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Strengthened inequalities for the mean width and the ℓ -norm of origin symmetric convex bodies

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Abstract
Barthe, Schechtman and Schmuckenschläger proved that the cube maximizes the mean width of symmetric convex bodies whose John ellipsoid (maximal volume ellipsoid contained in the body) is the Euclidean unit ball, and the regular crosspolytope minimizes the mean width of symmetric convex bodies whose Löwner ellipsoid is the Euclidean unit ball. Here we prove close-to-be optimal stronger stability versions of these results, together with their counterparts about the ℓ-norm based on Gaussian integrals. We also consider related stability results for the mean width and the ℓ-norm of the convex hull of the support of even isotropic measures on the unit sphere.

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1 Introduction

Geometric inequalities and extremal problems constitute a central topic in geometry and geometric analysis. Perhaps the best known example is the isoperimetric inequality which states that Euclidean balls have smallest surface area among sets of given (finite) volume in Euclidean space $\mathbb{R}^n$. If we restrict to compact convex sets with nonempty interior (convex bodies), then Euclidean balls are the only minimizers. Another important example is the Urysohn inequality according to which Euclidean balls minimize the mean width of convex bodies of given volume. While for these two classical examples the extremizers (Euclidean balls) are rotationally symmetric, for some other extremal problems simplices, cubes and crosspolytopes naturally arise as extremizers.

In the following, we mainly focus on geometric inequalities and extremal problems for origin symmetric (o-symmetric) convex bodies (see [16] for the non-symmetric setting) in Euclidean space $\mathbb{R}^n$, $n \geq 2$, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We write B_p^n to denote the unit ball of the l_p norm in $\mathbb{R}^n$ for $p \in [1, \infty]$, in particular B_1^n is a regular crosspolytope inscribed into B_2^n, and $(B_1^n)^\circ = B_\infty^n$ is a cube circumscribed around B_2^n, where $K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \, \forall \, y \in K\}$ is the polar dual of an origin symmetric convex body $K \subset \mathbb{R}^n$.

An intriguing geometric extremal problem for which cubes and crosspolytopes are extremizers has been discovered and studied much more recently. Let *K* denote some origin symmetric convex body in $\mathbb{R}^n$. It is well known that there exists a unique ellipsoid of maximal volume contained in *K* (the John ellipsoid of *K*), and a unique ellipsoid of minimal volume containing *K* (the Löwner ellipsoid of *K*). It has been proved by Ball [4] that cubes maximize the volume of *K* given the volume of the John ellipsoid of *K*, which can be paraphrased by saying that cubes determine the extremal "inner" volume ratio for origin symmetric convex bodies. For the dual problem, Barthe [8] showed that crosspolytopes minimize the volume of *K* given the volume of the Löwner ellipsoid of *K*, hence crosspolytopes determine the extremal "outer" volume ratio (see also Lutwak, Yang, Zhang [34, 35]). In all these cases, the understanding of the equality cases essentially relies on Barthe's fundamental work [8] on the Brascamp–Lieb inequality and its reverse inequality.

Since the mean width plays a key role in the present work, we provide an explicit definition. Let κ_n denote the volume of the unit ball B_2^n and S^{n-1} its boundary (the Euclidean unit sphere). For a convex body $K \subset \mathbb{R}^n$, the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of *K* is defined by $h_K(x) = \max_{y \in K} \langle x, y \rangle$ for $x \in \mathbb{R}^n$. Then the mean width of *K* is given by

$$W(K) = \frac{1}{n\kappa_n} \int_{S^{n-1}} (h_K(u) + h_K(-u)) \, \mathrm{d}u,$$

where the integration is with respect to the $(n-1)$-dimensional spherical Lebesgue measure. As a functional on convex bodies, the mean width is isometry invariant, continuous, additive (a valuation) and (positively) homogeneous of degree one. Moreover the mean width is characterized by these properties.

In addition to the (translation invariant) mean width we consider the ℓ-norm of origin symmetric convex bodies. To define the latter, for an o-symmetric convex body $K \subset \mathbb{R}^n$, we set

$$\|x\|_K = \min\{t \geq 0 : x \in tK\}, \qquad x \in \mathbb{R}^n.$$

Let γ_n denote the standard Gaussian measure in $\mathbb{R}^n$, which has the density function $x \mapsto \frac{1}{\sqrt{2\pi}^n} e^{-\|x\|^2/2}$, $x \in \mathbb{R}^n$, with respect to Lebesgue measure. Then the ℓ-norm of K is defined by

$$\ell(K) = \int_{\mathbb{R}^n} \|x\|_K \, \gamma_n(\mathrm{d}x) = \mathbb{E}\|X\|_K,$$

where X is a Gaussian random vector with distribution γ_n. Using polar coordinates, the relation $h_{K^\circ} = \|\cdot\|_K$ and denoting by $\Gamma(\cdot)$ the Gamma function, we get

$$\ell(K) = \frac{\ell(B_2^n)}{2} W(K^\circ) \tag{1}$$

with $\ell(B_2^n) = \sqrt{2}\Gamma(\frac{n}{2})^{-1}\Gamma(\frac{n+1}{2})$, hence

$$\lim_{n\to\infty} \frac{\ell(B_2^n)}{\sqrt{n}} = 1.$$

Clearly, the ℓ-norm of K can be rewritten in the form

$$\ell(K) = \int_0^\infty \mathbb{P}(\|X\|_K > t) \, \mathrm{d}t = \int_0^\infty (1 - \gamma_n(tK)) \, \mathrm{d}t. \tag{2}$$

The following Theorem 1.1 for the ℓ-norm of symmetric convex bodies was first stated by Schechtman, Schmuckenschläger [37] (without discussing the equality cases). Part (i) is only briefly mentioned by Schechtman, Schmuckenschläger [37, p. 270], an explicit proof including a characterization of the equality case was provided by Barthe [9, Theorem 2]. Part (ii) was proved by Schechtman, Schmuckenschläger [37, Proposition 4.11] as an application of the Brascamp–Lieb inequality. The equality case follows from Barthe's general analysis of the equality conditions in the Brascamp–Lieb inequality [8]. The non-symmetric cases of these statements are treated by Barthe [9] and Schmuckenschläger [38] (see [16] for strengthened inequalities in the non-symmetric case).

Theorem 1.1 *(Barthe '98, Schechtman, Schmuckenschläger '95) Let K be an origin symmetric convex body in $\mathbb{R}^n$.*

(i) If $B_2^n \supset K$ is the Löwner ellipsoid of K, then $\ell(K) \leq \ell(B_1^n)$ and $W(K) \geq W(B_1^n)$. Equality holds in either case if and only if K is a regular crosspolytope.

(ii) If $B_2^n \subset K$ is the John ellipsoid of K, then $W(K) \le W(B_\infty^n)$ and $\ell(K) \ge \ell(B_\infty^n)$. Equality holds in either case if and only if K is a cube.

It follows from (1) and the duality of Löwner and John ellipsoids that the first statement of (i) is equivalent to the first statement of (ii), and the second statement of (i) is equivalent to the second statement of (ii).

Let us briefly discuss the range of $W(K)$ (and thus that of $\ell(K)$) in Theorem 1.1. Note that $W(K) = \frac{2\kappa_{n-1}}{n\kappa_n} \cdot V_1(K)$ (*cf.* Schneider [39, (5.31) and (5.57)] or [26, (3.18)]), where $V_1(K)$ is the first intrinsic volume of K, and $\frac{\kappa_{n-1}}{\kappa_n} \sim \sqrt{\frac{n}{2\pi}}$, as n tends to infinity. If $K \subset \mathbb{R}^n$ is an o-symmetric convex body whose Löwner ellipsoid is B_2^n, then the monotonicity of the mean width and Theorem 1.1 (i) yield

$$W(B_1^n) \le W(K) \le W(B_2^n) = 2,$$

where Lemma 3.1 in Böröczky, Henk [17] yields that

$$W(B_1^n) \sim \sqrt{\frac{2\ln n}{n}} \quad \text{as } n \to \infty.$$

In addition, $V_1(B_\infty^n) = 2n$ as V_1 is additive. If K is a convex body in $\mathbb{R}^n$ whose John ellipsoid is B_2^n, then

$$2 = W(B_2^n) \le W(K) \le W(B_\infty^n),$$

where $W(B_\infty^n) \sim \sqrt{8n/\pi}$. As a consequence, we also have $\ell(B_\infty^n) \sim \sqrt{\frac{\ln n}{2}}$ as $n \to \infty$ and $\ell(B_1^n) = \sqrt{\frac{2}{\pi}} \cdot n$ for $n \ge 2$.

The main goal of this paper is to prove a close-to-be optimal stability version of Theorem 1.1. In the stronger version of Theorem 1.1, closeness of two non-empty compact sets $X, Y \subset \mathbb{R}^n$ is measured in terms of their Hausdorff distance

$$\delta_{\mathrm{H}}(X, Y) = \max\{\max_{y \in Y} d(y, X), \max_{x \in X} d(x, Y)\},$$

where $d(x, Y) = \min\{\|x - y\| : y \in Y\}$ is the distance of x from Y. The Hausdorff distance defines a metric on the set of non-empty compact subsets of $\mathbb{R}^n$. Let $O(n)$ denote the orthogonal group of $\mathbb{R}^n$. Since we are interested in the Hausdorff distance up to orthogonal transformations, we also consider the metric

$$\delta_{\mathrm{HO}}(X, Y) = \min\{\delta_{\mathrm{H}}(X, \Phi Y) : \Phi \in O(n)\}.$$

We start with the case when B_2^n is the Löwner ellipsoid of a convex body $K \subset B_2^n$, and then state the stability result when B_2^n is the John ellipsoid of a convex body $K \subset \mathbb{R}^n$.

Theorem 1.2 *If B_2^n is the Löwner ellipsoid of an origin symmetric convex body $K \subset B_2^n$, then*

$$\ell(K) \leq (1 - \gamma \cdot \delta_{\mathrm{HO}}(K, B_1^n))\, \ell(B_1^n), \tag{3}$$

$$W(K) \geq \left(1 + \gamma \cdot \delta_{\mathrm{HO}}(K, B_1^n)^{4n}\right) W(B_1^n), \tag{4}$$

where $\gamma = n^{-cn^2}$ with an absolute constant $c > 0$.

Theorem 1.3 *If B_2^n is the John ellipsoid of an origin symmetric convex body $K \subset \mathbb{R}^n$, then*

$$\ell(K) \geq \left(1 + \gamma \cdot \delta_{\mathrm{HO}}(K, B_\infty^n)^{4n}\right) \ell(B_\infty^n), \tag{5}$$

$$W(K) \leq (1 - \gamma \cdot \delta_{\mathrm{HO}}(K, B_\infty^n))\, W(B_\infty^n), \tag{6}$$

where $\gamma = n^{-cn^2}$ with an absolute constant $c > 0$.

Concerning the optimality of Theorem 1.3, let K_ε, for sufficiently small $\varepsilon > 0$, be obtained from the cube B_∞^n by cutting off each vertex *v* of B_∞^n by the half space $\{x \in \mathbb{R}^n : \langle x, v\rangle \leq (1-\varepsilon)\langle v, v\rangle\}$. The resulting body satisfies that $\delta_{\mathrm{HO}}(K_\varepsilon, B_\infty^n) \geq \gamma_1\varepsilon, \ell(K_\varepsilon) \leq (1 + \gamma_2\varepsilon^n)\, \ell(B_\infty^n)$ and $W(K_\varepsilon) \geq (1 - \gamma_3\varepsilon)\, W(B_\infty^n)$ for some $\gamma_1, \gamma_2, \gamma_3 > 0$ depending on *n*; therefore,

- the estimate (6) is optimal up to the factor depending on *n*, and
- the exponent 4*n* of $\delta_{\mathrm{HO}}(K, B_\infty^n)$ in (5) cannot be replaced by anything smaller than *n*.

For the optimality of Theorem 1.2, we consider the polar K_ε° for sufficiently small $\varepsilon > 0$, and hence the estimate (3) is optimal up to the factor depending on *n*, and the exponent 4*n* of $\delta_{\mathrm{HO}}(K, B_1^n)$ in (4) cannot be replaced by anything smaller than *n*.

Finally, in the statement of relation (4) the factor γ can be chosen of the order n^{-cn} with an absolute constant $c > 0$ and $\delta_{\mathrm{HO}}(K, B_1^n)^{4n}$ can be replaced by $\delta_{\mathrm{volO}}(K, B_1^n)^4$, where for origin symmetric convex bodies $X, Y \subset \mathbb{R}^n$ we define

$$\delta_{\mathrm{volO}}(X, Y) = \min\{\delta_{\mathrm{vol}}(X, \Phi Y) : \Phi \in O(n)\}$$

and δ_{vol} is the symmetric (volume) difference metric. Similar comments apply to relation (5).

An important concept underlying the proof of Theorem 1.1 is the notion of an isotropic measure on the unit sphere. A Borel measure μ on the unit sphere S^{n-1} is called isotropic (*cf.* [22, 35]) if

$$\int_{S^{n-1}} u \otimes u\, \mu(\mathrm{d}u) = \mathrm{I}_n, \tag{7}$$

where I_n is the identity map (or the identity matrix). Equating traces of the two sides of (7), we get $\mu(S^{n-1}) = n$. If μ is an even isotropic measure on S^{n-1}, then the cardinality of its support satisfies $|\operatorname{supp}\mu| \geq 2n$, with equality if and only if μ is concentrated on the vertices of some regular crosspolytope and each vertex has measure $\frac{1}{2}$. If $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfy $u_i \neq \pm u_j$ for $i \neq j$ and

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n,$$

then the discrete measure μ on S^{n-1} with support $\{\pm u_1, \ldots, \pm u_k\}$ and $\mu(\{u_i\}) = \mu(\{-u_i\}) = c_i/2$ for $i = 1, \ldots, k$ is isotropic and even. If $k = n$, $u_1, \ldots, u_n$ form an orthonormal basis of $\mathbb{R}^n$ and each $c_i = 1$, then such a measure is called a *cross measure*, and is characterized by the properties that it is an even isotropic measure whose support consists of the vertices of a regular crosspolytope.

We recall that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture [29] and in the analysis of Bourgain's recently resolved (cf. Klartag, Lehec [31]) hyperplane conjecture (slicing problem) (see, for instance, [1, 2, 10, 25, 30]). The relevance of isotropic measures on S^{n-1} in the present context is due to Ball's crucial insight that John's characteristic condition [27, 28] (see Sect. 2) for a convex body to have the unit ball as its John or Löwner ellipsoid (see [3, 4]) can be used to give the Brascamp–Lieb inequality a convenient form which is perfectly suited for many geometric applications (see Sect. 3).

We write $\operatorname{conv} X$ to denote the convex hull of a set $X \subset \mathbb{R}^n$, and for an even isotropic measure μ on S^{n-1}, we consider the o-symmetric convex bodies

$$Z_\infty(\mu) = \operatorname{conv}\operatorname{supp}\mu \subset B_2^n \quad \text{and} \quad Z_\infty^*(\mu) = Z_\infty(\mu)^\circ \supset B_2^n.$$

In particular, if ν is a cross measure on S^{n-1}, then

$$Z_\infty(\nu) = B_1^n \quad \text{and} \quad Z_\infty^*(\nu) = B_\infty^n.$$

Li, Leng [32] proved the following version of Theorem 1.1. The result can be obtained as an immediate consequence of Theorem 1.1 and Lemma 7.1.

Theorem 1.4 *(Li, Leng [32]) Let ν be a cross measure on S^{n-1}. If μ is an even isotropic measure, then*

(i) $\ell(Z_\infty(\mu)) \leq \ell(Z_\infty(\nu))$ and $W(Z_\infty(\mu)) \geq W(Z_\infty(\nu))$, and equality holds in either case if and only if μ is a cross measure.
(ii) $\ell(Z_\infty^*(\mu)) \geq \ell(Z_\infty^*(\nu))$ and $W(Z_\infty^*(\mu)) \leq W(Z_\infty^*(\nu))$, and equality holds in either case if and only if μ is a cross measure.

In order to state a stability version of Theorem 1.4, we need the notion of Wasserstein distance $\delta_{\mathrm{W}}(\mu, \nu)$ of two isotropic measures μ and ν on S^{n-1} (also called the Kantorovich–Monge–Rubinstein distance). To define it, we write $\angle(v, w)$

to denote the angle of $v, w \in S^{n-1}$; that is, the geodesic distance of v and w on S^{n-1}. Let $\mathrm{Lip}_1(S^{n-1})$ denote the family of Lipschitz functions with Lipschitz constant 1; namely, $f : S^{n-1} \to \mathbb{R}$ is in $\mathrm{Lip}_1(S^{n-1})$ if $\|f(x) - f(y)\| \le \angle(x, y)$ for $x, y \in S^{n-1}$. Then

$$\delta_{\mathrm{W}}(\mu, \nu) = \max \left\{ \int_{S^{n-1}} f\, d\mu - \int_{S^{n-1}} f\, d\nu : \ f \in \mathrm{Lip}_1(S^{n-1}) \right\}.$$

Since we allow rotation of one of the measures, the actual notion of distance that we use is

$$\delta_{\mathrm{WO}}(\mu, \nu) = \min \left\{ \delta_{\mathrm{W}}(\mu, \Phi_* \nu) : \ \Phi \in O(n) \right\}$$

where $\Phi_* \nu$ denotes the pushforward of ν by $\Phi : S^{n-1} \to S^{n-1}$. In turn, we have the following stronger form of Theorem 1.4.

Theorem 1.5 *Let μ be an even isotropic measure on S^{n-1}. There exist an absolute constant $c > 0$ and a cross measure ν on S^{n-1} such that for $\gamma = n^{-cn^2}$, for (a) and (d), and $\gamma = n^{-cn}$, for (b) and (c), and $\tilde{\gamma} = \frac{\gamma}{30n^3}$, we have*

(a) $$\begin{aligned} \ell(Z_\infty(\mu)) &\le (1 - \gamma \cdot \delta_{\mathrm{HO}}(\operatorname{supp} \mu, \operatorname{supp} \nu))\, \ell(Z_\infty(\nu)) \\ &\le (1 - \tilde{\gamma} \cdot \delta_{\mathrm{WO}}(\mu, \nu))\, \ell(Z_\infty(\nu)). \end{aligned}$$

(b) $$\begin{aligned} W(Z_\infty(\mu)) &\ge \left(1 + \gamma \cdot \delta_{\mathrm{HO}}(\operatorname{supp} \mu, \operatorname{supp} \nu)^4\right) W(Z_\infty(\nu)) \\ &\ge \left(1 + \tilde{\gamma} \cdot \delta_{\mathrm{WO}}(\mu, \nu)^4\right) W(Z_\infty(\nu)). \end{aligned}$$

(c) $$\begin{aligned} \ell(Z^*_\infty(\mu)) &\ge \left(1 + \gamma \cdot \delta_{\mathrm{HO}}(\operatorname{supp} \mu, \operatorname{supp} \nu)^4\right) \ell(Z^*_\infty(\nu)) \\ &\ge \left(1 + \tilde{\gamma} \cdot \delta_{\mathrm{WO}}(\mu, \nu)^4\right) \ell(Z^*_\infty(\nu)). \end{aligned}$$

(d) $$\begin{aligned} W(Z^*_\infty(\mu)) &\le (1 - \gamma \cdot \delta_{\mathrm{HO}}(\operatorname{supp} \mu, \operatorname{supp} \nu))\, W(Z^*_\infty(\nu)) \\ &\le (1 - \tilde{\gamma} \cdot \delta_{\mathrm{WO}}(\mu, \nu))\, W(Z^*_\infty(\nu)). \end{aligned}$$

The main task of the present work is to estimate the ℓ-norm. In Sect. 2 we deal with the case when the Löwner ellipsoid is a ball. This part of the argument avoids the use of Barthe's reverse form of the Brascamp–Lieb inequality and proceeds by a more direct reasoning. To estimate the ℓ-norm when the John ellipsoid is a ball, first we review the use of optimal transport and the Brascamp–Lieb inequality in Sect. 3, and secondly we provide some crucial estimates for the transport functions arising in this context in Sect. 4. Stability of the ℓ-norm around the cube when the John ellipsoid is a ball is verified in Sect. 5 (see Theorem 5.6), and we prove Theorem 1.2 and Theorem 1.3 in Sect. 6. Finally, Theorem 1.5 is established in Sect. 7 via Theorem 7.4 and Theorem 7.7.

Our arguments are partly based on the rank one geometric Brascamp–Lieb inequality and its stability version in a special case (see Sect. 5). No quantitative stability version of the Brascamp–Lieb inequality (or of the reverse Brascamp–Lieb inequality) is known in general (see Bennett, Bez, Flock, Lee [12] for a certain weak stability version for higher ranks). On the other hand, in the case of the Borell–Brascamp–Lieb inequaliy (see Borell [14], Brascamp, Lieb [19] and Balogh, Kristály [6]), stability versions were obtained by Ghilli, Salani [21] and Rossi, Salani [36].

2 The ℓ-norm when the Löwner ellipsoid is a ball

Theorem 2.1 below is part (i) of Theorem 1.1. We start by recalling the argument of Barthe (see [9, Theorem 2]) to prove Theorem 2.1.

In order to efficiently use the hypothesis that the unit ball is the John (or Löwner) ellipsoid of a symmetric convex body K, John's [27, 28] characteristic condition is employed. It states that B_2^n is the John ellipsoid (or Löwner ellipsoid) of an o-symmetric convex body K if and only if $B_2^n \subset K$ (or $K \subset B_2^n$), and there exist distinct unit vectors $u_1, \dots, u_k \in \partial K \cap S^{n-1}$ and $c_1, \dots, c_k > 0$ such that $u_i \neq \pm u_j$ for $i \neq j$ and

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{8}$$

with the proof of the equivalence completed by Ball [5] (see also [24]).

It follows that B_2^n is the John ellipsoid of an o-symmetric convex body $K \subset \mathbb{R}^n$ if and only if B_2^n is the Löwner ellipsoid of K°.

Furthermore, there exists some m element subset $I \subset \{1, \dots, k\}$ with $m \leq \frac{n(n+1)}{2}$ and some $\tilde{c}_i > 0$ for $i \in I$ such that $\sum_{i \in I} \tilde{c}_i u_i \otimes u_i = \mathrm{I}_n$.

In the following we fix an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. We also note that if $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfy

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{9}$$

then this data and any $x \in \mathbb{R}^n$ satisfy

$$x = \sum_{i=1}^{k} c_i \langle x, u_i \rangle u_i, \tag{10}$$

$$\|x\|_2^2 = \sum_{i=1}^{k} c_i \langle x, u_i \rangle^2, \tag{11}$$

$$\sum_{i=1}^{k} c_i = n, \tag{12}$$

$$c_j \leq 1 \quad \text{for } j = 1, \dots, k. \tag{13}$$

Here (10) and (11) directly follow from (9), (12) follows from equating traces, and (11) yields (13) by taking $x = u_j$.

Theorem 2.1 *(Barthe, Schechtman, Schmuckenschläger) If $K \subset \mathbb{R}^n$ is an origin symmetric convex body such that $B_2^n \supset K$ is the Löwner ellipsoid of K, then $\ell(K) \le \ell(B_1^n)$. Equality holds if and only if K is a regular crosspolytope inscribed to B_2^n.*

Proof According to John's characteristic condition, there exist distinct unit vectors $u_1, \dots, u_k \in \partial K \cap S^{n-1}$ and $c_1, \dots, c_k > 0$ such that (9) holds. Moreover, we can assume that $u_i \ne -u_j$ for $i \ne j$. We deduce from $\sum_{i=1}^k c_i = n$ (*cf.* (12)) and

$$\int_{\mathbb{R}^n} |\langle x, e_1\rangle| \, \gamma_n(\mathrm{d}x) = \int_{\mathbb{R}^n} |\langle x, v\rangle| \, \gamma_n(\mathrm{d}x)$$

for any $v \in S^{n-1}$ that

$$\begin{aligned} \ell(B_1^n) &= \int_{\mathbb{R}^n} \|x\|_{B_1^n} \, \gamma_n(\mathrm{d}x) = \int_{\mathbb{R}^n} \sum_{i=1}^n |\langle x, e_i\rangle| \, \gamma_n(\mathrm{d}x) = n \int_{\mathbb{R}^n} |\langle x, e_1\rangle| \, \gamma_n(\mathrm{d}x) \\ &= \left(\sum_{i=1}^k c_i\right) \int_{\mathbb{R}^n} |\langle x, e_1\rangle| \, \gamma_n(\mathrm{d}x) = \int_{\mathbb{R}^n} \sum_{i=1}^k c_i |\langle x, u_i\rangle| \, \gamma_n(\mathrm{d}x). \end{aligned} \tag{14}$$

The convex hull $M \subset K$ of $\pm u_1, \dots, \pm u_k$ is an o-symmetric polytope, and (10) yields that

$$\|x\|_K \le \|x\|_M = \inf\left\{\sum_{i=1}^k |\alpha_i| : x = \sum_{i=1}^k \alpha_i u_i\right\} \le \sum_{i=1}^k c_i |\langle x, u_i\rangle|. \tag{15}$$

It follows from (14) that

$$\begin{aligned} \ell(K) &= \int_{\mathbb{R}^n} \|x\|_K \, \gamma_n(\mathrm{d}x) \le \int_{\mathbb{R}^n} \|x\|_M \, \gamma_n(\mathrm{d}x) \\ &\le \int_{\mathbb{R}^n} \sum_{i=1}^k c_i |\langle x, u_i\rangle| \, \gamma_n(\mathrm{d}x) = \ell(B_1^n). \end{aligned}$$

Next we assume that $\ell(K) = \ell(B_1^n)$. Then equality holds in (15), hence $K = M$, and equality holds in the right inequality of (15). Applying this fact to $x = u_j$ and the estimate $|\langle u_j, u_i\rangle| \le 1$ yield

$$1 \ge \sum_{i=1}^k c_i |\langle u_j, u_i\rangle| \ge \sum_{i=1}^k c_i \langle u_j, u_i\rangle^2 = 1.$$

This shows that $|\langle u_i, u_j\rangle| \in \{0, 1\}$ for $i, j \in \{1, \dots, k\}$. Therefore, we have $k = n$, $u_1, \dots, u_n$ is an orthonormal basis and $c_1 = \dots = c_n = 1$. □

In order to have a stability version of Theorem 2.1, we need some observations on a system of vectors satisfying (9). We observe that setting $v_i = \sqrt{c_i}\, u_i$ in (9), we have $\sum_{i=1}^{k} v_i \otimes v_i = \mathrm{I}_n$. We deduce the following from the Cauchy–Binet formula.

Lemma 2.2 *If $v_1, \ldots, v_k \in \mathbb{R}^n$ satisfy $\sum_{i=1}^{k} v_i \otimes v_i = \mathrm{Id}_n$, then there exist $1 \le i_1 < \ldots < i_n \le k$ such that*

$$\det[v_{i_1}, \ldots, v_{i_n}]^2 \ge \binom{k}{n}^{-1}.$$

For non-zero vectors *v* and *w*, we write $\angle(v, w)$ to denote their angle, that is, the geodesic distance of the unit vectors $\|v\|^{-1}v$ and $\|w\|^{-1}w$ on the unit sphere. The following lemma is a variant of [18, Lemma 3.2], which follows by a similar argument as in [18].

Lemma 2.3 *Let $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$, $k > n$, satisfy (9). Let $0 < \eta \le 1/(3n)$. If for any $j \in \{n+1, \ldots, k\}$ there exists some $i \in \{1, \ldots, n\}$ satisfying $|\langle u_i, u_j \rangle| \ge \cos\eta$, then there exist an orthonormal basis $w_1, \ldots, w_n$ and $\xi_j \in \{-1, 1\}$ for $j = 1, \ldots, k$ such that*

$$\delta_{\mathrm{H}}\left(\{w_1, \ldots, w_n\}, \{\xi_1 u_1, \ldots, \xi_k u_k\}\right) < 4\sqrt{n}\,\eta.$$

Proof We partition the index set $\{1, \ldots, k\}$ into sets $\mathcal{V}_1, \ldots, \mathcal{V}_n$ such that $i \in \mathcal{V}_i$ for $i = 1, \ldots, n$, and if $j \ge n+1$ and $j \in \mathcal{V}_i$ for some $i \in \{1, \ldots, n\}$,then $|\langle u_i, u_j \rangle| \ge \cos\eta$. For $i = 1, \ldots, n$, (11) yields

$$1 = \|u_i\|^2 \ge \sum_{j \in \mathcal{V}_i} c_j \langle u_i, u_j \rangle^2 \ge \sum_{j \in \mathcal{V}_i} c_j (\cos\eta)^2,$$

and hence

$$\sum_{j \in \mathcal{V}_i} c_j \le (\cos\eta)^{-2}. \tag{16}$$

For $j \in \mathcal{V}_i$, we may replace u_j with $-u_j$ if necessary to ensure $\langle u_i, u_j \rangle \ge 0$, and hence

$$\angle(u_i, u_j) \le \eta \quad \text{ for } j \in \mathcal{V}_i. \tag{17}$$

For $i = 1, \ldots, n$, let $\tilde{w}_i \in S^{n-1}$ be orthogonal to u_m, $m \in \{1, \ldots, n\} \backslash \{i\}$, and satisfy $\langle \tilde{w}_i, u_i \rangle \ge 0$. In addition, let $\alpha_i \in [0, \pi/2]$ be defined by

$$\cos\alpha_i = \max\{|\langle \tilde{w}_i, u_j \rangle| : j \in \mathcal{V}_i\}.$$

We aim at bounding α_i from above. For a fixed $i \in \{1, \ldots, n\}$, we observe that if $j \in \mathcal{V}_i$, then $\langle \tilde{w}_i, u_j \rangle^2 \leq \cos^2 \alpha_i$, and if $j \in \mathcal{V}_m$ for some $m \in \{1, \ldots, n\} \setminus \{i\}$, then $\angle(u_m, u_j) \leq \eta$ and $\langle \tilde{w}_i, u_m \rangle = 0$ imply

$$\langle u_j, \tilde{w}_i \rangle^2 + \cos^2 \eta \leq \langle u_j, \tilde{w}_i \rangle^2 + \langle u_j, u_m \rangle^2 \leq \|u_j\|^2 = 1,$$

and hence $\langle \tilde{w}_i, u_j \rangle^2 \leq \sin^2 \eta$. Using these facts and (16), we deduce

$$\sum_{j \in \mathcal{V}_m} c_j \langle \tilde{w}_i, u_j \rangle^2 \leq \sin^2 \eta \sum_{j \in \mathcal{V}_m} c_j \leq \frac{\sin^2 \eta}{\cos^2 \eta}, \qquad \text{for } m \in \{1, \ldots, n\} \setminus \{i\},$$

$$\sum_{j \in \mathcal{V}_i} c_j \langle \tilde{w}_i, u_j \rangle^2 \leq \cos^2 \alpha_i \sum_{j \in \mathcal{V}_i} c_j \leq \frac{\cos^2 \alpha_i}{\cos^2 \eta}.$$

It follows from (11) that

$$1 = \|\tilde{w}_i\|^2 = \sum_{j=1}^{k} c_j \langle \tilde{w}_i, u_j \rangle^2 \leq \frac{(n-1)\sin^2 \eta}{\cos^2 \eta} + \frac{\cos^2 \alpha_i}{\cos^2 \eta},$$

and hence

$$\sin^2 \alpha_i = 1 - \cos^2 \alpha_i \leq 1 - \cos^2 \eta + (n-1)\sin^2 \eta = n \sin^2 \eta.$$

This shows that

$$\left(\frac{2}{\pi}\right)^2 \alpha_i^2 \leq n\eta^2 \leq \frac{1}{9n}, \tag{18}$$

thus $0 \leq \alpha_i \leq \frac{\pi}{6\sqrt{n}}$. Since $\eta < 1/(3n)$, we also get $\alpha_i + \eta < \pi/2$. By the definition of α_i, there is some $j \in \mathcal{V}_i$ such that $\cos \alpha_i = \pm \langle \tilde{w}_i, u_j \rangle$. First, suppose that $\cos \alpha_i = -\langle \tilde{w}_i, u_j \rangle$. Then $\angle(\tilde{w}_i, -u_j) = \alpha_i$, hence (17) yields

$$\angle(\tilde{w}_i, -u_i) \leq \angle(\tilde{w}_i, -u_j) + \angle(-u_j, -u_i) \leq \alpha_i + \eta < \frac{\pi}{2},$$

which is a contradiction to $\langle \tilde{w}_i, u_i \rangle \geq 0$. This shows that $\cos \alpha_i = \langle \tilde{w}_i, u_j \rangle$ and therefore $\angle(\tilde{w}_i, u_j) = \alpha_i$. As before, (17) now yields

$$\angle(\tilde{w}_i, u_i) \leq \alpha_i + \eta. \tag{19}$$

Therefore, (19), (18) and $\eta < 1/(3\sqrt{n})$ imply

$$\angle(\tilde{w}_i, u_i) \leq \alpha_i + \eta \leq 2\sqrt{n}\,\eta + \eta < 3\sqrt{n}\,\eta < 1, \qquad i = 1, \ldots, n. \tag{20}$$

Supposing that u_i is in the linear hull of the vectors u_m, $m \in \{1,\dots,n\} \setminus \{i\}$, we deduce that $\langle u_i, \tilde{w}_i \rangle = 0$, which contradicts (20). It follows that $u_1, \dots, u_n$ are linearly independent.

We define $w_1 = u_1$, and for $i = 2, \dots, n$, we let w_i be the unit vector in $\text{lin}\{u_1, \dots, u_i\}$ which is orthogonal to $u_1, \dots, u_{i-1}$ and satisfies $\langle w_i, u_i \rangle > 0$. As $w_1, \dots, w_n$ form an orthonormal basis, $\text{lin}\{u_1, \dots, u_m\} = \text{lin}\{w_1, \dots, w_m\}$ and $\langle \tilde{w}_i, u_j \rangle = 0$ for $j \leq i-1$, it follows that $u_i = \sum_{j=1}^{i} \beta_j w_j$ and $\tilde{w}_i = \sum_{m=i}^{n} \gamma_m w_m$ for $\beta_j, \gamma_m \in \mathbb{R}$ where $\langle w_i, u_i \rangle > 0$ and $\langle \tilde{w}_i, u_i \rangle \geq 0$ yield $\beta_i, \gamma_i \in [0,1]$. Thus $\langle u_i, w_i \rangle = \beta_i \geq \beta_i \gamma_i = \langle u_i, \tilde{w}_i \rangle$ or in other words $\angle(w_i, u_i) \leq \angle(\tilde{w}_i, u_i) < 3\sqrt{n}\,\eta$, and hence if $j \in \mathcal{V}_i$, then

$$\angle(w_i, u_j) \leq \angle(w_i, u_i) + \angle(u_i, u_j) < 4\sqrt{n}\,\eta,$$

which completes the argument. □

We also need an estimate about the Gaussian measure of cones.

Lemma 2.4 *If $\alpha \in (0, \frac{\pi}{2})$ and $w \in S^{n-1}$, then the convex cone $C = \{x \in \mathbb{R}^n : \langle x, w \rangle \geq \|x\| \cos \alpha\}$ satisfies*

$$\gamma_n(C) \geq \frac{\sin^{n-1}(\alpha)}{\sqrt{2\pi n}}.$$

Proof We write $\mathcal{H}^{n-1}(\cdot)$ to denote the $(n-1)$-dimensional Hausdorff measure normalized in such a way that it coincides with the $(n-1)$-dimensional Lebesgues measure on subsets of $w^\perp$. We observe that the orthogonal projection of $C \cap S^{n-1}$ onto $w^\perp$ is an $(n-1)$-ball of radius $\sin \alpha$; therefore,

$$\gamma_n(C) = \frac{\mathcal{H}^{n-1}(C \cap S^{n-1})}{\mathcal{H}^{n-1}(S^{n-1})} \geq \frac{\sin^{n-1}(\alpha)\kappa_{n-1}}{n\kappa_n}.$$

The logarithmic convexity of the Gamma function yields $(\kappa_{n-1})^2 \geq \kappa_{n-2}\kappa_n$. Since $\kappa_{n-2}\kappa_n^{-1} = \frac{n}{2\pi}$ we get

$$\frac{\kappa_{n-1}}{\kappa_n} \geq \sqrt{\frac{n}{2\pi}},$$

from which the assertion follows. □

Next we provide a quantitative estimate about polytopes whose vertices are close the vertices of a regular crosspolytope.

Lemma 2.5 *If $p_1, \dots, p_{2n}$ are the vertices of B_1^n, and $M = \text{conv}\{q_1, \dots, q_k\} \subset \mathbb{R}^n$ with $\delta_H(\{p_1, \dots, p_{2n}\}, \{q_1, \dots, q_k\}) \leq \alpha$ for some $\alpha \in (0, 1/\sqrt{n})$, then*

$$(1-\sqrt{n}\alpha)B_1^n \subset M \subset (1+\sqrt{n}\alpha)B_1^n.$$

Proof We have $(B_1^n)^\circ = B_\infty^n \subset \sqrt{n}B_2^n$, hence $B_2^n \subset \sqrt{n}B_1^n$. By assumption, $B_1^n \subset M + \alpha B_2^n$. It follows that

$$(1-\sqrt{n}\alpha)B_1^n + \sqrt{n}\alpha B_1^n \subset M + \sqrt{n}\alpha B_1^n,$$

which yields the left inclusion.

The right inclusion follows from $M \subset B_1^n + \alpha B_2^n \subset B_1^n + \alpha\sqrt{n}B_1^n$. □

The last auxiliary statement before Theorem 2.7 helps to estimate the variation of the ℓ function.

Lemma 2.6 *If $R > 0$ and $K \subset C \subset RB_2^n$ are o-symmetric convex bodies, then*

$$\ell(K) - \ell(C) \geq \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} R^{-(n+1)} V(C\backslash K).$$

Proof It follows from (2), and using substitutions like $x = ty$ and $t = s/R$ that

$$\begin{aligned}\ell(K) - \ell(C) &= \int_0^\infty (\gamma_n(tC) - \gamma_n(tK))\,dt = \frac{1}{\sqrt{2\pi}^n}\int_0^\infty \int_{t(C\backslash K)} e^{-\frac{\|x\|^2}{2}}\,dx\,dt \\ &= \frac{1}{\sqrt{2\pi}^n}\int_0^\infty \int_{C\backslash K} t^n e^{-\frac{t^2\|y\|^2}{2}}\,dy\,dt \geq \frac{V(C\setminus K)}{\sqrt{2\pi}^n}\int_0^\infty t^n e^{-\frac{t^2R^2}{2}}\,dt \\ &= \frac{V(C\setminus K)}{R^{n+1}\sqrt{2\pi}^n}\int_0^\infty s^n e^{-\frac{s^2}{2}}\,ds = \frac{V(C\setminus K)}{R^{n+1}\sqrt{2\pi}^n} 2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right).\end{aligned}$$

Since

$$\Gamma(x+1) \geq \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \quad x \geq 0,$$

we get

$$\frac{1}{\sqrt{2\pi}^n} 2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right) \geq \sqrt{e\pi}\left(1-\frac{1}{n}\right)^{\frac{n}{2}}\left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} \geq \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}}, \quad n \geq 2,$$

which proves the assertion. □

Now we are ready to prove a stability version of Theorem 2.1.

Theorem 2.7 *There exists some explicitly calculable absolute constant $c \geq 1$ with the following properties. If K is an origin symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \supset K$ is the Löwner ellipsoid of K, and $\ell(K) \geq (1-\varepsilon)\ell(B_1^n)$ for some $\varepsilon \in (0, \varepsilon_0)$ for $\varepsilon_0 = n^{-cn^2}$, then there exists an orthogonal transformation $\Phi \in O(n)$ such that $\delta_{\mathrm{H}}(K, \Phi B_1^n) \leq n^{cn^2}\cdot\varepsilon$.*

Proof According to John's characteristic condition, there exist $u_1,\dots,u_k \in S^{n-1} \cap \partial K$ and $c_1,\dots,c_k > 0$ with $n \le k \le \frac{n(n+1)}{2}$ that satisfy (9).

We note that up to (36), we do not use the condition $\ell(K) \ge (1-\varepsilon)\ell(B_1^n)$, only that (9) holds and $k \le n^2$. Applying Lemma 2.2 with $v_i = \sqrt{c_i}\, u_i$, and using that $\left(\begin{array}{c} k \\ n \end{array}\right) \le \left(\begin{array}{c} n^2 \\ n \end{array}\right) \le (\frac{e\cdot n^2}{n})^n = (en)^n$, we may assume after re-indexing that

$$c_1 \cdots c_n \cdot \Big|\det[u_1,\dots,u_n]\Big|^2 \ge (en)^{-n}.$$

We have $|\det[u_1,\dots,u_n]| \le 1$ as $u_1,\dots,u_n$ are unit vectors, and $c_i \le 1$ for $i = 1,\dots,n$ by (13); therefore,

$$c_i \ge (en)^{-n} \quad \text{for } i = 1,\dots,n, \text{ and } \quad \Big|\det[u_1,\dots,u_n]\Big| \ge (en)^{-n/2}. \tag{21}$$

We define $M = \mathrm{conv}\{\pm u_1,\dots,\pm u_k\}$. The core claim of our argument for proving Theorem 2.7 is that there exist $\Phi \in O(n)$ and $\gamma > 0$ depending on *n* such that

$$(1-\gamma\varepsilon)\Phi B_1^n \subset M \subset K. \tag{22}$$

If $k = n$, then $u_1,\dots,u_n$ form an orthonormal basis, and hence $\Phi B_1^n = M$ for some $\Phi \in O(n)$, and thus (22) readily holds. Therefore, we assume that $k > n$.

Let $\tilde\eta \in [0, \frac{\pi}{2})$ be minimal with the property that for any $j = n+1,\dots,k$ there exists $i \in \{1,\dots,n\}$ satisfying $|\langle u_i, u_j\rangle| \ge \cos\tilde\eta$ (here $\tilde\eta < \frac{\pi}{2}$ follows from (21)), and let

$$\eta = \min\left\{\tilde\eta, \frac{1}{3n\cdot(en)^{n/2}}\right\}. \tag{23}$$

In particular, we may assume that

$$|\langle u_i, u_k\rangle| \le \cos\tilde\eta \quad \text{for } i = 1,\dots,n. \tag{24}$$

Possibly replacing u_i by $-u_i$ if $i = 1,\dots,n$, we may assume that

$$u_k = \sum_{i=1}^n \lambda_i u_i \quad \text{for } \lambda_1,\dots,\lambda_n \ge 0 \text{where at least two } \lambda_i > 0. \tag{25}$$

We observe that

$$v = \frac{1}{\lambda_1+\dots+\lambda_n}\cdot u_k = \sum_{i=1}^n \frac{\lambda_i}{\lambda_1+\dots+\lambda_n}\cdot u_i \in (\mathrm{int}\, B_2^n) \cap \mathrm{conv}\{u_1,\dots,u_n\},$$

and hence $\|v\| < 1$. We claim that

$$\lambda_1 + \cdots + \lambda_n = \frac{1}{\|v\|} \geq 1 + \frac{1}{3 \cdot (en)^{n/2}} \cdot \eta. \tag{26}$$

For $x \in \mathbb{R}^n$ and a linear subspace $L \subset \mathbb{R}^n$, we write $\mathrm{dist}(x, L)$ for the Euclidean distance of x from L. In addition, for $i = 1, \ldots, n$, we write $u_1, \ldots, \check{u}_i, \ldots, u_n$ to denote the list of $n-1$ vectors where u_i is removed from the list $u_1, \ldots, u_n$. In particular, for $i = 1, \ldots, n$, (21) yields that the linear $(n-1)$-space $L_i = \mathrm{lin}\{u_1, \ldots, \check{u}_i, \ldots, u_n\}$ satisfies

$$\mathrm{dist}(u_i, L_i) = \frac{\Big|\det[u_1, \ldots, u_n]\Big|}{\Big|\det_{n-1}[u_1, \ldots, \check{u}_i, \ldots, u_n]\Big|} \geq (en)^{-n/2}. \tag{27}$$

We may assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and hence $\lambda_2 > 0$ and

$$\mathrm{dist}(v, L_1) \geq \frac{1}{n} \cdot \mathrm{dist}(u_1, L_1) \geq \frac{1}{n \cdot (en)^{n/2}}. \tag{28}$$

Let $v' \in S^{n-1}$ be the other endpoint of the secant of B_2^n such that $v \in \mathrm{conv}\{u_1, v'\}$, and hence $v' \neq -u_1$ by $\lambda_2 > 0$, and there exists a $v'' \in L_1 \cap \mathrm{conv}\{v, v'\}$ with $v'' \neq o$ as $v' \neq -u_1$. It follows from (28) that the distance of v from the line of o and v'' is at least $\frac{1}{n \cdot (en)^{n/2}}$, and hence $\|v\| \leq 1$ yields that $\sin \angle(v, v'') \geq \frac{1}{n \cdot (en)^{n/2}}$. We deduce that

$$\angle(v, v') \geq \angle(v, v'') \geq \sin \angle(v, v'') \geq \frac{1}{n \cdot (en)^{n/2}} \geq \eta \tag{29}$$

$$\cos \angle(v', u_1, o) = \frac{\|u_1 - v'\|}{2} \geq \frac{\mathrm{dist}(u_1, L_1)}{2} \geq \frac{1}{2 \cdot (en)^{n/2}}. \tag{30}$$

On the other hand, (24) yields that

$$\angle(v, u_1) = \angle(u_k, u_1) \geq \tilde{\eta} \geq \eta. \tag{31}$$

It follows from (29) and (31) that there exists $v_0 \in \mathrm{conv}\{v, u_1\}$ with $\angle(u_1, v_0) = \eta$, and it satisfies $\|v\| \leq \|v_0\|$ and $\angle(v_0, u_1, o) + \eta \leq \frac{\pi}{2}$; therefore, applying the Law of Sine in the triangle $\mathrm{conv}\{v_0, u_1, 0\}$ implies

$$\begin{aligned}\lambda_1 + \cdots + \lambda_n = \frac{1}{\|v\|} \geq \frac{1}{\|v_0\|} = \frac{\|u_1 - o\|}{\|v_0 - o\|} = \frac{\sin \angle(u_1, v_0, o)}{\sin \angle(v_0, u_1, o)} \\ = \frac{\sin(\angle(v', u_1, o) + \eta)}{\sin \angle(v', u_1, o)} = \cos \eta + \frac{\cos \angle(v', u_1, o)}{\sin \angle(v', u_1, o)} \cdot \sin \eta,\end{aligned}$$

and hence (30) yields that

$$\lambda_1 + \cdots + \lambda_n \geq \cos\eta + \frac{1}{2\cdot(en)^{n/2}}\cdot\sin\eta \geq 1 + \frac{1}{3\cdot(en)^{n/2}}\cdot\eta$$

where we used that $\eta \leq \frac{1}{3n\cdot(en)^{n/2}} \leq \frac{1}{6}c$ with $c = (en)^{-n/2}$, and that by basic calculus we have $\frac{\partial}{\partial t}(\cos t + \frac{\bar{c}}{2}\cdot\sin t) \geq \frac{\bar{c}}{3}$ whenever $\bar{c} \in (0,1]$ and $t \in (0, \frac{\bar{c}}{6}]$. In turn, we conclude the claim (26).

Next let the $(n-1)$-ball $B_2^n \cap \mathrm{aff}\{u_1,\ldots,u_n\}$ have center w_0 and radius ϱ, and let $w = w_0/\|w_0\| \in S^{n-1}$. Since the $(n-1)$-dimensional volume of $\mathrm{conv}\{u_1,\ldots,u_n\}$ is at most the $(n-1)$-dimensional volume $(\frac{n}{n-1})^{\frac{n-1}{2}}\frac{\sqrt{n}}{(n-1)!}$ of the regular $(n-1)$-dimensional simplex of circumradius 1, and the distance of o from $\mathrm{aff}\{u_1,\ldots,u_n\}$ is $\|w_0\|$, we deduce from (21) that

$$\begin{aligned}\frac{1}{n}\cdot\|w_0\|\cdot\sqrt{e}\cdot\frac{\sqrt{n}}{(n-1)!} &> \frac{1}{n}\cdot\|w_0\|\cdot\left(\frac{n}{n-1}\right)^{\frac{n-1}{2}}\frac{\sqrt{n}}{(n-1)!}\\ &> V(\mathrm{conv}\{o,u_1,\ldots,u_n\}) > \frac{1}{n!}\cdot(en)^{-n/2}.\end{aligned}$$

It follows that

$$\|w_0\| > (en)^{-\frac{n+1}{2}}.$$

Writing $\beta = \angle(w,u_i) \in (0,\frac{\pi}{2})$ for $i = 1,\ldots,n$, we thus get

$$\sin\left(\frac{\pi}{2}-\beta\right) = \cos\beta = \|w_0\| > (en)^{-\frac{n+1}{2}},$$

which in turn yields that

$$\beta < \frac{\pi}{2} - \frac{1}{(en)^{\frac{n+1}{2}}}. \tag{32}$$

Let C be the cone of vectors whose angle with w is at most $\frac{1}{2(en)^{\frac{n+1}{2}}}$; namely,

$$C = \left\{x \in \mathbb{R}^n : \langle x,w\rangle \geq \|x\|\cdot\cos\frac{1}{2(en)^{\frac{n+1}{2}}}\right\}.$$

We deduce from (32) that if $x \in C\setminus\{o\}$ and $i = 1,\ldots,n$, then

$$\angle(x,u_i) \leq \frac{\pi}{2} - \frac{1}{2(en)^{\frac{n+1}{2}}},$$

and hence (21) implies that

$$c_i\langle x, u_i\rangle \geq \frac{\|x\|}{(en)^n}\cdot \sin\frac{1}{2(en)^{\frac{n+1}{2}}} > \frac{\|x\|}{(en)^n}\cdot\frac{1}{4(en)^{\frac{n+1}{2}}} > \frac{\|x\|}{4(en)^{\frac{3n+1}{2}}}. \tag{33}$$

We also need an upper bound on λ_i for $i = 1, \ldots, n$. Let $u_1^*, \ldots, u_n^*$ be the dual basis to $u_1, \ldots, u_n$; namely, $\langle u_i^*, u_i\rangle = 1$ for $i = 1, \ldots, n$, and $\langle u_i^*, u_j\rangle = 0$ if $i \neq j$. It follows that if $i = 1, \ldots, n$, then (27) yields that

$$\|u_i^*\| = \frac{\left|\det_{n-1}[u_1, \ldots, \check{u}_i, \ldots, u_n]\right|}{\left|\det[u_1, \ldots, u_n]\right|} \leq (en)^{n/2};$$

therefore,

$$\lambda_i = \langle u_i^*, u_k\rangle \leq \|u_i^*\| \leq (en)^{n/2}. \tag{34}$$

We consider

$$\theta = \frac{1}{4(en)^{2n+1}}.$$

It follows from (33) and (34) that if $x \in C$ and $i = 1, \ldots, n$, then

$$c_i\langle x, u_i\rangle - \theta\cdot\lambda_i\|x\| \geq 0. \tag{35}$$

Now (10) and (25) imply that if $x \in C$, then

$$x = (c_k\langle x, u_k\rangle + \theta\cdot\|x\|)\, u_k + \sum_{i=1}^{n}(c_i\langle x, u_i\rangle - \theta\cdot\lambda_i\|x\|)\, u_i + \sum_{n<i<k} c_i\langle x, u_i\rangle u_i,$$

where the last term is void if $k = n + 1$, and hence applying first (35) and later (26) we get

$$\begin{aligned}\left(\sum_{i=1}^{k} c_i|\langle x, u_i\rangle|\right) - \|x\|_M &= \left(\sum_{i=1}^{k} c_i|\langle x, u_i\rangle|\right) - \inf\left\{\sum_{i=1}^{k}|\alpha_i| \,:\, x = \sum_{i=1}^{k}\alpha_i u_i\right\}\\ &\geq -\theta\cdot\|x\| + \sum_{i=1}^{n}\theta\cdot\lambda_i\|x\| \geq \theta\cdot\|x\|\cdot\frac{1}{3(en)^{n/2}}\cdot\eta\\ &\geq \frac{\|x\|}{12(en)^{3n}}\cdot\eta.\end{aligned}$$

We observe that if $x \in C\cup(-C)$ and $y \notin C\cup(-C)$, then $|\langle w, x\rangle| \geq |\langle w, y\rangle|$, and hence

$$\int_{C\cup(-C)} \|x\|\,\gamma_n(\mathrm{d}x) \geq \int_{C\cup(-C)} |\langle w,x\rangle|\,\gamma_n(\mathrm{d}x) \\ \geq 2\gamma_n(C)\int_{\mathbb{R}^n} |\langle w,x\rangle|\,\gamma_n(\mathrm{d}x). \tag{36}$$

Now we start to use the condition $\ell(K) \geq (1-\varepsilon)\ell(B_1^n)$. It follows from (15) that $\|x\|_M \leq \sum_{i=1}^k c_i|\langle x,u_i\rangle|$ for all $x\in\mathbb{R}^n$; therefore, (14), Lemma 2.4 and $\ell(M)\geq\ell(K)\geq(1-\varepsilon)\ell(B_1^n)$ yield

$$\begin{aligned}\varepsilon\ell(B_1^n) &\geq \ell(B_1^n)-\ell(M) = \int_{\mathbb{R}^n}\sum_{i=1}^k c_i|\langle x,u_i\rangle|\,\gamma_n(\mathrm{d}x) - \int_{\mathbb{R}^n}\|x\|_M\,\gamma_n(\mathrm{d}x)\\ &\geq \int_{C\cup(-C)}\frac{\|x\|}{12(en)^{3n}}\cdot\eta\,\gamma_n(\mathrm{d}x) \geq \frac{\gamma_n(C)}{6(en)^{3n}}\int_{\mathbb{R}^n}|\langle w,x\rangle|\,\gamma_n(\mathrm{d}x)\cdot\eta\\ &= \frac{\gamma_n(C)}{6n(en)^{3n}}\cdot\ell(B_1^n)\cdot\eta \geq \left(\sin\frac{1}{2(en)^{\frac{n+1}{2}}}\right)^{n-1}\cdot\frac{1}{18n^2(en)^{3n}}\cdot\ell(B_1^n)\cdot\eta.\end{aligned}$$

It follows that there exists some absolute constant $c(1)\geq 1$ such that

$$\eta \leq n^{c(1)n^2}\varepsilon.$$

We choose an absolute constant $c(2)\geq 4c(1)$ such that $n^{(c(1)-c(2))n^2} < \frac{1}{3n\cdot(en)^{n/2}}$ for $n\geq 2$, thus

$$n^{c(1)n^2}\varepsilon < \frac{1}{3n\cdot(en)^{n/2}} \text{ in (23) if } 0<\varepsilon<n^{-c(2)n^2}.$$

Therefore, if $\varepsilon_0 = n^{-c(2)n^2}$, then

$$\eta = \tilde{\eta}.$$

In particular, for any $j=n+1,\ldots,k$ there exists $i\in\{1,\ldots,n\}$ satisfying $|\langle u_i,u_j\rangle|\geq\cos\eta$, and hence Lemma 2.3 yields the existence of an orthonormal basis $w_1,\ldots,w_n$ such that

$$\delta_{\mathrm{H}}\left(\{\pm w_1,\ldots,\pm w_n\},\{\pm u_1,\ldots,\pm u_k\}\right) < 4\sqrt{n}\,\eta < n^{2c(1)n^2}\varepsilon. \tag{37}$$

In turn, we conclude the claim (22) by Lemma 2.5; namely,

$$(1-\gamma\varepsilon)B_1^n \subset M$$

for $\gamma = n^{3c(1)n^2}$ as $n^{3c(1)n^2}\varepsilon < n^{(3c(1)-c(2))n^2} \leq n^{-n^2} < n^{-1}$ by $c(2)\geq 4c(1)$ and $c(1)\geq 1$. Since $M\subset K$, we deduce that $(1-\gamma\varepsilon)B_1^n\subset K$.

Now if $z \in K \subset B_2^n$ and $z \notin (1-\gamma\varepsilon)B_1^n$, then $\|z\|_{B_1^n} = (1+t)(1-\gamma\varepsilon)$ for some $t > 0$. There exists a facet F of $(1-\gamma\varepsilon)B_1^n$ such that $z_0 = \frac{1}{1+t}\, z \in F$. Hence the polytope

$$P_z = \text{conv}\left(\{\pm z\} \cup (1-\gamma\varepsilon)B_1^n\right)$$

satisfies

$$\begin{aligned} V(P_z) - V\left((1-\gamma\varepsilon)B_1^n\right) &\geq 2t \cdot V\left(\text{conv}\{o, F\}\right) = 2t(1-\gamma\varepsilon)^n \frac{V(B_1^n)}{2^n} \\ &> \nu_1(n)\ell(B_1^n) \cdot t \end{aligned}$$

with $\nu_1(n) = (1-n^{-1})^n (n!\ell(B_1^n))^{-1} \geq (4n \cdot n!)^{-1}$, since $\ell(B_1^n) \leq n$. As $P_z \subset B_2^n$, Lemma 2.6 yields that

$$\ell\left((1-\gamma\varepsilon)B_1^n\right) - \ell(P_z) \geq \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} \nu_1(n)\ell(B_1^n) \cdot t \geq n^{-4n}\ell(B_1^n) \cdot t.$$

Here we used that $n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot 1.2$ for $n \geq 2$ and hence

$$\left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} 4n \cdot n! \leq n^{4n}. \tag{38}$$

We conclude that

$$\begin{aligned} (1-\varepsilon)\ell(B_1^n) \leq \ell(K) \leq \ell(P_z) &\leq \ell\Big((1-\gamma\varepsilon)B_1^n\Big) - n^{-4n} t \ell(B_1^n) \\ &= \Big((1-\gamma\varepsilon)^{-1} - n^{-4n} t\Big)\ell(B_1^n) \leq (1 + 2\gamma\varepsilon - n^{-4n} t)\ell(B_1^n), \end{aligned}$$

and therefore $t \leq (2\gamma+1)n^{4n} \cdot \varepsilon \leq n^{c(3)n^2} \cdot \varepsilon$ with an absolute constant $c(3) > 0$. If we set $\widehat{\gamma} = n^{c(3)n^2}$, then

$$K \subset (1 + \widehat{\gamma}\varepsilon)B_1^n.$$

Choosing $c = \max\{c(2), c(3)\}$, the assertion follows. □

3 Extremal ℓ-norm when the John ellipsoid is a ball and the rank one Brascamp–Lieb inequality

One of our main tools in this section is Barthe's proof of the rank one geometric Brascamp–Lieb inequality by means of optimal transport of probability measures (see [7, 8]). The geometric form of the corresponding general analytic inequality, verified originally by Brascamp and Lieb [19], was identified by Ball [3] as an important case that is perfectly suited for various geometric applications. For a more detailed discussion of the Brascamp–Lieb inequality (including higher ranks), we refer to

Carlen, Cordero-Erausquin [20], Lieb [33], Barthe [8], Valdimarsson [40], Bennett et al. [13], and Barthe et al. [11].

To set up the form (44) of the Brascamp–Lieb inequality, let the unit vectors $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfy $u_i \neq \pm u_j$ for $i \neq j$ and (9), i.e.,

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n. \tag{39}$$

During the argument in Barthe [8], the following four consequences of (39) observed by Ball [3] (see also [8] for a simpler proof of (40)) play crucial roles: If $k \geq n$, $c_1, \ldots, c_k > 0$ and $u_1, \ldots, u_k \in S^{n-1}$ satisfy (39), then we have the following properties:

Ball–Barthe inequality: For any $t_1, \ldots, t_k > 0$, we have

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i}. \tag{40}$$

Quadratic inequality: For $z = \sum_{i=1}^{k} c_i \theta_i u_i$ with $\theta_1, \ldots, \theta_k \in \mathbb{R}$, we have

$$\|z\|^2 = \sum_{i=1}^{k} c_i \langle z, u_i \rangle^2 \leq \sum_{i=1}^{k} c_i \theta_i^2. \tag{41}$$

Properties of $c_1, \ldots, c_k$:

$$c_i \leq 1 \quad \text{for } i = 1, \ldots, k, \tag{42}$$

$$c_1 + \cdots + c_k = n. \tag{43}$$

We only need the Brascamp–Lieb inequality in the following special case:

Theorem 3.1 *Let $b > 0$. Let f be a non-negative log-concave function on $\mathbb{R}$ such that $f(t) > 0$ if and only if $|t| \leq b$. Let $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfy $u_i \neq \pm u_j$ for $i \neq j$ and (39). Then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f(\langle x, u_i \rangle)^{c_i} \, \mathrm{d}x \leq \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f(t) \, \mathrm{d}t\right)^{c_i}. \tag{44}$$

Remark It was proved by Barthe [8] that equality in (44) holds if and only if $k = n$ and $u_1, \ldots, u_n$ form an orthonormal basis of $\mathbb{R}^n$. We carry out the proof of Theorem 3.1 since we will refer to the argument in our proof of Proposition 5.4.

Proof For the proof of (44), by scaling we may assume that

$$\int_{\mathbb{R}} f(t)\mathrm{d}t = \int_{-b}^{b} f(t)\,\mathrm{d}t = 1.$$

We follow Barthe in using a transport of measure argument. We write $\gamma_1(t) = \sqrt{2\pi}^{-1} e^{-t^2/2}$, $t \in \mathbb{R}$, for the standard one-dimensional Gaussian density (there is no danger of confusing it with the corresponding measure although we use the same symbol). Let $\varphi : (-b, b) \to \mathbb{R}$ be the transport map determined by the property

$$\int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{\varphi(x)} \gamma_1(t)\,\mathrm{d}t$$

for $x \in (-b, b)$. Here φ is C^1 as f is continuous on $(-b, b)$, and

$$f(x) = \gamma_1(\varphi(x)) \cdot \varphi'(x). \tag{45}$$

For

$$\mathcal{C} = \{x \in \mathbb{R}^n : \langle u_i, x\rangle \in (-b, b) \ \text{ for } i = 1, \ldots, k\},$$

we consider the transformation $\Theta : \mathcal{C} \to \mathbb{R}^n$ with

$$\Theta(x) = \sum_{i=1}^{k} c_i\, \varphi(\langle u_i, x\rangle)\, u_i, \qquad x \in \mathcal{C},$$

which satisfies

$$d\Theta(x) = \sum_{i=1}^{k} c_i\, \varphi'(\langle u_i, x\rangle)\, u_i \otimes u_i.$$

It is known that $d\Theta$ is positive definite and $\Theta : \mathcal{C} \to \mathbb{R}^n$ is injective (see [7, 8]). Therefore, using first (45), then the Ball–Barthe inequality (40) with $t_i = \varphi'(\langle u_i, x\rangle)$, and then the definition of Θ and (41), the following argument leads to the Brascamp–Lieb inequality in this special case:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{i=1}^k f(\langle u_i, x\rangle)^{c_i}\,\mathrm{d}x = \int_{\mathcal{C}} \prod_{i=1}^k f(\langle u_i, x\rangle)^{c_i}\,\mathrm{d}x \\
&= \int_{\mathcal{C}} \left(\prod_{i=1}^k \gamma_1(\varphi(\langle u_i, x\rangle))^{c_i}\right)\left(\prod_{i=1}^k \varphi'(\langle u_i, x\rangle)^{c_i}\right)\mathrm{d}x \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{C}} \left(\prod_{i=1}^k e^{-c_i\varphi(\langle u_i,x\rangle)^2/2}\right)\det\left(\sum_{i=1}^k c_i\varphi'(\langle u_i, x\rangle)\, u_i\otimes u_i\right)\mathrm{d}x \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{C}} e^{-\|\Theta(x)\|^2/2}\det\left(d\Theta(x)\right)\,\mathrm{d}x
\end{aligned} \tag{46}$$

$$\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\|y\|^2/2}\,\mathrm{d}y = 1, \tag{47}$$

which proves the assertion. □

We now use the Brascamp–Lieb inequality to prove the following estimate:

Proposition 3.2 *Let $b > 0$. For $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfying (39), the o-symmetric polytope $P = \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \le 1,\ i = 1, \ldots, k\}$ satisfies*

$$\gamma_n(bP) \le \gamma_n(bB^n_\infty). \tag{48}$$

Equality holds if and only if $k = n$, $u_1, \ldots, u_n$ is an orthonormal basis and hence P is a rotated copy of B^n_∞.

Proof Let

$$f_{(b)}(t) = \begin{cases} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}} & \text{if } |t| \le b, \\ 0 & \text{if } |t| > b. \end{cases}$$

It follows that

$$\gamma_n([-b, b]^n) = \left(\int_{\mathbb{R}} f_{(b)}(t)\,\mathrm{d}t\right)^n = \prod_{i=1}^k \left(\int_{\mathbb{R}} f_{(b)}(t)\,\mathrm{d}t\right)^{c_i}, \tag{49}$$

and

$$\gamma_n(bP) = \frac{1}{\sqrt{2\pi}^n}\int_{bP} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^n}\left(\frac{1}{\sqrt{2\pi}}\right)^{\sum_{i=1}^k c_i}\prod_{i=1}^k 1\{x\in\mathbb{R}^n : |\langle x,u_i\rangle|\le b\}e^{-\frac{1}{2}\sum_{i=1}^k c_i\langle x,u_i\rangle^2}\,\mathrm{d}x \tag{50}$$
$$= \int_{\mathbb{R}^n}\prod_{i=1}^k f_{(b)}(\langle x,u_i\rangle)^{c_i}\,\mathrm{d}x,$$

and hence the Brascamp–Lieb inequality (44) yields (48).

If equality holds, then equality must holds in (44) for the function $f_{(b)}$. By the remark after Theorem 3.1, the assertion about the equality case follows. □

We recall that, according to (2), we have

$$\ell(K) = \int_0^\infty (1-\gamma_n(tK))\,\mathrm{d}t. \tag{51}$$

Theorem 3.3 *(Schechtman, Schmuckenschläger) If K is an o-symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subset K$ is the John ellipsoid of K, then $\ell(K) \ge \ell(B_\infty^n)$. Equality holds if and only if K is a cube circumscribed around B_2^n.*

Proof Let $u_1,\dots,u_k \in S^{n-1}$ and $c_1,\dots,c_k > 0$ such that (39) holds. If

$$P = \{x\in\mathbb{R}^n : |\langle x,u_i\rangle| \le 1,\ i=1,\dots,k\},$$

then $K \subset P$ and $\ell(K) \ge \ell(P)$. On the other hand, an application of the estimate (48) in the formula (51) yields that $\ell(P) \ge \ell(B_\infty^n)$.

If equality holds, then clearly $K = P$ and P is a rotated copy of B_∞^n by the equality case in Proposition 3.2. □

4 Auxiliary statements to strengthen theorem 3.3

In order to obtain a stability version of Theorem 3.3, let

$$\Gamma_b = \frac{1}{\sqrt{2\pi}}\int_{-b}^{b} e^{-\frac{s^2}{2}}\,\mathrm{d}s < 1,$$

for $b > 0$, and consider the transport map $\varphi_b : (-b,b) \to \mathbb{R}$ such that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\varphi_b(t)} e^{-\frac{s^2}{2}}\,\mathrm{d}s = \frac{1}{\Gamma_b\sqrt{2\pi}}\int_{-b}^{t} e^{-\frac{s^2}{2}}\,\mathrm{d}s. \tag{52}$$

Readily, φ_b is an odd function, and hence $\varphi_b(0)=0$, $\varphi_b'(s)$ is even and $\varphi_b''(s)$ is odd. Differentiation of (52) yields

$$\varphi_b'(t)\cdot e^{-\frac{\varphi_b(t)^2}{2}} = \frac{1}{\Gamma_b}\cdot e^{-\frac{t^2}{2}}, \tag{53}$$

$$\varphi_b''(t)\cdot e^{-\frac{\varphi_b(t)^2}{2}} = \varphi_b(t)\cdot\varphi_b'(t)^2\cdot e^{-\frac{\varphi_b(t)^2}{2}} - \frac{t}{\Gamma_b}\cdot e^{-\frac{t^2}{2}}. \tag{54}$$

We collect various properties of the map φ_b for later use.

Lemma 4.1 *Let $b \in [1,2]$ and $t_b := (2\log(1/\Gamma_b))^{1/2}$. Then*

(a) $0.3 \le t_b \le b$,
(b) $1 \le \varphi_b'(t)$ for $t \in [0, t_b]$ and $\varphi_b'(t) \le 1.6$ for $t \in [0, 0.3]$,
(c) $t \le \varphi_b(t) \le 1.6t$ for $t \in [0, 0.3]$,
(d) $\varphi_b''(t) \ge (1-\Gamma_b)\Gamma_b^{-2}\cdot t \ge 0.049\cdot t$ for $t \in [0, t_b]$.

Proof (a) Since $b \mapsto \Gamma_b$ is increasing, we get $t_b \ge t_2 \ge 0.3$. The assertion $t_b \le b$ is equivalent to

$$h(b) := \sqrt{\frac{2}{\pi}}\int_0^b e^{-\frac{s^2}{2}}\,\mathrm{d}s - e^{-\frac{b^2}{2}} \ge 0.$$

Since $h(1) > 0$ and h is increasing, the assertion follows.

(b) Note that for $t \ge 0$ the condition $t \le t_b$ is equivalent to $\frac{1}{\Gamma_b}e^{-\frac{t^2}{2}} \ge 1$. Hence, if $t \in [0, t_b]$ then

$$\varphi_b'(t) = \frac{1}{\Gamma_b}e^{-\frac{t^2}{2}}e^{\frac{\varphi_b(t)^2}{2}} \ge \frac{1}{\Gamma_b}e^{-\frac{t^2}{2}} \ge 1.$$

To prove the upper bound for $\varphi_b'(t)$, first note that $t \mapsto \varphi_b(t)^2 - t^2$ is non-decreasing for $t \in [0, t_b]$, since

$$\frac{d}{dt}\left(\varphi_b(t)^2 - t^2\right) = 2\varphi_b(t)\varphi_b'(t) - 2t \ge 2t\cdot 1 - 2t \ge 0,$$

and $b \mapsto \varphi_b(t)$ is non-increasing for $0 \le t \le 0.3 \le t_b \le b$. To see this, let t be fixed. We show that the right side of (52), considered as a function of b, is non-increasing. Then the same is true for the left side of (52), hence $b \mapsto \varphi_b(t)$ must be non-increasing as well. For this, observe that

$$\frac{\mathrm{d}}{\mathrm{d}b}\left(\int_{-b}^{t} e^{-\frac{s^2}{2}}\,\mathrm{d}s\cdot\left(\int_0^b e^{-\frac{s^2}{2}}\,\mathrm{d}s\right)^{-1}\right)$$
$$= e^{-\frac{b^2}{2}}\left(\int_0^b e^{-\frac{s^2}{2}}\,\mathrm{d}s\right)^{-2}\left(\int_{-b}^{0} e^{-\frac{s^2}{2}}\,\mathrm{d}s-\int_{-b}^{t} e^{-\frac{s^2}{2}}\,\mathrm{d}s\right)\le 0,$$

since $t\ge 0$.

Using these facts, we get

$$\varphi_b'(t)=\frac{1}{\Gamma_b}e^{\frac{1}{2}\left(\varphi_b(t)^2-t^2\right)}\le\frac{1}{\Gamma_1}e^{\frac{1}{2}\left(\varphi_b(0.3)^2-0.3^2\right)}\le\frac{1}{\Gamma_1}e^{\frac{1}{2}\left(\varphi_1(0.3)^2-0.3^2\right)}\le 1.548.$$

(c) directly follows from (b) and $\varphi_b(0)=0$.

(d) Substituting $\varphi_b'(t)$ from (53) into (54) and using that $\varphi_b(t)\ge t$, we get

$$\begin{aligned}\varphi_b''(t)&=\varphi_b(t)e^{\varphi_b(t)^2}\frac{1}{\Gamma_b^2}e^{-t^2}-\frac{t}{\Gamma_b}e^{-\frac{t^2}{2}}e^{\frac{\varphi_b(t)^2}{2}}\\&\ge t\cdot\frac{1}{\Gamma_b}e^{\frac{1}{2}\left(\varphi_b(t)^2-t^2\right)}\left(\frac{1}{\Gamma_b}e^{\frac{1}{2}\left(\varphi_b(t)^2-t^2\right)}-1\right)\\&\ge\frac{1-\Gamma_b}{\Gamma_b^2}\cdot t\ge\frac{1-\Gamma_2}{\Gamma_2^2}\cdot t\ge 0.049\cdot t,\end{aligned}$$

which completes the argument. □

We also need the following stability version of the Ball–Barthe inequality (40) proved in Böröczky, Fodor, Hug [16]:

Lemma 4.2 *If $k\ge n+1$, $t_1,\ldots,t_k>0$, $c_1,\ldots,c_k>0$ and $u_1,\ldots,u_k\in S^{n-1}$ satisfy (39), and there exist $\beta>0$ and $n+1$ indices $\{i_1,\ldots,i_{n+1}\}\subset\{1,\ldots,k\}$ such that*

$$c_{i_1}\cdots c_{i_n}\det[u_{i_1},\ldots,u_{i_n}]^2\ge\beta,$$
$$c_{i_2}\cdots c_{i_{n+1}}\det[u_{i_2},\ldots,u_{i_{n+1}}]^2\ge\beta,$$

then

$$\det\left(\sum_{i=1}^k t_ic_iu_i\otimes u_i\right)\ge\left(1+\frac{\beta(t_{i_1}-t_{i_{n+1}})^2}{4(t_{i_1}+t_{i_{n+1}})^2}\right)\prod_{i=1}^k t_i^{c_i}.$$

5 Stability around the cube when the John ellipsoid is a ball

To prove Proposition 5.4, we recall Lemma 3.2 in Böröczky, Hug [18]:

Lemma 5.1 *Let* $v_1, \dots, v_k \in \mathbb{R}^n \setminus \{0\}$ *satisfy* $\sum_{i=1}^k v_i \otimes v_i = \mathrm{I}_n$, *and let* $0 < \eta < 1/(3\sqrt{k})$. *For any* $i \in \{1, \dots, k\}$, *we assume that* $\|v_i\| \le \eta$ *or there is some* $j \in \{1, \dots, n\}$ *with* $\angle(v_i, v_j) \le \eta$. *Then there exists an orthonormal basis* $w_1, \dots, w_n$ *such that* $\angle(v_i, w_i) < 3\sqrt{k}\,\eta$ *for* $i = 1, \dots, n$.

We will need the following consequence of Lemma 5.1.

Corollary 5.2 *Let* $u_1, \dots, u_k \in S^{n-1}$ *and* $c_1, \dots, c_k \in S^{n-1}$ *satisfy (39), and let* $0 < \eta < 1/(3\sqrt{k})$. *For any* $i \in \{1, \dots, k\}$, *we assume that* $c_i \le \eta^2$ *or there is some* $j \in \{1, \dots, n\}$ *with* $|\langle u_i, u_j\rangle| \ge \cos\eta$. *Then there exists an orthonormal basis* $w_1, \dots, w_n$ *such that* $\angle(u_i, w_i) < 3\sqrt{k}\,\eta$ *for* $i = 1, \dots, n$.

Proof For $i = 1, \dots, k$ we define $v_i = \sqrt{c_i} u_i$. Then $v_1, \dots, v_k \in \mathbb{R}^n \setminus \{0\}$ and $\sum_{i=1}^k v_i \otimes v_i = \mathrm{I}_n$. For any $i \in \{1, \dots, k\}$, by assumption we have $\|v_i\| = \sqrt{c_i} \le \eta$ or there is some $j \in \{1, \dots, n\}$ such that $0 \le \angle(v_i, v_j) \le \eta$ or $\pi \ge \angle(v_i, v_j) \ge \pi - \eta$. If $1 \le i \le n$, then we can choose $j = i$ so that $\angle(v_i, v_j) = 0 \le \eta$. If $n + 1 \le i \le k$ and $\|v_i\| > \eta$, then there is some $j \in \{1, \dots, n\}$ such that $0 \le \angle(v_i, v_j) \le \eta$ or $0 \le \angle(-v_i, v_j) \le \eta$. In the latter case, we simply replace v_i by $-v_i$. The possibly modified sequence $v_1, \dots, v_k$ satisfies all requirements of Lemma 5.1. Since $v_1, \dots, v_n$ remain unchanged, the assertion follows by an application of Lemma 5.1. $\square$

The technical statements Lemma 5.3 and Lemma 5.5 will be needed in the proof of Proposition 5.4:

Lemma 5.3 *Let* $\eta \in (0, \frac{\pi}{8})$, *and let* $u, u' \in S^1$ *be such that* $|\langle u, u'\rangle| \le \cos\eta$. *If* $v \in S^1$ *satisfies* $\cos\frac{3\pi}{8} \le |\langle v, \frac{u-u'}{\|u-u'\|}\rangle| \le \cos\frac{\pi}{8}$ *and* $\cos\frac{3\pi}{8} \le |\langle v, \frac{u+u'}{\|u+u'\|}\rangle| \le \cos\frac{\pi}{8}$, *then*

$$\Big|\, |\langle v, u\rangle| - |\langle v, u'\rangle| \,\Big| \ge \eta/5. \tag{55}$$

Proof Since (55) considers $|\langle v, u\rangle|$ and $|\langle v, u'\rangle|$, we may assume, by symmetry, that $u = (\cos\alpha, \sin\alpha)^\top$ and $u' = (-\cos\alpha, \sin\alpha)^\top$, where $\frac{\eta}{2} \le \alpha \le \frac{\pi}{2} - \frac{\eta}{2}$, and hence $\frac{u+u'}{\|u+u'\|} = (0, 1)$ and $\frac{u-u'}{\|u-u'\|} = (1, 0)$, and in addition, $v = (\cos\beta, \sin\beta)^\top$ where $\frac{\pi}{8} \le |\beta| \le \frac{3\pi}{8}$.

If $\beta \ge \frac{\pi}{2} - \alpha$, then

$$
\begin{aligned}
|\langle v,u\rangle| - |\langle v,u'\rangle| = \langle v,u\rangle - \langle v,u'\rangle &= 2\cos\alpha\cos\beta \\
&\geq 2\cos\left(\frac{\pi}{2}-\frac{\eta}{2}\right)\cos\frac{3\pi}{8} = 2\sin\frac{\eta}{2}\sin\frac{\pi}{8} \\
&> 2\frac{2}{\pi}\frac{\eta}{2}\sin\frac{\pi}{8} > 0.2436\cdot\eta > \frac{\eta}{5}.
\end{aligned}
$$

If $\frac{\pi}{8} \leq \beta \leq \frac{\pi}{2} - \alpha$, then

$$|\langle v,u\rangle| - |\langle v,u'\rangle| = \langle v,u\rangle + \langle v,u'\rangle = 2\sin\alpha\sin\beta \geq 2\sin\frac{\eta}{2}\sin\frac{\pi}{8} > \frac{\eta}{5}.$$

If $\alpha - \frac{\pi}{2} \leq \beta \leq -\frac{\pi}{8}$, then

$$|\langle v,u\rangle| - |\langle v,u'\rangle| = \langle v,u\rangle + \langle v,u'\rangle = 2\sin\alpha\sin\beta \leq -2\sin\frac{\eta}{2}\sin\frac{\pi}{8} < -\frac{\eta}{5}.$$

Finally, if $\beta \leq \alpha - \frac{\pi}{2}$, then

$$|\langle v,u\rangle| - |\langle v,u'\rangle| = -\langle v,u\rangle + \langle v,u'\rangle = -2\cos\alpha\cos\beta \leq -2\sin\frac{\eta}{2}\cos\frac{3\pi}{8} < -\frac{\eta}{5},$$

proving (55). □

For origin symmetric convex bodies $K, L \subset \mathbb{R}^n$, we consider the symmetric (volume) difference

$$\delta_{\text{vol}}(K,L) = V(K\Delta L).$$

In the following proposition, we first provide local bounds in terms of the volume difference for the Gaussian measure.

Proposition 5.4 *For $b \in [1,2]$, $n \leq k \leq \frac{n(n+1)}{2}$ and $u_1,\ldots,u_k \in S^{n-1}$ and $c_1,\ldots,c_k > 0$ satisfying (39), there exists $\Phi \in O(n)$ such that the polytope $P = \{x \in \mathbb{R}^n : |\langle x,u_i\rangle| \leq 1,\ i = 1,\ldots,k\}$ satisfies*

$$\gamma_n(bP) \leq \gamma_n(bB_\infty^n) - 2^{-80}n^{-40n}\delta_{\text{vol}}(P,\Phi B_\infty^n)^4. \tag{56}$$

Proof For $x \in P$, (39) and (43) yield

$$\|x\|^2 = \sum_{i=1}^k c_i\langle x,u_i\rangle^2 \leq \sum_{i=1}^k c_i \leq n, \quad \text{and hence } P \subset \sqrt{n}\,B_2^n. \tag{57}$$

In particular, for any $\Phi \in O(n)$, we have $\delta_{\text{vol}}(P,\Phi B_\infty^n) \leq V(P) + V(\Phi B_\infty^n) \leq 2\cdot 2^n$ according to Ball [3]. Therefore, Proposition 5.4 is implied by the following statement: If

$$\gamma_n(bP) \geq \gamma_n(bB_\infty^n) - \varepsilon \tag{58}$$

for $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 = 2^{-68}n^{-28n}$, then there exists $\Phi \in O(n)$ such that

$$\delta_{\mathrm{vol}}(P, \Phi B_\infty^n) \leq 2^{20}n^{10n}\varepsilon^{\frac{1}{4}}. \tag{59}$$

Hence (56) follows, since for $\varepsilon \geq \varepsilon_0$ the asserted inequality holds by the preceding rough bound.

In order to prove (59), the core claim is that (58) yields a $\Phi \in O(n)$ such that

$$P \subset (1 + \aleph\sqrt[4]{\varepsilon})\Phi B_\infty^n \tag{60}$$

for $\aleph = 2^{14}n^{6n}$.

As at the beginning of the proof of Theorem 2.7, applying Lemma 2.2 with $v_i = \sqrt{c_i}\, u_i$, and using that $\left(\begin{array}{c} k \\ n \end{array}\right) \leq (en)^n$, we may assume (after re-indexing if needed), that

$$c_1 \cdots c_n \cdot \Big|\det[u_1, \ldots, u_n]\Big|^2 \geq (en)^{-n}. \tag{61}$$

We set

$$\eta = 2^{14}n^{4n} \cdot \varepsilon^{1/4} < \frac{1}{8}, \tag{62}$$

and claim that if $i \in \{1, \ldots, k\}$, then

$$c_i \leq \eta^2, \text{ or there exists some } j \in \{1, \ldots, n\} \text{ with } |\langle u_i, u_j\rangle| \geq \cos\eta. \tag{63}$$

We suppose, on the contrary, that (63) does not hold and seek a contradiction. Hence we may assume

$$c_{n+1} > \eta^2 \quad \text{and} \quad |\langle u_j, u_{n+1}\rangle| < \cos\eta \quad \text{for } j = 1, \ldots, n,$$

Now $u_{n+1} = \sum_{i=1}^n \lambda_i u_i$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are uniquely determined. We may assume, by reordering, that $|\lambda_1| \geq \ldots \geq |\lambda_n|$. Choose $\theta_i \in \{-1, 1\}$ such that $\theta_i\lambda_i = |\lambda_i|$ for $i = 1, \ldots, n$, and hence $u_{n+1} = \sum_{i=1}^n |\lambda_i|\theta_i u_i$. As $u_{n+1} \notin \mathrm{conv}\{o, \theta_1 u_1, \ldots, \theta_n u_n\}$, we have $\sum_{i=}^n |\lambda_i| > 1$, thus $|\lambda_1| \geq \frac{1}{n}$. Therefore, $c_{n+1} > \eta^2$, $c_1 \leq 1$ (due to (42)) and (61) imply

$$c_2 \cdots c_{n+1} \det[u_2, \ldots, u_{n+1}]^2 \geq (en)^{-n} \cdot \frac{\eta^2}{n^2} = e^{-n}n^{-n-2}\eta^2. \tag{64}$$

Next we observe that $\angle(u_1 - u_{n+1}, u_1 + u_{n+1}) = \frac{\pi}{2}$, and let $\tilde{w} \in S^{n-1} \cap L$ for $L = \mathrm{lin}\{u_1, u_{n+1}\}$ satisfy that $\angle(\tilde{w}, u_1 - u_{n+1}) = \angle(\tilde{w}, u_1 + u_{n+1}) = \frac{\pi}{4}$. If

$v \in S^{n-1}$ with $\angle(v, \tilde{w}) \leq \frac{\pi}{8}$, then the orthogonal projection v' of v to L satisfies that $\|v'\| \geq \cos\frac{\pi}{8} > \frac{1}{2}$ and $\angle(v', \tilde{w}) \leq \frac{\pi}{8}$; therefore, $|\langle u_1, u_{n+1}\rangle| < \cos\eta$ and Lemma 5.3 yield that

$$\Big|\ |\langle v, u_1\rangle| - |\langle v, u_{n+1}\rangle|\ \Big| = \Big|\ |\langle v', u_1\rangle| - |\langle v', u_{n+1}\rangle|\ \Big| > \eta/8, \tag{65}$$

provided $\angle(v, \tilde{w}) \leq \frac{\pi}{8}$ for $v \in S^{n-1}$.

Now there exists a $w \in L \cap S^{n-1}$ with $\angle(w, \tilde{w}) \leq \frac{\pi}{16}$ such that $|\frac{\pi}{2} - \angle(w, u_1)| \geq \frac{\pi}{32}$ and $|\frac{\pi}{2} - \angle(w, u_{n+1})| \geq \frac{\pi}{32}$, and we consider the ball

$$\Xi = \frac{w}{8} + 2^{-9}B_2^n = \frac{w}{8} + \frac{1}{8\cdot 64}B_2^n \subset bP.$$

If $x \in \Xi$, then $\frac{1}{16} \leq \|x\| \leq \frac{1}{4}$ and $\angle(x, w) \leq \frac{\pi}{64}$, and hence also $|\frac{\pi}{2} - \angle(x, u_1)| \geq \frac{\pi}{64}$ and $|\frac{\pi}{2} - \angle(x, u_{n+1})| \geq \frac{\pi}{64}$. From the choice of w and $\frac{1}{16}\cdot\frac{1}{64} = 2^{-10}$ we deduce that if $x \in \Xi$, then

$$\begin{array}{rcll} |\langle x, u_1\rangle| & \geq & 2^{-10}, & \\ |\langle x, u_{n+1}\rangle| & \geq & 2^{-10}, & \\ |\langle x, u_i\rangle| & \leq & 2^{-2}, & i = 1, \ldots, k. \end{array} \tag{66}$$

We have $\angle(x, \tilde{w}) \leq \frac{\pi}{8}$ as $\angle(x, w) \leq \frac{\pi}{64}$, and hence (65) yields that

$$\Big|\ |\langle x, u_1\rangle| - |\langle x, u_{n+1}\rangle|\ \Big| > 2^{-7}\eta \quad \text{for } x \in \Xi. \tag{67}$$

As in the proof of Proposition 3.2, let

$$f_{(b)}(t) = \begin{cases} \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}, & \text{if } |t| \leq b, \\ 0, & \text{if } |t| > b. \end{cases}$$

Hence (*cf.* (49) and (50))

$$\gamma_n([-b, b]^n) = \prod_{i=1}^{k}\left(\int_{\mathbb{R}} f_{(b)}(t)\,\mathrm{d}t\right)^{c_i}, \tag{68}$$

$$\gamma_n(bP) = \int_{\mathbb{R}^n}\prod_{i=1}^{k} f_{(b)}(\langle x, u_i\rangle)^{c_i}\,\mathrm{d}x. \tag{69}$$

We consider the probability density (*cf.* Sect. 4)

$$\tilde{f}_{(b)} = \frac{1}{\Gamma_b} f_{(b)} \quad \text{for } \Gamma_b = \int_{\mathbb{R}} f_{(b)}(s)\,\mathrm{d}s = \int_{-b}^{b} f_{(b)}(s)\,\mathrm{d}s < 1,$$

and the corresponding transport map $\varphi_b : (-b, b) \to \mathbb{R}$ such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varphi_b(t)} e^{-\frac{s^2}{2}}\,\mathrm{d}s = \int_{-b}^{t} \tilde{f}_{(b)}(s)\,\mathrm{d}s.$$

It follows from Lemma 4.1, (66) and $0.049 > 2^{-5}$ that if $x \in \Xi$, then

$$\begin{array}{rcll} |\varphi_b(\langle x, u_i\rangle)| & \leq & 1, & i = 1, \ldots, k, \\ \varphi_b'(\langle x, u_i\rangle) & \geq & 1, & i = 1, \ldots, k, \\ \varphi_b'(\langle x, u_i\rangle) & \leq & 2, & i = 1, \ldots, k, \\ \varphi_b''(t) & \geq & 2^{-15}, & \text{if } t \text{ is between } |\langle x, u_1\rangle| \text{ and } |\langle x, u_{n+1}\rangle|. \end{array} \tag{70}$$

Therefore, (67) yields that if $x \in \Xi$, then

$$\begin{aligned} &\left|\varphi_b'\left(\langle x, u_1\rangle\right) - \varphi_b'\left(\langle x, u_{n+1}\rangle\right)\right| \\ &\quad = \left|\varphi_b'\left(|\langle x, u_1\rangle|\right) - \varphi_b'\left(|\langle x, u_{n+1}\rangle|\right)\right| > 2^{-22}\eta. \end{aligned} \tag{71}$$

We apply Lemma 4.2, the stability version of the Ball–Barthe inequality (40), with $\beta = n^{-4n}\eta^2 < e^{-n}n^{-n-2}\eta^2$, based on (61) and (64), and using the estimates (70) and (71) to conclude that if $x \in \Xi$, then

$$\begin{aligned} &\prod_{i=1}^{k} \varphi_b'(\langle x, u_i\rangle)^{c_i} \\ &\leq \det\left(\sum_{i=1}^{k} \varphi_b'(\langle x, u_i\rangle) c_i u_i \otimes u_i\right) \left(1 + \frac{\beta(\varphi_b'(\langle x, u_1\rangle) - \varphi_b'(\langle x, u_{n+1}\rangle))^2}{4(\varphi_b'(\langle x, u_1\rangle) + \varphi_b'(\langle x, u_{n+1}\rangle))^2}\right)^{-1} \\ &\leq \det\left(\sum_{i=1}^{k} \varphi_b'(\langle x, u_i\rangle) c_i u_i \otimes u_i\right) - 2^{-51} n^{-4n} \eta^4, \end{aligned} \tag{72}$$

where we also used that $(1+s)^{-1} < 1 - \frac{s}{2}$ if $s \in (0, \frac{1}{2})$ and the Ball–Barthe inequality (40) and (70) imply that

$$\det\left(\sum_{i=1}^{k} \varphi_b'(\langle x, u_i\rangle) c_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} \varphi_b'(\langle x, u_i\rangle)^{c_i} \geq 1.$$

We observe that the $\mathcal{C}$ in (46) is just bP, and use (72) for $x \in \Xi$ instead of (40) in (46). We deduce from (68) and (69) that

$$\begin{aligned}
\frac{\gamma_n(bP)}{\gamma_n([-b,b]^n)} &= \int_{\mathbb{R}^n} \prod_{i=1}^k \tilde{f}_{(b)}(\langle u_i, x\rangle)^{c_i}\,\mathrm{d}x \\
&= \int_{bP} \left(\prod_{i=1}^k \gamma_1(\varphi_b(\langle u_i, x\rangle))^{c_i}\right)\left(\prod_{i=1}^k \varphi_b'(\langle u_i, x\rangle)^{c_i}\right)\mathrm{d}x \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{bP\setminus\Xi}\left(\prod_{i=1}^k e^{-c_i\varphi_b(\langle u_i,x\rangle)^2/2}\right)\det\left(\sum_{i=1}^k c_i\varphi_b'(\langle u_i, x\rangle)\,u_i\otimes u_i\right)\mathrm{d}x \\
&\quad + \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Xi}\left(\prod_{i=1}^k e^{-c_i\varphi_b(\langle u_i,x\rangle)^2/2}\right) \\
&\quad \times\left(\det\left(\sum_{i=1}^k c_i\varphi_b'(\langle u_i, x\rangle)\,u_i\otimes u_i\right) - 2^{-51}n^{-4n}\eta^4\right)\mathrm{d}x.
\end{aligned}$$

Here $V(\Xi) = 2^{-9n}\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} > 2^{-9n}n^{-\frac{n}{2}-\frac{1}{2}}\frac{(e\pi)^{n/2}}{4}$, and (70) yields that if $x \in \Xi$, then

$$\prod_{i=1}^k e^{-c_i\varphi(\langle u_i,x\rangle)^2/2} \ge e^{-\frac{1}{2}\sum_{i=1}^k c_i} = e^{-n/2};$$

therefore, we deduce from (47) that

$$\frac{\gamma_n(bP)}{\gamma_n([-b,b]^n)} \le 1 - \frac{V(\Xi)}{(2\pi)^{\frac{n}{2}}}\cdot e^{-n/2}2^{-51}n^{-4n}\eta^4 < 1 - 2^{-53}n^{-15n}\eta^4.$$

We have $\gamma_n([-b,b]^n) = (2\Phi(b)-1)^n \ge (2\Phi(1)-1)^n > 2^{-n}$ for the cumulative distribution function $\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-\frac{s^2}{2}}\,\mathrm{d}s$, and hence, by (58)

$$\gamma_n([-b,b]^n) - \varepsilon < \gamma_n(bP) < \gamma_n([-b,b]^n) - 2^{-53}n^{-16n}\eta^4.$$

These inequalities contradict (62), and in turn prove the claim (63).

Since $\eta < 1/(3\sqrt{k})$ and due to (63), Corollary 5.2 yields that there exists an orthonormal basis $w_1, \ldots, w_n$ such that

$$\angle(u_i, w_i) < 3\sqrt{k}\,\eta < 3n\eta \quad \text{for } i = 1, \ldots, n. \tag{73}$$

Let $\Phi \in O(n)$ satisfy that ΦB_∞^n is the cube $\{x \in \mathbb{R}^n : |\langle x, w_i\rangle| \le 1\}$. For $x \in P$, (57) and (73) imply that if $i = 1, \ldots n$, then

$$|\langle x, w_i\rangle| \le |\langle x, u_i\rangle| + |\langle x, w_i - u_i\rangle| \le 1 + 3n^{3/2}\eta,$$

and hence

$$P \subset (1 + \aleph\sqrt[4]{\varepsilon})\Phi B_\infty^n$$

for $\aleph = 2^{14}n^{6n}$, as claimed in (60).

Since $(1+t)^n < e^{nt} < 1+2nt$ for $0<t<1/n$, it follows that $(1+\aleph\sqrt[4]{\varepsilon})^n < 1+2n\aleph\sqrt[4]{\varepsilon}$ if $\varepsilon < \varepsilon_0$. As $V(b\Phi B_\infty^n) \le 4^n$, we observe that

$$\begin{aligned}\gamma_n\left((1+\aleph\sqrt[4]{\varepsilon})b\Phi B_\infty^n \setminus (b\Phi B_\infty^n)\right) &\le (2\pi)^{-n/2} V\left((1+\aleph\sqrt[4]{\varepsilon})b\Phi B_\infty^n \setminus (b\Phi B_\infty^n)\right)\\ &\le (2\pi)^{-n/2} 2n\aleph\sqrt[4]{\varepsilon}4^n.\end{aligned} \tag{74}$$

We have $V(P) \le V(B_\infty^n) \le V(\Phi B_\infty^n)$ according to Ball [3], and hence $2V(\Phi B_\infty^n \setminus P) \ge \delta_{\rm vol}(\Phi B_\infty^n, P)$. In addition, $e^{-\|x\|^2/2} \ge e^{-2n}$ if $x \in bP \cup b\Phi B_\infty^n$ by (57); therefore, (60) and (74), and finally (58) and $\aleph = 2^{14}n^{6n}$ yield

$$\begin{aligned}\delta_{\rm vol}(\Phi B_\infty^n, P) &\le 2V(b\Phi B_\infty^n \setminus (bP)) \le 2\cdot(2\pi)^{n/2}e^{2n}\gamma_n(b\Phi B_\infty^n \setminus (bP))\\ &\le 2\cdot(2\pi)^{n/2}e^{2n}\gamma_n\left((1+\aleph\sqrt[4]{\varepsilon})b\Phi B_\infty^n \setminus (bP)\right)\\ &\le 2\cdot(2\pi)^{n/2}e^{2n}\left(\gamma_n\left((1+\aleph\sqrt[4]{\varepsilon})b\Phi B_\infty^n\right) - \gamma_n\left(b\Phi B_\infty^n\right)+\varepsilon\right)\\ &= 2\cdot(2\pi)^{n/2}e^{2n}\left(\gamma_n\left((1+\aleph\sqrt[4]{\varepsilon})b\Phi B_\infty^n \setminus (b\Phi B_\infty^n)\right)+\varepsilon\right)\\ &\le 2\cdot(2n)\cdot e^{2n}\cdot\aleph\cdot 4^n\cdot\sqrt[4]{\varepsilon} + 2\cdot 2^{2n}\cdot 2^{3n}\varepsilon \le 2^{20}\cdot n^{10n}\sqrt[4]{\varepsilon},\end{aligned}$$

which proves (59). □

The following lemma allows us to compare the Hausdorff distance and the volume difference. The lemma is a straightforward consequence of the proof of a result due to Groemer [23, Theorem (ii)] in the case of symmetric convex bodies.

Lemma 5.5 *Let $K, L \subset \mathbb{R}^n$ be o-symmetric convex bodies. If $rB_2^n \subset K, L \subset RB_2^n$, then*

$$\delta_{\rm H}(K,L) \le \left(\frac{n}{\kappa_{n-1}}\right)^{\frac{1}{n}}\left(\frac{R}{r}\right)^{\frac{n-1}{n}}\delta_{\rm vol}(K,L)^{\frac{1}{n}}.$$

If $r=1$, $R=\sqrt{n}$, then

$$\delta_{\rm H}(K,L) \le n\cdot n^{\frac{1}{2n}}\delta_{\rm vol}(K,L)^{\frac{1}{n}}.$$

Proof The first assertion follows from the proof of [23, Theorem (ii)], where in the case of symmetric convex bodies instead of the diameter of K, L the circumradius can be used in the argument.

The second part follows from the first part by basic calculus, using that

$$\Gamma\left(\frac{n+1}{2}\right) \le \sqrt{\pi(n-1)}\left(\frac{n-1}{2e}\right)^{\frac{n-1}{2}}\frac{3}{2}$$

for $n \geq 2$. □

Now we are ready to prove a stability version of Theorem 3.3.

Theorem 5.6 *If K is an origin symmetric convex body in* $\mathbb{R}^n$ *such that* $B_2^n \subset K$ *is the John ellipsoid of K, then there exists* $\Phi \in O(n)$ *such that*

$$\ell(K) \geq \ell(B_\infty^n) + 2^{-88} n^{-40n} \delta_{\mathrm{vol}}(K, \Phi B_\infty^n)^4, \tag{75}$$

$$\ell(K) \geq \ell(B_\infty^n) + 2^{-90} n^{-44n} \delta_{\mathrm{H}}(K, \Phi B_\infty^n)^{4n}. \tag{76}$$

Proof According to John's characteristic condition (8), there exist $u_1, \ldots, u_k \in S^{n-1} \cap \partial K$ and $c_1, \ldots, c_k > 0$ with $n \leq k \leq \frac{n(n+1)}{2}$ that satisfy (39).

For $P = \{x \in \mathbb{R}^n : |\langle x, u_i \rangle| \leq 1,\ i = 1, \ldots, k\}$, we have

$$K \subset P \subset \sqrt{n} B_2^n \text{ and } \ell(tK) \geq \ell(tP) \text{ for } t > 0. \tag{77}$$

We deduce from (2) that

$$\ell(K) - \ell(P) = \int_0^\infty (\gamma_n(tP) - \gamma_n(tK))\, \mathrm{d}t. \tag{78}$$

To estimate $\ell(P)$, first observe that (48) yields

$$\gamma_n(tP) \leq \gamma_n(tB_\infty^n) \quad \text{for } t > 0. \tag{79}$$

Next, let $\Phi \in O(n)$ be such that

$$\delta_{\mathrm{vol}}(P, \Phi B_\infty^n) = \min_{\Phi' \in O(n)} \delta_{\mathrm{vol}}(P, \Phi' B_\infty^n).$$

Hence if $t \in [1, 2]$, then Proposition 5.4 implies that

$$\gamma_n(tP) \leq \gamma_n(tB_\infty^n) - 2^{-80} n^{-40n} \delta_{\mathrm{vol}}(P, \Phi B_\infty^n)^4. \tag{80}$$

Combining (78), (79) and (80), we obtain

$$\ell(P) \geq \ell(B_\infty^n) + 2^{-80} n^{-40n} \delta_{\mathrm{vol}}(P, \Phi B_\infty^n)^4.$$

In the following, we use again [3] to get

$$V(K) \leq V(P) \leq V(B_\infty^n) = V(\Phi B_\infty^n). \tag{81}$$

We finish the proof by distinguishing two cases:

- If $V(\Phi B_\infty^n \setminus K) \leq 2V(\Phi B_\infty^n \setminus P)$, then (81) implies that

$$\delta_{\mathrm{vol}}(K, \Phi B_\infty^n) \leq 2V(\Phi B_\infty^n \backslash K) \leq 4V(\Phi B_\infty^n \backslash P) \leq 4\delta_{\mathrm{vol}}(P, \Phi B_\infty^n),$$

and hence

$$\ell(K) \geq \ell(B_\infty^n) + 2^{-88}n^{-40n}\delta_{\mathrm{vol}}(K, \Phi B_\infty^n)^4. \tag{82}$$

- If $V(\Phi B_\infty^n \setminus K) \geq 2V(\Phi B_\infty^n \setminus P)$ and $t \in [1,2]$, then, as $e^{-\frac{1}{2}\|x\|^2} \geq e^{-2n}$ by (77) for $x \in tP$, we obtain

$$\begin{aligned}
\gamma_n(tP) - \gamma_n(tK) &\geq \gamma_n\Big((t\Phi B_\infty^n) \cap [(tP) \setminus (tK)]\Big) \\
&\geq (2\pi)^{-\frac{n}{2}} e^{-2n} \cdot V\Big((t\Phi B_\infty^n) \cap [(tP) \setminus (tK)]\Big) \\
&\geq \frac{(2\pi)^{-\frac{n}{2}} e^{-2n}}{2} \cdot V(\Phi B_\infty^n \setminus K) \\
&\geq \frac{(2\pi)^{-\frac{n}{2}} e^{-2n}}{4} \cdot \delta_{\mathrm{vol}}(K, \Phi B_\infty^n).
\end{aligned}$$

For the third inequality, we used that

$$\begin{aligned}
V\left(\Phi B_\infty^n \setminus K\right) &= V\left(\Phi B_\infty^n \cap P \setminus K\right) + V\left(\Phi B_\infty^n \cap P^{\mathrm{c}} \setminus K\right) \\
&\leq V\left(\Phi B_\infty^n \cap P \setminus K\right) + V\left(\Phi B_\infty^n \setminus P\right) \\
&\leq V\left(\Phi B_\infty^n \cap P \setminus K\right) + \frac{1}{2} V\left(\Phi B_\infty^n \setminus K\right),
\end{aligned}$$

the last inequality follows since $V(K) \leq V(\Phi B_\infty^n)$. Hence

$$\gamma_n(tP) - \gamma_n(tK) \geq 2^{-6n}\delta_{\mathrm{vol}}(K, \Phi B_\infty^n) \geq n^{-6n}\delta_{\mathrm{vol}}(K, \Phi B_\infty^n).$$

From (78) we deduce that

$$\begin{aligned}
\ell(K) - \ell(B_\infty^n) &\geq \ell(K) - \ell(P) \geq n^{-6n}\delta_{\mathrm{vol}}(K, \Phi B_\infty^n) \\
&\geq 2^{-88}n^{-40n}\delta_{\mathrm{vol}}(K, \Phi_v B_\infty^n)^4.
\end{aligned} \tag{83}$$

We conclude (75) from (82) and (83).

Finally, (76) is implied by (75) and Lemma 5.5. □

6 Proof of theorems 1.2 and 1.3

The following trivial observation relates Hausdorff distance to the "dilation distance".

Lemma 6.1 *For convex bodies $K, C \subset \mathbb{R}^n$, if $n^{\frac{-1}{2}} B_2^n \subset K, C \subset \sqrt{n}\, B_2^n$, then*

$$(1+\sqrt{n}\delta_{\rm H}(K,C))^{-1}C\subset K\subset(1+\sqrt{n}\delta_{\rm H}(K,C))C,$$

and if $(1+t)^{-1}C\subset K\subset(1+t)C$ for $t\geq 0$, then $\delta_{\rm H}(K,C)\leq\sqrt{n}\,t$.

Proof We use that $\delta_{\rm H}(K,C)$ is the minimum of $\varrho\geq 0$ such that $C\subset K+\varrho B_2^n$ and $K\subset C+\varrho B_2^n$. □

Proof of Theorems 1.2 and 1.3 The statements about the ℓ-norm follow from Theorems 2.7 and 5.6. In turn, the statements about the mean width follow from polarity, (1) and Lemma 6.1. □

7 Proof of theorem 1.5

We need some auxiliary statements. The first lemma is a counterpart to [18, Lem. 10.1] for even isotropic measures.

Lemma 7.1 *If μ is an even isotropic measure on S^{n-1}, $n\geq 2$, then there exists a discrete even isotropic measure μ_0 on S^{n-1} such that* $\operatorname{supp}\mu_0\subset\operatorname{supp}\mu$ *and* $|\operatorname{supp}\mu_0|\leq n(n+1)/2$.

Proof We set $d=\frac{n(n+1)}{2}$, and consider the d-dimensional real vectorspace $\mathcal{M}^d$ of symmetric $n\times n$ matrices. Basic elements of $\mathcal{M}^d$ are the $n\times n$ identity matrix I_n, and the rank one matrices $u\otimes u=uu^\top$ for a $u\in S^{n-1}$. We equip $\mathcal{M}^d$ with a scalar product; namely, $\langle A,B\rangle=\operatorname{tr}AB^\top$ for $A,B\in\mathcal{M}^d$, and hence if $A=[a_{ij}]$ and $B=[b_{ij}]$, then $\langle A,B\rangle=\sum_{i,j=1,\ldots,n}a_{ij}b_{ij}$. We claim that

$$\frac{1}{n}\,\mathrm{I}_n\in\operatorname{conv}\{u\otimes u:\,u\in\operatorname{supp}\mu\}.\tag{84}$$

It is equivalent to prove that for any $M\in\mathcal{M}^d$, we have

$$\frac{1}{n}\,\langle\mathrm{I}_n,M\rangle\leq\max\{\langle u\otimes u,M\rangle:\,u\in\operatorname{supp}\mu\}.\tag{85}$$

What is known about μ is that $\frac{1}{n}\,\mu$ is a probability measure, and

$$\frac{1}{n}\int_{S^{n-1}}\langle u\otimes u,M\rangle\,d\mu(u)=\frac{1}{n}\,\langle I_n,M\rangle,$$

which in turn yields (85), and hence also (84).

Writing $\mathcal{M}_1^{d-1}$ to denote the affine $(d-1)$-dimensional subspace of $\mathcal{M}^d$ consisting of matrices with trace 1, we observe that $u\otimes u\in\mathcal{M}_1^{d-1}$ for $u\in\operatorname{supp}\mu$ and $\frac{1}{n}\,\mathrm{I}_n\in\mathcal{M}_1^{d-1}$. According to the Charatheodory theorem applied to (84) in $\mathcal{M}_1^{d-1}$, there exist $u_1,\ldots,u_k\in\operatorname{supp}\mu$ with $u_j\neq\pm u_i$ for $i\neq j$ and $k\leq d$ such that

$$\frac{1}{n}\,\mathrm{I}_n \in \mathrm{conv}\{u_i \otimes u_i : i = 1, \ldots, k\}.$$

It follows that there exist $\tilde{c}_1, \ldots \tilde{c}_k \geq 0$ with $\tilde{c}_1 + \cdots + \tilde{c}_k = 1$ such that

$$\sum_{i=1}^{k} n\tilde{c}_i\, u_i \otimes u_i = \mathrm{I}_n.$$

Therefore, we can define the even measure μ_0 so that $\mathrm{supp}\,\mu_0 \subset \{\pm u_1, \ldots, \pm u_k\}$, and $\mu_0(u_i) = n\tilde{c}_i/2$. □

The following lemma implies the bounds involving the δ_{WO} distance in Theorem 1.5, once the corresponding bounds for the δ_{HO} are established.

Lemma 7.2 *For any isotropic measures μ and ν on S^{n-1}, we have*

$$\delta_{\mathrm{W}}(\mu, \nu) \leq 7\pi n^3 \delta_{\mathrm{H}}(\mathrm{supp}\,\mu, \mathrm{supp}\,\nu). \tag{86}$$

Proof If $\delta_{\mathrm{H}}(\mathrm{supp}\,\mu, \mathrm{supp}\,\nu) \leq \frac{1}{7n^2}$, then [15, Cor. 6.2] yields that

$$\delta_{\mathrm{W}}(\mu, \nu) \leq 2n \cdot \delta_{\mathrm{H}}(\mathrm{supp}\,\mu, \mathrm{supp}\,\nu).$$

On the other hand, for any $f \in \mathrm{Lip}_1(S^{n-1})$, we may assume that $-\frac{\pi}{2} \leq f(u) \leq \frac{\pi}{2}$ for $u \in S^{n-1}$. As $\mu(S^{n-1}) = \nu(S^{n-1}) = n$, we have $\delta_{\mathrm{W}}(\mu, \nu) \leq \pi n$. Therefore, if $\delta_{\mathrm{H}}(\mathrm{supp}\,\mu, \mathrm{supp}\,\nu) \geq \frac{1}{7n^2}$, then (86) readily holds. □

In the next geometric lemma, $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$ (or any orthonormal basis).

Lemma 7.3 *Let $\varepsilon \in (0, 1/2)$ and $x \in \mathrm{conv}\,\{e_1, \ldots, e_n\}$. If $\angle(x, e_i) \geq \varepsilon$ for $i = 1, \ldots, n$, then $\|x\| \leq 1 - 4^{1-n} \cdot \varepsilon$.*

Proof We proceed by induction on the dimension $n \geq 2$. Let $n = 2$. Then there are $t \in (0, 1)$, $s \in [\varepsilon, \pi/2 - \varepsilon]$ and $\lambda \in (0, 1)$ such that

$$x = \lambda\,(\cos(s)e_2 + \sin(s)e_1) = (1 - t)e_2 + te_1,$$

hence

$$\lambda = (\sin(s) + \cos(s))^{-1} \leq (\sin(\varepsilon) + \cos(\varepsilon))^{-1} \leq 1 - 4^{-1}\varepsilon,$$

which implies that $\|x\| = \lambda \leq 1 - 4^{1-2} \cdot \varepsilon$.

Now we assume that $n \geq 3$ and that the assertion holds in smaller dimensions. Let x be as in the statement of the lemma. Then there are $t \in (0, 1)$ and $e \in \mathrm{conv}\,\{e_1, \ldots, e_{n-1}\}$ such that $x = (1 - t)e_n + te$. We distinguish two cases.

Case 1: $\angle(e, e_i) \geq \varepsilon/2$ for $i = 1, \dots, n-1$. An application of the induction hypothesis to e in the linear subspace spanned by $e_1, \dots, e_{n-1}$ then yields that $\|e\| \leq 1 - 4^{2-n} \cdot \varepsilon/2$. If $t \in [1/2, 1)$, then

$$\|x\| \leq 1 - t + t\|e\| \leq 1 - t + t - \frac{1}{2}4^{2-n}t\varepsilon \leq 1 - 4^{1-n}\varepsilon.$$

Now let $t \in (0, 1/2)$. Since $\angle(x, e_n) \geq \varepsilon$, we have

$$\left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}} \geq 1 - \frac{1}{4}\varepsilon^2 \geq \cos(\varepsilon) \geq \langle x, e_n\rangle \geq \frac{1-t}{\sqrt{(1-t)^2 + t^2}},$$

hence

$$t^2 \geq \frac{1}{4}\varepsilon^2\left(1 - 2t + 2t^2\right) \geq \frac{1}{8}\varepsilon^2.$$

This shows that $t \geq \frac{1}{4}\varepsilon$. Therefore

$$\|x\|^2 \leq (1-t)^2 + t^2 = 1 - t \cdot 2(1-t) \leq 1 - t \leq 1 - \frac{1}{4}\varepsilon,$$

which leads to

$$\|x\| \leq 1 - \frac{1}{16}\varepsilon \leq 1 - 4^{1-n}\varepsilon.$$

Case 2: $\angle(e, e_i) < \varepsilon/2$ for some $i \in \{1, \dots, n-1\}$. We may assume that $\angle(e, e_1) < \varepsilon/2$. Then $\angle(x, e_1) \geq \varepsilon$ implies that $\angle(x, e) \geq \varepsilon/2$. In the two-dimensional subspace spanned by e_n and e, we define $\tilde{x}$ by $\{\tilde{x}\} = [0, \infty)x \cap \operatorname{conv}\{e_n, \|e\|^{-1}e\}$. Since $\angle(x, e_n) = \angle(\tilde{x}, e_n) \geq \varepsilon/2$ and $\angle(x, e) = \angle(\tilde{x}, e) \geq \varepsilon/2$, we conclude that

$$\|x\| \leq \|\tilde{x}\| \leq 1 - \frac{1}{4}\varepsilon/2 \leq 1 - 4^{1-n}\varepsilon,$$

which proves the assertion. □

The statements (a) and (d) of Theorem 1.5 are implied by the following theorem. The estimates in terms of the Wasserstein distance in Theorem 1.5 follow from Lemma 7.2. By ν we denote a cross measure on S^{n-1}.

Theorem 7.4 *Let μ be an even isotropic measure on S^{n-1}. Let $c \geq 3$ be an absolute constant as in Theorem 2.7. If*

$$\ell(Z_\infty(\mu)) \geq (1-\varepsilon)\ell(Z_\infty(\nu)), \tag{87}$$

for some $\varepsilon \in (0, \varepsilon_0)$ for $\varepsilon_0 = \frac{1}{2}n^{-cn^2}$, then

$$\delta_{\rm HO}(\operatorname{supp}\mu, \operatorname{supp}\nu) \le n^{cn^2}\varepsilon.$$

Proof By Lemma 7.1 there exists a discrete even isotropic measure μ_0 on S^{n-1} such that $\operatorname{supp}\mu_0 \subset \operatorname{supp}\mu$ and $|\operatorname{supp}\mu_0| \le n(n+1)/2$. Then $Z_\infty(\mu_0) \subset Z_\infty(\mu)$ and B_2^n is the Löwner ellipsoid for $Z_\infty(\mu_0)$ and $Z_\infty(\mu)$ (*cf.* (8)). Hence, the assumption implies that

$$\ell(Z_\infty(\mu_0)) \ge (1-\varepsilon)\ell(Z_\infty(\nu)).$$

The proof of Theorem 2.7 shows that there exists an orthogonal transformation $\Phi \in O(n)$ such that

$$\delta_{\rm H}(\operatorname{supp}\mu_0, \Phi B_1^n) \le n^{cn^2}\varepsilon;$$

see (37) and the condition $c \ge c(2) \ge 4c(1)$ at the end of the proof of Theorem 2.7. If ν is the cross measure corresponding to $\Phi B_1^n = \operatorname{conv}\{\pm w_1, \ldots, \pm w_n\}$ with an orthonormal basis $w_1, \ldots, w_n$ of $\mathbb{R}^n$, then

$$\operatorname{supp}\nu \subset \operatorname{supp}\mu + n^{cn^2}\varepsilon B_2^n,$$

since $\operatorname{supp}\mu_0 \subset \operatorname{supp}\mu$. Moreover, we have

$$\operatorname{supp}\mu_0 \subset \operatorname{supp}\nu + n^{cn^2}\varepsilon B_2^n.$$

In order to show that in fact $\operatorname{supp}\mu \subset \operatorname{supp}\nu + n^{cn^2}\varepsilon B_2^n$, we assume that there is some $z \in \operatorname{supp}\mu \subset Z_\infty(\mu) \subset B_2^n$ and $z \notin \operatorname{supp}\nu + n^{cn^2}\varepsilon B_2^n$, aiming at a contradiction. If the assumption holds, then $z \in S^{n-1}$ and $\angle(z, w_i) \ge n^{cn^2}\varepsilon \in (0, 1/2)$ for $i = 1, \ldots, n$. An application of Lemma 7.3 shows that $\|z\|_{B_1^n} = 1+t$ satisfies

$$1+t \ge \left(1 - 4^{1-n}n^{cn^2}\varepsilon\right)^{-1} \ge 1 + 4^{1-n}n^{cn^2}\varepsilon,$$

hence $t \ge 4^{1-n}n^{cn^2}\varepsilon$. Similarly as in the final part of the proof of Theorem 2.7 it follows that there is a facet F of B_1^n such that $z_0 = (1+t)^{-1}z \in F$. Setting $P_z := \operatorname{conv}(\{\pm z\} \cup B_1^n)$, we obtain

$$V(P_z) - V(B_1^n) \ge 2tV(\operatorname{conv}\{o, F\}) \ge \frac{2}{n \cdot n!}t\ell(B_1^n). \tag{88}$$

Using Lemma 2.6 and (38), we deduce from (88) that

$$\ell(B_1^n) - \ell(P_z) \ge \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}}(V(P_z) - V(B_1^n)) \ge n^{-4n}t\ell(B_1^n). \tag{89}$$

On the other hand, we have

$$\ell(B_1^n) - \ell(P_z) \leq \ell(B_1^n) - (1-\varepsilon)\ell(B_1^n) = \varepsilon\ell(B_1^n),$$

which together with (89) implies that $t \leq n^{4n}\varepsilon$. But this is in conflict with $t \geq 4^{1-n}n^{cn^2}\varepsilon$, since $c \geq 3$. □

In the proof of Theorem 1.5 (b), (c), we need two lemmas. The first lemma is a dual counterpart to Lemma 2.5.

Lemma 7.5 *Let $u_1, \ldots, u_k \in S^{n-1}$, and let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. Let $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq 1,\ i = 1, \ldots, n\}$ be a polytope, and let $B_\infty^n = \{x \in \mathbb{R}^n : |\langle x, e_i \rangle| \leq 1,\ i = 1, \ldots, n\}$. Fix $\eta \in (0, 1/(2\sqrt{n}))$. If $\delta_{\mathrm{H}}(\{u_1, \ldots, u_k\}, \{\pm e_1, \ldots, \pm e_n\}) \leq \eta$, then*

$$(1-\sqrt{n}\eta)B_\infty^n \subset P \subset (1+2\sqrt{n}\eta)B_\infty^n.$$

Proof We obtain

$$P^\circ = \operatorname{conv}\{u_1, \ldots, u_k\}, \quad B_1^n = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n\},$$

and by assumption

$$P^\circ \subset B_1^n + \eta B_2^n, \quad B_1^n \subset P^\circ + \eta B_2^n.$$

Since $B_\infty^n \subset \sqrt{n}B_2^n$, we have $(1/\sqrt{n})B_2^n \subset B_1^n$. Hence

$$\frac{1}{\sqrt{n}}B_2^n \subset P^\circ + \eta B_2^n, \quad \left(\frac{1}{\sqrt{n}} - \eta\right) B_2^n \subset P^\circ,$$

and hence

$$B_1^n \subset P^\circ + \frac{\eta}{\frac{1}{\sqrt{n}} - \eta}P^\circ.$$

Since $\eta < 1/(2\sqrt{n})$, we deduce that

$$B_1^n \subset (1+2\sqrt{n}\eta)P^\circ.$$

Furthermore, from $B_2^n \subset \sqrt{n}B_1^n$ we deduce that

$$P^\circ \subset B_1^n + \eta B_2^n \subset B_1^n + \eta\sqrt{n}B_1^n = (1+\sqrt{n}\eta)B_1^n,$$

and therefore

$$(1-\sqrt{n}\eta)B_\infty^n \subset (1+\sqrt{n}\eta)^{-1}B_\infty^n \subset P,$$

which completes the proof. □

The second lemma implies a lower bound for the volume that is cut off from a cube by an additional hyperplane provided that the hyperplane is not too close to a facet hyperplane of the cube. For $u \in S^{n-1}$, we denote by $H^-(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle \le 1\}$ the halfspace touching B_2^n at u that contains the origin o, and $H^+(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle \ge 1\}$ is the closure of the complement of $H^-(u)$.

Lemma 7.6 *Let $u_1, \dots, u_k \in S^{n-1}$, let $w_1, \dots, w_n$ be an orthonormal basis of $\mathbb{R}^n$, and let $\eta > 0$. Assume that $\angle(u_i, w_i) \le \eta$ for $i = 1, \dots, n$ and $\angle(u_k, \epsilon_i w_i) \ge c(n)\eta$ for $\epsilon_i \in \{-1, 1\}$ and $i = 1, \dots, n$ with $c(n) = 2^{n+4} \cdot n^{n+3}$. If*

$$P = \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \le 1,\ i = 1, \dots, n\} \cap H^-(u_k) \quad \textit{and}$$
$$B_\infty^n = \{x \in \mathbb{R}^n : |\langle x, w_i\rangle| \le 1,\ i = 1, \dots, n\},$$

then

$$V(P) \le \left(1 - \frac{\Delta}{2^4 \cdot n^{n+1}}\right) V(B_\infty^n),$$

where $\Delta := \min\{\angle(u_k, \epsilon_i w_i) : \epsilon_i \in \{-1, 1\},\ i = 1, \dots, n\}$.

Proof By assumption, $\eta < c(n)^{-1} < 1/(2\sqrt{n})$, since $\angle(u_k, \epsilon_i w_i) < \pi/4$ for some $\epsilon_i \in \{-1, 1\}$ and some $i \in \{1, \dots, n\}$. Let $H^+ := H^+(u_k)$. We may assume that u_k is in the positive hull of $w_1, \dots, w_n$ and $\omega_0 := \angle(u_k, w_1) = \min\{\angle(u_k, w_i) : i = 1, \dots, n\}$. Note that $c(n)\eta \le \omega_0 \le \omega_0(n)$, where $\cos(\omega_0(n)) = n^{-\frac{1}{2}}$. To see this, assume that $\omega_0 > \omega_0(n)$, which implies that

$$1 = \|u_k\|^2 = \sum_{i=1}^n \langle u_k, w_i\rangle^2 < n \cdot \frac{1}{n} = 1,$$

a contradiction.

Let $F_1 = B_\infty^n \cap H^+(w_1)$. If z denotes the point of $H^+ \cap F_1$ closest to w_1, then $\|z - w_1\| = \tan\left(\frac{\omega_0}{2}\right)$, and therefore $H^+ \cap F_1$ contains an $(n-1)$-dimensional ball of radius $1 - \tan\left(\frac{\omega_0}{2}\right)$. Moreover, $H^+ \cap B_\infty^n$ contains a point whose distance from the affine hull of F_1 is at least

$$\begin{aligned} f(\omega_0) :&= \left(1 - \tan\left(\frac{\omega_0}{2}\right)\right)\tan(\omega_0) \\ &= \frac{\sin(\omega_0) + \cos(\omega_0) - 1}{\cos(\omega_0)} \ge \frac{1}{2}\omega_0, \quad \omega_0 \in [0, \pi/2]. \end{aligned} \tag{90}$$

For the proof of (90), we consider

$$g(x)=\sin(x)+\cos(x)-1-\frac{1}{2}x\cos(x),\quad x\in[0,\pi/2].$$

Since $g''(x)=\cos(x)\left(\frac{1}{2}x-1\right)\le 0$ for $x\in[0,\pi/2]$, g is a concave function. The assertion follows, since $g(0)=g(\pi/2)=0$.

From

$$\tan\left(\frac{\omega_0}{2}\right)\le\tan\left(\frac{\omega_0(n)}{2}\right)=\frac{\sqrt{1-n^{-1/2}}}{\sqrt{1+n^{-1/2}}}<1-\frac{1}{2\sqrt{n}},$$

we obtain

$$1-\tan\left(\frac{\omega_0}{2}\right)\ge\frac{1}{2\sqrt{n}}.$$

We thus conclude that

$$V(H^+\cap B_\infty^n)\ge\frac{1}{n}\kappa_{n-1}\left(\frac{1}{2\sqrt{n}}\right)^{n-1}\frac{1}{2}\omega_0\ge 2^{-3}n^{-\frac{1}{2}}n^{-n}\omega_0 V(B_\infty^n).$$

Lemma 7.5 implies that $\widetilde{B}_\infty^n=\{x\in\mathbb{R}^n:|\langle x,u_i\rangle|\le 1,\ i=1,\ldots,n\}$ satisfies

$$H^+\cap B_\infty^n\subset\left(H^+\cap\widetilde{B}_\infty^n\right)\cup\left(B_\infty^n\setminus(1-\sqrt{n}\eta)B_\infty^n\right)$$

and therefore

$$\begin{aligned}V(H^+\cap\widetilde{B}_\infty^n)&\ge V(H^+\cap B_\infty^n)-\left(1-(1-\sqrt{n}\eta)^n\right)V(B_\infty^n)\\&\ge\left(2^{-3}n^{-\frac{1}{2}}n^{-n}\omega_0-\sqrt{n}n\eta\right)V(B_\infty^n).\end{aligned}$$

Another application of Lemma 7.5 yields

$$\begin{aligned}V(P)&\le V(\widetilde{B}_\infty^n)-V(H^+\cap\widetilde{B}_\infty^n)\\&\le V((1+2\sqrt{n}\eta)B_\infty^n)-\left(2^{-3}n^{-\frac{1}{2}}n^{-n}\omega_0-\sqrt{n}n\eta\right)V(B_\infty^n)\\&\le\left(1+2^n\sqrt{n}n\eta+\sqrt{n}n\eta-2^{-3}n^{-\frac{1}{2}}n^{-n}\omega_0\right)V(B_\infty^n)\\&\le\left(1+2^n n^2\eta-2^{-3}n^{-\frac{1}{2}}n^{-n}\omega_0\right)V(B_\infty^n)\\&\le\left(1+\frac{2^n n^2}{c(n)}\omega_0-2^{-3}n^{-\frac{1}{2}}n^{-n}\omega_0\right)V(B_\infty^n)\\&\le\left(1-2^{-4}n^{-\frac{1}{2}}n^{-n}\omega_0\right)V(B_\infty^n),\end{aligned}$$

which implies the asserted volume bound. □

After these preparations, parts (b) and (c) of Theorem 1.5 follow from the following theorem.

Theorem 7.7 *Let μ be an even isotropic measure on S^{n-1}. If*

$$\ell(Z^*_\infty(\mu)) \le (1+\varepsilon)\ell(Z^*_\infty(\nu)), \tag{91}$$

for some $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 = 2^{-70} n^{-30n}$, then

$$\delta_{\mathrm{HO}}(\operatorname{supp}\mu, \operatorname{supp}\nu) \le 2^{20} n^{13n} \varepsilon^{\frac{1}{4}}.$$

Proof For the given measure μ, we choose an even discrete isotropic measure on S^{n-1} according to Lemma 7.1 with $\operatorname{supp}\mu_0 = \{u_1, \dots, u_k\} \subset \operatorname{supp}\mu$. In particular, we then have $Z^*_\infty(\mu) \subset Z^*_\infty(\mu_0)$ and

$$\ell(Z^*_\infty(\mu_0)) \le (1+\varepsilon)\ell(Z^*_\infty(\nu)). \tag{92}$$

We claim that

$$\gamma_n(bZ^*_\infty(\mu_0)) \ge \gamma_n(bZ^*_\infty(\nu)) - \varepsilon\gamma_n(Z^*_\infty(\nu)) \tag{93}$$

for some $b \in [1, 2]$.

For the proof, assume to the contrary that

$$\gamma_n(bZ^*_\infty(\mu_0)) < \gamma_n(bZ^*_\infty(\nu)) - \varepsilon\gamma_n(Z^*_\infty(\nu))$$

or all $b \in [1, 2]$. By Proposition 3.2,

$$\gamma_n(tZ^*_\infty(\mu_0)) \le \gamma_n(tZ^*_\infty(\nu)) \quad \text{for } t > 0.$$

Hence

$$\begin{aligned}\ell(Z^*_\infty(\nu)) - \ell(Z^*_\infty(\mu_0)) &= \int_0^\infty \left(\gamma_n(tZ^*_\infty(\mu_0)) - \gamma_n(tZ^*_\infty(\nu))\right) \mathrm{d}t \\ &< -\varepsilon\ell(Z^*_\infty(\nu)),\end{aligned}$$

which contradicts (92).

Thus relation (58) holds for some $b \in [1, 2]$ with ε replaced by $\varepsilon\ell(Z^*_\infty(\nu))$. In the proof of relation (59) it is shown that there exists an orthonormal basis $w_1, \dots, w_n$ of $\mathbb{R}^n$ such that (w.l.o.g.)

$$\angle(u_i, w_i) < c_1(n)\varepsilon^{\frac{1}{4}} \quad \text{for } i = 1, \dots, n$$

with $c_1(n) := 3n2^{14}n^{4n}n^{\frac{1}{8}}$, where we used that $\ell(Z_\infty^*(\nu)) = \frac{1}{2}\ell(B_2^n)W(B_1^n) \leq \ell(B_2^n) \leq \sqrt{n}$ (cf. the proof of [16, Lemma 5.7]). If ν is the cross measure associated with $\pm w_1, \dots, \pm w_n$, then

$$\operatorname{supp}\nu \subset \operatorname{supp}\mu_0 + c_1(n)\varepsilon^{\frac{1}{4}}B_2^n \subset \operatorname{supp}\mu + 2^{20}n^{13n}\varepsilon^{\frac{1}{4}}B_2^n.$$

It remains to be shown that

$$\operatorname{supp}\mu \subset \operatorname{supp}\nu + 2^{20}n^{13n}\varepsilon^{\frac{1}{4}}B_2^n. \tag{94}$$

For the proof, we set $\eta = c_1(n)\varepsilon^{\frac{1}{4}}$. Clearly, $\pm u_i$ is contained in the set on the right-hand side of (94) for $i \in \{1, \dots, n\}$. Recall that $c(n) = 2^{n+4}n^{n+3}$ and define

$$\Delta = \max_{u \in \operatorname{supp}\mu} \min_{i=1,\dots,n} \angle(u, w_i).$$

If $\Delta < c(n)\eta$, then relation (94) holds. Henceforth we consider the case where $\Delta \geq c(n)\eta$. Then we may assume that the maximum in the definition of Δ is realized by $u_0 \in (\operatorname{supp}\mu) \setminus \{\pm u_1, \dots, \pm u_n\}$. We define

$$\begin{aligned} P &= \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \leq 1,\ i = 0, 1, \dots, n\}, \\ \widetilde{B}_\infty^n &= \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \leq 1,\ i = 1, \dots, n\}, \\ B_\infty^n &= \{x \in \mathbb{R}^n : |\langle x, w_i\rangle| \leq 1,\ i = 1, \dots, n\}, \end{aligned}$$

hence $Z_\infty^*(\mu) \subset P \subset \widetilde{B}_\infty^n$. By Lemma 7.6 we have

$$V(P) \leq \left(1 - \frac{\Delta}{2^4 n^{n+1}}\right) V(B_\infty^n),$$

hence

$$\delta_{\mathrm{vol}}(P, B_\infty^n) \geq \frac{\Delta}{2^4 n^{n+1}} V(B_\infty^n).$$

Lemma 7.5 implies that

$$\delta_{\mathrm{vol}}(B_\infty^n, \widetilde{B}_\infty^n) \leq \left[(1 + 2\sqrt{n}\eta)^n - (1 - \sqrt{n}\eta)^n\right] V(B_\infty^n) \leq 4\sqrt{n}n\eta V(B_\infty^n),$$

since $(1 + 2\sqrt{n}\eta)^n \leq 4/3$. The triangle inequality yields

$$\begin{aligned} V(\widetilde{B}_\infty^n \setminus P) = \delta_{\mathrm{vol}}(\widetilde{B}_\infty^n, P) &\geq \delta_{\mathrm{vol}}(B_\infty^n, P) - \delta_{\mathrm{vol}}(B_\infty^n, \widetilde{B}_\infty^n) \\ &\geq \left(\frac{\Delta}{2^4 n^{n+1}} - 4\sqrt{n}n\eta\right) V(B_\infty^n). \end{aligned} \tag{95}$$

We claim that

$$V(\widetilde{B}_\infty^n \setminus P) \le 2^{17} n^{11n} \varepsilon^{\frac{1}{4}}. \tag{96}$$

Combination of (95) and (96) shows that

$$\Delta \le 2^4 n^{n+1} \left(4\sqrt{n} n\eta + 2^{-n} 2^{17} n^{11n} \varepsilon^{\frac{1}{4}}\right) \le 2^{20} n^{13n} \varepsilon^{\frac{1}{4}},$$

which completes the proof of (94), once (96) has been established.

We finally verify (96). Using Lemma 7.5, we get $\widetilde{B}_\infty^n \subset (1 + 2\sqrt{n}\eta) B_\infty^n \subset 2B_\infty^n$, hence $1/(2\sqrt{n})\widetilde{B}_\infty^n \subset B_2^n$. Then

$$\begin{aligned} \ell(Z_\infty^*(\mu)) - \ell(\widetilde{B}_\infty^n)) &= \int_0^\infty \left(\gamma_n(t\widetilde{B}_\infty^n) - \gamma_n(tZ_\infty^*(\mu))\right) \mathrm{d}t \\ &\ge \int_0^{1/(2\sqrt{n})} \frac{e^{-1/2}}{(2\pi)^{n/2}} t^n V(\widetilde{B}_\infty^n \setminus Z_\infty^*(\mu)) \, \mathrm{d}t \\ &\ge \frac{1}{n+1} \left(\frac{1}{2\sqrt{n}}\right)^{n+1} \frac{e^{-1/2}}{(2\pi)^{n/2}} V(\widetilde{B}_\infty^n \setminus Z_\infty^*(\mu)) \frac{\ell(B_\infty^n)}{\sqrt{n}} \\ &\ge n^{-6n} V(\widetilde{B}_\infty^n \setminus Z_\infty^*(\mu)) \ell(B_\infty^n). \end{aligned} \tag{97}$$

Again by an application of Lemma 7.5, we obtain

$$\begin{aligned} \ell(\widetilde{B}_\infty^n)) - \ell(B_\infty^n) &\ge \ell((1 + 2\sqrt{n}\eta) B_\infty^n)) - \ell(B_\infty^n) \\ &= \left[(1 + 2\sqrt{n}\eta)^{-1} - 1\right] \ell(B_\infty^n) \\ &\ge -2\sqrt{n}\eta \ell(B_\infty^n). \end{aligned} \tag{98}$$

Combining (91), (97) and (98), we arrive at

$$\varepsilon \ell(B_\infty^n) \ge \ell(Z_\infty^*(\mu)) - \ell(B_\infty^n) \ge \left(\frac{V(\widetilde{B}_\infty^n \setminus Z_\infty^*(\mu))}{n^{6n}} - 2\sqrt{n}\eta\right) \ell(B_\infty^n),$$

hence

$$\begin{aligned} V(\widetilde{B}_\infty^n \setminus Z_\infty^*(\mu)) &\le n^{6n}(\varepsilon + 2\sqrt{n}\eta) \le n^{6n} \left(1 + 2\sqrt{n} 3n 2^{14} n^{4n} n^{\frac{1}{8}}\right) \varepsilon^{\frac{1}{4}} \\ &\le 2^{17} n^{11n} \varepsilon^{\frac{1}{4}}, \end{aligned}$$

which proves the claim, and thus the theorem. □

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