vector Q_i to a four-vector Q_f

$$Q_f^\mu = [\Lambda_{\rm gen}(Q_f, Q_i)]^\mu{}_\nu \, Q_i^\nu, \tag{A.1}$$

reads

$$[\Lambda_{\rm gen}(Q_f, Q_i)]^\mu{}_\nu = g^\mu{}_\nu - \frac{2(Q_f+Q_i)^\mu (Q_f+Q_i)_\nu}{(Q_f+Q_i)^2} + \frac{2Q_f^\mu Q_{i,\nu}}{Q_f^2}. \tag{A.2}$$

The above equation is only valid if $Q_f^2 = Q_i^2$. We use this formula to compute expressions for the Lorentz transformations in the various limits.

A.1 Case $\vec{k}||\vec{p}_a$

We begin with the discussion of the collinear boosts in the case when the gluon is emitted along the direction of the incoming quark with the momentum p_a, $\vec{k}||\vec{p}_a$. Then,

$$Q_f = Q_a = xp_a + p_b, \quad Q_i = p_a + p_b - k, \quad k = (1-x)p_a + \tilde{k}_a, \tag{A.3}$$

and we need to expand the Lorentz boost in Eq. (A.2) to second order in $\tilde{k}_a$. We find

$$\Lambda_a^{\mu\nu}(Q_f, Q_i) = g^{\mu\nu} + \frac{\tilde{k}_a^\mu Q_a^\nu - \tilde{k}_a^\nu Q_a^\mu}{\tilde{Q}_a^2} - \frac{1}{2}\frac{Q_a^\mu Q_a^\nu}{Q_a^4}\tilde{k}_a^2 - \frac{1}{2}\frac{\tilde{k}_a^\mu \tilde{k}_a^\nu}{Q_a^2} + \mathcal{O}(\tilde{k}_a^3). \tag{A.4}$$

We will also need the inverse of Λ_a. It is easy to see that, to the required order, Λ_a^{-1} is obtained from Λ_a by replacing $\tilde{k}_a \to -\tilde{k}_a$. Then

$$\left[\Lambda_a^{-1}\right]^{\mu\nu}(Q_i, Q_f) = g^{\mu\nu} - \frac{\tilde{k}_a^\mu Q_a^\nu - \tilde{k}_a^\nu Q_a^\mu}{Q_a^2} - \frac{1}{2}\frac{Q_a^\mu Q_a^\nu}{Q_a^4}\tilde{k}_a^2 - \frac{1}{2}\frac{\tilde{k}_a^\mu \tilde{k}_a^\nu}{Q_a^2}. \tag{A.5}$$

This transformation needs to be applied to p_a, p_b and k. The calculation of $\Lambda_a p_{a,b}$ and $\Lambda_a k$ requires us to compute scalar products $Q_a \cdot p_{a,b}$ and $\tilde{k}_a \cdot p_{a,b}$. Since $Q_a = xp_a + p_b$, we find

$$Q_a^2 = xs, \quad Q_a \cdot p_a = s/2, \quad Q_a \cdot p_b = xs/2, \tag{A.6}$$

where $s = 2p_a \cdot p_b$. Furthermore,

$$\frac{Q_a \cdot p_a}{Q_a^2} = \frac{1}{2x}, \quad \frac{Q_a \cdot p_b}{Q_a^2} = \frac{1}{2}. \tag{A.7}$$

Since $k = (1-x)p_a + \tilde{k}_a$, we find

$$k \cdot p_a = \tilde{k}_a \cdot p_a, \tag{A.8}$$

and, using $k^2 = 0$, we obtain

$$\tilde{k}_a^2 = -2(1-x)\tilde{k}_a \cdot p_a = -2(1-x)k \cdot p_a. \tag{A.9}$$

It follows from Eq. (3.48) that $\tilde{k}_a \cdot p_b = -\tilde{k}_a \cdot p_a$.

Combining these formulas, we find the following expressions for the boosted momenta

$$\begin{aligned}
\Lambda_a p_a &= p_a + \frac{1}{2x}\,\tilde{k}_a + Q_a\,\frac{kp_a}{Q_a^2}\left(\frac{1-3x}{2x}\right), \\
\Lambda_a p_b &= p_b + \frac{1}{2}\,\tilde{k}_a + Q_a\,\frac{kp_a}{Q_a^2}\left(\frac{3-x}{2}\right), \\
\Lambda_a k &= (1-x)p_a + \frac{1+x}{2x}\,\tilde{k}_a + Q_a\,\frac{kp_a}{Q_a^2}\left(\frac{1-x^2}{2x}\right).
\end{aligned} \tag{A.10}$$

Additionally, we write the formula for the Lorentz transformation of $\tilde{k}_a$ and of $p_a - k$. We obtain

$$\Lambda_a(p_a - k) = xp_a - \frac{1}{2}\tilde{k}_a - Q_a\,\frac{k \cdot p_a}{Q_a^2}\frac{3-x}{2}, \quad \Lambda_a \tilde{k}_a = \tilde{k}_a - \frac{\tilde{k}_a^2}{Q_a^2}\,Q_a. \tag{A.11}$$

To extract soft singularities from the collinear case $\vec{k}||\vec{p}_a$, we need a Lorentz boost Λ_{ax}^{-1}. It reads

$$[\Lambda_{ax}^{-1}]^{\mu\nu} = g^{\mu\nu} - \frac{2(xp_a + p_b + \sqrt{x}P_{ab})^\mu (xp_a + p_b + \sqrt{x}P_{ab})^\nu}{s(2x + \sqrt{x}(x+1))} + \frac{2(xp_a + p_b)^\mu P_{ab}^\nu}{s\sqrt{x}}. \quad (A.12)$$

The expansion of Λ_{ax}^{-1} around $x = 1$ is given by the following formula

$$[\Lambda_{ax}^{-1}]^{\mu\nu} = g^{\mu\nu} - \frac{1-x}{2}\omega_{ab}^{\mu\nu} + \mathcal{O}((1-x)^2), \quad (A.13)$$

where

$$\omega_{ab}^{\mu\nu} = \frac{p_a^\mu p_b^\nu - p_a^\nu p_b^\mu}{p_a \cdot p_b}. \quad (A.14)$$

A.2 Case $\vec{k}||\vec{p}_b$

We continue with the discussion of the collinear boosts in case the gluon is emitted along the direction of the incoming anti-quark with momentum p_b, $\vec{k}||\vec{p}_b$. Then,

$$Q_f = Q_b = p_a + xp_b, \quad Q_i = p_a + p_b - k, \quad k = (1-x)p_b + \tilde{k}_b, \quad (A.15)$$

and we need to expand the Lorentz boost in Eq. (A.2) to second order in $\tilde{k}_b$. We find

$$\Lambda_b^{\mu\nu}(Q_f, Q_i) = g^{\mu\nu} + \frac{\tilde{k}_b^\mu Q_b^\nu - \tilde{k}_b^\nu Q_b^\mu}{\tilde{Q}_b^2} - \frac{1}{2}\frac{Q_b^\mu Q_b^\nu}{Q_b^4}\tilde{k}_b^2 - \frac{1}{2}\frac{\tilde{k}_b^\mu \tilde{k}_b^\nu}{Q_b^2} + \mathcal{O}(\tilde{k}_b^3). \quad (A.16)$$

To the required order, the inverse Λ_b^{-1} is obtained from Λ_b by replacing $\tilde{k}_b \to -\tilde{k}_b$. Then

$$\left[\Lambda_b^{-1}\right]^{\mu\nu}(Q_f, Q_i) = g^{\mu\nu} - \frac{\tilde{k}_b^\mu Q_b^\nu - \tilde{k}_b^\nu Q_b^\mu}{Q_b^2} - \frac{1}{2}\frac{Q_b^\mu Q_b^\nu}{Q_b^4}\tilde{k}_b^2 - \frac{1}{2}\frac{\tilde{k}_b^\mu \tilde{k}_b^\nu}{Q_b^2}. \quad (A.17)$$

The calculation of $\Lambda_b p_{a,b}$ and $\Lambda_b k$ requires the scalar products $Q_b \cdot p_{a,b}$ and $\tilde{k}_b \cdot p_{a,b}$. Since $Q_b = xp_b + p_a$, we find

$$Q_b^2 = xs, \quad Q_b \cdot p_a = xs/2, \quad Q_b \cdot p_b = s/2, \quad (A.18)$$

so that

$$\frac{Q_b \cdot p_a}{Q_b^2} = \frac{1}{2}, \quad \frac{Q_b \cdot p_b}{Q_b^2} = \frac{1}{2x}. \quad (A.19)$$

Since $k = (1-x)p_b + \tilde{k}_b$, we find

$$k \cdot p_b = \tilde{k}_b \cdot p_b, \quad (A.20)$$

and because $k^2 = 0$, we obtain

$$\tilde{k}_b^2 = -2(1-x)\tilde{k}_b \cdot p_a = -2(1-x)k \cdot p_b. \quad (A.21)$$

It follows from Eq. (3.48) that $\tilde{k}_b \cdot p_a = -\tilde{k}_b \cdot p_b$.

Using the above results, we find the following expressions for the boosted momenta

$$\begin{aligned}\Lambda_b p_b &= p_b + \frac{1}{2x}\,\tilde{k}_b + Q_b\,\frac{kp_b}{Q_b^2}\left(\frac{1-3x}{2x}\right),\\ \Lambda_b p_a &= p_a + \frac{1}{2}\,\tilde{k}_b + Q_b\,\frac{kp_b}{Q_b^2}\left(\frac{3-x}{2}\right),\\ \Lambda_b k &= (1-x)p_b + \frac{1+x}{2x}\,\tilde{k}_b + Q_b\,\frac{kp_b}{Q_b^2}\left(\frac{1-x^2}{2x}\right).\end{aligned} \tag{A.22}$$

We also find

$$\Lambda_b(p_b-k) = xp_b - \frac{1}{2}\tilde{k}_b - Q_b\,\frac{kp_b}{Q_b^2}\frac{3-x}{2}, \quad \Lambda_b\tilde{k}_b = \tilde{k}_b - \frac{\tilde{k}_b^2}{Q_b^2}\,Q_b. \tag{A.23}$$

The Lorentz boost Λ_{bx}^{-1} reads

$$\left[\Lambda_{bx}^{-1}\right]^{\mu\nu} = g^{\mu\nu} - \frac{2(p_a+xp_b+\sqrt{x}P_{ab})^\mu(p_a+xp_b+\sqrt{x}P_{ab})^\nu}{s(2x+\sqrt{x}(x+1))} + \frac{2(p_a+xp_b)^\mu P_{ab}^\nu}{s\sqrt{x}}. \tag{A.24}$$

The expansion of Λ_{bx}^{-1} around $x=1$ is given by

$$[\Lambda_{bx}^{-1}]^{\mu\nu} = g^{\mu\nu} + \frac{1-x}{2}\omega_{ab}^{\mu\nu} + \mathcal{O}((1-x)^2), \tag{A.25}$$

where the tensor $\omega_{ab}^{\mu\nu}$ can be found in Eq. (A.14).

B Formulas for remainders

In Eq. (4.73), we have defined a remainder for the $\vec{k}||\vec{p}_a$ case

$$F_{\text{fin},a} = F_{\text{rem},a} + \frac{s}{2}F_{rr,a} + \frac{s}{2}\left(C_{1a}^k + C_{2a}^k + C_{3a}^k\right), \tag{B.1}$$

where

$$\begin{aligned}F_{\text{rem},a} =& -|\mathcal{M}|^2(p_b, xp_a, ...) + \frac{1}{2sx}\text{Tr}\left[N_a\hat{p}_b\gamma^\nu\hat{p}_a N_b^+\hat{p}_a\gamma_\nu\hat{p}_b\right] + \text{c.c.}\\ &- \frac{1}{2x}\text{Tr}\left[N_a\hat{p}_b\gamma_\nu\hat{p}_a R_{\text{fin}}^{\nu,+}\hat{p}_b\right] + \text{c.c.} + \frac{1}{x}\text{Tr}\left[N_a\hat{p}_b N_a^+\hat{p}_b\right],\end{aligned} \tag{B.2}$$

and

$$F_{rr,a} = \frac{1}{s}\text{Tr}\left[R_{\text{fin}}^\mu\hat{p}_a N_b^+\hat{p}_a\gamma_\mu\hat{p}_b\right] + \text{c.c.} + \frac{2}{s}\text{Tr}\left[N_b\hat{p}_a N_b^+\hat{p}_a\right] - \text{Tr}\left[R_{\text{fin}}^\nu\hat{p}_a R_{\text{fin}}^{\mu,+}\hat{p}_b\right] g_{\perp,\mu\nu}. \tag{B.3}$$

When computing the collinear expansion of the matrix element squared in the $\vec{k}||\vec{p}_a$ limit, we pointed out that three terms need to be expanded to second order in the transverse momentum $k_\perp$, after the Lorentz boost is applied. They are

$$\begin{aligned}
C_{1a} &= \frac{2}{2p_a\cdot k}\mathrm{Tr}\left[N_a\hat{\kappa}_a N_a^+\hat{p}_b\right]_{\Lambda_a},\\
C_{2a} &= \mathrm{Tr}\left[\frac{N_a\hat{\kappa}_a\gamma_\nu\hat{p}_a N_{\mathrm{fin},a}^{+,\nu}\hat{p}_b}{(-2p_a\cdot k)}\right]_{\Lambda_a} + \mathrm{c.c.},\\
C_{3a} &= \frac{2\kappa_{a,\nu}}{(1-x)(-2p_a\cdot k)}\mathrm{Tr}\left[N_a x\hat{p}_a\left(R_{\mathrm{fin}}^{+,\nu} - N_b^+\frac{\hat{p}_a\gamma^\nu}{s}\right)\hat{p}_b\right]_{\Lambda_a} + \mathrm{c.c.}\,.
\end{aligned} \tag{B.4}$$

Since the Lorentz boost of κ_a gives the transverse momentum $k_\perp$ and since two powers of $k_\perp$ are needed to obtain the non-vanishing contribution we define quantities that contribute to the cross section in the collinear limit

$$C_{ia}^k = \lim_{k_\perp^2\to 0}\langle C_{ik}\rangle_{k_\perp}. \tag{B.5}$$

In the above formula the two brackets indicate that one averages over $k_\perp$ directions. To compute C_{ia}^k, $i=1,2,3$, we require the expansion of N_a, N_b and R_{fin}^ν after the boost Λ_a to first order in $k_\perp$. Performing the boost, and expanding in $k_\perp$, we find

$$\begin{aligned}
&N_a\left(p_b + \frac{k_\perp}{2}, xp_a - \frac{k_\perp}{2}, P_X\right) = N_a^{(0)} - \frac{k_{\perp,\mu}}{2}N_a^{(1),\mu} + \dots,\\
&N_b\left(p_b - (1-x)p_a - \frac{k_\perp}{2x}, p_a + \frac{k_\perp}{2x}, P_X\right) = N_b^{(0)} + \frac{k_{\perp,\mu}}{2x}N_b^{(1),\mu} + \dots,\\
&R_{\mathrm{fin}}^\nu\left(p_b + \frac{k_\perp}{2}, p_a + \frac{k_\perp}{2x}, p_a(1-x) + \frac{k_\perp(1+x)}{2x}, P_X\right) = R_{\mathrm{fin}}^{(0),\nu} + R_{\mathrm{fin}}^{(1),\nu\mu}k_{\perp,\mu} + \dots,
\end{aligned} \tag{B.6}$$

where ellipses stand for $\mathcal{O}(k_\perp^2)$ contributions. The quantities $N_a^{(1),\mu}, N_b^{(1),\mu}$ and $R_{\mathrm{fin}}^{(1),\nu\mu}$ are particular Green's functions that can be computed following the discussion in Sec. 5. For our purposes here, we assume that they are known. We find

$$C_{1a}^k = \frac{1-x}{2}g_\perp^{\mu\nu}D_\nu^{xa,b}\,\mathrm{Tr}\left[N_a(xp_a,\dots)\gamma_\mu N_a^+(xp_a,\dots)\hat{p}_b\right], \tag{B.7}$$

where $D_\nu^{xa,b} = x^{-1}\partial/\partial p_a^\nu - \partial/\partial p_b^\nu$.

The second limit is more complex. To write it in the compact form, we introduce two matrix functions

$$\begin{aligned}
X^{(0),\alpha,+} &= R_{\mathrm{fin}}^{(0),\alpha,+}(1-x) + N_b^{(0),+}\frac{\hat{q}_{ba}\gamma^\alpha}{s},\\
X^{(1),\alpha\mu,+} &= R_{\mathrm{fin}}^{(1),\alpha\mu,+}(1-x) + N_b^{(1),\mu,+}\frac{\hat{q}_{ba}\gamma^\alpha}{2xs},
\end{aligned} \tag{B.8}$$

where $\hat{q}_{ba} = \hat{p}_b - (1-x)\hat{p}_a$, and write

$$\begin{aligned}
C_{2a}^k = &-\frac{g_\perp^{\mu\nu}}{4}\mathrm{Tr}\left[N_a^{(0)}\gamma_\nu\gamma^\alpha\hat{p}_a\left(\frac{N_b^{(0),+}\gamma_\mu\gamma_\alpha\hat{p}_b}{xs} - X_\alpha^{(0),+}\gamma_\mu - 2X_{\alpha\mu}^{(1),+}\hat{p}_b\right)\right]\\
&+\frac{g_\perp^{\mu\nu}}{4}\mathrm{Tr}\left[\left(-N_{a,\mu}^{(1)}\gamma_\nu\gamma_\alpha\hat{p}_a + \frac{1}{x}N_a^{(0)}\gamma_\nu\gamma_\alpha\gamma_\mu\right)X^{(0),\alpha,+}\hat{p}_b\right] + \mathrm{c.c.}.
\end{aligned} \tag{B.9}$$

Finally, the third term reads

$$
\begin{aligned}
C_{3a}^{k} =& \frac{g_{\perp,\mu\nu}}{2}\mathrm{Tr}\left[\left(-N_a^{(1),\mu}x\hat{p}_a + N_a^{(0)}\gamma^\mu\right)\left(R_{\text{fin}}^{(0),\nu,+} - N_b^{(0),+}\frac{\hat{p}_a\gamma^\nu}{s}\right)\hat{p}_b\right] \\
&+ g_{\perp,\mu\nu}\mathrm{Tr}\left[N_a^{(0)}x\hat{p}_a\left(R_{\text{fin}}^{(1),\nu\mu,+} - N_b^{(1),\mu,+}\frac{\hat{p}_a\gamma^\nu}{2xs}\right)\hat{p}_b\right] \\
&+ \frac{g_{\perp,\mu\nu}}{2}\mathrm{Tr}\left[N_a^{(0)}x\hat{p}_a R_{\text{fin}}^{(0),\nu,+}\gamma^\mu\right] - \frac{1}{s}\mathrm{Tr}\left[N_a^{(0)}\hat{p}_a N_b^{(0),+}(x\hat{p}_a + \hat{p}_b)\right] + \text{c.c.}.
\end{aligned}
\tag{B.10}
$$

C Derivation of Eq. (4.40)

When simplifying Eq. (4.40), we wrote the integral of the function $W_3^{(a)}$ defined in Eq. (4.39) in the following way

$$
\begin{aligned}
&\int \mathrm{d}x\ \mathrm{d}\Phi_m^{ab}\frac{W_3^{(a)}(x,p_a,p_b,P_X,\mathcal{O}(P_X))}{(1-x)^2} = \\
&-\frac{1}{2}\left(\kappa_m + 2p_a^\mu\frac{\partial}{\partial p_a^\mu} + \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma}\right)L_{\rho\sigma}\right)\mathcal{O}(P_X)\ |\mathcal{M}|^2(p_b,xp_a,P_X) \\
&-\frac{1}{2}\int \mathrm{d}x\ \mathrm{d}\Phi(xp_a,p_b,P_X)\ \frac{1}{(1-x)_+} \\
&\times\left(\kappa_m + 2p_a^\mu\frac{\partial}{\partial p_a^\mu} + \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma}\right)L_{\rho\sigma}\right)\mathcal{O}(P_X)\ |\mathcal{M}|^2(p_b,xp_a,P_X).
\end{aligned}
\tag{C.1}
$$

In this appendix, we explain how to derive Eq. (C.1). To this end, we note that the function $W_3^{(a)}$ (c.f. Eq. (4.39)) is written as a difference of three terms, i.e.

$$
W_3^{(a)}(x) = F(x) - F(1) + (1-x)F'(1),
\tag{C.2}
$$

where $F'(1) = \mathrm{d}F(x)/\mathrm{d}x$ at $x=1$. Then, using integration by parts, it is easy to see that the following equation holds

$$
\int_0^1 \mathrm{d}x\frac{W_3^{(a)}(x)}{(1-x)^2} = -F'(1) - \int_0^1 \frac{\mathrm{d}x}{1-x}\left(xF'(x) - F'(1)\right).
\tag{C.3}
$$

Hence, we have

$$
\int_0^1 \mathrm{d}x\ \mathrm{d}\Phi_m^{ab}\ \frac{W_3^{(a)}(x)}{(1-x)^2} = -F'(1)\ \mathrm{d}\Phi_m^{ab} - \int_0^1 \mathrm{d}x\ \frac{\mathrm{d}\Phi_m^{ab}}{1-x}\left(xF'(x) - F'(1)\right).
\tag{C.4}
$$

Comparing Eq. (C.2) and Eq. (4.40), we find

$$
\begin{aligned}
F(x) &= \lambda^{\kappa_m}\mathcal{O}(\lambda\Lambda_{ax}^{-1}P_X)|\mathcal{M}|^2(xp_a,p_b,\lambda\Lambda_{ax}^{-1}P_X), \\
F'(1) &= \frac{1}{2}\left(\kappa_m + 2p_a^\mu\frac{\partial}{\partial p_a^\mu} + \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma}\right)L_{\rho\sigma}\right)|\mathcal{M}|^2\left(p_a,p_b,P_X\right)\mathcal{O}(P_X).
\end{aligned}
\tag{C.5}
$$

To simplify Eq. (C.4), we need to compute $F'(x)$. To this end, we introduce $x_1 = x + \Delta x$ and note that

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x_1) - F(x)}{\Delta x}. \tag{C.6}$$

We also note that because of the nature of the Lorentz boosts with x_1 and x, the following relation holds

$$\lambda_{x_1} \Lambda_{ax_1}^{-1} = (I + \delta K) \lambda_x \Lambda_{ax}^{-1}, \tag{C.7}$$

where $I^{\mu\nu} = g^{\mu\nu}$ and

$$\delta K^{\mu\nu} = \frac{\Delta x}{2x} \left(g^{\mu\nu} + \omega_{ab}^{\mu\nu} \right). \tag{C.8}$$

Hence,

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Bigg[\lambda_1^{\kappa_m} \mathcal{O}\left((I + \delta K) \lambda \Lambda_{ax}^{-1} P_X \right) |\mathcal{M}|^2 \left(x_1 p_a, p_b, (I + \delta K) \lambda \Lambda_{ax}^{-1} P_X \right) \\ &\quad - \lambda^{\kappa_m} \mathcal{O}(\lambda \Lambda_{ax}^{-1} P_X) |\mathcal{M}|^2 (x p_a, p_b, \lambda \Lambda_{ax}^{-1} P_X) \Bigg] \\ &= \frac{\lambda^{\kappa_m}}{2x} \left[\kappa_m + 2 p_a^\mu \frac{\partial}{\partial p_a^\mu} + \left(g^{\rho\sigma} + \omega_{ab}^{\rho\sigma} \right) L_{\rho\sigma} \right] \mathcal{O}(Q_X) |\mathcal{M}|^2 (x p_a, p_b, Q_X), \end{aligned} \tag{C.9}$$

where derivatives that appear in $L_{\rho\sigma}$ are computed with respect to momenta $Q_X \lambda = \Lambda_{ax}^{-1} P_X$. Finally, we change the momentum of the colorless system $P_X \to \lambda^{-1} \Lambda_{ax} P_X$ in the phase space, and obtain the result shown in Eq. (C.1).

D Comparison to results in the literature for the Drell-Yan process

In this appendix, we show how to derive results for zero-jettiness power corrections to the process $q\bar{q} \to Vg$ obtained in Ref. [16], from our Drell-Yan master formula in Eq. (6.19). At first glance the two results look very different, since we work with partonic cross sections for $q\bar{q} \to l^+ l^-$, where as authors of Ref. [16] work with hadronic production cross section of a fixed-mass vector boson, and study its rapidity distribution. Furthermore, the definition of the zero-jettiness variable itself is different, as Born-projected momenta are employed in Ref. [16] to calculate it. Nevertheless, as discussed below, it is possible to start with Eq. (6.19), and derive Eq. (5.36) in Ref. [16], by taking into account two important points.

First, we incorporate the constraint on the dilepton rapidity Y (which is equivalent to the rapidity of the vector boson) and the dilepton invariant mass M as the observable function in study. It reads

$$\mathcal{O}^{\rm DY}(p_1, p_2) = \delta\left(2 p_1 \cdot p_2 - M^2 \right) \, \delta\left(\frac{1}{2} \ln \frac{P_b \cdot (p_1 + p_2)}{P_a \cdot (p_1 + p_2)} - Y \right), \tag{D.1}$$

where $P_{a,b}$ are the momenta of incoming hadrons. Denoting the momenta fractions of q and $\bar{q}$ as $\xi_{a,b}$, one may expect that the subleading power correction to the hadronic cross section computed in Ref. [16] can be obtained by integrating the subleading partonic cross section, obtained with the help of our master formula, with parton distribution functions $f_{a,b}$

$$\left.\frac{\mathrm{d}\sigma_{\mathrm{had}}^{\mathrm{DY,NLP}}}{\mathrm{d}\tau\ \mathrm{d}Y\ \mathrm{d}M^2}\right|_{[16]} = \int_0^1 \mathrm{d}\xi_a\ \mathrm{d}\xi_b\ f_a(\xi_a) f_b(\xi_b) \left.\frac{\mathrm{d}\sigma^{\mathrm{DY,NLP}}}{\mathrm{d}\tau}\right|_{\mathcal{O}=\mathcal{O}^{\mathrm{DY}}}, \tag{D.2}$$

where we also need to write the partonic center-of-mass energy squared s as $\xi_a \xi_b S$, with $S = 2P_a \cdot P_b$.

We note that, depending on the collinear sector, our variable x corresponds to variables $z_{a,b}$ in Ref. [16]. We also note that $\mathrm{d}\sigma^{\mathrm{DY,NLP}}$ is described by our master formula and includes derivatives with respect to lepton momenta that act on the observable $\mathcal{O}^{\mathrm{DY}}$. These derivatives are re-written using integration by parts, resulting in derivatives of parton distribution functions.

Upon doing that, we find that the result that follows from Eq. (D.2) *does not agree* with the result for subleading power corrections presented in Ref. [16]. The reason for this is the difference in the zero-jettiness definition, as we now explain. Indeed, the definition employed in this paper uses partonic momenta of the process, while in Ref. [16] Born-projected initial-state momenta are used. These are obtained using the Born-like momentum fractions constructed from vector boson mass and its rapidity, as opposed to $\xi_{a,b}$ momenta fractions that we would use to define zero-jettiness. Hence, to match the result of Ref. [16], we need to account for this fact.

To this end, we compute the difference between to definitions. Suppose we find a gluon momentum k that satisfies the zero-jettiness constraint $T_0(k) = \tau$, where $T_0(k)$ is the definition of zero-jettiness adopted in this paper. Then, when the same momentum is used to evaluate zero-jettiness as defined in Ref. [16] yields

$$\left. T_0(k) \right|_{[16]} = \begin{cases} \tau \left[1 - (2-\psi)\, \dfrac{\omega_k}{\sqrt{s}} + \mathcal{O}(w_k^2)\right], & \text{soft}, \\ \tau\ z_a \left[1 + \dfrac{1+z_a}{z_a s} p_a k + \mathcal{O}((p_a k)^2)\right], & \text{collinear } a, \\ \tau\ z_b \left[1 + \dfrac{1+z_b}{z_b s} p_b k + \mathcal{O}((p_b k)^2)\right], & \text{collinear } b. \end{cases} \tag{D.3}$$

We observe that in addition to subleading terms, there is also a re-scaling at leading power in the collinear contributions. Given the relations of Eq. (D.3), we calculate how each of the three contributions (soft, *ca* and *cb*) changes. Although this is rather straightforward, one should exercise significant care when switching from one definition of the zero-jettiness to the other. For example, in our formula the term from Eq. (4.40), along with the manipulations presented in Appendix C, are affected by the presence of the extra factor $1/(z_{a,b})$ and have to be carefully re-analyzed.

Finally, putting everything together, we can express the result from Ref. [16] for Drell-Yan in terms of our master formula in the following manner

$$\left.\frac{\mathrm{d}\sigma_{\mathrm{had}}^{\mathrm{DY,NLP}}}{\mathrm{d}\tau\ \mathrm{d}Y\ \mathrm{d}M^2}\right|_{[16]} = \int_0^1 \mathrm{d}\xi_a\ \mathrm{d}\xi_b\ f_a(\xi_a) f_b(\xi_b) \left.\frac{\mathrm{d}\sigma^{\mathrm{DY,NLP}}}{\mathrm{d}\tau}\right|_{\mathcal{O}=\mathcal{O}^{\mathrm{DY}}} \tag{D.4}$$

where

$$\frac{\mathrm{d}\sigma^{\mathrm{DY,NLP}}}{\mathrm{d}\tau} = \left[\frac{\mathrm{d}\sigma^{\mathrm{DY,NLP}}}{\mathrm{d}\tau}\Big|_{\text{Eq. (6.19)}} - \frac{Q\tau}{2s}\left(\frac{z_a+z_b}{z_a z_b}\right)\frac{\mathrm{d}\sigma^{\mathrm{DY,LP}}}{\mathrm{d}\tau}\right]\left(\frac{1}{z_a z_b}\right) + \frac{4[\alpha_s]C_F Q}{s}\int\limits_0^1\int\limits_0^1 \frac{\mathrm{d}\sigma_0}{2z_a z_b}\left[\frac{\delta(1-z_b)}{(1-z_a)_+} + \frac{\delta(1-z_a)}{(1-z_b)_+}\right]\mathcal{O}(p_1,p_2)\,\mathrm{d}z_a\,\mathrm{d}z_b, \tag{D.5}$$

and

$$\frac{\mathrm{d}\sigma^{\mathrm{DY,LP}}}{\mathrm{d}\tau} = \frac{[\alpha_s]C_F}{\tau}\int\limits_0^1\int\limits_0^1 \mathrm{d}\sigma_0\left[4\delta(1-z_a)\delta(1-z_b) + \frac{1+z_a^2}{z_a(1-z_a)_+}\delta(1-z_b) + \frac{1+z_b^2}{z_b(1-z_b)_+}\delta(1-z_a)\right]\mathcal{O}(p_1,p_2)\,\mathrm{d}z_a\,\mathrm{d}z_b, \tag{D.6}$$

is the finite contribution arising from the leading power terms. The Drell-Yan leading order cross section $\mathrm{d}\sigma_0$ is given in Eq. (6.6). In Eq. (D.4) we should further integrate over the leptonic angle β, and take $Q = M$ to account for the definition of the zero-jettiness in Ref. [16]. The result obtained with the help of Eq. (D.5) is in complete agreement with Eq. (5.36) of Ref. [16].

E The Drell-Yan process with dilepton final states

In this appendix, we consider a photon-mediated dilepton production in quark-antiquark annihilation

$$q(p_a) + \bar{q}(p_b) \to l(p_1) + \bar{l}(p_2), \tag{E.1}$$

and compute zero-jettiness power corrections by directly expanding the phase space and the matrix element squared of the process. As we will see, this procedure allows us to compute power corrections to the rapidity distribution of a single charged lepton.

The Born cross section can be written as

$$\mathrm{d}\sigma_0 = \bar{\sigma}_0\,\left(1-2\epsilon+\cos\theta_1^2\right)[\mathrm{d}\Omega_1^{(d-1)}], \tag{E.2}$$

where θ_1 is the angle between the lepton with the momentum p_1 and the initial quark with the momentum p_a,

$$\bar{\sigma}_0 = \frac{\pi Q_q^2 \alpha_{\mathrm{QED}}^2 N_\epsilon}{2N_c\, s^{1+\epsilon}}, \tag{E.3}$$

and $[\Omega_1^{(d-1)}]$ is the solid angle of p_1 defined in the center-of-mass frame of the colliding quark and anti-quark, normalized to the solid angle in $d-2$ dimensions. Furthermore, Q_q is the quark charge in units of the positron charge, α_{QED} is the fine structure constant, $N_c = 3$ is the number of colors, and

$$N_\epsilon = 2^{2\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}, \tag{E.4}$$

is the normalization factor.

To compute power correction in the zero-jettiness, we consider the real-emission process

$$q(p_a) + \bar{q}(p_b) \to l(p_1) + \bar{l}(p_2) + g(k), \tag{E.5}$$

and impose the appropriate constraint on the final-state gluon. We write

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} &= \frac{1}{8sN_c^2}\int [\mathrm{d}p_1][\mathrm{d}p_2][\mathrm{d}k](2\pi)^d\delta(p_a+p_b-k-p_1-p_2)\\ &\times \delta(\tau - T_0(p_a,p_b,k))\sum_{\text{col,pol}}|\mathcal{M}|^2(p_a,p_b,p_1,p_2,k).\end{aligned} \tag{E.6}$$

We then integrate over the momentum of the anti-lepton p_2, and find

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} &= \frac{2\pi}{8sN_c^2}\int [\mathrm{d}p_2][\mathrm{d}k]\delta((p_a+p_b-k-p_1)^2)\\ &\times \delta(\tau - T_0(p_a,p_b,k))\sum_{\text{col,pol}}|\mathcal{M}|^2(p_a,p_b,p_1,p_2,k).\end{aligned} \tag{E.7}$$

Since

$$\delta((p_a+p_b-k-p_1)^2) = \delta(s - 2\sqrt{s}(\omega_k + E_1) + 2\omega_k E_1\rho_{1k}), \tag{E.8}$$

we can remove this δ-function by integrating over the lepton energy E_1 and then remove the zero-jettiness δ-function by integrating over the gluon energy ω_k. We find

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} &= \bar{\sigma}_0\,\frac{[\alpha_s]C_F}{\tau^{1+2\epsilon}}\int [\mathrm{d}\Omega_1^{(d-1)}]\,[\mathrm{d}\Omega_k^{(d-1)}]\,\frac{\left(1-2\frac{\omega_k^*}{\sqrt{s}}\right)^{1-2\epsilon}}{\left(1-\frac{\omega_k^*}{\sqrt{s}}\rho_{1k}\right)^{2-2\epsilon}}\left(\frac{Q}{\sqrt{s}\psi_k}\right)^{2-2\epsilon}\\ &\times \sum_{\text{col,pol}}\frac{\tau^2\,|\mathcal{M}|^2(p_a,p_b,p_1^*,p_2,k^*)}{4g_s^2N_cC_FQ_q^2e^4},\end{aligned} \tag{E.9}$$

where

$$\omega_k = \frac{\tau Q}{\sqrt{s}\psi_k}, \quad E_1 = \frac{\sqrt{s}}{2}\frac{1-\frac{2\omega_k}{\sqrt{s}}}{\left(1-\frac{\omega_k}{\sqrt{s}}\rho_{1k}\right)}. \tag{E.10}$$

Since $E_1 \geq 0$ is required, we derive the following constraint

$$\omega_k \leq \frac{\sqrt{s}}{2} \;\Rightarrow\; \psi_k \geq \frac{2\tau Q}{s}. \tag{E.11}$$

Although the constraints appear to be more complex than what has been discussed in Sec. 2, the situation is rather similar in that there are two distinct cases for the gluon emission angle: $\theta_k \sim 1$, $\omega_k \sim \tau$ and $\theta_k \sim \sqrt{\tau/Q}$, $\omega_k \sim \sqrt{s}$. The first case describes the soft emission and the second case – the collinear one.

We then expand the integrand in the soft region and in the two collinear regions. We begin with the discussion of the soft expansion. Computing the matrix element squared, expanding in $\omega_k \sim \tau$ and using the fact that $\rho_{bk} = 2-\rho_{ak}$, we find that two angular integrals, given in Eq. (3.31), are required to calculate the soft contribution to the cross section through next-to-leading power. Keeping τ-suppressed terms through next-to-leading power and using the explicit expression for the matrix element squared $|\mathcal{M}|^2(p_1,p_2,p_3,p_4,k)$ in Eq. (E.9), we find

$$\frac{\mathrm{d}\sigma^{(s)}}{\mathrm{d}\tau} = \frac{4[\alpha_s]C_F}{\tau^{1+2\epsilon}}\left(\frac{Q^2}{s}\right)^{-\epsilon}\mathrm{d}\sigma_0\left[\frac{1}{\epsilon}+\frac{1-2\epsilon}{1-\epsilon}\frac{Q\tau}{s}\right]. \tag{E.12}$$

An interesting feature of this result is that $\mathcal{O}(\tau)$ power corrections to the cross section are ϵ-finite in this case.

We continue with computing the collinear contribution, and focus on the case when the gluon is emitted along the incoming quark with momentum p_a. Then

$$\psi_k = \rho_{ak} = \rho. \tag{E.13}$$

We write

$$\rho = \frac{2\tau Q}{s(1-z)}, \tag{E.14}$$

with $0 < z < 1$ so that

$$\omega_k = \frac{\tau Q}{\sqrt{s}\psi_k} = \frac{\sqrt{s}}{2}(1-z). \tag{E.15}$$

It follows that

$$E_1 = \frac{\sqrt{s}z}{2-(1-z)\rho_{1k}}, \tag{E.16}$$

and, furthermore,

$$\frac{\tau Q}{s\rho} = \frac{(1-z)}{2}. \tag{E.17}$$

The expansion in the collinear case is the expansion in $\rho \sim \tau$ at fixed z. The scalar products are

$$2p_a\cdot k = \sqrt{s}\omega_k\rho = \frac{s}{2}(1-z)\rho, \quad 2p_b\cdot k = \sqrt{s}\omega_k(2-\rho) = \frac{s(1-z)}{2}\left(2-\frac{2\tau Q}{1-z}\right). \tag{E.18}$$

We will also need to account for the scalar product between the momenta of the emitted gluon and the outgoing lepton, and write it in such a way that its expansion in the collinear limit becomes possible. To do this, we consider the unit vector $\vec{k}/\omega_k = \vec{n}_k$ and write it as

$$\vec{n}_k = (1-\rho)\vec{n}_1 + \sqrt{\rho(2-\rho)}\vec{n}_{k,\perp}, \tag{E.19}$$

where $\vec{n}_1$ is the unit vector in the direction of the momentum $\vec{p}_1$, and $\vec{n}_{k\perp}$ is a unit vector which is orthogonal to $\vec{n}_1$. With this, we find

$$\rho_{1k} = 1-\vec{n}_1\cdot\vec{n}_k = \rho_{1a} + \rho\,(1-\rho_{1a}) - \sqrt{\rho(2-\rho)}(\vec{n}_{k,\perp}\cdot\vec{n}_1). \tag{E.20}$$

Hence, expansion of this quantity in τ is straightforward but, because of the last term in Eq. (E.20) the expansion proceeds in powers of $\sqrt{\rho}\sim\sqrt{\tau}$.

Upon the expansion in τ around collinear limit, we will average over directions of $\vec{n}_{k,\perp}$. This removes all contributions that contain odd powers of $\vec{n}_{k,\perp}$. The angular integration that is needed to compute the contribution at next-to-leading power is given by the standard formula

$$(\vec{n}_{k,\perp}\cdot\vec{n}_1)^2 \to \frac{\rho_{a1}(2-\rho_{a1})}{d-2}. \tag{E.21}$$

We find it to be convenient to combine both collinear contributions. We then write

$$\frac{\mathrm{d}\sigma^{(ca+cb)}}{\mathrm{d}\tau} = \bar{\sigma}_0\,\frac{[\alpha_s]C_F}{\tau^{1+\epsilon}}Q^{-\epsilon}\int\limits_0^1 \mathrm{d}z\,\left(g_c^{(1)}(z,c_1)+\frac{\tau Q}{s}g_c^{(2)}(z,c_1)\right). \tag{E.22}$$

The function $g_c^{(1)}$ has the following structure

$$g_c^{(1)}(z,c_1) = (1-z)^{-\epsilon}\tilde{P}_{qq}(z)\;\tilde{g}_c^{(1)}(z,c_1), \tag{E.23}$$

where $\tilde{P}_{qq}(z)$ is related to $q\to q^*g$ collinear splitting. The function $g_c^{(2)}$ describes the power correction. We expose its structure by writing it as

$$g_c^{(2)}(z,c_1) = 4(1+\epsilon)(1-2\epsilon+c_1^2)\left(\frac{1}{(1-z)^{2+\epsilon}}+\frac{\epsilon}{(1-z)^{1+\epsilon}}\right)+g_{c,\mathrm{fin}}^{(2)}(z,c_1), \tag{E.24}$$

which separates divergent $z\to 1$ contributions from terms that do not contain non-integrable singularities on the interval $0<z<1$; such contributions are described by the function $g_{c,\mathrm{fin}}^{(2)}(z)$. We note that a direct integration of the function $g_c^{(2)}$ over z at fixed c_1 is possible. Integrating over z, and discarding all terms that vanish in the $\epsilon\to 0$ limit, we find

$$\int\limits_0^1 \mathrm{d}z\; g_c^{(2)}(z,c_1) = \frac{2\left(3+37c_1^2-11c_1^4+3c_1^6\right)}{3(1-c_1^2)^2} - 4\,c_1^3\ln\frac{1+c_1}{1-c_1}. \tag{E.25}$$

We observe that, similar to the soft contribution, the next-to-leading power collinear contribution is ϵ-finite as well.

We are now in position to write the next-to-leading power zero-jettiness correction for the process in Eq. (E.1) at fixed c_1. Combining the soft and collinear contributions using Eqs (E.12, E.22, E.24), we obtain

$$\frac{\mathrm{d}\sigma^{\mathrm{NLP}}}{\mathrm{d}\tau\mathrm{d}c_1} = \frac{\alpha_s C_F}{2\pi}\bar{\sigma}_0\frac{Q}{s}\left[\frac{2\left(9+31c_1^2-17c_1^4+9c_1^6\right)}{3(1-c_1^2)^2} - 4c_1^3\ln\frac{1+c_1}{1-c_1}\right]. \tag{E.26}$$

Finally, we note that we can write this result using the lepton rapidity in the partonic center-of-mass frame as a variable. The relation between rapidity and the scattering angle is

$$y = \frac{1}{2}\ln\frac{p_b p_1}{p_a p_1} = \frac{1}{2}\ln\frac{1+\cos\theta_1}{1-\cos\theta_1}. \tag{E.27}$$

It follows that

$$c_1 = \cos\theta_1 = \frac{\mathrm{sh}(y)}{\mathrm{ch}(y)}. \tag{E.28}$$

Changing the variables, we obtain

$$\frac{\mathrm{d}\sigma^{\rm NLP}}{\mathrm{d}\tau \mathrm{d}y} = \frac{\alpha_s C_F}{2\pi}\bar{\sigma}_0 \frac{Q}{s}\left[\frac{32}{3}\mathrm{ch}(2y) - \frac{16}{3} + \frac{20}{3\mathrm{ch}^2(y)} - \frac{6}{\mathrm{ch}^4(y)} - 8y\frac{\mathrm{sh}^3(y)}{\mathrm{ch}^5(y)}\right]. \tag{E.29}$$

We observe the known fact [14, 16] that zero-jettiness power corrections exhibit an exponential growth at large lepton rapidities. Ways to address this problem were discussed in Ref. [16].

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