Measures, curvatures and currents in convex geometry

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Introduction

This work is devoted to the investigation of the basic interrelations between the geometry of convex sets and certain measures (or functionals), curvatures and currents which are associated with such sets. Indeed, measures have always played a decisive rôle in convex geometry and related fields. Subsequently, we shall roughly outline why this is so.

First and foremost, a general convex body can have a fairly wild, intricate and even counterintuitive boundary structure. This is in contrast to the tame case of a convex body which is bounded by a smooth hypersurface (of class C^2) having positive Gauss curvature everywhere. The discrepancy between the cases of a general and such a special convex body makes it clear that characteristics of and information about the general situation cannot always be obtained by a straightforward approximation argument, and even facts concerning smooth convex bodies are not always easy to visualize. Clearly, general convex bodies are distinguished from smooth ones by the appearance of various kinds of singularities. Thus it is of particular interest to describe these singularities in a quantitative way or, e.g., to study obstructions for such singularities. Here, of course, measures are naturally involved. For instance, although the set of singular boundary points of a convex body can form a dense subset of its boundary, it still has boundary measure zero. Even more interesting is the second order boundary structure. By a classical theorem of Aleksandrov, for almost all boundary points of a given convex surface there exists a paraboloid which yields a second order approximation of the surface in a neighbourhood of such a point. This basic fact allows one to introduce curvatures at almost all boundary points of a convex body. However, these are not the curvatures which we are interested in, since they do not contain sufficient information about the shape of the convex body.

A second context, in which measures are naturally involved, concerns exceptional relative positions, say, of pairs of convex bodies. Here one can imagine several situations in which a pair (K, L) of convex bodies in $\mathbb{R}^d$ is said to be in exceptional relative position. For example, K and L might be called in exceptional relative position if they contain parallel segments lying in parallel supporting hyperplanes (compare [123], §2.3), or else if K and L contain a common boundary point in which the linear hulls of the respective normal cones intersect in a non-trivial way (see [129]). Variants of the latter notion have for instance been treated in [158] and [77]. Now let $\mathbf{G}(K, L)$ denote the set of all rigid motions g for which K and gL are in exceptional relative position. Then, in each of the cases just mentioned, one can show that the intersection of $\mathbf{G}(K, L)$ with a suitable subgroup of the group of rigid motions has Haar measure zero. This is important, since the exceptional relative positions present major problems in integral-geometric or topological arguments. Thus appropriate Haar measures are used to show that certain situations are negligible from an integral-geometric point of view. Recent applications in this spirit can for instance be found in [110], [76], [77] (or in the literature cited there), but also in Sections 3 and 4 of the present work.

On the other hand, Haar measures appear in the description of basic functionals such as the ordinary (harmonic, affine, or dual affine) quermassintegrals as mean values of sections or projections of convex bodies. Quite often such averages give rise to challenging geometric inequalities some of which are still unsolved. The quermassintegrals of a convex body K also appear as coefficients of a (global) Steiner formula, that is, as coefficients of a polynomial expansion for the volumes of global outer parallel sets of K. By localizing the notion of an outer parallel set, one is led to a fruitful generalization of a global concept. Local Steiner formulae, arising in this way, perhaps provide the easiest route to the curvature and surface area measures, or to their common generalization, the support measures, of convex sets. In fact, the definition of these measures only requires a few basic facts of measure theory and convex geometry. The study of these measures and their extensions represents a central theme in convex geometry, and it is also a principal subject of the present work.

The basic rôle of the support measures (and of their image measures under projection maps) is, to a certain degree, explained by the fact that they can be characterized by a number of simple properties similar to Hadwiger's famous characterization theorem for the intrinsic volumes. Characterization theorems for curvature, surface area and support measures, which are similar in spirit, have been established in [119], [118], [157], [49], a recent application of such results and methods is given in [125] and [49]. Quite naturally, these measures are involved in the classification of all continuous, additive maps on the space of convex bodies satisfying suitable invariance hypotheses (compare [101], [54], [102], [127], [103], [80], [3], [4]). But this subject is still far from being completely understood.

Mixed volumes represent another important subject in which measures are employed to describe global geometric functionals. In fact, as a consequence of the Riesz representation theorem, the mixed volume $V(K_1, \dots, K_d)$ of convex bodies $K_1, \dots, K_d$ in $\mathbb{R}^d$ can be represented in terms of the mixed surface area measure $S(K_2, \dots, K_d, \cdot)$ of $K_2, \dots, K_d$ and the support function of K_1. Other mixed functionals and measures arise in the context of translative integral geometry and, in a certain sense, they may again be viewed as far reaching generalizations of intrinsic volumes. Such functionals are an indispensible tool in stochastic geometry, especially when anisotropic random structures are studied. We shall consider these subjects in some more detail in Section 4.

The variety of subjects and methods in convex geometry, which are based on or are at least related to measures, comprises many other topics such as the theory of zonoids (either as an interesting object of study or as an important tool for related areas), the theory of integral transforms and its conceptual generalizations (say to the level of distributions or to generalized bodies) or the theory of projection functions. Measures also form a constituent part of geometric probability, of the theory of random and deterministic approximation, and quite generally measures are involved when random methods are employed in the construction of examples or in proving existence results. The connection to applied disciplines becomes apparent, for instance, in the subjects treated in geometric tomography and stochastic geometry. In fact, especially the latter may be viewed as a branch of mathematics which has strong connections to all the fields mentioned so far, and thus it reveals its genuinely interdisciplinary character. On a more technical level, the support measures play a similar rôle.

We should not close this first part of the introduction without having mentioned that probabilistic and general measure theoretic methods are used in the local theory of Banach spaces. Recently, the exploration of isotropic positions of convex bodies and related extremal problems provided a new link of this part of convexity theory to the classical Brunn-Minkowski theory via the surface area measures of convex bodies (compare [47] and [46]). Moreover, Brenier's gradient map (see [18], [100]), arising from the Monge-Kantorovich mass transportation problem [107], has been combined with parts of Caffarelli's regularity theory to yield geometric and analytic inequalities; compare [13], [5].

Clearly, the present work cannot contribute to all of the subjects to which we alluded above, but there are strong links to most of them. Subsequently, we give a short description of the contents of each section and indicate some underlying connections. Then again, at the beginning of each section, we give a more detailed account of the material covered and provide relevant background information.

Section 1 is devoted to the investigation of curvature and surface area measures. The main problem, which we address, is the exploration of the connection between measure theoretic properties of the curvature and surface area measures and geometric properties of the underlying convex body. The specific measure theoretic property, which we pursue here, is the absolute continuity with respect to a suitable Hausdorff measure. The preface to Section 1 puts this particular problem into a broader context and relates it to the literature. In particular, the preceding two papers [72], [74] prepared the present research. Subsection 1.1 gives a detailed report on results of these papers which now serve as a starting point, and at the same time we take the opportunity to remark some minor improvements. Subsection 1.2 provides a collection of selected results, to be established in Subsections 1.3 – 1.7, which are designed to convey the flavour of our contribution in this part of the work and to which we refer for further details. Moreover, Section 1, and here in particular Subsection 1.2, also introduces to some of the basic concepts which we use and extend in this work. For instance, an important technical and geometric device, which we use repeatedly, is the notion of generalized curvatures that are defined almost everywhere on the unit normal bundle of a given convex body (compare [155]).

The last two subsections of Section 1 are more loosely connected to the remaining part of the section. In Subsection 1.8 we provide a stability estimate for Minkowski's uniqueness problem of optimal order, which applies to a large class of convex bodies. The proof is obtained by an extension and modification of arguments which are implicit in contributions by Diskant. Subsection 1.9 offers an alternative route to two results which have recently been obtained by different methods in [30]. Here we obtain representations of support measures of convex bodies on sets of σ-finite Hausdorff measure as weighted Hausdorff measures, where the strength of the singularities of the convex bodies is taken into account. The present approach to one of these theorems is based on a version of Federer's structure theorem in spherical space, for which we give a new integral-geometric proof that is in the spirit of B. White's recent proof for Federer's structure theorem in Euclidean space.

In Section 2 we extend our framework in an essential way by replacing the Euclidean unit ball as the fundamental gauge body by a rather arbitrary convex body B containing the origin. Some of our investigations, however, will require additional assumptions on B such as strict convexity or smoothness and non-vanishing Gauss curvature. In such a setting of relative geometry all measurements and definitions have to refer to the gauge body alone and should essentially be independent of an auxiliary Euclidean structure. Recently, support measures have been introduced in relative geometry for pairs of convex bodies (K, B) such that $o \in B$ and K and B are in general relative position; compare [80], and [76] for the case of a strictly convex body B. In particular, if K and B are polytopes in general relative position, then the B-support measures of K can be represented in a simple way. Other representations, extending results in [155], are obtained if either the support function of B or the support function of K is of class C^2. In fact, for a convex body of class C^2_+ one can even go one step further and introduce generalized relative curvatures which are defined on the relative normal bundle of K with respect to B. On the basis of these new concepts of relative geometry we find another representation for the B-support measures which extends a corresponding representation that is known from the Euclidean setting.

Subsections 2.3 and 2.4 are devoted to the investigation of relative normals and of certain Euler-type formulae involving relative support measures. The basic question, asking for estimates of the average number of relative normals passing through a point in a convex body, can be traced back to a classical paper by Santaló. The relevant literature on this kind of problem and references concerning Euler-type formulae are reviewed in [59], [60], [61], [62], [69], [51]. The case of the Minkowski plane deserves to be treated separately, since here stronger results are available due to the surprising fact that the B-projection onto K is Lipschitz for all pairs of convex bodies (K, B) in general relative position and for which $o \in \operatorname{int} B$. In the remaining part of the section we treat characterizations of gauge bodies by linear relations between relative support measures, related stability results and a splitting theorem in three-dimensional symmetric Minkowski spaces. These investigations heavily rely on results which are provided in Section 1.

Thus Section 2 may be viewed as a general contribution to an overall attempt to transfer concepts and results of Euclidean geometry to general finite-dimensional normed vector spaces. For progress in this spirit, concerning a different problem, see [48] (compare also [90]), where John's theorem is extended to arbitrary pairs of convex bodies and estimates for volume ratios are derived as a consequence.

Applications of the theory of relative support measures to stochastic geometry were recently given in [76]. There further results on additive as well as non-negative extensions of relative support measures are established, too. These results were then again applied in [75] to the study of the Boolean model with polyconvex grains. In Section 3 we combine results about support measures and probabilistic methods of stochastic geometry to make substantial progress on the structure of contact distributions of random closed sets in the extended convex ring. For the technical details we refer to Section 3 and to [76], [75], here we just include a short informal discussion of the very special case of a stationary Boolean model to provide some motivation.

Let X be a stationary Poisson particle process in the space of convex bodies. Then

$$Z_X := \bigcup_{K \in X} K$$

is a stationary Boolean model. In principle, the union set Z_X is a directly observable quantity in contrast to the underlying point process X. It is a basic problem of stochastic geometry to retrieve information about X, or about certain mean values which are associated with X, from knowledge about the associated Boolean model. One strategy to approach this problem is to consider the conditional probabilities

$$H_B(r) := \mathbb{P}\left(d_B(Z_X, o) \leq r | o \notin Z_X\right) ,$$

where $(\Omega, \mathbb{A}, \mathbb{P})$ is an underlying probability space, $o \in \operatorname{int}$ B, and

$$d_B(Z_X, o) = \min\{r \geq 0 : o \in Z_X + rB\}$$

is the distance of o from Z_X measured in terms of B. Now the special mean mixed volumes $\overline{V}_j(X, B)$, which are mean values that are associated with the particle process X, are related to the contact distribution function H_B by a probabilistic version of a Steiner formula:

$$H_B(r) = \sum_{j=0}^{d-1} d \binom{d-1}{j} \int_0^r t^{d-1-j}(1 - H_B(t)) dt \, \overline{V}_j(X, B) \,. \tag{1}$$

Hence, once we know H_B, we also know the densities $\overline{V}_j(X, B)$. Since B is not restricted to be either a ball or a segment (as in the case of the spherical or the linear contact distribution), we can for instance determine the mean surface measure $\overline{S}_{d-1}(X,\cdot)$ from knowledge of $\overline{V}_j(X,B)$ for a suitably large class of convex bodies B with $o \in \operatorname{int} B$. On the other hand, equation (1) yields the structural information that H_B is absolutely continuous and the density h_B satisfies

$$\frac{h_B(t)}{1-H_B(t)} = \sum_{j=0}^{d-1} d\binom{d-1}{j} t^{d-1-j}\overline{V}_j(X,B)\,. \tag{2}$$

The quantity on the left-hand side of (2) is called the hazard rate of Z_X. Estimators for the hazard rate have been explored in [9], [10], [64], [26], [63] (see also the survey in [8]) on the basis of the analogy to corresponding notions in survival analysis. Of course, the simple relations (1) and (2) are a particular feature of the Poisson process and the assumption of stationarity. However, relation (1) can be properly extended in various directions under substantially less restrictive assumptions.

For the case where B is the Euclidean unit ball, a local version of (1) has been established in [92] for a general stationary particle process of convex bodies (compare also [93] and Section 5 in [76] for further developments). In fact, the authors of [92] consider the distribution of the contact vector (see also [76]) rather than merely the distribution of the distance function. More generally, in a non-stationary setup, the contact distribution will be a function of a variable reference point from which distances are measured with respect to a general structuring element. This is the framework that is considered in [76]. There, for instance, a random closed set in the extended convex ring is studied which is derived from a fairly general marked point process. Then, for strictly convex gauge bodies, it is shown that the local contact distribution (for almost all reference points) is absolutely continuous and the density is explicitly described in terms of the Palm probabilities of the underlying marked point process. The main concern of Section 3 is to provide technical improvements and simplifications of the approach in [76]. These improvements finally result in various extensions of two of the main theorems in [76]. In fact, we are even able to replace the reference point, from which all distances are measured, by a convex body L, which can be viewed as a "blown-up reference point". Therefore it makes sense to consider points of contact in L at which the distance (with respect to B) between a random closed set in the extended convex ring and L is realized.

Another main subject, contained in Subsection 3.5, is the investigation of certain intensity measures which are associated with the random measures $\Theta_j^{+}(\Xi,B;\cdot)$. These random measures are constructed in [76] (with a different notation) as non-negative extensions of relative support measures that are associated with a random closed set Ξ in the extended convex ring. The connection of these measures to contact distributions and to other related intensity measures has been investigated in [76]; compare also the comments and illustrating examples in [75]. Our present objective is to establish the absolute continuity of these intensity measures and to determine the explicit form of the densities for general strictly convex gauge bodies. The case of strictly convex and *smooth* gauge bodies is covered in [76]. The remaining part of Section 3 (Subsections 3.2 – 3.4) is devoted to the derivation of an iterated translative integral formula for relative support measures. Even in the Euclidean case the corresponding formula is more general than the corresponding results in [131] and [150]. Such an integral formula involves certain relative mixed curvature measures which have been of great use in a Euclidean context for applications in stochastic geometry; compare e.g. [150], [41], [152]. Now a corresponding tool is available in the framework of relative geometry.

In Section 4 we return to a deterministic and Euclidean setting. Our primary interest in this section is the investigation of mixed volumes and of the mixed curvature measures of translative integral geometry. The latter were already treated in Section 3 in the setting of relative geometry, but from a different point of view and by employing other techniques. These and related mixed functionals and measures represent a central topic in convex and integral geometry. A survey of various approaches and contributions to these concepts is outlined in the preface to Section 4. The present primary concern is to describe these mixed functionals and measures by using the normal bundles and generalized curvatures of the bodies involved. The great success of this method in the study of support measures in integral and stochastic geometry is demonstrated by various previous contributions (compare [113], [110], [157], [108], [109], [29], [77]) starting with [155]. The latter paper also suggests a current representation of support measures which is particularly useful. In fact, we adopt this viewpoint in Subsection 4.3, since it greatly facilitates our arguments, and we apply it again in Subsections 4.4 and 4.5. Subsection 4.5 essentially relies on a current representaton for mixed curvature measures which was provided by J. Rataj in [108]. Further details of our methods of proof are given in the preface to Section 4.

It is to be expected that the representatons obtained in Section 4 have a definite impact on future research related to mixed functionals and measures of translative integral geometry, which then again would result in further progress in stochastic geometry. One obvious advantage of our method is that it clearly leads to new results and relationships which otherwise might only be obtained by delicate approximation arguments. Here the ultimate hope might be that the present framework eventually leads to a unified understanding of various concepts and tools of convex and integral geometry including zonoids, projection functions, support functions, mixed volumes or mixed curvature measures.

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1 Curvature measures and the shape of convex bodies

A central and challenging problem in geometry is to explore the basic relationships between suitably defined curvatures of a geometric object and the local geometric shape and topology of the object which is considered. In one direction, one asks for geometric properties of a set which can be retrieved, provided some specific information is available about the curvatures which are associated with the set. But it is also important to obtain inferences in the reverse direction. Here one wishes to find characteristic properties of the curvatures which can be deduced from knowledge of the local geometric shape of the sets involved. For example, from the point of view of Riemannian geometry, various facets of this interplay are highlighted in Berger's recent encyclopaedical survey [15].

In convex geometry, and of course in other branches of mathematics as well, it is a guiding philosophy to avoid as far as possible any smoothness assumptions on the geometric objects (here convex sets) which are considered. Indeed, the assumption of convexity implies a certain, though weak, degree of first and second order smoothness. This already allows one to introduce *principle curvatures* which are defined almost everywhere, in a measure theoretic sense, on the boundary of a convex set. On the other hand, for a general convex surface these curvatures only contain a limited amount of information. Therefore one introduces curvature measures of arbitrary closed convex sets, which now replace the pointwise defined elementary symmetric functions of the principle curvatures of smooth convex surfaces; the latter correspond to the framework of classical differential geometry. In spite of the lack of differentiability assumptions, at least in principle the curvature measures encapsulate all relevant information about the sets with which they are associated. In order to investigate these measures, it turned out that the methods and tools of convex and integral geometry, certain *generalized curvatures* which live on *generalized normal bundles* and Federer's *coarea formula* play a crucial rôle.

Our general setting is determined by the geometry of convex sets in Euclidean space $\mathbb{R}^d$ ($d \geq 2$). In this setting, *local Steiner formulae* are traditionally used to introduce the *curvature measures* $C_r(K, \cdot)$ of a (non-empty) closed convex set $K \subset \mathbb{R}^d$, for $r \in \{0, \ldots, d-1\}$, as Radon measures on the σ-algebra of Borel subsets of $\mathbb{R}^d$. These measures, as well as their spherical counterparts, the (intermediate) *surface area measures* $S_r(K, \cdot)$, have been the subject of numerous investigations over the last 30 years. This can be seen, e.g., from the books of Schneider [123] and Schneider & Weil [132], which are recommended for an introduction to this subject, as well as from the surveys by Schneider [124] and Schneider & Wieacker [135]. A considerable number of these investigations can be understood as contributions to the following fundamental question, which has also been pointed out in [128] and which certainly represents an instance of the general problem which we described initially.

Which geometric consequences can be inferred for a closed convex set K, provided some specific measure theoretic information on the curvature measure $C_r(K, \cdot)$, for some $r \in \{0, \ldots, d-1\}$, is available? For example, what can be said about the set of *singular boundary points* of a closed convex set K if the singular part of some curvature measure of K vanishes?

Clearly, similar questions can be asked for surface area measures as well, but in this introduction we shall restrict ourselves mainly to curvature measures. The general concept which underlies the preceding question is much related in spirit to the ideas which form the heart of *geometric tomography* as described in the preface to Gardner's book [44]. The similarities will become even clearer as soon as first answers to this question will be given; in fact, these answers naturally involve the sections and projections of convex sets.

Of course, the curvature measures of special classes of convex bodies (non-empty compact convex sets) such as bodies with smooth boundaries (of differentiability class C^2) or polytopes are fairly well understood. For arbitrary closed convex sets, a systematic investigation was initiated in [72], which aims at establishing a precise connection between the *local geometric shape*, in particular the *boundary structure*, of a given convex set K and the *absolute continuity* of some curvature measure $C_r(K,\cdot)$, $r \in \{0,\ldots,d-2\}$, of K with respect to the *boundary measure* $C_{d-1}(K,\cdot)$ of K (see Section 2 for some definitions). There, based on the previous work [73], the interplay between the absolute continuity of some curvature measure of a convex set and the measure theoretic size of the set of singular boundary points of this set has been elucidated. It is the purpose of the present section to continue this line of research.

One of the basic roots of the present research can be traced back to a result of Aleksandrov. Let $K \subset \mathbb{R}^3$ be a full-dimensional convex body, and suppose that the *specific curvature* of K is bounded, that is, there is a constant $\lambda \in \mathbb{R}$ such that $C_0(K,\cdot) \leq \lambda\, C_2(K,\cdot)$. Then K is *smooth* (has a unique *support plane* through each boundary point); see [1] or [2, p. 445]. Obviously, the assumption of bounded specific curvature precisely means that the Gaussian curvature measure $C_0(K,\cdot)$ is absolutely continuous with respect to the boundary measure $C_2(K,\cdot)$ and the density function is bounded by a constant. Aleksandrov's result has been discussed in the books by Busemann [23, pp. 32–34] and Pogorelov [105, pp. 57–60] or in Schneider's survey [121]. These authors also raised the question whether suitable generalizations of this result could be established in higher dimensions. But only recently, an extension of Aleksandrov's result to higher dimensions and all curvature measures has been found by Burago & Kalinin [22]. As a consequence of their result, it follows that the assumption

$$C_r(K,\cdot) \leq \lambda\, C_{d-1}(K,\cdot)\,, \tag{3}$$

for a closed convex set $K \subset \mathbb{R}^d$ with non-empty interior, a constant $\lambda \in \mathbb{R}$ and some $r \in \{0,\ldots,d-2\}$, implies that the dimension of the normal cone of K at an arbitrary boundary point x of K is $d-1-r$ at the most. In the important case of the mean curvature measure, that is for $r = d-2$, Bangert [12] and the present author have independently (and by different approaches) obtained a much stronger characterization, saying that condition (3) is satisfied if and only if a suitable ball rolls freely inside K. Thus it becomes apparent that the absolute continuity (with bounded density) of some curvature measure of a convex body K with respect to the boundary measure of K allows one to deduce a certain degree of regularity for the boundary surface of K.

The much more restrictive assumption

$$C_r(K,\cdot) = \lambda\, C_{d-1}(K,\cdot)\,, \tag{4}$$

for a convex body $K \subset \mathbb{R}^d$ with non-empty interior, a constant $\lambda \in \mathbb{R}$ and some $r \in \{0,\ldots,d-2\}$, yields that K must be a ball. This result, which in this generality was first proved by Schneider [120], represents a substantial generalization of the classical Liebmann-Süss theorem to the non-smooth setting of convex geometry. A different proof and extensions to spaces of constant curvature or to certain combinations of curvature measures have recently been found by Kohlmann [83], [84]. For closed convex sets with non-empty interiors, Kohlmann (see [81],

[85], [86]) has also studied *weak stability* and *splitting results* under pinching conditions of the form

$$\alpha\, C_{d-1}(K,\cdot) \leq C_r(K,\cdot) \leq \beta\, C_{d-1}(K,\cdot)\ , \tag{5}$$

where $\alpha, \beta \in \mathbb{R}$ are properly chosen positive constants. Furthermore, Bangert [12] (see also [97], [98]) has established an optimal splitting result in the case $r = d-2$. In some special situations, diameter bounds have been obtained; see, e.g., the contributions by Diskant [33], Lang [89], and Bangert [12]. It is worth pointing out that, e.g., the contributions by Bangert and Kohlmann are new even for convex sets with sufficiently smooth boundaries. For such sets condition (5) is equivalent to

$$\alpha \leq H_{d-1-r}(K,x) \leq \beta\,,$$

for all $x \in \mathrm{bd}\ K$, where $H_j(K,x)$, $j \in \{0,\dots,d-1\}$, is the normalized elementary symmetric function of order j of the principle curvatures of K at x. Conditions of the form (5) can be used to state stability results, which have been explored by various authors; see Diskant [32], Schneider [122], Arnold [6], Kohlmann [81], [85], and the literature cited there. Actually, in some of these papers arguments are implicitly used which involve the absolute continuity of some curvature measure. It is the purpose of the present section to investigate the relationship between the rather weak measure theoretic assumption of the absolute continuity of some curvature or surface area measure and the geometry of the associated convex set. In particular, we are concerned with *regularity results*. Essentially we continue the line of research of the two preceding papers [72], [74], but now we focus on the investigation of absolutely continuous measures with *bounded densities*. Some applications to *stability results* are also treated. A particular feature of our method is that we study the interplay between results for curvature measures and results for surface area measures.

In the smooth setting, various conditions on functions of curvatures of a convex set, which for instance ensure that K is a ball, have been investigated in the literature. The problems which arise in this context and methods of solution which have been invented for that case (see e.g. [143], [87], [88] and the literature cited there) are, however, not the subject of this section. To illustrate the fundamental difference between the smooth and the non-smooth setting, we consider Schneider's method to show that condition (4) (for $\lambda = 1$) implies that K is a ball. In a first step it is shown that (4) implies that

$$S_r(K,\cdot) = S_{d-1}(K,\cdot)\,.$$

The only convex bodies which satisfy the last condition are r-tangential bodies of a unit ball; this follows from deep results of the Brunn-Minkowski theory. Now, if it were already known that K is smooth, then we would obtain that K is a ball. However, the proof that K indeed must be smooth, represents a major part of the work in [120].

For some of the results mentioned in the preceding paragraphs corresponding theorems are known for surface area measures. The degree of similarity between statements of results and methods of proof for curvature and surface area measures depends on the particular case which is considered. For example, recent approaches to characterizations of balls or stability results for curvature measures differ considerably from the proofs of corresponding results for surface area measures. Moreover, surface area measures are distinguished by their connection to mixed volumes. Results for surface area measures which are in the spirit of the above mentioned theorems of Aleksandrov and Burago & Kalinin will be presented in this section for the first time. In fact, the strong interplay and analogy between surface area and curvature measures is exploited as a technique of proof. Clearly, curvature and surface area measures can be obtained as image measures of the more general *support measures* under the natural coordinate projections of the cartesian product $\mathbb{R}^d \times S^{d-1}$ over which the support measures are defined. However, it is not this kind of correspondence which we pursue here. A careful analysis of the true nature of the analogy suggests an underlying *duality*, which will be analysed thoroughly in the following and which will enhance our understanding of both types of measures.

The results and methods of this section are relevant for the remaining parts of this work as well. There we shall develop, for instance, a theory of *relative support measures* in a vector space with a smooth gauge body which may be different from a Euclidean ball. This theory is then applied to obtain characterization, stability and splitting results for relative curvature measures. Applications of relative support measures to contact distributions in stochastic geometry are contained in Section 3. We should also emphasize that another new view on the relationship between curvature and surface area measures is provided in the recent paper [30], where both sequences of measures are obtained as Hessian measures of special convex functions.

1.1 Notation and background information

The starting point for the present investigation was an explicit description of the Lebesgue decomposition for the curvature and surface area measures of convex sets in $\mathbb{R}^d$ with respect to the appropriate $(d-1)$-dimensional Hausdorff measures. As a preparation for this result and its consequences, we introduce some terminology, which will be used in the sequel. However, we shall assume that the reader is already familiar with curvature and surface area measures as introduced in [123]. Subsequently, we shall sketch the main results for curvature and surface area measures contained in the previous paper [74]. In particular, we shall try to emphasize the dual nature of the results obtained for these two sequences of measures. Thus we hope to motivate and prepare some of the new results of this section.

Let $\mathcal{C}^d$ be the set of all non-empty closed convex sets $K \subset \mathbb{R}^d$ with $K \neq \mathbb{R}^d$. Let $\mathcal{H}^s$, $s \geq 0$, denote the s-dimensional Hausdorff measure in a Euclidean space. Which space is meant, will be clear from the context. The unit sphere of $\mathbb{R}^d$ with respect to the Euclidean norm $|\cdot|$ is denoted by S^{d-1}, the unit ball centred at the origin o is denoted by B^d. Furthermore, we write $B^d(x,r)$ instead of $x + rB^d$. The scalar product is denoted by $\langle \cdot , \cdot \rangle$. If $K \in \mathcal{C}^d$ and $x \in \text{bd } K$ (the boundary of K), then the *normal cone* of K at x is denoted by $N(K,x)$; see [123] for notions of convex geometry which are not explicitly defined here. For our approach, the (generalized) *unit normal bundle* $\mathcal{N}(K)$ of a convex set $K \in \mathcal{C}^d$ plays an important rôle. It is defined as the set of all pairs $(x,u) \in \text{bd } K \times S^{d-1}$ such that $u \in N(K,x)$. Walter (see [141] or [142]) has shown that this set represents a strong $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^d \times \mathbb{R}^d$. For $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$, one can introduce *(generalized) curvatures* $k_i(x,u)$, $i \in \{1, \ldots , d-1\}$ (which we also call curvatures on the normal bundle). These generalized curvatures can be obtained as limits of curvatures which are defined with respect to the boundaries of the outer parallel sets of K. They are non-negative, since K is convex. But they are merely defined almost everywhere on $\mathcal{N}(K)$, since the boundaries of the outer parallel sets of K are submanifolds which are of class $C^{1,1}$ (that is they are submanifolds of class C^1 and the spherical image map is Lipschitz), but need not be of class C^2.

It is appropriate to describe the construction more explicitly, since the details will become relevant in the following. For that purpose, we write $p(K, \cdot)$ for the metric projection onto K, we set $d(K,y) := |y - p(K,y)|$ and define $u(K,y) := d(K,y)^{-1}(y - p(K,y))$ for $y \in \mathbb{R}^d \setminus K$. For any $\epsilon > 0$, let K_ϵ be the set of all points whose distance from K is at most ϵ. Then, for all $\epsilon > 0$, the map $(p(K,\cdot), u(K,\cdot))|_{\text{bd } K_\epsilon}$ provides a bi-Lipschitz homeomorphism between bd K_ϵ and $\mathcal{N}(K)$. Furthermore, let $\mathcal{D}_K$ denote the set of all $y \in \mathbb{R}^d \setminus K$ for which $p(K, \cdot)$ is differentiable at y. It is known that if $y \in \mathbb{R}^d \setminus K$, then $y \in \mathcal{D}_K$ if and only if $p(K,y) + [0,\infty)u(K,y) \subset \mathcal{D}_K$. For $y \in \text{bd } K_\epsilon$ and $\epsilon > 0$, let $\sigma_K(y) = u(K,y)$ denote the exterior unit normal vector of K_ϵ at y. Then, for any $(x,u) \in \mathcal{N}(K)$ such that $x + [0,\infty)u \subset \mathcal{D}_K$ (and thus for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$), the spherical image map $\sigma_K|_{\text{bd } K_\epsilon}$ is differentiable at $x + \epsilon u$ for all $\epsilon > 0$ (see [141]), and therefore curvatures $k_1(x+\epsilon u), \dots, k_{d-1}(x+\epsilon u)$ are defined as the eigenvalues of the symmetric linear map $D\sigma_K(x+\epsilon u)|_{u^\perp}$. The corresponding eigenvectors will be denoted by $u_1, \dots, u_{d-1}$. It is easy to see that they can be chosen in such a way that they do not depend on ϵ; of course, they depend on (x,u), but we shall often omit the argument. Hence, especially for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ and any $\epsilon > 0$, we can define the generalized curvatures

$$
\begin{aligned}
k_i(x,u) :&= \lim_{t \downarrow 0} \frac{k_i(x+\epsilon u)}{1 + (t-\epsilon)k_i(x+\epsilon u)} \\
&= \begin{cases} \frac{k_i(x+\epsilon u)}{1-\epsilon k_i(x+\epsilon u)} & \text{if } k_i(x+\epsilon u) < \epsilon^{-1} \,, \\ \infty & \text{if } k_i(x+\epsilon u) = \epsilon^{-1} \,, \end{cases}
\end{aligned}
$$

$i \in \{1, \dots, d-1\}$, independent of the particular choice of $\epsilon > 0$ (see [155]). We shall always assume that the ordering of these curvatures is such that

$$
0 \le k_1(x,u) \le \ldots \le k_{d-1}(x,u) \le \infty \,. \tag{6}
$$

In addition, we set $k_0(x,u) := 0$ and $k_d(x,u) := \infty$ for all $(x,u) \in \mathcal{N}(K)$. Finally, note that the preceding notation does not make explicit the dependence of the various curvature functions on the convex set K. If necessary, however, we shall be more precise and explain the appropriate notation as the need arises. Further details of this construction, in the general context of sets with positive reach, can be found in M. Zähle [155] and in [73], [72].

The curvature measures of an arbitrary convex set K cannot be expressed in terms of functions of the principle curvatures which are defined (almost everywhere) on the boundary of K; a similar remark applies to surface area measures and *principal radii of curvature*. However, the generalized curvatures can be used to describe curvature and surface area measures (and also the more general support measures) in an appropriate way. This is the reason why, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$, we define certain weighted elementary symmetric functions of generalized curvatures on $\mathcal{N}(K)$ by

$$
\mathbb{H}_j(K,(x,u)) := \binom{d-1}{j}^{-1} \sum_{|I|=j} \frac{\prod_{i \in I} k_i(x,u)}{\prod_{i=1}^{d-1} \sqrt{1 + k_i(x,u)^2}} \,,
$$

$j \in \{0, \dots, d-1\}$, where the sum extends over all sets $I \subset \{1, \dots, d-1\}$ of cardinality j. (For $j = 0$ the product over the empty set has to be interpreted as one.)

Let X be a locally compact Hausdorff space with a countable base. (Essentially, in Section 1 we are interested in the cases $X = \mathbb{R}^d$ and $X = S^{d-1}$). In the following, we refer to Chapter

1 of [39] for the basic notation and results concerning measure theory. However, there is one minor difference. For us a Radon measure over X will be defined on the Borel subsets of X, whereas in [39] Radon measures are understood to be outer measures defined on all subsets of X. The simple connection between these two points of view is as follows. A Radon measure μ in the sense of [39] yields a Radon measure in our sense simply by restricting μ to the σ-algebra of Borel sets. On the other hand, a Radon measure μ on the Borel sets of X can be extended as a Radon measure $\bar{\mu}$ to all subsets of X by setting

$$\bar{\mu}(A) := \inf \{\mu(B) : A \subset B\,, B \in \mathfrak{B}(X)\}\ .$$

Here and subsequently, we write $\mathfrak{B}(Y)$ for the σ-algebra of Borel sets of an arbitrary topological space Y. The preceding discussion shows that we can simply refer to Radon measures (over X) without further explanations.

Now let μ and ν be two Radon measures over X. If $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \in \mathfrak{B}(X)$, then we say that μ is *absolutely continuous* with respect to ν, and we write $\mu \ll \nu$. By the Radon-Nikodym theorem, $\mu \ll \nu$ if and only if there is a non-negative Borel measurable function $f : X \to \mathbb{R}$ such that

$$\mu(A) = \int_A f(x)\,\nu(dx)$$

for all $A \in \mathfrak{B}(X)$. In particular, the *density function* f is locally integrable with respect to ν. Furthermore, we say that μ is *singular* with respect to ν if there is a Borel set $B \subset X$ such that $\mu(X \setminus B) = 0 = \nu(B)$, and in this case we write $\mu \perp \nu$. Certainly, this is a symmetric relationship. A version of the Lebesgue decomposition theorem says that for arbitrary Radon measures μ and ν there are two Radon measures μ^a and μ^s such that $\mu = \mu^a + \mu^s$, $\mu^a \ll \nu$ and $\mu^s \perp \nu$. Moreover, the absolutely continuous part μ^a and the singular part μ^s (of μ with respect to ν) are uniquely determined by these conditions. We shall also consider the restriction $(\mu \llcorner A)(\cdot) := \mu(A \cap \cdot)$ of a Radon measure μ to a set $A \in \mathfrak{B}(X)$, which is again a Radon measure.

These notions and results will now be applied to the curvature and surface area measures of a convex set $K \in \mathcal{C}^d$. As the curvature measures are Borel measures over $\mathbb{R}^d$ which are locally finite and concentrated on bd K, the curvature measure $C_r(K,\cdot)$, for any $r \in \{0,\dots,d-1\}$, can be written as the sum of two measures, that is,

$$C_r(K,\cdot) = C_r^a(K,\cdot) + C_r^s(K,\cdot)\ ,$$

where $C_r^a(K,\cdot)$ is absolutely continuous and $C_r^s(K,\cdot)$ is singular with respect to the boundary measure $C_{d-1}(K,\cdot)$. Recall that if $K \in \mathcal{C}^d$, then

$$C_{d-1}(K,\cdot) = \mathcal{H}^{d-1} \llcorner \operatorname{bd} K$$

if K has non-empty interior or $\dim K \le d-2$. If $\dim K = d-1$, then

$$C_{d-1}(K,\cdot) = 2(\mathcal{H}^{d-1} \llcorner \operatorname{bd} K)\ .$$

Subsequently, we often say that the r-th curvature measure of a convex set is absolutely continuous, by which we wish to express that this measure is absolutely continuous with respect to the boundary measure of the set.

The surface area measures $S_r(K,\cdot)$ of a compact convex set $K \subset \mathbb{R}^d$ are finite Borel measures over S^{d-1}. Hence, if $K \subset \mathbb{R}^d$ is a convex body and $r \in \{0,\dots,d-1\}$, then we can write

$$S_r(K,\cdot) = S_r^a(K,\cdot) + S_r^s(K,\cdot)\ ,$$

where $S_r^a(K,\cdot)$ is absolutely continuous and $S_r^s(K,\cdot)$ is singular with respect to $S_0(K,\cdot)$. In this case, the surface area measure of order 0 is just the restriction of the $(d-1)$-dimensional Hausdorff measure to the Borel sets of the unit sphere, and thus it is independent of the convex body K.

In the remainder of this subsection, we recall various results and some notation from [72] and [74]. Let $\mathcal{K}^d$ denote the set of all convex bodies in $\mathbb{R}^d$. We write $\mathcal{C}_o^d$ ($\mathcal{K}_o^d$) for the set of all $K \in \mathcal{C}^d$ ($K \in \mathcal{K}^d$) for which int $K \neq \emptyset$. The following two results, which were proved in [72], Theorems 3.2 and 3.5, give an explicit description of the singular parts of the curvature and surface area measures of a convex set K in terms of properties of the generalized curvature functions of the unit normal bundle of K. Although these two theorems do not seem to be much related to geometry, they are indeed the first and essential step towards geometric results. Moreover, they already served as a main ingredient in the regularity theorems contained in [72].

Theorem 1.1 *For a convex set $K \in \mathcal{C}^d$, $r \in \{0,\dots,d-1\}$, and $\beta \in \mathfrak{B}(\mathbb{R}^d)$,*

$$C_r^s(K,\beta) = \int_{\mathcal{N}^s(K)} \mathbf{1}_\beta(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathcal{H}^{d-1}(d(x,u))\,, \tag{7}$$

where $\mathcal{N}^s(K)$ is the set of all $(x,u) \in \mathcal{N}(K)$ such that $k_{d-1}(x,u) = \infty$.

Theorem 1.2 *For a convex body $K \in \mathcal{K}^d$, $r \in \{0,\dots,d-1\}$, and $\omega \in \mathfrak{B}(S^{d-1})$,*

$$S_r^s(K,\omega) = \int_{\mathcal{N}_s(K)} \mathbf{1}_\omega(u)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathcal{H}^{d-1}(d(x,u))\,, \tag{8}$$

where $\mathcal{N}_s(K)$ is the set of all $(x,u) \in \mathcal{N}(K)$ such that $k_1(x,u) = 0$.

In [72], the absolutely continuous parts of these measures were recovered as well. To describe these, we write $k_1(K,x),\dots,k_{d-1}(K,x)$ for the principal curvatures of K at a normal boundary point $x \in$ bd K; thus these curvatures are defined for $\mathcal{H}^{d-1}$ almost all boundary points. Then the density function of $C_r^a(K,\cdot)$ with respect to $C_{d-1}(K,\cdot)$ is given by

$$H_{d-1-r}(K,x) = \binom{d-1}{r}^{-1} \sum_{|I|=d-1-r} \prod_{i\in I} k_i(K,x)\,,$$

where the summation extends over all sets $I \subset \{1,\dots,d-1\}$ of cardinality $d-1-r$.

Similarly, the principal radii of curvature $r_1(K,u),\dots,r_{d-1}(K,u)$ of K at $u \in S^{d-1}$ are defined for all $u \in S^{d-1}$ such that $h_K = h(K,\cdot)$ is second order differentiable at u as the eigenvalues of the restriction to $u^\perp$ of the second order differential of h_K at u, that is, as the eigenvalues of $d^2h_K(u)|_{u^\perp}$. Then the density function of $S_r^a(K,\cdot)$ with respect to $S_0(K,\cdot)$ is

$$P_r(K,u) = \binom{d-1}{r}^{-1} \sum_{|I|=r} \prod_{i\in I} r_i(K,u)\,,$$

where the summation is defined as before. Instead of $P_r(K,u)$ we presently prefer to write $D_r h(K,\cdot)$; see [72] for further comments and references.

Theorems 1.1 and 1.2 were used in [74] to prove useful conditions which are necessary and sufficient for the absolute continuity of a particular curvature or surface area measure of a convex set. These conditions allow one to express the measure theoretic property of absolute continuity in terms of properties of the generalized curvatures which are easier to treat analytically. This was the starting point of the recent paper [74]. In fact, the following two theorems also play a key rôle in the investigation of the present section. It is appropriate to state such characterizations (Theorem 1.3 and 1.4) as local results.

Theorem 1.3 *Let $K \in \mathcal{C}^d$, $r \in \{0,\dots,d-1\}$, and $\beta \in \mathfrak{B}(\mathbb{R}^d)$. Then*

$$C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta \tag{9}$$

if and only if

$$k_{d-1}(x,u) < \infty \quad \text{or} \quad k_{r+1}(x,u) = 0 \quad \text{or} \quad k_r(x,u) = \infty\ , \tag{10}$$

for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ such that $x \in \beta$.

Theorem 1.4 *Let $K \in \mathcal{K}^d$, $r \in \{0,\dots,d-1\}$, and $\omega \in \mathfrak{B}(S^{d-1})$. Then*

$$S_r(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega \tag{11}$$

if and only if

$$k_1(x,u) > 0 \quad \text{or} \quad k_{r+1}(x,u) = 0 \quad \text{or} \quad k_r(x,u) = \infty\ ,$$

for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ such that $u \in \omega$.

Obviously, the conditions (10) and (11) can be checked by simply counting the number of curvatures which satisfy $k_i(x,u) = 0$ or $k_i(x,u) = \infty$. Also note that in the present situation it is possible, for instance, to paraphrase condition (9) by saying that the Radon measure $C_r(K,\cdot)$ on $\mathbb{R}^d$ is $(d-1)$-rectifiable. This terminology is used in [106, p. 603], [99, p. 228], or [40], where the $(d-1)$-rectifiability of a general Radon measure μ is characterized in terms of properties of the $(d-1)$-dimensional densities of μ. However, these investigations do not seem to be directly related to the present work.

In the special but important case of the curvature measure $C_0(K,\cdot)$ of a convex body K, a characterization of absolute continuity can be stated which involves a spherical supporting property of K. This property will be described by using the set $\operatorname{expn}^* K$ of *directions of nearest boundary points* of K. For any $K \in \mathcal{C}^d$, $\operatorname{expn}^* K$ is defined as the set of all unit vectors $u \in S^{d-1}$ for which there exist points $x \in \operatorname{int} K$ and $y \in \operatorname{bd} K$ such that $|y-x| = \operatorname{dist}(x, \operatorname{bd} K)$ and $y - x = |y-x|u$. In other words, $u \in \operatorname{expn}^* K$ if and only if a non-degenerate ball which is contained in K contains a boundary point of K with exterior unit normal vector u. In the following, we shall say that $\mathcal{K} \in \mathcal{C}^d$ is *supported from inside by a d-dimensional ball in direction u* if and only if $u \in \operatorname{expn}^* K$. Since we are dealing with a local result, we shall also need the *spherical image* $\sigma(K,\beta)$ of $K \in \mathcal{C}^d$ at $\beta \subset \mathbb{R}^d$. It is defined as the union of the normal cones $N(K,x)$ with $x \in \beta$. The following result was established in [74].

Theorem 1.5 *For a convex body $K \in \mathcal{K}^d$ and $\beta \in \mathfrak{B}(\mathbb{R}^d)$, the following three conditions are equivalent:*

(a) $C_0(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$;

(b) $D_{d-1}h(K,u) > 0$ *for* $\mathcal{H}^{d-1}$ *almost all* $u \in \sigma(K,\beta)$;

(c) $\mathcal{H}^{d-1}(\sigma(K,\beta) \setminus \operatorname{expn}^* K) = 0$.

In addition, for $\gamma \in \mathfrak{B}(\mathbb{R}^d)$,

$$C_0^s(K,\gamma) = \mathcal{H}^{d-1}\left(\{u \in \sigma(K,\gamma) : D_{d-1}h(K,u) = 0\}\right)$$

and

$$C_0^a(K,\gamma) = \mathcal{H}^{d-1}\left(\{u \in \sigma(K,\gamma) : D_{d-1}h(K,u) > 0\}\right) .$$

Statement (b) of Theorem 1.5 is an analytic and statement (c) a geometric way of characterizing the absolute continuity of the Gaussian curvature measure. In fact, the geometric condition (c) can be viewed as a substantially weakened form of a condition requiring a suitable ball to roll freely inside K. Thus this theorem and the following counterpart for surface area measures represent first examples of results which correspond to the general programm which we described initially. These results indicate how local geometric properties of a convex set are related to measure theoretic properties of curvature and surface area measures which are associated with this set.

As in the case of the Gauss curvature measure $C_0(K,\cdot)$, the absolute continuity of $S_{d-1}(K,\cdot)$, for some $K \in \mathcal{K}^d$, can be characterized by a spherical supporting property. The situation here is 'dual' to the previous one. Condition (c) of Theorem 1.6 below can be interpreted as a substantially weakened form of a condition demanding K to roll freely inside a ball. The statement of this theorem involves the set $\exp^* K$ of *farthest boundary points* of a convex body $K \in \mathcal{K}^d$ (see [70]). This definition implies that $x \in \exp^* K$ if and only if the boundary of a ball which contains K passes through x. In the following, we say that $K \in \mathcal{K}^d$ is *supported from outside by a d-dimensional ball at* x if and only if $x \in \exp^* K$. Moreover, recall from [123, §2.2] that $\tau(K,\omega)$ denotes the *reverse spherical image* of K at $\omega \subset S^{d-1}$. By definition, $\tau(K,\omega)$ is equal to the union of the *support sets* $F(K,u)$ with $u \in \omega$.

Theorem 1.6 *Let $K \in \mathcal{K}^d$ and $\omega \in \mathfrak{B}(S^{d-1})$. Then the following three conditions are equivalent:*

(a) $S_{d-1}(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$;

(b) $H_{d-1}(K,x) > 0$ *for* $\mathcal{H}^{d-1}$ *almost all* $x \in \tau(K,\omega)$;

(c) $\mathcal{H}^{d-1}(\tau(K,\omega) \setminus \exp^* K) = 0$.

In addition, for $\alpha \in \mathfrak{B}(S^{d-1})$,

$$S_{d-1}^s(K,\alpha) = \mathcal{H}^{d-1}\left(\{x \in \tau(K,\alpha) : H_{d-1}(K,x) = 0\}\right)$$

and

$$S_{d-1}^a(K,\alpha) = \mathcal{H}^{d-1}\left(\{x \in \tau(K,\alpha) : H_{d-1}(K,x) > 0\}\right) .$$

Using a *Crofton intersection formula* and various integral-geometric transformations, Theorem 1.5 can be extended to curvature measures of any order. The corresponding result, Theorem 1.7, was proved in [74]. It can be interpreted as a two-step procedure for verifying the absolute continuity of curvature measures of convex bodies with non-empty interiors; see [74]. The precise formulation involves the conveniently normalized motion invariant Haar measure μ_r on the homogeneous space $\mathbf{A}(d,r)$ of r-dimensional affine subspaces in $\mathbb{R}^d$; compare [123]. Furthermore, here and in the following a prime which is attached to a quantity indicates that this quantity has to be calculated with respect to an appropriate affine or linear subspace, which will be clear from the context. Finally, we write $U(E)$ for the unique linear subspace which is parallel to a given affine subspace E.

Theorem 1.7 *Let $K \in \mathcal{C}_o^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and $r \in \{2, \dots, d\}$. Then*

$$C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

if and only if, for μ_r almost all $E \in \mathbf{A}(d,r)$ such that $E \cap \operatorname{int} K \neq \emptyset$, and in $\mathcal{H}^{r-1}$ almost all directions of the set $\sigma'(K \cap E, \beta \cap E) \subset U(E)$, the intersection $K \cap E$ is supported from inside by an r-dimensional ball contained in E.

In fact, Theorem 1.7 was stated in [74] for $K \in \mathcal{K}_o^d$, but for unbounded sets the assertion follows immediately, since the curvature measures are locally defined. The main tool for the proof of Theorem 1.7 is the special case $s = r$ of the following theorem, which is cited from [74].

Theorem 1.8 *Let $K \in \mathcal{C}_o^d$, let $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and assume that $r \in \{2, \dots, d-1\}$ and $s \in \{r, \dots, d-1\}$. Then*

$$C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

if and only if

$$C'_{s-r}(K \cap E, \cdot) \llcorner (\beta \cap E) \ll C'_{s-1}(K \cap E, \cdot) \llcorner (\beta \cap E) ,$$

for μ_s almost all $E \in \mathbf{A}(d,s)$ such that $E \cap \operatorname{int} K \neq \emptyset$.

Thus, for $s = r$, Theorem 1.8 especially says that in the mean curvature case ($r = 2$) absolute continuity can be verified by investigating planar sections of K.

Analogous results for surface area measures have been established in [74] as well. One of the basic tools which one uses now are integral-geometric *projection formulae*. Such formulae involve the Grassmann space $\mathbf{G}(d,j)$ of j-dimensional linear subspaces of $\mathbb{R}^d$ and the normalized rotation invariant Haar measure ν_j over $\mathbf{G}(d,j)$. In the following, we write $K|V$ for the orthogonal projection of a convex body K onto $V \in \mathbf{G}(d,j)$; moreover, a superscript $V \in \mathbf{G}(d,j)$, which is attached to a particular quantity, indicates that this quantity has to be considered in V and not in the surrounding space $\mathbb{R}^d$. The result corresponding to Theorem 1.8 is the following.

Theorem 1.9 *Let $K \in \mathcal{K}^d$, $i \in \{1, \dots, d-2\}$, $j \in \{i+1, \dots, d-1\}$, and let $\omega \in \mathfrak{B}(S^{d-1})$. Then*

$$S_i(K, \cdot) \llcorner \omega \ll S_0(K, \cdot) \llcorner \omega$$

if and only if

$$S_i^V(K|V,\cdot) \llcorner (\omega \cap V) \ll S_0^V(K|V,\cdot) \llcorner (\omega \cap V) \ ,$$

for ν_j almost all linear subspaces $V \in \mathbf{G}(d,j)$.

As before, Theorem 1.9 leads to the subsequent characterization for surface area measures.

Theorem 1.10 *Let $K \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, and $i \in \{1,\ldots,d-1\}$. Then*

$$S_i(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

if and only if, for ν_{i+1} almost all $U \in \mathbf{G}(d,i+1)$, and at $\mathcal{H}^i$ almost all points of the set $\tau(K|U,\omega\cap U)$, the projection $K|U$ is supported from outside by an $(i+1)$-dimensional ball contained in U.

With regard to Theorems 1.7 and 1.10 it is natural to ask for one-step procedures which allow one to decide whether a particular curvature or surface area measure of a convex body is absolutely continuous or not. For curvature measures, a result which leads to such a procedure is contained in the ensuing Theorem 1.11 which is taken from [74].

First, however, we point out that in [74] the following considerations concerning curvature measures were restricted to sets $K \in \mathcal{K}_o^d$, although they extend to the case $K \in \mathcal{C}_o^d$. To see this, one has to use that the curvature measures are locally defined, that the definition of the contact measures described subsequently can be extended consistently to unbounded closed convex sets (compare [113], equations (1) and (2)), and in addition a result of Zalgaller [159].

Let us fix a convex body $K \in \mathcal{C}^d$ and some $r \in \{0,\ldots,d-1\}$. For a unit vector $v \in S^{d-1}$ let $H(K,v)$ denote the support plane of K with exterior normal vector v. An *affine subspace* $E \in \mathbf{A}(d,r)$ is said to *touch* K if $E\cap K \neq \emptyset$ and $E \subset H(K,v)$ for some $v \in S^{d-1}$. Furthermore, we write $\mathrm{A}(K,d,r)$ for the $((d-r)(r+1)-1)$-rectifiable set of r-dimensional affine subspaces of $\mathbb{R}^d$ which touch K. Several authors (see [146], [43], [156], [113], [132]) have introduced naturally defined measures on $\mathrm{A}(K,d,r)$. For convex bodies all these measures are essentially equivalent. These *contact measures* have been used for calculating collision probabilities [119], [149], and they are related to absolute or total curvature measures; compare [116], [7], [134]. Let us denote such a measure by $\mu_r(K,\cdot)$.

Next we define the *spherical image of order* r of $K \in \mathcal{C}^d$ at $\beta \in \mathfrak{B}(\mathbb{R}^d)$ for any $r \in \{0,\ldots,d-1\}$ by

$$\sigma_r(K,\beta) := \{E \in \mathrm{A}(K,d,r) : \beta \cap \mathrm{bd}\, K \cap E \neq \emptyset\} \ .$$

The case $r = d-1$ leads to the ordinary spherical image, since $\mathrm{A}(K,d,d-1)$ is the set of supporting hyperplanes of K each of which can be identified with its exterior unit normal vector. Let ω_i denote the surface area of the $(i-1)$-dimensional unit sphere. Then the measure $\mu_r(K,\cdot)$ can be normalized so that the relation

$$C_{d-1-r}(K,\beta) = \frac{\omega_d}{\omega_{d-r}}\, \mu_r(K,\sigma_r(K,\beta)) \ , \tag{12}$$

which for convex bodies is due to Weil [146], holds for all $\beta \in \mathfrak{B}(\mathbb{R}^d)$. Set $u^- := \{tu : t \leq 0\}$ if $u \in \mathbb{R}^d \setminus \{o\}$ and let $r \in \{2,\ldots,d\}$. Then we say that $K \in \mathcal{C}^d$ is *supported from inside by an*

r-dimensional ball at $E \in \mathbf{A}(K, d, r-1)$ if there is some $p \in K \cap E$, some $u \in S^{d-1} \cap U(E)^{\perp}$ with $(E + u^{-}) \cap \operatorname{int} K \neq \emptyset$, and some $\rho > 0$ such that $B^d(p - \rho u, \rho) \cap (E + u^{-}) \subset K$.

Equation (12) provides an integral-geometric interpretation for curvature measures of convex sets. In the present context, it also suggests a characterization of absolute continuity involving touching planes.

Theorem 1.11 *Let $K \in \mathcal{C}_o^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and $r \in \{2, \dots, d\}$. Then*

$$C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

if and only if K is supported from inside by an r-dimensional ball at $\mu_{r-1}(K, \cdot)$ almost all $E \in \sigma_{r-1}(K, \beta)$.

Essentially, Theorem 1.11 is deduced from Theorem 1.7 through a succession of auxiliary results. The proof includes arguments from convexity, geometric measure theory and also some basic results about Haar measures. The key idea is to associate with an r-dimensional affine subspace E meeting int K and a unit vector $u \in U(E)$ the $(r-1)$-dimensional support plane of $K \cap E$ relative to E with exterior unit normal vector u. This support plane then represents an $(r-1)$-dimensional affine subspace which touches K.

In order to complete the picture, we present the analogous result for surface area measures as well. Again we introduce some terminology which is designed to underscore the duality of the situation. Instead of lower-dimensional balls which support a given convex body from inside, we now introduce the notion of an orthogonal spherical cylinder which supports a convex body from outside. To be explicit, we say that a convex body K is *supported from outside by an orthogonal spherical cylinder at $E \in \mathrm{A}(K, d, r)$* if there is some $R > 0$ and some $u \in S^{d-1}$ with $E \subset H(K, u)$ such that $K \subset E + B^d(-Ru, R)$. Moreover, we set

$$\tau_r(K, \omega) := \{E \in \mathbf{A}(K, d, r) : E \subset H(K, u) \text{ for some } u \in \omega\}$$

and call this the *reverse spherical image of order r of K at ω*. Thus the reverse spherical image of order $r = 0$ is just the ordinary reverse spherical image. Again we quote from [74].

Theorem 1.12 *Let $K \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, and $i \in \{1, \dots, d-1\}$. Then*

$$S_i(K, \cdot) \llcorner \omega \ll S_0(K, \cdot) \llcorner \omega$$

if and only if K is supported from outside by an orthogonal spherical cylinder at $\mu_{d-1-i}(K, \cdot)$ almost all $E \in \tau_{d-1-i}(K, \omega)$.

It has already become apparent that the boundary of a convex set $K \in \mathcal{C}_o^d$ one of whose curvature measures is absolutely continuous with respect to the boundary measure cannot be too irregular. A precise and in a certain sense optimal result in this spirit is stated as Theorem 4.6 in [72]. Another regularity result, which complements the picture, is provided by the following theorem. As usual, we say that $x \in \operatorname{bd} K$ is a *regular boundary point* of $K \in \mathcal{C}_o^d$ if there exists precisely one support plane of K passing through x.

Theorem 1.13 *Let $K \in \mathcal{C}_o^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, $r \in \{2, \dots, d-1\}$, and assume that*

$$C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta \, .$$

Then, for $\mu_{r-1}(K, \cdot)$ almost all $E \in \sigma_{r-1}(K, \beta)$, every boundary point of K which lies in E is regular.

The dual result demonstrates that the rectifiability of some surface area measure of a convex body K leads to a certain degree of strict convexity for K. Another precise statement in this direction was established in [72, Theorem 4.8]. Recall that a support plane $H(K,u)$, $u \in S^{d-1}$, of a convex body K is said to be regular if u is a regular normal vector of K, that is, if $F(K,u)$ contains precisely one point.

Theorem 1.14 *Let $K \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, $i \in \{1,\ldots,d-1\}$, and assume that*

$$S_i(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega \ .$$

Then, for $\mu_{d-1-i}(K,\cdot)$ almost all $E \in \tau_{d-1-i}(K,\omega)$, every support plane of K which contains E is regular.

1.2 Selected results

The review of results on the absolute continuity of curvature and surface area measures clearly indicates that there should be a general principle by which results for curvature and surface area measures are related. Indeed, corresponding pairs of notions such as boundary point – normal vector, support set – normal cone, principal curvatures – radii of curvature, intersection by an affine plane – projection onto a linear subspace, are at least to a certain extent connected by polarity; compare [123], p. 75, and [71]. Therefore it is natural to conjecture that the characterizations of absolute continuity of curvature measures correspond in a precise sense to the characterizations of absolute continuity of surface area measures via polarity; on an intuitive level, this kind of duality has already served as a guiding rule, although formal statements of results, which support this intuition, were not available so far.

The formation of the polar body of a given convex body certainly is a highly non-linear operation and it requires the non-canonical choice of a reference point (compare §1.6 in [123]). Subsequently, it will be convenient to fix the origin o as the reference point, but this does not restrict the generality of our statements; moreover, we shall see that often the choice of a reference point is immaterial for geometric consequences which appear in a translation invariant setting. Most of the results, which we intend to discuss, thus refer to the set $K \in \mathcal{K}^d_{oo}$ of all convex bodies $K \in \mathcal{K}^d$ for which $o \in \text{int } K$. For a given convex body $K \in \mathcal{K}^d_{oo}$, we introduce the map

$$f : S^{d-1} \to \text{bd } K^* \,, \quad u \mapsto h(K,u)^{-1}u \,.$$

This map precisely provides the required correspondence between normal vectors of K and boundry points of K^*.

Now we are prepared to state our *first transfer principle*, which allows us to transfer properties connected with the absolute continuity of the rth curvature measure $C_r(K,\cdot)$ of a convex body K to dual properties connected with the absolute continuity of the $(d-1-r)$th surface area measure $S_{d-1-r}(K^*,\cdot)$ of the polar body K^*, and conversely.

Theorem 1.15 *Let $K \in \mathcal{K}^d_{oo}$, $\omega \in \mathfrak{B}(S^{d-1})$, and choose $r \in \{0,\ldots,d-1\}$. Then*

$$S_r(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

if and only if

$$C_{d-1-r}(K^*,\cdot) \llcorner f(\omega) \ll C_{d-1}(K^*,\cdot) \llcorner f(\omega) \,.$$

Clearly, by the bi-polar theorem, the rôles of K and K^* are interchangeable. The proof of this result uses the characterization of absolute continuity for curvature and surface area measures from [74] involving the curvatures on the unit normal bundle in an essential way. Therefore, Theorem 1.15 cannot be used to deduce Theorems 1.3 and 1.4 from each other. Presumably, except for parts of Theorems 1.5 and 1.6, it is also not possible to deduce the remaining corresponding pairs of results of the preceding section in a rigorous and straightforward manner from each other. However, we shall encounter a number of other applications of Theorem 1.15 in the following subsections. In particular, Theorem 1.15 is an important ingredient for the proof of our *second transfer principle.*

In spherical space, the connection between curvature measures of a convex set and surface area measures of the polar set is much simpler and actually extends to support measures; see Glasauer [49]. This is due to the fact that polarity on the sphere essentially is the duality of cones, which is much easier to treat from a technical point of view. A similar phenomenon can be observed when one tries to extend certain integral-geometric results from the sphere to Euclidean space; compare the discussion in [49], [50], [52], [53]. Still another kind of duality for Hessian measures of convex functions was discovered in [30]. In this context the right notion of duality turned out to be the classical formation of the conjugate function. However, the theory developed in [30] does not seem to be applicable to the present situation.

Up to now we considered the case of absolutely continuous curvature or surface area measures. The next step and the primary concern of the following subsections is to study absolutely continuous measures with bounded densities. We say that a particular curvature or surface area measure is absolutely continuous with *bounded density*, if it is absolutely continuous and the density is bounded from above by a constant. Clearly, if the density of a measure with respect to another measure is bounded, then the former need not be absolutely continuous with respect to the latter. Again the natural task is to find conditions which characterize the absolute continuity with bounded density of a particular curvature or surface area measure or, at least, to find geometric consequences concerning the structure of the set of singular points or the set of singular normal vectors.

A first general result concerning bounded densities is given by our *second transfer principle*, which we state as Theorem 1.16 and which represents the analogue of Theorem 1.15 for the case of bounded densities. It is implied by Theorem 1.15 and by new estimates between elementary symmetric functions of principle curvatures of K^* and elementary symmetric functions of radii of curvature of K at corresponding points; see Corollary 1.22 below. These estimates again are consequences of a more general connection between elementary symmetric functions of principle curvatures of K^* and suitably weighted elementary symmetric functions of radii of curvature of K. A very special instance of such a relationship was found in [71], Theorem 2.2, but the present approach is completely different.

Theorem 1.16 *Let $K \in \mathcal{K}^d_{oo}$, $\omega \in \mathfrak{B}(S^{d-1})$, and $r \in \{0, \dots, d-1\}$. Then there is a constant c such that*

$$S_r(K,\cdot) \llcorner \omega \le c\, S_0(K,\cdot) \llcorner \omega$$

if and only if there is a constant c^* such that

$$C_{d-1-r}(K^*,\cdot) \llcorner f(\omega) \le c^*\, C_{d-1}(K^*,\cdot) \llcorner f(\omega)\,.$$

In order to demonstrate how Theorem 1.16 together with Corollary 1.22 can be applied effectively to obtain new results, we transfer a theorem of Weil [144] concerning surface area measures to a new theorem about curvature measures. Part (a) of Theorem 1.17 shows how an integrability assumption on the density of the mean curvature measure $C_{d-2}(K,\cdot)$ of a convex set K implies the absolute continuity of certain lower-dimensional curvature measures with precise information about the integrability of the corresponding densities. In a certain sense this result is optimal as an example demonstrates. We should also emphasize that in general the absolute continuity of the rth curvature measure of a convex body K does not imply the absolute continuity of any other curvature measure of order s ($s \neq r$) of K as shown by examples.

Theorem 1.17 *Let $K \in \mathcal{C}^d_o$, and let $\beta \subset \mathbb{R}^d$ be open.*

(a) *Assume that*

$$C_{d-2}(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta\,,$$

and further assume that $H_1(K,\cdot) \in \mathrm{L}^p_{loc}(\mathrm{bd}\, K \cap \beta)$ for some $p \in [1,\infty)$. Then

$$C_{d-1-j}(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

and $H_j(K,\cdot) \in \mathrm{L}^{\left[\frac{p}{j}\right]}_{loc}(\mathrm{bd}\, K \cap \beta)$ for $j \in \{1,\ldots,[p]\}$.

(b) *Assume that*

$$C_{d-2}(K,\cdot) \llcorner \beta \leq \bar{c}\, C_{d-1}(K,\cdot) \llcorner \beta$$

for some constant $\bar{c} > 0$. Then

$$C_{d-1-j}(K,\cdot) \llcorner \beta \leq \bar{c}^j\, C_{d-1}(K,\cdot) \llcorner \beta$$

for $j \in \{1,\ldots,d-1\}$.

In view of the inequality between the arithmetic-mean and the geometric-mean, it is not surprising that treating absolutely continuous curvature and surface area measures having bounded densities, the strongest implications can be obtained if one assumes that

$$S_1(K,\cdot) \leq c\, S_0(K,\cdot) \tag{13}$$

or

$$C_{d-2}(K,\cdot) \leq c\, C_{d-1}(K,\cdot)\,. \tag{14}$$

Under the assumption (13), we show that K is a summand of a ball with radius $(d-1)c$. This essentially is a known fact; see Weil [148]. The present proof is based on a new integral-geometric characterization of absolute continuity with bounded density for surface area measures, which is applied to condition (13); this integral-geometric characterization is analogous to Theorem 1.9 and is of interest in its own right.

Dually, the assumption (14) on the curvature measure of order $d-2$ implies that a ball with radius $((d-1)c)^{-1}$ rolls freely inside K. In fact, this variant and strengthening of *Blaschke's rolling theorem* can be extended to unbounded convex sets the boundary of which is connected. Independently, Bangert [12] has arrived at the same conclusion by a completely different method of proof. In addition, we show by examples that the bound for the radius of the ball as given above cannot be increased further in general.

Note that no a priori smoothness or strict convexity assumptions are required for these results. Moreover, local variants of these statements are established, too. In contrast to the proof of the characterization result which is related to (13), the proof of the analogous result for curvature measures is *not* based on an integral-geometric characterization of absolute continuity with bounded density for curvature measures. In fact, although such an integral-geometric characterization and thus an analogue of Theorem 1.8 does exist for curvature measures as well, it is more involved than its counterpart for surface area measures, since the angles of a section plane with tangent planes of the body have to be taken into account.

Our version of Blaschke's rolling theorem will eventually enable us to remove the smoothness assumptions in stability results of Schneider [122] and Arnold [6] for the $(d-2)$nd (mean) curvature measure. More surprisingly, the following stability result for the $(d-1)$st surface area measure improves a theorem of Diskant [33] who merely showed, under the same assumptions, that K lies in a $\gamma\epsilon^{1/(d-1)}$-neighbourhood of a unit ball. Our result shows that the exponent $1/(d-1)$ can be improved to 1, which is the right order.

Theorem 1.18 *Let $K \in \mathcal{K}^d$ and $0 \leq \epsilon < 1/4$. Assume that*

$$(1-\epsilon)S_0(K,\cdot) \leq S_{d-1}(K,\cdot) \leq (1+\epsilon)S_0(K,\cdot)\,.$$

Then K lies in a $\gamma\epsilon$-neighbourhood of a unit ball, where the constant γ depends only on the dimension d.

Finally, we make some comments on regularity results. Recall that the set $\Sigma^r(K)$ of r-singular boundary points of K is defined by

$$\Sigma^r(K) := \{x \in \operatorname{bd} K : \dim N(K,x) \geq d-r\}\,.$$

A point $x \in K$ is called an r-extreme boundary points of K if it is not the centre of an $(r+1)$-dimensional ball which is contained in K; compare [123]. We write $\operatorname{ext}_r K$ for the set of r-extreme boundary points of K. The following collection of results describes how the size of the set of r-singular points of a convex set K is restricted by an assumption of absolute continuity on the rth curvature measure of K:

1. For all $K \in \mathcal{K}^d$, the set $\Sigma^r(K)$ has σ-finite r-dimensional Hausdorff measure.

2. Let $K \in \mathcal{C}^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$ and $r \in \{0,\dots,d-2\}$. Assume that

   $$C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta\,.$$

   Then

   $$\mathcal{H}^r(\Sigma^r(K) \cap \beta) = 0\,.$$

   This was shown in [72].

3. Let $K \in \mathcal{C}^d_o$, let $\beta \subset \mathbb{R}^d$ be open, and assume that there is a constant c such that

   $$C_r(K,\cdot) \llcorner \beta \leq c\, C_{d-1}(K,\cdot) \llcorner \beta\,.$$

Then, for any $x \in \operatorname{bd} K \cap \beta$,

$$x \notin \operatorname{ext}_r K \qquad \text{or} \qquad \dim N(K,x) < (d-r+1)/2\,;$$

in particular, $\Sigma^r(K) \cap \beta = \emptyset$. This was first proved by Burago & Kalinin; compare Theorem 1.30 below.

Examples show that each of these three conclusions cannot be improved in general. We already mentioned in Subsection 1.1 that the third conclusion generalizes a classical result of Aleksandrov [1] which describes the influence of the Gauss curvature measure $C_0(K,\cdot)$ on the local shape of a non-smooth convex surface in $\mathbb{R}^3$. Such an extension has been asked for by Busemann [23] and Pogorelov [104]. It is perhaps worth pointing out that the third conclusion can be combined with the first part of the proof in [120] to yield a proof for the characterization of balls by condition (4).

Results for surface area measures which correspond to the first and the second conclusion have already been established; see [73] and [72]. The dual result for (intermediate) surface area measures, which corresponds to the third conclusion and thus to the theorem of Burago & Kalinin, is contained in the following new result. The set $\operatorname{extn}_r K$ of r-extreme normal vectors of K, which appears in the statement of the theorem and which by polarity corresponds to the set of r-extreme boundary points of K^*, is defined as the set of all $u \in S^{d-1}$ for which $\dim T(K,u) \le r+1$, where $T(K,u)$ is the touching cone of K at u. Again we refer to [123] for the details.

Theorem 1.19 *Let $K \in \mathcal{K}^d_o$ and $r \in \{1,\dots,d-1\}$. Let $\omega \subset S^{d-1}$ be open, and assume that there is a constant c such that*

$$S_r(K,\cdot) \llcorner \omega \le c\, S_0(K,\cdot) \llcorner \omega\,.$$

Then, for any $u \in \omega$,

$$u \notin \operatorname{extn}_{d-1-r} K \qquad \text{or} \qquad \dim F(K,u) < r/2\,.$$

Essentially two different proofs will be provided for Theorem 1.19, one of which is based on an integral-geometric characterization of absolute continuity with bounded density for surface area measures. In our opinion, this approach also leads to a simplified proof of the result of Burago and Kalinin. In any case, the second transfer principle plays a major rôle in both arguments, which will be provided.

In the last two subsections of Section 1 we are concerned with results that can be read independently of the preceding subsections, although they are related either by the methods of proof or the statements of results.

In Subsection 1.8 we improve the order of the stability function in Minkowski's uniqueness problem for a large class of convex bodies. For a detailed description of this subject and a review of the relevant literature we refer to the beginning of Subsection 1.8.

The investigation of absolute continuity which we described so far referred to $(d-1)$-dimensional Hausdorff measure as the dominating measure. However, the curvature measures of order r are naturally dominated by the r-dimensional Hausdorff measure, and the surface area measures of order r are dominated by the $(d-1-r)$-dimensional Hausdorff measure. This observation motivates the subjects which are investigated in Subsection 1.9. In particular, we present two theorems which show that support measures of convex sets can be viewed as suitably weighted Hausdorff measures, the weights depending on the strength of the singularities of the underlying convex set, at least as long as, roughly speaking, these measures are restricted to certain sets of σ-finite Hausdorff measure. Again we refer to the beginning of Subsection 1.9 for detailed references and background information. Included in this subsection is also a new integral-geometric proof of a version of the Besicovitch-Federer structure theorem in spherical space. A "simple" proof of the structure theorem in Euclidean space has recently been found by White [154]. The main new tool in our approach to the spherical version is an integral-geometric transformation formula which may have potential applications in stereology, too.

1.3 Absolute continuity and polarity

In the present subsection, we give a proof of the first transfer principle. The major problem here in treating polarity is that the map $K \mapsto K^*$ cannot be described by a tractable analytic expression. Therefore the idea is to pass to the normal bundles and to study instead a certain map $T : \mathcal{N}(K) \mapsto \mathcal{N}(K^*)$, which turns out to be much more convenient. The same map has independently been used for a different problem in [129].

For a convex body $K \in \mathcal{K}^d_{oo}$, we define the mapping

$$T : \mathcal{N}(K) \to \mathcal{N}(K^*)\,, \quad (x,u) \mapsto \left(\langle x,u\rangle^{-1}u, \frac{x}{|x|}\right)\,,$$

which is a bi-Lipschitz homeomorphism, and which is differentiable, if considered as a map from a neighbourhood of $\mathcal{N}(K)$ in $\mathbb{R}^{2d}$ into $\mathbb{R}^{2d}$. The mapping T is properly defined. To check this, let $\rho(L,\cdot)$ denote the radial function of $L \in \mathcal{K}^d_{oo}$. Choose any $(x,u) \in \mathcal{N}(K)$. Then $\langle x,u\rangle = h(K,u) = \rho(K^*,u)^{-1}$, and hence

$$\langle x,u\rangle^{-1}u = \rho(K^*,u)u \in \operatorname{bd} K^*\,.$$

In addition, we have

$$\left\langle \frac{x}{|x|}, \langle x,u\rangle^{-1}u \right\rangle = h\left(K^*, \frac{x}{|x|}\right)\,,$$

since this is equivalent to $\rho(K,x) = 1$. Thus $|x|^{-1}x \in N(K^*, \langle x,u\rangle^{-1}u) \cap S^{d-1}$. It is also easy to check that the inverse of T is given by

$$T^* : \mathcal{N}(K^*) \to \mathcal{N}(K)\,, \quad (x^*,u^*) \mapsto \left(\langle x^*,u^*\rangle^{-1}u^*, \frac{x^*}{|x^*|}\right)\,.$$

In the following, as a rule we will attach an asterisk to quantities which are associated with K^*. For example, we write $k_1^*(\cdot),\ldots,k_{d-1}^*(\cdot)$ for the (generalized) curvatures of K^* instead of the more elaborate notation $k_1(K^*,\cdot),\ldots,k_{d-1}(K^*,\cdot)$. Finally, we set $\mathrm{I}_{d-1} := \{1,\ldots,d-1\}$.

Now we are prepared for Proposition 1.20, which relates generalized curvatures of K to those of K^*. Basically, the equations which are asserted in this proposition result from counting one and the same quantity in two different ways.

Proposition 1.20 *Let $K \in \mathcal{K}^d_{oo}$. Then, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$,*

$$\operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i(x,u) = \infty\} = \operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i^*(T(x,u)) = 0\}$$

and

$$\operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i(x,u) = 0\} = \operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i^*(T(x,u)) = \infty\}\,.$$

Proof. In the proof we consider a pair $(x,u) \in \mathcal{N}(K)$ such that $x + \epsilon u \in \mathcal{D}_K$ for all $\epsilon > 0$. This condition is fulfilled for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$. For any such pair (x,u) an orthonormal basis of the $(d-1)$-dimensional linear subspace $\mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), (x,u)) \subset \mathbb{R}^d \times \mathbb{R}^d$ of $(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), d-1)$ approximate tangent vectors of $\mathcal{N}(K)$ at (x,u) is given by

$$w_i := \left(\frac{1}{\sqrt{1 + k_i(x,u)^2}} u_i, \frac{k_i(x,u)}{\sqrt{1 + k_i(x,u)^2}} u_i \right) , \quad i \in \{1, \dots, d-1\} ,$$

where the vectors $u_1, \dots, u_{d-1} \in S^{d-1}$ constitute a suitable orthonormal basis of $u^\perp$, and $k_1(x,u), \dots, k_{d-1}(x,u) \in [0, \infty]$ are the generalized curvatures of the unit normal bundle $\mathcal{N}(K)$. The generalized curvatures of $\mathcal{N}(K^*)$ at $T(x,u)$ are denoted by $k_1^*(T(x,u)), \dots, k_{d-1}^*(T(x,u))$. Since T is bi-Lipschitz, we can assume that $(x^*, u^*) := T(x,u)$ is such that $x^* + \epsilon u^* \in \mathcal{D}_{K^*}$ for all $\epsilon > 0$.

Let (x,u) be chosen as described. We also write T for the canonical extension of T to a neighbourhood of $\mathcal{N}(K)$ in $\mathbb{R}^{2d}$. In order to determine the special basis $DT(x,u)(w_i)$, $i \in \{1, \dots, d-1\}$, of $\mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K^*), T(x,u))$, we first determine the values $DT(x,u)(v,o)$ and $DT(x,u)(o,\bar{v})$ with $v, \bar{v} \in \mathbb{R}^d$ for the extended map T. By elementary calculus,

$$DT(x,u)(v,o) = \left(-\frac{\langle v,u \rangle}{\langle x,u \rangle^2} u, \frac{1}{|x|} \left[v - \left\langle \frac{x}{|x|}, v \right\rangle \frac{x}{|x|} \right] \right)$$

and

$$DT(x,u)(o,\bar{v}) = \left(\frac{1}{\langle x,u \rangle} \left[\bar{v} - \frac{\langle x, \bar{v} \rangle}{\langle x,u \rangle} u \right], o \right) .$$

Since $\langle u_i, u \rangle = 0$, we obtain for $i \in \{1, \dots, d-1\}$ that

$$DT(x,u)(w_i) = \left(\frac{k_i}{\sqrt{1+k_i^2}} \frac{1}{\langle x,u \rangle} \left[u_i - \frac{\langle x, u_i \rangle}{\langle x,u \rangle} u \right], \frac{1}{\sqrt{1+k_i^2}} \frac{1}{|x|} \left[u_i - \left\langle \frac{x}{|x|}, u_i \right\rangle \frac{x}{|x|} \right] \right) ,$$

where the argument (x,u) of k_i has been omitted. If we attach an asterisk to the corresponding expressions for K^*, another basis of $\mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K^*), T(x,u))$, in fact, the one which is usually considered, is given by

$$w_i^* = \left(\frac{1}{\sqrt{1+(k_i^*)^2}} u_i^*, \frac{k_i^*}{\sqrt{1+(k_i^*)^2}} u_i^* \right) , \quad i \in \{1, \dots, d-1\} ,$$

where the argument $T(x,u)$ of k_i^* has been omitted, and $(u_1^*, \dots, u_{d-1}^*)$ is a suitable orthonormal basis of $x^\perp$. From this representation it is easy to see that the integer

$$\mathrm{card}\, \{ i \in \mathrm{I}_{d-1} : k_i^*(T(x,u)) = \infty \}$$

equals the dimension of the kernel of the linear map π_1 which is given by

$$\pi_1 : \mathrm{lin}\{w_1^*, \dots, w_{d-1}^*\} \to x^\perp , \quad (y,z) \mapsto y .$$

Since the vectors

$$u_i - \frac{\langle x, u_i \rangle}{\langle x,u \rangle} u , \quad i \in \{1, \dots, d-1\} ,$$

are linearly independent, and since

$$\frac{k_i}{\sqrt{1+k_i^2}} \in (0,\infty)\,, \quad \text{if } k_i \in (0,\infty]\,,$$

the dimension of the kernel of π_1 is also equal to

$$\operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i(x,u) = 0\}\,.$$

To see this, recall that

$$\operatorname{lin}\{w_1^*,\dots,w_{d-1}^*\} = \operatorname{lin}\{DT(x,u)(w_1),\dots,DT(x,u)(w_{d-1})\}\,.$$

Now the remaining statement of the lemma follows since $K^{**} = K$. □

By combining the preceding proposition with results from [74], we can now establish the first transfer principle which was announced in the introduction.

Proof of Theorem 1.15. We continue to use the notation introduced in the proof of Proposition 1.20 and in the preceding remarks. Let as assume that

$$S_r(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega\,.$$

Hence, by Theorem 1.4, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ such that $u \in \omega$,

$$k_1(x,u) > 0 \quad \text{or} \quad k_{r+1}(x,u) = 0 \quad \text{or} \quad k_r(x,u) = \infty\,. \tag{15}$$

Denote by $\mathcal{N}_1 \subset \mathcal{N}(K)$ the set of all $(x,u) \in \mathcal{N}(K)$ such that $u \in \omega$ and (15) is violated. Then $\mathcal{H}^{d-1}(\mathcal{N}_1) = \mathcal{H}^{d-1}(T(\mathcal{N}_1)) = 0$. Let $\mathcal{N}_2$ be the set of all $(x,u) \in \mathcal{N}(K)$ such that at least one of the two relations of Proposition 1.20 is not satisfied. Again $\mathcal{H}^{d-1}(\mathcal{N}_2) = \mathcal{H}^{d-1}(T(\mathcal{N}_2)) = 0$, since T is bi-Lipschitz.

Recall the definition of the map

$$f : S^{d-1} \to \operatorname{bd} K^*\,, \quad u \mapsto h(K,u)^{-1}u\,,$$

choose $(x^*,u^*) \in \mathcal{N}(K^*) \setminus T(\mathcal{N}_1 \cup \mathcal{N}_2)$ such that $x^* \in f(\omega)$, and set $(x,u) := T^{-1}(x^*,u^*)$. Then $(x,u) \in \mathcal{N}(K) \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ and $u \in \omega$, because f is bijective and $f(u) \in f(\omega)$.

By relation (15) and using Proposition 1.20 thrice, we conclude that

$$k_{d-1}^*(x^*,u^*) < \infty \quad \text{or} \quad k_{d-r}^*(x^*,u^*) = 0 \quad \text{or} \quad k_{d-1-r}^*(x^*,u^*) = \infty\,.$$

Since $\mathcal{H}^{d-1}(T(\mathcal{N}_1 \cup \mathcal{N}_2)) = 0$, an application of Theorem 1.3 to the polar body K^* now yields that

$$C_{d-1-r}(K^*,\cdot) \llcorner f(\omega) \ll C_{d-1}(K^*,\cdot) \llcorner f(\omega)\,.$$

The reverse implication is proved in a similar manner. □

1.4 Bounded densities and polarity

In order to prove the second transfer principle, which deals with the case of bounded densities, it will be necessary to have sharp inequalities between elementary symmetric functions of principle curvatures of K^* and elementary symmetric functions of radii of curvature of K, at corresponding points. Such inequalities will be derived from the following more general theorem. Instead of an elementary symmetric function of radii of curvature, it involves a weighted sum of products of radii of curvature.

It is remarkable that although the assertion of Theorem 1.21 does not involve generalized curvatures on unit normal bundles, the proof essentially uses this concept. Furthermore, recall that for a convex body $K \in \mathcal{K}^d$ the reverse spherical image $\tau(K,u) = \tau_K(u)$ is well-defined for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$; see [123], pp. 77–78.

Theorem 1.21 *Let $K \in \mathcal{K}^d_{oo}$ and $l \in \{1,\dots,d-1\}$. Then, for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$,*

$$\binom{d-1}{l} H_l(K^*, h(K,u)^{-1}u) = \left\langle \frac{x}{|x|}, u \right\rangle^l \sum_{|I|=l} \left[1 - \sum_{i\in I} \left\langle \frac{x}{|x|}, u_i \right\rangle^2 \right] \prod_{i \in I} r_i(K,u)\,,$$

if $(u_1,\dots,u_{d-1})$ is a suitable orthonormal basis of $u^\perp$, $x := \tau_K(u)$, and the summation extends over all subsets $I \subset \{1,\dots,d-1\}$ of cardinality l.

Proof. Again we use the notation of the proof of Proposition 1.20. From the proofs of Lemma 3.1 in [72], applied to K^*, and Lemma 3.4 in [72], applied to K, as well as from the fact that $u \mapsto h(K,u)^{-1}u$, $u \in S^{d-1}$, is a bi-Lipschitz homeomorphism from S^{d-1} onto $\operatorname{bd} K^*$, we infer that for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$ the following conditions are simultaneously satisfied:

1. The support function $h(K,\cdot)$ of K is second order differentiable at u and $(\tau_K(u),u) \in \mathcal{N}(K)$ is such that $\tau_K(u) + \epsilon u \in \mathcal{D}_K$ for all $\epsilon > 0$.

2. The point $h(K,u)^{-1}u = \langle \tau_K(u), u\rangle^{-1}u$ of K^* is a normal boundary point, and hence $\langle x,u\rangle^{-1}u + \epsilon|x|^{-1}x \in \mathcal{D}_{K^*}$ for all $\epsilon > 0$, if $x := \tau_K(u)$.

Let us fix one such $u \in S^{d-1}$, and set $x := \tau_K(u)$ and $(x^*,u^*) := T(x,u)$. Then by the proof of Lemma 3.4 in [72], we especially get that

$$k_i := k_i(x,u) > 0\,, \quad i \in \{1,\dots,d-1\}\,;$$

moreover, by Lemma 3.1 in [72],

$$k_i^* := k_i^*(x^*,u^*) < \infty\,, \quad i \in \{1,\dots,d-1\}\,.$$

Also note that again by Lemmas 3.1 and 3.4 in [72],

$$\mathbb{H}_l(K^*, T(x,u)) = H_l(K^*, x^*)\,, \quad x^* = h(K,u)^{-1}u\,, \tag{16}$$

and

$$k_i(x,u)^{-1} = r_i(K,u)\,, \quad i \in \{1,\dots,d-1\}\,. \tag{17}$$

Hence the proof of Proposition 1.20 implies that

$$\left(u_i - \frac{\langle x, u_i\rangle}{\langle x, u\rangle}u, \frac{1}{k_i}\left\langle \frac{x}{|x|}, u\right\rangle \left[u_i - \left\langle \frac{x}{|x|}, u_i\right\rangle \frac{x}{|x|}\right]\right), \quad i \in \{1, \ldots, d-1\},$$

is a basis of $\operatorname{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K^*), (x^*, u^*))$. Observe that the case $k_i = \infty$ is not excluded. Define

$$a_i := u_i - \frac{\langle x, u_i\rangle}{\langle x, u\rangle}u \quad \text{and} \quad b_i := \frac{1}{k_i}\left\langle \frac{x}{|x|}, u\right\rangle \left[u_i - \left\langle \frac{x}{|x|}, u_i\right\rangle \frac{x}{|x|}\right],$$

for $i \in \{1, \ldots, d-1\}$. Note that the vectors $a_1, \ldots, a_{d-1} \in x^\perp$ are linearly independent. The linear mapping $\varphi : x^\perp \to x^\perp$, defined by

$$\varphi(a_i) := b_i, \quad i \in \{1, \ldots, d-1\},$$

can also be determined by prescribing that

$$\varphi\left(\frac{1}{\sqrt{1+(k_i^*)^2}}u_i^*\right) = \frac{k_i^*}{\sqrt{1+(k_i^*)^2}}u_i^*, \quad i \in \{1, \ldots, d-1\}.$$

To check this one can use that $(a_1, \ldots, a_{d-1})$ and

$$\left(\left(1+(k_1^*)^2\right)^{-\frac{1}{2}} u_1^*, \ldots, \left(1+(k_{d-1}^*)^2\right)^{-\frac{1}{2}} u_{d-1}^*\right)$$

are two bases of $x^\perp$ and that

$$\operatorname{lin}\{w_1^*, \ldots, w_{d-1}^*\} = \operatorname{lin}\{(a_1, b_1), \ldots, (a_{d-1}, b_{d-1})\}.$$

Therefore, the linear mapping φ has the eigenvalues $k_1^*, \ldots, k_{d-1}^*$. These eigenvalues are the zeros of the characteristic polynomial

$$t \mapsto \det(B - t\,E_{d-1}), \quad t \in \mathbb{R},$$

where E_{d-1} is the unit $(d-1)$-by-$(d-1)$ matrix, and the matrix $B = (\beta_{ij})$, $i, j \in \{1, \ldots, d-1\}$, is defined by the relations

$$b_j = \sum_{i=1}^{d-1} \beta_{ij} a_i, \quad j \in \{1, \ldots, d-1\}. \tag{18}$$

Substituting the expressions for a_i and b_j into (18), we arrive at

$$\frac{1}{k_j}\left\langle \frac{x}{|x|}, u\right\rangle \left[u_j - \left\langle \frac{x}{|x|}, u_j\right\rangle \frac{x}{|x|}\right] = \sum_{i=1}^{d-1} \beta_{ij}\left[u_i - \frac{\langle x, u_i\rangle}{\langle x, u\rangle}u\right]. \tag{19}$$

Since $(u_1, \ldots, u_{d-1}, u)$ is an orthonormal basis of $\mathbb{R}^d$, we have

$$\frac{x}{|x|} = \sum_{k=1}^{d-1}\left\langle \frac{x}{|x|}, u_k\right\rangle u_k + \left\langle \frac{x}{|x|}, u\right\rangle u. \tag{20}$$

If we use (20) for the unit vector $|x|^{-1}x$ within the bracket on the left-hand side of equation (19), a comparison of the coefficients of $u_1, \dots, u_{d-1}$ then yields, for $i, j \in \{1, \dots, d-1\}$, that

$$\beta_{ij} = \frac{\left\langle \frac{x}{|x|}, u \right\rangle}{k_j} \left\{ \delta_{ij} - \left\langle \frac{x}{|x|}, u_i \right\rangle \left\langle \frac{x}{|x|}, u_j \right\rangle \right\} .$$

Here, as usual, δ_{ij} denotes the Kronecker symbol. Moreover, for an arbitrary subset $I \subset \{1, \dots, d-1\}$ with $|I| = l$, we set

$$B_I := (\beta_{jk})_{j,\, k \in I} \; ;$$

thus the matrices B_I are the principal minors of order l of the matrix B. Furthermore, we know that $\binom{d-1}{l}\mathbb{H}_l(K^*, (x^*, u^*))$ can be calculated as the sum of these principal minors. Thus we obtain from (16) that

$$\begin{aligned} \binom{d-1}{l} H_l(K^*, x^*) &= \sum_{|I|=l} B_I \\ &= \sum_{|I|=l} \left\langle \frac{x}{|x|}, u \right\rangle^l \left(\prod_{i \in I} k_i \right)^{-1} \det \left(\left(\delta_{jk} - \left\langle \frac{x}{|x|}, u_j \right\rangle \left\langle \frac{x}{|x|}, u_k \right\rangle \right)_{j,\, k \in I} \right) \\ &= \sum_{|I|=l} \left\langle \frac{x}{|x|}, u \right\rangle^l \left(\prod_{i \in I} k_i \right)^{-1} \left[1 - \sum_{j \in I} \left\langle \frac{x}{|x|}, u_j \right\rangle^2 \right] . \end{aligned}$$

An application of (17) then implies the theorem. □

The following Corollary 1.22, which is an immediate consequence of Theorem 1.21, can not only be used to characterize absolute continuity with bounded density in terms of polarity, but it also leads to a characterization of the case where the measures are purely singular; see Corollary 1.23 below.

Corollary 1.22 *Let $K \in \mathcal{K}^d_{oo}$ and $l \in \{0, \dots, d-2\}$. Then, for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$,*

$$\left\langle \frac{x}{|x|}, u \right\rangle^{l+2} D_l h(K, u) \le H_l(K^*, h(K,u)^{-1}u) \le \left\langle \frac{x}{|x|}, u \right\rangle^{l} D_l h(K, u) ,$$

where $x := \tau_K(u)$. In addition, for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$,

$$H_{d-1}(K^*, h(K,u)^{-1}u) = \left\langle \frac{x}{|x|}, u \right\rangle^{d+1} D_{d-1} h(K, u) .$$

Remark. The special case $l = d-1$ of the preceding theorem and its corollary has already been established in [71] by a completely different method of proof. However, it does not seem to be possible to extend the approach of [71] to cover the present situation.

Corollary 1.23 *Let $K \in \mathcal{K}^d_{oo}$, $\omega \in \mathfrak{B}(S^{d-1})$, and $r \in \{0, \dots, d-1\}$. Then*

$$S_r(K, \cdot) \llcorner \omega = S^s_r(K, \cdot) \llcorner \omega$$

if and only if

$$C_{d-1-r}(K^*, \cdot) \llcorner f(\omega) = C^s_{d-1-r}(K^*, \cdot) \llcorner f(\omega) .$$

After these preparations it is now easy to provide a proof for the second transfer principle by just combining what we have proved so far.

Proof of Theorem 1.16. Assume that there is a constant c such that

$$S_r(K,\cdot) \llcorner \omega \le c\, S_0(K,\cdot) \llcorner \omega \,.$$

Then $S_r(K,\cdot) \llcorner \omega$ is absolutely continuous with respect to $S_0(K,\cdot) \llcorner \omega$, and $D_r h(K,u) \le c$ is satisfied for $\mathcal{H}^{d-1}$ almost all $u \in \omega$. According to Theorem 1.15, $C_{d-1-r}(K^*,\cdot) \llcorner f(\omega)$ is absolutely continuous with respect to $C_{d-1}(K^*,\cdot) \llcorner f(\omega)$ and, for $\mathcal{H}^{d-1}$ almost all $x^* \in f(\omega)$, the density is given by $H_r(K^*,x^*)$. Now Corollary 1.22 implies that $H_r(K^*,x^*) \le c$ for $\mathcal{H}^{d-1}$ almost all $x^* \in f(\omega)$. This finally shows that

$$C_{d-1-r}(K^*,\cdot) \llcorner f(\omega) \le c^*\, C_{d-1}(K^*,\cdot) \llcorner f(\omega)$$

is satisfied with $c^* = c$.

Similarly, the reverse implication follows from the inequality on the left-hand side of Corollary 1.22. In fact, let $r, R \in (0,\infty)$ be chosen in such a way that $B^d(o,r) \subset K \subset B^d(o,R)$; hence $\langle |x|^{-1}x, u\rangle \ge r/R$, for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$ and $x = \tau_K(u)$, and we can proceed as before. □

The following theorem has been established by Weil [144]. Its proof is based on a sophisticated convolution procedure which is applied to the restriction of the support function of a given convex body to properly chosen hyperplanes. Such a procedure is necessary in order to be able to exert control over the radii of curvature of a suitably constructed sequence of approximating smooth convex bodies.

Theorem 1.24 (Weil) *Let $K \in \mathcal{K}^d_o$, and let ω be an open subset of S^{d-1}.*

(a) *Assume that*

$$S_1(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

and further assume that $D_1 h(K,\cdot) \in \mathrm{L}^p(\omega)$ for some $p \in [1,\infty)$. Then

$$S_j(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

and $D_j h(K,\cdot) \in \mathrm{L}^{\left[\frac{p}{j}\right]}(\omega)$ for $j \in \{1,\dots,[p]\}$.

(b) *Assume that*

$$S_1(K,\cdot) \llcorner \omega \le c\, S_0(K,\cdot) \llcorner \omega$$

for some constant $c > 0$. Then

$$S_j(K,\cdot) \llcorner \omega \le c^j\, S_0(K,\cdot) \llcorner \omega$$

for $j \in \{1,\dots,d-1\}$.

The corresponding result for curvature measures is stated as Theorem 1.17 in Subsection 1.2. It will be implied by our first transfer principle, Corollary 1.22 and Theorem 1.24. Of course, it is conceivable to deduce Theorem 1.17 by a more direct application of a convolution procedure to the distance function of the convex body K. In this way, Theorem 1.24 might then be obtained by arguments similiar to those presented in the proof of Theorem 1.17 below. On the other hand, such a direct proof of Theorem 1.17 probably requires results analogous to Satz 1.1 and Satz 4.1 in [144]. But then one encounters the problem that the curvatures of a sequence of smooth convex bodies, which approximate a given convex body, are defined on different domains. Furthermore, the proof of Satz 4.1 in [144] exploits the connection of surface area measures to mixed volumes and such a relationship is not available for curvature measures.

Proof of Theorem 1.17. It is sufficient to consider the case $K \in \mathcal{K}^d_o$, since the curvature measures are locally defined. Furthermore, since all notions involved in Theorem 1.24 and Theorem 1.17 are invariant with respect to translations, we can assume that $o \in \operatorname{int} K$. Consider the maps

$$\eta : \operatorname{bd} K \to S^{d-1}\,, \quad x \mapsto |x|^{-1}x\,,$$

and

$$f^* : S^{d-1} \to \operatorname{bd} K\,, \quad u \mapsto \rho(K,u)u\,,$$

which are bi-Lipschitz homeomorphisms that are inverse to each other. Let $\omega := \eta(\operatorname{bd} K \cap \beta)$. Then $\omega \subset S^{d-1}$ is an open subset of S^{d-1}. The assumptions of (a) and Theorem 1.15, applied to K^*, imply that $S_1(K^*,\cdot) \llcorner \omega \ll S_0(K^*,\cdot) \llcorner \omega$. Moreover, an application of Corollary 1.22 to K^* yields, for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$ and $l \in \{0, \dots, d-1\}$, that

$$\left\langle \frac{x}{|x|}, \sigma_K(x) \right\rangle^{l+2} D_l h\left(K^*, \frac{x}{|x|}\right) \le H_l(K,x) \le D_l h\left(K^*, \frac{x}{|x|}\right). \tag{21}$$

Let $r, R \in (0,\infty)$ be such that $B^d(o,r) \subset K \subset B^d(o,R)$. Then, for $(x,u) \in \mathcal{N}(K)$,

$$\left(\left\langle \frac{x}{|x|}, u \right\rangle\right)^{-1} \le \frac{R}{r} =: c\,,$$

and hence we obtain, for $l \in \{0, \dots, d-1\}$ and $q > 0$, that

$$\begin{aligned}
&\int_\omega D_l h(K^*, u)^q \, \mathcal{H}^{d-1}(du) \\
&\le c^{(l+2)q} \int_\omega H_l(K, f^*(u))^q \, \mathcal{H}^{d-1}(du) \\
&= c^{(l+2)q} \int_\omega H_l(K, f^*(u))^q \frac{\langle u, \sigma_K(f^*(u))\rangle}{\rho(K,u)^{d-1}} J_{d-1} f^*(u) \, \mathcal{H}^{d-1}(du) \\
&= c^{(l+2)q} \int_{\mathrm{bd}\, K \cap \beta} H_l(K, x)^q \frac{\left\langle \frac{x}{|x|}, \sigma_K(x) \right\rangle}{|x|^{d-1}} \, \mathcal{H}^{d-1}(dx) \\
&\le \frac{c^{(l+2)q}}{r^{d-1}} \int_{\mathrm{bd}\, K \cap \beta} H_l(K, x)^q \, \mathcal{H}^{d-1}(dx) \\
&\le \frac{c^{(l+2)q}}{r^{d-1}} \int_{\mathrm{bd}\, K \cap \beta} D_l h\left(K^*, \frac{x}{|x|}\right)^q \, \mathcal{H}^{d-1}(dx) \\
&= \frac{c^{(l+2)q}}{r^{d-1}} \int_\omega D_l h(K^*, u)^q J_{d-1} f^*(u) \, \mathcal{H}^{d-1}(du) \\
&\le \frac{c^{(l+2)q+1} R^{d-1}}{r^{d-1}} \int_\omega D_l h(K^*, u)^q \, \mathcal{H}^{d-1}(du) \,.
\end{aligned}$$

Here we have used Lemma 3.1 from [71]. This shows that

$$D_l h(K^*, \cdot) \in \mathrm{L}^q(\omega) \Longleftrightarrow H_l(K, \cdot) \in \mathrm{L}^q(\mathrm{bd}\, K \cap \beta) \,. \tag{22}$$

Hence we get that $D_1 h(K^*, \cdot) \in \mathrm{L}^p(\omega)$, and thus Weil's result (Theorem 1.24) yields that, for $j \in \{1, \dots, [p]\}$,

$$S_j(K^*, \cdot) \llcorner \omega \ll S_0(K^*, \cdot) \llcorner \omega \quad \text{and} \quad D_j h(K^*, \cdot) \in \mathrm{L}^{\left[\frac{p}{j}\right]}(\omega) \,.$$

Again from Theorem 1.15 we then conclude that, for $j \in \{1, \dots, [p]\}$,

$$C_{d-1-j}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta \,,$$

and another application of equation (22) then completes the proof of (a).

The proof of (b) follows from (a) and from Newton's inequalities for elementary symmetric functions (see [65], [112]). □

Example. The result of Theorem 1.24 (a) cannot be improved in general. To see this, let $r, R > 0$ and define $X : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

$$X((\vartheta, \varphi)) := ((R + \sin\vartheta)\cos\varphi, (R + \sin\vartheta)\sin\varphi, \cos\vartheta) \,.$$

Then $K := \mathrm{conv}(X([0, \pi] \times [0, 2\pi]))$ is the convex hull of a torus. From Theorem 1.4 it easily follows that $S_1(K, \cdot)$ is absolutely continuous with respect to $S_0(K, \cdot)$. In addition,

$D_1 h(K,\cdot) \in \mathrm{L}^p(S^2)$ for each $p \in [1,2)$. In fact, the principal radii of curvature of K are given by

$$r_1(T(\vartheta,\varphi)) = r\,, \quad r_2(T(\vartheta,\varphi)) = \frac{R + r\sin\vartheta}{\sin\vartheta}\,,$$

where $T : (0,\pi) \times [0,2\pi] \to \mathbb{R}^3$ is defined by

$$T(\vartheta,\varphi) := (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\,.$$

Now we obtain

$$\begin{aligned} I(p) &:= \int_{S^2} (r_1(u) + r_2(u))^p\, \mathcal{H}^2(du) \\ &= 2\pi \int_0^\pi \left[2r(\sin\vartheta)^{\frac{1}{p}} + R(\sin\vartheta)^{\frac{1-p}{p}}\right]^p d\vartheta\,. \end{aligned}$$

If $p \in [1,2)$, it follows that

$$I(p) \le 2^p\pi \int_0^\pi \left[(2r)^p \sin\vartheta + R^p(\sin\vartheta)^{1-p}\right] d\vartheta\,.$$

The integral on the right hand side is finite, since $p-1<1$ and

$$\lim_{\vartheta\to 0} \left[(\sin\vartheta)^{1-p}\vartheta^{p-1}\right] = 1\,.$$

But for $p \ge 2$ one obtains $I(p) = \infty$. On the other hand, $S_2(K,\cdot)$ even has point masses.

By polarity a corresponding example for curvature measures is obtained.

1.5 Surface area measures: bounded densities

This section is mainly devoted to the investigation of absolute continuity with bounded density for surface area measures of convex bodies. In Theorem 1.26 we will provide an integral-geometric characterization of absolute continuity with bounded density which is strongly related to Theorem 1.9. As a preliminary step we need a lemma on the weak convergence of intermediate surface area measures of projected convex bodies. In order to indicate that the ith surface area measure of the projection $K|V$ of K onto $V \in \mathbf{G}(d,j)$ has to be calculated with respect to the subspace V, we write $S_i^V(K|V,\cdot)$, which is a measure on $\mathfrak{B}(S^{d-1}\cap V)$.

Lemma 1.25 *Let $K \in \mathcal{K}^d$, $i \in \{0,\dots,d-1\}$, and $j \in \{i+1,\dots,d\}$. Then, for each $V \in \mathbf{G}(d,j)$, the map $\omega \mapsto S_i^V(K|V,\omega\cap V)$, $\omega \in \mathfrak{B}(S^{d-1})$, is a Borel measure over S^{d-1}. In addition, if $V_l, V \in \mathbf{G}(d,j)$, $K_l \in \mathcal{K}^d$, $l \in \mathbb{N}$, $V_l \to V$ and $K_l \to K$, as $l\to\infty$, then*

$$S_i^{V_l}(K_l|V_l,\cdot\cap V_l) \xrightarrow{w} S_i^V(K|V,\cdot\cap V)\,,$$

as $l\to\infty$, in the sense of the weak convergence of measures over S^{d-1}.

Proof. Fix $V \in \mathbf{G}(d,j)$ and consider $\rho_l, \rho \in \mathbf{O}(d)$, $l \in \mathbb{N}$, with $\rho_l \to \rho$ as $l\to\infty$. We have to show that

$$S_i^{\rho_l V}(K_l|\rho_l V,\cdot\cap\rho_l V) \xrightarrow{w} S_i^{\rho V}(K|\rho V,\cdot\cap\rho V)$$

as Borel measures over S^{d-1}; in other words, it has to be proved that

$$\lim_{l\to\infty}\int_{S^{d-1}\cap\rho_l V} f(u)\,S_i^{\rho_l V}(K_l|\rho_l V, du) = \int_{S^{d-1}\cap\rho V} f(u)\,S_i^{\rho V}(K|\rho V, du) \tag{23}$$

is satisfied for all $f \in C(S^{d-1})$. Let $\omega \in \mathfrak{B}(S^{d-1})$ and $L \in \mathcal{K}^d$; then

$$\begin{aligned} S_i^{\rho V}(L|\rho V, \omega\cap\rho V) &= S_i^V(\rho^{-1}L|V, \rho^{-1}\omega\cap V) \\ &= \int_{S^{d-1}\cap V} \mathbf{1}_\omega(\rho u)\,S_i^V(\rho^{-1}L|V, du)\,, \end{aligned}$$

and hence, for any $f \in C(S^{d-1})$,

$$\int_{S^{d-1}\cap\rho V} f(u)\,S_i^{\rho V}(L|\rho V, du) = \int_{S^{d-1}\cap V} f(\rho u)\,S_i^V(\rho^{-1}L|V, du)\,. \tag{24}$$

Note that $\rho_l^{-1}K_l|V \to \rho^{-1}K|V$, as $l\to\infty$, with respect to the Hausdorff metric in V. This yields

$$S_i^V(\rho_l^{-1}K_l|V,\cdot) \xrightarrow{w} S_i^V(\rho^{-1}K|V,\cdot) \tag{25}$$

as Borel measures over $S^{d-1}\cap V$, and thus

$$\lim_{l\to\infty}\int_{S^{d-1}\cap V} f(\rho u)\,S_i^V(\rho_l^{-1}K_l|V, du) = \int_{S^{d-1}\cap V} f(\rho u)\,S_i^V(\rho^{-1}K|V, du)\,, \tag{26}$$

since $u \mapsto f(\rho u)$, $u \in S^{d-1}\cap V$, is a continuous function independent of $l \in \mathbb{N}$.

From (25) and the fact that f is uniformly continuous on S^{d-1}, it follows that

$$\lim_{l\to\infty}\int_{S^{d-1}\cap V} |f(\rho_l u) - f(\rho u)|\,S_i^V(\rho_l^{-1}K_l|V, du) = 0\,. \tag{27}$$

But then (23) is implied by (24), (26), (27) and the triangle inequality. □

Theorem 1.26 *Let $K \in \mathcal{K}^d$, $i \in \{1,\dots,d-1\}$, $j \in \{i+1,\dots,d\}$, and let $\omega \subset S^{d-1}$ be open. Then the following conditions are equivalent.*

(a) *There is a constant c such that*

$$S_i(K,\cdot)\llcorner\omega \le c\,S_0(K,\cdot)\llcorner\omega\,.$$

(b) *There is a constant $\tilde c$ such that, for ν_j almost all $V \in \mathbf{G}(d,j)$,*

$$S_i^V(K|V,\cdot)\llcorner(\omega\cap V) \le \tilde c\,S_0^V(K|V,\cdot)\llcorner(\omega\cap V)\,.$$

(c) *There is a constant $\tilde c$ such that, for all $V \in \mathbf{G}(d,j)$,*

$$S_i^V(K|V,\cdot)\llcorner(\omega\cap V) \le \tilde c\,S_0^V(K|V,\cdot)\llcorner(\omega\cap V)\,.$$

Remarks.

1. If the constant c in (a) is given, then $\tilde{c}$ in (b) and (c) can be chosen so that
$$\tilde{c} \leq \frac{\binom{d-1}{i}}{\binom{j-1}{i}} c\,,$$
and if the constant $\tilde{c}$ in (b) or (c) is given, then we can choose the constant c in (a) so that $c \leq \tilde{c}$.

2. It should be emphasized that a result for curvature measures, which is strictly analogous to Theorem 1.26, in the sense that the analogue of condition (c) applies to *all* planes of a certain dimension intersecting the interior of the convex body, is not true. A related problem arises in the proof of Theorem 1.30 which is provided in [22]. To cope with this difficulty, Burago and Kalinin introduce the notion of an "α-transversal" hyperplane. We will return to this subject in the following subsection.

Proof of Theorem 1.26. We can assume that $i \leq d-2$ and $j \leq d-1$.

(a) $\Rightarrow$ (b). By Theorem 1.9 we obtain that, for ν_j almost all $V \in \mathbf{G}(d,j)$,
$$S_i^V(K|V,\cdot) \llcorner (\omega \cap V) \ll S_0^V(K|V,\cdot) \llcorner (\omega \cap V)\,.$$
For ν_j almost all $V \in \mathbf{G}(d,j)$, the support function h_K is second order differentiable at $\mathcal{H}^{j-1}$ almost all $u \in S^{d-1} \cap V$. This can be deduced from Lemma 4.1 in [74]. Let u be one such unit vector; then we get
$$\binom{d-1}{i} D_i h(K,u) \geq \binom{j-1}{i} D_i^V h(K|V,u)\,.$$
This inequality is a consequence of the Courant-Fischer "min-max theorem" (see Theorem 4.3.15 in [67]) and the fact that all radii of curvature are nonnegative. Hence, (a) implies that (b) holds with a positive constant $\tilde{c}$, as described in the preceding remark.

(b) $\Rightarrow$ (c). Assume that condition (b) is fulfilled. Let $V_l, V \in \mathbf{G}(d,j)$, $l \in \mathbb{N}$, be such that $V_l \to V$ as $l \to \infty$ and (b) is satisfied for all V_l, $l \in \mathbb{N}$. Let $f \in C(S^{d-1})$ be nonnegative and assume that $\operatorname{spt} f \subset \omega$. Then we have
$$\int_{S^{d-1}\cap V_l} f(u)\, S_i^{V_l}(K|V_l,du) \leq \tilde{c} \int_{S^{d-1}\cap V_l} f(u)\, S_0^{V_l}(K|V_l,du)\,,$$
for all $l \in \mathbb{N}$, and hence Lemma 1.25 with $K_l = K$ for $l \in \mathbb{N}$ implies that the same estimate holds, if V is substituted for V_l. But then Lemma 7.2.7 in [28] yields that
$$S_i^V(K|V,\alpha \cap V) \leq \tilde{c}\, S_0^V(K|V,\alpha \cap V)$$
is true for arbitrary open subsets α of ω. Condition (c) then follows from another application of the Borel regularity of these measures.

(c) $\Rightarrow$ (a). This is immediately implied by relation (4.5.26) in [123]. $\square$

Now the first part of the following theorem can easily be inferred. The converse statement, saying that
$$S_1(K,\cdot) \leq c\, S_0(K,\cdot)$$
if K is a summand of the ball $B^d(o,c)$, immediately follows from the additivity of the first surface area measure; therefore we do not include it as part of the theorem.

Theorem 1.27 *Let $K \in \mathcal{K}^d$, and assume that*

$$S_1(K,\cdot) \le c\, S_0(K,\cdot)\,,$$

where c is a constant. Then K is a summand of the ball $B^d(o,(d-1)c)$; in general, the radius of this ball cannot be reduced further.

Proof. By an application of Theorem 1.26 and by the preceding remark we obtain that, for each $V \in \mathbf{G}(d,2)$,

$$S_1^V(K|V,\cdot\cap V) \le (d-1)c\, S_0^V(K|V,\cdot\cap V)\,.$$

Then Minkowski's theorem (see Theorem 7.1.1 in [123]), applied in the plane V, yields that $K|V$ is a summand of the ball $B^d(o,(d-1)c)|V$. Since this is true for all $V \in \mathbf{G}(d,2)$, Lemma 3.2.6 in [123] implies that K is a summand of $B^d(o,(d-1)c)$.

In order to see that, as far as the radius of this ball is concerned, the result cannot be improved in general, we consider an ellipsoid $\mathcal{E}(a,b)$ of revolution such that the lengths of its semiaxes are $a_1 = a$ and $a_2 = \ldots = a_d = b$, where $a,b>0$. By choosing suitable coordinates we obtain that $\mathcal{E}(a,b) = A\cdot B^d(o,1)$, where $A=(a_{ij})$ is a real d-by-d matrix with $a_{ii}=a_i$, for $i \in \{1,\ldots,d\}$, and $a_{ij}=0$, for $i\neq j$ and $i,j \in \{1,\ldots,d\}$. An elementary calculation then yields for the second derivatives of the support function $h := h(\mathcal{E}(a,b),\cdot)$ at $x \in \mathbb{R}^d \setminus \{o\}$ that

$$\frac{\partial^2 h}{\partial x_i \partial x_j}(x) = \frac{a^2b^2}{|Ax|^3}\cdot \begin{cases} \displaystyle\sum_{k=2}^{d} x_k^2\,, & i=j=1\,.\\ \displaystyle x_1^2 + \frac{b^2}{a^2}\sum_{k=2,\,k\neq i}^{d} x_k^2\,, & i=j\neq 1\,.\\ -x_1x_j\,, & i=1\neq j\,.\\ \displaystyle -\frac{b^2}{a^2}x_ix_j\,, & 1\neq i\neq j\neq 1\,. \end{cases}$$

In order to determine the principal radii of curvature of $\mathcal{E}(a,b)$, we can restrict ourselves to the case where $x=(x_1,x_2,0,\ldots,0)$ and $|x|=1$, since $\mathcal{E}(a,b)$ has rotational symmetry. In addition, the eigenvectors $u_2,\ldots,u_d \in S^{d-1}$ of the reverse Weingarten map $\overline{W}_x$ of $\mathcal{E}(a,b)$ at x are equal to the eigenvectors of the Weingarten map of $\mathcal{E}(a,b)$ at the uniquely determined boundary point of $\mathcal{E}(a,b)$ with exterior unit normal vector u (compare [123, §2.5]). The latter are given by

$$u_2 = (x_2,-x_1,0,\ldots,0) \quad \text{and} \quad u_i = e_i\,, \quad i\in\{3,\ldots,d\}\,;$$

see, e.g., [137, Chapter 3, IV]. Finally, Lemma 2.5.1 in [123] and some further calculations yield for the corresponding principal radii of curvature $r_2(x),\ldots,r_d(x)$ of $\mathcal{E}(a,b)$ at x that

$$r_2(x) = \overline{\mathrm{II}}_x(u_2,u_2) = \frac{a^2b^2}{\sqrt{a^2x_1^2+b^2x_2^2}^{\,3}}$$

and

$$r_i(x) = \overline{\mathrm{II}}_x(u_i,u_i) = \frac{b^2}{\sqrt{a^2x_1^2+b^2x_2^2}}\,, \quad i\in\{3,\ldots,d\}\,.$$

Choosing $a > b$, we obtain that the principal radii of curvature of this ellipsoid can be estimated by

$$r_2(x) \le \frac{a^2}{b} \quad \text{and} \quad r_i(x) \le b\,, \quad i \in \{3, \dots, d\}\,,$$

for any $x \in S^{d-1}$, and the upper bounds are simultaneously attained for a suitable choice of $x \in S^{d-1}$. Therefore

$$c = \frac{1}{d-1}\left[\frac{a^2}{b} + (d-2)b\right]$$

is the smallest positive constant such that $S_1(K,\cdot) \le c\,S_0(K,\cdot)$ is satisfied. In addition, the ellipsoid $\mathcal{E}(a,b)$ is a summand of the ball $B^d(o, a^2/b)$ and no smaller radius can be chosen. The ratio of this smallest radius and $(d-1)c$ is always less or equal 1, and it approaches 1 as $a \to \infty$. This justifies the second statement of the theorem. □

The following theorem extends a result of Weil [144], Satz 4.7 (a), to a local situation. We start with a simple lemma which is based on the following definition. Let $K \in \mathcal{K}^d_o$ and $R > 0$; then we define the closed set

$$S(K,R) := \left\{u \in S^{d-1} : K \subset B^d(x - Ru, R) \text{ for some } x \in F(K,u)\right\}.$$

On $S(K,R)$ the support function of K is differentiable and its gradient coincides with the reverse spherical image map of K. A dual notion will be introduced in the following subsection.

Lemma 1.28 *Let $K \in \mathcal{K}^d_o$ and $R > 0$. Then the restriction of the reverse spherical image map τ_K to $S(K,R)$ is Lipschitz with Lipschitz constant less or equal R.*

Proof. Let $u, v \in S(K,R)$, and set $x := \nabla h_K(u)$, $y := \nabla h_K(v)$. Then we get $y \in B^d(x - Ru, R)$. This implies

$$|y - x|^2 \le 2R\langle x - y, u\rangle\,; \tag{28}$$

by symmetry we also have

$$|x - y|^2 \le 2R\langle y - x, v\rangle\,. \tag{29}$$

Addition of (28) and (29) yields

$$2|y-x|^2 \le 2R\langle x - y, u - v\rangle \le 2R|x-y|\,|u-v|\,,$$

and this proves the statement. □

Theorem 1.29 *Let $K \in \mathcal{K}^d$, let $\omega \subset S^{d-1}$ be open, and assume that there is a constant c such that*

$$S_1(K,\cdot) \llcorner \omega \le c\,S_0(K,\cdot) \llcorner \omega\,.$$

Then, for each compact subset ω' of ω, there is some $R < \infty$ such that $K \subset B^d(\tau_K(u) - Ru, R)$ is satisfied for all $u \in \omega'$; in particular, h_K is locally of class $C^{1,1}$ on ω.

Proof. Throughout the proof, we write $\bar{\alpha}$ for the topological closure of the set α. First, let ω_1 be an open spherically convex set with $\bar{\omega}_1 \subset \omega$ such that there are open spherically convex sets ω_2, ω_3 with $\bar{\omega}_1 \subset \omega_2$, $\bar{\omega}_2 \subset \omega_3$, and $\bar{\omega}_3 \subset \omega$. For $i \in \{2, 3\}$ define $\Omega_i := \{tu : t \geq 0,\, u \in \omega_i\}$. By the proof of Satz 4.7 in [144] or by Satz 7.1 and the subsequent remark in Weil [145], it follows that there is a positive constant R such that the function $f : \Omega_3 \to \mathbb{R}$, defined by $f := h_{B^d(o,R)} - h_K$ on Ω_3, is convex (we can choose $R = (d-1)c$). In particular, f is positively homogeneous of degree 1 on the convex cone Ω_3. In the following we use the notion of subdifferential as introduced by Clarke [27]. The results of §§2.2-3 in [27] imply that the relation

$$\partial h_K + \partial f = \partial h_{B^d(o,R)}$$

is satisfied on $\Omega_3 \setminus \{o\}$. But then Proposition 2.2.4 in [27] yields that h_K and f are continuously differentiable on $\Omega_3 \setminus \{o\}$.

Now, for $y \in \mathbb{R}^d$, we define the function

$$g(y) := \sup\{\langle y, \nabla f(x)\rangle : x \in \omega_2\}\,.$$

Since ∇f is continuous on $\bar{\omega}_2 \subset \omega_3$, we have $|g(y)| < \infty$ for all $y \in \mathbb{R}^d$. By definition, g is convex and positively homogeneous of degree 1. Since f is positively homogeneous of degree 1 on Ω_3 and since

$$f(tx) - f(x) \geq \langle tx - x, \nabla f(x)\rangle\,,$$

for all $t > 0$ and $x \in \Omega_3 \setminus \{o\}$, we obtain that

$$f(x) = \langle x, \nabla f(x)\rangle\,, \quad x \in \Omega_3 \setminus \{o\}\,.$$

But this implies that $g|_{\Omega_2} = f|_{\Omega_2}$. In fact, let $x \in \omega_2$; then

$$g(x) \geq \langle x, \nabla f(x)\rangle = f(x)\,,$$

and thus $g \geq f$ on Ω_2. In addition, for any $x, \tilde{x} \in \omega_2$,

$$\langle x, \nabla f(\tilde{x})\rangle = \langle x - \tilde{x}, \nabla f(\tilde{x})\rangle + f(\tilde{x}) \leq f(x)\,,$$

and hence $g \leq f$ on Ω_2.

Thus we have shown that g is the support function of a convex body L and $f|_{\Omega_2} = g|_{\Omega_2}$. Define $M := K + L \in \mathcal{K}^d$. Then K slides freely inside M. Moreover, $h_M = h_{B^d(o,R)}$ on Ω_2. This implies the first statement of the theorem for $\bar{\omega}_1$, provided that R is suitably enlarged if necessary. The general case then follows from an obvious compactness argument. In particular, the preceding proof yields that h_K is continuously differentiable on ω.

The remaining assertion is implied by Lemma 1.28. □

In order to have a precise reference, we state the result of Burago & Kalinin [22], which was mentioned in the introduction, as a theorem. Note that in [22] a different terminology is used.

Theorem 1.30 (Burago & Kalinin) *Let $K \in \mathcal{C}^d_o$, $r \in \{0, \ldots, d-2\}$, let $\beta \subset \mathbb{R}^d$ be open, and assume that there is a constant c such that*

$$C_r(K, \cdot) \llcorner \beta \leq c\, C_{d-1}(K, \cdot) \llcorner \beta\,.$$

Then, for any $x \in \beta \cap \operatorname{bd} K$,

$$x \notin \operatorname{ext}_r K \quad \text{or} \quad \dim N(K, x) < (d - r + 1)/2\,.$$

To obtain Theorem 1.30, simply note that the support cone $S(K,x) = N(K,x)^*$ (compare [123, (2.2.1)]) contains a linear subspace of dimension l if and only if the normal cone $N(K,x)$ is contained in a linear subspace of dimension $d-l$. Then the theorem is seen to be just a restatement of the corresponding Theorem of Burago & Kalinin [22].

In [22], the proof of Theorem 1.30 is based on the following lemma, which is just the special case $r = 0$ of Theorem 1.30. We state it separately as a lemma to emphasize the logical dependence of the subsequent arguments.

Lemma 1.31 *Let $K \in \mathcal{C}^d_o$, let $\beta \subset \mathbb{R}^d$ be open, and assume that there is a constant c such that*

$$C_0(K,\cdot) \llcorner \beta \le c\, C_{d-1}(K,\cdot) \llcorner \beta .$$

Then, for any $x \in \beta \cap \operatorname{bd} K$,

$$x \notin \operatorname{ext}_0 K \quad \text{or} \quad \dim N(K,x) < (d+1)/2 .$$

The following corollary is an easy consequence of Theorem 1.30.

Corollary 1.32 *Let $K \in \mathcal{C}^d_o$, $r \in \{0, \dots, d-2\}$, let $\beta \subset \mathbb{R}^d$ be open, and assume that there is a constant c such that*

$$C_r(K,\cdot) \llcorner \beta \le c\, C_{d-1}(K,\cdot) \llcorner \beta .$$

Then, for any $x \in \beta \cap \operatorname{bd} K$,

$$\dim N(K,x) \le d-1-r .$$

For $r = d-2$, Corollary 1.32 implies that $\beta \cap \operatorname{bd} K$ is a continuously differentiable hypersurface.

Next we shall treat Theorem 1.19 which is the dual counterpart of Theorem 1.30. In fact, we will offer two proofs for this result. The first one is particularly short, but it relies essentially on Theorem 1.30. The second proof has the advantage of assuming merely the validity of Lemma 1.31. Thus, by this approach and by another application of our transfer principle, we essentially obtain a new proof for Theorem 1.30. In particular, in this way we can avoid the introduction of α-transversal hyperplanes.

Lemma 1.33 *The statement of Theorem 1.19 holds for a convex body $K \in \mathcal{K}^d_{oo}$, an open set $\omega \subset S^{d-1}$ and the rth surface area measure of K, for some $r \in \{1, \dots, d-1\}$, if and only if the statement of Theorem 1.30 holds for K^*, $\beta := f(\omega) \cup (\mathbb{R}^d \setminus \operatorname{bd} K^*)$ and the $(d-1-r)$th curvature measure of K^*.*

Proof. Let $K \in \mathcal{K}^d_{oo}$ be a given convex body, let $\omega \subset S^{d-1}$ be open, and let $r \in \{1, \dots, d-1\}$.

First, let us assume that if

$$S_r(K,\cdot) \llcorner \omega \le c\, S_0(K,\cdot) \llcorner \omega ,$$

for some constant c, then

$$u \notin \operatorname{extn}_{d-1-r} K \quad \text{or} \quad \dim F(K,u) < r/2 ,$$

for all $u \in \omega$. Now suppose that there is a constant c^* such that

$$C_{d-1-r}(K^*,\cdot) \llcorner \beta \le c^* \, C_{d-1}(K^*,\cdot) \llcorner \beta \, .$$

Let $x^* \in \operatorname{bd} K^* \cap \beta$ be arbitrarily chosen. From Theorem 1.16 we obtain that there is a constant c such that

$$S_r(K,\cdot) \llcorner \omega \le c \, S_0(K,\cdot) \llcorner \omega \, .$$

Define $u := |x^*|^{-1} x^* \in \omega$; our assumption then implies that

$$u \notin \operatorname{extn}_{d-1-r} K \quad \text{ or } \quad \dim F(K,u) < r/2 \, .$$

If $u \notin \operatorname{extn}_{d-1-r} K$, then the argument on page 75 in [123] shows that $x^* \notin \operatorname{ext}_{d-1-r} K^*$. Now assume that $\dim F(K,u) < r/2$. It is easy to see that

$$N(K^*,x^*) \cap S^{d-1} = \left\{ |x|^{-1} x \in S^{d-1} : x \in F(K,u) \right\} \, , \tag{30}$$

and hence we obtain that

$$\dim N(K^*,x^*) = \dim F(K,u) + 1 < (r+2)/2 \, .$$

In any case, we have

$$x^* \notin \operatorname{ext}_{d-1-r} K^* \quad \text{ or } \quad \dim N(K^*,x^*) < (r+2)/2 \, .$$

The reverse implication follows in a similar way, if instead of (30) the relation

$$F(K,u) = \left\{ h(K^*,u^*)^{-1} u^* : u^* \in N(K^*,x^*) \cap S^{d-1} \right\}$$

is used, which holds for any $u \in \omega$ and $x^* := f(u)$. □

First proof of Theorem 1.19. The theorem is an immediate consequence of Theorem 1.30 and Lemma 1.33. □

Second proof of Theorem 1.19. We proceed by induction on the dimension d of the Euclidean space. For $d = 2$ the theorem is obviously true. Assume now that the statement of the theorem is true in dimensions less than d, and let $d \ge 3$.

By Lemma 1.33, the case $r = d-1$ of Theorem 1.19 follows from Lemma 1.31. Henceforth we assume $r \in \{1, \ldots, d-2\}$. Let the assumption of Theorem 1.19 be fulfilled and assume that $u \in \omega$ is such that

$$u \in \operatorname{extn}_{d-1-r} K \quad \text{and} \quad \dim F(K,u) \ge \frac{r}{2} \, .$$

The theorem is proved by showing that this assumption eventually leads to a contradiction.

Since $u \in \operatorname{extn}_{d-1-r} K$, we have $l := \dim T(K,u) \le d-r$. Define the linear subspace $V := \operatorname{lin}\{u, T(K,u)^\perp\}$, and hence $\dim V = 1 + d - l \ge r + 1$.

First, assume that $l \ge 2$. Then we have $\dim V \le d-1$. In addition, we know that $\operatorname{lin} F(K,u) \subset T(K,u)^\perp$ and $\operatorname{lin} F(K,u) \subset \operatorname{lin} F(K|V,u)$. From Theorem 1.26 we obtain that there is a positive constant $\tilde{c}$ such that

$$S_r^V(K|V,\cdot) \llcorner (\omega \cap V) \le \tilde{c} \, S_0^V(K|V,\cdot) \llcorner (\omega \cap V) \, .$$

Furthermore, $u \in \omega \cap V$, $\omega \cap V$ is open relative to V, and

$$\dim F(K|V,u) \geq \dim F(K,u) \geq \frac{r}{2}\,.$$

Therefore, the inductive assumption implies that $u \notin \operatorname{extn}_{\dim V-1-r}(K|V)$, and hence $u \notin \operatorname{extn}_0(K|V)$. In other words, $\dim T(K|V,u) \geq 2$, and thus we obtain linearly independent vectors $u_1, u_2 \in T(K|V,u)$. But then also $u_1, u_2 \in T(K,u) \cap V$, which contradicts the definition of V.

Now we consider the case $l = 1$. If $\dim F(K,u) = d-1$, then let U be an arbitrary $(d-2)$-dimensional linear subspace of $u^\perp$. If $\dim F(K,u) \leq d-2$, then choose a linear subspace $U \in \mathbf{G}(d,d-2)$, $U \subset u^\perp$, with $\operatorname{lin} F(K,u) \subset U$. Set $V := \operatorname{lin}\{u, U\} \in \mathbf{G}(d,d-1)$, and note that $\dim V - 1 - r = d - 2 - r \geq 0$. In any case we get

$$\dim F(K|V,u) \geq \frac{r}{2}\,.$$

Again from Theorem 1.26 and the inductive assumption we obtain linearly independent vectors $u_1, u_2 \in T(K,u)$, and this contradicts $l = \dim T(K,u) = 1$. □

Corollary 1.34 *Let $K \in \mathcal{K}^d_o$, $r \in \{1, \ldots, d-1\}$, let ω be an open subset of S^{d-1}, and assume that there is a constant c such that*

$$S_r(K,\cdot) \llcorner \omega \leq c\, S_0(K,\cdot) \llcorner \omega\,.$$

Then, for any $u \in \omega$,

$$\dim F(K,u) \leq r - 1\,.$$

For $r = 1$, Corollary 1.34 implies that $h(K,\cdot)$ is continuously differentiable on ω.

Corollary 1.35 *Let $K \in \mathcal{K}^3_o$, and assume that there is a positive constant c such that*

$$S_2(K,\cdot) \llcorner \omega \leq c\, S_0(K,\cdot) \llcorner \omega\,.$$

Then K is strictly convex.

Remarks.

1. Let $\beta \subset \mathbb{R}^d$ be Borel measurable; for $r = 0$ and $x \in \beta$, the assumption

   $$C_0(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

   already implies that

   $$\dim N(K,x) \leq d - 1\,.$$

   This can easily be seen, e.g., from the relation $C_0(K,\{x\}) = \mathcal{H}^{d-1}(\sigma(K,\{x\}))$, $x \in \operatorname{bd} K$. Similarly, let ω be a Borel measurable subset of S^{d-1}; for $r = d-1$ and $u \in \omega$, we already obtain

   $$\dim F(K,u) \leq d-2\,,$$

   if merely

   $$S_{d-1}(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

   is assumed. To see this, simply note that $S_{d-1}(K,\{u\}) = \mathcal{H}^{d-1}(F(K,u))$, $u \in S^{d-1}$.

2. Corollary 1.32 should be compared with Theorem 4.6 from [72], and Corollary 1.34 with Theorem 4.8 from [72]. Thus the stronger assumption of absolute continuity with bounded densities implies a stronger obstruction to singularities.

   The case $r = 1$ of Corollary 1.34 is contained in Satz 4.7 (b3) in [144]. Moreover, using our second transfer principle, we can derive Corollary 1.32, for $r = d - 2$, from Weil's result.

3. Burago & Kalinin have given examples which show that the assumption of Theorem 1.30 can be fulfilled and still one of the two conditions which appear in the implication of Theorem 1.30 can be violated. By duality corresponding examples can be obtained for Theorem 1.19 as well.

1.6 Curvature measures: bounded densities

This section is mainly devoted to the mean curvature measure of convex sets. Recall from the preceding section that if $K \in \mathcal{C}_o^d$ and

$$C_{d-2}(K, \cdot) \leq c\, C_{d-1}(K, \cdot)\,,$$

for some constant c, then Theorem 1.30 implies that $\operatorname{bd} K$ is of class C^1. In the following, we will show that a much stronger result can be inferred. In fact, the same assumption already implies that a ball rolls freely inside K if $\operatorname{bd} K$ is connected. In a certain sense, the main difficulty is to demonstrate that $\operatorname{bd} K$ is of class $C^{1,1}$.

First, however, we wish to provide a partial analogue of Theorem 1.26 for curvature measures; this will extend a result of Burago & Kalinin in [22]. Thus we will also obtain an integral-geometric characterization of absolute continuity with bounded density corresponding to Theorem 1.8.

In the following, we will adopt the notation introduced in [74]. In particular, we write $T_x K$ for the linear tangent space of K at $x \in \operatorname{bd} K$, where it is defined uniquely; in fact, whenever we write $T_x K$, this will be justified at least "almost everywhere" with respect to x.

Theorem 1.36 *Let $K \in \mathcal{C}_o^d$, let $\beta \subset \mathbb{R}^d$ be open, and assume that $r \in \{2, \dots, d-1\}$ and $s \in \{r, \dots, d-1\}$. Then the following conditions are equivalent.*

(a) *There is a constant c such that*

$$C_{d-r}(K, \cdot) \llcorner \beta \leq c\, C_{d-1}(K, \cdot) \llcorner \beta\,.$$

(b) *There is a constant $\tilde{c}$ such that, for μ_s almost all $E \in \mathbf{A}(d, s)$ with $E \cap \operatorname{int} K \neq \emptyset$,*

$$C'_{s-r}(K \cap E, \cdot) \llcorner (\beta \cap E) \leq \tilde{c} \int_\beta \mathbf{1}\{x \in \cdot\}[T_x K, U(E)]^{-r+1} C'_{s-1}(K \cap E, dx)\,.$$

Remarks.

1. If the constant c in (a) is given, then $\tilde{c}$ in (b) can be chosen so that

$$\tilde{c} \leq \frac{\binom{d-1}{r-1}}{\binom{s-1}{r-1}}\, c\,,$$

   and if the constant $\tilde{c}$ in (b) is given, then we can shoose the constant c in (a) so that $c \leq \tilde{c}$.

2. It does not seem to be possible to infer that condition (b) is satisfied for all $E \in \mathbf{A}(d,s)$ with $E \cap \text{int } K \neq \emptyset$ if K is not smooth. Although the map $E \mapsto C'_{s-1}(K \cap E, \cdot)$ is weakly continous as a map from the set of all $E \in \mathbf{A}(d,s)$ with $E \cap \text{int } K \neq \emptyset$ into the space of Radon measures on $\mathbb{R}^d$, the integrand $[T_x K, U(E)]^{-r+1}$ is discontinuous as a function of x for fixed E. Moreover, the restriction of this map to the set of regular boundary points of K is not equicontinuous with respect to E, and the set of regular boundary points is not closed.

Proof. (a) $\Rightarrow$ (b). By Theorem 1.8 we know that

$$C'_{s-r}(K \cap E, \cdot) \llcorner (\beta \cap E) \ll C'_{s-1}(K \cap E, \cdot) \llcorner (\beta \cap E)\,, \tag{31}$$

for μ_s almost all $E \in \mathbf{A}(d,s)$ for which $E \cap \text{int } K \neq \emptyset$. Further, by Lemmas 5.2-4 in [74], we conclude that, for μ_s almost all $E \in \mathbf{A}(d,s)$ satisfying $E \cap \text{int } K \neq \emptyset$ and for $\mathcal{H}^{s-1}$ almost all $x \in E \cap \text{bd } K$, we have $x \in \mathcal{M}(K) \cap \mathcal{M}(K \cap E)$,

$$H'_{r-1}(K \cap E, x) = [T_x K, U(E)]^{1-r} H'_{r-1}(K \cap (x + U_0(E)), x)\,,$$

where $U_0(E) = \text{lin}\{\sigma_K(x), U(E) \cap T_x K\}$, and

$$\binom{d-1}{r-1} H_{r-1}(K,x) \geq \binom{s-1}{r-1} H'_{r-1}(K \cap (x + U_0(E)), x)\,.$$

The last inequality follows again from Theorem 4.3.15 in [67] and since the principle curvatures are non-negative (whenever they are defined). This finally implies that

$$H'_{r-1}(K \cap E, x) \leq \frac{\binom{d-1}{r-1}}{\binom{s-1}{r-1}} [T_x K, U(E)]^{1-r} H_{r-1}(K,x)\,, \tag{32}$$

for μ_s almost all $E \in \mathbf{A}(d,s)$ for which $E \cap \text{int } K \neq \emptyset$, and for $\mathcal{H}^{s-1}$ almost all $x \in E \cap \text{bd } K$. From (a) we further obtain that $H_{r-1}(K,x) \leq c$ for $\mathcal{H}^{d-1}$ almost all $x \in \text{bd } K$. Hence, by Lemma 5.4 in [74] we have

$$H_{r-1}(K,x) \leq c\,, \tag{33}$$

for μ_s almost all $E \in \mathbf{A}(d,s)$ for which $E \cap \text{int } K \neq \emptyset$, and for $\mathcal{H}^{s-1}$ almost all $x \in E \cap \text{bd } K$. Now (b) follows from (31), (32), and (33).

(b) $\Rightarrow$ (a). Let $\gamma \subset \beta$ be Borel measurable. For the following argument, we have to recall some terminology from [74]. We write κ_n for the volume of an n-dimensional unit ball and set $a_{dsr} := (\kappa_{d-r}\omega_s)^{-1}\kappa_{s-r}\omega_d$. If $T \in \mathbf{G}(d,d-1)$, then we write $\mathbf{G}(T,s-1)$ for the Grassmann space of $(s-1)$-dimensional linear subspaces of T. Further, we write ν^T_{s-1} for the rotation invariant normalized Haar measure over $\mathbf{G}(T,s-1)$; here rotations refer to the linear space T with the induced scalar product. For any $V \in \mathbf{G}(d,s-1)$, we set $\mathbf{G}^V(d,s) := \{U \in \mathbf{G}(d,s) : V \subset U\}$, and we let ν^V_s denote the corresponding normalized Haar measure over $\mathbf{G}^V(d,s)$; compare [74]. Finally, if $T \in \mathbf{G}(d,d-1)$ and $U \in \mathbf{G}(d,s)$ are such that $T+U = \mathbb{R}^d$, then $[T,U] := |\langle e,u\rangle|$, where $e \in S^{d-1} \cap T^\perp$ and $u \in U \cap (T \cap U)^\perp \cap S^{d-1}$.

Using successively Theorem 4.5.5 in [123] (the constant a_{dsr} is defined as in [74], Proposition 5.8), assumption (b), Lemma 5.4 in [74], Lemma 5.6 in [74] , and Lemma 5.7 in [74], we obtain

$$\begin{aligned}
C_{d-r}(K,\gamma) &= a_{dsr} \int_{\mathbf{A}(d,s)} C'_{s-r}(K\cap E, \gamma\cap E)\, \mu_s(dE) \\
&\le a_{dsr}\,\tilde{c} \int_{\mathbf{A}(d,s)} \int_{E\cap\gamma\cap \mathrm{bd}\ K} [T_xK, U(E)]^{-r+1}\, \mathcal{H}^{s-1}(dx)\, \mu_s(dE) \\
&= a_{dsr}\,\tilde{c} \int_{\gamma\cap \mathrm{bd}\ K} \int_{\mathbf{G}(d,s)} [T_xK, U(E)]^{-r+2}\, \nu_s(dU)\, \mathcal{H}^{d-1}(dx) \\
&= \frac{\omega_s\omega_{d-s+1}}{2\omega_d}\, a_{dsr}\,\tilde{c} \int_{\gamma\cap \mathrm{bd}\ K} \int_{\mathbf{G}(T_xK,s-1)} \int_{\mathbf{G}^V(d,s)} \\
&\qquad\qquad [T_xK, U(E)]^{s-r+1}\, \nu_s^V(dU) \nu_{s-1}^{T_xK}(dV) \mathcal{H}^{d-1}(dx) \\
&= \frac{\omega_s}{\omega_d} a_{dsr}\,\tilde{c}\, \frac{\kappa_{d-r}}{\kappa_{s-r}}\, (\mathcal{H}^{d-1} \llcorner \mathrm{bd}\ K)(\gamma) \\
&= \tilde{c}\, C_{d-1}(K,\gamma)\,.
\end{aligned}$$

This shows that (a) is satisfied. □

Corollary 1.37 *Let $K \in \mathcal{C}_o^d$, let $\beta \subset \mathbb{R}^d$ be open, and let $r \in \{2, \dots, d-1\}$ and $s \in \{r, \dots, d-1\}$. Assume that there is a constant c such that*

$$C_{d-r}(K,\cdot) \llcorner \beta \le c\, C_{d-1}(K,\cdot) \llcorner \beta\,.$$

Then, for each $E \in \mathbf{A}(d,s)$ with $E \cap \mathrm{int}\ K \neq \emptyset$, there is a constant $c(E)$ such that

$$C'_{s-r}(K\cap E,\cdot) \llcorner (\beta\cap E) \le c(E)\, C'_{s-1}(K\cap E,\cdot) \llcorner (\beta\cap E)\,.$$

Proof. Let $E \in \mathbf{A}(d,s)$ with $E \cap \mathrm{int}\ K \neq \emptyset$ be fixed. Then there is some $y_0 \in E$ and some $r_0 > 0$ such that $B^d(y_0, 2r_0) \subset K$. We can choose a sequence $E_i \in \mathbf{A}(d,s)$, $i \in \mathbb{N}$, for which the estimate of condition (b) in Theorem 1.36 is fulfilled and such that $E_i \cap B^d(y_0, r_0) \neq \emptyset$; thus

$$[T_xK, U(E_i)] \ge \frac{r_0}{\mathrm{diam}\ K}$$

for all $x \in E_i \cap \mathrm{bd}\ K$ and $i \in \mathbb{N}$. Therefore we obtain

$$C'_{s-r}(K\cap E_i,\cdot) \llcorner (\beta\cap E_i) \le \tilde{c} \left(\frac{\mathrm{diam}\ K}{r_0}\right)^{r-1} C'_{s-1}(K\cap E_i,\cdot) \llcorner (\beta\cap E_i)$$

for all $i \in \mathbb{N}$. An argument similar to the one used to establish (b) ⇒ (c) in the proof of Theorem 1.26 then completes the proof. The analogue of Lemma 1.25, which is needed for such an argument, follows from Theorem 1.8.8 in [123], the weak continuity of curvature measures and the fact that up to proper normalization the curvature measures are independent of the dimension of the surrounding space (compare [123], p. 205). □

Remarks. On Corollary 1.37 one can base an inductive proof of Theorem 1.30, starting from Lemma 1.31. In fact, such an approach was essentially used by Burago & Kalinin.

Our next aim is to show that if $C_{d-2}(K,\cdot) \le c\,C_{d-1}(K,\cdot)$ for a constant $c > 0$, then K is of class $C^{1,1}$. We shall deduce this theorem from Theorem 1.27 by means of the second transfer principle through a succession of geometric results, which will be established first. These geometric facts are designed to prove that if a convex body $K \in \mathcal{K}^d_{oo}$ is a summand of a ball of radius $R > 0$, then a ball of radius R^{-1} rolls freely inside the polar K^*. For the proofs of two preparatory lemmas, we have to accomplish some explicit calculations for ellipsoids.

Lemma 1.38 *Let $R > 0$, $t \in \mathbb{R}^d$ and $|t| < R$. Then $B^d(t,R)^*$ is an ellipsoid of revolution. The lengths of its semiaxes are*

$$a_1 = \frac{R}{R^2 - |t|^2} \quad \text{and} \quad a_2 = \cdots = a_d = \frac{1}{\sqrt{R^2 - |t|^2}}\,.$$

Proof. We can assume that $t = |t|e_1$, where $(e_1, \ldots, e_d)$ is the standard basis of $\mathbb{R}^d$. Then, for $u \in S^{d-1}$, we have

$$\rho(B^d(t,R)^*, u) = h(B^d(t,R), u)^{-1} = (\langle t, u\rangle + R)^{-1}\,.$$

If we set $x := \rho(B^d(t,R)^*, u)u$, $u \in S^{d-1}$, then we find (compare [44])

$$|x|^2 = \frac{1}{R^2}\left(1 - \frac{\langle t,u\rangle}{\langle t,u\rangle + R}\right)^2 = \frac{1}{R^2}(1 - \langle t,x\rangle)^2\,.$$

Hence, any boundary point x of $B^d(t,R)^*$ satisfies the equation

$$x_1^2 - \left(\frac{|t|}{R}\right)^2 x_1^2 + \frac{2|t|}{R^2}x_1 + \sum_{i=2}^{d} x_i^2 = \frac{1}{R^2}\,,$$

where $x_1, \ldots, x_d$ are the coordinates of x with respect to the standard basis $(e_1, \ldots, e_d)$. Some elementary calculations finally show that this is equivalent to

$$\frac{\left(x_1 + \frac{|t|}{R^2 - |t|^2}\right)^2}{\left(\frac{R}{R^2 - |t|^2}\right)^2} + \sum_{i=2}^{d} \frac{x_i^2}{\left(\frac{1}{\sqrt{R^2 - |t|^2}}\right)^2} = 1\,.$$

This proves that $\mathrm{bd}\, B^d(t,R)^*$ is contained in and thus coincides with the boundary of an ellipsoid with semiaxes as described in the statement of the lemma. □

Lemma 1.39 *Let $R > 0$ be fixed. Let $\mathcal{E}_t$, $t \in [0,R)$, be an ellipsoid of revolution in $\mathbb{R}^d$ the semiaxes of which have the lengths $a_1 = R\omega^2$ and $a_2 = \ldots = a_d = \omega$, where $\omega := (R^2 - t^2)^{-1/2}$. Then the principal radii of curvature of $\mathcal{E}_t$ can be uniformly bounded from below by R^{-1}.*

Proof. From the calculations in the proof of Theorem 1.27 we obtain for the principal radii of curvature $r_2(x), \ldots, r_d(x)$ of the ellipsoid $\mathcal{E}_t$ in direction $x = (x_1, x_2, 0, \ldots, 0) \in S^{d-1}$ that

$$r_2(x) = \overline{\mathrm{II}}_x(u_2, u_2) = R^{-1}\left(1 - \left(\frac{t}{R}\right)^2 + \left(\frac{t}{R}\right)^2 x_1^2\right)^{-\frac{3}{2}}$$

and

$$r_i(x) = \overline{\mathrm{II}}_x(u_i, u_i) = R^{-1}\left(1 - \left(\frac{t}{R}\right)^2 + \left(\frac{t}{R}\right)^2 x_1^2\right)^{-\frac{1}{2}}, \quad i \in \{3, \ldots, d\}.$$

This proves the lemma, since $x_1^2 \in [0, 1]$. □

Proposition 1.40 *Let $K \in \mathcal{K}^d_{oo}$ and assume that the polar body K^* is a summand of the ball $B^d(o, R)$ with $R > 0$. Then $B^d(o, R^{-1})$ rolls freely inside K.*

Proof. By Theorem 3.2.2 in [123], the assumption implies that K^* slides freely inside $B^d(o, R)$. Hence for each $u \in S^{d-1}$ there is a (uniquely determined) point $x^* = \tau_{K^*}(u) \in \operatorname{bd} K^*$ with $K^* \subset B^d(x^* - Ru, R)$. Thus we get $B^d(x^* - Ru, R)^* \subset K$. In particular, we have $F(K^*, u) = \{x^*\}$ and $h(K^*, u) = h(B^d(x^* - Ru, R), u)$. Therefore, $h(K^*, u)^{-1}u \in \operatorname{bd} K$ and also $h(K^*, u)^{-1}u \in \operatorname{bd}(B^d(x^* - Ru, R)^*)$. This shows that

$$h(K^*, u)^{-1}u \in \operatorname{bd} K \cap \operatorname{bd}(B^d(x^* - Ru, R)^*).$$

In other words, for each $x \in \operatorname{bd} K$ there is some $u \in S^{d-1}$ such that

$$x \in B^d(\tau_{K^*}(u) - Ru, R)^* \subset K.$$

From Lemma 1.38 we know that $B^d(\tau_{K^*}(u) - Ru, R)^*$ is an ellipsoid of revolution. Since $o \in \operatorname{int} K^*$ and $K^* \subset B^d(\tau_{K^*}(u) - Ru, R)$, we obtain $|\tau_{K^*}(u) - Ru| < R$. Now Lemma 1.38, Lemma 1.39 and a special case of Corollary 3.2.10 in [123] imply that $B^d(o, R^{-1})$ rolls freely inside any of the ellipsoids $B^d(\tau_{K^*}(u) - Ru, R)^*$. But then $B^d(o, R^{-1})$ rolls freely inside K. □

The proof of Proposition 1.40 actually yields the following local result.

Corollary 1.41 *Let $K \in \mathcal{K}^d_{oo}$, $u \in S^{d-1}$, and $R > 0$. Assume that $x \in F(K, u)$ is such that $K \subset B^d(x - Ru, R)$, and set $x^* := \rho(K^*, u)u \in \operatorname{bd} K^*$. Then $x^* \in B^d(x^* - R^{-1}\sigma_{K^*}(x^*), R^{-1}) \subset K^*$.*

Theorem 1.42 *Let $K \in \mathcal{K}^d_o$. Then there is a constant c such that*

$$C_{d-2}(K, \cdot) \leq c\, C_{d-1}(K, \cdot)$$

if and only if there is some $r > 0$ such that the ball $B^d(o, r)$ rolls freely inside K.

Proof. First, let us assume that

$$C_{d-2}(K, \cdot) \leq c\, C_{d-1}(K, \cdot)$$

is satisfied for some constant c. We may also assume that $o \in \operatorname{int} K$. Then Theorem 1.16 implies that there is a constant c^* such that

$$S_1(K^*, \cdot) \le c^* \, S_0(K^*, \cdot) \, .$$

Hence, Theorem 1.27 implies that K^* is a summand of a ball. But then Proposition 1.40 yields the assertion.

The reverse implication follows, e.g., from equation (12) of [69], since K is a parallel body of some body $L \in \mathcal{K}^d$ if $B^d(o,r)$ rolls freely inside K; see [123, Theorem 3.2.2]. □

Remark. Alternatively, the last assertion can also be seen from Lemma 3.1 in [72] and Theorem 1.3. The boundedness of the principal curvatures of K from above by r^{-1} at $\mathcal{H}^{d-1}$ almost all boundary points can easily be checked. Still another way to prove the reverse statement is to use Proposition 1.45 below, Satz 4.7 (a2) ⇒ (a1) in [144], and Theorem 1.16 of the present work.

In order to see that Theorem 1.27 and Theorem 1.42 are actually equivalent, we first provide a geometric result which is a converse of Proposition 1.40. We need two preparatory geometric lemmas.

Let $r, l > 0$, and let $e \in S^{d-1}$ be an arbitrary unit vector; then we define

$$K(r,l,e) := \operatorname{conv}\{B^d(o,r) \cup (le + B^d(o,r))\} \, .$$

We first calculate the radial function of the polar body of $K(r,l,e)$.

Lemma 1.43 *The radial function $\rho(K(r,l,e)^*, \cdot)$ of the polar body of $K(r,l,e)$ is given by*

$$\rho(K(r,l,e)^*, u) = \begin{cases} (r + l\langle u, e\rangle)^{-1} \, , & \text{if } \langle u, e\rangle \ge 0 \, , \\ r^{-1} \, , & \text{if } \langle u, e\rangle < 0 \, , \end{cases}$$

where $u \in S^{d-1}$.

Proof. The proof follows from $\rho(K(r,l,e)^*, u) = h(K(r,l,e), u)^{-1}$, $u \in S^{d-1}$, and elementary geometric considerations. □

Lemma 1.44 *Let $r > 0$, and let $l \in [0, L]$ for some $L > 0$. Then there is a finite number $R = R(r, L)$ such that $K(r,l,e)^*$ is a summand of $B^d(o,R)$.*

Proof. For the proof we show that for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K(r,l,e)^*$ the principal curvatures $k_1(x), \ldots, k_{d-1}(x)$ of $K(r,l,e)^*$ can be estimated from below by a positive number $R_0 = R_0(r,L)$. In the terminology of Dekster [31, p. 248], this implies that $K(r,l,e)^*$ has an R-support at $\mathcal{H}^{d-1}$ almost all of its boundary points if (say) we set $R := 2R_0$. Then Theorem 1.9 in [31] and Theorem 3.2.2 in [123] yield that $K(r,l,e)^*$ is a summand of $B^d(o,R)$.

It remains to establish the bounds for the principal curvatures of $K(r,l,e)^*$ at $\mathcal{H}^{d-1}$ almost all boundary points. If $x \in \operatorname{bd} K(r,l,e)^*$ is such that $\langle x, e\rangle < 0$, then trivially

$$k_i(K(r,l,e)^*, x) = r^{-1} \, , \quad i \in \{1, \ldots, d-1\} \, .$$

Now consider the case where $\langle x, e\rangle > 0$. By Lemma 1.43 we have to estimate the principal curvatures of the smooth hypersurface of revolution which is parametrized by the map $X : (0, \pi/2] \times S^{d-2} \to \mathbb{R}^d$ defined by

$$X(t, v) := \frac{\cos t}{r + l \sin t}\, v + \frac{\sin t}{r + l \sin t}\, e\,,$$

where $S^{d-2} := e^{\perp} \cap S^{d-1}$.

The exterior unit normal vector N of X is given by

$$N(t, v) = \frac{r \cos t}{\sqrt{r^2 + l^2 + 2rl \sin t}}\, v + \frac{l + r \sin t}{\sqrt{r^2 + l^2 + 2rl \sin t}}\, e\,,$$

where $(t, v) \in (0, \pi/2] \times S^{d-2}$. A straightforward calculation (compare [123, §2.5, p. 105]) yields for the principal curvatures of $K(r, l, e)^*$ at the boundary point $x = X(t, v)$, $(t, v) \in (0, \pi/2] \times S^{d-2}$,

$$k_1(K(r, l, e)^*, x) = \frac{r(r + l \sin t)^3}{\sqrt{r^2 + l^2 + 2rl \sin t}^3} \geq \frac{r^4}{(r + l)^3} \geq \frac{r^4}{(r + L)^3}$$

and

$$k_2(K(r, l, e)^*, x) = \ldots = k_{d-1}(K(r, l, e)^*, x) = \frac{r(r + l \sin t)}{\sqrt{r^2 + l^2 + 2rl \sin t}} \geq \frac{r^2}{r + L}\,.$$

Since $(r + L)^{-1} r^2 \geq (r + L)^{-3} r^4$, we can define R_0 as the minimum of r^{-1} and $(r + L)^{-3} r^4$, and this completes the proof of the lemma. □

Proposition 1.45 *Let $K \in \mathcal{K}^d_{oo}$, and assume that $B^d(o, r)$ rolls freely inside K for some $r > 0$. Then there is some $R < \infty$ such that K^* is a summand of $B^d(o, R)$.*

Proof. We may assume that $B^d(o, r) \subset K$. We set $L := \mathrm{diam}(K)$ and choose R as in the assertion of Lemma 1.44. By Theorem 3.2.2 in [123] it is sufficient to show that for each $u \in S^{d-1}$ there is some $t \in \mathbb{R}^d$ such that $Ru \in K^* + t \subset B^d(o, R)$. Let $u \in S^{d-1}$ be given. Define $x := \rho(K, u)u$ and

$$K(x) := \mathrm{conv}\{B^d(x - r\sigma_K(x), r) \cup B^d(o, r)\}\,.$$

Then $K(x) \subset K$, since $B^d(o, r)$ rolls freely inside K. In addition, $\rho(K, u) = \rho(K(x), u)$, and hence also

$$h(K^*, u) = h(K(x)^*, u)\,. \tag{34}$$

Choose $x^* \in \mathrm{bd}\, K^*$ such that $h(K^*, u) = \langle x^*, u\rangle$. By (34) and since $x^* \in K^* \subset K(x)^*$, we now obtain $x^* \in F(K(x)^*, u)$. But then Lemma 1.44 implies that

$$Ru \in K(x)^* + (Ru - x^*) \subset B^d(o, R)\,.$$

This shows that the definition $t := Ru - x^*$ in fact yields $Ru \in K^* + t \subset B^d(o, R)$. □

From the proof of Proposition 1.45 we can extract the following local result.

Corollary 1.46 *Let $K \in \mathcal{K}^d_{oo}$, $x \in \mathrm{bd}\, K$, and $r > 0$. Assume that $u \in N(K,x) \cap S^{d-1}$ is such that $B^d(x - ru, r) \subset K$, and set $u := |x|^{-1}x \in S^{d-1}$. Then there is some $R > 0$ such that $K^* \subset B^d(\tau_{K^*}(u) - Ru, R)$; moreover, R depends only on r and the diameter of K.*

The next theorem refines the statement of Theorem 1.42. But first we state a lemma which improves a less explicit assertion in [94], Hilfssatz 1. For this reason, we recall the following notion. Let $K \in \mathcal{K}^d_o$ and $r > 0$; then

$$(\mathrm{bd}\, K)_r := \{x \in \mathrm{bd}\, K : B^d(x - ru, r) \subset K \text{ for some } u \in N(K,x)\}$$

is the closed set of r-smooth boundary points of K.

Lemma 1.47 *Let $K \in \mathcal{K}^d_o$ and $r > 0$. Then the restriction of the spherical image map σ_K to $(\mathrm{bd}\, K)_r$ is Lipschitz with Lipschitz constant $1/r$.*

Proof. Let $x, y \in (\mathrm{bd}\, K)_r$, and define $u := \sigma_K(x)$, $v := \sigma_K(y)$. Then we have

$$x - ru + rv \in B^d(x - ru, r) \subset K\,,$$

and hence $\langle x - ru + rv - y, v\rangle \le 0$. This yields

$$r\langle v - u, v\rangle \le \langle y - x, v\rangle\,; \tag{35}$$

by symmetry we also have

$$r\langle v - u, -u\rangle \le \langle y - x, -u\rangle\,. \tag{36}$$

Addition of (35) and (36) implies

$$r|v - u|^2 \le \langle y - x, v - u\rangle \le |y - x|\,|v - u|\,,$$

which completes the proof. □

Theorem 1.48 *Let $K \in \mathcal{K}^d_o$, and assume that there exists a constant c such that*

$$C_{d-2}(K,\cdot) \le c\, C_{d-1}(K,\cdot)\,.$$

Then the ball $B^d\left(o, ((d-1)c)^{-1}\right)$ rolls freely inside K; in general, the radius of this ball cannot be increased further.

Proof. It follows from Theorem 1.42 and Lemma 1.47 that the spherical image map $\sigma_K : \mathrm{bd}\, K \to S^{d-1}$ is Lipschitz. The assumption implies that

$$k_i(K,x) \le (d-1)H_1(K,x) \le (d-1)c\,, \quad i \in \{1, \dots, d-1\}\,,$$

for $\mathcal{H}^{d-1}$ almost all $x \in \mathrm{bd}\, K$. Since σ_K is a Lipschitz map, Remark 2 after Lemma 2.5 in [70] yields that, for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, all eigenvalues of $d^2 h_K(u)|_{u^\perp}$ can be estimated from below by $((d-1)c)^{-1}$. One way to complete the proof is to simply cite Theorem 1 in [148] (or Satz 4.9 in [11]).

A different (and perhaps more elementary) argument proceeds as follows. By Lemma 4.1 in [72] we deduce that, for ν_2 almost all $U \in \mathbf{G}(d,2)$, and for $\mathcal{H}^1$ almost all $u \in S^{d-1} \cap U$, the inequality $r(K|U,u) \geq 1/[(d-1)c]$ is satisfied. This shows that

$$S_1^U(K|U,\cdot) \geq S_1^U(B^d(o,((d-1)c)^{-1})|U,\cdot)\,,$$

and hence by a special case of Minkowski's theorem (see Theorem 7.1.1 in [123]) we deduce that $B^d(o,((d-1)c)^{-1})|U$ is a summand of $K|U$, for ν_2 almost all $U \in \mathbf{G}(d,2)$. But then the assertion follows from Lemma 3.2.6 in [123].

For the second part, consider an ellipsoid $\mathcal{E}(a,b)$ whose semiaxes have lengths $a_1 = a$ and $a_2 = \ldots = a_{d-1} = b$, where $b \geq a > 0$. After some calculations (compare the proof of Theorem 1.27), one obtains that the principal curvatures of $\mathcal{E}(a,b)$ satisfy

$$k_1(x) \leq \frac{b}{a^2}\,, \quad k_i(x) \leq \frac{1}{b}\,, \quad i \in \{2,\ldots,d-1\}\,,$$

for all $x \in \mathrm{bd}\,\mathcal{E}(a,b)$. The upper bounds are simultaneously attained for one boundary point. Therefore,

$$c(a,b) := \frac{1}{d-1}\left(\frac{b}{a^2} + (d-2)\frac{1}{b}\right)$$

is the smallest constant such that

$$C_{d-2}(\mathcal{E}(a,b),\cdot) \leq c(a,b)\, C_{d-1}(\mathcal{E}(a,b),\cdot)$$

is fulfilled. Moreover, $r(a,b) := a^2/b$ is the largest radius of a ball which rolls freely inside $\mathcal{E}(a,b)$. This shows that

$$r(a,b)\, c(a,b) = \frac{1}{d-1}\left(1 + (d-2)\frac{a^2}{b^2}\right)\,, \tag{37}$$

and the expression in (37) is arbitrarily close to $(d-1)^{-1}$ for $b/a \to \infty$. □

Remarks.

1. In the plane ($d = 2$), a slight modification of the proof for Theorem 1.48 leads to a direct proof of this Theorem for $d = 2$. In fact, let $K \in \mathcal{K}_o^2$, and assume that

   $$C_0(K,\cdot) \leq c\, C_1(K,\cdot)$$

   for some constant c. Then, for $\mathcal{H}^1$ almost all $x \in \mathrm{bd}\,K$, we obtain $k_1(x) \leq c$. On the other hand, if $\beta \subset \mathrm{bd}\,K$ is a Borel set and $\mathcal{H}^1(\beta) = 0$, then

   $$\mathcal{H}^1(\sigma(K,\beta)) = C_0(K,\beta) \leq c\, C_1(K,\beta) = c\,\mathcal{H}^1(\beta) = 0\,,$$

   and therefore, as in the proof of Theorem 1.48, we have $r_1(u) \geq c^{-1}$, for $\mathcal{H}^1$ almost all $u \in S^1$. Now an application of Minkowski's theorem yields that $B^d(o,c^{-1})$ rolls freely inside K. (Instead of Minkowski's theorem, one can also use very special cases of Theorem 1 in [148] or Satz 4.9 in [11].)

2. The preceding results can be used to extend stability results for convex bodies of class C^2_+ to the nonsmooth case. In the plane, in particular, a sharp stability result can easily be obtained. In fact, let $K \in \mathcal{K}^2_o$, $0 \le \epsilon < 1$, and assume that
$$(1-\epsilon)C_1(K,\cdot) \le C_0(K,\cdot) \le (1+\epsilon)C_1(K,\cdot)\,.$$
Then a disc of radius $(1+\epsilon)^{-1}$ rolls freely inside K, and K rolls freely inside a disc of radius $(1-\epsilon)^{-1}$.

Corollary 1.49 *Let $K \in \mathcal{C}^d_o$, let $\beta \subset \mathbb{R}^d$ be open, and assume that there is a constant c such that*
$$C_{d-2}(K,\cdot)\llcorner\beta \le c\,C_{d-1}(K,\cdot)\llcorner\beta\,.$$
Then, for each compact subset $\bar\beta$ of β, there is some $r > 0$ such that $B^d(x - r\sigma_K(x), r) \subset K$ for all $x \in \bar\beta \cap \operatorname{bd} K$; in particular, all points in $\beta \cap \operatorname{bd} K$ are regular boundary points and σ_K is locally Lipschitz on $\beta \cap \operatorname{bd} K$.

Proof. It can be assumed that $K \in \mathcal{K}^d_{oo}$, since the curvature measures are translation invariant and locally defined. The proof of the first statement then follows from Theorem 1.16, Theorem 1.29, and Corollary 1.41.

The remaining assertion is implied by Lemma 1.47. □

Remark. Conversely, one can deduce Theorem 1.29 from Corollary 1.49 by using Theorem 1.16 and Corollary 1.46. Thus, in view of the second transfer principle, Theorem 1.29 and Corollary 1.49 are equivalent up to rather elementary additional geometric considerations.

Corollary 1.50 *Let $K \in \mathcal{C}^d_o$, let $\operatorname{bd} K$ be connected, and assume that there is a constant c such that*
$$C_{d-2}(K,\cdot) \le c\,C_{d-1}(K,\cdot)\,.$$
Then the ball $B^d\left(o, ((d-1)c)^{-1}\right)$ rolls freely inside K.

Proof. We already know that $\operatorname{bd} K$ is locally of class $C^{1,1}$. Let $\bar c > c$ be arbitrarily chosen. Further, choose $x_0 \in \operatorname{bd} K$ and set $u_0 := \sigma_K(x_0)$. Set $\bar K := K \cap B^d(x_0, 1) \in \mathcal{K}^d_o$, and let $\epsilon > 0$ be such that $x_0 - \epsilon u_0 \in \operatorname{int} \bar K$ and $\epsilon < ((d-1)\bar c)^{-1}$. We can choose the origin so that $o = x_0 - \epsilon u_0$. The assumption and equation (21) imply that
$$D_1 h(\bar K^*, u) \le \bar c$$
for all $u \in \omega$, where $\omega \subset S^{d-1}$ is a sufficiently small, spherically convex neighbourhood of u_0. In particular, we can assume that $\rho(K,u) = \rho(\bar K, u)$ for all $u \in \omega$. From Theorem 1.16 we first infer that
$$S_1(\bar K^*,\cdot)\llcorner\omega \ll S_0(\bar K^*,\cdot)\llcorner\omega\,,$$
and hence
$$S_1(\bar K^*,\cdot)\llcorner\omega \le \bar c\,S_0(\bar K^*,\cdot)\llcorner\omega\,.$$

Let $\omega_0 \subset S^{d-1}$ be a spherically convex, open neighbourhood of u_0 whose closure is contained in ω. As in the proof of Theorem 1.29 it then follows that

$$h(\bar{K}^*, u) \leq h\left(B^d\left(\tau_{\bar{K}^*}(u_0) - (d-1)\bar{c}u_0, (d-1)\bar{c}\right), u\right)$$

is satisfied for all $u \in \omega_0$. But then an application of Corollary 1.41 shows that

$$\rho\left(B^d\left(x_0 - ((d-1)\bar{c})^{-1}u_0, ((d-1)\bar{c})^{-1}\right), u\right) \leq \rho(K, u)$$

is true for all $u \in \omega_0$. In other words, we have proved that for all $x \in \operatorname{bd} K$ the ball $B^d\left(x - ((d-1)\bar{c})^{-1}\sigma_K(x), ((d-1)\bar{c})^{-1}\right)$ locally touches $\operatorname{bd} K$ from inside. By applying Theorem 4.3.2 or Theorem 9.3.2 of Brooks & Strantzen [19], we deduce that $B^d\left(o, ((d-1)\bar{c})^{-1}\right)$ rolls freely inside K. But this implies the desired result, since $\bar{c} > c$ was arbitrarily chosen.□

Remark. A completely different proof of Corollary 1.50 has independently been found by V. Bangert [12]. There one can also find references to the literature, where special cases of Corollary 1.50 have been treated in the setting of smooth hypersurfaces. The author gratefully acknowledges discussions with V. Bangert, which motivated the treatment of the noncompact case in the present work.

1.7 Applications to stability results

A familiar way of establishing stability and uniqueness results for balls is to use symmetrization techniques. This is also the method which was used by Diskant in order to prove stability results for convex bodies K for which $S_{d-1}(K, \cdot)$ or $C_0(K, \cdot)$ are close to the corresponding measures of the unit ball $B^d(o, 1)$. It is surprising, however, that it is possible to improve Diskant's result for the $(d-1)$st surface area measure by means of Diskant's stability result for the Gauss curvature measure $C_0(K, \cdot)$.

Proof of Theorem 1.18. We can assume that $\epsilon > 0$. The assumption implies that $\operatorname{int} K \neq \emptyset$ and

$$1 - \epsilon \leq D_{d-1}h(K, u) \leq 1 + \epsilon\,, \tag{38}$$

for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$. By Theorem 1.5, the left-hand side of (38) yields that

$$C_0(K, \cdot) \ll C_{d-1}(K, \cdot)\,; \tag{39}$$

moreover, the density function is given by $H_{d-1}(K, \cdot)$. Let $\omega_0 \subset S^{d-1}$ be the set of all $u \in S^{d-1}$ such that h_K is not second order differentiable at u or (38) is not satisfied. Hence, we get

$$0 = S_0(K, \omega_0) \geq (1+\epsilon)^{-1} S_{d-1}(K, \omega_0) = (1+\epsilon)^{-1}\mathcal{H}^{d-1}(\tau(K, \omega_0)) \geq 0\,.$$

For $x \in \mathcal{M}(K) \setminus \tau(K, \omega_0)$, and hence for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$, we obtain that

$$H_{d-1}(K, x) D_{d-1}h(K, \sigma_K(x)) = 1\,; \tag{40}$$

see Remark 2 after Lemma 2.5 in [70]. From (38) and (40) we deduce

$$1 - \epsilon \leq (1+\epsilon)^{-1} \leq H_{d-1}(K, x) \leq (1-\epsilon)^{-1} \leq 1 + 2\epsilon\,, \tag{41}$$

since $0 < \epsilon < 1/2$, and thus (39) and (41) imply

$$(1-2\epsilon)C_{d-1}(K,\cdot) \leq C_0(K,\cdot) \leq (1+2\epsilon)C_{d-1}(K,\cdot)\,.$$

Now the proof is completed by applying Theorem 1 of Diskant [32]; compare Theorem 7.2.11 in [123]. □

Remark. Let $K := B^d\left(o,(1+\epsilon)^{1/(d-1)}\right)$, $0 < \epsilon < 1/4$. Then the assumptions of Theorem 1.18 are fulfilled, but the Hausdorff distance of K to an arbitrary unit ball is greater or equal

$$\frac{1}{d-1}\left(\frac{4}{5}\right)^{\frac{d-2}{d-1}}\epsilon\,.$$

Therefore, the exponent of ϵ (namely 1) in the conclusion of Theorem 1.18 cannot be improved in general.

The proof of Theorem 1.18 also suggests the following consequence, which we include for the sake of completeness.

Corollary 1.51 *Let $K \in \mathcal{K}^d_o$, let $\omega \subset S^{d-1}$ be Borel measurable, and let $0 < \alpha \leq \beta < \infty$. Then the following conditions are equivalent:*

(a) $\alpha\, S_0(K,\cdot) \llcorner \omega \leq S_{d-1}(K,\cdot) \llcorner \omega \leq \beta\, S_0(K,\cdot) \llcorner \omega\,;$

(b) $1/\beta\, C_{d-1}(K,\cdot) \llcorner \tau(K,\omega) \leq C_0(K,\cdot) \llcorner \tau(K,\omega) \leq 1/\alpha\, C_{d-1}(K,\cdot) \llcorner \tau(K,\omega)\,.$

The second consequence of our general results on the absolute continuity of curvature measures, which we discuss in this subsection, concerns a stability result for the mean curvature measure. The following terminology is appropriate. For a given convex body K, let B_K be the Steiner ball of K. This is the ball which has the same Steiner point and the same mean width as K. Further, denote by $\delta_2(K,B_K)$ and $\delta(K,B_K)$ the L^2-distance of the support functions of K and B_K and the Hausdorff distance of K and B_K, respectively.

Theorem 1.52 *Let $K \in \mathcal{K}^d$, $\epsilon \in [0,1)$, and assume that*

$$(1-\epsilon)C_{d-1}(K,\cdot) \leq C_{d-2}(K,\cdot) \leq (1+\epsilon)C_{d-1}(K,\cdot)\,. \tag{42}$$

Then

$$\delta_2\left(K,B_K\right) \leq c_1(d)\, V(K)\, \epsilon^{\frac{1}{2}}\,, \tag{43}$$

where

$$c_1(d) = \sqrt{\frac{d}{\kappa_d}}\left(\frac{2^{3d-4}(d-1)^{2d-3}}{d+1}\right)^{\frac{1}{2}}\,.$$

Remarks.

1. By considering the convex hull of $\{x \in B^d(o,1) : \langle x, e_d\rangle \geq 0\}$ and $B^d(o,1-\epsilon)$ for $\epsilon \downarrow 0$ one can see that the exponent $1/2$ of ϵ on the right-hand side of (43) cannot be increased in general.

2. Under the assumptions of Theorem 1.52 one obtains from Lemma 2 in [58] that

$$\delta\left(K, B_K\right) \leq c_2(d)\, V(K)^{\frac{4}{d+3}}\, b(K)^{\frac{d-1}{d+3}}\, \epsilon^{\frac{2}{d+3}}\,,$$

   where $c_2(d)$ is a constant which merely depends on the dimension d and which can be calculated explicitly. In particular, by the same arguments as in [122], we get under the present weaker assumptions that

$$\delta(K, B^d(o,1)) \leq c_3(d, \operatorname{diam} K)\, \epsilon^{\frac{2}{d+3}}\,,$$

   where $c_3(d, \operatorname{diam} K)$ is a constant which depends on d and $\operatorname{diam} K$ and which can be determined explicitly.

3. Provided that (42) holds for some $\epsilon_0 \in [0, 1/(2d-3))$, we get for all $\epsilon \in [0, \epsilon_0]$ that

$$\delta(K, B^d(o,1)) \leq c_4(d, \epsilon_0)\, \epsilon^{\frac{2}{d+3}}\,,$$

   where $c_4(d, \epsilon_0)$ is a constant which depends on d and ϵ_0. This follows from Corollary 4.3 in Bangert [12], which yields a diameter bound $D_d(\epsilon_0)$ for convex bodies K such that

$$(1-\epsilon_0)C_{d-1}(K,\cdot) \leq C_{d-2}(K,\cdot) \leq (1+\epsilon_0)C_{d-1}(K,\cdot)\,. \tag{44}$$

   The constant $D_d(\epsilon_0)$, however, cannot be calculated explicitly.

4. A close inspection of the arguments in a recent preprint by Kohlmann [85] shows that for $\epsilon_0 \leq \epsilon(d)$ a diameter bound $D(d)$ can be established for all convex bodies satisfying (44). The constants $\epsilon(d)$ and $D(d)$ can be calculated explicitly, but certainly they are not optimal. The explicit determination of suitable numbers $\epsilon(d)$ and $D(d)$ is carried out in Section 2 in the more general context of relative curvature measures.

5. Theorem 1.52 will be extended to a statement about relative curvature measures in Section 2.

Proof of Theorem 1.52. The assumption (42) implies that K is of class $C^{1,1}$. Therefore, by Lemma 4.2.3 in [123], we get

$$(1-\epsilon)S_{d-1}(K,\cdot) \leq S_{d-2}(K,\cdot) \leq (1+\epsilon)S_{d-1}(K,\cdot)\,.$$

From this and from the symmetry of the mixed volumes we deduce that

$$W_1(K)^2 - W_0(K)W_2(K) \leq 4\,V(K)^2\,\epsilon\,,$$

where $W_i(K)$ is the i-th quermassintegral of K; compare the contributions by Schneider [122] and Arnold [6]. Moreover, Theorem 1.48 implies that the ball $B^d\left(o, ((d-1)(1+\epsilon))^{-1}\right)$ rolls freely inside K. But then the proof can be completed as in [6]. (Note that the definition of δ_2 in [6] differs from our definition by a normalizing factor.) □

1.8 Stability in Minkowski's uniqueness problem

In this section, we establish a stability result for Minkowski's fundamental (existence and) uniqueness theorem (see Theorem 7.1.1 in [123]) concerning measures over the unit sphere of $\mathbb{R}^d$. Our aim is to provide a stability estimate with optimal exponent for the surface area measures of order $d-1$ of convex bodies, which may be chosen from a large class of convex bodies.

In order to describe this class of convex bodies, we first recall that the surface are measure of order $d-1$ of a convex body $M \in \mathbb{R}^d$ can be decomposed with respect to the $(d-1)$-dimensional Hausdorff measure into an absolutely continuous part $S^a_{d-1}(M,\cdot)$ and a singular part $S^s_{d-1}(M,\cdot)$. The singular part can again be decomposed into two measures, $S^n_{d-1}(M,\cdot)$ and $S^c_{d-1}(M,\cdot)$, where the first of these does not have atoms and the second is a counting measure, that is an at most countable sum of point measures. More explicitly, we define

$$S^c_{d-1}(M,\cdot) := \sum_{u\in S^{d-1}} S_{d-1}(M,\{u\})\delta_u, \tag{45}$$

where $\delta_u(\omega)=1$ if $u\in\omega$, and $\delta_u(\omega)=0$ otherwise, for any Borel set $\omega\subset S^{d-1}$, and we set

$$S^n_{d-1}(M,\cdot) := S^s_{d-1}(M,\cdot) - S^c_{d-1}(M,\cdot)\,.$$

It is easy to see that in (45) at most countably many summands are non-zero; moreover, $S_{d-1}(M,\{u\}) = \mathcal{H}^{d-1}(F(M,u))$ for all $u\in S^{d-1}$. Thus we have the following decomposition

$$S_{d-1}(M,\cdot) = S^a_{d-1}(M,\cdot) + S^c_{d-1}(M,\cdot) + S^n_{d-1}(M,\cdot)$$

of the surface area measure of order $d-1$ into an absolutely continuous component, a component which is a counting measure, and a singular component which does not have point masses. In the following theorem, we shall consider $K,L\in\mathcal{K}^d$ for which the condition

$$S^n_{d-1}((1-t)K+tL,\cdot) \ge \max\{S^n_{d-1}(K,\cdot), S^n_{d-1}(L,\cdot)\}\,, \tag{46}$$

is satisfied for all $t\in[0,1]$. In particular, this condition is fulfilled if K,L are both polytopes or if the surface area measures of order $d-1$ of K and L are absolutely continuous; for instance, the latter is true if the support functions of K and L are of class C^2. Note, however, that $S_{d-1}((1-t)K+tL,\cdot)$, $t\in(0,1)$, is not necessarily absolutely continuous even if $S_{d-1}(K,\cdot)$ and $S_{d-1}(L,\cdot)$ are absolutely continuous for some convex bodies $K,L\in\mathcal{K}^d$; see Example 1 in [72].

Finally, for $0<r<R$ we denote by $\mathcal{K}^d(r,R)$ the set of all convex bodies in $\mathbb{R}^d$ which contain a ball of radius r and are contained in a ball of radius R.

Theorem 1.53 *Let $0<r<R$. Then there exist constants $\epsilon_0\in(0,1)$ and γ, which depend only on d,r,R and can be calculated explicitly, with the following property. If $K,L\in\mathcal{K}^d(r,R)$ satisfy condition* (46) *and fulfill*

$$|S_{d-1}(K,\cdot) - S_{d-1}(L,\cdot)| \le \epsilon \tag{47}$$

for some $\epsilon\in[0,\epsilon_0]$, then

$$\delta(K,L+x) \le \gamma\,\epsilon^{\frac{1}{d-1}}$$

for a suitable vector $x\in\mathbb{R}^d$.

It is appropriate to compare this theorem with the literature. A stability result which holds for all convex bodies in $\mathcal{K}^d(r,R)$, with the smaller exponent $1/d$, was proved by Diskant in [34]. He thus improved a previous result by Volkov [140] who had found a corresponding result with exponent $1/(d+2)$. Alternative approaches, leading to smaller exponents, but implying stability results for projection bodies, have been provided by Bourgain & Lindenstrauss [17] and Campi [24].

For convex bodies with support functions of class C^2 and under a more restrictive assumption than (47), Diskant [36], [37] has shown that the exponent can be increased to $1/(d-1)$; but he could not decide whether this is the optimal value. That the exponent in Theorem 1.53 is indeed optimal for the class of convex bodies considered, can be seen by choosing for K a unit cube and for L the convex body which is obtaind from K by cutting off a vertex of K in such a way that the section plane meets K in a regular $(n-1)$-simplex whose edges have lengths equal to $\epsilon^{\frac{1}{d-1}}$.

Finally, in Theorem 1.18 we treat a more special situation, since there one of the bodies is a ball and also the basic assumption is more restrictive than condition (47). Therefore it is not surprising that a better exponent can be obtained in the situation of Theorem 1.18.

Before we begin with the proof of Theorem 1.53, we make some preliminary remarks. For brevity, we set

$$V_1(M,N) := V(M[d-1],N)\,,$$

where M,N are convex bodies and, as usual, $V(M[d-1],N)$ is the mixed volume with $d-1$ entries M and one entry N. Therefore, we have

$$V_1(M,N) = \frac{1}{d}\int_{S^{d-1}} h(N,u)\,S_{d-1}(M,du)\,.$$

Further, if $M \in \mathcal{K}^d$ and $v \in S^{d-1}$, then we denote by M^v the orthogonal projection of M onto $v^\perp$, and we write $V_{d-1}(\cdot)$ for the volume functional in $(d-1)$-dimensional hyperplanes. Then a special case of equation (5.3.32) in [123] shows that

$$V_{d-1}(M^v) = \frac{1}{2}\int_{S^{d-1}} |\langle u,v\rangle|\,S_{d-1}(M,du)\,.$$

These relationships will be used repeatedly in the following.

Proof of Theorem 1.53. For $d=2$ the theorem follows easily from equation (3.1) in [14]. Henceforth, we consider the case $d \geq 3$.

The proof is divided into six steps. Subsequently, we denote by c_i, $i \in \mathbb{N}$, constants which merely depend on d,r,R. Furthermore, B^d denotes the d-dimensional Euclidean unit ball.

Step I. By Lemma 7.2.3 in [123], the following estimates are satisfied for all $\epsilon \geq 0$ under the present assumptions:

$$|V(K) - V_1(L,K)| \leq \frac{2R}{d}\,\epsilon\,, \tag{48}$$

$$|V(L) - V_1(K,L)| \leq \frac{2R}{d}\,\epsilon\,, \tag{49}$$

$$0 \leq V_1(K,L) - V(K)^{\frac{d-1}{d}}V(L)^{\frac{1}{d}} \leq \frac{2R}{d}\left(\frac{R}{r}+1\right)\epsilon\,, \tag{50}$$

$$0 \leq V_1(L,K) - V(L)^{\frac{d-1}{d}}V(K)^{\frac{1}{d}} \leq \frac{2R}{d}\left(\frac{R}{r}+1\right)\epsilon\,. \tag{51}$$

From $V_1(K,L)^d \geq V(K)^{d-1}V(L)$ and (49) we deduce

$$V(L)^d\left(1+\frac{2R}{d}\frac{\epsilon}{V(L)}\right)^d = \left(V(L)+\frac{2R}{d}\epsilon\right)^d \geq V(K)^{d-1}V(L)\,,$$

and hence

$$V(K) \leq V(L)(1+c_1\,\epsilon)^{\frac{d}{d-1}} \leq V(L) + c_2\,\epsilon\,.$$

By symmetry, we infer that

$$|V(K) - V(L)| \leq c_2\,\epsilon\,. \tag{52}$$

Since $K, L \in \mathcal{K}^d(r,R)$, the mean value theorem and (52) imply that

$$|V(K)^{\frac{1}{d}} - V(L)^{\frac{1}{d}}| \leq c_3\,\epsilon\,. \tag{53}$$

Step II. For $t \in [0,1]$ we set $H_t := (1-t)K + tL$; obviously, $H_t \in \mathcal{K}^d(r,R)$. We shall show that

$$|V(H_t) - V(K)| \leq c_4\,\epsilon \qquad \text{and} \qquad |V(H_t) - V(L)| \leq c_4\,\epsilon \tag{54}$$

for all $t \in [0,1]$.

For the proof, we first define

$$\phi(t) := V(H_t)^{\frac{1}{d}} - (1-t)V(K)^{\frac{1}{d}} - tV(L)^{\frac{1}{d}}\,, \quad t \in [0,1]\,.$$

Then ϕ is concave, smooth, and $\phi(0) = \phi(1) = 0$; in particular,

$$\phi'(0) \geq \phi(t) \geq 0\,, \quad t \in [0,1]\,. \tag{55}$$

A straightforward calculation shows that

$$\phi'(0) = \frac{V_1(K,L) - V(K)^{\frac{d-1}{d}}V(L)^{\frac{1}{d}}}{V(K)^{\frac{d-1}{d}}} \leq c_5\,\epsilon\,, \tag{56}$$

where (50) and $K \in \mathcal{K}^d(r,R)$ were used. Hence, by (53), (55) and (56) we obtain

$$|V(H_t)^{\frac{1}{d}} - V(K)^{\frac{1}{d}}| \leq |V(L)^{\frac{1}{d}} - V(K)^{\frac{1}{d}}| + c_5\,\epsilon \leq c_6\,\epsilon\,,$$

and thus

$$\begin{aligned} V(H_t) &\leq \left(V(K)^{\frac{1}{d}} + c_6\,\epsilon\right)^d \leq V(K)(1+c_7\,\epsilon)^d \\ &\leq V(K) + c_8\,\epsilon\,, \end{aligned}$$

for all $t \in [0,1]$. A similar estimate also gives

$$V(K) \leq V(H_t) + c_9\,\epsilon\,, \quad t \in [0,1]\,.$$

Since the rôles of K and L can be interchanged, the proof of (54) is completed.

Step III. For the remaining part of the proof, we will need two essential estimates, which can be obtained as consequences of (47). To derive these estimates, we introduce the following notation. If $u \in S^{d-1}$ and $M \in \mathcal{K}^d$, then we set

$$v(M,u) := V_{d-1}(F(M,u))$$

and

$$m(u) := \min\{v(K,u), v(L,u)\}\,.$$

The Brunn-Minkowski inequality implies that

$$v(H_t,u)^{\frac{1}{d-1}} \geq (1-t)v(K,u)^{\frac{1}{d-1}} + tv(L,u)^{\frac{1}{d-1}}\,, \quad u \in S^{d-1}\,,$$

and therefore

$$v(H_t,u) \geq m(u)\,, \quad u \in S^{d-1}\,, \tag{57}$$

for all $t \in [0,1]$. Furthermore, for all $u \in S^{d-1}$ such that $h(K,\cdot)$ and $h(L,\cdot)$ are second order differentiable at u, that is for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, we set

$$\overline{m}(u) := \min\{D_{d-1}h(K,u), D_{d-1}h(L,u)\}\,.$$

By known properties of mixed discriminants we then obtain

$$D_{d-1}h(H_t,u)^{\frac{1}{d-1}} \geq (1-t)D_{d-1}h(K,u)^{\frac{1}{d-1}} + tD_{d-1}h(L,u)^{\frac{1}{d-1}}\,,$$

and thus

$$D_{d-1}h(H_t,u) \geq \overline{m}(u)\,, \tag{58}$$

for all $u \in S^{d-1}$ such that $h(K,\cdot)$ and $h(L,\cdot)$ are second order differentiable at u, and thus for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$ and all $t \in [0,1]$.

Let ω_1 denote the set of all $u \in S^{d-1}$ such that $v(K,u) > m(u)$. Hence, ω_1 is an at most countable set, and for $u \in \omega_1$ we have $m(u) = v(L,u)$ and

$$v(K,u) - m(u) = S_{d-1}(K,\{u\}) - S_{d-1}(L,\{u\})\,.$$

This shows that

$$0 \le \sum_{u \in S^{d-1}} (v(K,u) - m(u)) = \sum_{u \in \omega_1} (v(K,u) - m(u))$$
$$= S_{d-1}(K,\omega_1) - S_{d-1}(L,\omega_1) \le \epsilon\,. \tag{59}$$

Further, let ω_2 denote the set of all $u \in S^{d-1}$ such that $h(K,\cdot)$ and $h(L,\cdot)$ are second order differentiable at u and $D_{d-1}h(K,u) > \overline{m}(u)$. Then $\overline{m}(u) = D_{d-1}h(L,u)$ and

$$0 \le \int_{S^{d-1}} (D_{d-1}h(K,u) - \overline{m}(u))\, \mathcal{H}^{d-1}(du) = \int_{\omega_2} (D_{d-1}h(K,u) - \overline{m}(u))\, \mathcal{H}^{d-1}(du)$$
$$= S_{d-1}(K,\omega_2) - S_{d-1}(L,\omega_2) \le \epsilon\,. \tag{60}$$

Here we have used the fact that if $M \in \mathcal{K}^d$ and ω is the set of all $u \in S^{d-1}$ such that $h(M,\cdot)$ is second order differentiable at u, then

$$S_{d-1}(M,\cdot) \llcorner \omega \ll \mathcal{H}^{d-1} \llcorner \omega\,.$$

This can be verified as follows. Choose any normal boundary point $x \in \tau(M,\omega)$ of M, that is $x = \tau_M(u)$ for a uniquely determined $u \in \omega$. Then by Lemma 3.4 in [72], we obtain $k_i(x,u) > 0$ for $i \in \{1,\dots,d-1\}$; moreover, by Lemma 3.1 in [72], we find $k_i(x) = k_i(x,u)$ for $i \in \{1,\dots,d-1\}$. This shows that $H_{d-1}(M,x) > 0$ for $\mathcal{H}^{d-1}$ almost all $x \in \tau(M,\omega)$. Now the assertion is implied by Theorem 1.6.

Step IV. For all $t \in [0,1]$, the estimates

$$|V_1(H_t,B^d) - V_1(K,B^d)| \le c_{10}\,\epsilon \qquad \text{and} \qquad |V_1(H_t,B^d) - V_1(L,B^d)| \le c_{10}\,\epsilon \tag{61}$$

are satisfied.

For the proof, we consider the sum $I = I_K$ which is defined by

$$I := V(H_t) - \frac{1}{d} \sum_{u \in S^{d-1}} h(H_t,u)m(u) - \frac{1}{d} \int_{S^{d-1}} h(H_t,u)\overline{m}(u)\, \mathcal{H}^{d-1}(du)$$
$$- \frac{1}{d} \int_{S^{d-1}} h(H_t,u)\, S^n_{d-1}(K,du)\,,$$

for $t \in [0,1]$. Note that here and in the following summations effectively extend only over countably many $u \in S^{d-1}$, since $v(H_t,u)$, $v(K,u)$ and $v(L,u)$ are non-zero for at most countably many $u \in S^{d-1}$. Therefore, we can simply write $\sum$ instead of $\sum_{u \in S^{d-1}}$. Our next aim is to estimate I from above and from below. Similarly, we simply write $\int$ instead of $\int_{S^{d-1}}$ if the integration is extended over S^{d-1}.

Let $t \in [0,1]$ be fixed for the moment. Using (54), (59) and (60), we obtain

$$
\begin{aligned}
I =& V(H_t) - \frac{1}{d}\int h(H_t,u)\, S_{d-1}(K,du) \\
&+ \frac{1}{d}\sum h(H_t,u)(v(K,u)-m(u)) \\
&+ \frac{1}{d}\int h(H_t,u)(D_{d-1}h(K,u)-\overline{m}(u))\,\mathcal{H}^{d-1}(du) \\
\le& (1-t)(V(H_t)-V(K)) + t(V(H_t)-V_1(K,L)) \\
&+ \frac{R}{d}\left[\sum (v(K,u)-m(u)) + \int (D_{d-1}h(K,u)-\overline{m}(u))\,\mathcal{H}^{d-1}(du)\right] \\
\le& t(V(H_t)-V_1(K,L)) + c_{11}\,\epsilon\,. \qquad (62)
\end{aligned}
$$

In addition, by (46), (57) and (58) we can estimate

$$
\begin{aligned}
I \ge& \frac{1}{d}\int h(H_t,u)\,(S^n_{d-1}(H_t,du)-S^n_{d-1}(K,du)) \\
&+ \frac{1}{d}\sum h(H_t,u)(v(H_t,u)-m(u)) \\
&+ \frac{1}{d}\int h(H_t,u)(D_{d-1}h(H_t,u)-\overline{m}(u))\,\mathcal{H}^{d-1}(du) \\
\ge& \frac{r}{d}\Big\{ S^n_{d-1}(H_t,S^{d-1}) - S^n_{d-1}(K,S^{d-1}) \\
&\quad + \sum (v(H_t,u)-m(u)) + \int (D_{d-1}h(H_t,u)-\overline{m}(u))\,\mathcal{H}^{d-1}(du)\Big\} \\
\ge& r(V_1(H_t,B^d)-V_1(K,B^d))\,. \qquad (63)
\end{aligned}
$$

Combining (62) and (63), we find

$$
V_1(H_t,B^d)-V_1(K,B^d) \le \frac{t}{r}(V(H_t)-V_1(K,L)) + c_{12}\,\epsilon\,.
$$

By (54) and (49),

$$
\begin{aligned}
|V(H_t)-V_1(K,L)| &\le |V(H_t)-V(L)| + |V(L)-V_1(K,L)| \\
&\le c_{13}\,\epsilon\,,
\end{aligned}
$$

and thus

$$
V_1(H_t,B^d)-V_1(K,B^d) \le c_{14}\,\epsilon\,, \quad t\in[0,1]\,. \qquad (64)
$$

On the other hand, from (57), (58), (46), (59) and (60) we deduce

$$\begin{aligned}
V_1(H_t,B) \geq& \frac{1}{d} S^n_{d-1}(H_t,S^{d-1}) + \frac{1}{d}\sum m(u) + \frac{1}{d}\int \overline{m}(u)\,\mathcal{H}^{d-1}(du)\\
\geq& \frac{1}{d} S_{d-1}(K,S^{d-1}) + \frac{1}{d}\sum v(K,u) + \frac{1}{d}\int D_{d-1}h(K,u)\,\mathcal{H}^{d-1}(du)\\
&- \frac{1}{d}\sum (v(K,u)-m(u)) - \frac{1}{d}\int (D_{d-1}h(K,u)-\overline{m}(u))\,\mathcal{H}^{d-1}(du)\\
\geq& V_1(K,B^d) - \epsilon\,.
\end{aligned} \tag{65}$$

Now (64) and (65) yield the first estimate of (61), and the second follows by interchanging K and L.

Step V. If $v \in S^{d-1}$ and $t \in [0,1]$, then

$$|V_{d-1}(H_t^v) - V_{d-1}(K^v)| \leq c_{15}\,\epsilon \qquad \text{and} \qquad |V_{d-1}(H_t^v) - V_{d-1}(L^v)| \leq c_{15}\,\epsilon\,. \tag{66}$$

For the proof, we define $J = J_K$ by

$$\begin{aligned}
J := V_{d-1}(H_t^v) &- \frac{1}{2}\sum_{u\in S^{d-1}} |\langle u,v\rangle| m(u) - \frac{1}{2}\int_{S^{d-1}} |\langle u,v\rangle| \overline{m}(u)\,\mathcal{H}^{d-1}(du)\\
&- \frac{1}{2}\int_{S^{d-1}} |\langle u,v\rangle|\, S^n_{d-1}(K,du)\,.
\end{aligned}$$

Since

$$\begin{aligned}
J =& \frac{1}{2}\int |\langle u,v\rangle|\,(S^n_{d-1}(H_t,du) - S^n_{d-1}(K,du))\\
&+ \frac{1}{2}\sum |\langle u,v\rangle| (v(H_t,u) - m(u))\\
&+ \frac{1}{2}\int |\langle u,v\rangle| (D_{d-1}h(H_t,u) - \overline{m}(u))\,\mathcal{H}^{d-1}(du) \geq 0\,,
\end{aligned}$$

we can deduce that

$$\begin{aligned}
V_{d-1}(H_t^v) - V_{d-1}(K^v) \geq& -\frac{1}{2}\sum |\langle u,v\rangle| (v(K,u) - m(u))\\
&- \frac{1}{2}\int |\langle u,v\rangle| (D_{d-1}h(K,u) - \overline{m}(u))\,\mathcal{H}^{d-1}(du)\\
\geq& -\epsilon\,.
\end{aligned} \tag{67}$$

On the other hand, by (61)

$$\begin{aligned} V_{d-1}(H_t^v) - V_{d-1}(K^v) \leq & J \\ \leq & S_{d-1}^n(H_t, S^{d-1}) - S_{d-1}^n(K, S^{d-1}) \\ & + \sum (v(H_t,u) - m(u)) + \int (D_{d-1}h(H_t,u) - \overline{m}(u))\, \mathcal{H}^{d-1}(du) \\ = & dV_1(H_t, B^d) - dV_1(K, B^d) \\ & + \sum (v(K,u) - m(u)) + \int (D_{d-1}h(K,u) - \overline{m}(u))\, \mathcal{H}^{d-1}(du) \\ \leq & c_{16}\, \epsilon\,. \end{aligned} \tag{68}$$

The estimates (67) and (68) yield the first estimate in (66), and the second estimate follows by symmetry.

Step VI. Now we consider the function

$$\Phi_v(K,L,t) := V_{d-1}(H_t^v)^{\frac{1}{d-1}} - (1-t)V_{d-1}(K^v)^{\frac{1}{d-1}} - tV_{d-1}(L^v)^{\frac{1}{d-1}}\,, \quad t \in [0,1]\,,$$

for an arbitrary but fixed $v \in S^{d-1}$. Using (66) and the mean value theorem, we can estimate

$$\begin{aligned} \Phi_v(K,L,t) &= (1-t)\left[V_{d-1}(H_t^v)^{\frac{1}{d-1}} - V_{d-1}(K^v)^{\frac{1}{d-1}}\right] + t\left[V_{d-1}(H_t^v)^{\frac{1}{d-1}} - V_{d-1}(L^v)^{\frac{1}{d-1}}\right] \\ &\leq c_{17}\, \epsilon\,. \end{aligned} \tag{69}$$

Choose $\lambda > 0$ so that $V_{d-1}(\lambda K^v) = V_{d-1}(L^v)$; then by (66)

$$\lambda_1 := 1 - c_{18}\,\epsilon \leq \left(1 - \frac{c_{15}}{\kappa_{d-1} r^{d-1}}\,\epsilon\right)^{\frac{1}{d-1}} \leq \lambda \leq \left(1 + \frac{c_{15}}{\kappa_{d-1} r^{d-1}}\,\epsilon\right)^{\frac{1}{d-1}} \leq 1 + c_{18}\,\epsilon =: \lambda_2$$

if $0 < \epsilon \leq \kappa_{d-1} r^{d-1} c_{15}^{-1}$. But then

$$\begin{aligned} \Phi_v(\lambda K, L, t) &\leq \lambda_2 \Phi_v(K,L,t) + (\lambda_2 - \lambda_1)\left\{(1-t)V(K^v)^{\frac{1}{d-1}} + tV(L^v)^{\frac{1}{d-1}}\right\} \\ &\leq c_{19}\, \epsilon\,. \end{aligned}$$

Now the main Theorem in [35] shows that there exist ϵ_0 and $\bar{\gamma}$, depending only on d, r, R, such that

$$\delta(\lambda K^v, L^v + x(v)) \leq \bar{\gamma}\, \epsilon^{\frac{1}{d-1}}$$

for some $x(v) \in v^{\perp}$, and therefore

$$\delta(K^v, L^v + x(v)) \leq \bar{\gamma}\, \epsilon^{\frac{1}{d-1}}\,.$$

Thus the assertion of the theorem is implied by Theorem 4.3.4 in [44]. □

1.9 A measure geometric structure theorem and applications

For a long time the Besicovitch-Federer structure theorem (see Chapter 3.3 in [42]) has played a central rôle in geometric measure theory. We refer to the survey paper [20] for some general comments on this theorem and its impact. Although Ross [111] has developed a simplified approach to the structure theorem, the proof of this key result still remained long and involved. This fact (and other reasons) have eventually led to the development of alternative approaches to some parts of geometric measure theory which previously did depend on the structure theorem. The recent discovery of a relatively simple integral-geometric proof of the structure theorem by White [154] may well change the general attitude again.

The structure theorem has been extended to certain homogeneous spaces by Brothers [21]. In this general situation a simple proof has not yet been worked out (if possible at all). Subsequently, we establish an integral-geometric transformation formula, which should be interesting in its own right and from which the structure theorem in spherical spaces follows easily. In fact, the argument actually shows that the essential statements of the structure theorem in Euclidean and spherical space are equivalent. We should emphasize that the latter result is not just a simple corollary of the Euclidean version, since global integral-geometric considerations are required in the argument.

Our original interest in this subject was motivated by certain applications to the investigation of the absolute continuity of surface area measures, which are also included in this section. This part of the present work has to be seen in connection with recent work on Hessian measures of convex functions [29], [30]. There a general strategy is described which allows one to deduce certain results on curvature or surface area measures of convex sets from corresponding more general theorems about convex functions.

We start with some preparatory remarks. Let $d \geq 2$ and $m \in \{0, \dots, d-1\}$. Let ν denote the (complete) normalized invariant measure over the group $\mathbf{O}(d)$ of rotations in Euclidean space $\mathbb{R}^d$. Recall that $\langle \cdot , \cdot \rangle$ and $| \cdot |$ are the scalar product and the norm, respectively. Let $U_0 \in \mathbf{G}(d, d-m)$ be arbitrarily chosen and fixed, denote by κ_n the volume of the n-dimensional unit ball B^n for $n \in \mathbb{N}$, and define $\omega_n := n\kappa_n$ (as before). Then the m-dimensional spherical integral-geometric measure is defined by

$$\mathcal{I}_S^m(A) := \frac{\omega_{m+1}}{2} \int_{\mathbf{O}(d)} \operatorname{card}(A \cap \rho U_0) \nu(d\rho) , \tag{70}$$

if $A \in \mathfrak{B}(S^{d-1})$, and

$$\mathcal{I}_S^m(M) := \inf \left\{ \mathcal{I}_S^m(A) : M \subset A \in \mathfrak{B}(S^{d-1}) \right\} \tag{71}$$

for $M \subset S^{d-1}$; compare the remarks in [20], [21]. It is easy to check that $\mathcal{I}_S^m$ is a Borel regular outer measure over S^{d-1}. In order to show the measurability of the map

$$\mathbf{O}(d) \to [0, \infty] , \quad \rho \mapsto \operatorname{card}(A \cap \rho U_0) ,$$

for each $A \in \mathfrak{B}(S^{d-1})$, one can either refer to Theorem 3.2.48 of Federer's book [42] or, more explicitly, proceed as follows. Let $\pi_3 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{O}(d) \to \mathbf{O}(d)$ denote the projection onto the third component. If $S \subset S^{d-1}$ is a Borel set, then the set

$$\begin{aligned} &\{\rho \in \mathbf{O}(d) : S \cap \rho U_0 \neq \emptyset\} \\ &\quad = \pi_3(\{(x, u, \rho) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{O}(d) : x \in S, u \in U_0, \rho u = x\}) , \end{aligned}$$

is measurable with respect to the completion of the measure space $(\mathbf{O}(d), \mathfrak{B}(\mathbf{O}(d)), \nu)$. This follows from the theory of Suslin sets; compare [42, §2.2], [28, §8.4]. Set

$$\chi_{S \cap \rho U_0} := \begin{cases} 0\,, & \text{if } S \cap \rho U_0 = \emptyset\,, \\ 1\,, & \text{otherwise}\,. \end{cases}$$

Then, as $j \to \infty$,

$$\sum_{S \in \mathcal{H}_j} \chi_{S \cap \rho U_0} \uparrow \operatorname{card}(A \cap \rho U_0)$$

if $(\mathcal{H}_j)_{j \in \mathbb{N}}$ is a sequence of Borel partitions of A such that

$$\lim_{j \to \infty} \sup\{\operatorname{diam}(H) : H \in \mathcal{H}_j\} = 0\,.$$

This yields the asserted measurability statement.

The assumption of the subsequent theorem involves a finiteness condition on Hausdorff measures over S^{d-1}. The metric with respect to which the Hausdorff measures are constructed can either be the interior geodesic metric of S^{d-1} or the metric of the surrounding Euclidean space $\mathbb{R}^d$. It is easy to see that the value of the two measures thus obtained is independent of the chosen metric, i.e., for all $r \geq 0$,

$$\mathcal{H}^r_{S^{d-1}} = \mathcal{H}^r_{\mathbb{R}^d}\,.$$

Therefore we shall simply write $\mathcal{H}^r$ in the following.

Theorem 1.54 *Let $d \geq 2$ and $m \in \{0, \ldots, d-1\}$. Assume that $A \subset S^{d-1}$ and $\mathcal{H}^m(A) < \infty$. Then there is a countably m-rectifiable Borel subset R of S^{d-1} such that $A \setminus R$ is purely $(\mathcal{H}^m, m)$-unrectifiable and*

$$\mathcal{I}^m_S(A \setminus R) = 0\,;$$

moreover, $\mathcal{H}^m(A) = \mathcal{I}^m_S(A)$ if and only if A is $(\mathcal{H}^m, m)$-rectifiable.

With such a structure theorem for subsets of the unit sphere, we can now remove an assumption in the statement of a result on surface area measures $S_m(K, \cdot)$ of convex bodies $K \in \mathcal{K}^d$; compare Chapter 4.6 in Schneider's book [123].

Corollary 1.55 *Let $K \in \mathcal{K}^d$ and $m \in \{0, \ldots, d-1\}$. Then*

$$S_m(K, \omega) \leq a_2\, \mathcal{H}^{d-1-m}(\omega)$$

for each $\omega \in \mathfrak{B}(S^{d-1})$, with some constant a_2 depending only on d, m and on the surface areas of the projections of K onto $(m+1)$-dimensional subspaces of $\mathbb{R}^d$.

A corresponding result for curvature measures was found by Schneider (compare [123]). However, for curvature measures a simpler approach, based on an idea of Fallert [41], is provided in [30]. A result similar to Corollary 1.55 is outlined in the Notes to Section 4.6 in [123].

The following two theorems, which are suggested by Corollary 1.55 and its counterpart for curvature measures, represent the final step in a number of preliminary results (actually, Theorem 1.56 can be extended to sets with positive reach as shown in [30]). We refer to [73] for a discussion of the relevant literature; there one can also find the most general versions of Theorems 1.56 and 1.57 which were previously known (see Theorems 3.2 and 4.3 in [73]).

Theorem 1.56 *Let $K \subset \mathbb{R}^d$ be a convex body and $j \in \{0, \dots, d-1\}$. Further, let $\alpha \subset \mathbb{R}^d$ be a Borel set having σ-finite j-dimensional Hausdorff measure, and let $\eta \subset \alpha \times \mathbb{R}^d$ be Borel measurable. Then*

$$\binom{d-1}{j}\Theta_j(K,\eta) = \int_{\mathbb{R}^d} \mathcal{H}^{d-1-j}(N(K,x) \cap \eta_x)\mathcal{H}^j(dx)\,,$$

where $\eta_x := \{u \in S^{d-1} : (x,u) \in \eta\}$.

Theorem 1.57 *Let $K \subset \mathbb{R}^d$ be a convex body and $j \in \{0, \dots, d-1\}$. Further, let $\omega \subset S^{d-1}$ be a Borel set having σ-finite $(d-1-j)$-dimensional Hausdorff measure, and let $\eta \subset \mathbb{R}^d \times \omega$ be Borel measurable. Then*

$$\binom{d-1}{j}\Theta_j(K,\eta) = \int_{S^{d-1}} \mathcal{H}^{j}(F(K,u) \cap \eta^u)\mathcal{H}^{d-1-j}(du)\,,$$

where $\eta^u := \{x \in \mathbb{R}^d : (x,u) \in \eta\}$.

In [30], these results are deduced from a corresponding theorem involving semi-convex functions. The proof of this theorem is essentially based on Federer's structure theorem in Euclidean space. Following such an approach, the proof of Theorem 1.57 turns out to be more involved than the argument required for Theorem 1.56, since it additionally requires a duality theorem for Hessian measures of convex functions. On the other hand, Theorems 1.56 and 1.57 both admit a direct approach which avoids the use of Hessian measures. We shall demonstrate this for Theorem 1.57, and this will provide another (though related) application of the structure theorem in spherical space. A proof for Theorem 1.56 can be given along similar lines.

The proof of Theorem 1.54 will be based on an integral-geometric theorem and on the Euclidean version of the structure theorem. For the statement and the proof of the integral-geometric result we need some additional notation. Let $U \in \mathbf{G}(d,j)$ and $V \in \mathbf{G}(d,k)$ be in general position, that is, assume that

$$n := \dim(U \cap V) = \max\{j+k-d, 0\}\,.$$

Choose vectors $x_1, \dots, x_n$ (if $n > 0$), $y_{n+1}, \dots, y_j, z_{n+1}, \dots, z_k$ so that $(x_1, \dots, x_n)$ is an orthonormal basis of $U \cap V$, $(x_1, \dots, x_n, y_{n+1}, \dots, y_j)$ is an orthonormal basis of U and $(x_1, \dots, x_n, z_{n+1}, \dots, z_k)$ is an orthonormal basis of V. Then we set

$$[U,V] := |\det(x_1, \dots, x_n, y_{n+1}, \dots, y_j, z_{n+1}, \dots, z_k)|\,.$$

This definition is independent of the particular choice of bases. Furthermore, we set $H_e := \{x \in \mathbb{R}^d : \langle x, e\rangle = 1\}$ if $e \in \mathbb{R}^d \setminus \{o\}$, and we define $\mathbf{A}(H_e, m)$ to be the set of all m-dimensional affine subspaces of H_e.

Theorem 1.58 *Let $d \geq 2$, $m \in \{0, \dots, d-2\}$, and $A \in \mathfrak{B}(S^{d-1})$. Let $e \in S^{d-1}$ be fixed. Then, for any measurable function $f : \mathbf{G}(d, m+1) \to [0,\infty]$,*

$$\int_{\mathbf{G}(d,m+1)} f(U)\mathcal{H}^{(m+1)(d-1-m)}(dU)$$
$$= \int_{\mathbf{A}(H_e,m)} f(\text{lin}\{E\})[\text{lin}\{E\}, e^\perp]^d \mathcal{H}^{(m+1)(d-1-m)}(dE)\,. \qquad (72)$$

Before we provide a proof for Theorem 1.58 some remarks are appropriate.

First, the Hausdorff measure on the left-hand side of equation (72) is, up to a constant multiplier, equal to the $\mathbf{O}(d)$-invariant normalized Haar measure on $\mathbf{G}(d, m+1)$. In particular,

$$\mathcal{H}^{(m+1)(d-1-m)}(\mathbf{G}(d, m+1)) = \frac{\omega_d \cdots \omega_{d-m}}{\omega_{m+1} \cdots \omega_1} =: c_{d,m+1}\,,$$

where $\omega_n = \mathcal{H}^{n-1}(S^{n-1}) = n\kappa_n$ for $n \in \mathbb{N}$. The Hausdorff measure on the right-hand side is up to a constant multiplier equal to the usual motion invariant Haar measure on $\mathbf{A}(H_e, m)$. Especially, it satisfies the relationship

$$\begin{aligned}
&\int_{\mathbf{A}(H_e, m)} g(E)\mathcal{H}^{(m+1)(d-1-m)}(dE) \\
&= \int_{\mathbf{G}(e^\perp, m)} \int_{e^\perp \cap U^\perp} g(U + t + e)\mathcal{H}^{d-1-m}(dt)\mathcal{H}^{m(d-1-m)}(dU)\,,
\end{aligned}$$

where $e \in S^{d-1}$ is fixed, $g : \mathbf{A}(H_e, m) \to [0, \infty]$ is an arbitrary measurable function, and $\mathbf{G}(e^\perp, m)$ is the Grassmannian of m-dimensional linear subspaces of the linear subspace $e^\perp := \{v \in \mathbb{R}^d : \langle v, e\rangle = 0\}$.

Second, there is a slightly more general variant of Theorem 1.58 for oriented subspaces and oriented planes. It can be proved by using the coarea formula and some calculations involving the Grassmann algebra. In contrast to this method, the approach which we prefer presently mainly uses classical formulae of integral geometry. A nice exposition of this method is provided in Chapter 6 of the book by Schneider and Weil [132].

Proof of Theorem 1.58. The proof of the case $m = 0$ requires only relations (73) and (74) of the subsequent argument. Therefore we can assume $m \geq 1$. Recall that ν_{m+1} denotes the $\mathbf{O}(d)$-invariant probability measure over $\mathbf{G}(d, m+1)$ and set $c_1 := \mathcal{H}^{d-1}(S^{d-1})^{-(m+1)}$. Then we have

$$\begin{aligned}
&\int_{\mathbf{G}(d, m+1)} f(U)\nu_{m+1}(dU) \\
&= c_1 \int_{S^{d-1}} \cdots \int_{S^{d-1}} f(\mathrm{lin}\{u_0, \dots, u_m\})\mathcal{H}^{d-1}(du_0) \dots \mathcal{H}^{d-1}(du_m)\,. \qquad (73)
\end{aligned}$$

In fact, if f is specialized to indicator functions of Borel subsets of $\mathbf{G}(d, m+1)$, then the integrals on both sides of equation (73) define invariant measures with respect to $\mathbf{O}(d)$. Hence, they must be equal up to a constant multiplier. This constant c_1 can be determined explicitly by setting $f \equiv 1$.

Now consider the transformation

$$\varphi : S^{d-1} \setminus e^\perp \to H_e\,, \quad u \mapsto \langle u, e\rangle^{-1} u\,.$$

Let $(u, v_2, \dots, v_d)$ be an orthonormal basis of $\mathbb{R}^d$. A straightforward calculation then yields for the Jacobian of this map

$$
\begin{aligned}
J_{d-1}\varphi(u) &= \left|\det\left(\frac{1}{\langle u,e\rangle}\left(v_2-\frac{\langle v_2,e\rangle}{\langle u,e\rangle}u\right),\dots,\frac{1}{\langle u,e\rangle}\left(v_d-\frac{\langle v_d,e\rangle}{\langle u,e\rangle}u\right)\right)\right| \\
&= |\langle u,e\rangle|^{-(d-1)}\left\{1+\sum_{i=2}^{d}\left(\frac{\langle v_i,e\rangle}{\langle u,e\rangle}\right)^2\right\}^{\frac{1}{2}} \\
&= |\langle u,e\rangle|^{-d} = |\varphi(u)|^d .
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\int_{S^{d-1}}\dots\int_{S^{d-1}} f(\mathrm{lin}\{u_0,\dots,u_m\})\mathcal{H}^{d-1}(du_0)\dots\mathcal{H}^{d-1}(du_m) \\
&= 2^{m+1}\int_{H_e}\dots\int_{H_e} f(\mathrm{lin}\{x_0,\dots,x_m\})\prod_{i=0}^{m}|x_i|^{-d}\mathcal{H}^{d-1}(dx_0)\dots\mathcal{H}^{d-1}(dx_m) . \qquad (74)
\end{aligned}
$$

For the next step we apply the affine Blaschke-Petkantschin formula in the $(d-1)$-dimensional affine subspace H_e; see Satz 6.1.5 in [132]. Hence,

$$
\begin{aligned}
&\int_{H_e}\dots\int_{H_e} f(\mathrm{lin}\{x_0,\dots,x_m\})\prod_{i=0}^{m}|x_i|^{-d}\mathcal{H}^{d-1}(dx_0)\dots\mathcal{H}^{d-1}(dx_m) \\
&= c_{d-1,m}\int_{\mathbf{A}(H_e,m)}\int_E\dots\int_E f(\mathrm{lin}\{E\})\prod_{i=0}^{m}|x_i|^{-d} \\
&\quad\times|\det(x_1-x_0,\dots,x_m-x_0)|^{d-1-m}\mathcal{H}^m(dx_0)\dots\mathcal{H}^m(dx_m)\ \mu_m^{H_e}(dE) , \qquad (75)
\end{aligned}
$$

where $\mu_m^{H_e}$ is the natural motion-invariant measure on $\mathbf{A}(H_e, m)$, normalized as in [132], and

$$
c_{d-1,m} := \frac{\omega_{d-1}\cdots\omega_{d-m}}{\omega_m\cdots\omega_1} .
$$

Observe that

$$
|\det(x_1-x_0,\dots,x_m-x_0)| = [\mathrm{lin}\{x_0,\dots,x_m\},e^{\perp}]\,|\det(x_0,\dots,x_m)| \qquad (76)
$$

for $\otimes^{m+1}\mathcal{H}^m$ almost all $(x_0,\dots,x_m)\in E^{m+1}$. To verify this, we can assume that $\mathrm{lin}\{x_0,\dots,x_m\}\in\mathbf{G}(d,m+1)$. Let $(u_1,\dots,u_{m+1})$ be an orthonormal basis of the linear subspace $\mathrm{lin}\{x_0,\dots,x_m\}$ so that $\langle u_{m+1},e\rangle>0$ and

$$
\mathrm{lin}\{x_0,\dots,x_m\}\cap e^{\perp} = \mathrm{lin}\{u_1,\dots,u_m\} .
$$

But then, for any $x\in\mathrm{lin}\{x_0,\dots,x_m\}\cap H_e$,

$$
\begin{aligned}
[\mathrm{lin}\{x_0,\dots,x_m\},e^{\perp}] &= \langle u_{m+1},e\rangle \\
&= \left|x-\sum_{i=1}^{m}\langle x,u_i\rangle u_i\right|^{-1} ,
\end{aligned}
$$

since

$$u_{m+1} = \frac{x - \sum_{i=1}^m \langle x, u_i \rangle u_i}{|x - \sum_{i=1}^m \langle x, u_i \rangle u_i|} .$$

This finally implies (76), if also

$$\begin{aligned} |\det(x_0, x_1, \dots, x_m)| &= |\det(x_1 - x_0, \dots, x_m - x_0)| \, |\det(x_0, u_1, \dots, u_m)| \\ &= |\det(x_1 - x_0, \dots, x_m - x_0)| \left| x_0 - \sum_{i=1}^m \langle x_0, u_i \rangle u_i \right| \end{aligned}$$

is used.

Thus the right-hand side of (75) is equal to

$$\begin{aligned} & c_{d-1,m} \int_{\mathbf{A}(H_e,m)} \int_E \dots \int_E f(\operatorname{lin}\{E\}) \\ & \times \left| \det\left(\frac{x_0}{|x_0|}, \dots, \frac{x_m}{|x_m|} \right) \right|^{d-1-m} \prod_{j=0}^m \left(|x_j|^{m+1} [\operatorname{lin}\{E\}, e^\perp] \right)^{-1} \\ & \times [\operatorname{lin}\{E\}, e^\perp]^d \mathcal{H}^m(dx_0) \dots \mathcal{H}^m(dx_m) \mu_m^{H_e}(dE) \\ = \quad & c_{d-1,m} \int_{\mathbf{A}(H_e,m)} f(\operatorname{lin}\{E\}) [\operatorname{lin}\{E\}, e^\perp]^d \\ & \times \int_E \dots \int_E \left| \det\left(\frac{x_0}{|x_0|}, \dots, \frac{x_m}{|x_m|} \right) \right|^{d-1-m} \prod_{j=0}^m \left(|x_j|^{m+1} [\operatorname{lin}\{E\}, e^\perp] \right)^{-1} \\ & \times \mathcal{H}^m(dx_0) \dots \mathcal{H}^m(dx_m) \mu_m^{H_e}(dE) . \end{aligned} \tag{77}$$

Define the map

$$\psi^E : E \to S^{d-1} \cap \operatorname{lin}\{E\} , \quad x \mapsto |x|^{-1} x .$$

If the previous notation is used, then an elementary calculation leads to

$$\begin{aligned} J_m \psi^E(x) &= \left| \det\left(\frac{1}{|x|} \left(u_1 - \left\langle \frac{x}{|x|}, u_1 \right\rangle \frac{x}{|x|} \right), \dots, \frac{1}{|x|} \left(u_m - \left\langle \frac{x}{|x|}, u_m \right\rangle \frac{x}{|x|} \right) \right) \right| \\ &= |x|^{-(m+1)} |\det(u_1, \dots, u_m, x)| \\ &= \left(|x|^{m+1} [\operatorname{lin}\{E\}, e^\perp] \right)^{-1} . \end{aligned}$$

Thus

$$\begin{aligned} & \int_E \dots \int_E \left| \det\left(\frac{x_0}{|x_0|}, \dots, \frac{x_m}{|x_m|} \right) \right|^{d-1-m} \prod_{j=0}^m \left(|x_j|^{m+1} [\operatorname{lin}\{E\}, e^\perp] \right)^{-1} \mathcal{H}^m(dx_0) \dots \mathcal{H}^m(dx_m) \\ & = 2^{-(m+1)} \int_{S^{d-1} \cap \operatorname{lin}\{E\}} \dots \int_{S^{d-1} \cap \operatorname{lin}\{E\}} |\det(u_0, \dots, u_m)|^{d-1-m} \mathcal{H}^m(du_0) \dots \mathcal{H}^m(du_m) \\ & = 2^{-(m+1)} (c_1 c_{d,m+1})^{-1} . \end{aligned} \tag{78}$$

In order to derive the last equality one can proceed as follows. A special case of the linear Blaschke-Petkantschin formula (see Satz 6.1.3 in [132]) yields

$$\kappa_d^{m+1} = c_{d,\,m+1} \int_{B^d \cap U} \cdots \int_{B^d \cap U} |\det(x_0, \dots, x_m)|^{d-1-m} \mathcal{H}^{m+1}(dx_0) \dots \mathcal{H}^{m+1}(dx_m)\,, \qquad (79)$$

if $U \in \mathbf{G}(d, m+1)$ is arbitrarily chosen. Introducing polar coordinates on the right-hand side of equation (79), we obtain the desired relation.

Finally, from relations (73)–(78) we deduce

$$\int_{\mathbf{G}(d,\,m+1)} f(U)\nu_{m+1}(dU) = \frac{c_{d-1,\,m}}{c_{d,\,m+1}} \int_{\mathbf{A}(H_e,\,m)} f(\operatorname{lin}\{E\})[\operatorname{lin}\{E\}, e^\perp]^d \mu_m^{H_e}(dE)\,,$$

and this is equivalent to the statement of the theorem. □

Proof of Theorem 1.54. The case $m = 0$ can easily be checked directly. Henceforth we assume $m \in \{1, \dots, d-1\}$. The set $A \subset S^{d-1}$ can also be conceived as a subset of $\mathbb{R}^d$. According to our previous remark concerning Hausdorff measures, Federer's structure theorem can be applied. Thus there is a countably m-rectifiable Borel set $R \subset \mathbb{R}^d$ such that $A \setminus R$ is purely $(\mathcal{H}^m, m)$-unrectifiable. We can assume that $R \subset S^{d-1}$ by replacing R by $R \cap S^{d-1}$. Next we show that an arbitrary purely $(\mathcal{H}^m, m)$-unrectifiable subset M of S^{d-1} with $\mathcal{H}^m(M) < \infty$ fulfills

$$\mathcal{I}_S^m(M) = 0\,.$$

Since $\mathcal{H}^m$ is Borel regular, it is sufficient to consider the case where $M \in \mathfrak{B}(S^{d-1})$. Furthermore, by decomposing M into finitely many subsets we can assume that

$$M \subset \left\{ u \in S^{d-1} : \langle u, e \rangle \geq \frac{1}{2} \right\},$$

for some $e \in S^{d-1}$. Let $H_e := \{x \in \mathbb{R}^d : \langle x, e \rangle = 1\}$ be defined as before, and define the homeomorphism

$$\varphi : \{u \in S^{d-1} : \langle u, e \rangle > 0\} \to H_e\,, \quad u \mapsto \langle u, e \rangle^{-1} u\,,$$

which is locally bi-Lipschitz. In particular, $\varphi(M)$ is a purely $(\mathcal{H}^m, m)$-unrectifiable subset of the $(d-1)$-dimensional affine subspace H_e and $\mathcal{H}^m(\varphi(M)) < \infty$. Hence, from Federer's structure theorem, now applied to $\varphi(M)$ in H_e, we obtain

$$\int_{\mathbf{A}(H_e,\,d-1-m)} \operatorname{card}(\varphi(M) \cap E)\mathcal{H}^{(d-m)m}(dE) = 0\,,$$

and hence also

$$\int_{\mathbf{A}(H_e,\,d-1-m)} \operatorname{card}(\varphi(M) \cap \operatorname{lin}\{E\})[\operatorname{lin}\{E\}, e^\perp]^d \mathcal{H}^{(d-m)m}(dE) = 0\,.$$

But now Proposition 1.58 implies

$$\int_{\mathbf{G}(d,\,d-m)} \operatorname{card}(\varphi(M) \cap U)\mathcal{H}^{(d-m)m}(dU)\,,$$

which is equivalent to $\mathcal{I}_S^m(M) = 0$, since

$$\operatorname{card}(M \cap U) = \operatorname{card}(\varphi(M) \cap U)$$

for all $U \in \mathbf{G}(d, d-m)$.

On the other hand, if A is a $(\mathcal{H}^m, m)$-rectifiable subset of S^{d-1}, then

$$\mathcal{H}^m(A) = \mathcal{I}_S^m(A)\,. \tag{80}$$

This follows from a special case of Theorem 3.2.48 in [42].

Finally, let us assume that equation (80) holds for some $A \subset S^{d-1}$. Then

$$\begin{aligned}
\mathcal{H}^m(A) &= \mathcal{H}^m(A \cap R) + \mathcal{H}^m(A \setminus R) \\
&= \mathcal{I}_S^m(A \cap R) + \mathcal{H}^m(A \setminus R) \\
&= \mathcal{I}_S^m(A) + \mathcal{H}^m(A \setminus R) \\
&= \mathcal{H}^m(A) + \mathcal{H}^m(A \setminus R)\,.
\end{aligned} \tag{81}$$

Recall that the set R is a countably m-rectifiable Borel subset of S^{d-1} such that $A \setminus R$ is purely $(\mathcal{H}^m, m)$-unrectifiable. Moreover we have used

$$\begin{aligned}
\mathcal{I}_S^m(A) &= \mathcal{I}_S^m(A \cap R) + \mathcal{I}_S^m(A \setminus R) \\
&= \mathcal{I}_S^m(A \cap R)\,,
\end{aligned}$$

and also the results which have been verified in the preceding part of the proof. Relation (81) then implies $\mathcal{H}^m(A \setminus R) = 0$, and thus A is $(\mathcal{H}^m, m)$-rectifiable. □

Proof of Corollary 1.55. The proof is an immediate consequence of the proof for Theorem 4.6.5 in [123] and of Theorem 1.54. □

Proof of Theorem 1.57. For the proof we can assume that $\mathcal{H}^{d-1-j}(\omega) < \infty$. By Theorem 4.3 in [73] und using the definition of the set

$$\Sigma_{d-1-j}(K) = \{u \in S^{d-1} : \dim F(K,u) \geq j\}\,,$$

we obtain

$$\begin{aligned}
\binom{d-1}{j}\Theta_j(K, \eta \cap (\mathbb{R}^d \times \Sigma_{d-1-j}(K))) &= \int_{\Sigma_{d-1-j}(K)} \mathcal{H}^j(F(K,u) \cap \eta^u)\mathcal{H}^{d-1-j}(du) \\
&= \int_{S^{d-1}} \mathcal{H}^j(F(K,u) \cap \eta^u)\mathcal{H}^{d-1-j}(du)\,.
\end{aligned}$$

Hence it is sufficient to show that

$$S_j(K, \omega \setminus \Sigma_{d-1-j}(K)) = 0\,.$$

By the structure Theorem 1.54, we can write $\omega \setminus \Sigma_{d-1-j}(K) = \omega_r \cup \omega_u$ as a disjoint union of two Borel sets, where ω_r is countably $(d-1-j)$-rectifiable and ω_u is purely $(\mathcal{H}^{d-1-j}, d-1-j)$-unrectifiable. We show that $S_j(K, \omega_r) = 0$. Indeed, we have

$$\frac{\omega_{d-j}}{2}\int_{\mathbf{O}(d)} \operatorname{card}(\omega_r \cap \rho U_0)\nu(d\rho) = \mathcal{H}^{d-1-j}(\omega_r) < \infty\,,$$

where $U_0 \in \mathbf{G}(d, j+1)$ is arbitrarily fixed. This shows that $\omega_r \cap \rho U_0$ is a finite set for ν almost all $\rho \in \mathbf{O}(d)$. For any such ρ, we have $\omega_r \cap \rho U_0 = \{u_1, \dots, u_p\}$, where the vectors u_i depend on ρ. Hence, by the definition of ω_r and using equation (4.2.24) in [123], we deduce that

$$S_j^{\rho U_0}(K|\rho U_0, \omega_r \cap \rho U_0) = 0\,.$$

Now the assertion follows from equation (4.5.26) in [123].

The proof is completed once we have proved that $S_j(K, \omega_u) = 0$. By the structure Theorem 1.54, we first obtain

$$0 = \mathcal{I}_S^{d-1-j}(\omega_u) = \int_{\mathbf{O}(d)} \operatorname{card}(\omega_u \cap \rho U_0)\nu(d\rho)\,,$$

and hence $\omega_u \cap \rho U_0 = \emptyset$ for ν almost all $\rho \in \mathbf{O}(d)$. Thus the assertion follows again from equation (4.5.26) in [123]. □

2 Curvatures and normals in Minkowski spaces

The basic objective of the present section is to provide extensions of various results of the *Brunn-Minkowski theory* in a Euclidean space to general *Minkowski spaces.* The results which we have in mind use *support measures* of convex sets at a certain point. In fact, these measures form a central subject in convexity and, in particular, they are an essential link to related fields. Therefore it is an important task to extend the existing theory of Euclidean support measures to the setting of Minkowski (or relative) geometry. Extensions of other concepts of Euclidean geometry to finite dimensional normed linear spaces have recently been pushed ahead by specialists working in the geometry of Banach spaces with great success. Subsequently, we merely give a short survey over the subjects treated in this section. More detailed comments and references are provided at the beginning of each subsection.

The first part of the present section is formed by Subsections 2.1 and 2.2. There we provide some of the main results which are required for studying *relative support measures.* Then we proceed (in the second part) to establish sharp bounds on the mean number of Minkowski (relative) normals through a point in a convex body. Thus we extend previous results in [59], [69], concerning the Euclidean case, and in [61], [62], concerning certain special Minkowski planes. At the same time, this particular problem, which is classical in Euclidean spaces, provides an excellent opportunity for introducing the appropriate notions of *relative normals* and *relative curvatures* in Minkowski spaces. Moreover, we prove an *Euler-type formula* in general Minkowski spaces, which complements a local Steiner formula in Minkowski spaces; see [59], [69] and [51] for previous contributions in Euclidean spaces. In the third part of the present section we investigate characterizations of *gauge bodies* in Minkowski spaces via linear relations for relative curvature measures, and we also study stability and splitting results. The results obtained in this final part are essentially based on a theory of *generalized relative curvatures* which live on *relative normal bundles.* Such an extension of the existing Euclidean theory is developed in Subsections 2.5 and 2.6.

The results of this section are also developed in view of applications to *stochastic geometry.* The recent contributions [76], [75] already show how support measures in Minkowski spaces can be used as an important tool for the investigation of *local contact distributions* of random closed sets.

2.1 Preliminaries from Minkowski geometry

We start with a brief description of some facts from Minkowski geometry, but familiarity with notation and basic results of the geometry of convex bodies (see [123]) will be assumed. Furthermore, we essentially adopt the notation of [80] and, in particular, we shall freely use results from that paper as well as from [76]. Specific information on Minkowski geometry is also contained in Thompson's recent book [138].

Let $\mathcal{K}^d$ be the set of convex bodies. We fix a convex body $B \in \mathcal{K}^d$ with $o \in B$ and denote its distance function by d_B, that is

$$d_B(x) := \inf\{\alpha \geq 0 : x \in \alpha B\},$$

$x \in \mathbb{R}^d$ (with $\inf \emptyset := \infty$). If $o \in \operatorname{int} B$, then $d_B(\cdot)$ equals the *gauge function* $g(B, \cdot)$ of B. Usually, B is called gauge body or structuring element. If B is d-dimensional and centrally symmetric with respect to o, then there exists a uniquely determined norm $\| \cdot \|_B = g(B, \cdot)$ with unit ball B. The pair $(\mathbb{R}^d, \| \cdot \|_B)$ or, equivalently, the pair $(\mathbb{R}^d, B)$ is called a Minkowski space with gauge body B. More generally, we shall also adopt the same terminology in the more general situation where B is not necessarily symmetric. For a nonempty closed set $K \subset \mathbb{R}^d$, $K \neq \mathbb{R}^d$, we define the *B-distance of $x \in \mathbb{R}^d$ from K* (or the distance of x from K with respect to B) by

$$d(K, B, x) := \inf\{d_B(x - y) : y \in K\}.$$

It is easy to check that if $d(K, B, x) < \infty$, then

$$\begin{aligned} d(K, B, x) &= \min\{r \geq 0 : x \in K + rB\} \\ &= \min\{r \geq 0 : (x + r\check{B}) \cap K \neq \emptyset\}, \end{aligned}$$

where $\check{B} := \{-b : b \in B\}$. Let $\langle \cdot , \cdot \rangle$ be an auxiliary scalar product, let $| \cdot |$ be the induced norm, let B^d be the Euclidean unit ball, and set $S^{d-1} := \text{bd } B^d$ (as in Section 1). Clearly, if $o \in \text{int } B$, then

$$|d(K, B, x) - d(K, B, y)| \leq \max\{g(B, y - x), g(B, x - y)\},$$

and hence

$$|d(K, B, x) - d(K, B, y)| \leq \frac{1}{r_0}|y - x|,$$

if $r_0 B^d \subset B$. Thus $d(K, B, \cdot)$ is a Lipschitz map. In particular, $d(K, B, \cdot)$ is differentiable $\mathcal{H}^d$ almost everywhere in $\mathbb{R}^d$. We remark that $d(K, B, \cdot)$ is a convex function if K is convex, which can be proved without the use of Euclidean notions.

In the following, we write $\mathcal{K}^d_{2,gp}$ for the set of all pairs $(K, L) \in \mathcal{K}^d \times \mathcal{K}^d$ which are in *general relative position*; this means that K, L are such that

$$\dim F(K + L, u) = \dim F(K, u) + \dim F(L, u),$$

for all $u \in \mathbb{R}^d \setminus \{o\}$. Further, we write $(\mathcal{K}^d_{2,gp})_o$ for the set of all $(K, L) \in \mathcal{K}^d_{2,gp}$ with $o \in \text{int } L$. We fix a pair $(K, B) \in \mathcal{K}^d_{2,gp}$ and choose a point $x \in (K + \text{pos } B) \setminus K$. Then there are uniquely determined points $p(K, B, x) \in \text{bd}\, K$ and $u(K, B, x) \in \text{bd}\, B$ such that

$$x = p(K, B, x) + d(K, B, x) u(K, B, x).$$

We call $p(K, B, x)$ the *B-projection* and $u(K, B, x)$ the *B-projection direction* of x; compare [80].

Now let $K, B \in \mathcal{K}^d$ with $o \in \text{int } B$, $x \in \mathbb{R}^d \setminus K$, and $\epsilon \in [0, t)$, where $t := d(K, B, x)$. We can choose $y \in \text{bd}\, K$ and $b \in \text{bd}\, B$ such that $x = y + tb$. Since $x - \epsilon b = y + (t - \epsilon) b \in K + (t - \epsilon) B$ and $x - \epsilon b \notin K + (s - \epsilon) B$, for $s \in [\epsilon, t)$, we obtain $d(K, B, x - \epsilon b) = t - \epsilon$, and thus the directional derivative of $g := d(K, B, \cdot)$ at x in direction b satisfies

$$g'(x; b) = 1. \tag{82}$$

In the preceding argument we did not use the convexity of K. In addition, using the convexity of K and a separation argument, we find that $d(K, B, x + \epsilon b) = t + \epsilon$, for any $\epsilon \geq 0$, and hence the restriction of $d(K, B, \cdot)$ to the line $x + \mathbb{R} b$ is differentiable at x.

Lemma 2.1 *Let $K, B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$, $x \in \mathbb{R}^d \setminus K$, and set $t := d(K, B, x)$. Then $d(K, B, \cdot)$ is differentiable at x if and only if x is a regular boundary point of $K + tB$. If either of these two conditions is fulfilled, then $Dd(K, B, x) = \langle h(B, u)^{-1}u, \cdot\rangle$ for any $u \in N(K + tB, x) \setminus \{o\}$.*

Proof. First, let us assume that $d(K, B, \cdot)$ is differentiable at x. Since $x \in \operatorname{bd}(K + tB)$, there is some vector $u \in N(K + tB, x) \cap S^{d-1}$ such that $(x + \mathbb{R}u) \cap \operatorname{int}(K + tB) \neq \emptyset$. Note that $d(K, B, y) = t$ for all $y \in \operatorname{bd}(K + tB)$. Furthermore, there exists a nonnegative convex function f such that

$$\gamma_v(s) := x + sv - f(x + sv)u \in \operatorname{bd}(K + tB)\,,$$

if $v \in S^{d-1} \cap u^\perp$ and $|s|$ is small enough. The convexity of f implies the existence of the limit

$$\gamma_v'(0; 1) = \lim_{s \downarrow 0} \frac{\gamma_v(s) - \gamma_v(0)}{s} = v - f'(x; v)u\,.$$

But since $d(K, B, \cdot) \circ \gamma_v(s) = t$, for $|s|$ sufficiently small, we obtain that

$$Dd(K, B, x)(v - f'(x; v)u) = 0\,, \tag{83}$$

for all $v \in u^\perp \cap S^{d-1}$. Now we assume that $f'(x; v) \neq 0$, and hence $f'(x; v) > 0$, for some $v \in u^\perp \cap S^{d-1}$. Set $u_1 := v$, let $(u_1, \ldots, u_{d-1})$ be a basis of $u^\perp$, and define $u_d := -\sum_{i=1}^{d-1} u_i$. Then we set $a_j := u_j - f'(x; u_j)u$ for $j \in \{1, \ldots, d\}$. The vectors $a_1, \ldots, a_d$ are linearly independent, and therefore (83) implies that $Dd(K, B, x) = 0$, which contradicts (82). Hence we obtain that $f'(x; v) = 0$ for all $v \in u^\perp$. This proves the first statement of the lemma.

In addition, we also get that $Dd(K, B, x)(v) = 0$ for all $v \in u^\perp$. Now let $y \in \operatorname{bd} K$ and $b \in \operatorname{bd} B$ be such that $x = y + tb$. By the considerations preceding Lemma 2.1 we find $Dd(K, B, x)(b) = 1$, which implies

$$Dd(K, B, x)(u) = \frac{1}{\langle b, u\rangle} = h(B, u)^{-1}\,.$$

This remained to be proved.

Finally, assume that x is a regular boundary point of $K + tB$. Let $u \in N(K + tB, x) \setminus \{o\}$ and $v \in u^\perp \setminus \{o\}$. We continue to use the notation from the first part of the proof. Let $r_0 > 0$ be such that $r_0 B^d \subset B$. Then, for $|s| > 0$ sufficiently small, the estimate

$$\begin{aligned}\left|\frac{d(K, B, x + sv) - d(K, B, x)}{s}\right| &= \left|\frac{d(K, B, x + sv) - d(K, B, \gamma_v(s))}{s}\right| \\ &\leq \frac{1}{r_0}\left|\frac{x + sv - \gamma_v(s)}{s}\right| \\ &= \frac{1}{r_0}\left|\frac{f(x + sv)}{s}\right|\end{aligned}$$

implies that the partial derivatives of the convex function $d(K, B, \cdot)|_{(x + u^\perp)}$ at x exist. In addition, we already know that $d(K, B, \cdot)|_{(x + \mathbb{R}\, b)}$ is differentiable at x. Let $u_1, \ldots, u_{d-1}$ be a basis of $u^\perp$. Then we consider the function

$$g : \mathbb{R}^d \to \mathbb{R}, \quad (\alpha_1, \dots, \alpha_d) \mapsto d\left(K, B, x + \sum_{i=1}^{d-1} \alpha_i u_i + \alpha_d b\right),$$

which is convex and the partial derivatives $\partial_i g(o)$, $i = 1, \dots, d$, of which exist. Hence by Theorem 1.5.6 in [123], g is differentiable at o. But then $d_B(K, \cdot)$ is differentiable at x. □

The following lemma simplifies and clarifies an argument (check the inclusion on page 256, lines -8 to -9) in [45]. Moreover, the presentation in [45] is restricted to norms which correspond to symmetric gauge bodies.

Lemma 2.2 *Let $K \subset \mathbb{R}^d$ be closed, let $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$ be strictly convex, and let $x \in \mathbb{R}^d \setminus K$. Assume that $d(K, B, \cdot)$ is differentiable at x. Then*

$$\operatorname{card}\{b \in B : x - d(K, B, x)b \in K\} = 1. \tag{84}$$

Proof. Choose any $b \in B$ such that $x = y + d(K, B, x)b$ for some $y \in K$; set $f := d(K, B, \cdot)$. By the considerations preceding Lemma 2.1, we have $Df(x)(b) = 1$. For any two such vectors $b_1, b_2 \in B$, we deduce that $Df(x)((b_1 + b_2)/2) = 1$. On the other hand, for any $v \in \mathbb{R}^d$,

$$d(K, B, x + v) - d(K, B, x) \le g_B(v),$$

and hence

$$Df(x)(v) = \lim_{\epsilon \downarrow 0} \frac{d(K, B, x + \epsilon v) - d(K, B, x)}{\epsilon} \le g_B(v).$$

This implies

$$1 = Df(x)((b_1 + b_2)/2) \le g_B((b_1 + b_2)/2) \le 1.$$

Since B is strictly convex, we deduce that $b_1 = b_2$. □

Using the notation of Lemma 2.2, we define the *B-exoskeleton of K*, $\operatorname{exo}_B(K)$, as the set of all $x \in \mathbb{R}^d \setminus K$ for which condition (84) is violated. The proof of the following consequence is based on the fact that $d(K, B, \cdot)$ is Lipschitz and hence differentiable at $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$.

Corollary 2.3 *Let $K \subset \mathbb{R}^d$ be closed, and let $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$ be strictly convex. Then $\mathcal{H}^d(\operatorname{exo}_B(K)) = 0$.*

For later use, we also mention the following lemma, which is essentially due to Erhard Schmidt [117], pp. 94-95.

Lemma 2.4 *Let $A \subset \mathbb{R}^d$, and let $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$. Then $\mathcal{H}^d(\operatorname{bd}(A + \rho B)) = 0$, for any $\rho > 0$.*

Proof. Let $x \in \operatorname{bd}(A + \rho B)$ be fixed for the moment. Then there is some $a \in \operatorname{clos} A$ such that $x \in \operatorname{bd}(a + \rho B)$; moreover,

$$\operatorname{int}(a + \rho B) \cap \operatorname{bd}(A + \rho B) = \emptyset.$$

Therefore,

$$\limsup_{r\downarrow 0} \frac{\mathcal{H}^d(\mathrm{bd}(A+\rho B)\cap(x+rB^d))}{\mathcal{H}^d(x+rB^d)} < 1\,,$$

for all $x \in \mathrm{bd}(A+\rho B)$, which implies the assertion; compare Corollary 2.14 (1) in [99]. □

In the following, we shall consider pairs (K, B) of convex bodies. The general assumption will be that $(K, B) \in \mathcal{K}^d_{2,gp}$ and $o \in \mathrm{int}\, B$, but sometimes we shall require, e.g., that K or B is strictly convex or impose more restrictive assumptions. For example, if B is strictly convex and $x \in \mathbb{R}^d \setminus K$, then $u(K, B, x) = \nabla h_B(u)$ for any $u \in N(K+d(K, B, x)B, x)\setminus\{o\}$. Similarly, if K is strictly convex, then $p(K, B, x) = \nabla h_K(u)$ for any $u \in N(K + d(K, B, x)B, x) \setminus \{o\}$. In these two cases, the points $\nabla h_B(u)$ and $\nabla h_K(u)$ are uniquely determined by x, but this is not true for $u \in N(K + d(K, B, x)B, x) \cap S^{d-1}$. However, Lemma 2.1 shows that $u \in N(K + d(K, B, x)B, x) \cap S^{d-1}$ is uniquely determined by x if $d(K, B, \cdot)$ is differentiable at $x \in \mathbb{R}^d \setminus K$, which is true for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d \setminus K$.

For $x \in \mathbb{R}^d \setminus K$ and $\lambda \geq 0$ we define $x(\lambda) := p(K, B, x) + \lambda u(K, B, x)$. It is easy to check that $p(K, B, x(\lambda)) = p(K, B, x)$. Therefore we define

$$\mathcal{N}(K, B) := \left\{(p(K, B, x), u(K, B, x)) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d \setminus K\right\}\,,$$

and call $\mathcal{N}(K, B)$ the *B-normal bundle of K*. Alternatively, we also call $\mathcal{N}(K, B)$ the relative normal bundle of K with respect to B. Since $p(K, B, \cdot)$ and $u(K, B, \cdot)$ are continuous, we deduce that $\mathcal{N}(K, B)$ is homeomorphic to $K + tB$, for any $t > 0$, and thus is a closed subset of $\mathrm{bd}\, K \times \mathrm{bd}\, B$.

2.2 Relative support measures

The aim of the first part of this subsection is to sketch the construction of support measures in Minkowski spaces; see [76], [80]. The arguments involved in the construction of these relative support measures are similar to and have been inspired by the ones in [123], §§4.1–2, and [125], though the present setting is more general. Our basic interest, however, will be in representations of these measures which can be obtained in each of the following three cases:

1. The convex bodies K and L are polytopes in general relative position.
2. The support function of B is of class C^2 and K is arbitrary.
3. The support function of K is of class C^2 and B is arbitrary.

For the subsequent discussion, we assume that $(K, B) \in \mathcal{K}^d_{2,gp}$, $o \in \mathrm{int}\, B$, and $\rho > 0$. Then the map

$$(K + \rho B) \setminus K \to \mathbb{R}^{2d}\,, \quad x \mapsto (p(K, B, x), u(K, B, x))\,,$$

is continuous and hence Borel measurable. Thus, for any measurable set $\eta \subset \mathbb{R}^{2d}$, the set

$$M_\rho(K, B, \eta) := \left\{x \in \mathbb{R}^d : 0 < d(K, B, x) \leq \rho,\ (p(K, B, x), u(K, B, x)) \in \eta\right\}$$

is a Borel set. Note that $M_\rho(K,B,\mathbb{R}^{2d}) = (K+\rho B)\setminus K$ and define

$$\mu_\rho(K,B,\eta) := \mathcal{H}^d(M_\rho(K,B,\eta))\,.$$

This, in particular, implies that

$$\mu_\rho(K,B,\mathbb{R}^{2d}) = \sum_{j=0}^{d-1} \rho^{d-j}\binom{d}{j} V(K[j],B[d-j])\,. \tag{85}$$

One can easily check that μ_ρ enjoys similar properties as in the Euclidean case, but now one has an additional dependence with respect to a gauge body B; compare Theorems 4.1.1-3 in [123], [125] and the discussion in [76] and [80], Lemma 5.2. We emphasize that the measure $\mu_\rho(K,B,\cdot)$ is concentrated on the Borel subsets of $\mathcal{N}(K,B)\subset\mathbb{R}^{2d}$.

In order to derive a local Steiner formula and to construct Minkowski (or relative) support measures, one first considers the case of polytopes (K,B) in general relative position. Once the desired representations have been established in this particular case, the general result follows by the usual approximation arguments, similar to the approach in Sections 4.1-2 of [123]. The following theorem can be found in [80] and essentially also in [76].

Theorem 2.5 *Let $(K,B)\in\mathcal{K}^d_{2,gp}$ with $o\in\operatorname{int} B$. Then there exist finite positive Borel measures $\Theta_0(K,B;\cdot),\dots,\Theta_{d-1}(K,B;\cdot)$ over $\mathbb{R}^{2d}$ such that*

$$\mu_\rho(K,B,\cdot) = \frac{1}{d}\sum_{j=0}^{d-1}\rho^{d-j}\binom{d}{j}\Theta_j(K,B;\cdot)\,, \tag{86}$$

for $\rho\ge 0$. The measure $\Theta_j(K,B;\cdot)$ is concentrated on $\mathcal{N}(K,B)$, homogeneous of degree j in K, and homogeneous of degree $d-j$ in B. The mapping $(K,B)\mapsto\Theta_j(K,B;\cdot)$ is weakly continuous and satisfies additivity properties (see [80], [76]).

Remarks.

1. In the construction of the measures $\Theta_j(K,B;\cdot)$ Euclidean notions are involved. Nevertheless, these measures clearly are Minkowski quantities, since the measures $\mu_\rho(K,B,\cdot)$ are intrinsically defined and equation (86) holds for all $\rho>0$.
2. The assumption $o\in\operatorname{int} B$ in Theorem 2.5 is unnecessarily restrictive; it is sufficient to assume that $o\in B$.
3. The proof of Theorem 2.5 in [80] leads to an explicit description of the support measures for a pair $(K,B)\in\mathcal{K}^d_{2,gp}$ of polytopes with $o\in B$. In fact, one obtains

$$\binom{d-1}{j}\Theta_j(K,B;\cdot) \tag{87}$$
$$= \sum_{F\in\mathcal{F}_j(K)}\ \sum_{G\in\mathcal{F}_{d-1-j}(B)} \det(F,G)h(B,u(F+G,K+B))\lambda_F\otimes\lambda_G\,,$$

where λ_F is the j-dimensional Lebesgue measure restricted to F, $\det(F,G)$ is the volume of the parallelepiped spanned by an orthonormal basis of the linear subspace parallel to F and an orthonormal basis of the linear subspace parallel to G, and $u(A,L)$, for a convex body L and a set $A\subset\mathbb{R}^d$, is defined as the outer unit normal of L at A if A is a facet of L and as o otherwise.

4. The measures $\Theta_j(K, B; \cdot)$, $j \in \{0, \ldots, d-1\}$, are called the Minkowski (the relative) support measures of K with respect to B or, more specifically, the *B-support measures of K*.

The next corollary is a straightforward extension of (86).

Corollary 2.6 *Let $(K, B) \in \mathcal{K}^d_{2,gp}$ with $o \in \operatorname{int} B$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel measurable function with compact support. Then*

$$\int_{\mathbb{R}^d \setminus K} f(y) \mathcal{H}^d(dy) = \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty t^{d-1-j} \int_{\mathcal{N}(K,B)} f(x+tb) \Theta_j(K, B; d(x,b)) dt \, .$$

Proof. The map

$$\mathcal{N}(K, B) \times (0, \infty) \to \mathbb{R}^d \setminus K \, , \quad (x, b, t) \mapsto x + tb \, ,$$

is a homeomorphism with inverse

$$\mathbb{R}^d \setminus K \to \mathcal{N}(K, B) \times (0, \infty) \, , \quad y \mapsto (p(K, B, y), u(K, B, y), d(K, B, y)) \, .$$

Hence, the desired result can be deduced from (86) by the usual measure theoretic arguments. □

The relative support measures also satisfy a polynomial expansion with respect to the formation of B-parallel bodies. Indeed, for any $\rho > 0$, let us denote by p_ρ the map

$$p_\rho : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \, , \quad (x, b) \mapsto (x + \rho b, b) \, .$$

By an obvious modification of the proof for Theorem 4.2.2 in [123], the following theorem can be established.

Theorem 2.7 *Let $K, B \in \mathcal{K}^d$, $o \in \operatorname{int} B$, let $\eta \subset \mathbb{R}^{2d}$ be measurable, $\rho > 0$, and let $m \in \{0, \ldots, d-1\}$. Assume that B is strictly convex. Then*

$$\Theta_m (K + \rho B, B; p_\rho \eta) = \sum_{j=0}^{m} \rho^j \binom{m}{j} \Theta_{m-j}(K, B; \eta) \, .$$

As in the Euclidean case, it is appropriate to introduce two sequences of measures which are obtained by specialization from $\Theta_j(K, B; \cdot)$, $j \in \{0, \ldots, d-1\}$, if $(K, B) \in \mathcal{K}^d_{2,gp}$ and $o \in \operatorname{int} B$. We define the Minkowski curvature measures (compare [125])

$$C_j(K, B; \beta) := \Theta_j(K, B; \beta \times \mathbb{R}^d) \, , \qquad \beta \in \mathfrak{B}(\mathbb{R}^d) \, ,$$

and the Minkowski surface area measures

$$S_j(K, B; \gamma) := \Theta_j(K, B; \mathbb{R}^d \times \gamma) \, , \qquad \gamma \in \mathfrak{B}(\mathbb{R}^d) \, ,$$

where $j \in \{0, \ldots, d-1\}$.

By combining Corollary 2.6 and Theorem 2.7 (with $m = d-1$), one can easily establish the following disintegration of Lebesgue measure; see [146, Lemmas 4.1-2] or [132, Hilfssatz 5.3.1] for the Euclidean case.

Corollary 2.8 *Let $K, B \in \mathcal{K}^d$, $o \in \operatorname{int} B$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel measurable function with compact support. Assume that B is strictly convex. Then*

$$\int_{\mathbb{R}^d \setminus K} f(x)\mathcal{H}^d(dx) = \int_0^\infty \int f(x) C_{d-1}(K + tB, B; dx)dt\,.$$

The construction of the Minkowski support measures already led to an explicit description of these measures for pairs of polytopes in general relative position. We shall now derive such representations in the two remaining cases which were mentioned at the beginning of this subsection. As in the first part of this work, we write $k_i(x,u)$, $i = 1, \ldots, d-1$, for the generalized Euclidean curvatures of the unit normal bundle $\mathcal{N}(K)$, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$; the convex body K, which is meant, will be clear from the context or we shall use a more specific notation. The unit vectors which are associated with these curvatures are denoted by $u_1, \ldots, u_{d-1}$. Here and in the following, we usually do not indicate the dependence of these vectors on (x,u). To simplify the notation, we introduce the abbreviations

$$\mathbb{H}_I(K; x, u) := \frac{\prod_{i \in I} k_i(x,u)}{\prod_{i=1}^{d-1} \sqrt{1 + k_i(x,u)^2}}$$

and

$$R_I(B; u) := \det\left(\left(d^2 h_B(u)(u_k, u_l) \right)_{k,l \in I} \right),$$

where $I \subset \{1, \ldots, d-1\}$, provided that these expressions are defined. Moreover, we define

$$\mathbb{H}_j(K, B; x, u) := \binom{d-1}{j}^{-1} \sum_{|I|=j} \mathbb{H}_I(K; x, u) R_I(B; u)\,,$$

for $j \in \{0, \ldots, d-1\}$. Here the summation is extended over all sets $I \subset \{1, \ldots, d-1\}$ of cardinality j.

For m-vectors $x_1 \wedge \ldots \wedge x_m \in \bigwedge_m \mathbb{R}^d$, $m \in \{1, \ldots, d\}$, we write $|x_1 \wedge \ldots \wedge x_m|$ for the norm which is induced by the auxiliary Euclidean scalar product; compare Chapter 1 of [42].

Theorem 2.9 *Let $K, B \in \mathcal{K}^d$, $o \in \operatorname{int} B$, and assume that h_B is of class C^2. Then, for $j \in \{0, \ldots, d-1\}$ and $\eta \in \mathfrak{B}(\mathbb{R}^{2d})$,*

$$\Theta_j(K, B; \eta) = \int_{\mathcal{N}(K)} \mathbf{1}_\eta(x, \nabla h_B(u)) h(B, u)\, \mathbb{H}_{d-1-j}(K, B; x, u) \mathcal{H}^{d-1}(d(x,u))\,.$$

Proof. Let $\eta \in \mathfrak{B}(\mathbb{R}^d \times \mathbb{R}^d)$, set $\mathbb{R}^d_* := \mathbb{R}^d \setminus \{o\}$, and define the Borel measurable map

$$\tilde{T}_B : \mathbb{R}^d \times \mathbb{R}^d_* \to \mathbb{R}^d \times \mathbb{R}^d\,, \quad (x,u) \mapsto (x, \nabla h_B(u))\,,$$

as well as the locally Lipschitz map

$$\tilde{F}_B : \mathcal{N}(K) \times (0, \infty) \to \mathbb{R}^d\,, \quad (x,u,t) \mapsto x + t\nabla h_B(u)\,.$$

Then $\tilde\eta := \tilde T_B^{-1}(\eta) \in \mathfrak{B}(\mathbb{R}^d \times \mathbb{R}^d_*)$ and, by Lemma 2.1, $\operatorname{card}\left(\tilde F_B^{-1}(\{z\})\right) = 1$ for $\mathcal{H}^d$ almost all $z \in \mathbb{R}^d \setminus K$. Furthermore, we obtain for the approximate Jacobian

$$\begin{aligned}
&\text{ap } J_d \tilde F_B(x,u,t)\\
&= \left| \bigwedge_{i=1}^{d-1} \left(\frac{1}{\sqrt{1+k_i(x,u)^2}} u_i + t \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} d^2 h_B(u)(u_i) \right) \wedge \nabla h_B(u) \right|\\
&= h(B,u) \sum_{j=0}^{d-1} t^{d-1-j} \sum_{|I|=j} \frac{\prod_{i\in I^c} k_i(x,u)}{\prod_{i=1}^{d-1} \sqrt{1+k_i(x,u)^2}} \det\left(\left(d^2 h_B(u)(u_k,u_l)\right)_{k,l\in I^c}\right)\\
&= h(B,u) \sum_{j=0}^{d-1} t^{d-1-j} \binom{d-1}{j} \mathbb{H}_{d-1-j}(K,B;x,u)\,,
\end{aligned}$$

for $\mathcal{H}^d$ almost all $(x,u,t) \in \mathcal{N}(K) \times (0,\infty)$. From Federer's coarea formula we hence deduce, for any $\rho \in (0,\infty)$,

$$\begin{aligned}
&\mathcal{H}^d(M_\rho(K,B,\eta))\\
&= \int_{(K+\rho B)\setminus K} \int_{\tilde F_B^{-1}(\{z\})} \mathbf{1}_{\tilde\eta}(x,u) \mathcal{H}^0(d(x,u,t)) \mathcal{H}^d(dz)\\
&= \int_{\mathcal{N}(K)} \int_0^\rho \mathbf{1}_{\tilde\eta}(x,u) \text{ ap } J_d \tilde F_B(x,u,t) dt \mathcal{H}^{d-1}(d(x,u))\\
&= \int_{\mathcal{N}(K)} \int_0^\rho \mathbf{1}_\eta \circ \tilde T_B(x,u) \sum_{j=0}^{d-1} t^{d-1-j} h(B,u) \binom{d-1}{j} \mathbb{H}_{d-1-j}(K,B;x,u) dt \mathcal{H}^{d-1}(d(x,u))\\
&= \sum_{j=0}^{d-1} \frac{\rho^{d-j}}{d} \binom{d}{j} \int_{\mathcal{N}(K)} \mathbf{1}_\eta(x, \nabla h_B(u)) h(B,u) \mathbb{H}_{d-1-j}(K,B;x,u) \mathcal{H}^{d-1}(d(x,u))\,. \qquad (88)
\end{aligned}$$

A comparison of this expansion with equation (86) then yields the result. □

The rôles of K and B in Theorem 2.9 are not completely symmetric. In view of the applications in the next subsection we provide a version of this result with the rôles of K and B interchanged. For that purpose, we henceforth denote the generalized curvatures on $\mathcal{N}(B)$ by $l_1(b,v),\dots,l_{d-1}(b,v)$ and the associated directions by $v_1,\dots,v_{d-1} \in v^\perp$; of course, these directions are functions of (b,v).

Theorem 2.10 *Let $K, B \in \mathcal{K}^d$, $o \in \operatorname{int} B$, and assume that h_K is of class C^2. Then, for $j \in \{0,\dots,d-1\}$ and $\eta \in \mathfrak{B}(\mathbb{R}^{2d})$,*

$$\Theta_j(K,B;\eta) = \int_{\mathcal{N}(B)} \mathbf{1}_\eta(\nabla h_K(v), b) h(B,v) \mathbb{H}_j(B,K;b,v) \mathcal{H}^{d-1}(d(b,v))\,.$$

Proof. We define

$$\tilde T_K : \mathbb{R}^d \times \mathbb{R}^d_* \to \mathbb{R}^d \times \mathbb{R}^d\,, \quad (b,v) \mapsto (\nabla h_K(v), b)\,,$$

as well as

$$\tilde{F}_K : \mathcal{N}(B) \times (0,\infty) \to \mathbb{R}^d\,, \quad (b,v,t) \mapsto \nabla h_K(v) + tb\,,$$

and then we proceed as in the proof of Theorem 2.9; in particular,

$$\text{ap } J_d\tilde{F}_K(b,v,t) = h(B,v)\sum_{j=0}^{d-1} t^{d-1-j}\binom{d-1}{j}\,\mathbb{H}_j(B,K;b,v)\,,$$

for $\mathcal{H}^d$ almost all $(b,v,t) \in \mathcal{N}(B) \times (0,\infty)$. □

The preceding theorem suggests the following definition. Namely, under the assumptions of Theorem 2.9, we set

$$\tilde{\Theta}_j(K,B;\tilde{\eta}) := \int_{\mathcal{N}(K)} \mathbf{1}_{\tilde{\eta}}(x,u)h_B(u)\,\mathbb{H}_{d-1-j}(K,B;x,u)\mathcal{H}^{d-1}(d(x,u))$$

for $j \in \{0,\dots,d-1\}$ and $\tilde{\eta} \in \mathfrak{B}(\mathbb{R}^{2d})$. Thus, Theorem 2.9 can be paraphrased by saying that $\Theta_j(K,B;\cdot)$ is the image measure of $\tilde{\Theta}_j(K,B;\cdot)$ under $\tilde{T}_B$, that is,

$$\Theta_j(K,B;\cdot) = (\tilde{T}_B)_\#\tilde{\Theta}_j(K,B;\cdot) := \tilde{\Theta}_j(K,B;\cdot)\circ\tilde{T}_B^{-1}$$

on $\mathfrak{B}(\mathbb{R}^{2d})$.

Let us consider Theorem 2.9 in the special cases where $j=0$ or $j=d-1$. If $j=d-1$, then

$$\begin{aligned}
&\int g(x,b)\Theta_{d-1}(K,B;d(x,b))\\
&= \int_{\mathcal{N}(K)} g\circ\tilde{T}_B(x,u)h(B,u)\prod_{i=1}^{d-1}\frac{1}{\sqrt{1+k_i(x,u)^2}}\mathcal{H}^{d-1}(d(x,u))\\
&= \int g\circ\tilde{T}_B(x,u)h(B,u)\Theta_{d-1}(K,d(x,u))\,.
\end{aligned}$$

This is true for all $g \in C_0(\mathbb{R}^{2d})$, and for all $K,B \in \mathcal{K}^d$, $o \in \text{int } B$, for which h_B is of class C^2. An approximation argument shows that the additional smoothness assumption can be weakened.

Corollary 2.11 *Let $K,B \in \mathcal{K}^d$, $o \in \text{int } B$, and assume that B is strictly convex. Then, for any $\eta \in \mathfrak{B}(\mathbb{R}^{2d})$,*

$$\Theta_{d-1}(K,B;\eta) = \int \mathbf{1}_\eta(x,\nabla h_B(u))h(B,u)\Theta_{d-1}(K,d(x,u))\,.$$

In particular,

$$C_{d-1}(K,B;\beta) = \int_{\text{bd } K}\mathbf{1}_\beta(x)h(B,\sigma_K(x))\mathcal{H}^{d-1}(dx)\,, \quad \beta\in\mathfrak{B}(\mathbb{R}^d)\,,$$

for all $(K,B) \in \mathcal{K}^d_{2,gp}$ with $o \in \text{int } B$, and

$$S_{d-1}(K,B;\gamma) = \int_{\text{bd } K}\mathbf{1}_\gamma(\nabla h_B(\sigma_K(x)))h(B,\sigma_K(x))\mathcal{H}^{d-1}(dx)\,, \quad \gamma\in\mathfrak{B}(\text{bd } B)\,.$$

Under the assumptions of Corollary 2.8 we thus obtain

$$\int_{\mathbb{R}^d\setminus K} f(x)\mathcal{H}^d(dx) = \int_0^\infty \int_{\mathrm{bd}(K+tB)} f(x)h(B,\sigma_{K+tB}(x))\mathcal{H}^{d-1}(dx)dt\,.$$

More directly, this can be inferred from Lemma 2.1 by applying Federer's coarea formula to the Lipschitz map $d(K,B,\cdot)$.

If $j=0$, then again under the assumptions of Theorem 2.9 we deduce

$$\begin{aligned}\Theta_0(K,B;\eta) &= \int_{\mathcal{N}(K)} \mathbf{1}_\eta \circ \tilde{T}_B(x,u)h(B,u)\,\mathbb{H}_{d-1}(K,B;x,u)\mathcal{H}^{d-1}(d(x,u))\\ &= \int \mathbf{1}_\eta \circ \tilde{T}_B(x,u)h(B,u)\det\left(d^2h_B(u)|_{u^\perp}\right)\Theta_0(K,d(x,u))\,.\end{aligned}$$

This implies that, for $\beta\in\mathfrak{B}(\mathbb{R}^d)$,

$$\begin{aligned}C_0(K,B;\beta) &= \int_{S^{d-1}} \mathbf{1}_\beta(\tau_K(u))h(B,u)\det\left(d^2h_B(u)|_{u^\perp}\right)\mathcal{H}^{d-1}(du)\\ &= \int_{S^{d-1}} \mathbf{1}_\beta(\tau_K(u))h(B,u)S_{d-1}(B,u)\,.\end{aligned}$$

The weak continuity of the Euclidean surface area measures thus shows that

$$C_0(K,B;\beta) = \int_{S^{d-1}} \mathbf{1}_\beta(\tau_K(u))h(B,u)S_{d-1}(B,du)$$

for all convex bodies K,B such that K is strictly convex and $o\in\mathrm{int}\ B$.

The corresponding statement for the Minkowski surface area measure $S_0(K,B;\cdot)$ is a special case of Theorem 2.14 below. In fact, Minkowski surface area measures can be described in terms of certain mixed surface area measures. To see this, we start with a lemma.

Lemma 2.12 *Let $K\in\mathcal{K}^d$, and let $B\in\mathcal{K}^d_o$ be such that h_B is of class C^2. Then, for any $\omega\in\mathfrak{B}(S^{d-1})$ and $j\in\{0,\dots,d-1\}$,*

$$S(K[j],B[d-1-j],\omega) = \int_{\mathcal{N}(K)} \mathbf{1}_\omega(u)\,\mathbb{H}_{d-1-j}(K,B;x,u)\mathcal{H}^{d-1}(d(x,u))\,.$$

Remark. As usual, this result can be extended to mixed surface area measures $S(K[j],B_{j+1},\dots,B_{d-1},\cdot)$, where the convex bodies $B_{j+1},\dots,B_{d-1}\in\mathcal{K}^d_o$ are assumed to have support functions of class C^2. This can be verified by replacing the determinant under the integral by a suitable mixed discriminant and by polynomial expansion.

Proof. For the proof, we can assume that $o\in\mathrm{int}\ B$. Fix $\epsilon>0$ and consider the map

$$\tilde{F}^B_\epsilon : \mathcal{N}(K)\to\mathbb{R}^d\,,\quad (x,u)\mapsto x+\epsilon\nabla h_B(u)\,.$$

It is easy to check that

$$\tilde{F}^B_\epsilon(\{(x,u)\in\mathcal{N}(K) : u\in\omega\}) = \bigcup_{u\in\omega} F(K+\epsilon B,u) \tag{89}$$

and that card $\left((\tilde{F}_\epsilon^B)^{-1}(\{y\})\right) = 1$ is satisfied for $\mathcal{H}^{d-1}$ almost all $y \in \tilde{F}_\epsilon^B(\mathcal{N}(K))$. Thus, Theorem 4.2.5 in [123], (89) and the coarea formula imply that

$$\begin{aligned}
&S_{d-1}(K+\epsilon B,\omega)\\
&= \mathcal{H}^{d-1}\left(\bigcup_{u\in\omega} F(K+\epsilon B,u)\right)\\
&= \int_{\mathcal{N}(K)} \mathbf{1}_\omega(u) \text{ ap } J_{d-1}\tilde{F}_\epsilon^B(x,u)\mathcal{H}^{d-1}(d(x,u))\\
&= \int_{\mathcal{N}(K)} \mathbf{1}_\omega(u)\left|\bigwedge_{i=1}^{d-1}\left(\frac{1}{\sqrt{1+k_i(x,u)^2}}u_i + \epsilon\frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}d^2h_B(u)(u_i)\right)\right| \mathcal{H}^{d-1}(d(x,u))\\
&= \sum_{j=0}^{d-1}\epsilon^{d-1-j}\int_{\mathcal{N}(K)} \mathbf{1}_\omega(u)\binom{d-1}{j}\mathbb{H}_{d-1-j}(K,B;x,u)\mathcal{H}^{d-1}(d(x,u))\,. \qquad (90)
\end{aligned}$$

Comparing the coefficients of ϵ^{d-1-j} in the expansion

$$S_{d-1}(K+\epsilon B,\omega) = \sum_{j=0}^{d-1}\epsilon^{d-1-j}\binom{d-1}{j}S(K[j],B[d-1-j],\omega)$$

and in equation (90), we derive the desired conclusion. □

The special case where K is a polytope deserves to be mentioned separately.

Corollary 2.13 *Let $P \in \mathcal{K}^d$ be a polytope, and let $B \in \mathcal{K}_o^d$ be such that h_B is of class $C^{1,1}$. Then*

$$\begin{aligned}
&\binom{d-1}{j}S(P[j],B[d-1-j],\omega)\\
&\qquad = \sum_{F\in\mathcal{F}_j(P)}\mathcal{H}^j(F)\int_{N(P,F)\cap\omega}\det\left(d^2h_B(u)|_{u^\perp\cap F^\perp}\right)\mathcal{H}^{d-1-j}(du)\,,
\end{aligned}$$

for $\omega \in \mathfrak{B}(S^{d-1})$ and $j \in \{0,\dots,d-1\}$.

If $K, B \in \mathcal{K}^d$, $o \in \text{int } B$, and if h_B is of class C^2, then Lemma 2.12 and Theorem 2.9 imply that

$$\int f(b)S_j(K,B;db) = \int_{S^{d-1}} f\circ\nabla h_B(u)h(B,u)S(K[j],B[d-1-j],du)\,,$$

where $f \in C_0(\mathbb{R}^d)$. The weak continuity of the mixed surface area measures then implies that this equation holds in greater generality.

Theorem 2.14 *Let $K, B \in \mathcal{K}^d$, $o \in$ int B, and let B be strictly convex. Then, for $j \in \{0,\dots,d-1\}$ and $\gamma \in \mathfrak{B}(\mathbb{R}^d)$,*

$$S_j(K,B;\gamma) = \int_{S^{d-1}}\mathbf{1}_\gamma(\nabla h_B(u))h(B,u)S(K[j],B[d-1-j],du)\,.$$

The preceding theorem allows one to reduce the assertions of the following two corollaries to known results.

Corollary 2.15 *Let $K, L \in \mathcal{K}^d$, let $B \in \mathcal{K}^d$, $o \in \operatorname{int} B$, be strictly convex and smooth (of class C^1), and let $j \in \{1, \dots, d-1\}$. Further, assume that* $\dim K \geq j+1$ *and* $\dim B \geq j+1$. *Then*

$$S_j(K, B; \cdot) = S_j(L, B; \cdot)$$

if and only if K and L are translates of each other.

Proof. Use Theorem 2.14 and Theorem 7.2.4 in [123]. □

Corollary 2.16 *Let $K, B \in \mathcal{K}^d$,* $\operatorname{int} K \neq \emptyset$, *$o \in \operatorname{int} B$, assume that B is strictly convex, and let $j \in \{0, \dots, d-2\}$. If K is homothetic to a j-tangential body of B, then*

$$S_j(K, B; \cdot) = \alpha S_{d-1}(K, B; \cdot)\,, \tag{91}$$

where α is a positive constant. Conversely, if condition (91) *is satisfied for some positive constant α, and if B is smooth (of class C^1), then K is a translate of a j-tangential body of $\alpha^{-1/(d-1-j)}B$.*

Proof. Use Theorem 2.14. For the second assertion, one can adapt the proof of Theorem 7.2.9 in [123]. □

Our final result in this subsection will be applied in the course of the investigation of the mean number of normals which pass through a point in a convex body. A Euclidean version of this result was proved in [69]. Subsequently, we write $V(M)$ for the d-dimensional volume of a measurable set $M \subset \mathbb{R}^d$. If $L \subset \mathbb{R}^d$ is a star body containing the origin, then we write $\rho(L, \cdot)$ for its radial function (compare Section 1).

Proposition 2.17 *Let $(K, B) \in (\mathcal{K}^d_{2,gp})_o$, and let $L \subset \mathbb{R}^d$ be a compact star body with $o \in$* $\operatorname{int} L$. *Then*

$$\frac{1}{d}\sum_{r=0}^{d-1}\binom{d}{r+1}\int \rho(L, b)^{r+1} S_{d-1-r}(K, B; db) \leq V(K+L) - V(K)\,.$$

Proof. It is sufficient to prove the result for *polytopes* K, B in general relative position. We use the notation which was introduced in Remark 3 after Theorem 2.5.

Let $\mathcal{F}_r(K, B)$ denote the set of all pairs $(F, G) \in \mathcal{F}_r(K) \times \mathcal{F}_{d-1-r}(B)$ for which $u(F + G, K + B) \neq o$. Then we obtain

$$\begin{aligned} &(K+L) \setminus K \qquad\qquad (92)\\ &= \bigcup_{r=0}^{d-1} \bigcup_{(F,G)\in\mathcal{F}_r(K,B)} \left[(K+L) \cap (p(K, B, \cdot), u(K, B, \cdot))^{-1}(\operatorname{relint} F \times \operatorname{relint} G)\right]\,, \end{aligned}$$

where the equality holds up to a set of $\mathcal{H}^d$ measure zero and the union on the right-hand side is disjoint.

Furthermore, for any such pair $(F,G) \in \mathcal{F}_r(K,B)$, we certainly have

$$\begin{aligned} A(F,G) :=& \{x+tb : t \in [0,\rho(L,b)], x \in \text{relint } F, b \in \text{relint } G\} \\ \subset& (K+L) \cap (p(K,B,\cdot), u(K,B,\cdot))^{-1}(\text{relint } F \times \text{relint } G) \,. \end{aligned} \tag{93}$$

Let $(F,G) \in \mathcal{F}_r(K,B)$ be fixed for the moment, let $f_1,\dots,f_r$ be an orthonormal basis of the linear subspace parallel to F, let $g_1,\dots,g_{d-1-r}$ be an orthonormal basis of the linear subspace parallel to G, and let $u_0 := u(F+G,K+B)$. Then an elementary version of the coarea formula shows that

$$\begin{aligned} & V(A(F,G)) \\ &= \int_F \int_G \int_0^{\rho(L,b)} |\det(f_1,\dots,f_r,tg_1,\dots,tg_{d-1-r},b)| dt \lambda_F(dx) \lambda_G(db) \\ &= \det(F,G)\lambda_F(F) \int_G \int_0^{\rho(L,b)} t^{d-1-r} h(B,u_0) dt \lambda_G(db) \\ &= h(B,u_0)\det(F,G)\lambda_F(F) \frac{1}{d-r} \int_G \rho(L,b)^{d-r} \lambda_G(db) \,. \end{aligned}$$

Consequently, by (87)

$$\sum_{(F,G)\in\mathcal{F}_r(K,B)} V(A(F,G)) = \frac{1}{d}\binom{d}{r} \int \rho(L,b)^{d-r} S_r(K,B;db) \,. \tag{94}$$

Hence the assertion follows from (92), (93), and (94). □

We remark that Proposition 2.17 can be proved by an application of the coarea formula if B has a support function of class C^2. The general case then again follows by approximation.

As an immediate consequence of Theorem 2.14 and Proposition 2.17, we obtain an extension of the first inequality in the main theorem in [115].

Corollary 2.18 *Let $K,B,L \in \mathcal{K}^d$, $o \in B$, $o \in \text{int } L$, and assume that B is strictly convex. Then*

$$\int_{S^{d-1}} \rho(L,\nabla h_B(u)) h(B,u) S_{d-1}(K,du) \le dV(K[d-1],L)$$

and

$$\int_{S^{d-1}} \rho(L,\nabla h_B(u))^{d-1} h(B,u) S(K,B[d-2],du) \le dV(K,L[d-1]) \,.$$

We also remark that Proposition 2.17 can be proved by an application of the coarea formula if B has a support function of class C^2. The general case then again follows by approximation.

2.3 Relative normals

In this subsection we investigate normals of a convex body K in a Minkowski space with gauge body B and $o \in \text{int } B$. A *B-normal of K* is any ray $x + (-\infty,0]b$, where $x \in \text{bd}\, K$

and $b \in F(B,u)$ for some $u \in N(K,x)$. Clearly, this definition is independent of auxiliary Euclidean notions. The underlying notion of orthogonality in Minkowski spaces is usually attributed to Birkhoff; see Section 3.2 in [138]. Basically, we are interested in the average number of B-normals of K. In a Minkowski plane, this number is always infinite if $(K,B) \notin \mathcal{K}^2_{2,gp}$. Therefore we assume that $(K,B) \in \mathcal{K}^d_{2,gp}$ in the following. But then the number of B-normals of K passing through a point $p \in \mathbb{R}^d$ is given by

$$n(K,B,p) = \operatorname{card}\{(x,b) \in \mathcal{N}(K,B) : p \in x - [0,\infty)b\}\,.$$

This functional has been investigated repeatedly from various points of view, especially in the Euclidean case; we refer to the literature cited in [59], [60], [61], [62], [69]. Here we shall be interested in the mean value of $n(K,B,\cdot)$ over K. To some extent for technical reasons, we essentially restrict our considerations to the following three cases:

1. K and B are polytopes in general relative position.
2. K is arbitrary and h_B is of class C^2.
3. B is arbitrary and h_K is of class C^2.

In each of these cases, the mapping $p \mapsto n(K,B,p)$ is $\mathcal{H}^d$ measurable. In fact, first note that $\mathcal{N}(K,B)$ is a $(d-1)$-rectifiable Borel set. In the first case, this follows since $\mathcal{N}(K,B)$ then is the set of all $(x,b) \in \mathbb{R}^{2d}$ for which there are faces F of K and G of B such that $F+G$ is a face of $K+B$ and $x \in F$, $b \in G$. In the second case, we have the representation

$$\mathcal{N}(K,B) = \{(x, \nabla h_B(u)) : (x,u) \in \mathcal{N}(K)\}\,,$$

and in the third case, we have

$$\mathcal{N}(K,B) = \{(\nabla h_K(v), b) : (b,v) \in \mathcal{N}(B)\}\,.$$

Furthermore,

$$G : \mathcal{N}(K,B) \times [0,\infty) \to \mathbb{R}^d\,, \quad (x,b,t) \mapsto x - tb\,,$$

is locally Lipschitz and, for any $p \in \mathbb{R}^d$,

$$n(K,B,p) = \operatorname{card}\left(G^{-1}(\{p\})\right)\,.$$

The required measurability thus follows from Section 2.10.26 in [42]. Therefore we can consider the mean value

$$n(K,B) := \frac{1}{V(K)} \int_K n(K,B,p)\mathcal{H}^d(dp)\,.$$

The following theorem provides sharp bounds for this average. It generalizes the corresponding Euclidean result (see [69]), and it extends partial results concerning the Minkowski plane to Minkowski spaces of arbitrary dimensions. We should also emphasize that the maps $(K,B) \mapsto n(K,B)$ and $(K,B,p) \mapsto n(K,B,p)$ apparently do not enjoy suitable continuity properties to admit an extension of the following theorem, for instance, to arbitrary pairs of convex bodies in general relative position.

Theorem 2.19 *Let $(K,B)\in\mathcal{K}^d_{2,gp}$, $\operatorname{int} K\neq\emptyset$, and $o\in\operatorname{int} B$. Then, in each of the above three cases,*

$$2\le n(K,B)\le\frac{V(K+DK)}{V(K)}-1\,,$$

where $DK:=K-K$ denotes the difference body of K.

Proof. We only have to prove the right inequality. Moreover, we shall consider each of the three cases separately.

We start with the first case. For $(x,b)\in\mathcal{N}(K,B)$ we define

$$\sigma_B(K,x,b):=\frac{\mathcal{H}^1(K\cap(x+\mathbb{R}b))}{|b|}$$

and

$$\bar{A}(F,G):=\{x-tb:t\in[0,\sigma_B(K,x,b)]\,,x\in\operatorname{relint} F,b\in\operatorname{relint} G\}\,,$$

where $(F,G)\in\mathcal{F}_r(K,B)$ for some $r\in\{0,\dots,d-1\}$. Then, for $\mathcal{H}^d$ almost all $p\in\mathbb{R}^d$,

$$n(K,B,p)=\sum_{r=0}^{d-1}\sum_{(F,G)\in\mathcal{F}_r(K,B)}\mathbf{1}\{p\in\bar{A}(F,G)\}\,.$$

Hence

$$\int_K n(K,B,p)\mathcal{H}^d(dp)=\sum_{r=0}^{d-1}\sum_{(F,G)\in\mathcal{F}_r(K,B)}V(\bar{A}(F,G))\,.$$

An elementary argument shows that, for $(x,b)\in\mathcal{N}(K,B)$,

$$\sigma_B(K,x,b)\le\rho(DK,b)\,. \tag{95}$$

Using (95) and a calculation similar to the one which led to equation (94) in the proof of Proposition 2.17, we obtain

$$V(\bar{A}(F,G))\le h(B,u_0)\det(F,G)\lambda_F(F)\frac{1}{d-r}\int_G\rho(DK,b)^{d-r}\mathcal{H}^{d-1-r}(db)\,,$$

where $u_0=u(F+G,K+B)$. This and Proposition 2.17 finally imply

$$\begin{aligned}\int_K n(K,B,p)\mathcal{H}^d(dp)&\le\sum_{r=0}^{d-1}\frac{1}{d}\binom{d}{r}\int\rho(DK,b)^{d-r}S_r(K,B;db)\\&\le V(K+DK)-V(K)\,,\end{aligned}$$

which completes the proof in the first case.

We now turn to the proof of the theorem in case two. Let us consider the Lipschitz map

$$\tilde{G}_B:\mathcal{N}(K)\times\mathbb{R}\to\mathbb{R}^d\,,\quad(x,u,t)\mapsto x-t\nabla h_B(u)\,,$$

and the set

$$\tilde{M}_B(K) := \{(x,u,t) \in \mathcal{N}(K) \times \mathbb{R} : t \in [0, \tilde{\sigma}_B(K,x,u)]\} ,$$

where

$$\tilde{\sigma}_B(K,x,u) := \frac{\mathcal{H}^1(K \cap (x + \mathbb{R}\nabla h_B(u)))}{|\nabla h_B(u)|} , \quad (x,u) \in \mathcal{N}(K) .$$

Note that $K \cap (x + (0,\infty)b) = \emptyset$ for all $(x,b) \in \mathcal{N}(K,B)$. Therefore, $\tilde{M}_B(K) = \tilde{G}_B^{-1}(K)$, and thus $\tilde{M}_B(K)$ is a Borel set. For $p \in \mathbb{R}^d$ we define

$$\tilde{n}_B(K,p) := \operatorname{card}\{(x,u) \in \mathcal{N}(K) : p \in x - [0,\infty)\nabla h_B(u)\} ,$$

and hence

$$n(K,B,p) \leq \tilde{n}_B(K,p) ,$$

for all $p \in \mathbb{R}^d$. Equality holds, for example, if B is smooth, but this will not be needed subsequently. Note that $\tilde{n}_B(K,p) = \operatorname{card}\left(\tilde{G}_B^{-1}(\{p\})\right)$, for all $p \in \mathbb{R}^d$, and

$$\int_{\tilde{G}_B^{-1}(\{p\})} \mathbf{1}_{\tilde{M}_B(K)}(x,u,t)\mathcal{H}^0(d(x,u,t)) = \begin{cases} \tilde{n}_B(K,p) , & \text{if } p \in K , \\ 0 , & \text{otherwise} . \end{cases}$$

After these preparations, Federer's coarea formula (applied twice) yields

$$\begin{aligned} V(K)n(K,B) &= \int_K n(K,B,p)\mathcal{H}^d(dp) \\ &\leq \int_K \tilde{n}_B(K,p)\mathcal{H}^d(dp) \\ &= \int_{\mathbb{R}^d} \int_{\tilde{G}_B^{-1}(\{p\})} \mathbf{1}_{\tilde{M}_B(K)}(x,u,t)\mathcal{H}^0(d(x,u,t))\mathcal{H}^d(dp) \\ &= \int_{\mathcal{N}(K)\times\mathbb{R}} \mathbf{1}_{\tilde{M}_B(K)}(x,u,t) \operatorname{ap} J_d\tilde{G}_B(x,u,t)\mathcal{H}^d(d(x,u,t)) \\ &= \int_{\mathcal{N}(K)} \int_0^{\tilde{\sigma}_B(K,x,u)} \operatorname{ap} J_d\tilde{G}_B(x,u,t)dt\mathcal{H}^{d-1}(d(x,u)) . \end{aligned}$$

Furthermore, for $\mathcal{H}^d$ almost all $(x,u,t) \in \mathcal{N}(K) \times \mathbb{R}$,

$$\begin{aligned} &\operatorname{ap} J_d\tilde{G}_B(x,u,t) \\ &\quad = \left| \bigwedge_{i=1}^{d-1} \left(\frac{1}{\sqrt{1+k_i(x,u)^2}} u_i - t\frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} d^2h_B(u)(u_i) \right) \wedge (-\nabla h_B(u)) \right| \\ &\quad = h_B(u) \left| \sum_{r=0}^{d-1} (-t)^r \sum_{|I|=r} \binom{d-1}{r} \mathbb{H}_I(K,B;x,u)R_I(B;u) \right| \\ &\quad \leq h_B(u) \sum_{r=0}^{d-1} t^r \binom{d-1}{r} \mathbb{H}_r(K,B;x,u) . \end{aligned} \tag{96}$$

Using (96), the fact that

$$\tilde{\sigma}_B(K,x,u) \leq \rho\left(DK, \nabla h_B(u)\right), \quad (x,u) \in \mathcal{N}(K),$$

and applying successively Theorem 2.9 and Proposition 2.17, we infer that

$$\begin{aligned}
&V(K)n(K,B)\\
&\leq \sum_{r=0}^{d-1} \int_{\mathcal{N}(K)} h_B(u) \frac{1}{r+1} \tilde{\sigma}_B(K,x,u)^{r+1} \binom{d-1}{r} \mathbb{H}_r(K,B;x,u)\mathcal{H}^{d-1}(d(x,u))\\
&\leq \sum_{r=0}^{d-1} \frac{1}{r+1} \int_{\mathcal{N}(K)} h_B(u)\rho\left(DK, \nabla h_B(u)\right)^{r+1} \binom{d-1}{r} \mathbb{H}_r(K,B;x,u)\mathcal{H}^{d-1}(d(x,u))\\
&= \sum_{r=0}^{d-1} \frac{1}{d} \binom{d}{r+1} \int \rho\left(DK, b\right)^{r+1} S_{d-1-r}(K,B;db)\\
&\leq V(K+DK) - V(K),
\end{aligned}$$

and this completes the proof in the second case.

Finally, we sketch the proof of the theorem in the third case. We define the Lipschitz map

$$\tilde{G}_K : \mathcal{N}(B) \times \mathbb{R} \to \mathbb{R}^d, \quad (b,v,t) \mapsto \nabla h_K(v) - tb,$$

and the set

$$\tilde{M}_K(B) := \{(b,v,t) \in \mathcal{N}(B) \times \mathbb{R} : t \in [0, \hat{\sigma}_B(K,b,v)]\},$$

where

$$\hat{\sigma}_B(K,b,v) := \frac{\mathcal{H}^1(K \cap (\nabla h_K(v) + \mathbb{R}b))}{|b|}, \quad (b,v) \in \mathcal{N}(B).$$

Again $\tilde{M}_K(B) = \tilde{G}_K^{-1}(K)$ is a Borel set. For $p \in \mathbb{R}^d$ we define

$$\tilde{n}_K(B,p) := \operatorname{card}\{(b,v) \in \mathcal{N}(B) : p \in \nabla h_K(v) - [0,\infty)b\},$$

and hence

$$n(K,B,p) \leq \tilde{n}_K(B,p),$$

for $p \in \mathbb{R}^d$, with equality if K is smooth. Similarly as before we thus obtain

$$V(K)n(K,B) \leq \int_{\mathcal{N}(B)} \int_0^{\hat{\sigma}_B(K,b,v)} \operatorname{ap} J_d\tilde{G}_K(b,v,t)\,dt\,\mathcal{H}^{d-1}(d(b,v)).$$

Furthermore, for $\mathcal{H}^d$ almost all $(b,v,t) \in \mathcal{N}(B) \times \mathbb{R}$,

$$\operatorname{ap} J_d\tilde{G}_K(b,v,t) \leq h_B(v) \sum_{r=0}^{d-1} t^r \binom{d-1}{r} \mathbb{H}_{d-1-r}(B,K;b,v).$$

The proof can then be completed by using Theorem 2.10 instead of Theorem 2.9. □

An inspection of the preceding proof yields the following corollary, which also shows the connection to the subject of the next subsection.

Corollary 2.20 *Let* $(K,B) \in \mathcal{K}^d_{2,gp}$, $\operatorname{int} K \neq \emptyset$, *and* $o \in \operatorname{int} B$. *Then, in the cases two and three,*

$$V(K)n(K,B) \le \sum_{r=0}^{d-1} \frac{1}{d}\binom{d}{r+1} \int \sigma_B(K,x,b)^{r+1}\Theta_{d-1-r}(K,B;d(x,b))\,,$$

where $\sigma_B(K,x,b) := \mathcal{H}^1(K \cap (x+\mathbb{R}b))/|b|$.

Note that $\sigma_B(K,x,b)$ is the Minkowski length of the secant of K which is determined by the line $x+\mathbb{R}b$.

In the two-dimensional case we shall obtain the same conclusions for all pairs of convex bodies in general relative position; see Subsection 2.5.

2.4 Euler-type formulae

In this section, we establish an Euler-type version of a Steiner formula in Minkowski spaces and deduce various consequences. In particular, we obtain relations of a combinatorial type, special cases of which have been used in [51] for an investigation of such results in Euclidean spaces. It is not clear, at the moment, whether the approach suggested in [51] can be extended to Minkowski spaces. Instead we reverse the sequence of arguments in [51] and prove directly the desired Steiner type formula by a measure geometric argument. An additional advantage of the present method is that it can be applied in other situations as well. However, both approaches employ a continuity arguments at essential points.

The following lemma will be used in the proof.

Lemma 2.21 *Let* $0 < r \le R < \infty$, *let* $K, L \in \mathcal{K}^d$ *with* $B^d(o,r) \subset K, L \subset B^d(o,R)$, *and let* $u, v \in S^{d-1}$. *Then*

$$|\rho(K,u) - \rho(L,v)| \le \frac{R}{r}\delta(K,L) + \frac{R^2}{r}|u-v|\,,$$

where δ *denotes the Hausdorff distance.*

Proof. We set $\epsilon := \delta(K,L)$; hence $K \subset L + B^d(o,\epsilon)$ and

$$\rho(K,u) \le \max\{t \ge 0 : tu \in L + B^d(o,\epsilon)\}\,,$$

for $u \in S^{d-1}$. For $a := \rho(L,u)u$ and $n \in N(L,a) \cap S^{d-1}$ we obtain

$$\rho(K,u) \le \rho(L,u) + \frac{\epsilon}{\langle u,n\rangle}\,.$$

By assumption we have $\langle u,n\rangle \ge r/R$, and thus

$$\rho(K,u) \le \rho(L,u) + \frac{R}{r}\epsilon\,.$$

By symmetry we obtain

$$|\rho(K,u) - \rho(L,u)| \le \frac{R}{r}\delta(K,L)\,.$$

Furthermore,

$$|\rho(L,u)-\rho(L,v)|=\frac{|h(L^*,v)-h(L^*,u)|}{h(L^*,u)h(L^*,v)}\le\frac{R^2}{r}|u-v|\,,$$

since $B^d(o,R^{-1})\subset L^*\subset B^d(o,r^{-1})$. □

Theorem 2.22 *Let $(K,B)\in\mathcal{K}^d_{2,gp}$, and assume that $o\in B$ and $\dim(K+B)=d$. Let $g:\mathbb{R}^d\to\mathbb{R}$ be a bounded, Borel measurable function with compact support. Then*

$$\sum_{r=0}^{d-1}\binom{d-1}{r}\int_{\mathbb{R}}t^{d-1-r}\int_{\mathcal{N}(K,B)}g(x+tb)\Theta_r(K,B;d(x,b))dt$$
$$=\left(1-(-1)^d\right)\int_{\mathbb{R}^d}g(x)\mathcal{H}^d(dx)\,.$$

Proof. It is sufficient to prove the theorem for an arbitrary convex body K and a convex body B the support function of which is of class C^∞ and which contains o as an interior point. As usual, for the proof we shall use Euclidean notions. In fact, we shall show that

$$\sum_{r=0}^{d-1}\binom{d-1}{r}\int_{\mathbb{R}}t^{d-1-r}\int_{\mathcal{N}(K)}g(x+t\nabla h_B(u))\tilde{\Theta}_r(K,B;d(x,u))dt$$
$$=\left(1-(-1)^d\right)\int_{\mathbb{R}^d}g(x)\mathcal{H}^d(dx)\,.$$

Choose some $r_0>0$ with $B^d(o,r_0)\subset B$, and let $R_0>0$ be such that K, B, $\operatorname{spt}(g)\subset\operatorname{int}B^d(o,R_0)$. Then we set $R:=2R_0/r_0$. In the following, we shall consider the push-forward of certain normal currents by the map

$$F:\mathbb{R}^d\times\mathbb{R}^d_*\times\mathbb{R}\to\mathbb{R}^d\,,\quad (x,u,t)\mapsto x+t\nabla h_B(u)\,,$$

which is of class C^∞. Since B is held fixed throughout the proof, we simply write F without indicating the dependence on B. For $\lambda\in[0,1]$ we set $K(\lambda):=(1-\lambda)K+\lambda B$, and by $N_{K(\lambda)}\in\mathbf{N}_{d-1}(\mathbb{R}^d\times\mathbb{R}^d_*)$ we denote the normal current (cycle) which is associated with $K(\lambda)$; see [155], [157]. Since $[-R,R]\in\mathbf{N}_1(\mathbb{R})$, we obtain that $N_{K(\lambda)}\times[-R,R]\in\mathbf{N}_d(\mathbb{R}^d\times\mathbb{R}^d_*\times\mathbb{R})$. Then, in particular, $N_{K(\lambda)}\times[-R,R]$ is an integral current with compact support. The support of the boundary of this current, $\partial(N_{K(\lambda)}\times[-R,R])$, is contained in the compact set $\mathcal{N}(K(\lambda))\times\{-R,R\}$, since $N_{K(\lambda)}$ is a cycle. According to Section 4.1.7 in [42], the push-forward

$$F_\#(N_{K(\lambda)}\times[-R,R])\in\mathcal{D}_d(\mathbb{R}^d)$$

is well-defined. In addition, by the Remarks 26.21 and 27.2 in [136], $F_\#(N_{K(\lambda)}\times[-R,R])$ is an integral current. Furthermore, we have

$$\operatorname{spt}\left(\partial\left(F_\#\left(N_{K(\lambda)}\times[-R,R]\right)\right)\right)\subset F\left(\mathcal{N}(K(\lambda))\times\{-R,R\}\right)\,,$$

and hence, according to the special choice of R,

$$\operatorname{spt}\left(\partial\left(F_{\#}\left(N_{K(\lambda)}\times[-R,R]\right)\right)\right)\subset\mathbb{R}^d\setminus\operatorname{int}B^d(o,R_0).$$

Federer's constancy theorem [42, 4.1.7] then implies that there exists a constant $c\in\mathbb{Z}$ such that

$$\operatorname{spt}\left(F_{\#}\left(N_{K(\lambda)}\times[-R,R]\right)-c\,\mathbf{E}^d\right)\subset\mathbb{R}^d\setminus\operatorname{int}B^d(o,R_0).$$

Note that c may depend on $K(\lambda)$, R, and F. Since K, B, R, and F are held fixed for the moment, we can write $c=c(\lambda)$ in order to indicate the dependence of c on $\lambda\in[0,1]$. Let us denote by Ω the standard volume form in $\mathbb{R}^d$. Then, for any $h\in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{spt}(h)\subset\operatorname{int}B^d(o,R_0)$, we deduce

$$F_{\#}\left(N_{K(\lambda)}\times[-R,R]\right)(h\,\Omega)=c(\lambda)\,\mathbf{E}^d(h\,\Omega)=c(\lambda)\int_{\mathbb{R}^d}h(x)\mathcal{H}^d(dx)\,. \tag{97}$$

In order to determine the constant $c(\lambda)$, we proceed as follows. First of all, observe that the map

$$H:[0,1]\to\mathbb{R},\quad \lambda\mapsto F_{\#}\left(N_{K(\lambda)}\times[-R,R]\right)(h\,\Omega)\,,$$

is continuous; compare [157] and Section 4.1.14 in [42]. This already shows that $c(0)=c(\lambda)$ for all $\lambda\in[0,1]$. Next we obtain, for $\lambda\in[0,1]$,

$$\begin{aligned}
&F_{\#}\left(N_{K(\lambda)}\times[-R,R]\right)(h\,\Omega)\\
&=\left(N_{K(\lambda)}\times[-R,R]\right)\left(F^{\#}(h\,\Omega)\right)\\
&=\left(N_{K(\lambda)}\times[-R,R]\right)(h\circ F\,F^{\#}\Omega)\\
&=\int_{\mathcal{N}(K(\lambda))}\int_{-R}^{R}h\circ F(x,u,t)\\
&\qquad\times\langle\wedge_d DF(x,u,t)\left[(\wedge_{d-1}P)a_\lambda(x,u)\wedge(\wedge_1 Q)e_1\right],\Omega\rangle dt\mathcal{H}^{d-1}(d(x,u))\\
&=\sum_{r=0}^{d-1}\binom{d-1}{r}\int_{\mathcal{N}(K(\lambda))}\int_{-R}^{R}t^{d-1-r}h(x+t\nabla h_B(u))dt\tilde{\Theta}_r(K(\lambda),B;d(x,u))\,,
\end{aligned}\tag{98}$$

where P, Q are canonical injections into $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$ (see [42, p. 360]), a_λ is a $(d-1)$-vectorfield orienting $\mathcal{N}(K(\lambda))$ (compare [155, p. 565]), and $e_1=1$ is the unit vector of $\mathbb{R}$.

Combining (97) and (98) and using that $c:=c(0)=c(\lambda)$, $\lambda\in[0,1]$, we obtain that

$$\begin{aligned}
&\sum_{r=0}^{d-1}\binom{d-1}{r}\int_{\mathcal{N}(K(\lambda))}\int_{\mathbb{R}}t^{d-1-r}h(x+t\nabla h_B(u))dt\tilde{\Theta}_r(K(\lambda),B;d(x,u))\\
&\qquad=c\int_{\mathbb{R}^d}h(x)\mathcal{H}^d(dx)\,,
\end{aligned}\tag{99}$$

for $\lambda \in [0,1]$. It remains to determine c. Relation (99) holds for any $h \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{spt}(h) \subset \operatorname{int} B^d(o, R_0)$, and hence it remains true if we merely assume that $h \in C_0(\mathbb{R}^d)$ with the same condition on the support. By considering a sequence $(h_n)_{n\in\mathbb{N}}$ of such functions with $h_n \downarrow \mathbf{1}_{K(\lambda)}$ pointwise as $n \to \infty$, we infer from the dominated convergence theorem that (99) is still satisfied if $\mathbf{1}_{K(\lambda)}$ is substituted for h. In the case $\lambda = 1$ we thus deduce that

$$\begin{aligned}\sum_{r=0}^{d-1}(-1)^{d-1-r}\frac{1}{d}\binom{d}{r}\int_{\mathcal{N}(B)} \tilde{\sigma}_B(B,x,u)^{d-r}\tilde{\Theta}_r(B,B;d(x,u))\\ = c\,V(B)\,,\end{aligned} \tag{100}$$

where $\tilde{\sigma}_B(B,x,u) = \mathcal{H}^1(B\cap(x+\mathbb{R}\,\nabla h_B(u)))/|\nabla h_B(u)|$ for $(x,u) \in \mathcal{N}(B)$. Using the definition of $\tilde{\Theta}_r$ and Lemma 2.12, we can simplify equation (100) to

$$(1-c(B))V(B) = \frac{1}{d}\int_{S^{d-1}}(1-\sigma_B(u))^d h(B,u) S_{d-1}(B,du)\,, \tag{101}$$

where

$$\sigma_B(u) = \frac{\mathcal{H}^1(B\cap\mathbb{R}\,\nabla h_B(u))}{|\nabla h_B(u)|}\,.$$

We write $c(B)$ instead of c to indicate the dependence of c on B. Thus, in particular, we have proved that if B is any convex body with support function of class C^∞ and $o \in \operatorname{int} B$, then there is a constant $c(B) \in \mathbb{Z}$ such that (101) is satisfied. In order to show that $c(B) = 1-(-1)^d$, we apply another continuity argument. For $\tau \in [0,1]$ we set $B(\tau) := (1-\tau)B+\tau B^d$. Since $h(B(\tau),\cdot)$ is of class C^∞ and $o \in \operatorname{int} B(\tau)$, there is a constant $c(\tau) \in \mathbb{Z}$ such that (101) is satisfied with B replaced by $B(\tau)$ and $c(B)$ replaced by $c(\tau)$, for all $\tau \in [0,1]$. Clearly, $V(B(\tau))$ depends continuously on τ. Moreover, by Lemma 2.21,

$$(\tau,u) \mapsto (1-\sigma_{B(\tau)}(u))^d h(B(\tau),u)$$

is Lipschitz, since

$$\sigma_{B(\tau)}(u) = \frac{\rho((1-\tau)B+\tau B^d, v(\tau,u)) + \rho((1-\tau)B+\tau B^d, -v(\tau,u))}{|v(\tau,u)|}\,,$$

where

$$v(\tau,u) = (1-\tau)\nabla h_B(u) + \tau u\,.$$

Here we also use that $v(\tau,u)$ is bounded away from zero. Hence,

$$\tau \mapsto \int_{S^{d-1}}(1-\sigma_{B(\tau)}(u))^d h(B(\tau),u) S_{d-1}(B(\tau),du)$$

is continuous as well. But $c(B(\tau)) \in \mathbb{Z}$ for all $\tau \in [0,1]$, and therefore $c(B) = c(B^d)$. Since $\sigma_{B^d}(u) = 2$ for all $u \in S^{d-1}$, we infer that $1-c(B^d) = (-1)^d$, which was to be shown.

The statement of the theorem now follows from the case $\lambda = 0$ of (99) by standard measure theoretic approximation arguments. □

Remark. Instead of using the preceding "current technique", Theorem 2.22 can also be proved by a mapping-degree argument; compare [59] and [69] for the Euclidean case. Of course, the two methods are essentially equivalent to each other. We preferred the present framework, because it is more general and since, in our opinion, it perfectly corresponds to the requirements of the situation.

Corollary 2.23 *Let $(K,B) \in \mathcal{K}^d_{2,gp}$, and assume that $o \in B$ and $\dim(K+B) = d$. Further, let $g : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel measurable function with compact support. Then*

$$\int_{\mathbb{R}^d} g(x)\mathcal{H}^d(dx) = \sum_{r=0}^{d-1}(-1)^r \binom{d-1}{r} \int_0^\infty t^{d-1-r} \int_{\mathcal{N}(K,B)} g(x-tb)\Theta_r(K,B;d(x,b))dt$$
$$+ (-1)^d \int_K g(x)\mathcal{H}^d(dx)\,.$$

Proof. From Theorem 2.22 we deduce

$$\left(1-(-1)^d\right)\int_{\mathbb{R}^d} g(x)\mathcal{H}^d(dx)$$
$$= \sum_{r=0}^{d-1}(-1)^{d-1-r}\binom{d-1}{r}\int_0^\infty t^{d-1-r}\int_{\mathcal{N}(K,B)} g(x-tb)\Theta_r(K,B;d(x,b))dt$$
$$+\sum_{r=0}^{d-1}\binom{d-1}{r}\int_0^\infty s^{d-1-r}\int_{\mathcal{N}(K,B)} g(x+sb)\Theta_r(K,B;d(x,b))ds$$
$$= \sum_{r=0}^{d-1}(-1)^{d-1-r}\binom{d-1}{r}\int_0^\infty t^{d-1-r}\int_{\mathcal{N}(K,B)} g(x-tb)\Theta_r(K,B;d(x,b))dt$$
$$+\int_{\mathbb{R}^d\setminus K} g(x)\mathcal{H}^d(dx)\,.$$

In the last step we have used the Steiner formula for Minkowski spaces in the form of Corollary 2.6. This immediately implies the result. □

Corollary 2.24 *Let $(K,B) \in \mathcal{K}^d_{2,gp}$, and assume that $o \in \operatorname{int} B$. Then*

$$\sum_{r=0}^{d-1}(-1)^r\frac{1}{d}\binom{d}{r}\int_{\mathcal{N}(K,B)}\sigma_B(K,x,b)^{d-r}\Theta_r(K,B;d(x,b))$$
$$= \left(1-(-1)^d\right)V(K)\,,$$

where $\sigma_B(K,x,b) := \mathcal{H}^1(K\cap(x+\mathbb{R}b))/|b|$ is the Minkowski length of the segment $K\cap(x+\mathbb{R}b)$ for all $(x,b) \in \mathcal{N}(K,B)$.

Proof. The assertion follows from Corollary 2.23 by setting $g = \mathbf{1}_K$. □

For a polytope $P \subset \mathbb{R}^d$ and a strictly convex compact set $B \subset \mathbb{R}^d$ with $o \in \operatorname{int} B$, we define the Minkowski normal cone $N_B(P,F)$ of P with respect to B at a proper face F of P as the set of all vectors tb for which $t \geq 0$ and such that there is a point $x \in \operatorname{relint} F$ with $(x,b) \in \mathcal{N}(P,B)$. If $F = P$, then we set $N_B(P,F) := \{o\}$. It is easy to see that if $F \neq P$, then

$$N_B(P,F) := \{t\nabla h_B(u) \in \mathbb{R}^d : t \geq 0,\, u \in N(P,F)\}\,.$$

In particular, if B is the Euclidean unit ball B^d, then $N_{B^d}(P,F) = N(P,F)$. But note that in general the cone $N_B(P,F)$ need not be convex.

Corollary 2.25 *Let $P \in \mathcal{K}^d$ be a polytope, let $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$, and assume that h_B is of class $C^{1,1}$. Then*

$$\sum_{F\in\mathcal{F}(P)\setminus\{\emptyset\}} (-1)^{\dim F}\,\mathbf{1}\{x \in F - N_B(P,\,F)\} = 1\,,$$

for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$.

Proof. This follows from a special case of Corollary 2.23. In fact, if $K = P$ is a polytope and $g : \mathbb{R}^d \to \mathbb{R}$ is an arbitrary bounded, Borel measurable function with compact support, then Corollary 2.23 and Theorem 2.9 imply

$$\begin{aligned}\int_{\mathbb{R}^d} g(x)\mathcal{H}^d(dx) =& \sum_{r=0}^{d-1}(-1)^r \int_0^\infty t^{d-1-r} \sum_{F\in\mathcal{F}_r(P)} \int_F \int_{N(P,\,F)\cap S^{d-1}} g(x - t\nabla h_B(u))h(B,u) \\ &\times \det\left(d^2h_B(u)|_{u^\perp\cap F^\perp}\right)\mathcal{H}^{d-1-r}(du)\mathcal{H}^r(dx)dt \\ &+ (-1)^d \int_K g(x)\mathcal{H}^d(dx)\,;\end{aligned}$$

compare Corollary 2.13. For each $r \in \{0,\dots,d-1\}$ and $F \in \mathcal{F}_r(P)$, consider the Lipschitz map

$$\tilde{H}_F : F \times (N(P,F) \cap S^{d-1}) \times (0,\infty) \to \mathbb{R}^d\,, \quad (x,u,t) \mapsto x - t\nabla h_B(u)\,.$$

Then $\operatorname{im}(\tilde{H}_F) = F - (N_B(P,F) \setminus \{o\})$ and, for $\mathcal{H}^d$ almost all $y \in \operatorname{im}(\tilde{H}_F)$, we have $\operatorname{card}\left(\tilde{H}_F^{-1}(\{y\})\right) = 1$. Moreover,

$$\operatorname{ap} J_d\tilde{H}_F(x,u,t) = h_B(u)t^{d-1-r}\det\left(d^2h_B(u)|_{u^\perp\cap F^\perp}\right)\,,$$

for $\mathcal{H}^d$ almost all $(x,u,t) \in F \times (N(P,F) \cap S^{d-1}) \times (0,\infty)$, and hence the area formula yields that

$$\begin{aligned}\int_{\mathbb{R}^d} g(x)\mathcal{H}^d(dx) =& \sum_{r=0}^{d-1}(-1)^r \sum_{F\in\mathcal{F}_r(P)} \int_{F-N_B(P,F)} g(x)\mathcal{H}^d(dx) + (-1)^d \int_K g(x)\mathcal{H}^d(dx) \\ =& \int_{\mathbb{R}^d}\left\{\sum_{F\in\mathcal{F}(P)\setminus\{\emptyset\}} (-1)^{\dim F}\,\mathbf{1}\{x \in F - N_B(P,\,F)\}\right\} g(x)\mathcal{H}^d(dx)\,.\end{aligned}$$

But this implies the asserted result. □

If one considers pairs of polytopes in general relative position, then one obtains a combinatorial relationship which is more symmetric. First, recall that if $(K,B) \in \mathcal{K}^d_{2,gp}$, $o \in B$ and $\dim(K+B) = d$, then $\mathcal{F}_r(K,B)$ denotes the set of all pairs $(F,G) \in \mathcal{F}_r(K) \times \mathcal{F}_{d-1-r}(B)$ for which $u(F+G, K+B) \neq o$; in addition, we set $\mathcal{F}_d(K,B) := \{K \times \{o\}\}$. Furthermore, we define

$$\mathcal{F}(K,B) := \bigcup_{r=0}^{d} \mathcal{F}_r(K,B)\,.$$

The proof of the following corollary is similar to the preceding proof, if one uses equation (87) as a starting point, so we omit it.

Corollary 2.26 *Let $(K,B) \in \mathcal{K}^d_{2,gp}$ be polytopes, $o \in B$ and $\dim(K+B) = d$. Then*

$$\sum_{(F,G)\in\mathcal{F}(K,B)} (-1)^{\dim F}\mathbf{1}\{x \in F - \operatorname{pos} G\} = 1\,,$$

for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$.

We also mention the following extension of Theorem 2 in [51] to Minkowski spaces. Recall that if $K, B \in \mathcal{K}^d$, $B = -B$ and $\operatorname{int} B \neq \emptyset$, then K is said to have *constant B-width $w \geq 0$* if $K + (-K) = wB$.

Corollary 2.27 *Let $(K,B) \in \mathcal{K}^d_{2,gp}$, let $B = -B$ and $\operatorname{int} B \neq \emptyset$. Let $\beta \subset \mathbb{R}^d$ be a Borel set, and assume that K has constant B-width $w \geq 0$. Then, for $j \in \{0,\dots,d-1\}$,*

$$S_j(K,B;-\beta) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} w^{j-i} S_i(K,B;\beta)\,.$$

A similar proof for this result can be given as in [51], which uses the preceding extensions of results in [51]. However, in the present situation additional approximation arguments are necessary, since the characterization of convex bodies K of constant B-width given in [25], p. 56 (VI'), is only valid if B is smooth and strictly convex. We do not give the details of such an approach, since the following theorem contains Corollary 2.27 as a special case.

Theorem 2.28 *Let $K, L, B \in \mathcal{K}^d$ with $o \in B$ and such that $(K,B), (L,B) \in \mathcal{K}^d_{2,gp}$. Assume that $K + L = B$. Then, for $j \in \{0,\dots,d-1\}$,*

$$S_j(K,B;\cdot) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} S_i(L,B;\cdot)\,.$$

Proof. It is well known that equation (6.7) in [25] implies that

$$S(K[j], B_1,\dots,B_{d-1-j},\cdot) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} S(L[i], M[j-i], B_1,\dots,B_{d-1-j},\cdot)\,, \qquad (102)$$

where $K, L, M, B_1, \ldots, B_{d-1-j} \in \mathcal{K}^d$ are such that $K + L = M$. By Theorem 3.3.1 in [123] we can find sequences $K_n, L_n, B_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, of strictly convex compact sets such that $K_n + L_n = B_n$ for all $n \in \mathbb{N}$ and $K_n \to K$, $L_n \to L$, $B_n \to B$ as $n \to \infty$. A special case of (102) and Theorem 2.14 imply that

$$S_j(K_n, B_n; \cdot) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} S_i(L_n, B_n; \cdot),$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$ yields the desired conclusion. □

2.5 Curvatures in a Minkowski plane

The Minkowski plane exhibits many special features, which cannot be observed in general Minkowski spaces. For example, it is not known under which assumptions precisely the metric projection in Minkowski spaces is a Lipschitz map; see also the subsequent remarks. In the plane, however, we can show that the B-projection onto K is a Lipschitz map $\mathbb{R}^2 \to K$ for any pair $(K, B) \in (\mathcal{K}^2_{2,gp})_o$. Using this result, we introduce generalized relative curvatures on the relative B-normal bundle of K by generalizing the Euclidean construction in a suitable way. These curvature functions can then be used to derive representations of the B-support measures $\Theta_o(K, B; \cdot)$ and $\Theta_1(K, B; \cdot)$ as integrals over $\mathcal{N}(K, B)$. Finally, we apply these results to extend Theorem 2.19 and Corollary 2.25 in the present two-dimensional case to the most general situation.

In the remaining part of this section, we change our notation by writing ∂K instead of bd K for the boundary of a convex set $K \subset \mathbb{R}^d$.

Theorem 2.29 *For $(K, B) \in (\mathcal{K}^2_{2,gp})_o$ the B-projection $p(K, B, \cdot) : \mathbb{R}^2 \to K$ is Lipschitz.*

Proof. If dim $K = 0$, there is nothing to prove. In order to prove the lemma if dim $K \geq 1$, we proceed by contradiction. Thus we assume that there is a sequence of points $x_n, y_n \in \mathbb{R}^2$, $n \in \mathbb{N}$, such that

$$|p(K, B, x_n) - p(K, B, y_n)| > n|x_n - y_n| \tag{103}$$

for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed for the moment. Then we can assume that

$$p(K, B, y_n) - p(K, B, x_n) = \alpha e_1,$$

where $\alpha > 0$ and $e_1 \in S^1$. Let $u_n := x_n - p(K, B, x_n)$ and $v_n := y_n - p(K, B, y_n)$. Choose $e_2 \in e_1^\perp \cap S^1$ such that $\langle u_n, e_2 \rangle \geq 0$. Now, up to symmetry, it is sufficient to consider the following three cases.

Case 1: $\langle u_n, e_1 \rangle \leq 0$, $\langle v_n, e_1 \rangle \geq 0$. But then

$$|x_n - y_n| \geq |p(K, B, x_n) - p(K, B, y_n)|,$$

and this together with (103) leads to a contradiction. Thus Case 1 cannot occur under the present assumption.

Case 2: $\langle u_n, e_1 \rangle \leq 0$, $\langle v_n, e_1 \rangle < 0$. Denote by $\bar{y}_n$ the orthogonal projection of $p(K, B, x_n)$

onto $p(K,B,y_n)+[0,\infty)v_n$. Since $(p(K,B,x_n)+[0,\infty)u_n)\cap(p(K,B,y_n)+[0,\infty)v_n)=\emptyset$, it easily follows that

$$|x_n-y_n|\geq|p(K,B,x_n)-\bar{y}_n|.$$

Thus we have

$$|p(K,B,x_n)-p(K,B,\bar{y}_n)|>n|p(K,B,x_n)-\bar{y}_n|,$$

since $p(K,B,y_n)=p(K,B,\bar{y}_n)$.

Case 3: $\langle u_n,e_1\rangle>0$, $\langle v_n,e_1\rangle<0$, $\langle v_n,e_2\rangle<0$. Let $\bar{y}_n$ be defined as in Case 2, and let $\bar{x}_n$ be the orthogonal projection of $p(K,B,y_n)$ onto $p(K,B,x_n)+[0,\infty)u_n$. If $|p(K,B,y_n)-\bar{x}_n|\leq|p(K,B,x_n)-\bar{y}_n|$, then it follows that

$$|x_n-y_n|\geq|p(K,B,y_n)-\bar{x}_n|,$$

and hence

$$|p(K,B,\bar{x}_n)-p(K,B,y_n)|>n|\bar{x}_n-p(K,B,y_n)|.$$

If, however, $|p(K,B,y_n)-\bar{x}_n|>|p(K,B,x_n)-\bar{y}_n|$, then similarly

$$|p(K,B,x_n)-p(K,B,\bar{y}_n)|>n|p(K,B,x_n)-\bar{y}_n|.$$

In any case, after a change of notation (if necessary), we can assume that, for all $n\in\mathbb{N}$,

$$|p(K,B,x_n)-p(K,B,\bar{y}_n)|>n|p(K,B,x_n)-\bar{y}_n|. \tag{104}$$

But then (104) implies that $\measuredangle(p(K,B,x_n)-p(K,B,\bar{y}_n),\bar{y}_n-p_B(K,y_n))\to 0$ as $n\to\infty$. This shows that there is a sequence of points $b_n\in\partial B$ such that $\gamma_n\to 0$ as $n\to\infty$, if γ_n denotes the angle between the vector b_n and some (if there are several) support line of B at b_n. Since $o\in\text{int }B$, this yields a contradiction. □

Let $(K,B)\in(\mathcal{K}^2_{2,gp})_o$ be fixed. Then, for $\lambda>0$, we consider the map

$$h_\lambda:\mathbb{R}^2\setminus K\to\mathbb{R}^2\setminus K,\quad y\mapsto p(K,B,y)+\lambda(y-p(K,B,y)).$$

In writing h_λ we do not make explicit the dependence of h_λ on K and B. Obviously, $(h_\lambda)^{-1}=h_{\lambda^{-1}}$ and for $t>0$ we have $h_\lambda(\partial(K+tB))=\partial(K+t\lambda B)$.

By Theorem 2.29, h_λ is a bi-Lipschitz map. In the same way as in the Euclidean plane one can thus show that rang $Dh_\lambda(y)=2$ and $(h_\lambda)^{-1}$ is differentiable at $h_\lambda(y)$ provided that h_λ is differentiable at $y\in\mathbb{R}^2\setminus K$. This implies that $p(K,B,\cdot)$ is differentiable at $y\in\mathbb{R}^2\setminus K$ if and only if $p(K,B,\cdot)$ is differentiable at $\bar{y}$ for any $\bar{y}\in p(K,B,y)+(0,\infty)(y-p(K,B,y))$. We define $\mathcal{D}_{K,B}$ as the set of all $y\in\mathbb{R}^2\setminus K$ for which $y\in\text{reg}(K+d(K,B,y)B)$ and $p(K,B,\cdot)$ is differentiable at y. Thus, $y\in\mathcal{D}_{K,B}$ if and only if $p(K,B,y)+(0,\infty)(y-p(K,B,y))\subset\mathcal{D}_{K,B}$; moreover, Theorem 2.29 yields that

$$\mathcal{H}^2(\mathbb{R}^2\setminus(K\cup\mathcal{D}_{K,B}))=0$$

and, for all $t > 0$,

$$\mathcal{H}^1(\partial(K+tB) \setminus \mathcal{D}_{K,B}) = 0 .$$

Choose $y \in \mathcal{D}_{K,B}$ and set $t := d(K,B,y)$; then $u_t(K,B,\cdot) := u(K,B,\cdot)|_{\partial(K+tB)}$ is differentiable at y and $Du_t(K,B,y) : \mathrm{Tan}(\partial(K+tB),y) \to \mathbb{R}^2$. Set $u := \sigma_{K+tB}(y)$, and let $v \in u^\perp \cap S^1$. If $Du_t(K,B,y)(v) = o$, then $\mathrm{im}\, Du_t(K,B,y) = \{o\} \subset u^\perp$. If, however, $Du_t(K,B,y)(v) \neq o$, then we can choose a curve $\gamma : (-\epsilon,\epsilon) \to \partial(K+tB)$ which locally parametrizes $\partial(K+tB)$ and fulfills $\gamma(0) = y$ and $\gamma'(0) = v$. Hence, $u_t(K,B,\cdot)\circ\gamma$ is differentiable at 0, $(u_t(K,B,\cdot)\circ\gamma)'(0) \neq o$ and $\mathrm{im}(u_t(K,B,\cdot)\circ\gamma) \subset \partial B$. This shows that $u(K,B,y)$ is a smooth boundary point of B. But then $\mathrm{im}\, Du_t(K,B,y) = u^\perp$ and one can easily check that $Du(K,B,y)(v) = k(y)v$ with some $k(y) \geq 0$ (and hence also $k(y) > 0$). Thus we have established the following lemma.

Lemma 2.30 *Let $(K,B) \in (\mathcal{K}^2_{2,gp})_o$, let $y \in \mathcal{D}_{K,B}$, and set $u := \sigma_{K+d(K,B,y)B}(y)$. Then there is some $k(K,B,y) \geq 0$ such that, for all $v \in u^\perp$, $Du(K,B,y)(v) = k(K,B,y)v$.*

Now, for $y \in \mathcal{D}_{K,B}$ and $0 < t < d(K,B,y)$, we have

$$y - tu(K,B,y) \in \partial(K + (d(K,B,y)-t)B)$$

and

$$u(K,B,y-tu(K,B,y)) = u(K,B,y). \tag{105}$$

Choosing $v \in u^\perp \setminus \{o\}$, where

$$u := \sigma_{K+d(K,B,y)B}(y)^\perp = \sigma_{K+(d(K,B,y)-t)B}(y - tu(K,B,y))^\perp ,$$

we obtain from (105) that

$$(1 - tk(K,B,y))Du(K,B,y-tu(K,B,y))(v) = k(K,B,y)v ; \tag{106}$$

moreover

$$Du(K,B,y-tu(K,B,y))(v) = k(K,B,y-tu(K,B,y))v . \tag{107}$$

From (106) and (107) it follows that $k(K,B,y) < 1/t$ and

$$k(K,B,y-tu(K,B,y)) = \frac{k(K,B,y)}{1-tk(K,B,y)} ; \tag{108}$$

hence

$$0 \leq k(K,B,y) \leq d(K,B,y)^{-1} .$$

Using (108), we see that $k(K,B,y) = d(K,B,y)^{-1}$ implies

$$k(K,B,y-tu(K,B,y)) = d(K,B,y-tu(K,B,y))^{-1} ;$$

furthermore, $k(K,B,y) < d(K,B,y)^{-1}$ yields

$$\frac{k(K,B,y)}{1-d(K,B,y)k(K,B,y)} = \frac{k(K,B,y-tu(K,B,y))}{1-d(K,B,y-tu(K,B,y))k(K,B,y-tu(K,B,y))} .$$

The preceding considerations are summarized in the next lemma.

Lemma 2.31 *Let $(K,B) \in (\mathcal{K}^2_{2,gp})_o$, and let $(x,b) \in \mathcal{N}(K,B)$ be such that $y := x+tb \in \mathcal{D}_{K,B}$ for some (and hence for all) $t > 0$. Then the definition of the quantity*

$$k(K,B;x,b) := \frac{k(K,B,y)}{1 - d(K,B,y)k(K,B,y)} \in [0,\infty],$$

which we call the generalized B-curvature of K at (x,b), is independent of the particular choice of $t > 0$.

For fixed $t > 0$, the mappings

$$F_t^{K,B} : \mathcal{N}(K,B) \to \partial(K+tB)\,, \quad (x,b) \mapsto x + tb\,,$$

and

$$(F_t^{K,B})^{-1} : \partial(K+tB) \to \mathcal{N}(K,B)\,, \quad y \mapsto (p(K,B,y), u(K,B,y))\,,$$

are Lipschitz maps which are inverse to each other. In particular, this shows that $\mathcal{N}(K,B)$ is a 1-rectifiable subset of $\mathbb{R}^2 \times \mathbb{R}^2$ and that the generalized B-curvature $k(K,B;x,b)$ is defined for $\mathcal{H}^1$ almost all $(x,b) \in \mathcal{N}(K,B)$.

Let $y \in \mathcal{D}_{K,B} \cap \partial(K+tB)$, $t > 0$, and $(x,b) := (F_t^{K,B})^{-1}(y)$. Then, if $u := \sigma_{K+tB}(y)$ and $v \in S^1 \cap u^\perp$, we obtain

$$\begin{aligned}
&\mathrm{Tan}^1\left(\mathcal{N}(K,B), (F_t^{K,B})^{-1}(y)\right) = D(F_t^{K,B})^{-1}(y)(u^\perp) \\
&= \mathrm{lin}\,\{(Dp(K,B,y)(v), Du(K,B,y)(v))\} \\
&= \mathrm{lin}\,\{((1 - tk(K,B,y))v, k(K,B,y)v)\} \\
&= \mathrm{lin}\left\{\left(\frac{1}{\sqrt{1+k(K,B;x,b)^2}}\,v, \frac{k(K,B;x,b)}{\sqrt{1+k(K,B;x,b)^2}}\,v\right)\right\}.
\end{aligned}$$

Next we consider the bijective Lipschitz map

$$F^{K,B} : \mathcal{N}(K,B) \times (0,\infty) \to \mathbb{R}^d \setminus K\,, \quad (x,b,t) \mapsto x + tb\,.$$

A straightforward calculation shows that

$$\mathrm{ap}\ J_2F^{K,B}(x,b,t) = \langle b, u(x,b)\rangle \left(\frac{1}{\sqrt{1+k(K,B;x,b)^2}} + t\frac{k(K,B;x,b)}{\sqrt{1+k(K,B;x,b)^2}}\right),$$

for $\mathcal{H}^2$ almost all $(x,b,t) \in \mathcal{N}(K,B) \times (0,\infty)$, where $u(x,b) := \sigma_{K+tB}(x+tb)$ is defined for $\mathcal{H}^1$ almost all $(x,b) \in \mathcal{N}(K,B)$ (for each fixed $t > 0$). By a repeated application of Federer's coarea formula we thus arrive at the subsequent theorem.

Theorem 2.32 *Let $(K,B) \in (\mathcal{K}^2_{2,gp})_o$, and let $\eta \subset \mathbb{R}^2 \times \mathbb{R}^2$ be a Borel set. Then*

$$\Theta_0(K,B;\eta) = \int_{\mathcal{N}(K,B)\cap\eta} \langle b, u(x,b)\rangle \frac{k(K,B;x,b)}{\sqrt{1+k(K,B;x,b)^2}} \mathcal{H}^1(d(x,b))$$

and

$$\Theta_1(K,B;\eta) = \int_{\mathcal{N}(K,B)\cap\eta} \langle b, u(x,b)\rangle \frac{1}{\sqrt{1+k(K,B;x,b)^2}} \mathcal{H}^1(d(x,b))\,.$$

These results are already sufficiently general to yield an optimal version of Theorem 2.19 in a Minkowski plane.

Theorem 2.33 *Let* $(K, B) \in (\mathcal{K}^2_{2,gp})_o$. *Then*

$$2 \le n(K, B) \le \frac{V(K + DK)}{V(K)} - 1 .$$

Proof. First, note that $\mathcal{N}(K, B)$ is compact and $n(K, B, \cdot)$ is $\mathcal{H}^2$ measurable. Moreover $M_B(K)$ is a Borel set by the same reasoning as in the proof of Theorem 2.19. We define

$$M_B(K) := \{(x, b, t) \in \mathcal{N}(K, B) \times \mathbb{R} : t \in [0, \sigma_B(K, x, b)]\}$$

and maintain the previous notation; thus,

$$\int_{G^{-1}(\{p\})} \mathbf{1}_{M_B(K)}(x, b, t)\mathcal{H}^0(d(x, b, t)) = \begin{cases} n(K, B, p)\,, & \text{if } p \in K\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Hence,

$$\begin{aligned} V(K)n(K, B) &= \int_{\mathbb{R}^2} \int_{G^{-1}(\{p\})} \mathbf{1}_{M_B(K)}(x, b, t)\mathcal{H}^0(d(x, b, t))\mathcal{H}^2(dp) \\ &= \int_{\mathcal{N}(K,B)\times\mathbb{R}} \mathbf{1}_{M_B(K)}(x, b, t)\text{ap } J_2G(x, b, t)\mathcal{H}^2(d(x, b, t)) \\ &\le \int_{\mathcal{N}(K,B)} \int_0^{\sigma_B(K,x,b)} \frac{\langle b, u(x, b)\rangle}{\sqrt{1 + k(K, B; x, b)^2}}\mathcal{H}^1(d(x, b))dt \\ &\quad + \int_{\mathcal{N}(K,B)} \int_0^{\sigma_B(K,x,b)} t\frac{k(K, B; x, b)}{\sqrt{1 + k(K, B; x, b)^2}}\langle b, u(x, b)\rangle\mathcal{H}^1(d(x, b))dt \\ &= \int_{\mathcal{N}(K,B)} \sigma_B(K, x, b)\Theta_1(K, B; d(x, b)) + \frac{1}{2}\int_{\mathcal{N}(K,B)} \sigma_B(K, x, b)^2\Theta_0(K, B; d(x, b)) . \end{aligned}$$

In the last step, we used Theorem 2.32. From the estimate $\sigma_B(K, x, b) \le \rho(DK, b)$ we finally deduce

$$\begin{aligned} V(K)n(K, B) &\le \sum_{r=0}^{1} \frac{1}{2}\binom{2}{r+1} \int_{\mathcal{N}(K,B)} \rho(DK, b)^{r+1}\Theta_{2-1-r}(K, B; d(x, b)) \\ &= \sum_{r=0}^{1} \frac{1}{2}\binom{2}{r+1} \int_{\partial B} \rho(DK, b)^{r+1}S_{2-1-r}(K, B; db) \\ &\le V(K + DK) - V(K)\,, \end{aligned}$$

by a special case of Proposition 2.17. □

Remark. The bounds established in Theorems 2.19 and 2.33 can be improved under various special assumptions such as constant relative width or other curvature bounds. However, it is not the present purpose to present an exhaustive discussion of the subject. Rather we wish to provide the general techniques required to explore the field.

Corollary 2.34 *Let $(P,B) \in (\mathcal{K}^2_{2,gp})_o$, and assume that P is a polytope. Then*

$$\sum_{F\in\mathcal{F}(P)\setminus\{\emptyset\}} (-1)^{\dim F}\,\mathbf{1}_{F-N_B(P,F)}(x) = 1\,,$$

for $\mathcal{H}^2$ almost all $x \in \mathbb{R}^2$.

Proof. For a polytope $P \in \mathcal{K}^2$, $F \in \mathcal{F}_1(P)$ and $x \in \operatorname{relint} F$ it is easy to see that $k(P,B;x,b) = 0$ whenever $(x,b) \in \mathcal{N}(P,B)$. In addition, for $F = \{x\} \in \mathcal{F}_0(P)$ we have $k(P,B;x,b) = \infty$ for $\mathcal{H}^1$ almost all $(x,b) \in \mathcal{N}(P,B)$. Using a very special case of Corollary 2.23 and Theorem 2.32, we find

$$\begin{aligned}
&\int_{\mathbb{R}^2} g(x)\mathcal{H}^2(dx) - \int_K g(x)\mathcal{H}^2(dx)\\
&= \int_0^\infty t \int_{\mathcal{N}(P,B)} g(x-tb)\langle b,u(x,b)\rangle \frac{k(P,B;x,b)}{\sqrt{1+k(P,B;x,b)^2}}\mathcal{H}^1(d(x,b))dt\\
&\quad - \int_0^\infty \int_{\mathcal{N}(P,B)} g(x-tb)\frac{\langle b,u(x,b)\rangle}{\sqrt{1+k(P,B;x,b)^2}}\mathcal{H}^1(d(x,b))dt\\
&=: \mathrm{I} - \mathrm{II}\,.
\end{aligned}$$

For the first integral we obtain

$$\begin{aligned}
\mathrm{I} &= \sum_{\{x\}\in\mathcal{F}_o(P)} \int_0^\infty \int_{N_B(P,\{x\})\cap\partial B} g(x-tb)t\langle b,u(x,b)\rangle\mathcal{H}^1(db)dt\\
&= \sum_{\{x\}\in\mathcal{F}_o(P)} \int_{\{x\}-N_B(P,\{x\})} g(x)\mathcal{H}^2(dx),
\end{aligned}$$

since the Jacobian of the map

$$F_1 : (N_B(P,\{x\})\cap\partial B)\times(0,\infty) \to \{x\} - N_B(P,\{x\})\,, \quad (b,t)\mapsto x-tb\,,$$

is equal to

$$J_2F_1(b,t) = |\det(-b,-tv)| = t\langle b,u(x,b)\rangle\,,$$

for $\mathcal{H}^2$ almost all $(b,t) \in (N_B(P,\{x\})\cap\partial B)\times(0,\infty)$, where $v \in \operatorname{Tan}(\partial B,b)\cap S^1$.

Similarly,

$$\begin{aligned}
\mathrm{II} &= \sum_{F\in\mathcal{F}_1(P)} \int_0^\infty \int_F \int_{N_B(P,F)\cap\partial B} \langle b,\sigma_P(x)\rangle g(x-tb)\mathcal{H}^0(db)\mathcal{H}^1(dx)dt\\
&= \sum_{F\in\mathcal{F}_1(P)} \int_F \int_{N_B(P,F)} \big\langle \bar b/|\bar b|,\sigma_P(x)\big\rangle\, g(x-\bar b)\mathcal{H}^1(d\bar b)\mathcal{H}^1(dx)\\
&= \sum_{F\in\mathcal{F}_1(P)} \int_{F-N_B(P,F)} g(x)\mathcal{H}^2(dx)\,.
\end{aligned}$$

In the last step we used that the Jacobian of the map

$$F_2 : F \times N_B(P,F) \to F - N_B(P,F)\,, \quad (x,\bar{b}) \mapsto x - \bar{b}\,,$$

is equal to

$$J_2F_2(x,\bar{b}) = |\det(f, |\bar{b}|^{-1}\bar{b})| = \langle \bar{b}/|\bar{b}|, \sigma_P(x)\rangle\,,$$

for $\mathcal{H}^2$ almost all $(x,\bar{b}) \in F \times N_B(P,F)$, where $f \in \sigma_P(x)^\perp \cap S^1$. □

2.6 Curvatures in Minkowski spaces

In this subsection we develop the theory of support measures and generalized curvature functions in Minkowski spaces with a sufficiently regular gauge body. The constructions are clearly guided by the Euclidean situation and also partly by the intention to establish characterizations of gauge bodies via relations between Minkowski curvature measures. As a first major step in this direction, we extend a volumic formula involving interior reach, due to J. Sangwine-Yager [114], and a generalized Minkowski integral formula to the relative setting of this section.

In the following we write $\mathcal{K}^d_*$ for the set of all convex bodies of class C^2_+ which contain the origin as an interior point. The gauge bodies to be considered in this subsection and the remaining part of Section 2 will be assumed to satisfy $B \in \mathcal{K}^d_*$. It is certainly desirable to weaken this assumption, and for some of the following arguments this is indeed possible. But at a certain point of our arguments, we cannot avoid the C^2_+ hypothesis on the gauge bodies. Therefore we adopt $B \in \mathcal{K}^d_*$ as an overall assumption. In particular, without any additional assumptions, for a pair (K,B) of convex bodies in general relative position, the B-projection onto K is probably not Lipschitz in contrast to the two-dimensional situation. Finally, for $K \in \mathcal{K}^d$, $B \in \mathcal{K}^d_*$, and $t > 0$, we set $p_t(K,B,\cdot) := p(K,B,\cdot)|_{\partial(K+tB)}$ and $u_t(K,B,\cdot) := u(K,B,\cdot)|_{\partial(K+tB)}$.

Lemma 2.35 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Then the map $p_t(K,B,\cdot)$ is Lipschitz for any $t > 0$. Moreover, the Minkowski normal bundle $\mathcal{N}(K,B)$ is a $(d-1)$-rectifiable compact subset of $\mathbb{R}^d \times \mathbb{R}^d$.*

Proof. For $t > 0$, a ball of positive radius rolls freely inside the convex body $K + tB$, and hence σ_{K+tB} is Lipschitz. Therefore the assertions of the lemma are implied by the relation $u_t(K,B,\cdot) = \nabla h_B \circ \sigma_{K+tB}$. □

In particular, the preceding lemma implies that the map $h_\lambda|_{\partial(K+tB)} : \partial(K+tB) \to \partial(K+t\lambda B)$, which is defined in analogy to the two-dimensional case of the preceding section, constitutes a bi-Lipschitz map for any fixed $t, \lambda > 0$. Note that we do not make explicit the dependence of h_λ on K and B.

Lemma 2.36 *Let $K \in \mathcal{K}^d$, let $B \in \mathcal{K}^d_*$, and assume that $h_\lambda|_{\partial(K+tB)}$ is differentiable at $y \in \partial(K+tB)$. Then* rang $Dh_\lambda(y) = d-1$ *and the inverse map of $h_\lambda|_{\partial(K+tB)}$ is differentiable at $h_\lambda(y)$.*

Proof. For any $t > 0$, $\partial(K + tB)$ is a submanifold of class C^1. Using local charts, we can then proceed as in the proof of Lemma 4.1 in [141]. □

Next we define $\mathcal{D}_{K,B}$ as the set of all $y \in \mathbb{R}^d \setminus K$ for which $p(K, B, \cdot)|_{\partial(K+d(K,B,y)B)}$ is differentiable at y. By Lemma 2.36, $p(K, B, \cdot)|_{\partial(K+d(K,B,y_0)B)}$ is differentiable at y_0 if and only if $p(K, B, \cdot)|_{\partial(K+d(K,B,y)B)}$ is differentiable at y for any $y \in p(K, B, y_0) + (0, \infty)u(K, B, y_0)$.

Let $y \in \mathcal{D}_{K,B}$, $t := d(K, B, y)$, and set $u := \sigma_{K+tB}(y)$. Then

$$u_t(K, B, \cdot) : \partial(K + tB) \to \partial B$$

can be obtained as the composition of $\sigma_{K+tB} : \partial(K + tB) \to S^{d-1}$ and $\nabla h_B : S^{d-1} \to \partial B$, the latter being a diffeomorphism between C^1 manifolds. Clearly, σ_{K+tB} is differentiable at y, since $y \in \mathcal{D}_{K,B}$. Hence, $Du_t(K, B, y) : u^\perp \to u^\perp$ can be represented as

$$Du_t(K, B, y) = \left(D^2 h_B(u)|_{u^\perp}\right) \circ D\sigma_{K+tB}(y)\,,$$

where $D^2 h_B(u)|_{u^\perp}$ and $D\sigma_{K+tB}(y)$ will be viewed as symmetric linear maps of $u^\perp$ onto itself, where the symmetry property refers to the restriction to $u^\perp$ of our underlying auxiliary scalar product.

In order to describe the eigenvalues and eigenvectors of $Du_t(K, B, y)$ we introduce another scalar product $[\cdot\,,\cdot]_u$ on $u^\perp$ (here the formation of the orthogonal complement $u^\perp$ of u is meant with respect to the initial scalar product) by setting

$$[v, w]_u := \langle (D^2 h_B(u)|_{u^\perp})^{-1} v, w\rangle$$

for $v, w \in u^\perp$. Then $Du_t(K, B, y)$ is symmetric and positive semi-definit with respect to $[\cdot\,,\cdot]_u$; compare [78], Satz 7.22. Thus we arrive at the following lemma.

Lemma 2.37 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Then, for $y \in \mathcal{D}_{K,B}$ and $u := \sigma_{K+d(K,B,y)B}(y)$, there are numbers $k_i^B(y) \geq 0$, $i \in \{1, \ldots, d-1\}$, and corresponding vectors $u_1, \ldots, u_{d-1} \in u^\perp$, which form an orthonormal basis with respect to $[\cdot\,,\cdot]_u$, such that $Du_t(K, B, y)(u_i) = k_i^B(y) u_i$ for $i \in \{1, \ldots, d-1\}$ and $t := d(K, B, y)$.*

As in the Euclidean case, we usually do not indicate the dependence of the numbers $k_i^B(y)$, which are called the relative curvatures of K with respect to B (or simply the B-curvatures of K), on the convex body K.

Now we go one step further and introduce generalized relative curvatures. Indeed, in the same way as Lemma 2.31 was deduced from Lemma 2.30, we now obtain Lemma 2.38 as a consequence of Lemma 2.37. This argument also shows that the orthonormal basis $u_1, \ldots, u_{n-1} \in u^\perp$ with respect to $[\cdot\,,\cdot]_u$, which appears in Lemma 2.37 and which is associated with any point $y' \in \mathcal{D}_{K,B}$, can (and will) be chosen in such a way that it does not depend on the particular choice of the point y' on the ray $p(K, B, y) + (0, \infty)u(K, B, y)$, for some fixed $y \in \mathcal{D}_{K,B}$.

Lemma 2.38 *Let $K \in \mathcal{K}^d$, let $B \in \mathcal{K}^d_*$, and assume that $(x, b) \in \mathcal{N}(K, B)$ is such that $y := x + tb \in \mathcal{D}_{K,B}$ for some (and hence for all) $t > 0$. Then the definition of the quantities*

$$k_i^B(x, b) := \frac{k_i^B(y)}{1 - d(K, B, y) k_i^B(y)} \in [0, \infty]\,, \quad i \in \{1, \ldots, d-1\}\,,$$

which we call the generalized B-curvatures of K at (x, b), does not depend on the special choice of $t > 0$. In particular, $k_i^B(x, b)$ is defined for $\mathcal{H}^{d-1}$ almost all $(x, b) \in \mathcal{N}(K, B)$.

In the Euclidean case the support measures of a convex body can be expressed in terms of integrals of generalized curvatures which are integrated over the Euclidean normal bundle of K with respect to the $(d-1)$-dimensional Hausdorff measure over $\mathbb{R}^d \times \mathbb{R}^d$. A similar relationship will now be deduced in Minkowski spaces. However, the fact that the scalar product $[\cdot\,,\cdot]_u$, $u \in S^{d-1}$, does not coincide with the underlying scalar product and, in addition, depends on $u \in S^{d-1}$, slightly complicates the situation.

For that reason, we introduce a Riemannian metric on the C^2 submanifold $M := \mathbb{R}^d \times \partial B$ of $\mathbb{R}^d \times \mathbb{R}^d$, which is associated with B and which takes into account the anisotropy of B. Since B is of class C^2_+, the tangent space $T_b B$ of B at $b \in \partial B$ is uniquely determined. Let $v, w \in \mathbb{R}^d$ be arbitrarily given. Then there are unique decompositions $v = v_1 + v_2$ and $w = w_1 + w_2$ such that $v_2 = \lambda(v)b$, $w_2 = \lambda(w)b$ with $\lambda(v), \lambda(w) \in \mathbb{R}$ and $v_1, w_1 \in T_b B$; hence we can define

$$[v,w]_b^B := \langle D\sigma_B(b)v_1, w_1\rangle + \lambda(v)\lambda(w)\,.$$

Obviously,

$$[v,w]_b^B = [v,w]_u$$

if $v, w \in u^\perp$ and $b = \nabla h_B(u)$. Furthermore, a Riemannian metric g^B of class C^0 is defined on M by setting

$$g^B_{(x,b)}((v,w),(\bar{v},\bar{w})) := [v,\bar{v}]_b^B + [w,\bar{w}]_b^B\,,$$

where $(x,b) \in M$ and $(v,w),(\bar{v},\bar{w}) \in T_{(x,b)}M = \mathbb{R}^d \times T_b B$.

Now let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Then $\mathcal{N}(K,B)$ is a $(d-1)$-rectifiable subset (actually a strong Lipschitz submanifold) of M and

$$\begin{aligned}&\operatorname{Tan}^{d-1}(\mathcal{N}(K,B),(x,b))\\ &\quad= \operatorname{lin}\left\{\left(\frac{1}{\sqrt{1+k_i^B(x,b)^2}}\,u_i, \frac{k_i^B(x,b)}{\sqrt{1+k_i^B(x,b)^2}}\,u_i\right),\ i=1,\dots,d-1\right\}\,,\end{aligned}$$

whenever $x + tb \in \mathcal{D}_{K,B}$ for some $t > 0$. In this case, the vectors

$$\left(\frac{1}{\sqrt{1+k_i^B(x,b)^2}}\,u_i, \frac{k_i^B(x,b)}{\sqrt{1+k_i^B(x,b)^2}}\,u_i\right),\ i=1,\dots,d-1\,,$$

represent an orthonormal basis of $\operatorname{Tan}^{d-1}(\mathcal{N}(K,B),(x,b))$ with respect to $g^B_{(x,b)}$; moreover, $u_1,\dots,u_{d-1} \in u^\perp$ is an orthonormal system with respect to $[\cdot\,,\cdot]_b^B$.

Subsequently, we write $\mathcal{H}^r_B$, $r \geq 0$, for the r-dimensional Hausdorff measure over $\mathbb{R}^d \times \partial B$ which is induced by the intrinsic metric of the Riemannian space (M, g^B). Further, for $j \in \{0,\dots,d-1\}$ and whenever the right-hand side is defined, we set

$$\mathbb{H}_j^B(K,x,b) := \binom{d-1}{j}^{-1} \sum_{|I|=j} \frac{\prod_{i\in I} k_i^B(x,b)}{\prod_{i=1}^{d-1}\sqrt{1+k_i^B(x,b)^2}}\,,$$

where the summation extends over all subsets $I \subset \{1,\dots,d-1\}$ of cardinality j.

We need one more preparatory remark. Since ∂B is compact and $g^B_{(x,b)}$ does not depend on x, the intrinsic metric which is induced by g^B on M is equivalent to the intrinsic metric which is induced by the Riemannian metric which M inherits as a submanifold of $\mathbb{R}^{2d}$. This implies that there are constants $c, C > 0$ such that

$$c\,\mathcal{H}^{d-1}_B \leq \mathcal{H}^{d-1} \leq C\,\mathcal{H}^{d-1}_B$$

over M; especially, $\mathcal{H}^{d-1} \llcorner M$ and $\mathcal{H}^{d-1}_B$ have the same sets of measure zero. This fact will be used implicitly several times in the following.

Theorem 2.39 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Then*

$$\Theta_j(K,B;\eta) = \int_{\mathcal{N}(K,B)\cap\eta} \langle b, \sigma_B(b)\rangle\, \mathbb{H}^B_{d-1-j}(K,x,b)|u_1 \wedge \ldots \wedge u_{d-1}|\mathcal{H}^{d-1}_B(d(x,b))\,,$$

for $j \in \{0,\ldots,d-1\}$ and Borel sets $\eta \subset \mathbb{R}^{2d}$.

Proof. We consider the transformation

$$T : M \times (0,\infty) \to \mathbb{R}^d \setminus K\,, \quad (x,b,t) \mapsto x + tb\,,$$

as a map between Riemannian manifolds and apply the coarea formula (in this setting) twice to obtain

$$\mu_\rho(K,B,\eta) = \int_0^\rho \int_{\mathcal{N}(K,B)\cap\eta} \text{ap } J_d T(x,b,t)\mathcal{H}^{d-1}_B(d(x,b))dt\,.$$

Moreover, for $\mathcal{H}^{d-1}_B$ almost all $(x,b) \in \mathcal{N}(K,B)$ and all $t > 0$, we have

$$\begin{aligned}\text{ap } J_d T(x,b,t) &= \left|\bigwedge_{i=1}^{d-1}\left(\frac{1}{\sqrt{1+k_i^B(x,b)^2}}u_i + t\frac{k_i^B(x,b)}{\sqrt{1+k_i^B(x,b)^2}}u_i\right)\wedge b\right| \\ &= \langle b,\sigma_B(b)\rangle \sum_{j=0}^{d-1} t^{d-1-j}\binom{d-1}{j}\mathbb{H}^B_{d-1-j}(K,x,b)|u_1\wedge\ldots\wedge u_{d-1}|\,.\end{aligned}$$

A comparison of coefficients finally yields the desired result. □

Of course, in the case where $B = B^d$ the representation for $\Theta_j(K,B;\cdot)$, which is given in Theorem 2.39, yields the corresponding representation for the Euclidean generalized curvature measures due to Zähle [155].

Similarly to [155], we can provide a current representation of $\Theta_j(K,B;\cdot)$. In fact, let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$ be fixed. Let a^B_K be the simple $(d-1)$-vectorfield which is uniquely determined, for $\mathcal{H}^{d-1}$ almost all $(x,b) \in \mathcal{N}(K,B)$, by the requirements that

$$a^B_K(x,b) = a_1(x,b)\wedge\ldots\wedge a_{d-1}(x,b)\,,$$

where $a_i(x,b) \in \text{Tan}^{d-1}(\mathcal{N}(K,B),(x,b))$,

$$|a_1(x,b)\wedge\ldots\wedge a_{d-1}(x,b)| = 1 \quad \text{and} \quad \text{sgn}(a_1(x,b),\ldots,a_{d-1}(x,b),b) = 1\,;$$

compare [155] for further explanations.

Then we introduce the $(d-1)$-current

$$T_K^B := \left(\mathcal{H}_B^{d-1} \llcorner \mathcal{N}(K,B)\right) \wedge a_K^B ,$$

which is defined at least on $\mathcal{D}_{d-1}(\mathbb{R}^d \times \mathbb{R}^d)$; see Chapter 4 in [42] for the terminology. In fact, the domain of T_K^B naturally includes the $(d-1)$-forms $\mathbf{1}_A\, \varphi_j$, where $A \subset \mathbb{R}^{2d}$ is Borel measurable and the $(d-1)$-covector $\varphi_j(x,b)$, for $(x,b) \in \mathbb{R}^{2d}$, is defined by its values for simple $(d-1)$-vectors. In order to give an explicit definition, we introduce the coordinate projections $\Pi_0 : \mathbb{R}^{2d} \to \mathbb{R}^d$, $(x,y) \mapsto x$ and $\Pi_1 : \mathbb{R}^{2d} \to \mathbb{R}^d$, $(x,y) \mapsto y$. Furthermore, let Ω denote the standard volume form on $\mathbb{R}^d$, and choose any $\xi_1, \dots, \xi_{d-1} \in \mathbb{R}^{2d}$. Then we define

$$\langle \xi_1 \wedge \ldots \wedge \xi_{d-1}, \varphi_j(x,b)\rangle := \binom{d-1}{j}^{-1} \sum \langle \Pi_{\epsilon_1}\xi_1 \wedge \ldots \wedge \Pi_{\epsilon_{d-1}}\xi_{d-1} \wedge b, \Omega\rangle ,$$

where the summation is extended over all $\epsilon_i \in \{0,1\}$, $i = 1, \dots, d-1$, for which $\sum_{i=1}^{d-1} \epsilon_i = d-1-j$. Up to a normalizing factor, this definition coincides with the one given in [155]. Using this terminology, we obtain that

$$\Theta_j(K,B;A) = T_K^B(\mathbf{1}_A\, \varphi_j)$$

for $j \in \{0, \dots, d-1\}$ and Borel sets $A \subset \mathbb{R}^{2d}$.

In analogy to the Euclidean case, the value of the interior reach function of a convex body $K \in \mathcal{K}^d$ with respect to $B \in \mathcal{K}^d_*$ at the boundary point $x \in \partial K$ is given by

$$r_B(K,x) := \max\{\lambda \geq 0 : x \in z + \lambda B \subset K) \text{ for some } z \in K\} .$$

It is clear from the definition that the map $r_B(K,\cdot)$ is upper semicontinuous, and hence it is Borel measurable.

Finally, we shall provide one more auxiliary result concerning curvature functions. To state it, let us define

$$\mathcal{M}_B(K) := \{x \in \mathcal{M}(K) : x \in p(K,B,\cdot)(\mathcal{D}_{K,B})\}$$

as a subset of the set $\mathcal{M}(K)$ of normal boundary points of K. For any $x \in \mathcal{M}_B(K)$ and $u := \sigma_K(x)$, we denote by $k_i^B(x)$, $i \in \{1, \dots, d-1\}$, the eigenvalues of the linear map

$$D^2 h_B(u) \circ D\sigma_K(x) ; \tag{109}$$

see the arguments preceding Lemma 2.37. The linear map $D\sigma_K(x)$ coincides with $D^2 f(x) : u^\perp \to u^\perp$, where f is a convex function which locally represents ∂K at x (compare [70]) and which is second order differentiable at the normal boundary point x.

Lemma 2.40 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Then $\mathcal{H}^{d-1}(\partial K \setminus \mathcal{M}_B(K)) = 0$. If $x \in \mathcal{M}_B(K)$ and $b := \nabla h_B(\sigma_K(x))$, then $k_i^B(x,b) = k_i^B(x) \leq r_B(K,x)^{-1}$, for $i \in \{1, \dots, d-1\}$, provided the ordering is chosen properly.*

Proof. The first statement follows since $\mathcal{H}^{d-1}(\partial K \setminus \mathcal{M}(K)) = 0$, the restriction of $p(K, B, \cdot)$ to $\partial(K + tB)$ is Lipschitz, and $\mathcal{H}^{d-1}(\partial(K + tB) \setminus \mathcal{D}_{K,B}) = 0$ for any $t > 0$. Now let $x \in \mathcal{M}_B(K)$, $u := \sigma_K(x)$, and let $t > 0$ be fixed. For any $p \in \partial K$ we choose some $\sigma_K^*(p) \in N(K, p) \cap S^{d-1}$. Hence the map $\sigma_K^* : \partial K \to S^{d-1}$ represents a choice of an exterior unit normal vector field for K. But then we have

$$\sigma_K^*(p) = \sigma_{K+tB}(p + t\nabla h_B \circ \sigma_K^*(p)) . \tag{110}$$

In fact, this follows from

$$\begin{aligned} N(K + tB, p + t\nabla h_B(\sigma_K^*(p))) &= N(K, p) \cap N(B, \nabla h_B(\sigma_K^*(p))) \\ &= N(K, p) \cap \{r\sigma_K^*(p) : r \geq 0\} . \end{aligned}$$

By a result of Bangert [11] (see also Schneider [121]) and since $x \in \mathcal{M}_B(K)$, we thus obtain from (110) that

$$k_i^B(x)u_i = (1 + tk_i^B(x))D^2 h_B(u) \circ D\sigma_{K+tB}(x + t\nabla h_B(u))(u_i) ,$$

if $u_1, \ldots, u_{d-1} \in u^\perp$ denotes a suitable orthonormal basis of $u^\perp$ with respect to $[\cdot, \cdot]_u$, consisting of eigenvectors of the map described in (109). Thus

$$k_i^B(y) = \frac{k_i^B(x)}{1 + tk_i^B(x)} , \quad i \in \{1, \ldots, d-1\} ,$$

if the ordering is chosen properly and $y := x + t\nabla h_B(u)$. On the other hand, we know that $y \in \mathcal{D}_{K,B}$, since $x \in \mathcal{M}_B^*(K)$, and therefore we also have

$$k_i^B(y) = \frac{k_i^B(x, b)}{1 + tk_i^B(x, b)} , \quad i \in \{1, \ldots, d-1\} .$$

This obviously yields that $k_i^B(x) = k_i^B(x, b)$, since $k_i^B(x) < \infty$ and thus $k_i^B(y) < 1/t$.

Finally, assume that $x \in \mathcal{M}_B(K)$, $u := \sigma_K(x)$, and $x \in z + r_B(K, x)B \subset K$. If f_1 and f_2 are convex functions which locally represent $\partial(z + r_B(K, x)B)$ and ∂K at x, respectively, then

$$D^2 f_1(x) \geq D^2 f_2(x) ,$$

and therefore also

$$r_B(K, x)^{-1} \operatorname{id}_{u^\perp} = D^2 h_B(u) \circ D^2 f_1(x) \geq D^2 h_B(u) \circ D^2 f_2(x) ,$$

which completes the proof. □

Theorem 2.41 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}_*^d$. Then*

$$V(K) = \int_{\partial K} \int_0^{r_B(K,x)} h(B, \sigma_K(x)) \prod_{i=1}^{d-1} (1 - tk_i^B(x)) dt \mathcal{H}^{d-1}(dx)$$

and

$$V(K) = \frac{1}{d} \sum_{i=0}^{d-1} \binom{d}{i} (-1)^{d-1-i} \int_{\partial K} r_B(K, x)^{d-i} C_i(K, B; dx) .$$

Proof. Define the Borel set

$$M_1(K,B) := \{(x,b,t) \in \mathcal{N}(K,B) \times \mathbb{R} : 0 \le t \le r_B(K,x)\}$$

and the Lipschitz map

$$F : \mathcal{N}(K,B) \times \mathbb{R} \to \mathbb{R}^d \,, \quad (x,b,t) \mapsto x - tb\,.$$

Let $(x,b) \in \mathcal{N}(K,B)$ and $t \in [0, r_B(K,x)]$. Then there is some $u \in N(K,x) \cap S^{d-1}$ such that $b = \nabla h_B(u)$. By definition there is some $z \in K$ such that $x \in z + r_B(K,x)B \subset K$, and hence $x = z + r_B(K,x)b$. Thus we get $F(x,b,r_B(K,x)) = z \in K$. But this also implies that $F(M_1(K,B)) \subset K$.

Setting

$$M_2(K,B) := \{(x,b,t) \in \mathcal{N}(K,B) \times \mathbb{R} : 0 < t < r_B(K,x)\}\,,$$

we obtain that $F|M_2(K,B)$ is injective. To see this, assume that $y = x - tb$ for some $(x,b,t) \in M_2(K,B)$. It is sufficient to show that the following conditions are satisfied:

(i) $t = \min\{\lambda \ge 0 : (y + \lambda B) \cap \partial K \ne \emptyset\}$.

(ii) $\{x\} = (y + tB) \cap \partial K$.

First, we verify condition (i). Let λ_0 denote the right-hand side of (i) and let $\lambda \in (0, \lambda_0)$. Then $(y + \lambda B) \cap \partial K = \emptyset$, and hence $x - y \notin \lambda B$. Since $x - y = tb \in tB$, we infer that $\lambda < t$, and thus $\lambda_0 \le t$. For the proof of the reverse implication, we set $r := r_B(K,x)$. There is some $u \in N(K,x) \cap S^{d-1}$ such that $b = \nabla h_B(u)$. In addition, there is some $z \in K$ with $x \in z + rB \subset K$, and thus $x = z + rb$. For any $s \in (0,r)$ we have

$$(z + sb + (r-s)B) \setminus \{x\} \subset \text{int}(z + rB) \subset \text{int } K, \tag{111}$$

where the first inclusion is implied by the strict convexity of B. Therefore, for any $\lambda \in (0,t)$,

$$x - tb + \lambda B = z + (r-t)b + \lambda B \subset \text{int}(z + (r-t)b + tB) \subset \text{int } K.$$

But this implies that $t \le \lambda_0$. This proves (i).

From $y + tB = z + (r-t)b + tB$ and (111) we see that

$$(y + tB) \setminus \{x\} = (z + (r-t)b + tB) \setminus \{x\} \subset \text{int}(z + rB) \subset \text{int } K,$$

and this yields (ii).

Finally, let $y \in \text{int } K$ and set $\lambda_0 := \min\{\lambda \ge 0 : (y + \lambda B) \cap \partial K \ne \emptyset\}$. Choose some $x \in (y + \lambda_0 B) \cap \partial K$ and $u \in N(K,x) \cap S^{d-1}$. Then we also have $u \in N(y + \lambda_0 B, x)$, and thus $b := \nabla h_B(u) = (x-y)/\lambda_0$. This shows that $(x,b) \in \mathcal{N}(K,B)$ and $y = F(x,b,\lambda_0)$. Moreover, $x = y + \lambda_0 b \in y + \lambda_0 B \subset K$, which implies that $\lambda_0 \le r_B(K,x)$. Hence we have proved that $\text{int } K \subset F(M_1(K,B))$. From this we can infer that

$$\begin{aligned}\mathcal{H}^d(K \setminus F(M_2(K,B)) &= \mathcal{H}^d(F(M_1(K,B)) \setminus F(M_2(K,B))) \\ &\le (\text{Lip } F)^d \, \mathcal{H}^d(M_1(K,B) \setminus M_2(K,B)) = 0\,;\end{aligned}$$

the last equality can be justified by applying the coarea formula to a projection map.

The considerations involved in the proof of Theorem 2.39 yield that

$$\text{ap } J_d F(x,b,t) = \langle b, \sigma_B(b)\rangle \left| \prod_{i=1}^{d-1} \frac{1 - t k_i^B(x,b)}{\sqrt{1 + k_i^B(x,b)^2}} \right| |u_1 \wedge \ldots \wedge u_{d-1}| \,,$$

for $\mathcal{H}^d$ almost all $(x,b,t) \in \mathcal{N}(K,B) \times \mathbb{R}$. We assert that $tk_i^B(x,b) \le 1$ for $\mathcal{H}^{d-1}$ almost all $(x,b) \in \mathcal{N}(K,B)$, $t \in [0, r_B(K,x)]$, and $i \in \{1,\ldots,d-1\}$. In fact, we can assume that $r_B(K,x) > 0$ and $x + sb \in \mathcal{D}_{K,B}$ for all $s > 0$. Let $z \in K$ be such that $x \in z + r_B(K,x)B \subset K$. Then

$$x + sb \in z + (r_B(K,x) + s)B \subset K + sB \,, \quad \text{for } s > 0 \,,$$

$x + sb \in \partial(K + sB)$, and therefore

$$s^{-1} > (r_B(K,x) + s)^{-1} \ge k_i^B(x + sb) \,, \quad i \in \{1,\ldots,d-1\} \,.$$

Thus we get

$$k_i^B(x,b) < \infty \quad \text{and} \quad \frac{k_i^B(x,b)}{1 + s k_i^B(x,b)} (r_B(K,x) + s) \le 1 \,,$$

$i \in \{1,\ldots,d-1\}$, from which the conclusion follows.

Again from the coarea formula, applied to $F|M_2(K,B)$, we now obtain

$$V(K) = \int_{\mathcal{N}(K,B)} \int_0^{r_B(K,x)} \langle b, \sigma_B(b)\rangle |u_1 \wedge \ldots \wedge u_{d-1}| \prod_{i=1}^{d-1} \frac{1 - t k_i^B(x,b)}{\sqrt{1 + k_i^B(x,b)^2}} dt \mathcal{H}_B^{d-1}(d(x,b)) \,. \tag{112}$$

An expansion of the product and Theorem 2.39 then lead to the second equation of Theorem 2.41.

The first equation will also be deduced from (112). Define

$$\mathcal{N}_B^*(K) := \{(x,b) \in \mathcal{N}(K,B) : r_B(K,x) > 0\}$$

and

$$\partial^* K := \{x \in \partial K : r_B(K,x) > 0\} \,.$$

Further, note that $k_i^B(x,b) \le r_B(K,x)^{-1} < \infty$ for $\mathcal{H}^{d-1}$ almost all $(x,b) \in \mathcal{N}_B^*(K)$. Hence the coarea formula and Fubini's theorem yield

$$\begin{aligned} V(K) &= \int_{\mathcal{N}_B^*(K)} \int_0^{r_B(K,x)} \langle b, \sigma_B(b)\rangle |u_1 \wedge \ldots \wedge u_{d-1}| \prod_{i=1}^{d-1} \frac{1 - t k_i^B(x,b)}{\sqrt{1 + k_i^B(x,b)^2}} dt \mathcal{H}_B^{d-1}(d(x,b)) \\ &= \int_{\partial^* K} \int_0^{r_B(K,x)} h(B, \sigma_K(x)) \prod_{i=1}^{d-1} \left(1 - t k_i^B(x)\right) dt \mathcal{H}^{d-1}(dx) \,. \end{aligned}$$

In fact, the required Jacobian of $\pi_1 : \mathbb{R}^d \times \partial B \to \mathbb{R}^d$ is given by

$$\text{ap } J_{d-1}\left(\pi_1|\mathcal{N}(K,B)\right)(x,b) = |u_1 \wedge \ldots \wedge u_{d-1}| \prod_{i=1}^{d-1} \frac{1}{\sqrt{1+k_i^B(x,b)^2}}\,, \tag{113}$$

for $\mathcal{H}^{d-1}$ almost all $(x,b) \in \mathcal{N}(K,B)$. In addition, we used that $\mathcal{H}^{d-1}$ almost all boundary points of K are normal and the simple fact that if $(x,b) \in \mathcal{N}(K,B)$ and x is a regular boundary point of K, then $\sigma_B(b) = \sigma_K(x)$; hence

$$\langle b, \sigma_B(b)\rangle = \langle \nabla h_B(\sigma_K(x)), \sigma_K(x)\rangle = h(B, \sigma_K(x))\,.$$

Furthermore, for $\mathcal{H}^{d-1}$ almost all normal boundary points and $i \in \{1,\ldots,d-1\}$, Lemma 2.40 implies that $k_i^B(x,b) = k_i^B(x)$. This establishes the first equation. □

The last result in this subsection is a relative version of a *generalized Minkowski integral formula.* In the (Euclidean and) smooth setting, corresponding integral formulas represent an important tool for proving characterizations of balls. Recently, generalized Minkowski integral formulas, which explicitly involve generalized curvature expressions, were used to establish characterizations of balls in the non-smooth Euclidean case. The existing proofs (in Euclidean space) of generalized Minkowski formulas use the divergence theorem and approximation arguments. The approach which we provide in the more general framework of relative geometry, is essentially based on symmetry properties of mixed volumes and on the results which have been established so far in this section.

Theorem 2.42 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Then*

$$\begin{aligned}&\int_{\mathcal{N}(K,B)} \langle x, \sigma_B(b)\rangle\, \mathbb{H}_i^B(K,x,b)|u_1 \wedge \ldots \wedge u_{d-1}|\mathcal{H}_B^{d-1}(d(x,b))\\ &\quad= \int_{\mathcal{N}(K,B)} \langle b, \sigma_B(b)\rangle\, \mathbb{H}_{i-1}^B(K,x,b)|u_1 \wedge \ldots \wedge u_{d-1}|\mathcal{H}_B^{d-1}(d(x,b))\,,\end{aligned}$$

where $i \in \{1,\ldots,d-1\}$.

Proof. In the following, we shall use mixed volumes of convex bodies and their connection with mixed surface area measures; see [123, Chapter 5] for background information and notation.

Using a special case of Theorem 2.39 and Theorem 2.14, we obtain

$$\begin{aligned}&\int_{\mathcal{N}(K,B)} \langle b, \sigma_B(b)\rangle\, \mathbb{H}_{i-1}^B(K,x,b)|u_1 \wedge \ldots \wedge u_{d-1}|\mathcal{H}_B^{d-1}(d(x,b))\\ &\qquad= S_{d-i}(K,B;\mathbb{R}^d)\\ &\qquad= \int_{S^{d-1}} h(B,u) S(K[d-i], B[i-1], du)\\ &\qquad= dV(K[d-i], B[i])\,.\end{aligned}$$

The symmetry of mixed volumes yields

$$V(K[d-i], B[i]) = V(K[d-1-i], B[i], K)\,,$$

and therefore

$$
\begin{aligned}
&dV(K[d-i], B[i]) \\
&\quad = \int_{S^{d-1}} h(K,u) S(K[d-1-i], B[i], du) \\
&\quad = \int_{S^{d-1}} \frac{h_K}{h_B} \circ \sigma_B \circ \nabla h_B(u) h(B,u) S(K[d-1-i], B[i], du) \\
&\quad = \int_{\partial B} \frac{h_K}{h_B} \circ \sigma_B(b) S_{d-1-i}(K, B; db) \\
&\quad = \int_{\mathcal{N}(K,B)} \frac{h(K, \sigma_B(b))}{h(B, \sigma_B(b))} \Theta_{d-1-i}(K, B; d(x,b)) \\
&\quad = \int_{\mathcal{N}(K,B)} \frac{\langle x, \sigma_B(b)\rangle}{\langle b, \sigma_B(b)\rangle} \langle b, \sigma_B(b)\rangle \, \mathbb{H}_i^B(K,x,b) |u_1 \wedge \ldots \wedge u_{d-1}| \mathcal{H}_B^{d-1}(d(x,b)) \\
&\quad = \int_{\mathcal{N}(K,B)} \langle x, \sigma_B(b)\rangle \, \mathbb{H}_i^B(K,x,b) |u_1 \wedge \ldots \wedge u_{d-1}| \mathcal{H}_B^{d-1}(d(x,b)) \, .
\end{aligned}
$$

This yields the desired equality. □

2.7 A characterization of gauge bodies

In this section, we establish an extension to the setting of relative geometry of a recent characterization of Euclidean balls via certain linear relations between Euclidean curvature measures. A very special case of such a result was stated by Leichtweiß in [95], Theorem 6.1; see also the references cited there. The result mentioned by Leichtweiß extends the classical Liebmann-Süss Theorem to the smooth setting of relative differential geometry in the same way as our result generalizes contributions by Kohlmann [84] and Schneider [120] in the general setting of convex geometry.

Theorem 2.43 *Let $K \in \mathcal{K}_o^d$, $B \in \mathcal{K}_*^d$, and assume that there are constants $\lambda_0, \ldots, \lambda_{d-2} \geq 0$ such that*

$$
C_{d-1}(K, B; \cdot) = \sum_{j=0}^{d-2} \lambda_j C_j(K, B; \cdot) \, .
$$

Then K is homothetic to B.

For the proof we need an extension of the Lebesgue decomposition of Euclidean curvature measures (compare [72]). For a given convex body K, we write

$$
H_j^B(K, x) := \binom{d-1}{j}^{-1} \sum_{|I|=j} \prod_{i \in I} k_i^B(x)
$$

if $x \in \mathcal{M}_B(K)$ and $j \in \{0, \ldots, d-1\}$. We define $\mathcal{N}^s(K, B)$ as the set of all $(x, b) \in \mathcal{N}(K, B)$ for which $k_i^B(x, b) = \infty$ for some $i \in \{, \ldots, d-1\}$. Finally, we write $C_r^a(K, B; \cdot)$ for the absolutely continuous part of $C_r(K, B; \cdot)$ with respect to $(d-1)$-dimensional Hausdorff measure, and we let $C_r^s(K, B; \cdot)$ denote the singular part. Recall that we already know that $C_{d-1}(K, B; \cdot)$ and $\mathcal{H}^{d-1} \llcorner \partial K$ have the same Borel sets of measure zero.

Theorem 2.44 *Let $K \in \mathcal{K}^d$, $B \in \mathcal{K}^d_*$, and $r \in \{0, \dots, d-1\}$. Then*

$$C^a_r(K, B; \cdot) = \int_{\partial K} \mathbf{1}\{x \in \cdot\} h(B, \sigma_K(x)) H^B_{d-1-r}(K, x) \mathcal{H}^{d-1}(dx)$$

and

$$C^s_r(K, B; \cdot) = \int_{\mathcal{N}^s(K,B)} \mathbf{1}\{x \in \cdot\} \langle b, \sigma_B(b)\rangle \, \mathbb{H}^B_{d-1-r}(K, x, b) |u_1 \wedge \ldots \wedge u_{d-1}| \mathcal{H}^{d-1}_B(d(x, b)) \, .$$

Proof. The proof follows the lines of the proof of Theorem 3.2 in [72]. In particular, one has to use Theorem 2.39, Lemma 2.40 and formula (113). □

The essential ingredient of the proof of the characterization theorem is the following estimate for the volume of a convex body. We shall derive this sharp inequality from our representation of volume involving the interior reach function.

Lemma 2.45 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Then*

$$dV(K) \leq \int_{\partial K} \frac{h(B, \sigma_K(x))}{H^B_1(K, x)} \mathcal{H}^{d-1}(dx)$$

with equality if and only if $k^B_1(x) = \ldots = k^B_{d-1}(x) = r_B(K, x)^{-1}$ for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$.

Proof. We can assume that $0 \leq k^B_1(x) \leq \ldots \leq k^B_{d-1}(x) < \infty$, whenever these curvatures are defined. Recall that $r_B(K, x) \leq k^B_i(x)^{-1}$, $i \in \{1, \dots, d-1\}$, and note that

$$H^B_1(K, x) = \frac{1}{d-1} \left(k^B_1(x) + \ldots + k^B_{d-1}(x) \right) \leq k^B_{d-1}(x) \, ,$$

for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$. Therefore, for $0 \leq t < r_B(K, x)$ and for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$,

$$0 < \prod_{i=1}^{d-1} \left(1 - k^B_i(x) t\right) \leq \left[\frac{1}{d-1} \sum_{i=1}^{d-1} \left(1 - k^B_i(x) t\right) \right]^{d-1} = \left[1 - H^B_1(K, x) t\right]^{d-1} . \qquad (114)$$

Moreover, using Theorem 2.41, (114), and the fact that $r_B(K, x) \leq H^B_1(K, x)^{-1}$, we obtain

$$\begin{aligned}
dV(K) &= d \int_{\partial K} \int_0^{r_B(K,x)} h(B, \sigma_K(x)) \prod_{i=1}^{d-1} \left(1 - k^B_i(x) t\right) dt \mathcal{H}^{d-1}(dx) \\
&\leq d \int_{\partial K} \int_0^{r_B(K,x)} h(B, \sigma_K(x)) \left[1 - H^B_1(K, x) t\right]^{d-1} dt \mathcal{H}^{d-1}(dx) \\
&\leq d \int_{\partial K} \int_0^{H^B_1(K,x)^{-1}} h(B, \sigma_K(x)) \left[1 - H^B_1(K, x) t\right]^{d-1} dt \mathcal{H}^{d-1}(dx) \\
&= d \int_{\partial K} h(B, \sigma_K(x)) \left[-\frac{1}{d H^B_1(K, x)} \left(1 - H^B_1(K, x) t\right)^d \right]_0^{H^B_1(K,x)^{-1}} \mathcal{H}^{d-1}(dx) \\
&= \int_{\partial K} \frac{h(B, \sigma_K(x))}{H^B_1(K, x)} \mathcal{H}^{d-1}(dx) \, .
\end{aligned}$$

In particular, the preceding analysis is correct for those $x \in \partial K$ for which $H_1^B(K,x) = 0$. This proves the asserted inequality. Now assume that equality holds; then

$$1 - k_1^B(x)t = \ldots = 1 - k_{d-1}^B(x)t\,,$$

for $t \in (0, r_B(K,x))$, and

$$r_B(K,x) = H_1^B(K,x)^{-1}$$

hold for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$. This implies the statement on the case of equality, since the reverse conclusion is obvious from the preceding argument. □

Proof of Theorem 2.43. Because of the homogeneity of the Minkowski curvature measures, we can assume that $\lambda_0 + \ldots + \lambda_{d-2} = 1$. Let $i \in \{0, \ldots, d-2\}$ be such that $\lambda_i > 0$. Then

$$C_i(K,B;\cdot) \leq \lambda_i^{-1} C_{d-1}(K,B;\cdot)\,,$$

and hence by Theorem 2.44,

$$C_i(K,B;\cdot) = \int_{\partial K} \mathbf{1}\{x \in \cdot\} h(B, \sigma_K(x)) H_{d-1-i}^B(K,x) \mathcal{H}^{d-1}(dx) \tag{115}$$

and

$$C_{d-1}(K,B;\cdot) = \int_{\partial K} \mathbf{1}\{x \in \cdot\} h(B, \sigma_K(x)) \mathcal{H}^{d-1}(dx)\,.$$

The last formula also follows from Corollary 2.11. Therefore the assumption of the theorem implies that

$$\sum_{i=0}^{d-2} \lambda_i H_{d-1-i}^B(K,x) = 1 \tag{116}$$

for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$. But then

$$\begin{aligned}
dV(K) &= \int_{\partial K} \langle x, \sigma_K(x) \rangle \mathcal{H}^{d-1}(dx) \\
&= \sum_{i=0}^{d-2} \lambda_i \int_{\partial K} \langle x, \sigma_K(x) \rangle H_{d-1-i}^B(K,x) \mathcal{H}^{d-1}(dx) \\
&= \sum_{i=0}^{d-2} \lambda_i \int_{\mathcal{N}(K,B)} \langle x, \sigma_B(b) \rangle \, \mathbb{H}_{d-1-i}^B(K,x,b) |u_1 \wedge \ldots \wedge u_{d-1}| \mathcal{H}_B^{d-1}(d(x,b)) \\
&= \sum_{i=0}^{d-2} \lambda_i \int_{\mathcal{N}(K,B)} \langle b, \sigma_B(b) \rangle \, \mathbb{H}_{d-2-i}^B(K,x,b) |u_1 \wedge \ldots \wedge u_{d-1}| \mathcal{H}_B^{d-1}(d(x,b)) \\
&\geq \sum_{i=0}^{d-2} \lambda_i \int_{\partial K} h(B, \sigma_K(x)) H_{d-2-i}^B(K,x) \mathcal{H}^{d-1}(dx)\,. \qquad (117)
\end{aligned}$$

Here we have used (116), Theorem 2.44 (and its method of proof) and Theorem 2.42. From Lemma 2.45, (117) and (116) we infer

$$
\begin{aligned}
0 &\le \int_{\partial K} \frac{h(B,\sigma_K(x))}{H_1^B(K,x)} \mathcal{H}^{d-1}(dx) - dV(K) \\
&\le \int_{\partial K} \left\{ \frac{h(B,\sigma_K(x))}{H_1^B(K,x)} - \sum_{i=0}^{d-2} \lambda_i h(B,\sigma_K(x)) H_{d-2-i}^B(K,x) \right\} \mathcal{H}^{d-1}(dx) \\
&= \int_{\partial K} \frac{h(B,\sigma_K(x))}{H_1^B(K,x)} \sum_{i=0}^{d-2} \lambda_i \left(H_{d-1-i}^B(K,x) - H_{d-2-i}^B(K,x) H_1^B(K,x) \right) \mathcal{H}^{d-1}(dx) \le 0 \,,
\end{aligned}
$$

where Newton's inequalities were used to obtain the last estimate (compare [65], p. 52, or [112]). This shows that equality must hold in the inequality of Lemma 2.45, and thus we further deduce that

$$
\sum_{i=0}^{d-2} \lambda_i \left(k_j^B(x) \right)^{d-1-i} = 1 \,,
$$

for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$ and any $j \in \{1, \ldots, d-1\}$. Since $\lambda_0 + \ldots + \lambda_{d-2} = 1$ according to our initial assumption, this implies that

$$
k_1^B(x) = \ldots = k_{d-1}^B(x) = H_1^B(K,x) = r_B(K,x)^{-1} = 1 \,,
$$

for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$.

In particular, $H_{d-1}^B(K,x) = 1$ for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$ and B is contained in a translate of K. The latter shows that

$$
V(B) \le V(K) \,. \tag{118}
$$

Further, using Theorem 2.44 and a comparison of coefficients in (85) and (86), we deduce that

$$
\begin{aligned}
V(K[d-1], B) &= \frac{1}{d} \int_{S^{d-1}} h(B,u) S_{d-1}(K,du) \\
&= \frac{1}{d} \int_{\partial K} h(B,\sigma_K(x)) \mathcal{H}^{d-1}(dx) \\
&= \frac{1}{d} \int_{\partial K} h(B,\sigma_K(x)) H_{d-1}^B(K,x) \mathcal{H}^{d-1}(dx) \\
&\le \frac{1}{d} \Theta_0(K,B;\mathbb{R}^{2d}) = V(B) \,.
\end{aligned} \tag{119}
$$

On the other hand, Minkowski's inequality (see (6.2.2) in [123]) yields

$$
V(K[d-1], B)^d \ge V(K)^{d-1} V(B) \tag{120}
$$

with equality if and only if K and B are homothetic (since K and B are d-dimensional). But (118) and (119) show that equality holds in (120); hence K must be a translate of B, since the relative curvatures equal one. □

2.8 Stability and relative curvature measures

Stability results in convex geometry represent an important and challenging subject, since every stability result includes a corresponding uniqueness assertion as a special case. We have already addressed this subject in the preceding section. Until recently, stability results for Euclidean curvature measures have been rather exceptional and have only been obtained in restricted situations. Weak versions of stability theorems involving curvature measures have been established in [81] by means of the classical method of symmetrization (and other tools). However, a first satisfactory solution of the stability problem is of even more recent vintage (see [85]). It is based on a representation of the volume of a convex body K in terms of the interior reach function of K, the Gauss-Bonnet theorem, and an estimate for the isoperimetric deficit of K involving the circumradius and inradius of K.

As in the preceding subsection, we shall use the tools which are now available to establish corresponding stability results for relative curvature measures. Extensions of stability results for curvature measures to spaces forms have been treated in [86].

For a convex body K, we consider the normalization

$$\overline{K} := \frac{K - s(K)}{b(K)},$$

where $s(K)$ denotes the Steiner point of K and $b(K)$ is the mean width of K; compare §1.7 in [123]. For convex bodies $K, L \in \mathcal{K}^d$, the L_2-distance $\delta_2(K, L)$ is defined as

$$\delta_2(K, L) := \left\{ \int_{S^{d-1}} |h(K, u) - h(L, u)|^2 \mathcal{H}^{d-1}(du) \right\}^{\frac{1}{2}}.$$

It is known that estimates in terms of δ_2 can be transformed into estimates for the Hausdorff distance δ, and vice versa.

Lemma 2.46 *Let $K \in \mathcal{K}^d$ and $B \in \mathcal{K}^d_*$. Assume that ρB^d ($\rho > 0$) rolls freely inside B, let R denote the circumradius of B and let $r > 0$ be the inradius of K. Then*

$$\delta_2(\overline{K}, \overline{B})^2 \leq \alpha(d) \left(\frac{\operatorname{diam}(K)}{r} \right)^2 \left(\frac{R}{\rho} \right)^{2d-2} \left[\left(\frac{V(K[d-1], B)}{V(B)} \right)^d - \left(\frac{V(K)}{V(B)} \right)^{d-1} \right],$$

where $\alpha(d) := \kappa_d 4^{d-3} d(d-1)/(d+1)$.

Proof. An application of Theorem 3 in [6] shows that

$$V(K, B[d-1])^2 - V(B)V(K[2], B[d-2]) \geq \Delta(K, B) \left(\frac{\rho}{2} \right)^{2d-4}, \tag{121}$$

where

$$\Delta(K, B) := V(K, B, B^d[d-2]) - V(K[2], B^d[d-2]) V(B[2], B^d[d-2]).$$

Moreover, a repetition of the argument on pp. 355-6 in [123] with $W_k := V(K[d-k], B[k])$ yields that

$$\left(\frac{V(K[d-1], B)}{V(B)} \right)^d - \left(\frac{V(K)}{V(B)} \right)^{d-1} \geq \frac{V(K, B[d-1])^2 - V(B)V(K[2], B[d-2])}{V(B)V(K[2], B[d-2])}. \tag{122}$$

Finally, by Theorem 6.6.6 in [123],

$$\Delta(K,B) \geq \frac{d+1}{d(d-1)} b(K)^2 V(B[2], B^d[d-2]) \delta_2(\overline{K}, \overline{B})^2 . \tag{123}$$

Combining equations (121) – (123), we obtain the assertion by obvious estimates. □

For $s \in \{1, \dots, d-2\}$, we define the number

$$a(s,d) := \frac{d-1}{s} \left(1 - \left(\frac{d-1-s}{d-1}\right)^{\frac{1}{s}}\right)^{-d} + \frac{d-1-s}{s}(d-1)\left(\frac{d-1}{d-1-s}\right)^{\frac{d-1}{s}} > 1$$

and set

$$\epsilon_0(d,s) := \frac{1}{2} \frac{1}{1 + 2a(s,d)^d} .$$

If $s = d-1$, then we define $\epsilon_0(d, d-1) := 1/2$.

Theorem 2.47 *Let $K \in \mathcal{K}^d$, $B \in \mathcal{K}^d_*$, and $s \in \{1, \dots, d-1\}$. Assume that*

$$(1-\epsilon) C_{d-1}(K, B; \cdot) \leq C_{d-1-s}(K, B; \cdot) \leq (1+\epsilon) C_{d-1}(K, B; \cdot)$$

for some $\epsilon \in (0, \epsilon_0(d,s))$. Then there exist constants $c(B)$, $c(d)$, which merely depend on B, d respectively, such that

$$\delta_2(\overline{K}, \overline{B}) \leq c(B) c(d)\, \epsilon^{\frac{1}{2d}}$$

and, for some $x \in \mathbb{R}^d$,

$$\delta(K + x, B) \leq c(B) c(d)\, \epsilon^{\frac{1}{(d-1)d}} .$$

Indeed, one can also prove a stability version of Theorem 2.43. But then the determination of the constants is much less explicit than in the situation of Theorem 2.47. Such a result can be proved by combining the arguments of the proof of Theorem 2.47 with the approach in [86]. In particular, we emphasize that instead of proving a weak stability result (as in §8 in [86]) one should use the estimate of Lemma 9.1 in [86] (for $r = n$) to obtain a diameter bound as in the proof of Theorem 2.47. Furthermore, the proof of Lemma 9.4 in [86] seems to be restricted to the Euclidean situation; hence the improvement of the order of the stability function, which was achieved in [86] by means of this lemma, cannot be carried over to the present setting.

Proof. Let the assumptions of the theorem be satisfied. Recall that $k_i^B(x) \leq r_B(K,x)^{-1}$, and hence

$$r_B(K,x) \leq H_s^B(K,x)^{-\frac{1}{s}} \leq (1-\epsilon)^{-\frac{1}{s}} , \tag{124}$$

for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$. Using the assumption, Theorems 2.44 and 2.42, and the estimate

$$H_{s-1}^B(K,x) \geq H_s^B(K,x)^{\frac{s-1}{s}} \geq (1-\epsilon)^{\frac{s-1}{s}} ,$$

which is based on Newton's inequalities and again on the assumption, we obtain

$$\begin{aligned}
dV(K) &= \int_{\partial K} \langle x, \sigma_K(x)\rangle \mathcal{H}^{d-1}(dx) \\
&\geq \frac{1}{1+\epsilon} \int_{\partial K} \langle x, \sigma_K(x) H_s^B(K,x) \mathcal{H}^{d-1}(dx) \\
&= \frac{1}{1+\epsilon} \int_{\mathcal{N}(K,B)} \langle x, \sigma_B(b)\rangle \, \mathbb{H}_s^B(K,x,b) |u_1 \wedge \ldots \wedge u_{d-1}| \mathcal{H}_B^{d-1}(d(x,b)) \\
&= \frac{1}{1+\epsilon} \int_{\mathcal{N}(K,B)} h(B, \sigma_B(b)) \, \mathbb{H}_{s-1}^B(K,x,b) |u_1 \wedge \ldots \wedge u_{d-1}| \mathcal{H}_B^{d-1}(d(x,b)) \\
&\geq \frac{1}{1+\epsilon} \int_{\partial K} h(B, \sigma_K(x)) H_{s-1}^B(K,x) \mathcal{H}^{d-1}(dx) \\
&\geq \frac{(1-\epsilon)^{\frac{s-1}{s}}}{1+\epsilon} \int_{\partial K} h(B, \sigma_K(x)) \mathcal{H}^{d-1}(dx) \\
&\geq \frac{(1-\epsilon)^{1-\frac{1}{s}}}{1+\epsilon} dV(K[d-1], B) \,. \qquad (125)
\end{aligned}$$

This can be combined with Theorem 2.41 to yield a point $x_0 \in \partial K \cap \mathcal{M}_B(K)$ such that

$$\int_0^{r_B(K,x_0)} \prod_{i=1}^{d-1} (1 - t k_i^B(x_0)) dt \geq \frac{1}{d} \frac{1-\epsilon}{1+\epsilon} \,,$$

and thus by Lemma 1 (iii) in [85]

$$r_B(K,x) \geq 1 - (2\epsilon)^{\frac{1}{d}} \,.$$

Therefore we obtain

$$V(K) \geq \left(1 - (2\epsilon)^{\frac{1}{d}}\right)^d V(B) \,. \qquad (126)$$

In order to be able to apply Lemma 2.46, we now estimate $V(K[d-1], B)$ from above. Let us first consider the case $s = d-1$. Then $H_{d-1}(K,x) \geq 1 - \epsilon$ for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$, and hence

$$\begin{aligned}
dV(K[d-1], B) &= \int_{\partial K} h(B, \sigma_K(x)) \mathcal{H}^{d-1}(dx) \\
&\leq \frac{1}{1-\epsilon} \int_{\partial K} H_{d-1}^B(K,x) h(B, \sigma_K(x)) \mathcal{H}^{d-1}(dx) \\
&\leq \frac{1}{1-\epsilon} C_0(K, B; \mathbb{R}^d) \\
&\leq \frac{1}{1-\epsilon} dV(B) \,.
\end{aligned}$$

Thus for $s = d-1$ we have proved

$$\frac{V(K[d-1],B)}{V(B)} \le (1-\epsilon)^{-1} . \tag{127}$$

The remaining cases $s = 1, \dots, d-2$ are more involved. Here we proceed as follows. From (125), Theorem 2.41 and Lemma 1 (iii) in [85] we first deduce

$$\int_{\partial K} h(B,\sigma_K(x)) \left[1 - r_B(K,x)(1-\epsilon)^{\frac{1}{s}}\right]^d \mathcal{H}^{d-1}(dx) \le \frac{2\epsilon}{1-\epsilon} dV(K[d-1],B) . \tag{128}$$

Then Hölder's inequality, applied to the left-hand side of (129), and equation (128) yield

$$\int_{\partial K} h(B,\sigma_K(x)) \left(1 - r_B(K,x)(1-\epsilon)^{\frac{1}{s}}\right) \mathcal{H}^{d-1}(dx) \le \left(\frac{2\epsilon}{1-\epsilon}\right)^{\frac{1}{d}} dV(K[d-1],B) . \tag{129}$$

Next we define

$$\lambda_\epsilon := \left(\frac{d-1-s}{(d-1)(1-\epsilon)}\right)^{\frac{1}{s}}$$

and

$$\partial K_n := \{x \in \partial K : r_B(K,x) \le \lambda_\epsilon\} , \qquad \partial K_g := \partial K \setminus \partial K_n .$$

Similarly, we set

$$V_n := \frac{1}{d} \int_{\partial K_n} h(B,\sigma_K(x)) \mathcal{H}^{d-1}(dx) , \qquad V_g := V(K[d-1],B) - V_n .$$

We start with some preparatory estimates. Using (128), we obtain

$$\begin{aligned}
\frac{2\epsilon}{1-\epsilon} dV(K[d-1],B) &\ge \int_{\partial K} h(B,\sigma_K(x)) \left[1 - r_B(K,x)(1-\epsilon)^{\frac{1}{s}}\right]^d \mathcal{H}^{d-1}(dx) \\
&\ge \int_{\partial K_n} h(B,\sigma_K(x)) \left[1 - r_B(K,x)(1-\epsilon)^{\frac{1}{s}}\right]^d \mathcal{H}^{d-1}(dx) \\
&\ge \int_{\partial K_n} h(B,\sigma_K(x)) \left[1 - \lambda_\epsilon(1-\epsilon)^{\frac{1}{s}}\right]^d \mathcal{H}^{d-1}(dx) \\
&= \left[1 - \left(\frac{d-1-s}{d-1}\right)^{\frac{1}{s}}\right]^d dV_n ,
\end{aligned}$$

and hence

$$V_n \le \frac{2\epsilon}{1-\epsilon} \left[1 - \left(\frac{d-1-s}{d-1}\right)^{\frac{1}{s}}\right]^{-d} V(K[d-1],B) . \tag{130}$$

Using (124) and (130), we derive

$$\int_{\partial K_g} \frac{d-1}{s} r_B(K,x)^{s-d+1}(1-\epsilon)h(B,\sigma_K(x))\mathcal{H}^{d-1}(dx)$$

$$\geq \frac{d-1}{s}(1-\epsilon)(1-\epsilon)^{\frac{d-1-s}{s}} d(V(K[d-1],B)-V_n)$$

$$\geq \frac{d-1}{s}(1-\epsilon)^{\frac{d-1}{s}} \left(1-\frac{2\epsilon}{1-\epsilon}\left[1-\left(\frac{d-1-s}{d-1}\right)^{\frac{1}{s}}\right]^{-d}\right) dV(K[d-1],B)\,. \tag{131}$$

On the other hand, starting with (129), as in [85] we are led to

$$0 \leq \int_{\partial K_g} \left(\left[r_B(K,x)(1-\epsilon)^{\frac{1}{s}}\right]^{-(d-1)} - 1\right) h(B,\sigma_K(x))\mathcal{H}^{d-1}(dx)$$

$$\leq (d-1)\left(\frac{d-1}{d-1-s}\right)^{\frac{d-1}{s}} \left(\frac{2\epsilon}{1-\epsilon}\right)^{\frac{1}{d}} dV(K[d-1],B)\,,$$

and thus

$$-\int_{\partial K_g} \frac{d-1-s}{s} r_B(K,x)^{-(d-1)} h(B,\sigma_K(x))\mathcal{H}^{d-1}(dx) \tag{132}$$

$$\geq -\frac{d-1-s}{s}(1-\epsilon)^{\frac{d-1}{s}}\left[1+(d-1)\left(\frac{d-1}{d-1-s}\right)^{\frac{d-1}{s}}\left(\frac{2\epsilon}{1-\epsilon}\right)^{\frac{1}{d}}\right] dV(K[d-1],B)\,.$$

Next we use the crucial estimate

$$H^B_{d-1}(K,x) \geq \max\left\{\frac{d-1}{s} r_B(K,x)^{s-d+1}(1-\epsilon) - \frac{d-1-s}{s} r_B(K,x)^{-(d-1)}, 0\right\}\,, \tag{133}$$

which under the assumptions of the theorem is satisfied for $\mathcal{H}^{d-1}$ almost all $x \in \partial K$; compare Lemma 2 in [85]. Therefore, combining (131) – (133), we can conclude that

$$dV(B) = C_0(K,B;\mathbb{R}^d) \geq \int_{\partial K_g} H^B_{d-1}(K,x)h(B,\sigma_K(x))\mathcal{H}^{d-1}(dx)$$

$$\geq (1-\epsilon)^{\frac{d-1}{s}}\left(1-a(s,d)\left(\frac{2\epsilon}{1-\epsilon}\right)^{\frac{1}{d}}\right) dV(K[d-1],B)\,, \tag{134}$$

where also $0 < \epsilon < \epsilon_0(d,s) \leq 1/3$ was used.

By (126) and (134), we find that

$$\left(\frac{V(K[d-1],B)}{V(B)}\right)^d - \left(\frac{V(K)}{V(B)}\right)^{d-1} \tag{135}$$

$$\leq (1-\epsilon)^{-\frac{(d-1)d}{s}}\left(1-a(s,d)\left(\frac{2\epsilon}{1-\epsilon}\right)^{\frac{1}{d}}\right)^{-d} - \left(1-(2\epsilon)^{\frac{1}{d}}\right)^{(d-1)d}$$

$$\leq c(d)\,\epsilon^{\frac{1}{d}}\,. \tag{136}$$

The same conclusion can be obtained for $s = d-1$ by (126) and (127).

In order to see that K satisfies a diameter bound, it is sufficient to combine (134) (or (127) if $s = d-1$) with the Minkowski inequality

$$V(K[d-1], B) \geq V(K)^{\frac{d-1}{d}} V(B)^{\frac{1}{d}} \tag{137}$$

and with the elementary estimate

$$V(K) \geq \frac{1}{d}\left(1-(2\epsilon)^{\frac{1}{d}}\right)^{d-1} V^*(B)\mathrm{diam}(K)\,,$$

where

$$V^*(B) = \min\left\{V_{d-1}((B-c(B))\cap u^{\perp}) : u \in S^{d-1}\right\}$$

and $c(B)$ denotes the centre of mass of B. In fact, this shows that $\mathrm{diam}(K) \leq \alpha(B)\gamma(d)$ with $\alpha(B) = V(B)/V^*(B)$ and a constant $\gamma(d)$ which can be determined explicitly.

The assertions of the theorem now follow from (136), Lemma 2.46 and from a stability result of Groemer concerning the Minkowski inequality; compare [57] or Theorem 6.2.2 in [123]. For the application of the latter result, we also need that (126), (127), (134) and (137) imply that $|V(K)/V(B)-1| \leq \bar{\gamma}(d)\,\epsilon^{1/d}$. This finally completes the proof. □

In the special case of the relative mean curvature measure, we obtain a stability result of optimal order. This provides the announced extension of Theorem 1.52.

Theorem 2.48 *Let $K \in \mathcal{K}^d$, $B \in \mathcal{K}^d_*$ and $\epsilon \in [0, 1/3)$. Assume that*

$$(1-\epsilon)C_{d-1}(K, B; \cdot) \leq C_{d-2}(K, B; \cdot) \leq (1+\epsilon)C_{d-1}(K, B; \cdot)\,.$$

Then

$$\delta_2(\overline{K}, \overline{B}) \leq c(B)c(d)V(K)\sqrt{\epsilon}\,,$$

where $c(d)$ depends merely on d and $c(B)$ depends merely on the radius of the largest ball which rolls freely inside B.

Remark. If $\epsilon \leq \epsilon_0(d, d-2)$, then the factor $V(K)$ can be bounded from above as described in Theorem 2.47.

Proof. First, we remark that ∂K is of class $C^{1,1}$. In fact, by Theorem 2.9 and the assumption we obtain $C_{d-2}(K, \cdot) \leq \alpha\, C_{d-1}(K, \cdot)$ for a suitable constant $\alpha > 0$, and hence the assertion follows from Theorem 1.49. More precisely, let $\rho > 0$ be the radius of the largest ball which rolls freely inside B. Then

$$\mathbb{H}_1(K, B; x, u) \geq \mathbb{H}_1(K, (x, u))\rho\,,$$

and thus

$$\int_{\partial K} \mathbf{1}\{x \in \cdot\} h(B, \sigma_K(x)) H_1(K, x)\mathcal{H}^{d-1}(dx) \leq \frac{1+\epsilon}{\rho} \int_{\partial K} \mathbf{1}\{x \in \cdot\} h(B, \sigma_K(x))\mathcal{H}^{d-1}(dx)\,.$$

This shows that

$$C_{d-2}(K,\cdot) \leq \frac{1+\epsilon}{\rho} C_{d-1}(K,\cdot)\,,$$

and thus by Theorem 1.48, $\{(d-1)(1+\epsilon)/\rho\}^{-1} B^d$ rolls freely inside K.

Furthermore, similarly as in the Euclidean case, we can deduce that

$$(1-\epsilon)S_{d-1}(K,B;\cdot) \leq S_{d-2}(K,B;\cdot) \leq (1+\epsilon)S_{d-1}(K,B;\cdot)\,,$$

and hence

$$(1-\epsilon)S_{d-1}(K,\cdot) \leq S(K[d-2],B,\cdot) \leq (1+\epsilon)S_{d-1}(K,\cdot)\,. \tag{138}$$

From (138) we infer that

$$\begin{aligned} V(K[d-1],B) &\leq (1+\epsilon)V(K)\,,\\ V(K[d-1],B) &\leq (1-\epsilon)^{-1}V(K[d-2],B[2])\,, \end{aligned}$$

and therefore

$$\begin{aligned} V(K[d-1],B)^2 - V(K)V(K[d-2],B[2]) &\leq 2\epsilon V(K[d-1],B)V(K)\\ &\leq 4\epsilon V(K)^2\,. \end{aligned} \tag{139}$$

Now we apply Theorem 3 in [6]. Using the terminology of [6], we have in the present situation $R(B^d;B)^{-1} \geq \rho$,

$$r(K;B) \geq \max\{r_B(K,x) : x \in \partial K\} \geq 1-(2\epsilon)^{\frac{1}{d}} \geq 1-\left(\frac{2}{3}\right)^{\frac{1}{d}}$$

and

$$\eta \geq \frac{\rho}{(d-1)(1+\epsilon)} \geq \frac{\rho}{2(d-1)}\,.$$

Hence

$$V(K[d-1],B)^2 - V(K)V(K[d-2],B[2]) \geq c_1(d)\rho^{2d-4}\Delta(K,B)\,. \tag{140}$$

Thus (139) and (140) imply

$$\Delta(K,B) \leq c_2(d)\left(\frac{1}{\rho}\right)^{2d-4} V(K)^2\,\epsilon\,. \tag{141}$$

In addition,

$$\begin{aligned} \Delta(K,B) &\geq \frac{d+1}{d(d-1)} b(K)^2 V(B[2],B^d[d-2])\delta_2(\overline{K},\overline{B})^2\\ &\geq c_3(d)\rho^4\delta_2(\overline{K},\overline{B})^2\,. \end{aligned} \tag{142}$$

Thus (141) and (142) yield

$$\delta_2(\overline{K},\overline{B})^2 \leq c(d)\left(\frac{1}{\rho}\right)^{2d} V(K)^2\,\epsilon\,.$$

The remaining assertion follows from Theorem 2.47. □

2.9 A splitting result

In the previous sections, we have restricted our attention to the investigation of relative curvature measures which are associated with compact convex sets. The extension of relative curvature measures to closed convex sets is accomplished in the usual way; in fact, the results which are described in Theorems 2.5, 2.9 and 2.39 extend without essential changes to the non-compact case.

Once we consider unbounded closed convex sets $K \subset \mathbb{R}^d$, the question naturally arises whether a pinching condition for some curvature measure of K implies that K splitts, that is, K is a cylinder. For Euclidean curvature measures, a weak splitting theorem has recently been found by Kohlmann [82]. It states the existence of a constant $\epsilon_0 = \epsilon(d, j)$, $j \in \{1, \ldots, d-1\}$, such that if $K \subset \mathbb{R}^d$ is a closed convex set with $\emptyset \neq \text{int}\, K \neq \mathbb{R}^d$ and

$$(1-\epsilon)\, C_{d-1}(K, \cdot) \leq C_{d-1-j}(K, \cdot) \leq (1+\epsilon)\, C_{d-1}(K, \cdot)$$

for some $\epsilon \in (0, \epsilon_0)$, then K is isometric to $\mathbb{R}^i \times K'$, where $i \in \{0, \ldots, d-1-j\}$ and K' is a convex body in $\mathbb{R}^{d-i}$. In [82], however, an explicit (but presumably far from optimal) value for $\epsilon(d, j)$ is only provided for $j = d-2$. Of course, it would be interesting to know explicit or even optimal values for $\epsilon(d, j)$. In the Euclidean context and for $j = 1$ corresponding results have recently been discovered by Bangert [12] and Senchun Lin [97], [98]. The treatment in [97] is restricted to $d = 3$ and smooth convex bodies; in particular, the smoothness requirement rules out a phenomenon which is covered by the more general Theorem 4.2 in [12]. An alternative (and fairly involved) approach in [98] works in general dimensions, but again the presentation is restricted to smooth convex sets.

In view of the results of the previous sections, it is tempting to predict that splitting results can be obtained under pinching conditions on relative curvature measures as well. However, up to now any attempt to generalize the methods used in [12], [98], [82] led to unsurmountable problems. For example, although it seems to be possible to generalize the results of Sections 1 and 2 in [12] to the setting of relative geometry, we do not even have a plausible conjecture for a suitable extension of the "Cap Theorem" (Theorem 3.1) in [12], which represents a main ingredient for the proof of a splitting result for $d \geq 4$. On the other hand, the inductive procedure used in [82] is based on a structure theorem for curvature bounded sets, on the natural decomposition of Euclidean curvature measures of cylinders (also used in [12]), and on a symmetrization argument. Whereas the structure theorem can be extended, the decomposition property probably fails to be true, and the symmetrization argument seems to be restricted to the Euclidean situation, too.

In [97], another approach to a splitting theorem is described, which essentially uses the particular features of the three-dimensional situation. In the following, we describe how the basic ideas in [97] can be refined and combined with other results to yield an optimal splitting theorem for a convex set in $\mathbb{R}^3$ whose relative curvature measure of order $d-2$ (for $d = 3$) satisfies a pinching condition.

We fix a convex body $B \in \mathcal{K}^d_*$. Then we say that a closed set $K \subset \mathbb{R}^d$ with $\emptyset \neq \text{int}\, K \neq \mathbb{R}^d$ has a pinched $(d-1-j)$th B-curvature measure if there are positive constants $a, b > 0$ such that

$$a\, C_{d-1}(K, B; \cdot) \leq C_{d-1-j}(K, B; \cdot) \leq b\, C_{d-1}(K, B; \cdot)\,. \tag{143}$$

Furthermore, for a convex set $K \subset \mathbb{R}^d$ and $u \in \mathbb{R}^d \setminus \{o\}$, we write $K|u^\perp$ or K^u for the orthogonal projection of K onto $u^\perp$. For the sake of completeness, we state the following structur theorem.

Theorem 2.49 *Let $B \in \mathcal{K}^d_*$, let $K \subset \mathbb{R}^d$ be closed and convex with $\emptyset \neq \operatorname{int} K \neq \mathbb{R}^d$, and let $j \in \{1, \dots, d-1\}$. Assume that K satisfies (143). Then the following is true.*

(i) *Either K is compact or there exists a vector $v \in \mathbb{R}^d \setminus \{o\}$ of the recession cone of K such that $\operatorname{clos}(K|v^\perp)$ has a pinched $(d-1-j)$th B^v-curvature measure.*

(ii) *There exists a constant $\alpha > 0$, depending on a, b, d, j, such that K is isometric to a subset of $(B^d(o,\alpha) \cap \mathbb{R}^{j+1}) \times \mathbb{R}^{d-1-j}$; moreover, the dimension of the recession cone of K is less or equal $d-1-j$.*

Proof. By the representation of the relative curvature measures given in Theorem 2.9, the assertions immediately follow from the corresponding Euclidean result; see Theorem 2 in [82]. □

The main subject of this subsection is the proof of the following splitting result.

Theorem 2.50 *Let $K \subset \mathbb{R}^3$ be closed and convex with $\emptyset \neq \operatorname{int} K \neq \mathbb{R}^3$, let $B \in \mathcal{K}^3_*$ be centrally symmetric, let $\lambda > 0$, and assume that*

$$\frac{1}{2}\lambda\, C_2(K,B;\cdot) \leq C_1(K,B;\cdot) \leq \lambda\, C_2(K,B;\cdot)\,. \tag{144}$$

Then either K is compact or one of the following conditions is satisfied.

(i) *There is some $u \in \mathbb{R}^3 \setminus \{o\}$ and a two-dimensional compact convex set K' such that $K = K' \oplus \mathbb{R}u$.*

(ii) *There is some $u \in \mathbb{R}^3 \setminus \{o\}$ such that K is a translate of $\lambda^{-1}B + [0,\infty)u$.*

Proof. By the homogeneity properties of relative curvature measures, we can assume that $\lambda = 1$ in (144).

We assume that K is not compact. By Theorem 2.9 and since $B \in \mathcal{K}^3_*$, (144) implies that

$$a\, C_2(K,\cdot) \leq C_1(K,\cdot) \leq b\, C_2(K,\cdot) \tag{145}$$

for suitable constants $a, b > 0$; in particular, by Corollary 1.49, ∂K is locally of class $C^{1,1}$ (that is, ∂K is of class C^1 and σ_K is locally Lipschitz).

Clearly, the dimension of the lineality space of K cannot exceed one. If it is one, then $K = K' \oplus \mathbb{R}u$ for some $u \in \mathbb{R}^3 \setminus \{o\}$ and a closed convex set $K' \subset u^\perp$ which does not contain a line. But then (145) implies

$$\alpha\, C_1(K',\cdot) \leq C_0(K',\cdot) \leq \beta\, C_1(K',\cdot)\,, \tag{146}$$

where the curvature measures are considered in $u^\perp$ and $\alpha, \beta > 0$ are constants; compare the proof of Theorem 2.49. But then the left-hand side of (146) yields that K' is compact.

Next we consider the case that K does not contain a line. Then the argument on page 276 in [12] (or Theorem 2.49) and an estimate as on the left-hand side of (145) show that there is some $u \in \mathbb{R}^3 \setminus \{o\}$ such that $L := \operatorname{clos}(K|u^\perp)$ is compact and K is the epigraph of a convex function $f : L \to [0,\infty]$, that is

$$K = \{x + tu : x \in L, t \in [f(x),\infty)\}\,.$$

Subsequently, we state three claims from which the remaining assertion will be deduced. Volumes and mixed volumes in a two-dimensional subspace will be denoted by $v(\cdot)$ and $v(\cdot\,,\cdot)$, respectively.

Claim 1:

$$\int_{\partial K} \langle \sigma_K(x), -u \rangle H_1^B(K,x) \mathcal{H}^2(dx) \le v(L, B^u)\,.$$

Claim 2:

$$\int_{\partial K} \langle \sigma_K(x), -u \rangle H_2^B(K,x) \mathcal{H}^2(dx) = v(B^u)\,.$$

Claim 3: For all $v \in u^\perp$,

$$h(L,v) + h(L,-v) \le h(B^u, v) + h(B^u, -v)\,.$$

Let us assume that these assertions have already been verified. Using successively the assumption $H_1^B(K,x) \le 1$ for $\mathcal{H}^2$ almost all $x \in \partial K$, Claim 2, Claim 1, the central symmetry of B and Claim 3, we obtain

$$\begin{aligned}
0 &\le \int_{\partial K} \langle \sigma_K(x), -u \rangle \left[H_1^B(K,x)^2 - H_2^B(K,x) \right] \mathcal{H}^2(dx) \\
&\le \int_{\partial K} \langle \sigma_K(x), -u \rangle H_1^B(K,x) \mathcal{H}^2(dx) - v(B^u) \\
&\le v(L, B^u) - v(B^u) \\
&= \frac{1}{2} \int_{S^2 \cap u^\perp} h(L,v) S_1(B^u, dv) - v(B^u) \\
&= \frac{1}{4} \int_{S^2 \cap u^\perp} (h(L,v) + h(L,-v)) S_1(B^u, dv) - v(B^u) \\
&\le \frac{1}{2} \int_{S^2 \cap u^\perp} h(B^u, v) S_1(B^u, dv) - v(B^u) \\
&= v(B^u) - v(B^u) = 0\,.
\end{aligned}$$

Therefore, for $\mathcal{H}^2$ almost all $x \in \operatorname{relint}(L + \mathbb{R}u) \cap \partial K =: M$, we obtain that

$$k_1^B(K,x) = k_2^B(K,x) = 1\,.$$

To complete the proof, we define the function

$$F : M \to \mathbb{R}^3\,, \quad x \mapsto x - \nabla h_B \circ \sigma_K(x)\,.$$

Then, for $\mathcal{H}^2$ almost all $x \in M$, $DF(x) = o$. Since M is locally of class $C^{1,1}$, we deduce that $F \equiv c_0$ for a constant $c_0 \in \mathbb{R}^3$, and hence $x - c_0 = \nabla h_B \circ \sigma_K(x)$ for all $x \in M$. Thus $M - c_0 \subset \partial B$ and σ_K is injective on M, which implies the remaining assertion.

We still have to provide the proofs of the three Claims.

Proof of Claim 1. Let $S^2_u := \{v \in S^2 : \langle v, u\rangle \le 0\}$, and define

$$K^i := K \cap \{x \in \mathbb{R}^3 : \langle x, u\rangle \le i\}, \quad \partial K^{i*} := \partial K^i \cap \{x \in \mathbb{R}^3 : \langle x, u\rangle < i\},$$

for $i \in \mathbb{N}$. Note that $(K^i)^u \subset (K^j)^u \subset L^u$ for all $i < j$. Then we deduce

$$\begin{aligned}
&\int_{\partial K^{i*}} \langle \sigma_K(x), -u\rangle H_1^B(K,x)\mathcal{H}^2(dx)\\
&= \int_{\partial K^{i*}} \langle \sigma_{K^i}(x), -u\rangle H_1^B(K^i,x)\mathcal{H}^2(dx)\\
&\le \int_{\mathcal{N}(K^i,B)} \mathbf{1}\left\{\sigma_B(b) \in S^2_u\right\} \langle \sigma_B(b), -u\rangle\, \mathbb{H}_1^B(K^i,x,b)|u_1 \wedge u_2|\mathcal{H}^2_B(d(x,b))\\
&= \int_{\partial B} \mathbf{1}\left\{\sigma_B(b) \in S^2_u\right\} \frac{\langle \sigma_B(b), -u\rangle}{h(B,\sigma_B(b))} S_1(K^i,B;db)\\
&= \int_{S^2} \mathbf{1}\{v \in S^2_u\}\langle v, -u\rangle S(K^i,B,dv)\\
&= \frac{1}{2}\int_{S^2} |\langle v,u\rangle| S(K^i,B,dv)\\
&= v((K^i)^u, B^u) \le v(L, B^u),
\end{aligned}$$

where we used Theorems 2.44, 2.39, 2.14, and a special case of formula (5.3.31) in [123]. Consequently, the monotone convergence theorem implies that

$$\int_{\partial K} \langle \sigma_K(x), -u\rangle H_1^B(K,x)\mathcal{H}^2(dx) \le v(L, B^u),$$

which was to be proved.

Proof of Claim 2. For the proof we consider the map

$$T : \partial K \to B^u, \quad x \mapsto \nabla h_B \circ \sigma_K(x)|u^\perp,$$

where $|u^\perp$ denotes the orthogonal projection onto $u^\perp$; more explicitly,

$$T(x) = \nabla h_B \circ \sigma_K(x) - \langle \nabla h_B \circ \sigma_K(x), u\rangle u.$$

By Theorem 2.2.9 in [123], $\mathrm{card}(T^{-1}(\{y\})) = 1$ for $\mathcal{H}^2$ almost all $y \in B^u$ (note that $\sigma_K(\partial K) \supset \mathrm{int}\, S^2_u$). Then, for $\mathcal{H}^2$ almost all $x \in \partial K$,

$$DT(x)(u_i) = k_i^B(x)[u_i - \langle u_i, u\rangle u],$$

and hence

$$\begin{aligned}
J_2T(x) &= H_2^B(K,x)\frac{|[u_1 - \langle u_1,u\rangle u] \wedge [u_2 - \langle u_2,u\rangle u]|}{|u_1 \wedge u_2|}\\
&= H_2^B(K,x)\frac{|\det(u_1 - \langle u_1,u\rangle u, u_2 - \langle u_2,u\rangle u, u)|}{|\det(u_1,u_2,\sigma_K(x))|}\\
&= H_2^B(K,x)\langle \sigma_K(x), -u\rangle.
\end{aligned}$$

Therefore the coarea formula yields that

$$\begin{aligned}\int_{\partial K} \langle \sigma_K(x), -u\rangle H_2^B(K,x)\mathcal{H}^2(dx) &= \int_{\partial K} J_2T(x)\mathcal{H}^2(dx)\\ &= v(B^u)\,.\end{aligned}$$

This completes the proof of Claim 2.

Proof of Claim 3. Let $v \in S^2 \cap u^\perp$ and set $l := h(L,v) + h(L,-v)$. Then, for any $\epsilon > 0$ and any $k \in \mathbb{N}$, the assumptions imply the existence of a convex function g which is defined, say, over a neighbourhood of $D := [o,(l-\epsilon)v] \times [0,ku]$ and for which $H_1^B(\mathrm{epi}(g),(x,g(x))) \geq 1/2$ is satisfied for all $x \in D$. In fact, a vector $w \in \mathbb{R}^3$ can be found such that u, v, w are an orthonormal basis of $\mathbb{R}^3$ and $\{x + g(x)w : x \in D\}$ parametrizes a part of ∂K. We can assume that $g > 1$ on D by a proper choice of the origin. Then we define

$$\Omega_i := \{x + zw : x \in D, z \in [i, g(x)]\}\,,$$

for $i = 0, 1$, and

$$N_B(x,z) := \nabla h_B\left(\frac{\nabla g(x) - w}{\sqrt{1 + |\nabla g(x)|^2}}\right),$$

for $(x,z) \in D \times \mathbb{R}$. Then, by the general Gauss Theorem,

$$\int_{\Omega_i} \mathrm{div}\ N_B(x,z)\mathcal{H}^3(d(x,z)) = \int_{\partial\Omega_i} \langle N_B(x,z), \nu_i(x,z)\rangle \mathcal{H}^2(d(x,z))\,,$$

where $\nu_i(x,z)$ is an exterior unit normal vector of Ω_i at $(x,z) \in \partial\Omega_i$, $i = 0,1$. Since $N_B(x,z)$ is independent of $z \in \mathbb{R}$, we obtain

$$\int_D \mathrm{div}\ N_B(x,0)\mathcal{H}^2(dx) = \int_{\partial D} \langle N_B(x,0), \nu(x)\rangle \mathcal{H}^1(dx)\,, \tag{147}$$

where $\nu(x)$ is an exterior unit normal vector of D at $x \in \partial D$. But $N_B(x,0) \in B$, and hence

$$\langle N_B(x,0), \nu(x)\rangle \leq h(B, \nu(x)) \tag{148}$$

for all $x \in \partial D$; moreover, for $x \in D$ and $z \in \mathbb{R}$,

$$\begin{aligned}\mathrm{div}\ N_B(x,z) &= \sum_{i=1}^{3} \langle DN_B(x,z)(e_i), e_i\rangle = \mathrm{Trace}\ [DN_B(x,z)]\\ &= \sum_{i=1}^{2} k_i^B(\mathrm{epi}(g),(x,g(x))) + 0 = 2H_1^B(\mathrm{epi}(g),(x,g(x))) \geq 1\,,\end{aligned} \tag{149}$$

where e_1, e_2, e_3 denotes the standard basis of $\mathbb{R}^3$. From (147) – (149) we conclude that

$$(l-\epsilon)k \leq k(h(B,v) + h(B,-v)) + (l-\epsilon)(h(B,u) + h(B,-u))\,.$$

Since $\epsilon > 0$ and $k \in \mathbb{N}$ are arbitrary, we deduce

$$l \leq h(B,v) + h(B,-v) = h(B^u, v) + h(B^u, -v)\,,$$

where we used that u and v are orthonormal. This completes the proof of Claim 3, and hence also the proof of the theorem. □

3 Applications to stochastic geometry

This section represents a continuation of the recent work [76], where methods and results from the geometry of Minkowski spaces were developed and then applied to stochastic geometry. The subject of the paper [76] was the investigation of *contact distribution functions* of random closed sets, the study of *intensity measures* associated with certain random measures that are derived from these random closed sets, and the exploration of the natural interplay between these two fundamental concepts. One of the main objectives of that work was to avoid the assumptions of stationarity or isotropy, that is the translation or rotation invariance of the underlying probability distributions. Perhaps, this may be viewed as a distinguishing feature compared with related research, although instationary models have received much attention recently. Anisotropy has already been studied since about 15 years by methods of translative integral geometry. Without the assumption of rotational invariance the Euclidean unit ball is no longer distinguished as a reference (or gauge) body. Therefore, and also for other more practical reasons, the need arises to treat more general structuring elements.

The present aim is to generalize results and methods from the previous paper [76] by employing a modified approach. For an introduction to the particular questions considered here and for the basic terminology in this field we refer to [76] and to the literature cited there. Especially, the forthcoming book [133] is recommended for the geometric aspects of the theory. Some further discussion of the subject of this section in the more restricted, but practically relevant, situation of a Boolean model is contained in [75]. Stationary cluster models have recently been considered by Last and Holtmann [91].

3.1 Contact distributions

Let $\mathcal{K}^d_c$ denote the set of all convex bodies for which the centre of the circumscribed ball of miminal radius is the origin. Let Φ be a point process in $\mathbb{R}^d \times \mathcal{K}^d_c$, defined on an abstract probability space $(\Omega, \mathbb{A}, \mathbb{P})$, whose intensity measure $\alpha(\cdot) = \mathbb{E}[\Phi(\cdot)]$ can be represented in the form

$$\alpha(d(x,K)) = f(x,K)\mathcal{H}^d(dx)\mathbb{Q}(dK)\,, \tag{150}$$

where $\mathbb{Q}$ is a probability measure over $\mathcal{K}^d_c$ and f is a non-negative measurable function such that α is σ-finite (equivalently, $f(x,K) < \infty$ for $\mathcal{H}^d \otimes \mathbb{Q}$ almost all $(x,u) \in \mathbb{R}^d \times \mathcal{K}^d_c$). Further, we shall assume that there exists a σ-finite measure β over $\mathcal{K}^d_c \times \mathbb{R}^d \times \mathcal{K}^d_c$ such that the *second factorial moment measure* $\alpha^{(2)}$ of Φ, which is defined by

$$\alpha^{(2)}(\cdot) := \mathbb{E}\left[\iint \mathbf{1}\{(x_1,K_1,x_2,K_2) \in \cdot\}(\Phi \setminus \delta_{(x_1,K_1)})(d(x_2,K_2))\Phi(d(x_1,K_1))\right]\,,$$

where $\Phi \setminus \delta_{(x,K)} := \Phi - \mathbf{1}\{\Phi(\{(x,K)\}) > 0\}\delta_{(x,K)}$, satisfies

$$\alpha^{(2)} \ll \mathcal{H}^d \otimes \beta\,. \tag{151}$$

Subsequently, we shall assume that, for all compact sets $L \subset \mathbb{R}^d$,

$$\int \mathbf{1}\{(x+K) \cap L \neq \emptyset\}\Phi(d(x,K)) < \infty \tag{152}$$

is satisfied $\mathbb{P}$ almost surely. Under the assumption (152), the point process $\Phi(\cdot \times \mathcal{K}_c^d)$ is locally finite $\mathbb{P}$ almost surely. Therefore we can associate the random closed set

$$\Xi := \bigcup_{n=1}^{\infty} (\xi_n + Z_n)$$

with the point process Φ which is given by

$$\Phi = \sum_{n=1}^{\infty} \delta_{(\xi_n, Z_n)} ;$$

moreover, we set $\Xi_n := \xi_n + Z_n$ for $n \in \mathbb{N}$. Each point process on $\mathbb{R}^d \times \mathcal{K}_c^d$ can be represented in this way by sequences of random variables ξ_n and Z_n, $n \in \mathbb{N}$, as follows from [79]. In fact, the summation actually extends from $n = 1$ to $n = \tau$, where τ is a random variable taking values in $\mathbb{N}_0 \cup \{\infty\}$, although we do not make this explicit by notation. Of course, the map T which assigns Ξ to Φ does not depend on the particular enumeration of the point masses of Φ. Conversely, if Ξ is any random closed set in the extended convex ring, then there exists a point process Φ for which $T(\Phi) = \Xi$; compare [153].

Further, we denote by $\mathcal{K}_{3,gp}^d$ the set of all $(K_1, K_2, K_3) \in (\mathcal{K}^d)^3$ which are in general relative position, that is for which

$$\dim F(K_1 + K_2 + K_3, u) = \dim F(K_1, u) + \dim F(K_2, u) + \dim F(K_3, u)$$

is satisfied for all $u \in S^{d-1}$; see [80] for more details. Finally, let $L, B \in \mathcal{K}^d$ with $o \in B$ be given, and let $F \subset \mathbb{R}^d$ be closed. Then we define

$$d_B(F, L) := \inf\{t \geq 0 : (F + tB) \cap L \neq \emptyset\} ;$$

the infimum is attained if $d_B(F, L) < \infty$.

Lemma 3.1 *Let $L, B \in \mathcal{K}^d$, and let $\mathbb{Q}$ be a probability measure over $\mathcal{K}_c^d$. Then, for $\nu \otimes \nu$ almost all $(\rho_1, \rho_2) \in \mathbf{SO}(d) \times \mathbf{SO}(d)$, the condition $(\rho_1 B, \rho_2 L, K) \in \mathcal{K}_{3,gp}^d$ is satisfied for $\mathbb{Q}$ almost all $K \in \mathcal{K}_c^d$.*

Proof. The set

$$\{(K, \rho_1, \rho_2) \in \mathcal{K}_c^d \times \mathbf{SO}(d) \times \mathbf{SO}(d) : (K, \rho_1 B, \rho_2 L) \notin \mathcal{K}_{3,gp}^d\}$$

is Borel measurable. To see this, observe that

$$(K_1, K_2, K_3) \in \mathcal{K}_{3,gp}^d$$

if and only if

$$(K_1, K_2) \in \mathcal{K}_{2,gp}^d \qquad \text{and} \qquad (K_1 + K_2, K_3) \in \mathcal{K}_{2,gp}^d .$$

Moreover, $(K_1, K_2) \notin \mathcal{K}_{2,gp}^d$ if and only if K_1 and K_2 contain (non-degenerate) parallel segments lying in parallel supporting hyperplanes. Finally, one has to use that the set of all pairs $(K_1, K_2) \in (\mathcal{K}^d)^2$ such that K_1 and K_2 contain parallel segements of length greater or equal $1/m$ lying in parallel supporting hyperplanes is closed, for all $m \in \mathbb{N}$.

Hence the assertion follows by a repeated application of Theorem 2.3.10 in [123] and by means of Fubini's theorem. □

Subsequently, we freely use certain mixed curvature measures $\Theta_{i,j;k+1}(K, M; B; \cdot)$, for $i, j, k \in \{0, \dots, d-1\}$ with $i+j+k = d-1$ and $K, M, B \in \mathcal{K}^d_{3,gp}$, which have been introduced in [80]. These measures have to be distinguished from the mixed curvature measures of translative integral geometry which will be considered in the following section.

Proposition 3.2 *Let Φ be a Poisson process in $\mathbb{R}^d \times \mathcal{K}^d_c$ satisfying (150) and (152). Let $B \in \mathcal{K}^d$, $o \in B$, and let $L \in \mathcal{K}^d$ be such that $(K, \check{L}, B) \in \mathcal{K}^d_{3,gp}$ for $\mathbb{Q}$ almost all $K \in \mathcal{K}^d_c$. Then $\mathbb{P}(\Xi \cap M = \emptyset) > 0$ for all compact sets $M \subset \mathbb{R}^d$ and*

$$H_B(L, t) := \mathbb{P}(d_B(\Xi, L) \le t \mid \Xi \cap L = \emptyset), \quad t \ge 0,$$

satisfies

$$H_B(L, t) = 1 - \exp\left[-\int_0^t \rho_B(L, s)\, ds\right], \quad t \ge 0,$$

where

$$\rho_B(L, s) = \sum_{i,j,k=0}^{d-1} \binom{d-1}{i\, j\, k} s^k \int_{\mathcal{K}^d_c} \int f(-z_2 - z_1 - sb, K) \times \Theta_{i,j;k+1}(K, \check{L}; B; d(z_1, z_2, b))\, \mathbb{Q}(dK).$$

Proof. The first assertion was already proved in [76]. The second can be obtained as follows:

$$\begin{aligned}
-\ln \mathbb{P}(d_B(\Xi, L) > t) &= -\ln \mathbb{P}(\Xi \cap (L + t\check{B}) = \emptyset) \\
&= \int_{\mathbb{R}^d} \int_{\mathcal{K}^d_c} \mathbf{1}\{y \in \check{K} + L + t\check{B}\} f(y, K)\, \mathbb{Q}(dK)\, \mathcal{H}^d(dy) \\
&= \int_{\mathcal{K}^d_c} \int_{\mathbb{R}^d} \mathbf{1}\{y \in K + \check{L} + tB\} f(-y, K)\, \mathcal{H}^d(dy)\, \mathbb{Q}(dK) \\
&= \int_{\mathcal{K}^d_c} \int_{\mathbb{R}^d} f(-y, K) \mathbf{1}\{y \in K + \check{L}\}\, \mathcal{H}^d(dy)\, \mathbb{Q}(dK) \\
&\quad + \int_{\mathcal{K}^d_c} \int_{\mathbb{R}^d} f(-y, K) \mathbf{1}\{y \in (K + \check{L} + tB) \setminus (K + \check{L})\}\, \mathcal{H}^d(dy)\, \mathbb{Q}(dK) \\
&=: (a) + (b)\,.
\end{aligned}$$

Furthermore, using a consequence of Theorem 5.3 and Theorem 5.6 in [80], we obtain

$$
\begin{aligned}
(b) &= \int_{\mathcal{K}_c^d} \int_0^t \sum_{j=0}^{d-1} \frac{1}{d} \binom{d}{j} (d-j) s^{d-1-j} \\
&\qquad \times \int f(-y - sb, K)\, \Theta_{j;d-j}(K + \check{L}; B; d(y,b))\, ds\, \mathbb{Q}(dK) \\
&= \int_{\mathcal{K}_c^d} \int_0^t \sum_{j=0}^{d-1} \frac{1}{d} \binom{d}{j} (d-j) s^{d-1-j} \sum_{\substack{i_1, i_2 = 0 \\ i_1 + i_2 = j}}^{d-1} \binom{d}{i_1\, i_2\, d-j} \binom{d}{j}^{-1} f(-y_1 - y_2 - sb, K) \\
&\qquad \times \Theta_{i_1, i_2; d-j}(K, \check{L}; B; d(y_1, y_2, b))\, ds\, \mathbb{Q}(dK) \\
&= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i\, j\, k} \int_0^t t^k \int_{\mathcal{K}_c^d} \int f(-y_2 - y_1 - sb, K) \\
&\qquad \times \Theta_{i,j;k+1}(K, \check{L}; B; d(y_1, y_2, b))\, \mathbb{Q}(dK)\, ds\,.
\end{aligned}
$$

Now the proof can be completed as in [76]. □

The following theorem is the crucial result in this subsection. It provides a generalization of Theorem 4.16 in [76], which concerns contact distributions of random closed sets in a general setting, in various respects. Indeed, instead of calculating the distance of a point to a random set, we now determine the distance of a convex body L to a random set. About the possibility of such an extension of the results in [76] G. Last and W. Weil have speculated in a personal discussion, which Günter Last has kindly communicated to the present author. But even in the very special case where $L = \{x\}$, for some $x \in \mathbb{R}^d$, the new result is more general, since Theorem 4.16 in [76] was only established for $\mathcal{H}^d$ almost all points $x \in \mathbb{R}^d$ and for strictly convex and smooth gauge bodies B. All these restrictions can now be avoided, and indeed this is imperative for the further extension to the case of a general convex body L as a "blown-up reference point". The present progress is essentially due to a simplified method of proof. Note, however, that such a simplification and extension does not carry over, for instance, to the treatment of the intensity measure $\mathbb{E}\left[C_j^+(\Xi, \cdot)\right]$, which is included in [76] and which requires the derivation of a Steiner type formula for sets in the extended convex ring in a Minkowski space. Such intensity measures will be investigated more thoroughly in Subsection 3.5.

In the following, we do not assume that the structuring element B is strictly convex or a d-dimensional convex body. Another new feature of the present investigation is that we additionally consider the point of L at which the distance to the random set Ξ is realized, whereas previously we had just considered the direction of the shortest segment connecting a point of Ξ and a fixed point x, for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$, as well as the length of this segment.

We first provide some lemmas and then we state the announced theorem and give the proof. The result can be specified in various ways, as in [76], by specializing to Poisson processes, Cox processes, Gibbs processes or Poisson cluster processes. Further simplifications can be obtained in a stationary setting. A more detailed analysis of these particular cases as well as an investigation of potential applications will be carried out elsewhere. It should be emphasized, however, that the proof of Theorem 5.1 in [76] can probably not be simplified in a similar way.

Lemma 3.3 *Let $\Phi = \sum_{n=1}^{\infty} \delta_{(\xi_n, Z_n)}$ satisfy the conditions* (150) *and* (151), *and let* $B, L \in \mathcal{K}^d$, $o \in B$, *be given. Then,* $\mathbb{P}$ *almost surely,* $m \neq n$ *implies that*

$$d_B(\xi_n + Z_n, L) \neq d_B(\xi_m + Z_m, L)$$

provided that one of these two numbers is finite and $(\cup_{n=1}^{\infty}(\xi_n + Z_n)) \cap L = \emptyset$.

Proof. Let the assumptions of the lemma be fulfilled. Let ρ denote the density of $\alpha^{(2)}$ with respect to $\mathcal{H}^{d-1} \otimes \beta$. Then

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{m\neq n}\{d_B(\xi_n+Z_n,L)=d_B(\xi_m+Z_m,L)<\infty\}\cap\{L\cap\Xi=\emptyset\}\right)\\
&\leq \mathbb{E}\left\{\int\int \mathbf{1}\left\{0<d_B(x_1+K_1,L)=d_B(x_2+K_2,L)<\infty\right\}\right.\\
&\qquad\qquad \left.\times\ \left(\Phi\setminus\delta_{(x_1,K_1)}\right)(d(x_2,K_2))\,\Phi(d(x_1,K_1))\right\}\\
&=\int \mathbf{1}\left\{x_1\in \mathrm{bd}(L+\check{K}_1+d_B(x_2+K_2,L)\check{B})\right\}\\
&\qquad\qquad \times\ \mathbf{1}\{0<d_B(x_2+K_2,L)<\infty\}\,\alpha^{(2)}(d(x_1,K_1,x_2,K_2))\\
&=\int\int \mathbf{1}\left\{x_1\in \mathrm{bd}(L+\check{K}_1+d_B(x_2+K_2,L)\check{B})\right\}\rho(x_1,K_1,x_2,K_2)\\
&\qquad\qquad \times\ \mathbf{1}\{0<d_B(x_2+K_2,L)<\infty\}\,\mathcal{H}^d(dx_1)\,\beta(d(K_1,x_2,K_2))\\
&=0\,,
\end{aligned}$$

since the boundary of a convex body has $\mathcal{H}^d$ measures zero. □

Remark. By a similar but easier argument it follows that under the assumptions of Lemma 3.3, $\mathbb{P}$ almost surely, $m \neq n$ implies that $(\xi_m, Z_m) \neq (\xi_n, Z_n)$. Thus such a point process Φ is simple.

In the following, we always assume that Φ is a point process in $\mathbb{R}^d \times \mathcal{K}_c^d$ which satisfy (150) – (152).

Let $L, B \in \mathcal{K}^d$ with $o \in B$ be fixed. Then we define a measurable map $D : \Omega \to \mathbb{N}$ by distinguishing several cases.

1. If $d_B(\Xi(\omega), L) = 0$, then we define $D(\omega)$ as the smallest integer $n \in \mathbb{N}$ such that $(\xi_n(\omega) + Z_n(\omega)) \cap L \neq \emptyset$.

2. If $d_B(\Xi(\omega), L) = \infty$, then we define $D(\omega) := 1$.

3. If $0 < d_B(\Xi(\omega), L) < \infty$ and if there exists a unique $n \in \mathbb{N}$ such that
$$d_B(\xi_n(\omega) + Z_n(\omega), L) < d_B(\xi_m(\omega) + Z_m(\omega), L)$$
for all $m \in \mathbb{N} \setminus \{n\}$, then we set $D(\omega) := n$.
4. By Lemma 3.3, the remaining measurable subset Ω^* of Ω has $\mathbb{P}$ measure zero, and we set $D(\omega) := 1$ for each $\omega \in \Omega^*$.

Of course, D depends on L, B and on the choice of a representation for Φ, but this is immaterial.

The appropriate tool for formulating and proving our general result are the *Palm probabilities* $\{\mathbb{P}_{(x,K)} : (x, K) \in \mathbb{R}^d \times \mathcal{K}_c^d\}$ of Φ. Their definition requires that the intensity measure α of Φ is σ-finite, which we have assumed from the beginning. Then $(x, K) \mapsto \mathbb{P}_{(x,K)}(A)$ is for all $A \in \mathbb{A}$ a Radon-Nikodym derivative of the measure $E[\mathbf{1}_A \Phi(\cdot)]$ with respect to α. It is easy to see that this definition entails that
$$\iint H(\omega, x, K)\Phi(\omega, d(x, K))\mathbb{P}(d\omega) = \iint H(\omega, x, K)\mathbb{P}_{(x,K)}(d\omega)\alpha(d(x, K)), \tag{153}$$
where $H : \Omega \times \mathbb{R}^d \times \mathcal{K}_c^d \to [0, \infty]$ is an arbitrary measurable function. In the following, this equation will be used in an essential way. As in Kallenberg ([79], p. 84) we can assume without restricting generality that $(x, K) \mapsto \mathbb{P}_{(x,K)}(\cdot)$ is a stochastic kernel, since all of our random elements take their values in Polish spaces. Moreover, by Lemma 10.2 in [79] we can also assume that $\mathbb{P}_{(x,K)}(\Phi(\{(x, K)\}) \geq 1) = 1$ for all (x, K). Since Φ is a simple point process (as we have seen already), $\mathbb{P}_{(x,K)}(A)$ can be interpreted as the conditional probability of A given that $\Phi(\{(x, K)\}) = 1$.

Before we can state our general result, some more preparatory work is needed.

Lemma 3.4 *Let the assumptions of Lemma 3.3 be fulfilled. Assume that* $(B, \check{L}, K) \in \mathcal{K}_{3,gp}^d$ *for* $\mathbb{Q}$ *almost all* $K \in \mathcal{K}_c^d$. *Then* $(B, \check{L}, Z_D(\omega)) \in \mathcal{K}_{3,gp}^d$ *for* $\mathbb{P}$ *almost all* $\omega \in \Omega$*; moreover, for* $\nu \otimes \nu$ *almost all* $(\rho_1, \rho_2) \in (\mathbf{SO}(d))^2$, *the condition* $(\rho_1 B, \rho_2 \check{L}, Z_D(\omega)) \in \mathcal{K}_{3,gp}^d$ *is satisfied for* $\mathbb{P}$ *almost all* $\omega \in \Omega$.

Proof. We define a Borel measurable map $g : \mathbb{R}^d \times \mathcal{K}_c^d \times \Omega \to [0, \infty)$ by
$$g(y, K, \omega) := \mathbf{1}\left\{(y, K) = (\xi_D, Z_D)(\omega), (B, \check{L}, K) \notin \mathcal{K}_{3,gp}^d\right\}.$$
Using the remarks preceding the statement of the lemma, we obtain
$$\begin{aligned}
&\mathbb{P}\left(\left\{(B, \check{L}, Z_D) \notin \mathcal{K}_{3,gp}^d\right\}\right) \\
&= \int_\Omega \int g(y, K, \omega)\, \Phi(d(y, K))\, \mathbb{P}(d\omega) \\
&= \int_{\mathbb{R}^d} \int_{\mathcal{K}_c^d} \int_\Omega g(y, K, \omega)\, \mathbb{P}_{(y,K)}(d\omega)\, f(y, K)\, \mathbb{Q}(dK)\, \mathcal{H}^d(dy) \\
&= \int_{\mathbb{R}^d} \int_{\mathcal{K}_c^d} \mathbb{P}_{(y,K)}(\{(y, K) = (\xi_D, Z_D)\}) f(y, K) \mathbf{1}\left\{(B, \check{L}, K) \notin \mathcal{K}_{3,gp}^d\right\}\, \mathbb{Q}(dK)\, \mathcal{H}^d(dy) \\
&= 0,
\end{aligned}$$

since by assumption

$$\mathbb{Q}\left(\left\{K \in \mathcal{K}_c^d : (B, \check{L}, K) \notin \mathcal{K}_{3,gp}^d\right\}\right) = 0\,.$$

The remaining assertion is implied by Lemma 3.1. □

The preceding lemma shows that it is not particularly restrictive to assume that $(B, \check{L}, Z_D(\omega)) \in \mathcal{K}_{3,gp}^d$ for $\mathbb{P}$ almost all $\omega \in \Omega$. Especially, this condition is trivially satisfied for all $\omega \in \Omega$ if B and L are strictly convex. Moreover, if $(B, \check{L}, K) \in \mathcal{K}_{3,gp}^d$ and $d_B(K, L) \in (0, \infty)$, then there exist uniquely determined points $z_B(K, L) \in K$, $z_B(L, K) \in L$ and a uniquely determined vector $u_B(K, L) \in B$ such that

$$z_B(K, L) + d_B(K, L)u_B(K, L) = z_B(L, K)\,.$$

Now assume that $(B, \check{L}, Z_D(\omega)) \in \mathcal{K}_{3,gp}^d$ for $\mathbb{P}$ almost all $\omega \in \Omega$. Then, for $\mathbb{P}$ almost all $\omega \in \Omega$ for which $d_B(\Xi(\omega), L) \in (0, \infty)$, there exist uniquely determined points $z_B(\Xi(\omega), L) \in \Xi(\omega)$, $z_B(L, \Xi(\omega)) \in L$ and a uniquely determined vector $u_B(\Xi(\omega), L) \in B$ such that

$$z_B(\Xi(\omega), L) + d_B(\Xi(\omega), L)u_B(\Xi(\omega), L) = z_B(L, \Xi(\omega))\,;$$

this follows from Lemma 3.3. In fact, in this situation we have

$$z_B(L, \Xi(\omega)) = z_B(L, \xi_D(\omega) + Z_D(\omega))\,,$$

and similar relations hold for the other quantities. Hence, the preceding discussion shows that $u_B(\Xi, L)$ and $z_B(L, \Xi)$ are well-defined $\mathbb{P}$ almost surely if $\Xi \cap L = \emptyset$ and $(\Xi + [0, \infty)B) \cap L \neq \emptyset$.

Finally, for the map $g : [0, \infty) \times \partial B \times \partial L \to [0, \infty)$, which appears in the statement of Theorem 3.5, we set $g(d, u, z) := 0$ if $d = \infty$ to simplify the notation. This convention is consistent with the case where g is an indicator function.

Theorem 3.5 *Let Φ be a point process in $\mathbb{R}^d \times \mathcal{K}_c^d$ satisfying* (150) – (152)*. Let $B, L \in \mathcal{K}^d$, $o \in B$, be such that $(B, \check{L}, K) \in \mathcal{K}_{3,gp}^d$ for $\mathbb{Q}$ almost all $K \in \mathcal{K}_c^d$. Further, let $g : [0, \infty) \times \partial B \times \partial L \to [0, \infty)$ be measurable. Then*

$$\begin{aligned}
&\mathbb{E}[g(d_B(\Xi, L), u_B(\Xi, L), z_B(L, \Xi))\mathbf{1}\{\Xi \cap L = \emptyset\}] \\
&= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i\,j\,k} \int_0^\infty t^k \int_{\mathcal{K}_c^d} \int g(t, b, -z_2)\mathbb{P}_{(-z_2-z_1-tb,K)}(d_B(T(\Phi \setminus \delta_{(-z_2-z_1-tb,K)}), L) > t) \\
&\quad \times f(-z_2 - z_1 - tb, K)\, \Theta_{i,j;k+1}(K, \check{L}; B; d(z_1, z_2, b))\, \mathbb{Q}(dK)\, dt\,.
\end{aligned}$$

Proof. The discussion preceding the statement of the asserted theorem already shows that all expressions which appear in the formula are properly defined, since the integrand is defined to be zero if $d_B(\Xi, L) = \infty$. Subsequently, we often write y instead of $\{y\}$.

Similarly as in the proof of Lemma 3.4, we define a function $\tilde{g} : \mathbb{R}^d \times \mathcal{K}_c^d \times \Omega \to [0, \infty)$ by

$$\begin{aligned}
\tilde{g}(y, K, \omega) :=\,& g(d_B(y + K, L), u_B(y + K, L), z_B(L, y + K)) \\
&\times \mathbf{1}\{0 < d_B(\Xi(\omega), L) < \infty\}\mathbf{1}\{(B, \check{L}, Z_D(\omega)) \in \mathcal{K}_{3,gp}^d\} \\
&\times \mathbf{1}\{(y, K) = (\xi_D, Z_D)(\omega)\}\,.
\end{aligned}$$

More precisely, $\tilde{g}$ is defined to be zero if one of the indicator functions is zero; in the remaining cases the argument of g is well-defined. Recall that Φ is simple under $\mathbb{P}$. Then the defining equation (153) for the Palm probabilities shows that for α almost all $(y, K) \in \mathbb{R}^d \times \mathcal{K}_c^d$ the point process Φ is simple also under $\mathbb{P}_{(y,K)}$. This is used implicitly in the following argument. Using Lemma 3.3, Lemma 3.4, a straightforward extension of equation (2.6) in [76] and Theorem 5.6 in [80], we obtain

$$
\begin{aligned}
&\mathbb{E}[g(d_B(\Xi, L), u_B(\Xi, L), z_B(L, \Xi))\mathbf{1}\{\Xi \cap L = \emptyset\}] \\
&= \iint \tilde{g}(y, K, \omega)\, \Phi(d(y, K))\, \mathbb{P}(d\omega) \\
&= \iint \tilde{g}(y, K, \omega) f(y, K)\, \mathcal{H}^d(dy)\, \mathbb{Q}(dK) \\
&= \iint \mathbf{1}\{(y + K) \cap L = \emptyset\} g(d_B(y + K, L), u_B(y + K, L), z_B(L, y + K)) \\
&\qquad \times\ \mathbb{P}_{(y,K)}\left(\left\{\omega \in \Omega : d_B\left(T(\Phi(\omega) \setminus \delta_{(y,K)}), L\right) > d_B(K + y, L)\right\}\right) \\
&\qquad \times\ f(y, K)\, \mathcal{H}^d(dy)\, \mathbb{Q}(dK) \\
&= \iint \mathbf{1}\{K \cap (L + y) = \emptyset\} g(d_B(K, L + y), u_B(K, L + y), z_B(L + y, K) - y) \\
&\qquad \times\ \mathbb{P}_{(-y,K)}\left(d_B\left(T(\Phi \setminus \delta_{(-y,K)}), L\right) > d_B(K, L + y)\right) \\
&\qquad \times\ f(-y, K)\, \mathcal{H}^d(dy)\, \mathbb{Q}(dK) \\
&= \iint \mathbf{1}\{y \notin K + \check{L}\} g(d_B(K + \check{L}, y), u_B(K + \check{L}, y), z_B(L + y, K) - y) \\
&\qquad \times\ \mathbb{P}_{(-y,K)}\left(d_B\left(T(\Phi \setminus \delta_{(-y,K)}), L\right) > d_B(K + \check{L}, y)\right) \\
&\qquad \times\ f(-y, K)\, \mathcal{H}^d(dy)\, \mathbb{Q}(dK) \\
&= \int_{\mathcal{K}_c^d} \int_0^\infty \int \sum_{j=0}^{d-1} \frac{1}{d} \binom{d}{j} (d - j) s^{d-1-j} g(s, b, z_B(L + z + sb, K) - z - sb) \\
&\qquad \times\ \mathbb{P}_{(-z-sb,K)}\left(d_B\left(T(\Phi \setminus \delta_{(-z-sb,K)}), L\right) > s\right) f(-z - sb, K) \\
&\qquad \times\ \Theta_{j;d-j}(K + \check{L}; B; d(z, b))\, ds\, \mathbb{Q}(dK) \\
&= \int_0^\infty \int_{\mathcal{K}_c^d} \sum_{i,j,k=0}^{d-1} \binom{d-1}{i\ j\ k} t^k \int g(t, b, -z_2) \\
&\qquad \times\ \mathbb{P}_{(-z_1-z_2-sb,K)}\left(d_B\left(T(\Phi \setminus \delta_{(-z_1-z_2-sb,K)}), L\right) > t\right) f(-z_1 - z_2 - sb, K) \\
&\qquad \times\ \Theta_{i,j;k+1}(K, \check{L}; B; d(z_1, z_2, b))\, \mathbb{Q}(dK)\, dt\,.
\end{aligned}
$$

In order to justify the third equality one has to distinguish several cases, similar to the definition of the map D. This finally proves the theorem. □

As an immediate consequence of the previous general theorem, we obtain an extension of Theorem 4.16 in [76], where the case $L = \{x\}$ was considered. There, however, the conclusion was only obtained for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$. The subsequent assumption that K and B should be in general relative position is fulfilled, for instance, if B is strictly convex; also K and ρB are in general relative position for ν almost all $\rho \in \mathbf{SO}(d)$. Moreover, analogues of Lemmas 3.1 and 3.4 are clearly satisfied.

Corollary 3.6 *Let Φ be a point process in $\mathbb{R}^d \times \mathcal{K}_c^d$ satisfying* (150) – (152). *Let $B \in \mathcal{K}^d$, $o \in B$, be such that $(K, B) \in \mathcal{K}^d_{2,gp}$ for $\mathbb{Q}$ almost all $K \in \mathcal{K}_c^d$. Further, let $g : [0,\infty) \times \partial B \to [0,\infty)$ be measurable. Then*

$$\begin{aligned}&\mathbb{E}[g(d_B(\Xi,x),u_B(\Xi,x))\mathbf{1}\{x \notin \Xi\}]\\&= \sum_{i=0}^{d-1}\binom{d-1}{i}\int_0^\infty t^{d-1-i}\int_{\mathcal{K}_c^d}\int g(t,b)\mathbb{P}_{(x-z-tb,K)}(d_B(T(\Phi\setminus\delta_{(x-z-tb,K)}),x)>t)\\&\qquad\times f(x-z-tb,K)\,\Theta_{i;d-i}(K;B;d(z,b))\,\mathbb{Q}(dK)\,dt\,.\end{aligned}$$

Proof. The translation invariance of the relative support measures and Theorem 5.6 in [80] imply that

$$\Theta_{i,j;k+1}(K,\{x\};B;\cdot) = 0 \qquad \text{if } j > 0$$

and

$$\Theta_{i,0;k+1}(K,\{x\};B;\beta_1\times\beta_2\times\gamma) = \Theta_{i;k+1}(K;B;\beta_1\times\gamma)\mathbf{1}\{x\in\beta_2\}\,,$$

where $i, j, k \geq 0$ and $i + j + k = d - 1$. The assertion follows by combining these remarks with Theorem 3.5.

Alternatively, the assertion is also implied by a repetition of the proof of Theorem 3.5 in the special case considered. □

If Φ is a Poisson process, then by Slivnyak's theorem we have

$$\mathbb{P}_{(x,K)}(\Phi\setminus\delta_{(x,K)}\in\cdot) = \mathbb{P}(\Phi\in\cdot)$$

for α almost all $(x, K) \in \mathbb{R}^d \times \mathcal{K}_c^d$. Furthermore, the contact distribution function $H_B(L,t)$ has been determined in Proposition 3.2 and

$$\mathbb{P}(d_B(\Xi,L)>t) = \mathbb{P}(\Xi\cap L=\emptyset)(1-H_B(L,t))\,.$$

These remarks imply the next corollary.

Corollary 3.7 *Let Φ be a Poisson process in $\mathbb{R}^d \times \mathcal{K}_c^d$ satisfying* (150) *and* (152). *Let $B, L \in \mathcal{K}^d$, $o \in B$, be such that $(B, \check{L}, K) \in \mathcal{K}^d_{3,gp}$ for $\mathbb{Q}$ almost all $K \in \mathcal{K}_c^d$. Further, let $g : [0,\infty) \times \partial B \times \partial L \to [0,\infty)$ be measurable. Then*

$$\begin{aligned}&\mathbb{E}[g(d_B(\Xi,L),u_B(\Xi,L),z_B(L,\Xi))\mid \Xi\cap L=\emptyset]\\&= \sum_{i,j,k=0}^{d-1}\binom{d-1}{i\;j\;k}\int_0^\infty t^k\int_{\mathcal{K}_c^d}\int g(t,b,-z_2)(1-H_B(L,t))\\&\qquad\times f(-z_2-z_1-tb,K)\,\Theta_{i,j;k+1}(K,\check{L};B;d(z_1,z_2,b))\,\mathbb{Q}(dK)\,dt\,.\end{aligned}$$

Similar specializations in the spirit of the results in [76] can be obtained for other types of processes as well.

Corollary 3.8 *Let Φ be a stationary Poisson process in $\mathbb{R}^d \times \mathcal{K}_c^d$ with intensity γ and satisfying* (152)*. Let $B, L \in \mathcal{K}^d$, $o \in B$, be such that $(B, \check{L}, K) \in \mathcal{K}_{3,gp}^d$ for $\mathbb{Q}$ almost all $K \in \mathcal{K}_c^d$. Further, let $\alpha, \beta \subset \mathbb{R}^d$ be Borel measurable and $r \geq 0$. Then*

$$\mathbb{E}[d_B(\Xi, L) \leq r, u_B(\Xi, L) \in \beta, z_B(L, \Xi) \in \alpha \mid \Xi \cap L = \emptyset]$$

$$= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i\ j\ k} \int_0^r t^k (1 - H_B(L,t))dt \, \gamma \int_{\mathcal{K}_c^d} \Theta_{i,j;k+1}(K, \check{L}; B; \mathbb{R}^d \times \check{\alpha} \times \beta) \, \mathbb{Q}(dK) \,,$$

where

$$H_B(L,t) = 1 - \exp\left\{ \sum_{m,n,l}^{d-1} \binom{d}{m\ n\ l+1} t^{l+1} \gamma \int_{\mathcal{K}_c^d} V(K[m], \check{L}[n], B[l+1]) \mathbb{Q}(dK) \right\} .$$

Corollary 3.9 *Let Φ be a stationary Poisson process in $\mathbb{R}^d \times \mathcal{K}_c^d$ with intensity γ and satisfying* (152)*. Let $B \in \mathcal{K}^d$ with $o \in B$ be strictly convex. Further, let $\beta \subset \mathbb{R}^d$ be Borel measurable and $r \geq 0$. Then*

$$\mathbb{E}[d_B(\Xi, o) \leq r, u_B(\Xi, o) \in \beta \mid \Xi \cap L = \emptyset]$$

$$= \sum_{i=0}^{d-1} \binom{d-1}{i} \int_0^r t^{d-1-i} (1 - H_B(t))dt \, \gamma \int_{\mathcal{K}_c^d} \Theta_i(K; B; \mathbb{R}^d \times \beta) \, \mathbb{Q}(dK) \,,$$

where $H_B(t) := H_B(\{o\}, t)$.

Recall that in the special case $B = B^d$ one has

$$\Theta_{i,j;k+1}(K, \check{L}; B^d; \cdot) = C_{i,j,k}^{d-1}(K, \check{L}, B^d; \cdot) \,,$$

where the mixed measures on the right-hand side of the preceding equation are introduced in [80]. From the special case $\gamma = \mathbb{R}^d$ of Corollary 3.8 and the relationship

$$C_{i,j,k}^{d-1}(K, \check{L}, B^d; \mathbb{R}^d \times \mathbb{R}^d \times \cdot) = S(K[i], \check{L}[j], B^d[k], \cdot)$$

one could deduce the next corollary under the additional assumption $(\check{L}, K) \in \mathcal{K}_{2,grep}^d$. However, in order to obtain the result in full generality, it seems to be necessary to repeat the proof of Theorem 3.5 by using the simplifications which are available under the assumptions of the corollary. (An approximation argument, though conceivable in principle, does not seem to go through easily.)

Corollary 3.10 *Let Φ be a stationary Poisson process in $\mathbb{R}^d \times \mathcal{K}_c^d$ with intensity γ and satisfying* (152)*. Let $L \in \mathcal{K}^d$ be given. Further, let $\beta \subset \mathbb{R}^d$ be Borel measurable and $r \geq 0$. Then*

$$\mathbb{E}[d_{B^d}(\Xi, L) \leq r, u_{B^d}(\Xi, L) \in \beta \mid \Xi \cap L = \emptyset]$$

$$= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i\ j\ k} \int_0^r t^k (1 - H_{B^d}(L,t))dt \, \gamma \int_{\mathcal{K}_c^d} S(K[i], \check{L}[j], B^d[k], \beta) \, \mathbb{Q}(dK) \,.$$

3.2 Translative integral formulae – preparations

In order to establish translative integral formulae in the setting of relative geometry, we will extend the methods of Schneider and Weil [131] and Weil [150]. Another main ingredient comes from recent work of Kiderlen and Weil [80], where explicit representations of relative support measures are obtained for convex polytopes. It seems likely that an extension of the measure geometric method developed by Rataj and Zähle [110] can be used for an alternative approach as long as the gauge body is sufficiently smooth.

We introduce some notation. For $K \in \mathcal{K}^d$ let $L(K)$ denote the linear subspace which is parallel to the affine hull of K. Recall from [80] that $K_1, K_2 \in \mathcal{K}^d$ are said to be in general relative position if $L(F(K_1, u))$ and $L(F(K_2, u))$ are complementary linear subspaces for all $u \in \mathbb{R}^d \setminus \{o\}$. Let $K_1, \ldots, K_k, B \in \mathcal{K}^d$ and $o \in B$. We say that $(K_1, \ldots, K_k)$ and B are in general relative position if $K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k)$ and B are in general relative position for $\otimes_{i=2}^k \mathcal{H}^d$ almost all $(x_2, \ldots, x_k) \in (\mathbb{R}^d)^{k-1}$ for which the intersection is not empty. For $k = 1$ this definition coincides with the one given by Kiderlen and Weil [80]. We write $(\mathcal{K}^d)_{k,gp}$, $k \geq 1$, for the set of all $(K_1, \ldots, K_k, B) \in (\mathcal{K}^d)^{k+1}$ such that $o \in B$ and $(K_1, \ldots, K_k)$ and B are in general relative position. Theorem 2.3.10 in [123] and Fubini's theorem together imply that $(K_1, \ldots, K_k)$ and ρB are in general relative position for ν almost all $\rho \in \mathbf{SO}(d)$. If $K_1, \ldots, K_k, B$ are all polytopes, then $K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k)$ and ρB are in general relative position for all $x_2, \ldots, x_k \in \mathbb{R}^d$ and for ν almost all $\rho \in \mathbf{SO}(d)$. In fact, assume that $x_2, \ldots, x_k \in \mathbb{R}^d$ and $\rho \in \mathbf{SO}(d)$ are such that there is some $u \in S^{d-1}$ for which the linear subspaces parallel to the affine hulls of $F(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k), u)$ and $F(\rho B, u)$ are not complementary and hence contain a common one-dimensional linear subspace. The support set $F(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k), u)$ is a face of $K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k)$, and hence there are faces $F_1, \ldots, F_k$ of $K_1, \ldots, K_k$ such that

$$F(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k), u) = F_1 \cap (F_2 + x_2) \cap \ldots \cap (F_k + x_k)$$

and

$$\text{relint } F_1 \cap (\text{relint } F_2 + x_2) \cap \ldots \cap (\text{relint } F_k + x_k) \neq \emptyset \, .$$

Therefore we obtain

$$L(F(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k), u)) = L(F_1) \cap \ldots \cap L(F_k) \, .$$

Since $K_1, \ldots, K_k, B$ have only finitely many faces, the assertion follows from Lemma 4.5.1 in [123]. Clearly, if $B \in \mathcal{K}^d$ is strictly convex and $o \in B$, then $(K_1, \ldots, K_k)$ and B are in general relative position for all $K_1, \ldots, K_k \in (\mathcal{K}^d)^k$ with non-empty interiors.

For linear subspaces $L_1, \ldots, L_k \subset \mathbb{R}^d$ with $\dim L_1 + \ldots + \dim L_k =: m \leq d$ we choose an orthonormal basis in each subspace L_j. Then we define

$$\det(L_1, \ldots, L_k)$$

as the m-dimensional volume of the parallelepiped which is spanned by the union of these orthonormal bases. Of course, the definition is independent of the particular choices involved. For linear subspaces $L_1, \ldots, L_k \subset \mathbb{R}^d$ with

$$\sum_{i=1}^{k} \dim L_i \geq (k-1)d$$

we define

$$[L_1, \ldots, L_k] := \det(L_1^{\perp}, \ldots, L_k^{\perp}) .$$

Moreover, if $A_1, \ldots, A_k$ are non-empty convex sets and $L(A_i)$ denotes the linear subspace which is parallel to aff A_i, then we define

$$[A_1, \ldots, A_k] := [L(A_1), \ldots, L(A_k)]$$

provided that $\dim A_1 + \ldots + \dim A_k \geq (k-1)d$. It is not hard to check that these definitions are consistent with the ones given in [123] for the case $k = 2$.

Now let $K_1, \ldots, K_k, B \in \mathcal{K}^d$ be polytopes with faces $F_1, \ldots, F_k, G$, respectively. Further, let $x_2, \ldots, x_k \in \mathbb{R}^d$ be chosen in such a way that

$$\operatorname{relint}(F_1) \cap \operatorname{relint}(F_2 + x_2) \cap \ldots \cap \operatorname{relint}(F_k + x_k) \neq \emptyset .$$

Then

$$\begin{aligned} &\det(L(F_1 \cap (F_2 + x_2) \cap \ldots \cap (F_k + x_k)), L(G)) \\ &\quad = \det(L(F_1) \cap (L(F_2) + x_2) \cap \ldots \cap (L(F_k) + x_k), L(G)) \\ &\quad = \det(L(F_1) \cap \ldots \cap L(F_k), L(G)) \end{aligned}$$

is independent of the particular choice of $x_2, \ldots, x_k$, and therefore we denote this quantity by

$$\det(F_1, \ldots, F_k; G) .$$

Finally, for a d-dimensional convex polytope $K \subset \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$, we define $u(A, K)$ as the outer unit normal vector of K at A if A is a facet of K, and as o otherwise. In order to extend this definition to the case of several convex bodies, we need the following lemma.

Lemma 3.11 *Let* $(K, B) \in \mathcal{K}^d_{2,gp}$ *with* $\dim(K + B) = d$, *and let* F, G *be faces of* K, B, *respectively. Then* $F + G$ *is a facet of* $K + B$ *if and only if* $\dim(F + G) = d - 1$ *and* $N(K, F) \cap N(B, G) \neq \{o\}$.

Proof. Let $F+G$ be a facet of $K+B$. Since $\dim(K+B) = d$, this implies $\dim(F+G) = d-1$. Choose $z \in \operatorname{relint}(F+G)$ and $u \in N(K+B, F+G) \setminus \{o\}$. By Lemma 1.3.12 in [123], there is some $x \in \operatorname{relint} F$ and some $y \in \operatorname{relint} G$ such that $z = x + y$. Hence, Theorem 2.2.1 in [123] yields

$$N(K+B, F+G) = N(K+B, x+y) = N(K, x) \cap N(B, y) = N(K, F) \cap N(B, G) ,$$

and thus

$$u \in N(K, F) \cap N(B, G) \setminus \{o\} .$$

Conversely, assume that $\dim(F + G) = d - 1$ and $u \in N(K, F) \cap N(B, G) \setminus \{o\}$. Then $F \subset F(K, u)$ and $G \subset F(B, u)$. Since K and B are in general relative position, we obtain

$$\dim F(K, u) + \dim F(B, u) = \dim F(K + B, u) \leq d - 1 ,$$

and hence

$$d-1=\dim(F+G)\leq \dim F+\dim G\leq \dim F(K,u)+\dim F(B,u)\leq d-1\,.$$

This proves that $F=F(K,u)$ and $G=F(B,u)$, and therefore $F+G=F(K+B,u)$. Thus we have shown that $F+G$ is a facet of $K+B$. □

Lemma 3.12 *Let $K_1,\dots,K_k,B\in\mathcal{K}^d$ be polytopes, and let $F_1,\dots,F_k,G$ be faces of $K_1,\dots,K_k,B\in\mathcal{K}^d$, respectively. Further, let $x_2,\dots,x_k\in\mathbb{R}^d$ be vectors such that the following conditions are satisfied:*

(a) $\operatorname{relint}(F_1)\cap\operatorname{relint}(F_2+x_2)\cap\ldots\cap\operatorname{relint}(F_k+x_k)\neq\emptyset$;

(b) $\dim(K_1\cap(K_2+x_2)\cap\ldots\cap(K_k+x_k))=d$;

(c) $K_1\cap(K_2+x_2)\cap\ldots\cap(K_k+x_k)$ *and B are in general relative position.*

Then we can unambiguously define the number

$$u(F_1,\dots,F_k;G;K_1,\dots,K_k;B)$$

as

$$u(F_1\cap(F_2+x_2)\cap\ldots\cap(F_k+x_k)+G;K_1\cap(K_2+x_2)\cap\ldots\cap(K_k+x_k)+B)\,.$$

If there do not exist vectors $x_2,\dots,x_k\in\mathbb{R}^d$ such that (a) – (c) are satisfied, then we define $u(F_1,\dots,F_k;G;K_1,\dots,K_k;B)$ as zero.

Proof. Assume that $x_2,\dots,x_k\in\mathbb{R}^d$ satisfy the assumptions of the lemma. By Lemma 3.11 we obtain that $F_1\cap\ldots\cap(F_k+x_k)+G$ is a facet of $K_1\cap\ldots\cap(K_k+x_k)+B$ if and only if

$$\dim(F_1\cap\ldots\cap(F_k+x_k)+G)=d-1 \tag{154}$$

and

$$N(K_1\cap\ldots\cap(K_k+x_k),F_1\cap\ldots\cap(F_k+x_k))\cap N(B,G)\neq\{o\}\,. \tag{155}$$

As

$$\begin{aligned}\dim(F_1\cap\ldots\cap(F_k+x_k)+G)&=\dim(L(F_1\cap\ldots\cap(F_k+x_k))+L(G))\\&=\dim(L(F_1)\cap\ldots\cap L(F_k)+L(G))\,,\end{aligned}$$

the first condition is independent of the particular choice involved. Furthermore, if $x_0\in\operatorname{relint}(F_1)\cap\ldots\cap\operatorname{relint}(F_k+x_k)$, then $x_0\in\operatorname{relint}(F_1\cap\ldots\cap(F_k+x_k))$. Thus

$$\begin{aligned}&N(K_1\cap\ldots\cap(K_k+x_k),F_1\cap\ldots\cap(F_k+x_k))\\&=N(K_1\cap\ldots\cap(K_k+x_k),x_0)\\&=N(K_1,x_0)+\ldots+N(K_k+x_k,x_0)\\&=N(K_1,F_1)+\ldots+N(K_k+x_k,F_k+x_k)\\&=N(K_1,F_1)+\ldots+N(K_k,F_k)\,.\end{aligned}$$

This establishes the independence of the second condition of the particular choice of $x_2, \ldots, x_k$.

Moreover, if (154) and (155) are satisfied, then

$$u(F_1 \cap (F_2 + x_2) \cap \ldots \cap (F_k + x_k) + G; K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k) + B)$$

is the uniquely determined unit vector which lies in

$$(N(K_1, F_1) + \ldots + N(K_k, F_k)) \cap N(B, G),$$

as follows from the proof of Lemma 3.11. □

3.3 The basic formula

For polytopes $K, B \in \mathcal{K}^d$ in general relative position and with $o \in B$, the relative support measures $\Phi_j(K; B; \cdot)$ are defined as $\binom{d-1}{j}\Theta_j(K, B; \cdot)$ (or $\binom{d-1}{j}\Theta_{j;d-j}(K; B; \cdot)$ in the notation of [80]), and hence

$$\Phi_j(K; B; \cdot) = \sum_{F \in \mathcal{F}_j(K)} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \det(F, G) h(B, u(F + G, K + B)) \lambda_F \otimes \lambda_G .$$

In the case where B is a Euclidean unit ball, a corresponding explicit representation for the ordinary Euclidean curvature measures was used as the starting point for the proof of the translative integral formula for curvature measures, which is due to Schneider and Weil [131].

Theorem 3.13 *Let $(K_1, K_2, B) \in (\mathcal{K}^d)_{2,gp}$, let $j \in \{0, \ldots, d-1\}$, and let $f : (\mathbb{R}^d)^3 \to \mathbb{R}$ be a non-negative Borel measurable function. Then there exist (uniquely determined) Borel measures $\Phi^j_{k,d+j-k}(K_1, K_2; B; \cdot)$, $k = j, \ldots, d$, over $(\mathbb{R}^d)^3$ such that*

$$\int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^2} f(z, z-x, b) \Phi_j(K_1 \cap (K_2 + x); B; d(z, b)) \mathcal{H}^d(dx)$$
$$= \sum_{k=j}^{d} \int_{(\mathbb{R}^d)^3} f(x, y, b) \Phi^j_{k,d+j-k}(K_1, K_2; B; d(x, y, b)) . \tag{156}$$

Let $A_1, A_2, C \subset \mathbb{R}^d$ and $D \subset (\mathbb{R}^d)^3$ be Borel sets. Then the following properties are satisfied:

(i) $\Phi^j_{k,d+j-k}(K_1, K_2; B; A_1 \times A_2 \times C) = \Phi^j_{d+j-k,k}(K_2, K_1; B; A_2 \times A_1 \times C)$;

(ii) $\Phi^j_{d,j}(K_1, K_2; B; A_1 \times A_2 \times C) = \mathcal{H}^d(K_1 \cap A_1)\Phi_j(K_2; B; A_2 \times C)$;

(iii) $\Phi^j_{k,d+j-k}(K_1, K_2; B; \cdot)$ *is a finite non-negative Borel measure over* $(\mathbb{R}^d)^3$ *which is supported by* $S_1 \times S_2 \times \mathrm{bd}\, B$, *where* $S_1 = K_1$ *if* $k = d$ *and* $S_1 = \mathrm{bd}\, K_1$ *otherwise and where* $S_2 = K_2$ *if* $k = j$ *and* $S_2 = \mathrm{bd}\, K_2$ *otherwise;*

(iv) *the maps* $\Phi^j_{k,d+j-k}(\cdot, K_2; B; \cdot \times A_2 \times C)$, $\Phi^j_{k,d+j-k}(K_1, \cdot; B; A_1 \times \cdot \times C)$ *and* $\Phi^j_{k,d+j-k}(K_1, K_2; \cdot; A_1 \times A_2 \times \cdot)$ *are positively homogeneous of degree* $k, d+j-k$ *and* $d-j$, *respectively;*

(v) *if K_1, K_2, B are polytopes such that rm dim K_1 = rm dim $K_2 = d$, then*

$$\Phi^j_{k,d+j-k}(K_1,K_2;B;\cdot) = \sum_{F_1\in\mathcal{F}_k(K_1)}\sum_{F_2\in\mathcal{F}_{d+j-k}(K_2)}\sum_{G\in\mathcal{F}_{d-1-j}(B)} \det(F_1,F_2;G)$$
$$\times\, h(B,u(F_1,F_2;G;K_1,K_2;B))[F_1,F_2]\lambda_{F_1}\otimes\lambda_{F_2}\otimes\lambda_G\,.$$

(vi) *the map $(K_1,K_2,B)\mapsto\Phi^j_{k,d+j-k}(K_1,K_2;B;\cdot)$ from $(\mathcal{K}^d)_{2,gp}$ into the space of finite Borel measures over $(\mathbb{R}^d)^3$ is weakly continuous;*

(vii) *the map $(K_1,K_2,B)\mapsto\Phi^j_{k,d+j-k}(K_1,K_2;B;D)$ defined over $(\mathcal{K}^d)_{2,gp}$ is measurable;*

(viii) *the maps $\Phi^j_{k,d+j-k}(\cdot,K_2;B;D)$, $\Phi^j_{k,d+j-k}(K_1,\cdot;B;D)$ and $\Phi^j_{k,d+j-k}(K_1,K_2;\cdot;D)$ are additive, where they are defined;*

Proof. First, we prove that for an arbitrary non-negative Borel measurable function $f:(\mathbb{R}^d)^3\to\mathbb{R}$ the map

$$x\mapsto\int_{(\mathbb{R}^d)^2} f(z,z-x,b)\Phi_j(K_1\cap(K_2+x);B;d(z,b))$$

is Lebesgue measurable and well-defined for $\mathcal{H}^d$ almost all $x\in\mathbb{R}^d$. Let $N\subset\mathbb{R}^d$ be a set of $\mathcal{H}^d$ measure zero such that $K_1\cap(K_2+x)$ and B are in general relative position for all $x\in\mathbb{R}^d\setminus N$. Then the integral is well-defined for all $x\in\mathbb{R}^d\setminus N$. Next we observe that Hilfssatz 7.2.3 in [132] remains true if the topological space T appearing in the statement of this lemma is merely assumed to have a countable base. In view of this, it is sufficient to show that the map

$$x\mapsto\int_{(\mathbb{R}^d)^2} g(z,b)\Phi_j(K_1\cap(K_2+x);B;d(z,b)) \tag{157}$$

is Borel measurable on $\mathbb{R}^d\setminus(N\cup\partial(K_1-K_2)) =: T$, for any continuous function $g:(\mathbb{R}^d)^2\to\mathbb{R}$. Of course, as a subspace of $\mathbb{R}^d$ the topological space T has a countable base. Thus the required measurability follows from the continuity of (157) as a map defined on T by the argument on p. 62 in [132] and by Theorem 5.3 in [80].

Let $K_1,K_2,B\subset\mathbb{R}^d$ be polytopes such that $(K_1,K_2,B)\in(K^d)_{2,gp}$ and $\dim K_i = d$, $i=1,2$. Let $\beta_1,\beta_2,\gamma\subset\mathbb{R}^d$ be Borel sets. Neglecting a set of translation vectors x of $\mathcal{H}^d$

measure zero (compare p. 231 in [123]), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Phi_j(K_1 \cap (K_2 + x); B; (\beta_1 \cap (\beta_2 + x)) \times \gamma)\mathcal{H}^d(dx) \\
&= \int_{\mathbb{R}^d} \sum_{F \in \mathcal{F}_j(K_1 \cap (K_2 + x))} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \det(F, G) \\
&\quad \times h(B, u(F + G, K_1 \cap (K_2 + x_2) + B))\lambda_F(\beta_1 \cap (\beta_2 + x))\lambda_G(\gamma)\mathcal{H}^d(dx) \\
&= \sum_{k=j}^{d} \sum_{F_1 \in \mathcal{F}_k(K_1)} \sum_{F_2 \in \mathcal{F}_{d+j-k}(K_2)} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \\
&\quad \times \int_{\mathbb{R}^d} \det(F_1 \cap (F_2 + x); G) h(B, u(F_1 \cap (F_2 + x) + G, K_1 \cap (K_2 + x) + B)) \\
&\quad \times \lambda_{F_1 \cap (F_2 + x)}(\beta_1 \cap (\beta_2 + x))\lambda_G(\gamma)\mathcal{H}^d(dx) \\
&= \sum_{k=j}^{d} \sum_{F_1 \in \mathcal{F}_k(K_1)} \sum_{F_2 \in \mathcal{F}_{d+j-k}(K_2)} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \\
&\quad \times \det(F_1, F_2; G) h(B, u(F_1, F_2; G; K_1, K_2; B))[F_1, F_2]\lambda_{F_1}(\beta_1)\lambda_{F_2}(\beta_2)\lambda_G(\gamma) \\
&= \sum_{k=j}^{d} \Phi^j_{k,d+j-k}(K_1, K_2; B; \beta_1 \times \beta_2 \times \gamma)
\end{aligned}$$

if the relation in (v) is used as a definition for $\Phi^j_{k,d+j-k}(K_1, K_2; B; \cdot)$ in the particular case considered. Here we have also used the considerations from the beginning of this subsection and the Fubini-type argument on p. 232 in [123].

Hence, for polytopes $K_1, K_2, B \subset \mathbb{R}^d$ such that (K_1, K_2) and B are in strongly general relative position and for an arbitrary non-negative Borel measurable function $f : (\mathbb{R}^d)^3 \to \mathbb{R}$, we deduce

$$\int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^2} f(z, z - x, b)\Phi_j(K_1 \cap (K_2 + x); B; d(z, b))\mathcal{H}^d(dx)$$

$$\sum_{k=j}^{d} \int_{(\mathbb{R}^d)^3} f(x, y, b)\Phi^j_{k,d+j-k}(K_1, K_2; B; d(x, y, b))\,;$$

moreover, in this case the assertions (i)–(iv) are easy to check.

In order to establish the general result, let (K_1, K_2) and B be convex bodies in $\mathbb{R}^d$ in general relative position. Choose sequences of polytopes K_1^i, K_2^i, B^i, $i \in \mathbb{N}$, such that (K_1^i, K_2^i) and B^i are in strongly general relative position and $K_1^i \to K_1$, $K_2^i \to K_2$ and $B^i \to B$ as $i \to \infty$. Let N be the union of $\partial(K_1 - K_2)$ and of the set of all $x \in \mathbb{R}^d$ such that $K_1 \cap (K_2 + x)$ and B are not in general relative position. Thus $\mathcal{H}^d(N) = 0$. For $x \in \mathbb{R}^d \setminus N$ we obtain

$$\Phi_j(K_1^i \cap (K_2^i + x); B^i; \cdot) \to \Phi_j(K_1 \cap (K_2 + x); B; \cdot)$$

for $i \to \infty$, in the sense of the weak convergence of measures. For an arbitrary continuous function $f : (\mathbb{R}^d)^3 \to \mathbb{R}$ and for all $x \in \mathbb{R}^d \setminus N$, we infer that

$$\int_{(\mathbb{R}^d)^2} f(z, z-x, b)\Phi_j(K_1^i \cap (K_2^i + x); B; d(z,b))$$
$$\longrightarrow \int_{(\mathbb{R}^d)^2} f(z, z-x, b)\Phi_j(K_1 \cap (K_2 + x); B; d(z,b)) \tag{158}$$

as $i \to \infty$. Since

$$\Phi_j(K_1^i \cap (K_2^i + x); B^i; (\mathbb{R}^d)^2) \leq d\binom{d-1}{j} V(K_1^i[j], (B + B^d)[n-j])\mathbf{1}_{K_1^i - K_2^i}(x),$$

for all $x \in \mathbb{R}^d$ and all $i \in \mathbb{N}$, by the monotoneity of mixed volumes, the bounded convergence theorem can be applied to yield

$$\lim_{i\to\infty} \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^2} f(z, z-x, b)\Phi_j(K_1^i \cap (K_2^i + x); B^i; d(z,b))\mathcal{H}^d(dx)$$
$$= \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^2} f(z, z-x, b)\Phi_j(K_1 \cap (K_2 + x); B; d(z,b))\mathcal{H}^d(dx), \tag{159}$$

where $f : (\mathbb{R}^d)^3 \to \mathbb{R}$ is an arbitrary continuous function.

Let again $f : (\mathbb{R}^d)^3 \to \mathbb{R}$ be an arbitrary continuous function. For $r, s, t > 0$ and $x, y, b \in \mathbb{R}^d$ we define

$$D_{r,s,t}(x, y, b) := \left(\frac{x}{r}, \frac{y}{s}, \frac{b}{t}\right)$$

and

$$D_{r,s,t}J(f, K_1, K_2, B) := \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^2} f\left(\frac{z}{r}, \frac{z-x}{s}, \frac{b}{t}\right) \Phi_j(K_1 \cap (K_2 + x); B; d(z,b))\mathcal{H}^d(dx).$$

To given convex bodies (K_1, K_2) and B in general relative position we now choose sequences of polytopes as described above. Then

$$D_{r,s,t}J(f, K_1^i, K_2^i, B^i) \to D_{r,s,t}J(f, K_1, K_2, B)$$

as $i \to \infty$ and

$$D_{r,s,t}J(f, K_1^i, K_2^i, B^i)$$
$$= \sum_{k=j}^{d} r^k s^{d+j-k} t^{d-j} \int_{(\mathbb{R}^d)^3} f(x, y, b)\Phi^j_{k,n+j-k}(K_1^i, K_2^i; B^i; d(x, y, b)). \tag{160}$$

Now the proof for the existence of the measures $\Phi^j_{k,d+j-k}(K_1, K_2; B; \cdot)$ with the asserted properties can be completed as in [132], pp. 69 – 70. In fact, properties (i), (ii) and (iv) extend from the polytopal to the general case by approximation. Properties (iii) and (vi) – (viii) follow from (156), since (160) extends to the general case by continuity and corresponding properties can be established for the left-hand side of (156). For instance, for the proof of (viii) we use that if (K_1, K_2) and B are in general relative position, (K_1', K_2) and B are in general relative position, and $K_1 \cup K_1'$ is convex, then $(K_1 \cup K_1', K_2)$ and B as well as $(K_1 \cap K_1', K_2)$ and B are in general relative position; moreover, we use Theorem 5.3 in [80]. This yields the additivity of $\Phi^j_{k,d+j-k}(\cdot, K_2; B; D)$. A similar argument works for the other arguments. The proof of (vi) follows again from (156) and an easy extension of the argument employed to establish (159). In fact, one merely has to exclude an additional set of translation vectors of $\mathcal{H}^d$ measure zero to obtain (158) for arbitrary sequences of convex bodies K_1^i, K_2^i, B^i, $i \in \mathbb{N}$, such that (K_1^i, K_2^i) and B^i are in general relative position and $K_1^i \to K_1$, $K_2^i \to K_2$ and $B^i \to B$ as $i \to \infty$, where K_1, K_2, B are convex bodies such that (K_1, K_2) and B are in general relative position. □

3.4 The iterated formula

The subsequent treatment of an iterated translative integral formula is based on ideas in [150] and [80].

Theorem 3.14 *Let $(K_1, \ldots, K_k, B) \in (\mathcal{K}^d)_{k,gp}$, $j \in \{0, \ldots, d-1\}$, $k \in \mathbb{N}$ with $k \geq 2$, and $m_1, \ldots, m_k \in \{j, \ldots, d\}$ with $m_1 + \ldots + m_k = (k-1)d + j$. Further, let $f : (\mathbb{R}^d)^{k+1} \to \mathbb{R}$ be a non-negative Borel measurable function. Then there exist (uniquely determined) Borel measures $\Phi^j_{m_1,\ldots,m_k}(K_1, \ldots, K_k; B; \cdot)$ over $(\mathbb{R}^d)^{k+1}$, for $m_1, \ldots, m_k \in \{j, \ldots, d\}$ with $m_1 + \ldots + m_k = (k-1)d + j$, such that*

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^2} f(z, z - x_2, \ldots, z - x_k, b)$$
$$\times \Phi_j(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k); B; d(z, b)) \mathcal{H}^d(dx_2) \ldots \mathcal{H}^d(dx_k)$$
$$= \sum \int_{(\mathbb{R}^d)^{k+1}} f(x_1, \ldots, x_k, b) \Phi^j_{m_1,\ldots,m_k}(K_1, \ldots, K_k; B; d(x_1, \ldots, x_k, b)), \quad (161)$$

where the summation extends over all $m_1, \ldots, m_k \in \{j, \ldots, d\}$ such that $m_1 + \ldots + m_k = (k-1)d + j$.

Let $A_i, C \subset \mathbb{R}^d$, $i \in \{1, \ldots, k\}$, let $D' \subset (\mathbb{R}^d)^k$, and let $D \subset (\mathbb{R}^d)^{k+1}$ be Borel sets. Then the following properties are satisfied:

(i) *$\Phi^j_{m_1,\ldots,m_k}(K_1, \ldots, K_k; B; A_1 \times \ldots \times A_k \times C)$ is symmetric with respect to permutations of $\{1, \ldots, k\}$;*

(ii) *$\Phi^j_{d,m_2,\ldots,m_k}(K_1, \ldots, K_k; B; A_1 \times D') = \mathcal{H}^d(K_1 \cap A_1) \Phi^j_{m_2,\ldots,m_k}(K_2, \ldots, K_k; B; D')$;*

(iii) *$\Phi^j_{m_1,\ldots,m_k}(K_1, \ldots, K_k; B; \cdot)$ is a finite non-negative Borel measure over $(\mathbb{R}^d)^{k+1}$ which is supported by $S_1 \times \ldots \times S_k \times \mathrm{bd}\, B$, where $S_i = K_i$ if $m_i = d$ and $S_i = \mathrm{bd}\, K_i$ otherwise;*

(iv) *the map $\Phi^j_{m_1,\ldots,m_k}(K_1, \ldots, K_k; B; A_1 \times \ldots \times A_k \times C)$ is positively homogeneous of degree m_i with respect to (K_i, A_i) and of degree $d - j$ with respect to (B, C);*

(v) *if* $K_1, \dots, K_k, B$ *are polytopes such that* $(K_1, \dots, K_k, B) \in (\mathcal{K}^d)_{k,gp}$ *and dim* $K_i = d$ *for* $i = 1, \dots, k$*, then*

$$\begin{aligned}
&\Phi^j_{m_1,\dots,m_k}(K_1, \dots, K_k; B; \cdot) \\
&\quad = \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \dots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \det(F_1, \dots, F_k; G) \\
&\quad h(B, u(F_1, \dots, F_k; G; K_1, \dots, K_k; B))[F_1, \dots, F_k] \lambda_{F_1} \otimes \dots \otimes \lambda_{F_k} \otimes \lambda_G \,.
\end{aligned}$$

(vi) *the map* $(K_1, \dots, K_k, B) \mapsto \Phi^j_{m_1,\dots,m_k}(K_1, \dots, K_k; B; \cdot)$ *from* $(\mathcal{K}^d)_{k,gp}$ *into the space of finite Borel measures on* $(\mathbb{R}^d)^{k+1}$ *is weakly continuous;*

(vii) *the map* $(K_1, \dots, K_k, B) \mapsto \Phi^j_{m_1,\dots,m_k}(K_1, \dots, K_k; B; D)$ *defined on* $(\mathcal{K}^d)_{k,gp}$ *is measurable;*

(viii) *the map* $\Phi^j_{m_1,\dots,m_k}(K_1, \dots, K_k; B; D)$ *is additive in each component as long as it is defined;*

(ix) *if* $(K'_1, \dots, K'_k, B') \in (\mathcal{K}^d)_{k,gp}$, $\beta_1, \dots, \beta_k, \gamma \subset \mathbb{R}^d$ *are open sets,* $K_i \cap \beta_i = K'_i \cap \beta_i$ *for* $i = 1, \dots, k$ *and* $B \cap \gamma = B' \cap \gamma$*, then*

$$\Phi^j_{m_1,\dots,m_k}(K_1, \dots, K_k; B; \cdot) = \Phi^j_{m_1,\dots,m_k}(K'_1, \dots, K'_k; B'; \cdot)$$

on Borel subsets of $\beta_1 \times \dots \times \beta_k \times \gamma$.

Proof. We start with an assertion of measurability. By Fubini's theorem and Theorem 1.8.8 in [123] it follows that there exists a set $N' \subset (\mathbb{R}^d)^{k-1}$ with $\otimes_{i=2}^k \mathcal{H}^d(N') = 0$ such that

$$(x_2, \dots, x_k) \mapsto K_1 \cap (K_2 + x_2) \cap \dots \cap (K_k + x_k)$$

is continuous on $(\mathbb{R}^d)^{k-1} \setminus N'$ with respect to the Hausdorff metric in the image space. Let $f : (\mathbb{R}^d)^k \to \mathbb{R}$ be an arbitrary non-negative Borel measurable function. Then an argument similar to the one used in the proof of Theorem 3.13 shows that the map

$$\begin{aligned}
&(x_2, \dots, x_k) \mapsto \\
&\int_{(\mathbb{R}^d)^2} f(z, z - x_2, \dots, z - x_k) \Phi_j(K_1 \cap (K_2 + x_2) \cap \dots \cap (K_k + x_k); B; d(z, b))
\end{aligned}$$

is Lebesgue measurable and well-defined for $\otimes_{i=2}^k \mathcal{H}^d$ almost all $(x_2, \dots, x_k) \in (\mathbb{R}^d)^{k-1}$.

First, we prove the translative integral formula for polytopes in strongly general relative position by induction on k. The case $k = 2$ has already been settled in Theorem 3.13. Now we assume that $k \geq 3$ and that the assumptions of (v) are satisfied. Subsequently, we use Fubini's theorem and the translation invariance of $\mathcal{H}^d$, the inductive assumption, arguments which have already been used in the proof of Theorem 3.13, and Proposition 1 in [151]. Let

$\beta_1, \ldots, \beta_k, \gamma \subset \mathbb{R}^d$ be Borel sets. Then we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \Phi_j(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_k + x_k); B; \\
&\quad [\beta_1 \cap (\beta_2 + x_2) \cap \ldots \cap (\beta_k + x_k)] \times \gamma) \mathcal{H}^d(dx_2) \ldots \mathcal{H}^d(dx_k) \\
&= \sum_{\substack{m_1, \ldots, m_{k-1} = j \\ m_1 + \ldots + m_{k-1} = (k-2)d + j}}^{d} \int_{\mathbb{R}^d} \Phi^j_{m_1, \ldots, m_{k-1}}(K_1, \ldots, K_{k-2}, K_{k-1} \cap (K_k + x_k); B; \\
&\quad [\beta_1 \times \ldots \times \beta_{k-2} \times (\beta_{k-1} \cap (\beta_k + x_k)) \times \gamma]) \mathcal{H}^d(dx_k) \\
&= \sum_{\substack{m_1, \ldots, m_{k-2}, m = j \\ m_1 + \ldots + m_{k-2} + m = (k-2)d + j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \ldots \sum_{F_{k-2} \in \mathcal{F}_{m_{k-2}}(K_{k-2})} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \\
&\quad \int_{\mathbb{R}^d} \sum_{F \in \mathcal{F}_m(K_{k-1} \cap (K_k + x))} \det(F_1, \ldots, F_{k-2}, F; G) \\
&\quad h(B, u(F_1, \ldots, F_{k-2}, F; G; K_1, \ldots, K_{k-2}, K_{k-1} \cap (K_k + x); B)) \\
&\quad [F_1, \ldots, F_{k-2}, F] \lambda_{F_1}(\beta_1) \ldots \lambda_{F_{k-2}}(\beta_{k-2}) \lambda_F(\beta_{k-1} \cap (\beta_k + x)) \lambda_G(\gamma) \mathcal{H}^d(dx) \\
&= \sum_{\substack{m_1, \ldots, m_k = j \\ m_1 + \ldots + m_k = (k-1)d + j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \ldots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \\
&\quad \det(F_1, \ldots, F_k; G) h(B, u(F_1, \ldots, F_k; G; K_1, \ldots, K_k; B)) \\
&\quad [F_1, \ldots, F_{k-2}, L(F_{k-1}) \cap L(F_k)] \lambda_{F_1}(\beta_1) \ldots \lambda_{F_{k-2}}(\beta_{k-2}) \lambda_G(\gamma) \\
&\quad \int_{\mathbb{R}^d} \lambda_{F_{k-1} \cap (F_k + x)}(\beta_{k-1} \cap (\beta_k + x)) \mathcal{H}^d(dx) \\
&= \sum_{\substack{m_1, \ldots, m_k = j \\ m_1 + \ldots + m_k = (k-1)d + j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \ldots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \sum_{G \in \mathcal{F}_{d-1-j}(B)} \\
&\quad \det(F_1, \ldots, F_k; G) h(B, u(F_1, \ldots, F_k; G; K_1, \ldots, K_k; B)) \\
&\quad [F_1, \ldots, F_k] \lambda_{F_1}(\beta_1) \ldots \lambda_{F_k}(\beta_k) \lambda_G(\gamma) \\
&= \sum_{\substack{m_1, \ldots, m_k = j \\ m_1 + \ldots + m_k = (k-1)d + j}}^{d} \Phi^j_{m_1, \ldots, m_k}(K_1, \ldots, K_k; B; \beta_1 \times \ldots \times \beta_k \times \gamma),
\end{aligned}$$

where we used Lemma 2.1 (b) in [150] and the relation in assertion (v) as a definition for the mixed measures $\Phi^j_{m_1,\dots,m_k}(K_1,\dots,K_k;B;\cdot)$. Moreover, we also used the fact that

$$\mathrm{relint}(F_1)\cap\ldots\cap\mathrm{relint}(F_{k-2}+x_{k-2})\cap(\mathrm{relint}(F_{k-1}\cap(F_k+x_k))+x_{k-1})\neq\emptyset\,,$$

$$\mathrm{relint}(F_{k-1})\cap\mathrm{relint}(F_k+x_k)\neq\emptyset\,,$$

and

$$\dim(K_1\cap\ldots\cap(K_{k-2}+x_{k-2})\cap(K_{k-1}\cap(K_k+x_k)+x_{k-1}))=d$$

together yield

$$\begin{aligned}u(F_1\cap\ldots\cap(F_{k-2}+x_{k-2})\cap(F_{k-1}\cap(F_k+x_k)+x_{k-1})+G;\\ K_1\cap\ldots\cap(K_{k-2}+x_{k-2})\cap(K_{k-1}\cap(K_k+x_k)+x_{k-1})+B)\\ =u(F_1,\dots,F_k;G;K_1,\dots,K_k;B)\,,\end{aligned}$$

provided that $K_1\cap\ldots\cap(K_{k-2}+x_{k-2})\cap(K_{k-1}\cap(K_k+x_k)+x_{k-1})$ and B are in general relative position; compare the proof of Lemma 3.12. A similar observation can be made for the expression $\det(F_1,\dots,F_k;G)$.

Finally, the integral formula can be extended to the general case and the asserted properties of the mixed measures can be justified in way which is similar to the proof of Theorem 3.13. □

3.5 Intensity measures

In this subsection, we establish a version of Theorem 4.17 in [76] which holds for all strictly convex gauge bodies $B\in\mathcal{K}^d$ with $o\in\mathrm{int}\ B$. In fact, we also obtain a variant which is true for every gauge body ρB, where $B\in\mathcal{K}^d$ with $o\in\mathrm{int}\ B$ is arbitrary and $\rho\in\mathbf{SO}(d)\setminus N(B)$ with a set $N(B)$ of proper rotations of Haar measure zero. Thus our aim is to remove the smoothness and strict convexity assumptions on B, which considerably simplified the argument in [76]. In fact, the smoothness (and strict convexity) of the gauge body B together with an absolute continuity assumption on the second factorial moment measure $\alpha^{(2)}$ of the underlying point process Φ implied that the random measure $\Theta^+_j(\Xi,B;\cdot)$ could be represented in a simple way; see equation (4.13) in [76]. Instead of the deterministic assumption of the smoothness of B, we impose absolute continuity assumptions for the higher order factorial moment measures $\alpha^{(n)}$ of Φ. These assumptions are automatically satisfied, for instance, for Poisson or Gibbs processes, which supports the present point of view.

We introduce some terminology. Let Φ be a point process in $\mathbb{R}^d\times\mathcal{K}^d_c$ as described in Subsection 3.1. Hence Φ is supposed to satisfy conditions (150) – (152). In particular, Φ takes its values in the space $\mathbf{N}$ of all $(\mathbb{N}_0\cup\{\infty\})$-valued measures φ over $\mathbb{R}^d\times\mathcal{K}^d_c$ for which $\varphi(\cdot\times\mathcal{K}^d_c)$ is locally finite. The space $\mathbf{N}$ is endowed with the vague topology, and $\mathcal{N}$ denotes the induced Borel σ-algebra.

Subsequently, we shall consider the non-negative random measure

$$\Lambda^+_j(B;\cdot):=\mathbb{E}\left[\Theta^+_j(\Xi,B;\cdot)\right]\,,$$

which was introduced and investigated in [76] (with a different normalization) and which is called the *intensity measure* of $\Theta_j^+(\Xi, B; \cdot)$. Here we first assume that $B \in \mathcal{K}^d$ is strictly convex and $o \in \text{int } B$. The non-negative extension $\Theta_j^+(K, B; \cdot)$ of the B-support measures to the local convex ring can be expressed in terms of B-support measures of convex bodies; see Theorem 3.4 in [76] and the literature cited there. In fact, if

$$K = \bigcup_{i=1}^{\infty} K_i$$

is in the local convex ring, $K_i \in \mathcal{K}^d$ for $i \in \mathbb{N}$, then

$$\Theta_j^+(K, B; \cdot) = \sum_{n=1}^{\infty} \sum_{i_1 < \ldots < i_n} \int \mathbf{1}\left\{z \notin K^{(i_1, \ldots, i_n)}\right\} \mathbf{1}\{(z, b) \in \cdot \cap \mathcal{N}_B(K_{i_1}) \cap \ldots \cap \mathcal{N}_B(K_{i_n})\} \\ \times \Theta_j(K_{i_1} \cap \ldots \cap K_{i_n}, B; d(z, b)) \,, \tag{162}$$

where

$$K^{(i_1, \ldots, i_n)} := \bigcup_{i \notin \{i_1, \ldots, i_n\}} K_i$$

and where $\mathcal{N}_B(K) := \mathcal{N}(K, B)$ is the relative normal bundle of K with respect to B. The structure of this formula suggest to introduce, for $\varphi \in \mathbf{N}$ and $n \in \mathbb{N}$, the factorial measures

$$\varphi^{(n)}(d(a_1, \ldots, a_n)) := \left(\varphi \setminus \delta_{a_1} \setminus \ldots \setminus \delta_{a_{n-1}}\right)(da_n) \ldots (\varphi \setminus \delta_{a_1})(da_2)\varphi(da_1) \,,$$

where $a_i \in \mathbb{R}^d \times \mathcal{K}_c^d$ for $i = 1, \ldots, n$, and the n-th order factorial moment measure

$$\alpha^{(n)}(\cdot) := \mathbb{E}\left[\int \mathbf{1}\{(a_1, \ldots, a_n) \in \cdot\}\Phi^{(n)}(d(a_1, \ldots, a_n))\right] .$$

In the following, we assume that there exists a σ-finite Borel measure $\beta^{(n)}$ over $\mathbb{R}^d \times \left(\mathcal{K}_c^d\right)^n$ such that

$$\alpha^{(n)}(d(x_1, K_1, \ldots, x_n, K_n)) = f_n(x_1, K_1, \ldots, x_n, K_n) \\ \times \mathcal{H}^d(dx_1) \ldots \mathcal{H}^d(dx_{n-1})\beta^{(n)}(d(x_n, K_1, \ldots, K_n)) \tag{163}$$

with a non-negative Borel measurable function $f_n < \infty$, for each $n \geq 2$. This assumption implies that $\alpha^{(n)}$ is σ-finite. Finally, we write $C^{(n)}$ for the n-th order reduced Campell measure of Φ, that is

$$C^{(n)}(\cdot) := \mathbb{E}\left[\int \mathbf{1}\left\{(\Phi \setminus \delta_{a_1} \setminus \ldots \setminus \delta_{a_n}, a_1, \ldots, a_n) \in \cdot\right\} \Phi^{(n)}(d(a_1, \ldots, a_n))\right] ,$$

which is a Borel measure over $\mathbf{N} \times (\mathbb{R}^d \times \mathcal{K}_c^d)^n$. Obviously, we have

$$\alpha^{(n)}(\cdot) = C^{(n)}(\mathbf{N} \times \cdot) \,,$$

and therefore there exists a probability kernel

$$q^{(n)}(a_1, \ldots, a_n, d\varphi)$$

from $(\mathbb{R}^d \times \mathcal{K}_c^d)^n$ to $\mathbf{N}$ such that

$$C^{(n)}(d(\varphi, a_1, \dots, a_n)) = q^{(n)}(a_1, \dots, a_n, d\varphi)\alpha^{(n)}(d(a_1, \dots, a_n)).$$

Note that $\alpha^{(1)} = \alpha$, $f_1 = f$ and $\beta^{(2)} = \beta$. For $n = 1$ we interpret condition (163) as condition (150).

For the proof of our main result, we need a deterministic lemma. Surprisingly, the only proof of which we know uses point processes in an essential way and thus is of a probabilistic nature. For $K_1, \dots, K_n \in \mathcal{K}^d$ and strictly convex $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$ we write $\mathcal{M}(K_1, \dots, K_n; B)$ for the set of all $(x_1, \dots, x_n, b) \in (\mathbb{R}^d)^{n+1}$ such that $(x_i, b) \in \mathcal{N}_B(K_i)$ for all $i = 1, \dots, n$.

Lemma 3.15 *Let $K_1, \dots, K_n \in \mathcal{K}^d$ $(n \geq 2)$, let $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$ be strictly convex, and let $j \in \{0, \dots, d-1\}$. Then*

$$\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \Theta_j(K_1 \cap (K_2 + y_2) \cap \dots \cap (K_n + y_n), B; \cdot) \llcorner \mathcal{N}_B(K_1) \cap \mathcal{N}_B(K_2 + y_2) \cap \dots \cap \mathcal{N}_B(K_n + y_n) \mathcal{H}^d(dy_2) \dots \mathcal{H}^d(y_n) = 0;$$

moreover,

$$\Phi^j_{m_1, \dots, m_n}(K_1, \dots, K_n; B; \cdot) \llcorner \mathcal{M}(K_1, \dots, K_n; B) = 0$$

for all $m_1, \dots, m_n \in \{j, \dots, d\}$ with $m_1 + \dots + m_n \geq (n-1)d + j$.

Proof. For the proof, we consider the stationary (marked) Poisson process Φ in $\mathbb{R}^d \times \mathcal{K}_c^d$ with intensity measure

$$\alpha(d(x, K)) = \mathcal{H}^d(dx)\mathbb{Q}(dK),$$

where $\mathbb{Q}$ is an arbitrary probability measure over $\mathcal{K}_c^d$. By an extension of the proof of Theorem 4.16 in [76] in the special case considered here, and by an argument based on formula (162), it follows that $H_B(t) := H_B(\{o\}, t)$ is absolutely continuous with density $h_B(t)$ and

$$\frac{h_B(t)}{1 - H_B(t)} = \sum_{j=0}^{d-1} \binom{d-1}{j} t^{d-1-j} \sum_{n=1}^{\infty} \frac{1}{n!} h_n(t, B),$$

where

$$\begin{aligned} h_n(t, B) := \int \dots \int & \mathbf{1}\{(z, b) \in \mathcal{N}_B(K_1) \cap \mathcal{N}_B(K_2 + y_2) \cap \dots \cap \mathcal{N}_B(K_n + y_n)\} \\ & \times \Theta_j(K_1 \cap (K_2 + y_2) \cap \dots \cap (K_n + y_n), B; d(z, b)) \\ & \times \mathcal{H}^d(dy_2) \dots \mathcal{H}^d(dy_n)\mathbb{Q}(dK_1) \dots \mathbb{Q}(dK_n). \end{aligned}$$

A comparison with Corollary 3.9 demonstrates that

$$h_n(t, B) = 0$$

for all $n \geq 2$. Choosing

$$\mathbb{Q} := \frac{1}{n} \sum_{i=1}^{n} \delta_{K_i} ,$$

we obtain the first assertion. The second assertion is then implied by Theorem 3.14. □

Now we are prepared for the proof of our main result in this subsection. Subsequently, we also write $d_B(\Phi, x)$ instead of $d_B(T(\Phi), x)$.

Theorem 3.16 *Let Φ be a point process in $\mathbb{R}^d \times \mathcal{K}_c^d$ such that its n-th order factorial moment measure satisfies conditions* (152) *and* (163) *for all $n \in \mathbb{N}$. Let $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$ be strictly convex, and let $A \subset \mathbb{R}^d$ be Borel measurable. Then $\Lambda_j^+(B; \cdot \times A)$, $j = 0, \ldots, d-1$, is absolutely continuous with density*

$$\lambda_j^+(B; x, A) = \int_{\mathcal{K}_c^d} \int \mathbb{P}_{(x-z,K)}(d_B\left(\Phi \setminus \delta_{(x-z,K)}, x) > 0\right) f(x-z, K) \Theta_j(K, B; dz \times A) \mathbb{Q}(dK) .$$

Proof. Using equation (162), the monotone convergence theorem, and the definitions of the various types of factorial measures and factorizations, we obtain

$$\begin{aligned}
\Lambda_j^+(B; \cdot) &= \mathbb{E}\left[\Theta_j^+(\Xi, B; \cdot)\right] \\
&= \mathbb{E}\Bigg[\sum_{n=1}^{\infty} \sum_{i_1 < \ldots < i_n} \int \mathbf{1}\left\{z \notin \Xi^{(i_1, \ldots, i_n)}\right\} \mathbf{1}\left\{(z, b) \in \cdot \cap \mathcal{N}_B(\Xi_{i_1}) \cap \ldots \cap \mathcal{N}_B(\Xi_{i_n})\right\} \\
&\qquad \times \Theta_j(\Xi_{i_1} \cap \ldots \cap \Xi_{i_n}, B; d(z, b))\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{n=1}^{\infty} \frac{1}{n!} \iint \mathbf{1}\left\{z \notin T\left(\Phi \setminus \delta_{(x_1,K_1)} \setminus \ldots \setminus \delta_{(x_n,K_n)}\right)\right\} \\
&\qquad \times \mathbf{1}\left\{(z, b) \in \cdot \cap \mathcal{N}_B(x_1 + K_1) \cap \ldots \cap \mathcal{N}_B(x_n + K_n)\right\} \\
&\qquad \times \Theta_j((x_1 + K_1) \cap \ldots \cap (x_n + K_n), B; d(z, b)) \Phi^{(n)}(d(x_1, K_1, \ldots, x_n, K_n))\Bigg] \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \iint \mathbf{1}\{z \notin T(\varphi)\} \mathbf{1}\left\{(z, b) \in \cdot \cap \mathcal{N}_B(x_1 + K_1) \cap \ldots \cap \mathcal{N}_B(x_n + K_n)\right\} \\
&\qquad \times \Theta_j((x_1 + K_1) \cap \ldots \cap (x_n + K_n), B; d(z, b)) C^{(n)}(d(\varphi, x_1, K_1, \ldots, x_n, K_n)) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d \times (\mathcal{K}_c^d)^n} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \int q^{(n)}(x_1, K_1, \ldots, x_n, K_n, \{\varphi : z \notin T(\varphi)\}) \\
&\qquad \times f_n(x_1, K_1, \ldots, x_n, K_n) \mathbf{1}\left\{(z, b) \in \cdot \cap \mathcal{N}_B(x_1 + K_1) \cap \ldots \cap \mathcal{N}_B(x_n + K_n)\right\} \\
&\qquad \times \Theta_j((x_1 + K_1) \cap \ldots \cap (x_n + K_n), B; d(z, b)) \\
&\qquad \times \mathcal{H}^d(dx_1) \ldots \mathcal{H}^d(dx_{n-1}) \beta^{(n)}(d(x_n, K_1, \ldots, K_n)) .
\end{aligned}$$

Hence, by an application of Lemma 3.15, we find that

$$\begin{aligned}\Lambda_j^+(B;\cdot) &= \int_{\mathcal{K}_c^d}\int_{\mathbb{R}^d}\int_{\mathcal{N}(K)} q^{(1)}(x,K,\{\varphi : z \notin T(\varphi)\})\\ &\qquad \times f(x,K)\mathbf{1}\{(z,b)\in\cdot\cap\mathcal{N}_B(x+K)\}\Theta_j(x+K,B;d(z,b))\mathcal{H}^d(dx)\mathbb{Q}(dK)\\ &= \int_{\mathcal{K}_c^d}\int_{\mathbb{R}^d}\int_{\mathcal{N}(K)} q^{(1)}(x,K,\{\varphi : z+x \notin T(\varphi)\})\\ &\qquad \times f(x,K)\mathbf{1}\{(z+x,b)\in\cdot\}\Theta_j(K,B;d(z,b))\mathcal{H}^d(dx)\mathbb{Q}(dK)\\ &= \int_{\mathcal{K}_c^d}\int_{\mathbb{R}^d}\int_{\mathcal{N}(K)} q^{(1)}(y-z,K,\{\varphi : y \notin T(\varphi)\})\\ &\qquad \times f(y-z,K)\mathbf{1}\{(y,b)\in\cdot\}\mathcal{H}^d(dy)\Theta_j(K,B;d(z,b))\mathbb{Q}(dK)\\ &= \int_{\mathbb{R}^d}\int_{\mathcal{K}_c^d}\int_{\mathcal{N}(K)} q^{(1)}(x-z,K,\{\varphi : x \notin T(\varphi)\})\\ &\qquad \times f(x-z,K)\mathbf{1}\{(x,b)\in\cdot\}\Theta_j(K,B;d(z,b))\mathbb{Q}(dK)\mathcal{H}^d(dx)\,.\end{aligned}$$

Since clearly

$$q^{(1)}(x,K,\cdot) = \mathbb{P}_{(x,K)}(\Phi\setminus\delta_{(x,K)}\in\cdot)$$

is satisfied for α-almost all $(x,K)\in\mathbb{R}^d\times\mathcal{K}_c^d$, the assertion follows. □

Examples. 1. If Φ is a Poisson process with intensity measure

$$\alpha(d(x,K)) = f(x,K)\mathcal{H}^d(dx)\mathbb{Q}(dK)$$

as described in Subsection 3.1, then

$$\begin{aligned}&\alpha^{(n)}(d(x_1,K_1,\dots,x_n,K_n))\\ &\qquad = f(x_1,K_1)\dots f(x_n,K_n)\mathcal{H}^d(dx_1)\dots\mathcal{H}^d(dx_n)\mathbb{Q}(dK_1)\dots\mathbb{Q}(dK_n)\,.\end{aligned}$$

Thus the assumptions of Theorem 3.16 are satisfied.

2. Let Φ be a Gibbs process with local energy function $-\ln\lambda$ as described in [76] (see also the literature cited there). Then it follows by a repeated application of equation (4.23) in [76] that

$$\begin{aligned}\alpha^{(n)}(\cdot) = \int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}\int_{\mathcal{K}_c^d}\dots\int_{\mathcal{K}_c^d} &\mathbf{1}\{(x_1,K_1,\dots,x_n,K_n)\in\cdot\}f_n(x_1,K_1,\dots,x_n,K_n)\\ &\times\mathcal{H}^d(dx_1)\dots\mathcal{H}^d(dx_n)\mathbb{Q}(dK_1)\dots\mathbb{Q}(dK_n)\,,\end{aligned}$$

where

$$\begin{aligned}f_n(x_1,K_1,\dots,x_n,K_n) = \mathbb{E}\Big[&\lambda\left(\Phi+\delta_{(x_n,K_n)}+\dots+\delta_{(x_2,K_2)},x_1,K_1\right)\\ &\times\lambda\left(\Phi+\delta_{(x_n,K_n)}+\dots+\delta_{(x_3,K_3)},x_2,K_2\right)\dots\\ &\dots\lambda\left(\Phi+\delta_{(x_n,K_n)},x_{n-1},K_{n-1}\right)\lambda\left(\Phi,x_n,K_n\right)\Big]\,.\end{aligned}$$

Thus the assumptions of Theorem 3.16 are satisfied whenever $f_n < \infty$ for all $n \in \mathbb{N}$.

Instead of considering a strictly convex body, we can also investigate the case of a general convex body $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$ in a similar way as in Subsection 3.1. (Presumably, it is sufficient to assume that $o \in B$, but the details remain to be checked.) Indeed, first one checks that the relative support measures can be extended to the local convex ring $\mathcal{S}^d$ in the following sense. Let $(K, B) \in \mathcal{S}^d \times \mathcal{K}^d$ and assume that K has a representation

$$K = \bigcup_{i=1}^{\infty} K_i$$

such that

$$(K_v, B) \in \mathcal{K}^d_{2,gp} \tag{164}$$

for all $v \in S(\mathbb{N})$, where $S(\mathbb{N})$ denotes the set of all non-empty finite subsets of $\mathbb{N}$ and, for $v \in S(\mathbb{N})$,

$$K_v := \bigcap_{i \in v} K_i \,.$$

Then we can define $\Theta_j(K, B; \cdot)$ and $\Theta_j^+(K, B; \cdot)$ as in [76], where now we only work with representations of K which satisfy (164). In the same way, the non-negative extensions of the relative support measures and Theorem 3.4 in [76] can be extended.

In order to obtain an appropriate extension of Lemma 3.15, we have to use Theorem 3.3 in [76] in the present generality, and therefore we need an extension of Theorem 3.2. Unfortunately, we do not know whether such a deterministic result is true. However, by the arguments of Subsection 3.1 it is easy to see that if Φ is a Poisson process (as in the proof of Lemma 3.15), then the exoskeleton $\operatorname{exo}_B(\Xi(\omega))$ of $\Xi(\omega)$ with respect to B has d-dimensional Hausdorff measure zero for $\mathbb{P}$ almost all $\omega \in \Omega$. This is sufficient for the proof of the required extension.

Moreover, under the assumptions of Theorem 3.16 and for $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$, we find (as in Subsection 3.1) that for ν almost all $\rho \in \mathbf{SO}(d)$, and for $\mathbb{P}$ almost all $\omega \in \Omega$, the random set

$$\Xi(\omega) = \bigcup_{i=1}^{\infty} \Xi_i(\omega)$$

satisfies

$$(\Xi_v, \rho B) \in \mathcal{K}^d_{2,gp}$$

for all $v \in S(\mathbb{N})$.

These considerations lead to the following supplement to Theorem 3.16.

Theorem 3.17 *Let Φ be a point process in $\mathbb{R}^d \times \mathcal{K}^d_c$ such that its n-th order factorial moment measure satisfies conditions* (152) *and* (163) *for all $n \in \mathbb{N}$. Let $B \in \mathcal{K}^d$ with $o \in \operatorname{int} B$, and let $A \subset \mathbb{R}^d$ be Borel measurable. Then, for ν almost all $\rho \in \mathbf{SO}(d)$ and $j = 0, \ldots, d-1$, $\Lambda_j^+(\rho B; \cdot \times A)$ is absolutely continuous with density $\lambda_j^+(\rho B; x, A)$.*

A combination of results of Subsections 3.1 and 3.5 can now be used to continue the investigation of conditions under which the surface density $\lambda_{d-1}^{+}(B; x, A)$ is the derivative of the contact distribution

$$(1 - p(x))H_B(x, t, A) = \mathbb{P}(d_B(\Xi, x) \leq t, u_B(\Xi, x) \in A, x \notin \Xi)\,;$$

compare [76] and [75]. Under the assumption of stationarity and for $A = \mathbb{R}^d$, a study of second order derivatives has recently been initiated by Last and Schassberger [93]. Finally, we wish to point out that the deterministic translative integral formula of Subsection 3.4 can be used to obtain immediate extensions of various results in [41], [150], [152] to the relative setting.

4 Mixed volumes and measures of convex bodies

A central subject in convex geometry is the investigation of the behaviour of various functionals of convex bodies with respect to basic operations for convex bodies. Perhaps, the most important functional is the *volume*, and a fundamental operation is the formation of *Minkowski sums*. The combination of these two concepts naturally leads to the theory of *mixed volumes* which represents the backbone of classical convexity theory. But the importance and usefulness of mixed volumes can only be appreciated properly if one also observes that mixed volumes provide a link to and represent an essential tool for other areas such as combinatorics, algebraic geometry or stochastic geometry.

Mixed volumes are introduced as coefficients of the homogeneous polynomial in $\lambda_1, \dots, \lambda_k \geq 0$ as which the d-dimensional volume

$$V_d(\lambda_1 K_1 + \ldots + \lambda_k K_k)$$

of the Minkowski combination $\lambda_1 K_1 + \ldots + \lambda_k K_k$, for convex bodies $K_1, \dots, K_k \in \mathcal{K}^d$, can be expressed; see [123] for an introduction to this subject and for the notation used here. The validity of the polynomial expansion

$$V_d(\lambda_1 K_1 + \ldots + \lambda_k K_k) = \sum_{\alpha_1, \dots, \alpha_k = 0}^{d} \binom{d}{\alpha_1 \ldots \alpha_k} \lambda_1^{\alpha_1} \ldots \lambda_k^{\alpha_k} V(K_1[\alpha_1], \dots, K_k[\alpha_k])$$

is established first for polytopes by a recursive argument, and then the general case is deduced by approximation. This argument also shows that the mixed volume of $K_1, \dots, K_d \in \mathcal{K}^d$ can be defined by

$$V(K_1, \dots, K_d) := \frac{1}{d!} \sum_{j=1}^{d} (-1)^{d+j} \sum_{1 \leq i_1 < \ldots < i_j \leq d} V_d(K_{i_1} + \ldots + K_{i_j})\,. \tag{165}$$

Clearly, such an approach to mixed volumes only requires basic results about convex bodies and it already yields simple proofs of various properties of mixed volumes. On the other hand, the definition (165) of mixed volumes is rather unsatisfactory from a computational point of view, and it provides little insight into how the interaction between different convex bodies $K_1, \dots, K_k \in \mathcal{K}^d$ influences the numerical value of the mixed volume

$$V(K_1[\alpha_1], \dots, K_k[\alpha_k])\,,$$

where $k \geq 2$, $\alpha_1, \dots, \alpha_k \in \{0, \dots, d\}$ and $\alpha_1 + \ldots + \alpha_k = d$. For convex polytopes, these drawbacks of the classical approaches to and of the formerly known representations of mixed volumes are avoided by a recent result of Schneider; see [126], [130], and [16] for a special case. This result has been discussed repeatedly in the context of computational convexity (see [68], [56], [139], [66], and [38]). Schneider's summation formula, as it was called in [66], shows that, for convex polytopes $K_1, \dots, K_k \in \mathcal{K}^d$,

$$\binom{d}{\alpha_1 \ldots \alpha_k} V(K_1[\alpha_1], \dots, K_k[\alpha_k]) = \sum_{(F_1, \dots, F_k)} [F_1, \dots, F_k] V_{\alpha_1}(F_1) \ldots V_{\alpha_k}(F_k)\,, \tag{166}$$

where the summation extends over certain faces $F_i \in \mathcal{F}_{\alpha_i}(K_i)$, $i = 1, \dots, k$, which are chosen according to an explicit summation rule which is based on specific information about the normal cones of the faces. The number $[F_1, \ldots, F_k]$ is just the volume of the parallelepiped that is the sum of the unit cubes in the affine hulls of $F_1, \ldots, F_k$, and thus this factor takes account of the relative position of the faces selected. Moreover, the α_i-dimensional volume of F_i is denoted by $V_{\alpha_i}(F_i)$. The expression on the right-hand side of (166) is symmetric in $K_1, \ldots, K_k$ and it contains only non-negative contributions. Recently, Schneider extended the construction described in [126] to more general functionals which are induced by a surjective linear transformation $f : (\mathbb{R}^d)^k \to (\mathbb{R}^d)^r$, where $k > r \geq 1$. In such a framework, representations of mixed volumes and of special total mixed curvature measures are derived for convex polytopes as special instances of a general result (see [130]). The main objective of this section is to establish extensions of such representations of mixed volumes and of general mixed curvature measures for arbitrary convex bodies. Then, of course, the summation which appears on the right-hand side of (166) will have to be replaced by an integration, and also the normal bundles of the convex bodies are naturally involved. In addition, curvatures on the normal bundles of the convex bodies involved appear in such integral representations. Thus, local geometric characteristics of convex bodies are related to the values of non-local mixed functionals.

For general convex bodies $K_1, \ldots, K_d \in \mathcal{K}^d$, one has a non-symmetric representation of a mixed volume as an integral over the unit sphere with respect to a mixed surface area measure, that is

$$V(K_1, \ldots, K_d) = \frac{1}{d} \int_{S^{d-1}} h(K_1, u) S(K_2, \ldots, K_d, du) . \tag{167}$$

Although this representation is particularly useful for many purposes and may be specified for convex bodies $K_2, \ldots, K_d$ with support functions of class C^2, it does not provide a description of the type we are aiming at, since for the mixed surface area measures no explicit description is available so far in the case of general convex bodies. On the other hand, equation (167) was used by Weil [147] as the starting point for a description of mixed volumes and mixed surface area measures as distributions (applied to tensor products of support functions). A corresponding representation of the mixed volume $V(K_1, \ldots, K_d)$ as a limit of d-fold integrals over $(S^{d-1})^d$ of the products of the support functions $h(K_1, \cdot), \ldots, h(K_d, \cdot)$ also involves certain functions f_k, $k \in \mathbb{N}$, of class C^∞ which change signs and which therefore are difficult to interpret geometrically.

A surprisingly simple expression for the mixed volume of convex bodies $K_1, \ldots, K_d$ was recently established by Alesker, Dar and Milman [5]. They show that

$$V(K_1, \ldots, K_d) = \int_{\mathbb{R}^d} D(d^2 f_1(x), \ldots, d^2 f_d(x)) \mathcal{H}^d(dx) , \tag{168}$$

where $f_1, \ldots, f_d$ are certain convex functions of class C^2 which are associated with $K_1, \ldots, K_d$ and $D(\cdot)$ is the mixed discriminant of the Hessian matrices $d^2 f_i(x)$, $i = 1, \ldots, d$. Indeed, the required smoothness of the functions f_i follows from Caffarelli's delicate regularity theory for the Brenier map. The expression on the right-hand side of (168) is symmetric, non-negative and it can be used to verify a very special case of the Aleksandrov-Fenchel inequality essentially by reducing the proof to a local calculation for special mixed discriminants. But even for very special convex bodies, it is not clear how the associated functions look like, since effectively they are obtained by an approximation argument. Moreover, for a given convex body K_i the construction of the function f_i does not seem to be canonical.

For the derivation of our new representation of mixed volumes we follow an idea of Schneider [126] (compare also the lifting theorem of Walkup and Wets) and first consider the orthogonal projection $p_L : \mathbb{R}^{kd} \to L$ (for $k \geq 2$), where $L \subset \mathbb{R}^{kd}$ is the "diagonal" d-dimensional linear subspace of $\mathbb{R}^d \times \ldots \times \mathbb{R}^d$ (k times). Thus we find that $V_d(K_1 + \ldots + K_k)$ can be expressed as the volume of the orthogonal projection of the cartesian product $K_1 \times \ldots \times K_k \subset \mathbb{R}^{kd}$ onto L. Now the volume $\mathcal{H}^d(K|L)$ of the projection $K|L$ of an arbitrary convex body $K \in \mathcal{K}^p$ onto a d-dimensional linear subspace $L \subset \mathbb{R}^p$ can be described as a very special mixed volume of two convex bodies. Moreover, this particular mixed volume can be represented as an integral over the normal bundle of K. To prove this, we use the close relationship between mixed volumes and mixed curvature measures for two convex bodies and exploit the fact that for the mixed curvature measures of two convex bodies a suitable integral representation, on which our argument is based, is provided in [110]. The integral representations for *projection functions* which are thus established may be of some interest for other purposes as well. It is a surprising feature of this approach that our derivation of a representation for mixed volumes essentially relies on notions and tools of translative integral geometry.

The final step then consists in transforming an integral over the normal bundle of the product $K_1 \times \ldots \times K_k$ into an integral over the product of the normal bundles of $K_1, \ldots, K_k$. To be able to carry out this transformation, we first derive a current representation for the volumes of projections of convex bodies onto linear subspaces, that is for projection functions on Grassmannians. In fact, this will enable us to avoid the explicit use of generalized curvatures on the normal bundle of $K_1 \times \ldots \times K_k$; instead we can use the area formula for currents which is much more convenient.

Furthermore, we shall establish similar representations for mixed surface area measures (involving a limit) and for general mixed curvature measures. The mixed surface area measures

$$S(K_1[\alpha_1], \ldots, K_k[\alpha_k], \cdot) ,$$

for convex bodies $K_1, \ldots, K_k \in \mathcal{K}^d$ and $\alpha_1, \ldots, \alpha_k \in \{0, \ldots, d-1\}$ with $\alpha_1 + \ldots + \alpha_k = d-1$, are intimately related to mixed volumes via the Riesz representation theorem, as equation (167) suggests. But these Borel measures over S^{d-1} can also be obtained by the polynomial expansion for the $(d-1)$st surface area measure of the Minkowski combinations of convex bodies $K_1, \ldots, K_k \in \mathcal{K}^d$. In fact, for $\lambda_1, \ldots, \lambda_k \geq 0$ we have

$$S_{d-1}(\lambda_1 K_1 + \ldots + \lambda_k K_k, \cdot) = \sum_{\alpha_1, \ldots, \alpha_k = 0}^{d-1} \binom{d-1}{\alpha_1 \ldots \alpha_k} \lambda_1^{\alpha_1} \ldots \lambda_k^{\alpha_k} S(K_1[\alpha_1], \ldots, K_k[\alpha_k], \cdot) ,$$

and, for $K_1, \ldots, K_{d-1} \in \mathcal{K}^d$,

$$S(K_1, \ldots, K_{d-1}, \cdot) = \frac{1}{(d-1)!} \sum_{j=1}^{d-1} (-1)^{d-1+j} \sum_{1 \leq i_1 < \ldots < i_j \leq d-1} S_{d-1}(K_{i_1} + \ldots + K_{i_j}, \cdot) ,$$

in correspondence to the analogous relations for mixed volumes. In particular, mixed volumes and mixed surface area measures both are based on the Minkowski addition as the underlying operation for convex bodies.

The mixed curvature measures

$$C_{r_1, \ldots, r_q}(K_1, \ldots, K_q; \cdot) ,$$

for $K_1, \dots, K_q \in \mathcal{K}^d$ and $r_1, \dots, r_q \in \{0, \dots, d\}$ with $r_1 + \dots + r_q \geq (q-1)d$, are Borel measures over $\mathbb{R}^{(q+1)d}$, which arise by forming intersections $K_1 \cap (K_2 + z_2) \cap \dots \cap (K_q + z_q)$, $z_2, \dots, z_q \in \mathbb{R}^d$, of translates of convex bodies, by considering the kth support measure of such intersections, and by subsequently averaging over all translations. More precisely, the defining relationship for the mixed curvature measures is the translative integral formula

$$\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \int h(x, x - z_2, \dots, x - z_q, u) C_k(\underline{K}(\underline{z}), d(x,u)) \mathcal{H}^d(dz_q) \dots \mathcal{H}^d(dz_2)$$

$$= \sum_{\substack{0 \leq r_1, \dots, r_q \leq d \\ r_1 + \dots + r_q = (q-1)d + k}} \int h(x_1, \dots, x_k, u) C_{r_1, \dots, r_q}(K_1, \dots, K_q; d(x_1, \dots, x_q, u)),$$

where

$$\underline{K}(\underline{z}) := K_1 \cap (K_2 + z_2) \cap \dots \cap (K_q + z_q),$$

$k \in \{0, \dots, d-1\}$, and $h : \mathbb{R}^{(q+1)d} \to \mathbb{R}$ is an arbitrary non-negative Borel measurable function. This iterated integral formula was first proved by Schneider & Weil [131] for $q = 2$ and by Weil [150] for $q \geq 2$ (in a slightly less general form, respectively) and later by Rataj [108] in the setting of sets with positive reach. An extension to relative curvature measures has been obtained in Section 3 of the present work. For polytopes these mixed curvature measures can be described in a simple and explicit way, but for arbitrary convex bodies and $q \geq 3$ only a current representation was available so far (see [108]). For the mixed curvature measures of two convex bodies, however, an integral formula has already been proved in [110], which we now extend to a finite sequence of convex bodies.

More research, however, will be required to obtain a complete picture of the various relationships (for instance via Crofton formulae) between these mixed functionals and measures, and to understand their appearance in the context of geometric inequalities; compare [151], [130], [55].

4.1 Preliminaries

In this section, we shall consider various Euclidean spaces. The basic space, in which we are interested, will be denoted by $\mathbb{R}^d$, $d \geq 2$. Scalar products and norms will always be denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively, independent of the particular dimension. For simplicity, we shall assume that $\langle \cdot, \cdot \rangle$ is the standard scalar product, but of course this specific choice can be avoided. In particular, we shall investigate cartesian products such as $\mathbb{R}^d \times \dots \times \mathbb{R}^d$ (k factors) for which we also write $\mathbb{R}^{kd}$. In this case, we endow each factor with the same scalar product, and the cartesian product will carry the natural scalar product which is derived from its components by summation. In view of their importance for the subsequent investigation, we shall recall some basic notions of convexity, which we have already used repeatedly in the preceding sections.

Let $\mathbb{R}^p$, $p \geq 2$, be a given Euclidean space, and let $K \in \mathcal{K}^p_o$ be a convex body in $\mathbb{R}^p$ with non-empty interior. Later we shall remove the assumption int $K \neq \emptyset$, but for the moment we preserve this hypothesis. The following definitions are well-known:

(i) *Support function:* For $u \in \mathbb{R}^d$,

$$h(K, u) := \max\{\langle x, u \rangle : x \in K\};$$

(ii) *Boundary:* bd K is the topological boundary of K;

(iii) *Normal cone:* For $x \in K$,

$$N(K,x) := \{u \in \mathbb{R}^p : \langle x,u\rangle = h(K,u)\}\,;$$

in particular, we always have $o \in N(K,x)$, $N(K,x) = \{o\}$ if $x \in \text{int } K$, and $\dim N(K,x) \geq 1$ if $x \in \text{bd } K$;

(iv) *Normal bundle:*

$$\mathcal{N}(K) := \{(x,u) \in \mathbb{R}^p \times S^{p-1} : x \in K, u \in N(K,x)\}\,;$$

hence $(x,u) \in \mathcal{N}(K)$ implies that $x \in \text{bd } K$.

These definitions will now be applied to the special situation where $p := k \cdot d$, $d \geq 2$, and $k \geq 1$. Let $K_i \in \mathcal{K}_o^d$, for $i = 1, \ldots, k$; then $K_1 \times \ldots \times K_k \in \mathcal{K}_o^p = \mathcal{K}_o^{kd}$. In particular, if the details are clear from the context, then we simply write $\underline{K}$ for $K_1 \times \ldots \times K_k$ and $\underline{x}$ for $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in \mathbb{R}^d$. Using these conventions, for the notions recalled in (i) – (iv) we thus obtain:

(i):

$$h(\underline{K},\underline{u}) = \sum_{i=1}^{k} h(K_i,u_i)\,.$$

(ii):

$$\begin{aligned}\text{bd } \underline{K} &= \{\underline{x} \in \underline{K} : x_i \in \text{bd } K_i \text{ for some } i \in \{1,\ldots,k\}\}\\ &= \bigcup_{j=1}^{k} \bigcup_{|\mathcal{I}|=j} \{\underline{x} \in \underline{K} : x_i \in \text{bd } K_i \text{ for } i \in \mathcal{I}, x_i \in \text{int } K_i \text{ for } i \notin \mathcal{I}\}\,,\end{aligned}$$

where $\mathcal{I}$ ranges over all subsets of $\{1, \ldots, k\}$ of cardinality j.

(iii): For $\underline{x} \in \underline{K}$,

$$N(\underline{K},\underline{x}) = \prod_{i=1}^{k} N(K_i,x_i)\,.$$

To see this, note that $\langle x_i,u_i\rangle \leq h(K_i,u_i)$ for $i = 1, \ldots, k$, since $x_i \in K_i$; hence

$$\langle \underline{x},\underline{u}\rangle = \sum_{i=1}^{k}\langle x_i,u_i\rangle \leq \sum_{i=1}^{k} h(K_i,u_i) = h(\underline{K},\underline{u})\,.$$

Equality is satisfied if and only if $\langle x_i,u_i\rangle = h(K_i,u_i)$ for all $i = 1, \ldots, k$. Thus we deduce

$$\begin{aligned}N(\underline{K},\underline{x}) &= \left\{\underline{u} \in \mathbb{R}^{kd} : \langle \underline{x},\underline{u}\rangle = h(\underline{K},\underline{u})\right\}\\ &= \left\{\underline{u} \in \mathbb{R}^{kd} : \langle x_i,u_i\rangle = h(K_i,u_i) \text{ for all } i \in \{1,\ldots,k\}\right\}\\ &= \prod_{i=1}^{k} N(K_i,x_i)\,.\end{aligned}$$

(iv): Let $\mathcal{I} \subset \{1,\ldots,k\} \setminus \{\emptyset\}$; then we define $\mathcal{N}_{\mathcal{I}}(\underline{K})$ as the set of all $(\underline{x},\underline{u}) \in \mathbb{R}^{kd} \times S^{kd-1}$ for which

$$x_i \in \begin{cases} \text{bd } K_i\,, & i \in \mathcal{I}\,, \\ \text{int } K_i\,, & i \notin \mathcal{I}\,, \end{cases} \qquad \text{and} \qquad u_i \in \begin{cases} N(K_i,x_i)\,, & i \in \mathcal{I}\,, \\ \{o\}\,, & i \notin \mathcal{I}\,. \end{cases}$$

Then we obtain

$$\begin{aligned} \mathcal{N}(\underline{K}) &= \left\{(\underline{x},\underline{u}) \in \mathbb{R}^{kd} \times S^{kd-1} : \underline{x} \in \underline{K}, \underline{u} \in N(\underline{K},\underline{x})\right\} \\ &= \left\{(\underline{x},\underline{u}) \in \mathbb{R}^{kd} \times S^{kd-1} : \underline{x} \in \underline{K}, u_i \in N(K_i,x_i) \text{ for } i = 1,\ldots,k\right\} \\ &= \bigcup_{j=1}^{k} \bigcup_{|\mathcal{I}|=j} \mathcal{N}_{\mathcal{I}}(\underline{K})\,. \end{aligned}$$

Subsequently, we shall repeatedly refer to the d-dimensional linear subspace $L \subset \mathbb{R}^{kd}$ which is defined by

$$L := \left\{(x_1,\ldots,x_k) \in \mathbb{R}^{kd} : x_1 = \ldots = x_k \in \mathbb{R}^d\right\}\,; \tag{169}$$

the integer k will always be clear from the context. We write $e_1,\ldots,e_d$ for the standard basis of $\mathbb{R}^d$. Then the mapping $\varphi : \mathbb{R}^d \to L \subset \mathbb{R}^{kd}$, $x \mapsto (x,\ldots,x)$ is an isomorphism. An orthonormal basis of L is given by

$$\underline{e_i} := \frac{1}{\sqrt{k}}(e_i,\ldots,e_i)\,, \quad i \in \{1,\ldots,d\}\,;$$

moreover,

$$L^{\perp} = \left\{(x_1,\ldots,x_k) \in \mathbb{R}^{kd} : \sum_{i=1}^{k} x_i = o\right\}$$

and dim $L^{\perp} = (k-1)d$. Let $p_L : \mathbb{R}^{kd} \to L$ denote the orthogonal projection onto L. Then we obtain

$$p_L(\underline{x}) = \frac{1}{k}\left(\sum_{j=1}^{k} x_j,\ldots,\sum_{j=1}^{k} x_j\right)$$

and

$$\begin{aligned} |p_{L^{\perp}}(\underline{x})|^2 = |\underline{x}|^2 - |p_L(\underline{x})|^2 &= \sum_{i=1}^{k} |x_i|^2 - \frac{1}{k}\left|\sum_{j=1}^{k} x_j\right|^2 \\ &= \frac{1}{k}\sum_{1 \le i < j \le k} |x_i - x_j|^2\,. \end{aligned}$$

The inverse of φ is given by $\pi : L \to \mathbb{R}^d$, $(x,\ldots,x) \mapsto x$, and $\sqrt{k}\pi$ is an isometry with respect to the scalar product which L inherits as a subspace of $\mathbb{R}^{kd}$. Therefore we deduce that

$$V_d(K_1 + \ldots + K_k) = k^{\frac{d}{2}}\mathcal{H}^d\left(p_L(K_1 \times \ldots \times K_k)\right)\,, \tag{170}$$

where $K_1, \ldots, K_k \in \mathcal{K}^d$ and V denotes the d-dimensional volume.

Next we choose any Borel set $\omega \subset S^{d-1}$ and set $\underline{\omega} := k^{-1/2}\varphi(\omega)$. Then $\underline{\omega}$ is a Borel subset of $S^{kd-1} \cap L$ and

$$S_{d-1}(K_1 + \ldots + K_k, \omega) = k^{\frac{d-1}{2}} S^L_{d-1}\left(p_L(K_1 \times \ldots \times K_k), \underline{\omega}\right) . \tag{171}$$

4.2 Volume and surface area measures of projections

The main objective of this subsection is to provide some auxiliary results on which the subsequent investigation is based.

We start with some remarks concerning notation. Let $p \in \mathbb{N}$, $p \geq 2$, and $d \in \{1, \ldots, p-1\}$. The bracket $[\cdot, \cdot]$ will be defined as in [110] for pairs of multi-vectors in R^p, which is consistent with our previous usage. For example, if $a_1, \ldots, a_p \in \mathbb{R}^p$ are linearly independent, $I \subset \{1, \ldots, p\}$ with $|I| = d$ and $L \in \mathbf{G}(p, d)$, then we write

$$\left[\bigwedge_I a_i, L^\perp\right] \tag{172}$$

for $[W, L^\perp]$, where W is the linear subspace which is associated with $\alpha := \wedge_{i \in I} a_i$. Alternatively, we can choose any simple unit $(p-d)$-vector β which is associated with $L^\perp$ and define (172) as $[\alpha, \beta]$; compare p. 260 in [110] and Chapter 1 in [42]. Furthermore, we write $\theta(m, n) \in (0, \pi)$ for the angle between two unit vectors m, n with $m \neq \pm n$.

In order to simplify our notation we occasionally omit arguments of functions; so we write k_i instead of $k_i(x, u) = k_i(K; x, u)$ for the generalized curvatures of a convex body K, if the convex body and the argument are clear from the context. A similar remark applies to the associated eigenvectors (see also Section 1), which now are denoted by a_i instead of the more precise notation $a_i(x, u)$ or even $a_i(K; x, u)$. In addition, we introduce the abbreviation

$$\mathbb{K} := \prod_{i=1}^{d-1} \sqrt{1 + k_i^2} \,,$$

where again the argument is usually omitted. Finally, we often write $\wedge_I a_i$ instead of $\wedge_{i \in I} a_i$ and a similar remark applies, e.g., to the formation of products.

Proposition 4.1 *Let $K \in \mathcal{K}^p$, $d \in \{1, \ldots, p-1\}$, and $L \in \mathbf{G}(p, d)$. Then*

$$\mathcal{H}^d(K|L) = \frac{1}{\omega_{p-d}} \int_{\mathcal{N}(K)} |p_{L^\perp}(u)|^{d-p} \sum_{|I|=d} \frac{\prod_{I^c} k_i(x, u)}{\mathbb{K}(x, u)} \left[\bigwedge_I a_i(x, u), L^\perp\right]^2 \mathcal{H}^{p-1}(d(x, u)) \,,$$

where I ranges over all subsets of $\{1, \ldots, p-1\}$ of cardinality d and $I^c := \{1, \ldots, p-1\} \setminus I$.

Proof. We set $U := B^p \cap L^\perp$. By [123], p. 294, and equation (4) in [110], we obtain

$$\begin{aligned}\mathcal{H}^d(K|L) &= \frac{\binom{p}{d}}{\kappa_{p-d}} V(K[d], U[p-d]) \\ &= \frac{1}{\kappa_{p-d}} \overline{\Psi}_{d,p-d}(K, U; \mathbb{R}^{2p}) \,,\end{aligned}$$

where $\overline{\Psi}_{d,p-d}(K,U;\cdot)$ is a particular mixed curvature measure of K and U; see [110] for further details. Hence we deduce from Theorem 2 in [110] that

$$\begin{aligned}\mathcal{H}^d(K|L) =& \frac{1}{\kappa_{p-d}} \int_{\mathcal{N}(K)} \int_{L^\perp} \int_{S^{p-1}\cap L} F_{d,p-d}(\theta(u,n)) \mathbf{1}\{y \in U\} \\ & \times \sum_{|I|=d} \frac{\prod_{I^c} k_i(x,u)}{\mathbb{K}(x,u)} \left[\bigwedge_I a_i(x,u), L^\perp \right]^2 \mathcal{H}^{p-1}(d(x,u)) \mathcal{H}^{p-d}(dy) \mathcal{H}^{d-1}(dn) \quad (173) \\ =& \int_{\mathcal{N}(K)} \sum_{|I|=d} \frac{\prod_{I^c} k_i(x,u)}{\mathbb{K}(x,u)} \left[\bigwedge_I a_i(x,u), L^\perp \right]^2 \\ & \times \int_{S^{p-1}\cap L} F_{d,p-d}(\theta(u,n)) \mathcal{H}^{d-1}(dn) \mathcal{H}^{p-1}(d(x,u)) \,, \quad (174)\end{aligned}$$

where

$$F_{d,p-d}(\theta) = \frac{1}{\omega_p} \frac{\theta}{\sin\theta} \int_0^1 \left(\frac{\sin(1-t)\theta}{\sin\theta} \right)^{p-1-d} \left(\frac{\sin t\theta}{\sin\theta} \right)^{d-1} dt \,.$$

Note that a boundary term in (173), which comes from $(y,n) \in \mathcal{N}(U)$ with $y \in \operatorname{relbd} U$, vanishes, since for $\mathcal{H}^{p-1}$ almost all such (y,n) precisely d of the curvatures $k_i(U;y,n)$ are infinite. Furthermore, if $u = \pm n$, then $[\,\wedge_I a_i(x,u), L^\perp] = 0$; in this case the integrand is defined to be zero and thus $F_{d,p-d}(\theta)$ need not be defined for $\theta \in \{0,\pi\}$.

An application of equation (4.1) in [109] shows that

$$\int_{S^{p-1}\cap L} F_{d,p-d}(\theta(u,n)) \mathcal{H}^{d-1}(dn) = \frac{1}{\omega_{p-d}} |p_{L^\perp}(u)|^{d-p} \,, \quad (175)$$

and thus the assertion is implied by (174) and (175). □

The result of the preceding proposition can be extended to a representation of the intrinsic volumes $V_r(K|L)$, $r \in \{0,\dots,d\}$, of the orthogonal projection $K|L$ of a convex body K onto a linear subspace L.

Proposition 4.2 *Let $K \in \mathcal{K}^p$, $d \in \{1,\dots,p-1\}$, $L \in \mathbf{G}(p,d)$ and $r \in \{0,\dots,d\}$. Then*

$$\begin{aligned}V_r(K|L) =& \frac{1}{\kappa_{d-r}\omega_{p-d}} \int_{\mathcal{N}(K)} |p_{L^\perp}(u)|^{d-p} \sum_{\substack{(N,J): J \subset N^c \\ |N|=r, |J|=d-r}} \frac{\prod_{N^c} k_i(x,u)}{\mathbb{K}(x,u)} \\ & \times \left[\bigwedge_{N\cup J} a_i(x,u), L^\perp \right]^2 \mathcal{H}^{p-1}(d(x,u)) \,,\end{aligned}$$

where the summation extends over all subsets $N, J \subset \{1,\dots,p-1\}$ fulfilling the prescribed restrictions.

Proof. We derive two representations for the polynomial expansion of $\mathcal{H}^d((K+\epsilon B^p)|L)$, where $\epsilon > 0$. A comparison of coefficients then yields the desired equality. First, for any $\epsilon > 0$ we have

$$\mathcal{H}^d((K+\epsilon B^p)|L) = \mathcal{H}^d(K|L+\epsilon B^p|L) = \sum_{r=0}^{d} \epsilon^{d-r}\kappa_{d-r}V_r(K|L)\,.$$

On the other hand, for any $\epsilon > 0$ Proposition 4.1 yields that

$$\begin{aligned}
\mathcal{H}^d((K+\epsilon B^p)|L) =& \frac{1}{\omega_{p-d}} \int_{\mathcal{N}(K+\epsilon B^p)} |p_{L^\perp}(u)|^{d-p} \sum_{|I|=d} \frac{\prod_{I^c} k_i(K+\epsilon B^p; y,u)}{\mathbb{K}(K+\epsilon B^p; y,u)} \\
&\times \left[\bigwedge_I a_i(K+\epsilon B^p; y,u), L^\perp\right]^2 \mathcal{H}^{p-1}(d(y,u)) \\
=& \frac{1}{\omega_{p-d}} \int_{\mathcal{N}(K)} \mathrm{ap}\ J_{p-1}F_\epsilon(x,u)|p_{L^\perp}(u)|^{d-p} \sum_{|I|=d} \frac{\prod_{I^c} k_i(K+\epsilon B^p; F_\epsilon(x,u))}{\mathbb{K}(K+\epsilon B^p; F_\epsilon(x,u))} \\
&\times \left[\bigwedge_I a_i(K+\epsilon B^p; F_\epsilon(x,u)), L^\perp\right]^2 \mathcal{H}^{p-1}(d(x,u))\,,
\end{aligned}$$

where

$$F_\epsilon : \mathcal{N}(K) \to \mathcal{N}(K+\epsilon B^p)\,, \quad (x,u) \mapsto (x+\epsilon u, u)\,.$$

It is easy to check that, for $\mathcal{H}^{p-1}$ almost all $(x,u) \in \mathcal{N}(K)$,

$$a_i(K+\epsilon B^p; F_\epsilon(x,u)) = a_i(K;x,u)\,,$$

$$k_i(K+\epsilon B^p; F_\epsilon(x,u)) = \frac{k_i(K;x,u)}{1+\epsilon k_i(K;x,u)}\,,$$

(if the ordering is chosen properly)

$$\mathbb{K}(K+\epsilon B^p; F_\epsilon(x,u)) = \prod_{i=1}^{p-1} \frac{\sqrt{(1+\epsilon k_i(K;x,u))^2 + k_i(K;x,u)^2}}{1+\epsilon k_i(K;x,u)}$$

and

$$\mathrm{ap}\ J_{p-1}F_\epsilon(x,u) = \prod_{i=1}^{p-1} \left\{ \frac{(1+\epsilon k_i(K;x,u))^2 + k_i(K;x,u)^2}{1+k_i(K;x,u)^2} \right\}^{\frac{1}{2}}\,.$$

Hence, by the coarea formula, we obtain

$$
\begin{aligned}
\mathcal{H}^d((K+\epsilon B^p)|L) =& \frac{1}{\omega_{p-d}} \int_{\mathcal{N}(K)} \left(\prod_{i=1}^{p-1} \frac{1+\epsilon k_i(K;x,u)}{\sqrt{1+k_i(K;x,u)^2}} \right) |p_{L^\perp}(u)|^{d-p} \\
& \times \sum_{|I|=d} \left(\prod_{I^c} \frac{k_i(K;x,u)}{1+\epsilon k_i(K;x,u)} \right) \left[\bigwedge_I a_i(K;x,u), L^\perp \right]^2 \mathcal{H}^{p-1}(d(x,u)) \\
=& \frac{1}{\omega_{p-d}} \int_{\mathcal{N}(K)} |p_{L^\perp}(u)|^{d-p} \sum_{|I|=d} \frac{\{\prod_I (1+\epsilon k_i(K;x,u))\} \{\prod_{I^c} k_i(K;x,u)\}}{\prod_{i=1}^{p-1} \sqrt{1+k_i(K;x,u)^2}} \\
& \times \left[\bigwedge_I a_i(K;x,u), L^\perp \right]^2 \mathcal{H}^{p-1}(d(x,u)) \\
=& \frac{1}{\omega_{p-d}} \sum_{r=0}^{d} \epsilon^{d-r} \int_{\mathcal{N}(K)} |p_{L^\perp}(u)|^{d-p} \sum_{\substack{(M,J), J\subset M \\ |J|=d-r, |M|=p-1-r}} \frac{\prod_M k_i(K;x,u)}{\mathbb{K}(K;x,u)} \\
& \times \left[\bigwedge_{M^c \cup J} a_i(K;x,u), L^\perp \right]^2 \mathcal{H}^{p-1}(d(x,u)) \\
=& \frac{1}{\omega_{p-d}} \sum_{r=0}^{d} \epsilon^{d-r} \int_{\mathcal{N}(K)} |p_{L^\perp}(u)|^{d-p} \sum_{\substack{(N,J), J\subset N^c \\ |J|=d-r, |N|=r}} \frac{\prod_{N^c} k_i(K;x,u)}{\mathbb{K}(K;x,u)} \\
& \times \left[\bigwedge_{N \cup J} a_i(K;x,u), L^\perp \right]^2 \mathcal{H}^{p-1}(d(x,u)) \,,
\end{aligned}
$$

which clearly completes the proof. □

To illustrate the result of Proposition 4.2, we first consider the special case $r = d-1$, where we have

$$
\begin{aligned}
S^L(K|L) &= 2V_{d-1}(K|L) \\
&= \frac{1}{\omega_{p-d}} \int_{\mathcal{N}(K)} |p_{L^\perp}(u)|^{d-p} \sum_{\substack{(N,J): J\subset N^c \\ |J|=1, |N|=d-1}} \frac{\prod_{N^c} k_i}{\mathbb{K}} \left[\bigwedge_{N\cup J} a_i, L^\perp \right]^2 \mathcal{H}^{p-1}(d(x,u)) \,;
\end{aligned}
$$

especially, if $d = p-1$ and $L = v^\perp$ for some $v \in S^{p-1}$, then we obtain

$$
\begin{aligned}
S^L(K|L) &= \frac{1}{2} \int_{\mathcal{N}(K)} |\langle u,v\rangle|^{-1} \sum_{|N|=p-2} \frac{\prod_{N^c} k_i}{\mathbb{K}} |\langle u,v\rangle|^2 \mathcal{H}^{p-1}(d(x,u)) \\
&= \frac{p-1}{2} \int_{\mathcal{N}(K)} |\langle u,v\rangle| S_{p-2}(K,du) \,,
\end{aligned}
$$

where the last step follows from the known representation of the $(p-2)$nd surface area measure.

Another special case is $d = p-1$, $r \in \{0,\dots,p-1\}$, and $L = v^\perp$ for some $v \in S^{p-1}$. Then we obtain

$$V_r(K|v^\perp) = \frac{1}{\kappa_{p-1-r}\omega_1}\int_{\mathcal{N}(K)} |\langle u,v\rangle|^{-1} \sum_{\substack{(N,J):J\subset N^c\\ |N|=r,|J|=p-1-r}} \frac{\prod_{N^c} k_i(x,u)}{\mathbb{K}(x,u)} |\langle u,v\rangle|^2 \mathcal{H}^{p-1}(d(x,u))$$

$$= \frac{1}{2\kappa_{p-1-r}}\int_{\mathcal{N}(K)} |\langle u,v\rangle| \sum_{|N|=r} \frac{\prod_{N^c} k_i(x,u)}{\mathbb{K}(x,u)} \mathcal{H}^{p-1}(d(x,u))$$

$$= \frac{1}{2\kappa_{p-1-r}}\binom{d-1}{r}\int_{S^{p-1}} |\langle u,v\rangle| S_r(K,du)\,.$$

Of course, these relationships are well-known and can be derived in a different way.

Finally, we mention that for $d = r = 1$ and $L = \mathrm{lin}\{v\}$ with $v \in S^{p-1}$, we obtain

$$V_1(K|L) = \frac{1}{\omega_{p-1}}\int_{\mathcal{N}(K)} \sqrt{1-\langle u,v\rangle^2}^{\,1-p} \sum_{|N|=1} \frac{\prod_{N^c} k_i(x,u)}{\mathbb{K}(x,u)} \langle a_i(x,u),v\rangle^2 \mathcal{H}^{p-1}(d(x,u))\,.$$

For convex bodies of class C^∞_+ and in three-dimensional Euclidean space, this formula was recently derived and used in [96] to prove a stability estimate for δ-umbilical ovaloids in $\mathbb{R}^3$.

Next we consider surface area measures of projections. Let again $d \in \{1,\dots,p-1\}$ and $L \in \mathbf{G}(p,d)$. Let $U := B^p \cap L^\perp$ and set $\tilde{U} := (\kappa_{p-d})^{-1/(p-d)}U$. Then we obtain, for any $K_1,\dots,K_d \in \mathcal{K}^p$,

$$v^L(K_1|L,\dots,K_d|L) = \binom{p}{d} V(K_1,\dots,K_d,\tilde{U}[p-d])$$

$$= \binom{p}{d}\frac{1}{p}\int_{S^{p-1}} h(K_d,u) S(K_1,\dots,K_{d-1},\tilde{U}[p-d],du)\,, \tag{176}$$

where $v^L(\cdot)$ denotes the mixed volume in L. On the other hand,

$$v^L(K_1|L,\dots,K_d|L) = \frac{1}{d}\int_{S^{p-1}\cap L} h(K_d|L,u) S^L(K_1|L,\dots,K_{d-1}|L,du)$$

$$= \frac{1}{d}\int_{S^{p-1}} \mathbf{1}\{u\in L\} h(K_d,u) S^L(K_1|L,\dots,K_{d-1}|L,du)\,. \tag{177}$$

From (176) and (177) we infer that

$$S^L(K_1|L,\dots,K_{d-1}|L,\cdot\cap L) = \frac{\binom{p-1}{d-1}}{\kappa_{p-d}} S(K_1,\dots,K_{d-1},U[p-d],\cdot)\,,$$

where both sides are conceived as measures over S^{p-1}. In particular, for all Borel sets $\omega \subset S^{p-1}$,

$$S^L_{d-1}(K|L,\omega\cap L) = \frac{\binom{p-1}{d-1}}{\kappa_{p-d}} S(K[d-1],U[p-d],\omega)\,; \tag{178}$$

note that the right-hand side does not change if ω is replaced by $\omega \cap L$.

Equation (178) shows that for surface area measures the situation is more involved than for volumes, if we aim at a representation of the left-hand side of (178) as an integral over the normal bundle of K. From our point of view, the main problem is that the support function of U in $\mathbb{R}^p$ is not of class C^2. Therefore we approximate $U = B^p \cap L^\perp$ by a sequence of convex bodies $U(\epsilon)$, $\epsilon > 0$, which have smooth support functions and satisfy $U(\epsilon) \to U$ as $\epsilon \downarrow 0$. For instance, we can choose

$$U(\epsilon) := \left\{ \sum_{i=1}^{p} x_i u_i \in \mathbb{R}^p : \sum_{i=1}^{p-d} (x_i)^2 + \sum_{i=p-d+1}^{p} (x_i/\epsilon)^2 \leq 1 \right\},$$

where $u_1, \dots, u_{p-d}$ is an orthonormal basis of $L^\perp$ and $u_{p-d+1}, \dots, u_p$ is an orthonormal basis of L, but we shall not make explicit use of such a specific choice.

For any such choice of a family $U(\epsilon)$, $\epsilon > 0$, Lemma 2.12 yields that

$$\begin{aligned} &\binom{p-1}{d-1} S(K[d-1], U(\epsilon)[p-d], \cdot) \\ &= \int_{\mathcal{N}(K)} \mathbf{1}\{u \in \cdot\} \sum_{|I|=d-1} \frac{\prod_{i \in I^c} k_i(x,u)}{\mathbb{K}(x,u)} \det\left(d^2 h(U(\epsilon),u)|a_{I^c}(x,u)\right) \mathcal{H}^{p-1}(d(x,u)), \end{aligned}$$

where

$$d^2 h(U(\epsilon),u)|a_{I^c} := \left(\langle d^2 h(U(\epsilon),u)(a_i), a_j\rangle\right)_{i,j \in I^c}$$

and $a_i = a_i(x,u)$. Let us denote by $r_l^\epsilon(u)$, $l = 1, \dots, p-1$, the principal radii of curvature of $U(\epsilon)$ at u and let $b_l^\epsilon(u)$ denote the associated eigenvectors of the reverse Weingarten map of $U(\epsilon)$. Then we can conclude (for $|I| = d-1$)

$$\begin{aligned} d^2 h(U(\epsilon),u)|a_{I^c} &= \sum_{|J|=p-d} \left(\prod_{l \in J} r_l^\epsilon(u)\right) \det\left(\langle b_s^\epsilon(u), a_t(x,u)\rangle_{s \in J, t \in I^c}\right)^2 \\ &= \sum_{|J|=p-d} \left(\prod_{l \in J} r_l^\epsilon(u)\right) \left| \bigwedge_{s \in J} b_s^\epsilon(u) \wedge \bigwedge_{t \in I} a_t(x,u) \right|^2 \\ &= \sum_{|J|=p-d} \left(\prod_{l \in J} r_l^\epsilon(u)\right) \left[\bigwedge_{s \in J} b_s^\epsilon(u), \bigwedge_{t \in I} a_t(x,u) \wedge u \right]^2 \\ &= \sum_{|J|=p-d} \left(\prod_{l \in J} r_l^\epsilon(u)\right) \left[\bigwedge_{s \in J^c} b_s^\epsilon(u) \wedge u, \bigwedge_{t \in I^c} a_t(x,u) \right]^2 . \end{aligned}$$

Thus we have established the following proposition.

Proposition 4.3 *Let $d \in \{1, \ldots, p-1\}$, $L \in \mathbf{G}(p,d)$ and $K \in \mathcal{K}^p$. Let $U(\epsilon)$, b_i^ϵ, r_i^ϵ be chosen as in the preceding discussion. Further, let $f \in C(S^{p-1})$. Then*

$$\int_{S^{p-1}\cap L} f(u) S_{d-1}^L(K|L, du) = \lim_{\epsilon \downarrow 0} \frac{1}{\kappa_{p-d}} \int_{\mathcal{N}(K)} f(u) \sum_{|I|=d-1} \sum_{|J|=p-d} \frac{\prod_{i \in I^c} k_i(x,u)}{\mathbb{K}(x,u)}$$

$$\times \left(\prod_{l \in J} r_l^\epsilon(u) \right) \left[\bigwedge_J b_s^\epsilon(u), \bigwedge_I a_t(x,u) \wedge u \right]^2 \mathcal{H}^{p-1}(d(x,u)) .$$

Certainly, one could find representations for the other intermediate surface area measures $S_j^L(K|L, \cdot)$, $j \in \{0, \ldots, d-2\}$, as well, by the same method which was used to establish Proposition 4.2.

4.3 Projection volumes and surface measures from a current point of view

In this subsection, we shall provide a current representation for the volume of the projection of a convex body onto a linear subspace. In the following subsection, this result will then be used in the derivation of a representation of mixed volumes in terms of generalized curvature functions.

Let us fix $p \geq 2$, $d \in \{1, \ldots, p-1\}$, $L \in \mathbf{G}(p,d)$ and $K \in \mathcal{K}^p$. A rectifiable $(p-1)$-current T_K in $\mathbb{R}^{2p}$ is defined by

$$T_K := \left(\mathcal{H}^{p-1} \llcorner \mathcal{N}(K) \right) \wedge a_K ,$$

where $a_K(x,u)$ is the simple unit $(p-1)$-vector which is associated with $\mathrm{Tan}^{p-1}(\mathcal{N}(K), (x,u))$, for $\mathcal{H}^{p-1}$ almost all $(x,u) \in \mathcal{N}(K)$; the orientation of $a_K(x,u)$ is determined in the following way. We set

$$A_i(x,u) := \left(\frac{1}{\sqrt{1+k_i(x,u)^2}} a_i(x,u), \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} a_i(x,u) \right) ,$$

$i = 1, \ldots, p-1$, where $k_1, \ldots, k_{p-1}$ are the generalized curvatures and $a_1, \ldots, a_{p-1}$ are the associated eigenvectors of K. We assume that $a_1(x,u), \ldots, a_{p-1}(x,u), u$ has the same orientation as the standard basis $e_1, \ldots, e_p$, whenever the former is defined. Then we define

$$a_K(x,u) := A_1(x,u) \wedge \ldots \wedge A_{p-1}(x,u) ,$$

for $\mathcal{H}^{p-1}$ almost all $(x,u) \in \mathcal{N}(K)$, and thus we also fix the orientation of a_K.

Let $v_1, \ldots, v_d$ be an orthonormal basis of L, and let $v_{d+1}, \ldots, v_p$ be an orthonormal basis of $L^\perp$. Then we define

$$V := v_1 \wedge \ldots \wedge v_d \qquad \text{and} \qquad V^\perp := v_{d+1} \wedge \ldots \wedge v_p .$$

We assume that $v_1, \ldots, v_p$ are chosen in such a way that $V^\perp \wedge V = e_1 \wedge \ldots \wedge e_p$. In addition, we write $e_1^*, \ldots, e_p^*$ for the basis of covectors which is dual to $e_1, \ldots, e_p$ and set $\Omega_p := e_1^* \wedge \ldots \wedge e_p^*$. Finally, we set $\Pi_0 : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, $\Pi_0(x,y) := x$, and $\Pi_1 : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, $\Pi_1(x,y) := y$. Using the notation of multilinear algebra as described in Chapter 1 in [42], we can define the $(p-1)$-covector $\psi_L(u)$ in $\mathbb{R}^{2p}$, for $u \in S^{p-1}$, by setting

$$\psi_L(u) := \frac{|V \wedge u|^{d-p}}{\omega_{p-d}} \left[\Pi_0^*(V^\perp \lrcorner \Omega_p) \right] \wedge \left[\Pi_1^*((u \wedge V) \lrcorner \Omega_p) \right]$$

if $u \in \mathbb{R}^p \setminus V$, and by $\psi_L(u) := 0$ if $u \in V$.

Thus we are prepared to state the following result.

Proposition 4.4 *Let $p \geq 2$, $d \in \{1, \ldots, p-1\}$, $L \in \mathbf{G}(p, d)$ and $K \in \mathcal{K}^p$. Then*

$$\mathcal{H}^d(K|L) = T_K(\psi_L)\,.$$

It is part of the assertion of Proposition 4.4 that T_K is defined for ψ_L; compare also the discussion in [155].

Proof. The proof is based on Proposition 4.1 and on some multilinear algebra which is needed to rewrite the integrand of Proposition 4.1.

Any set $I \subset \{1, \ldots, p-1\}$ with $|I| = d$ can be represented as $I = \{i_1, \ldots, i_d\}$, where $i_1 < \ldots < i_d$; let $I^c = \{j_1, \ldots, j_{p-1-d}\}$ with $j_1 < \ldots < j_{p-1-d}$. Then we set $\operatorname{sgn}(I) := \operatorname{sgn}(i_1, \ldots, i_d, j_1, \ldots, j_{p-1-d})$. Thus, omitting arguments, we obtain

$$\begin{aligned}\left\langle \bigwedge_I a_i \wedge V^\perp, \Omega_p \right\rangle &= \left\langle \bigwedge_I a_i \wedge \bigwedge_{I^c} a_i \wedge u, \Omega_p \right\rangle \langle V^\perp \wedge V, \Omega_p \rangle \left\langle \bigwedge_{I^c} a_i \wedge u \wedge V, \Omega_p \right\rangle \\ &= \operatorname{sgn}(I) \left\langle \bigwedge_{I^c} a_i \wedge u \wedge V, \Omega_p \right\rangle ,\end{aligned}$$

and therefore

$$\begin{aligned}&\sum_{|I|=d} \frac{\prod_{I^c} k_i}{\mathbb{K}} \left[\bigwedge_I a_i, L^\perp \right]^2 \\ &= \sum_{|I|=d} \operatorname{sgn}(I) \left\langle \bigwedge_{I^c} \frac{k_i}{\sqrt{1+k_i^2}} a_i \wedge u \wedge V, \Omega_p \right\rangle \left\langle \bigwedge_I \frac{1}{\sqrt{1+k_i^2}} a_i \wedge V^\perp, \Omega_p \right\rangle \\ &= \sum_{|I|=d} \operatorname{sgn}(I) \left\langle \bigwedge_{I^c} \Pi_1 A_i \wedge u \wedge V, \Omega_p \right\rangle \left\langle \bigwedge_I \Pi_0 A_i \wedge V^\perp, \Omega_p \right\rangle \\ &= \sum_{\sigma \in \operatorname{Sh}(d,p-1-d)} \operatorname{sgn}(\sigma) \left\langle \bigwedge_{i=1}^{d} \Pi_0 A_{\sigma(i)}, V^\perp \lrcorner\, \Omega_p \right\rangle \left\langle \bigwedge_{i=d+1}^{p-1} \Pi_1 A_{\sigma(i)}, (u \wedge V) \lrcorner\, \Omega_p \right\rangle \\ &= \sum_{\sigma \in \operatorname{Sh}(d,p-1-d)} \operatorname{sgn}(\sigma) \Pi_0^*(V^\perp \lrcorner\, \Omega_p) \left(A_{\sigma(1)}, \ldots, A_{\sigma(d)} \right) \\ &\qquad\qquad \times \Pi_1^*((u \wedge V) \lrcorner\, \Omega_p) \left(A_{\sigma(d+1)}, \ldots, A_{\sigma(p-1)} \right) \\ &= \left\langle a_K(x,u), \left[\Pi_0^*(V^\perp \lrcorner\, \Omega_p) \right] \wedge \left[\Pi_1^*((u \wedge V) \lrcorner\, \Omega_p) \right] \right\rangle ,\end{aligned}$$

where shuffles of type (a, b) are defined as in [42], p. 15, and $\operatorname{Sh}(a, b)$ denotes the set of all such shuffles. This proves the assertion of the proposition. □

Remarks.

1. The support of a convex body $K \in \mathcal{K}^p$ with Steiner point at the origin can be represented by

$$h(K,v) = \int_{\mathcal{N}(K)} F_{p-1,1}(\theta(v,u)) \sum_{|I|=1} \frac{\prod_{I^c} k_i}{\mathbb{K}} \langle a_i, v\rangle^2 \mathcal{H}^{p-1}(d(x,u)) \,.$$

Hence we obtain as before

$$h(K,v) = T_K(\psi_{K,v}) \,,$$

where

$$\psi_{K,v}(u) := F_{p-1,1}(\theta(v,u)) \left[\Pi_0^*(v^\perp \lrcorner \Omega_p)\right] \wedge \left[\Pi_1^*((u \wedge v) \lrcorner \Omega_p)\right]$$

if $v \notin \text{lin}\{u\}$ and $\psi_{K,v}(u) := 0$ otherwise. Here $v^\perp$ is defined as $v_1 \wedge \ldots \wedge v_{p-1}$, where $v_1, \ldots, v_{p-1}, v$ is an orthonormal basis which has the same orientation as the standard basis.

2. As a special case of Proposition 4.4, we obtain for the support function of the projection body ΠK of K the representation

$$h(\Pi K, v) \\ = \int_{\mathcal{N}(K)} \left\langle a_K(x,u), |\langle u,v\rangle|^{-1} \frac{1}{2} \left[\Pi_0^*(v \lrcorner \Omega_p)\right] \wedge \left[\Pi_1^*((u \wedge v^\perp) \lrcorner \Omega_p)\right] \right\rangle \mathcal{H}^{p-1}(d(x,u)) \,,$$

where $v^\perp := v_2 \wedge \ldots \wedge v_d$ and $v, v_2, \ldots, v_d$ is an orthonormal basis which has the same orientation as the standard basis. (Again the relevant covector is defined to be zero if $\langle u, v\rangle = 0$.)

3. Clearly, a current representation for $V_r(K|L)$, $r \in \{0, \ldots, d-1\}$, can be obtained as well; compare the arguments in the following subsection.

A similar current representation can be given for the surface area measures of projections. We use the notation of the preceding subsection and set

$$B_s^\epsilon(u) := (r_s^\epsilon(u) b_s^\epsilon(u), b_s^\epsilon(u)) \,, \quad s \in \{1, \ldots, p-1\} \,.$$

Here and in the following, we assume that $b_1^\epsilon(u), \ldots, b_{p-1}^\epsilon(u), u$ has the same orientation as the standard basis. Furthermore, for $\epsilon > 0$, $d \in \{1, \ldots, p-1\}$ and $L \in \mathbf{G}(d,p)$ we define

$$\psi_L^\epsilon(u) := \frac{1}{\kappa_{p-d}} \sum_{|J|=p-d} \text{sgn}(J) \left[\left(\bigwedge_J B_s^\epsilon(u)\right) \lrcorner \Pi_0^*(u \lrcorner \Omega_p)\right] \wedge \left[\left(\bigwedge_{J^c} B_s^\epsilon(u)\right) \lrcorner \Pi_1^*(u \lrcorner \Omega_p)\right] \,,$$

where $u \in S^{p-1}$.

Proposition 4.5 *Let $p \geq 2$, $d \in \{1, \ldots, p-1\}$, $L \in \mathbf{G}(p,d)$ and $K \in \mathcal{K}^p$. Then, for any $f \in C(S^{p-1})$,*

$$\int_{S^{p-1} \cap L} f(u) S_{d-1}^L(K|L, du) = \lim_{\epsilon \downarrow 0} T_K(f\, \psi_L^\epsilon) \,.$$

Proof. The assertion will be deduced from Proposition 4.3 by some additional multilinear algebra. In fact, omitting arguments, we derive

$$\sum_{|I|=d-1}\sum_{|J|=p-d}\frac{\prod_{I^c}k_i}{\mathbb{K}}\left(\prod_J r_l^\epsilon\right)\left[\bigwedge_J b_s^\epsilon,\bigwedge_I a_t\wedge u\right]^2$$

$$=\sum_{|J|=p-d}\left(\prod_J r_l^\epsilon\right)\operatorname{sgn}(J)\sum_{|I|=d-1}\operatorname{sgn}(I)\left\langle\bigwedge_J b_s^\epsilon\wedge\bigwedge_I\Pi_0 A_t\wedge u,\Omega_p\right\rangle$$

$$\times\left\langle\bigwedge_{J^c}b_s^\epsilon\wedge\bigwedge_{I^c}\Pi_1 A_t\wedge u,\Omega_p\right\rangle$$

$$=\sum_{|J|=p-d}\operatorname{sgn}(J)\sum_{|I|=d-1}\operatorname{sgn}(I)\left\langle\bigwedge_J B_s^\epsilon\wedge\bigwedge_I A_t,\Pi_0^*(u\,\lrcorner\,\Omega_p)\right\rangle\left\langle\bigwedge_{J^c}B_s^\epsilon\wedge\bigwedge_{I^c}A_t,\Pi_1^*(u\,\lrcorner\,\Omega_p)\right\rangle$$

$$=\sum_{|J|=p-d}\operatorname{sgn}(J)\sum_{|I|=d-1}\operatorname{sgn}(I)\left\langle\bigwedge_I A_t,\left(\bigwedge_J B_s^\epsilon\right)\lrcorner\,\Pi_0^*(u\,\lrcorner\,\Omega_p)\right\rangle$$

$$\times\left\langle\bigwedge_{I^c}A_t,\left(\bigwedge_{J^c}B_s^\epsilon\right)\lrcorner\,\Pi_1^*(u\,\lrcorner\,\Omega_p)\right\rangle$$

$$=\langle A_1\wedge\ldots\wedge A_{p-1},\kappa_{p-d}\psi_L^\epsilon\rangle\ ,$$

from which the assertion follows. ☐

4.4 Mixed volumes, surface measures and generalized curvatures

In this subsection, we derive particular representations of mixed volumes as integrals over products of unit normal bundles.

Let $d,k\geq 2$ and $K_1,\ldots,K_k\in\mathcal{K}_o^d$ be fixed, and let L be as described in (169). Then by equation (170) and Proposition 4.1 we obtain

$$\mathcal{H}^d(K_1+\ldots+K_k)=\frac{k^{\frac{d}{2}}}{\omega_{(k-1)d}}\int_{\mathcal{N}(\underline{K})}|p_{L^\perp}(\underline{u})|^{-(k-1)d}$$
$$\times\sum_{|I|=d}\frac{\prod_{I^c}k_i(\underline{K};\underline{x},\underline{u})}{\mathbb{K}(\underline{K};\underline{x},\underline{u})}\left[\bigwedge_I a_i(\underline{K};\underline{x},\underline{u}),L^\perp\right]^2\mathcal{H}^{kd-1}(d(\underline{x},\underline{u}))\,. \tag{179}$$

Our aim is to show that the right-hand side of (179) admits a polynomial expansion so that a comparison of coefficients (an inspection of the degree of homogeneity) allows one to derive the desired representation of mixed volumes.

In order to identify the mixed volume $V(K_1[\alpha_1],\ldots,K_k[\alpha_k])$ for $\alpha_1,\ldots,\alpha_k\in\{0,\ldots,d\}$ with $\alpha_1+\ldots+\alpha_k=d$, we first consider two cases separately. These cases arise, since by (iv) in Subsection 4.1 the normal bundle $\mathcal{N}(\underline{K})$ can be described as a disjoint union of sets

$\mathcal{N}_{\mathcal{I}}(\underline{K})$ with $|\mathcal{I}| = j$ and $j \in \{1, \dots, k\}$.

(a) We claim that for $\mathcal{H}^{kd-1}$ almost all $(\underline{x}, \underline{u}) \in \mathcal{N}_{\mathcal{I}}(\underline{K})$ the integrand on the right-hand side of (179) vanishes if $|\mathcal{I}| \leq k - 2$.

To see this, it is sufficient to consider the case where $\mathcal{I} \subset \{3, \dots, k\}$; hence $u_1 = u_2 = o$ if $(\underline{x}, \underline{u}) \in \mathcal{N}_{\mathcal{I}}(\underline{K})$. Moreover, we have $\underline{x} + \epsilon\underline{u} \in \mathrm{bd}(\underline{K}_\epsilon)$ and hence

$$\sigma_{\underline{K}_\epsilon}(\underline{x} + \epsilon\underline{u} + (te_j, se_l, o, \dots, o)) = \underline{u}\,,$$

for all $j, l = 1, \dots, d$, provided that $|t|, |s| > 0$ are sufficiently small. In fact, if $|t|, |s| > 0$ are small enough, then

$$\underline{x} + (te_j, se_l, o, \dots, o) \in \mathrm{bd}\,\underline{K}$$

and

$$\underline{u} \in N(\underline{K}, \underline{x} + (te_j, se_l, o, \dots, o))\,.$$

But this shows that $\sigma_{\underline{K}_\epsilon}$ has at least $2d$ eigenvectors $(e_j, o, \dots, o)$, $(o, e_l, o, \dots, o)$, $j, l \in \{1, \dots, d\}$ at $\underline{x} + \epsilon\underline{u}$ with corresponding eigenvalue zero (if they exist). Thus at least $2d$ of the $kd - 1$ generalized curvatures $k_i(\underline{K}; \underline{x}, \underline{u})$ vanish (if they are defined). Since $|I^c| = (k-1)d - 1$ in (179), we infer that each summand under the integral contains a factor which is zero. This verifies the assertion (a).

(b) The right-hand side of equation (179) can be written in the form

$$\int_{\mathcal{N}(\underline{K})} G(\underline{x}, \underline{u})\mathcal{H}^{kd-1}(d(\underline{x}, \underline{u}))\,.$$

We consider the integral of G over $\mathcal{N}_{\mathcal{I}}(\underline{K})$ with $|\mathcal{I}| = k - 1$. Let us choose $\mathcal{I} = \{2, \dots, k\}$ (say). Then we obtain that at least d of the generalized curvatures of $\underline{K}$ vanish, and from this we can deduce that

$$\int_{\mathcal{N}_{\{2,\dots,d\}}(\underline{K})} G(\underline{x}, \underline{u})\mathcal{H}^{kd-1}(d(\underline{x}, \underline{u})) = V(K_1)H(K_2, \dots, K_k)\,,$$

where $H(K_2, \dots, K_k)$ does not depend on K_1. The subsequent discussion will show that the integral

$$\int_{\mathcal{N}_{\{1,\dots,d\}}(\underline{K})} G(\underline{x}, \underline{u})\mathcal{H}^{kd-1}(d(\underline{x}, \underline{u}))$$

only produces contributions which have a degree of homogeneity with respect to K_1 which is strictly smaller than d. Hence we infer that in fact $H(K_2, \dots, K_k) = 1$.

(c) It thus remains to consider the case $\mathcal{I} = \{1, \dots, k\}$. We write $\mathcal{N}_*(\underline{K}) := \mathcal{N}_{\{1,\dots,k\}}(\underline{K})$. The integral

$$\int_{\mathcal{N}_*(\underline{K})} G(\underline{x}, \underline{u})\mathcal{H}^{kd-1}(d(\underline{x}, \underline{u}))$$

will produce the *mixed* expressions in the polynomial expansion of the volume of the Minkowski sum $K_1 + \ldots + K_k$.

First, recall that

$$\mathcal{N}_*(\underline{K}) = \left\{ (\underline{x}, \underline{u}) \in \mathbb{R}^{kd} \times S^{kd-1} : x_i \in \text{bd } K_i, u_i \in N(K_i, x_i) \right\} .$$

Furthermore, we write

$$\underline{\mathcal{N}(K)} := \mathcal{N}(K_1) \times \ldots \times \mathcal{N}(K_k) ,$$

$$\underline{(x,u)} := ((x_1, u_1), \ldots , (x_k, u_k)) \qquad \text{and} \qquad \underline{(tu)} := (t_1 u_1, \ldots , t_k u_k) ,$$

where $t_1, \ldots , t_k \geq 0$. Next we introduce the transformation

$$T : \underline{\mathcal{N}(K)} \times S_+^{k-1} \to \mathcal{N}_*(\underline{K}) , \quad (\underline{(x,u)}, t) \mapsto (\underline{x}, \underline{(tu)}) ,$$

where

$$S_+^{k-1} := \left\{ t \in S^{k-1} : \langle t, e_i \rangle > 0 \text{ for } i = 1, \ldots , k \right\} .$$

Obviously, T is Lipschitz, injective and $\mathcal{H}^{kd-1}(\mathcal{N}_*(\underline{K}) \backslash \text{im}(T)) = 0$. The generalized curvatures of $\mathcal{N}(K_j)$ in (x_j, u_j) are denoted by $k_1(j)((x_j, u_j)), \ldots , k_{d-1}(j)((x_j, u_j))$ and the associated eigenvectors are denoted by $a_1(j)((x_j, u_j)), \ldots , a_{d-1}(j)((x_j, u_j))$, whenever they are defined; but usually, the argument is omitted. Moreover, we write

$$\mathbb{K}_j := \prod_{i=1}^{d-1} \sqrt{1 + k_i(j)^2} ,$$

for $j = 1, \ldots , k$. We assume that $a_1(j)((x_j, u_j)), \ldots , a_{d-1}(j)((x_j, u_j)), u_j$ is positively oriented for $j = 1, \ldots , k$. Subsequently, we write $f_1, \ldots , f_{k-1}, t$ for an orthonormal basis of $\mathbb{R}^k$ which is oriented in such a way that

$$\det(f_1, \ldots , f_{k-1}, t) = (-1)^{(k-1)k(d-1)/2} .$$

For the components of f_i we write $f_i = (f_1^i, \ldots , f_k^i)^T$. Now we can define a simple unit $(kd-1)$-vector $\tilde{a}(\cdot)$, which is associated with the tangent space of $\underline{\mathcal{N}(K)} \times S_+^{k-1}$, by

$$\tilde{a}(\underline{(x,u)}, t) := \bigwedge_{j=1}^{k} \frac{1}{\mathbb{K}_j} \bigwedge_{i=1}^{d-1} (\underbrace{o, \ldots , o}_{2(j-1)}, a_i(j), k_i(j) a_i(j), \underbrace{o, \ldots , o}_{2(k-j)}, o) \wedge \bigwedge_{i=1}^{k-1} (\underbrace{o, \ldots , o}_{2k}, f_i) ,$$

for $\mathcal{H}^{kd-1}$ almost all $(\underline{(x,u)}, t) \in \underline{\mathcal{N}(K)} \times S_+^{k-1}$. An application of the area formula hence yields

$$\begin{aligned} &\int_{\mathcal{N}_*(\underline{K})} |u \wedge L|^{-(k-1)d} \mathbf{1}\{\underline{u} \notin L\} \left\langle a_{\underline{K}}(\underline{x}, \underline{u}), \psi_L(\underline{u}) \right\rangle \mathcal{H}^{kd-1}(d(\underline{x}, \underline{u})) \\ &= \int_{\underline{\mathcal{N}(K)} \times S_+^{k-1}} |\underline{(tu)} \wedge L|^{-(k-1)d} \mathbf{1}\{\underline{(tu)} \notin L\} \\ &\qquad\qquad \times \left\langle \tilde{a}(\underline{(x,u)}, t), (T^{\#}\psi_L)(\underline{(x,u)}, t) \right\rangle \mathcal{H}^{kd-1}(d(\underline{(x,u)}, t)) . \end{aligned}$$

Indeed, the orientations are chosen properly, since the quantities

$$\langle a_{\underline{K}}(\underline{x},\underline{u}), \psi_L(\underline{u})\rangle \tag{180}$$

and

$$\left\langle \tilde{a}(\underline{(x,u)},t), (T^{\#}\psi_L)(\underline{(x,u)},t)\right\rangle \tag{181}$$

are both non-negative. For the expression in (180) this is already known; for the quantity in (181) the assertion follows from the subsequent calculations.

In order to check the last assertion and to arrive at the desired expression for mixed volumes, we rewrite (181) in a more explicit way. In fact, we can deduce

$$\begin{aligned}
&\left\langle \tilde{a}(\underline{(x,u)},t), (T^{\#}\psi_L)(\underline{(x,u)},t)\right\rangle \\
&= \left\langle \wedge_{kd-1}\mathrm{ap}\ DT(\underline{(x,u)},t)\tilde{a}(\underline{(x,u)},t), \psi_L(\underline{(tu)})\right\rangle \\
&= \left\langle \bigwedge_{j=1}^{k}\frac{1}{\mathbb{K}_j}\bigwedge_{i=1}^{d-1}(\underbrace{o,\dots,o}_{j-1}, a_i(j), \underbrace{o,\dots,o}_{k-1}, t_j k_i(j) a_i(j), \underbrace{o,\dots,o}_{k-j})\wedge \right. \\
&\qquad\qquad \left. \wedge \bigwedge_{i=1}^{k-1}(\underbrace{o,\dots,o}_{k}, f_1^i u_1,\dots,f_k^i u_k), \psi_L(\underline{(tu)})\right\rangle .
\end{aligned} \tag{182}$$

The vectors appearing on the right-hand side of (182) are successively denoted by $\underline{A}_1,\dots,\underline{A}_{d-1},\dots,\underline{A}_{(k-1)(d-1)+1},\dots,\underline{A}_{k(d-1)},\underline{A}_{k(d-1)+1},\dots,\underline{A}_{kd-1}$. In addition, we define $\Pi_0:\mathbb{R}^{kd}\times\mathbb{R}^{kd}\to\mathbb{R}^{kd}$, $(\underline{x},\underline{y})\mapsto\underline{x}$, and $\Pi_1:\mathbb{R}^{kd}\times\mathbb{R}^{kd}\to\mathbb{R}^{kd}$, $(\underline{x},\underline{y})\mapsto\underline{y}$. Then, by the definition of the covector ψ_L, the preceding two equalities can be continued with

$$\begin{aligned}
&= \left(\prod_{j=1}^{k}\mathbb{K}_j\right)^{-1}\frac{|\underline{(tu)}\wedge L|^{-(k-1)d}}{\omega_{(k-1)d}}\sum_{\sigma\in\mathrm{Sh}(d,(k-1)d-1)}\mathrm{sgn}(\sigma) \\
&\quad\times\left\langle\bigwedge_{i=1}^{d}\Pi_0\underline{A}_{\sigma(i)}\wedge L^{\perp},\Omega_{kd}\right\rangle\left\langle\bigwedge_{i=d+1}^{kd-1}\Pi_1\underline{A}_{\sigma(i)}\wedge\underline{(tu)}\wedge L,\Omega_{kd}\right\rangle ;
\end{aligned}$$

compare the proof of Proposition 4.4. Next we subdivide the set of all shuffles over which we sum into finer subclasses. For $\alpha_1,\dots,\alpha_k\in\{0,\dots,d-1\}$ and $\alpha_1+\ldots+\alpha_k=d$, we define

$$\mathrm{Sh}(d(\alpha_1,\dots,\alpha_k),(k-1)d-1)$$

as the set of all $\sigma\in\mathrm{Sh}(d,(k-1)d-1)$ for which

$$\begin{aligned}
\sigma(\{1,\dots,\alpha_1\}) &\subset \{1,\dots,d-1\}, \\
&\vdots \\
\sigma(\{\alpha_1+\ldots+\alpha_{k-1}+1,\dots,\alpha_1+\ldots+\alpha_k\}) &\subset \{(k-1)(d-1)+1,\dots,k(d-1)\}.
\end{aligned}$$

The remaining shuffles, which do not lie in one of these subclasses, do not yield a contribution subsequently. Thus we obtain

$$
\begin{aligned}
&\left\langle \tilde{a}(\underline{(x,u)},t), (T^{\#}\psi_L)(\underline{(x,u)},t)\right\rangle \\
&= \frac{|\underline{(tu)}\wedge L|^{-(k-1)d}}{\omega_{(k-1)d}} \sum_{\substack{\alpha_1,\dots,\alpha_k=0\\ \alpha_1+\dots+\alpha_k=d}}^{d-1} \sum_{\sigma\in \mathrm{Sh}(d(\alpha_1,\dots,\alpha_k),(k-1)d-1)} \mathrm{sgn}(\sigma)\left(\prod_{j=1}^{k}\mathbb{K}_j\right)^{-1} \\
&\qquad \times \left\langle \bigwedge_{i=1}^{d} \Pi_0 \underline{A}_{\sigma(i)} \wedge L^{\perp}, \Omega_{kd}\right\rangle \left\langle \bigwedge_{i=d+1}^{kd-1} \Pi_1 \underline{A}_{\sigma(i)} \wedge \underline{(tu)}\wedge L, \Omega_{kd}\right\rangle .
\end{aligned}
$$

The preceding analysis leads to the following preliminary integral representation

$$
\begin{aligned}
&\binom{d}{\alpha_1\dots\alpha_k}\frac{\omega_{(k-1)d}}{k^{\frac{d}{2}}} V(K_1[\alpha_1],\dots,K_k[\alpha_k]) \\
&= \int_{\underline{\mathcal{N}(K)}}\int_{S_+^{k-1}} \mathbf{1}\{\underline{(tu)}\notin L\}|\underline{(tu)}\wedge L|^{-(k-1)d} \sum_{\sigma\in \mathrm{Sh}(d(\alpha_1,\dots,\alpha_k),(k-1)d-1)} \mathrm{sgn}(\sigma)\left(\prod_{j=1}^{k}\mathbb{K}_j\right)^{-1} \\
&\qquad \times \left\langle \bigwedge_{i=1}^{d} \Pi_0 \underline{A}_{\sigma(i)} \wedge L^{\perp}, \Omega_{kd}\right\rangle \left\langle \bigwedge_{i=d+1}^{kd-1} \Pi_1 \underline{A}_{\sigma(i)} \wedge \underline{(tu)}\wedge L, \Omega_{kd}\right\rangle \\
&\qquad \times \mathcal{H}^{k-1}(dt)\mathcal{H}^{k(d-1)}(d\underline{(x,u)}) . \qquad (183)
\end{aligned}
$$

In fact, the right-hand side of the preceding equation is homogeneous of degree α_i with respect to K_i. We postpone the proof of this assertion, which also justifies the remaining assertion in (b), to the proof of Theorem 4.6. In fact, this can also be seen more directly by considering, for $\lambda = (\lambda_1,\dots,\lambda_k)$ with $\lambda_i > 0$, the transformation

$$
T_\lambda : \underline{\mathcal{N}(K)}\times S_+^{k-1} \to \mathcal{N}(\lambda_1 K_1\times\dots\times\lambda_k K_k), \quad (\underline{(x,u)},t)\mapsto (\lambda_1 x_1,\dots,\lambda_k x_k,\underline{(tu)}),
$$

instead of T.

It remains to simplify the expression under the integral in (183). For that purpose we define

$$
\begin{aligned}
I_\sigma(1) &:= \sigma(\{1,\dots,\alpha_1\}) \subset \{1,\dots,d-1\}, \\
&\ \ \vdots \\
I_\sigma(k) &:= \sigma(\{\alpha_1+\dots+\alpha_{k-1}+1,\dots,\alpha_1+\dots+\alpha_k\}) - (k-1)(d-1) \subset \{1,\dots,d-1\}, \\
I_\sigma(j)^c &:= \{1,\dots,d-1\}\setminus I_\sigma(j), \quad j=1,\dots,k,
\end{aligned}
$$

where $\sigma \in \mathrm{Sh}(d(\alpha_1, \dots, \alpha_k), (k-1)d-1)$. Then we can deduce

$$
\begin{aligned}
&\mathrm{sgn}(\sigma)\left(\prod_{j=1}^{k}\mathbb{K}_j\right)^{-1}\left\langle\bigwedge_{i=1}^{d}\Pi_0\underline{A}_{\sigma(i)}\wedge L^\perp,\Omega_{kd}\right\rangle\left\langle\bigwedge_{i=d+1}^{kd-1}\Pi_1\underline{A}_{\sigma(i)}\wedge\underline{(tu)}\wedge L,\Omega_{kd}\right\rangle\\
&=\mathrm{sgn}(\sigma)\left(\prod_{j=1}^{k}\mathbb{K}_j\right)^{-1}\left\langle\bigwedge_{i\in I_\sigma(1)}\begin{pmatrix}a_i(1)\\o\\\vdots\\o\end{pmatrix}\wedge\dots\wedge\bigwedge_{i\in I_\sigma(k)}\begin{pmatrix}o\\\vdots\\o\\a_i(k)\end{pmatrix}\wedge L^\perp,\Omega_{kd}\right\rangle\\
&\quad\times\left\langle t_1^{d-1-\alpha_1}\left(\prod_{i\in I_\sigma(1)^c}k_i(1)\right)\wedge\bigwedge_{i\in I_\sigma(1)^c}\begin{pmatrix}a_i(1)\\o\\\vdots\\o\end{pmatrix}\wedge\dots\wedge t_k^{d-1-\alpha_k}\right.\\
&\qquad\left.\times\left(\prod_{i\in I_\sigma(k)^c}k_i(k)\right)\wedge\bigwedge_{i\in I_\sigma(k)^c}\begin{pmatrix}o\\\vdots\\o\\a_i(k)\end{pmatrix}\wedge\bigwedge_{i=1}^{k-1}\begin{pmatrix}f_1^iu_1\\\vdots\\f_k^iu_k\end{pmatrix}\wedge\begin{pmatrix}t_1u_1\\\vdots\\t_ku_k\end{pmatrix}\wedge L,\Omega_{kd}\right\rangle\\
&=\mathrm{sgn}(\sigma)t_1^{d-1-\alpha_1}\dots t_k^{d-1-\alpha_k}\left(\prod_{j=1}^{k}\mathbb{K}_j\right)^{-1}\left(\prod_{i\in I_\sigma(1)^c}k_i(1)\right)\dots\left(\prod_{i\in I_\sigma(k)^c}k_i(k)\right)\\
&\quad\times\left[\left\langle\bigwedge_{i\in I_\sigma(1)}\begin{pmatrix}a_i(1)\\o\\\vdots\\o\end{pmatrix}\wedge\dots\wedge\bigwedge_{i\in I_\sigma(k)}\begin{pmatrix}o\\\vdots\\o\\a_i(k)\end{pmatrix}\wedge L^\perp,\Omega_{kd}\right\rangle\right]^2\\
&\quad\times\mathrm{sgn}\left(\left\langle L^\perp\wedge L,\Omega_{kd}\right\rangle\right)\mathrm{sgn}\left(\left\langle\bigwedge_{i\in I_\sigma(1)}\begin{pmatrix}a_i(1)\\o\\\vdots\\o\end{pmatrix}\wedge\dots\wedge\bigwedge_{i\in I_\sigma(k)}\begin{pmatrix}o\\\vdots\\o\\a_i(k)\end{pmatrix}\wedge\right.\right.\\
&\qquad\left.\left.\wedge\bigwedge_{i\in I_\sigma(1)^c}\begin{pmatrix}a_i(1)\\o\\\vdots\\o\end{pmatrix}\wedge\dots\wedge\bigwedge_{i\in I_\sigma(k)^c}\begin{pmatrix}o\\\vdots\\o\\a_i(k)\end{pmatrix}\wedge\bigwedge_{i=1}^{k-1}\begin{pmatrix}f_1^iu_1\\\vdots\\f_k^iu_k\end{pmatrix}\wedge\begin{pmatrix}t_1u_1\\\vdots\\t_ku_k\end{pmatrix},\Omega_{kd}\right\rangle\right)
\end{aligned}
$$

$$= t_1^{d-1-\alpha_1} \dots t_k^{d-1-\alpha_k} \left(\prod_{j=1}^{k} \mathbb{K}_j \right)^{-1} \left(\prod_{i \in I_\sigma(1)^c} k_i(1) \right) \dots \left(\prod_{i \in I_\sigma(k)^c} k_i(k) \right)$$

$$\times \left[\left\langle \bigwedge_{i \in I_\sigma(1)} \begin{pmatrix} a_i(1) \\ o \\ \vdots \\ o \end{pmatrix} \wedge \dots \wedge \bigwedge_{i \in I_\sigma(k)} \begin{pmatrix} o \\ \vdots \\ o \\ a_i(k) \end{pmatrix} \wedge L^\perp, \Omega_{kd} \right\rangle \right]^2$$

$$\times \operatorname{sgn} \left(\left\langle \bigwedge_{i=1}^{d-1} \begin{pmatrix} a_i(1) \\ o \\ \vdots \\ o \end{pmatrix} \wedge \dots \wedge \bigwedge_{i=1}^{d-1} \begin{pmatrix} o \\ \vdots \\ o \\ a_i(k) \end{pmatrix} \wedge \bigwedge_{i=1}^{k-1} \begin{pmatrix} f_1^i u_1 \\ \vdots \\ f_k^i u_k \end{pmatrix} \wedge \begin{pmatrix} t_1 u_1 \\ \vdots \\ t_k u_k \end{pmatrix}, \Omega_{kd} \right\rangle \right).$$

By some manipulations with determinants, we find that

$$\operatorname{sgn} \left(\left\langle \bigwedge_{i=1}^{d-1} \begin{pmatrix} a_i(1) \\ o \\ \vdots \\ o \end{pmatrix} \wedge \dots \wedge \bigwedge_{i=1}^{d-1} \begin{pmatrix} o \\ \vdots \\ o \\ a_i(k) \end{pmatrix} \wedge \bigwedge_{i=1}^{k-1} \begin{pmatrix} f_1^i u_1 \\ \vdots \\ f_k^i u_k \end{pmatrix} \wedge \begin{pmatrix} t_1 u_1 \\ \vdots \\ t_k u_k \end{pmatrix}, \Omega_{kd} \right\rangle \right)$$

$$= (-1)^{(k-1)k(d-1)/2} \det \begin{pmatrix} E_{d-1} & & & \\ & \ddots & & \\ & & E_{d-1} & \\ & & & (f^1 \dots f^k t) \end{pmatrix} = 1\,.$$

In addition, we have

$$\left[\left\langle \bigwedge_{i \in I_\sigma(1)} \begin{pmatrix} a_i(1) \\ o \\ \vdots \\ o \end{pmatrix} \wedge \dots \wedge \bigwedge_{i \in I_\sigma(k)} \begin{pmatrix} o \\ \vdots \\ o \\ a_i(k) \end{pmatrix} \wedge L^\perp, \Omega_{kd} \right\rangle \right]^2$$

$$= \left[\det \left(\frac{1}{\sqrt{k}} \left(\langle a_i(1), e_j \rangle \right)_{j=1}^{d} | i \in I_\sigma(1), \dots, \frac{1}{\sqrt{k}} \left(\langle a_i(k), e_j \rangle \right)_{j=1}^{d} | i \in I_\sigma(k) \right) \right]^2$$

$$= \frac{1}{k^d} \left| \bigwedge_{j=1}^{k} \bigwedge_{i \in I_\sigma(j)} a_i(j) \right|^2 .$$

Moreover,

$$\left| \underline{(tu)} \wedge L \right|^{-(k-1)d} = k^{\frac{(k-1)d}{2}} \left\{ \sum_{1 \le i < j \le k} |t_i u_i - t_j u_j|^2 \right\}^{-(k-1)d/2},$$

and this number is zero if and only if $t_i u_i = t_j u_j$ for all $i \neq j$, that is, if and only if

$$t_1 = \dots = t_k = \frac{1}{\sqrt{k}} \qquad \text{and} \qquad u_1 = \dots = u_k\,.$$

In this case, we also have

$$\bigwedge_{j=1}^{k} \bigwedge_{i \in I_\sigma(j)} a_i(j) = o\,.$$

Finally, for $\alpha_1, \ldots, \alpha_k \in \{0, \ldots, d-1\}$ with $\alpha_1 + \ldots + \alpha_k = d$ and $u_1, \ldots, u_k \in S^{d-1}$, we define

$$F(\underline{\alpha}, \underline{u}) := \frac{k^{d(\frac{k}{2}-1)}}{\omega_{(k-1)d}} \int_{S_+^{k-1}} \left(\prod_{i=1}^{k} t_i^{d-1-\alpha_i} \right) \left\{ \sum_{1 \le i < j \le k} |t_i u_i - t_j u_j|^2 \right\}^{-(k-1)d/2} \mathcal{H}^{k-1}(dt)$$

if $(u_1, \ldots, u_k) \notin L$, and $F(\underline{\alpha}, \underline{u}) := 0$ otherwise. Thus we can state the following theorem.

Theorem 4.6 *Let* $k, d \geq 2$, $\alpha_1, \ldots, \alpha_k \in \{0, \ldots, d-1\}$ *with* $\alpha_1 + \ldots + \alpha_k = d$, *and* $K_1, \ldots, K_k \in \mathcal{K}^d$. *Then*

$$\begin{aligned} &\binom{d}{\alpha_1 \ldots \alpha_k} V(K_1[\alpha_1], \ldots, K_k[\alpha_k]) \\ &= \int_{\underline{\mathcal{N}(K)}} F(\underline{\alpha}, \underline{u}) \sum_{\substack{|I_j| = \alpha_j \\ j=1,\ldots,k}} \left\{ \prod_{j=1}^{k} \frac{\prod_{i \in I_j^c} k_i(j)}{\mathbb{K}_j} \right\} \left| \bigwedge_{j=1}^{k} \bigwedge_{i \in I_j} a_i(j) \right|^2 \mathcal{H}^{k(d-1)}(d\underline{(x,u)})\,, \end{aligned}$$

where the sum extends over all subsets $I_j \subset \{1, \ldots, d-1\}$, $j = 1, \ldots, k$, *of the prescribed cardinality.*

Proof. Almost all of the work has already been done. It remains to supply two further details. First, we indicate why the right-hand side of the asserted formula is homogeneous of degree α_j with respect to K_j. To justify this, we remark that, for a convex body $K \in \mathcal{K}^d$ and $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$,

$$k_i(\lambda K; \lambda x, u) = \lambda^{-1} k_i(K; x, u)\,,$$

where $\lambda > 0$ and $i \in \{1, \ldots, d-1\}$. Further, the Jacobian of the map

$$F_\lambda : \mathcal{N}(K) \to \mathcal{N}(\lambda K)\,, \quad (x, u) \mapsto (\lambda x, u)\,,$$

is given by

$$\text{ap } J_{d-1} F_\lambda(x, u) = \lambda^{d-1} \prod_{i=1}^{d-1} \frac{\sqrt{1 + (\lambda^{-1} k_i(K; x, u))^2}}{\sqrt{1 + k_i(K; x, u)^2}}\,.$$

Applying these two facts to $K_1, \ldots, K_k$, we obtain the assertion.

Thus the assertion of the theorem is already established for convex bodies with interior points. The general case follows from this by first considering $K_i + \delta B^d$ and using then the continuity of both sides with respect to the limit $\delta \downarrow 0$; compare for example pp. 335-6 in [69] for a similar argument. □

Remark. The last step of the proof shows that in principle it is sufficient to prove the theorem for convex bodies of class $C^{1,1}$. This class is stable with respect to Minkowski addition and projection onto subspaces. (For higher classes of differentiability, rather unexpected phenomena can occur; see the examples of Kiselman as reported in [123], Notes for Section 2.5.) However, these facts do not seem to lead to a simplification of the proof of Theorem 4.6. In fact, the transformation T, which we considered previously, cannot be replaced by a map from $\underline{\text{bd } K} \times S_+^{k-1}$ to bd $\underline{K}$. Moreover, the use of the normal bundles leads to a convenient linearization of the calculations.

A corresponding representation can be established for mixed surface area measures, too. Such a result, however, now involves a limit in the same way as Propositions 4.3 and 4.5.

Theorem 4.7 *Let $k, d \geq 2$, $\alpha_1, \ldots, \alpha_k \in \{0, \ldots, d-1\}$ with $\alpha_1 + \ldots + \alpha_k = d-1$, and $K_1, \ldots, K_k \in \mathcal{K}^d$. Further, let $f \in C(S^{kd-1})$. Then*

$$\binom{d-1}{\alpha_1 \ldots \alpha_k} \kappa_{(k-1)d} k^{-\frac{d-1}{2}} \int_{S^{d-1}} f\left(\frac{1}{\sqrt{k}}(u, \ldots, u)\right) S(K_1[\alpha_1], \ldots, K_k[\alpha_k], du)$$

$$= \lim_{\epsilon \downarrow 0} \int_{\underline{\mathcal{N}(K)}} \int_{S_+^{k-1}} f(\underline{(tu)}) \sum_{|J|=(k-1)d} \left(\prod_{j=1}^{k} t_j^{d-1-\alpha_j}\right) \left(\prod_{l \in J} r_l^\epsilon(\underline{(tu)})\right) \sum_{\substack{|I_j|=\alpha_j \\ j=1,\ldots,k}} \left\{\prod_{j=1}^{k} \frac{\prod_{i \in I_j^c} k_i(j)}{\mathbb{K}_j}\right\}$$

$$\times \left(\det\left(\left\langle \widetilde{\Pi}_j(b_s^\epsilon(\underline{(tu)})), a_{i_j}(j)\right\rangle_{\substack{s \in J^c, i_j \in I_j \\ j=1,\ldots,k}}\right)\right)^2 \mathcal{H}^{k-1}(dt)\mathcal{H}^{(k-1)d}(d\underline{(x,u)}),$$

where $\widetilde{\Pi}_j : \mathbb{R}^{kd} \to \mathbb{R}^d$, $(y_1, \ldots, y_k) \mapsto y_j$.

Proof. Subsequently, we omit some details which are similar to the proof of Theorem 4.6. Moreover, we adopt the same notation as in the proof of Theorem 4.6 and the discussion preceding it. By equation (171) and Proposition 4.5 we obtain

$$\int_{S^{d-1}} f\left(\frac{1}{\sqrt{k}}(u, \ldots, u)\right) S_{d-1}(K_1 + \ldots + K_k, du)$$

$$= k^{\frac{d-1}{2}} \int_{S^{kd-1} \cap L} f(\underline{u}) S_{d-1}^L(K_1 \times \ldots \times K_k | L, d\underline{u})$$

$$= \lim_{\epsilon \downarrow 0} k^{\frac{d-1}{2}} \int_{\mathcal{N}(\underline{K})} f(\underline{u}) \left\langle a_{\underline{K}}(\underline{x}, \underline{u}), \psi_L^\epsilon(\underline{u})\right\rangle \mathcal{H}^{kd-1}(d(\underline{x}, \underline{u}))$$

$$= \lim_{\epsilon \downarrow 0} \frac{k^{\frac{d-1}{2}}}{\kappa_{(k-1)d}} \int_{\mathcal{N}_{\{1,\ldots,k\}}(\underline{K})} f(\underline{u}) \sum_{|I|=d-1} \sum_{|J|=(k-1)d} \frac{\prod_{I^c} k_i(\underline{K}; \cdot)}{\mathbb{K}(\underline{K}; \cdot)}$$

$$\times \left(\prod_J r_l^\epsilon\right) \left[\bigwedge_J b_s^\epsilon, \bigwedge_I a_t(\underline{K}; \cdot) \wedge \underline{u}\right]^2 \mathcal{H}^{kd-1}(d(\underline{x}, \underline{u})).$$

In the last step, we have used that if $|\mathcal{I}| \leq k-1$, then at least d of the $kd-1$ generalized curvatures $k_i(\underline{K}; \cdot)$ are zero and therefore

$$\frac{\prod_{I^c} k_i(\underline{K}; \cdot)}{\mathbb{K}(\underline{K}; \cdot)} = 0$$

for all $I \subset \{1, \dots, kd-1\}$ with $|I| = d-1$.

Hence,

$$\int_{S^{d-1}} f\left(\frac{1}{\sqrt{k}}(u, \dots, u)\right) S_{d-1}(K_1 + \dots + K_k, du)$$
$$= \lim_{\epsilon \downarrow 0} k^{\frac{d-1}{2}} \int_{\underline{\mathcal{N}(K)} \times S_+^{k-1}} f(\underline{(tu)}) \left\langle \tilde{a}(\underline{(x,u)}, t) \left(T^{\#}\psi_L^\epsilon\right) (\underline{(x,u)}, t)\right\rangle \mathcal{H}^{kd-1}(d(\underline{(x,u)}, t)) \,.$$

In addition, we find

$$\left\langle \tilde{a}(\underline{(x,u)}, t), \left(T^{\#}\psi_L^\epsilon\right) (\underline{(x,u)}, t)\right\rangle$$
$$= \left(\prod_{j=1}^k \mathbb{K}_j\right)^{-1} \sum_{\sigma \in \mathrm{Sh}(d-1,(k-1)d)} \mathrm{sgn}(\sigma) \sum_{|J|=(k-1)d} \mathrm{sgn}(J) \left(\prod_J r_l^\epsilon(\underline{(tu)})\right)$$
$$\times \left\langle \bigwedge_J b_s^\epsilon \wedge \bigwedge_I \Pi_0 \underline{A}_t \wedge \underline{(tu)}, \Omega_{kd}\right\rangle \left\langle \bigwedge_{J^c} b_s^\epsilon \wedge \bigwedge_{I^c} \Pi_1 \underline{A}_t \wedge \underline{(tu)}, \Omega_{kd}\right\rangle \,.$$

This shows that

$$\binom{d-1}{\alpha_1 \dots \alpha_k} k^{-\frac{d-1}{2}} \kappa_{(k-1)d} \int_{S^{d-1}} f\left(\frac{1}{\sqrt{k}}(u, \dots, u)\right) S(K_1[\alpha_1], \dots, K_k[\alpha_k], du)$$
$$= \lim_{\epsilon \downarrow 0} \int_{\underline{\mathcal{N}(K)} \times S_+^{k-1}} f(\underline{(tu)}) \sum_{|J|=(k-1)d} \mathrm{sgn}(J) \left(\prod_J r_l^\epsilon(\underline{(tu)})\right) \left(\prod_{j=1}^k \mathbb{K}_j\right)^{-1}$$
$$\times \sum_{\sigma \in \mathrm{Sh}((d-1)(\alpha_1, \dots, \alpha_k),(k-1)d)} \mathrm{sgn}(\sigma) \left\langle \bigwedge_J b_s^\epsilon \wedge \bigwedge_{i=1}^{d-1} \Pi_0 \underline{A}_{\sigma(i)} \wedge \underline{(tu)}, \Omega_{kd}\right\rangle$$
$$\times \left\langle \bigwedge_{J^c} b_s^\epsilon \wedge \bigwedge_{i=d}^{kd-1} \Pi_1 \underline{A}_{\sigma(i)} \wedge \underline{(tu)}, \Omega_{kd}\right\rangle \mathcal{H}^{kd-1}(d(\underline{(x,u)}, t)) \,,$$

once it is clear that the right-hand side has the proper degrees of homogeneity with respect to $K_1, \dots, K_k$. Using some multilinear algebra, we deduce that the last integrand is equal to

$$\sum_{|J|=(k-1)d} \mathrm{sgn}(J) \left(\prod_J r_l^\epsilon(\underline{(tu)})\right) \sum_{\substack{|I_j|=\alpha_j \\ j=1,\dots,k}} \left(\prod_{j=1}^k t_j^{d-1-\alpha_j}\right) \left(\prod_{j=1}^k \mathbb{K}_j\right)^{-1}$$
$$\times \prod_{j=1}^k \left(\prod_{i \in I_j^c} k_i(j)\right) \left| \left\langle \bigwedge_J b_s^\epsilon(\underline{(tu)}) \wedge \bigwedge_{i \in I_1} \begin{pmatrix} a_i(1) \\ o \\ \vdots \\ o \end{pmatrix} \wedge \dots \wedge \bigwedge_{i \in I_k} \begin{pmatrix} o \\ \vdots \\ o \\ a_i(k) \end{pmatrix} \wedge \underline{(tu)}, \Omega_{kd}\right\rangle \right|^2 \,.$$

The proof can now be completed as in the proof of Theorem 4.6. □

Subsequently, we mention several special cases of Theorem 4.6 which illustrate the general result. Of course, a similar discussion could be given for Theorem 4.7.

(a) Obviously, Theorem 4.6 leads to various formulae for computing the mixed volume of a sequence of convex bodies. In particular, the result can be used for calculating volumes, since for a convex body K we have

$$V_d(K) = V(K[1], \ldots, K[1]) = V(K[\alpha], K[d-\alpha]),$$

to mention just a few possibilities.

(b) The case $k = 2$ plays a special rôle, since

$$\binom{d}{\alpha} V(K_1[\alpha], K_2[d-\alpha]) = \Psi_{\alpha,d-\alpha}(K_1, -K_2; \mathbb{R}^{2d}),$$

where $\Psi_{\alpha,d-\alpha}(K_1, -K_2; \cdot)$ denotes a particular mixed curvature measure and $\alpha \in \{1, \ldots, d-1\}$; see [110]. If one compares Theorem 2 in [110] and Theorem 1.6 in this special case one is led to expect the relationship (see the notation in [110])

$$F(\alpha, d-\alpha, u_1, u_2) = F_{\alpha,d-\alpha}(\theta(u_1, u_2)),$$

for $u_1 \neq \pm u_2$, which can indeed be verified.

(c) Suppose that the assumptions of Theorem 1.6 are satisfied. Assume further that $P_1, \ldots, P_k$ are polytopes in $\mathbb{R}^d$. Then

$$\begin{aligned} &\binom{d}{\alpha_1 \ldots \alpha_k} V(P_1[\alpha_1], \ldots, P_k[\alpha_k]) \\ &= \sum_{F_1 \in F_{\alpha_1}(P_1)} \cdots \sum_{F_k \in F_{\alpha_k}(P_k)} V(F_1 + \ldots + F_k)[F_1, \ldots, F_k] \\ &\quad \times \int_{\sigma(P_1,F_1)} \cdots \int_{\sigma(P_k,F_k)} F(\underline{\alpha}, \underline{u}) \mathcal{H}^{d-1-\alpha_1}(du_1) \ldots \mathcal{H}^{d-1-\alpha_k}(du_k), \end{aligned}$$

where $\sigma(P_i, F_i) := N(P_i, F_i) \cap S^{d-1}$ and $[F_1, \ldots, F_k]$ denotes the volume of the parallelepiped that is the sum of the unit cubes in the affine hulls of $F_1, \ldots, F_k$, respectively.

It is not clear, at present, whether this very special case of Theorem 1.6 can be deduced from Theorem 4.1 in [126] for $k > 2$. Theorem 4.1 in [126] describes a method of computing $V(P_1[\alpha_1], \ldots, P_k[\alpha_k])$ by summing the volumes $V(F_1 + \ldots + F_k)$, $F_i \in \mathcal{F}_{\alpha_i}(P_i)$ for $i = 1, \ldots, k$, of certain parallelotopes according to a given selection rule. In contrast to such a selective summation, in the preceding formula we sum over all possible faces $F_i \in \mathcal{F}_{\alpha_i}(P_i)$ for $i = 1, \ldots, k$, but now the volumes $V(F_1 + \ldots + F_k)$ have to be weighted in a suitable way.

(d) Suppose that the assumptions of Theorem 1.6 are satisfied. Assume further that

$K_1, \ldots, K_k$ are of class $C^{1,1}$. Then

$$\binom{d}{\alpha_1 \ldots \alpha_k} V(K_1[\alpha_1], \ldots, K_k[\alpha_k])$$
$$= \int_{\underline{\text{bd } K}} F(\underline{\alpha}, \sigma_{\underline{K}}(\underline{x})) \sum_{\substack{|I_j|=\alpha_j \\ j=1,\ldots,k}} \left\{ \prod_{j=1}^{k} \prod_{i \in I_j^c} k_i(j)(x_j) \right\} \left| \bigwedge_{j=1}^{k} \bigwedge_{i \in I_j} a_i(j) \right|^2 \left(\otimes_{j=1}^{k} \mathcal{H}^{d-1} \right) (d\underline{x}) ,$$

where $\underline{\text{bd } K} = \text{bd } K_1 \times \ldots \times \text{bd } K_k$ and $\sigma_{\underline{K}}(\underline{x}) = (\sigma_{K_1}(x_1), \ldots, \sigma_{K_k}(x_k))$. It does not seem to be possible to give a more direct proof of this special case.

(e) Suppose that the assumptions of Theorem 1.6 are satisfied. Assume further that $K_1, \ldots, K_k$ have support functions of class $C^{1,1}$. Then

$$\binom{d}{\alpha_1 \ldots \alpha_k} V(K_1[\alpha_1], \ldots, K_k[\alpha_k])$$
$$= \int_{(S^{d-1})^k} F(\underline{\alpha}, \underline{u}) \sum_{\substack{|I_j|=\alpha_j \\ j=1,\ldots,k}} \left\{ \prod_{j=1}^{k} \prod_{i \in I_j} r_i(j)(u_j) \right\} \left| \bigwedge_{j=1}^{k} \bigwedge_{i \in I_j} a_i(j) \right|^2 \left(\otimes_{j=1}^{k} \mathcal{H}^{d-1} \right) (d\underline{u}) .$$

Again a more direct approach to this formula does not seem to be known.

(f) Clearly, one also obtains simplified expressions for mixed volumes of convex bodies which belong to one of the classes considered in (c) – (e).

4.5 Mixed curvature measures

The basic aim of this last subsection is to derive representations for mixed curvature measures which are similar to those already obtained for mixed volumes.

Let $q, d \geq 2$ and $K_1, \ldots, K_q \in \mathcal{K}^d$. For unit vectors $u_1, \ldots, u_q \in S^{d-1}$ we set

$$\text{cone}\{u_1, \ldots, u_q\} := \left\{ \sum_{i=1}^{q} \lambda_i u_i : \lambda_i > 0 \text{ for } i = 1, \ldots, q \right\}$$

if o is not an element of the set on the right-hand side, and otherwise we define $\text{cone}\{u_1, \ldots, u_q\} := \{o\}$. Next we introduce the *joint unit normal bundle*

$$\mathcal{N}(K_1, \ldots, K_q) := \{(x_1, \ldots, x_q, u) \in \mathbb{R}^{qd} \times S^{d-1} \ : \ u \in \text{cone}\{u_1, \ldots, u_q\} \text{ for some } (x_i, u_i) \in \mathcal{N}(K_i), i = 1, \ldots, q\} ;$$

compare [108]. Further, we set

$$R^c := \{(x_1, u_1, \ldots, x_q, u_q) \in (\mathbb{R}^d \times S^{d-1})^q : o \notin \text{cone}\{u_1, \ldots, u_q\}\} ;$$

obviously, R^c is a Borel set. The map

$$T : [(\mathcal{N}(K_1) \times \ldots \times \mathcal{N}(K_q)) \cap R^c] \times S_+^{q-1} \to \mathcal{N}(K_1, \ldots, K_q)$$

is defined by

$$T(x_1,u_1,\dots,x_q,u_q,t) := \left(x_1,\dots,x_q,\frac{\sum_{i=1}^q t_i u_i}{|\sum_{i=1}^q t_i u_i|}\right).$$

It is easy to see that F is well-defined, locally Lipschitz and onto. Although F is not injective, the following lemma is sufficient for our purposes. The proof of Lemma 4.8 was suggested by J. Rataj.

Lemma 4.8 *For $\mathcal{H}^{qd-1}$ almost all elements of* $\mathrm{im}(T)$, *the pre-image under T is a single point.*

Proof. We write

$$\triangle(q-1) := \left\{(t_1,\dots,t_q)\in(0,1)^q : \sum_{i=1}^q t_i = 1\right\}$$

for the $(q-1)$-dimensional open simplex embedded in $\mathbb{R}^q$. Clearly, to prove the lemma it is sufficient to show that the map

$$\begin{aligned} G \ &: \ [(\mathcal{N}(K_1)\times\ldots\times\mathcal{N}(K_q))\cap R^c]\times\triangle(q-1)\to\mathbb{R}^{qd}\times S^{d-1}, \\ &(x_1,u_1,\dots,x_q,u_q,t_1,\dots,t_q)\mapsto\left(x_2-x_1,\dots,x_q-x_1,x_1,\frac{\sum_{i=1}^q t_i u_i}{|\sum_{i=1}^q t_i u_i|}\right), \end{aligned}$$

has a unique pre-image for $\mathcal{H}^{qd-1}$ almost all elements of $\mathrm{im}(G)$. For the proof we proceed by induction. The case $q=2$ has been established in [158]. Now we assume that the assertion has already been proved for $q-1$ convex bodies. Set

$$\bar{R}^c := \{(y_1,\dots,y_{q-1},u,y,v)\in\mathbb{R}^{(q-1)d}\times S^{d-1}\times\mathbb{R}^d\times S^{d-1} : o\notin\mathrm{cone}\{u,v\}\}.$$

To establish the assertion for q convex bodies $K_1,\dots,K_q$, $q\ge 3$, we introduce the maps

$$\varphi_q:\mathbb{R}^{qd}\times\mathbb{R}^d\to\mathbb{R}^{qd}\times\mathbb{R}^d,\quad (x_1,\dots,x_q,u)\mapsto(x_2-x_1,\dots,x_q-x_1,x_1,u),$$

$$\begin{aligned} G_2 \ &: \ [(\mathcal{N}(K_1)\times\ldots\times\mathcal{N}(K_q))\cap R^c]\times\triangle(q-1) \\ &\qquad\to([\varphi_{q-1}(\mathcal{N}(K_1,\dots,K_{q-1}))\times\mathcal{N}(K_q)]\cap\bar{R}^c)\times(0,\infty), \\ &(x_1,u_1,\dots,x_q,u_q,t_1,\dots,t_q) \\ &\qquad\mapsto\left(x_2-x_1,\dots,x_{q-1}-x_1,x_1,\frac{\sum_{i=1}^{q-1}t_iu_i}{|\sum_{i=1}^{q-1}t_iu_i|},x_q,u_q,\frac{t_q}{|\sum_{i=1}^{q-1}t_iu_i|}\right), \end{aligned}$$

and

$$\begin{aligned} G_1 \ &: \ ([\varphi_{q-1}(\mathcal{N}(K_1,\dots,K_{q-1}))\times\mathcal{N}(K_q)]\cap\bar{R}^c)\times(0,\infty)\to\varphi_q(\mathcal{N}(K_1,\dots,K_q)), \\ &(z_2,\dots,z_{q-1},x_1,v,x_q,u_q,s)\mapsto\left(z_2,\dots,z_{q-1},x_q-x_1,x_1,\frac{v+su_q}{|v+su_q|}\right); \end{aligned}$$

hence, $G=G_1\circ G_2$. By the inductive hypothesis it follows that, for $\mathcal{H}^{qd-1}$ almost all elements of $\mathrm{im}(G_2)$, the map G_2 has a unique pre-image. Thus, since G_1 is locally Lipschitz, the image under G_1 of the set of all elements of $\mathrm{im}(G_2)$ for which the pre-image under G_2 is not uniquely determined has $(qd-1)$-dimensional Hausdorff measure zero.

Furthermore, for $\mathcal{H}^{qd-1}$ almost all

$$(z_2,\dots,z_{q-1},x_1,v,x_q,u_q,s)\in([\varphi_{q-1}(\mathcal{N}(K_1,\dots,K_{q-1}))\times\mathcal{N}(K_q)]\cap\bar{R}^c)\times(0,\infty)$$

we have

$$(x_1,v)\in\mathcal{N}(K_1\cap(K_2-z_2)\cap\ldots\cap(K_{q-1}-z_{q-1}))\,,$$

and therefore the result in [158] shows that $\mathcal{H}^{qd-1}$ almost all elements of $\mathrm{im}(G_2)$ have a unique pre-image under G_1. In fact, here we use that

$$\frac{v+su_q}{|v+su_q|}=\frac{\frac{1}{s+1}v+\frac{s}{s+1}u_q}{|\frac{1}{s+1}v+\frac{s}{s+1}u_q|}$$

and $(0,\infty)\to(0,1)$, $s\mapsto(1+s)^{-1}s$, is locally bi-Lipschitz. Thus the assertion follows. □

In order to describe the current representation of mixed curvature measures, which is our starting point, we recall some notation from [108]. Let $q,d\geq 2$ and $r_1,\dots,r_q\in\{0,\dots,d-1\}$ with $r_1+\ldots+r_q\geq(q-1)d$; thus, necessarily $q\leq d$. Note that in view of Corollary 3.3 in [150], Proposition 4 in [151] or [108], we need not consider the case $r_i=d$ for some $i\in\{1,\dots,q\}$. We set $R_1:=r_1$, $R_2:=r_1+r_2$, ... , $R_q:=r_1+\ldots+r_q$, $r_{q+1}:=qd-1-R_q$ and $k:=r_1+\ldots+r_q-(q-1)d\in\{0,\dots,d-q\}$; hence $r_{q+1}=d-1-k$. Let $\mathrm{Sh}(r_1,\dots,r_{q+1})$ denote the set of all permutations of $\{1,\dots,qd-1\}$ which are increasing on each of the sets $\{1,\dots,R_1\}$, $\{R_1+1,\dots,R_2\}$, ... , $\{R_q+1,\dots,qd-1\}$. Following [108], we write $\varphi_{r_1,\dots,r_q}\in\mathcal{D}^{qd-1}(\mathbb{R}^{(q+1)d})$ for the differential form which is defined by

$$\begin{aligned}&\left\langle\bigwedge_{j=1}^{qd-1}\left(a_j^1,\dots,a_j^{q+1}\right),\varphi_{r_1,\dots,r_q}(x_1,\dots,x_q,u)\right\rangle\\&=\frac{1}{\omega_{d-k}}(-1)^{c_1(d,r_1,\dots,r_q)}\sum_{\sigma\in\mathrm{Sh}(r_1,\dots,r_{q+1})}\mathrm{sgn}(\sigma)\\&\qquad\times\left[\bigwedge_{i=1}^{R_1}a^1_{\sigma(i)},\bigwedge_{i=R_1+1}^{R_2}a^2_{\sigma(i)},\dots,\bigwedge_{i=R_{q-1}+1}^{R_q}a^q_{\sigma(i)},\bigwedge_{i=R_q+1}^{qd-1}a^{q+1}_{\sigma(i)}\wedge u\right],\end{aligned}$$

where $a_j^i\in\mathbb{R}^d$, for $i\in\{1,\dots,q+1\}$ and $j\in\{1,\dots,qd-1\}$, is arbitrarily chosen. For a definition of the bracket $[\,,\dots,\,]$ we refer to [108]. The numerical value of the constant $c_1(d,r_1,\dots,r_q)\in\mathbb{Z}$, as given in [108], does not seem to be correct. However, it is not necessary to specify the precise value in the following as long as it is clear that it merely depends on $d,r_1,\dots,r_q$.

Writing $a_{K_1,\dots,K_q}$ for the unit simple $(qd-1)$-vectorfield which is associated with $\mathcal{N}(K_1,\dots,K_q)$ almost everywhere, and with orientation as prescribed in [108], we obtain the representation

$$C_{r_1,\dots,r_q}(K_1,\dots,K_q;A):=\left[\left(\mathcal{H}^{qd-1}\llcorner\mathcal{N}(K_1,\dots,K_q)\right)\wedge a_{K_1,\dots,K_q}\right](\mathbf{1}_A\varphi_{r_1,\dots,r_q})$$

for the mixed curvature measure of $K_1, \ldots, K_q$ of order $r_1, \ldots, r_q$, where $A \subset \mathbb{R}^{qd} \times S^{d-1}$ is an arbitrary Borel measurable set. Furthermore, for $t \in S_+^{q-1}$ we define

$$\tilde{u}(t) := \sum_{i=1}^{q} t_i u_i \qquad \text{and} \qquad \underline{u}(t) := \frac{\sum_{i=1}^{q} t_i u_i}{|\sum_{i=1}^{q} t_i u_i|}$$

if $u_1, \ldots, u_q \in S^{d-1}$ are linearly independent,

$$\mu_{\underline{r}}(\underline{(x,u)}; A) := \frac{1}{\omega_{d-k}} \int_{S_+^{q-1}} \mathbf{1}_A(\underline{x}, \underline{u}(t)) \prod_{i=1}^{q} t_i^{d-1-r_i} \, |\tilde{u}(t)|^{-(d-k)} \, \mathcal{H}^{q-1}(dt)$$

if $u_1, \ldots, u_q \in S^{d-1}$ are linearly independent, and $\mu_{\underline{r}}(\underline{(x,u)}; A) := 0$ otherwise, where $\underline{(x,u)} \in \underline{\mathcal{N}(K)} := \mathcal{N}(K_1) \times \ldots \times \mathcal{N}(K_q)$ and $A \subset \mathbb{R}^{qd} \times S^{d-1}$ is a Borel measurable set.

Theorem 4.9 *Let $q, d \geq 2$, $K_1, \ldots, K_q \in \mathcal{K}^d$, $r_1, \ldots, r_q \in \{0, \ldots, d-1\}$ with $r_1 + \ldots + r_q \geq (q-1)d$ and $k := r_1 + \ldots + r_q - (q-1)d$. Further, let $A \subset \mathbb{R}^{qd} \times S^{d-1}$ be Borel measurable. Then*

$$C_{r_1,\ldots,r_q}(K_1, \ldots, K_q; A) = \int_{\underline{\mathcal{N}(K)}} \mu_{\underline{r}}(\underline{(x,u)}; A) \sum_{\substack{|I_j|=r_j \\ j=1,\ldots,q}} \prod_{j=1}^{q} \frac{\prod_{i \in I_j^c} k_i(j)}{\mathbb{K}_j}$$
$$\times \left| \bigwedge_{j=1}^{q} \bigwedge_{i \in I_j^c} a_i(j) \wedge u_1 \wedge \ldots \wedge u_q \right|^2 \mathcal{H}^{q(d-1)}(d\underline{(x,u)}) \,.$$

Proof. For given $t \in S_+^{q-1}$ we denote by $f_1, \ldots, f_{q-1}, t$ an orthonormal basis of $\mathbb{R}^q$ whose orientation is chosen in such a way that

$$\det(f_1, \ldots, f_{q-1}, t) = (-1)^{c_1(d,r_1,\ldots,r_q)+c_2(d,r_1,\ldots,r_q)+c_3(d,r_1,\ldots,r_q)} \,,$$

where $c_i(d, r_1, \ldots, r_q) \in \mathbb{Z}$, $i = 2, 3$, are constants depending merely on $d, r_1, \ldots, r_q$, and which are given explicitly in the course of the proof. All other conventions of the previous subsection concerning orientations of bases are preserved. Hence we obtain

$$C_{r_1,\ldots,r_q}(K_1, \ldots, K_q; A) = \int_{\underline{\mathcal{N}(K)}} \int_{S_+^{q-1}} \left\langle \wedge_{kd-1} \mathrm{ap}\ DT(\underline{(x,u)}, t) \tilde{a}(\underline{(x,u)}, t), \varphi_{r_1,\ldots,r_q}(\underline{u}(t)) \right\rangle$$
$$\times \mathbf{1}_A(\underline{x}, \underline{u}(t)) \mathcal{H}^{q-1}(dt) \mathcal{H}^{q(d-1)}(d\underline{(x,u)}) \,,$$

where $\tilde{a}(\underline{(x,u)}, t)$ is defined as in the previous subsection (with k replaced by q), and the orientation is properly chosen as we shall see.

A direct calculation shows that, for $\mathcal{H}^{qd-1}$ almost all $(\underline{(x,u)},t) \in \underline{\mathcal{N}(K)} \times S_+^{q-1}$,

$$\begin{aligned}\wedge_{qd-1}\mathrm{ap}\ DT\tilde{a}(\underline{(x,u)},t) = \ & \frac{1}{\mathbb{K}_1}\bigwedge_{i=1}^{d-1}\left(a_i(1), o, \ldots, o, \frac{t_1 k_i(1)}{|\tilde{u}(t)|}a_i(1) + \lambda_i(1)\tilde{u}(t)\right) \wedge \\ & \vdots \\ & \wedge \frac{1}{\mathbb{K}_q}\bigwedge_{i=1}^{d-1}\left(o, \ldots, o, a_i(q), \frac{t_q k_i(q)}{|\tilde{u}(t)|}a_i(q) + \lambda_i(q)\tilde{u}(t)\right) \\ & \wedge \bigwedge_{i=1}^{q-1}\left(o, \ldots, o, \frac{\tilde{u}(f_i)}{|\tilde{u}(t)|} + \lambda_i \tilde{u}(t)\right),\end{aligned}$$

where $\lambda_i(j)$, $i \in \{1, \ldots, d-1\}$ and $j \in \{1, \ldots, q\}$, and λ_i, $i \in \{1, \ldots, q-1\}$, are suitably chosen. We write $\mathrm{Sh}^*(r_1, \ldots, r_{q+1})$ for the set of all $\sigma \in \mathrm{Sh}(r_1, \ldots, r_{q+1})$ which satisfy

$$\begin{aligned}\sigma(\{1, \ldots, R_1\}) \ & \subset \ \{1, \ldots, d-1\} \\ & \vdots \\ \sigma(\{R_{q-1}+1, \ldots, R_q\}) \ & \subset \ \{(q-1)(d-1)+1, \ldots, q(d-1)\},\end{aligned}$$

and then we define $I_\sigma(i)$ and $I_\sigma(i)^c$, for $\sigma \in \mathrm{Sh}^*(r_1, \ldots, r_{q+1})$, as in the preceding subsection. Thus we arrive at

$$\begin{aligned}&\left\langle \wedge_{qd-1}\mathrm{ap}\ DT\tilde{a}(\underline{(x,u)},t), \varphi_{r_1,\ldots,r_q}(\underline{u}(t))\right\rangle \\ &= \frac{1}{\omega_{d-k}}(-1)^{c_1(d,r_1,\ldots,r_q)} \sum_{\sigma \in \mathrm{Sh}^*(r_1,\ldots,r_{q+1})} \mathrm{sgn}(\sigma)\left(\prod_{j=1}^{q} t_j^{d-1-r_j}\right)\prod_{j=1}^{q}\frac{\prod_{i\in I_\sigma(j)^c} k_i(j)}{\mathbb{K}_j}|\tilde{u}(t)|^{-(d-k)} \\ &\quad \times \left[\bigwedge_{i\in I_\sigma(1)} a_i(1), \ldots, \bigwedge_{i\in I_\sigma(q)} a_i(q), \bigwedge_{i\in I_\sigma(1)^c} a_i(1) \wedge \ldots \wedge \bigwedge_{i\in I_\sigma(q)^c} a_i(q) \wedge \bigwedge_{i=1}^{q-1}\tilde{u}(f_i) \wedge \tilde{u}(t)\right].\end{aligned} \tag{184}$$

Observe that

$$\bigwedge_{i=1}^{q-1}\tilde{u}(f_i) \wedge \tilde{u}(t) = \det(f_1, \ldots, f_{q-1}, t)u_1 \wedge \ldots \wedge u_q \tag{185}$$

and

$$\mathrm{sgn}(\sigma) = \left(\prod_{j=1}^{q}\mathrm{sgn}(I_\sigma(j)I_\sigma(j)^c)\right)(-1)^{c_2(d,r_1,\ldots,r_q)}. \tag{186}$$

Set

$$\lambda\alpha_0 := \bigwedge_{i\in I_\sigma(1)^c} a_i(1) \wedge \ldots \wedge \bigwedge_{i\in I_\sigma(q)^c} a_i(q) \wedge u_1 \wedge \ldots \wedge u_q,$$

where $\lambda > 0$ and $|\alpha_0| = 1$, and choose the unit simple $(r_1 + \ldots + r_q)$-vector α_1 in such a way that

$$\langle \alpha_0 \wedge \alpha_1, \Omega \rangle = 1\,;$$

hence

$$\begin{aligned}
&\left[\bigwedge_{i\in I_\sigma(1)} a_i(1), \ldots, \bigwedge_{i\in I_\sigma(q)} a_i(q), \bigwedge_{i\in I_\sigma(1)^c} a_i(1) \wedge \ldots \wedge \bigwedge_{i\in I_\sigma(q)^c} a_i(q) \wedge \bigwedge_{i=1}^{q-1} \tilde{u}(f_i) \wedge \tilde{u}(t) \right] \\
&= \lambda \left\langle \bigwedge_{j=1}^{q} \left[\operatorname{sgn}(I_\sigma(j) I_\sigma(j)^c) \bigwedge_{i\in I_\sigma(j)^c} a_i(j) \wedge u_j \right] \wedge \alpha_1, \Omega \right\rangle \\
&= \lambda \left(\prod_{j=1}^{q} \operatorname{sgn}(I_\sigma(j) I_\sigma(j)^c) \right) (-1)^{c_3(d,r_1,\ldots,r_q)} \left\langle \bigwedge_{j=1}^{q} \bigwedge_{i\in I_\sigma(j)^c} a_i(j) \wedge u_1 \wedge \ldots \wedge u_q \wedge \alpha_1, \Omega \right\rangle \\
&= \lambda^2 \left(\prod_{j=1}^{q} \operatorname{sgn}(I_\sigma(j) I_\sigma(j)^c) \right) (-1)^{c_3(d,r_1,\ldots,r_q)}\,. \qquad (187)
\end{aligned}$$

Combining (184) – (187), we finally obtain the required assertion. □

Remarks.

1. For convex bodies $K_1, K_2 \in \mathcal{K}^d$ and $\alpha \in \{1, \ldots, d-1\}$, the relationship

$$\binom{d}{\alpha} V(K_1[\alpha], K_2[d-\alpha]) = C_{\alpha,d-\alpha}(K_1, -K_2; \mathbb{R}^{2d} \times S^{d-1})$$

is well-known. This equality now can also be verified from the corresponding special cases of Theorems 4.6 and 4.9.

2. By definition and using the preceding notation, we have

$$\left| \bigwedge_{j=1}^{q} \bigwedge_{i\in I_j^c} a_i(j) \wedge u_1 \wedge \ldots \wedge u_q \right| = [\operatorname{lin}\{a_i(1) : i \in I_1\}, \ldots, \operatorname{lin}\{a_i(q) : i \in I_q\}]\,,$$

where the bracket on the right-hand side is defined as in [151] and is therefore non-negative.

Similar to the discussion in the preceding subsection, Theorem 4.9 can be specified in various ways. The case where $K_1, \ldots, K_q \in \mathcal{K}^d$ are polytopes is of particular interest, since it shows that the representation of mixed curvature measures given in Theorem 4.9 extends the defining relationship (3.1) in [151] in a natural way. For $F_i \in \mathcal{F}_{r_i}(K_i)$, the bracket $[F_1, \ldots, F_q]$ is defined as in [151] in the following.

Corollary 4.10 *Let $q, d \geq 2$, let $K_1, \ldots, K_q \in \mathcal{K}^d$ be polytopes (or polyhedral sets), let $r_1, \ldots, r_q \in \{0, \ldots, d-1\}$ with $r_1 + \ldots + r_q \geq (q-1)d$ and $k := r_1 + \ldots + r_q - (q-1)d$. Further, let $A \subset \mathbb{R}^{qd}$ and $C \subset S^{d-1}$ be Borel measurable. Then*

$$C_{r_1,\ldots,r_q}(K_1, \ldots, K_q; A \times C) = \sum_{F_1 \in \mathcal{F}_{r_1}(K_1)} \cdots \sum_{F_q \in \mathcal{F}_{r_q}(K_q)} \frac{\mathcal{H}^{d-1-k}\left(\left(\sum_{i=1}^q N(K_i, F_i)\right) \cap C\right)}{\omega_{d-k}} \times [F_1, \ldots, F_q] \left(\otimes_{i=1}^q \left(\mathcal{H}^{r_i} \llcorner F_i\right)\right)(A) .$$

In particular, Corollary 4.10 is an extension of the defining relation (3.1) in [151], since

$$\gamma(F_1, \ldots, F_q; K_1, \ldots, K_q) = \frac{\mathcal{H}^{d-1-k}\left(\left(\sum_{i=1}^q N(K_i, F_i)\right) \cap S^{d-1}\right)}{\omega_{d-k}} ,$$

provided that $\text{lin}\, N(K_1, F_1), \ldots, \text{lin}\, N(K_q, F_q)$ are linearly independent subspaces.

Proof. We continue to use the previous notation. Under the present special assumptions, the formula of Theorem 4.9 boils down to

$$\begin{aligned} C_{r_1,\ldots,r_q}(K_1, \ldots, K_q; A \times C) = \sum_{F_1 \in \mathcal{F}_{r_1}(K_1)} \cdots \sum_{F_q \in \mathcal{F}_{r_q}(K_q)} & [F_1, \ldots, F_q] \left(\otimes_{i=1}^q \left(\mathcal{H}^{r_i} \llcorner F_i\right)\right)(A) \\ & \times \int_{S_+^{q-1}} \int_{N(K_1,F_1)\cap S^{d-1}} \cdots \int_{N(K_q,F_q)\cap S^{d-1}} \left(\prod_{j=1}^q t_j^{d-1-r_j}\right) \\ & \times \left| \bigwedge_{j=1}^q \bigwedge_{i \in I_j^c} a_i(j) \wedge u_1 \wedge \ldots \wedge u_q \right| \mathbf{1}_C(\underline{u}(t)) |\tilde{u}(t)|^{-(d-k)} \\ & \times \mathcal{H}^{d-1-r_q}(du_q) \ldots \mathcal{H}^{d-1-r_1}(du_1) \mathcal{H}^{q-1}(dt) , \end{aligned}$$

where $\{a_i(j) : i \in I_j^c\}$ is an orthonormal basis of $\text{Tan}(N(K_j, F_j) \cap S^{d-1}, u_j)$ and $\{a_i(j) : i \in I_j\}$ spans $\text{lin}(F_j - F_j)$, $j = 1, \ldots, q$. Let $F_j \in \mathcal{F}_{r_j}(K_j)$, $j = 1, \ldots, q$, be fixed and assume that $\text{lin}\, N(K_1, F_1), \ldots, \text{lin}\, N(K_q, F_q)$ are linearly independent. Consider the map

$$T : N(K_1, F_1) \times \ldots \times N(K_q, F_q) \times S_+^{q-1} \to S^{d-1} , \quad (u_1, \ldots, u_q, t) \mapsto \underline{u}(t) .$$

Then the assertion of the corollary follows by an application of the area formula once we have checked that

$$JT(\underline{u}, t) = |\tilde{u}(t)|^{-(d-k)} \left(\prod_{j=1}^q t_j^{d-1-r_j}\right) \left| \bigwedge_{j=1}^q \bigwedge_{i \in I_j^c} a_i(j) \wedge u_1 \wedge \ldots \wedge u_q \right| .$$

In fact, using the previous notation, we find that

$$\frac{\partial T}{\partial a_i(j)}(\underline{u}, t) = \frac{t_j a_i(j)}{|\tilde{u}(t)|} + \lambda_i(j) \underline{u}(t) ,$$

where $i \in I_j^c$, $j \in \{1, \dots, q\}$, and $\lambda_i(j) \in \mathbb{R}$,

$$\frac{\partial T}{\partial f_l}(\underline{u}, t) = \frac{\tilde{u}(f_l)}{|\tilde{u}(t)|} + \lambda_l \underline{u}(t)\,,$$

where $l \in \{1, \dots, q-1\}$ and $\lambda_l \in \mathbb{R}$, and

$$\left\langle \frac{\partial T}{\partial a_i(j)}(\underline{u}, t), \underline{u}(t) \right\rangle = \left\langle \frac{\partial T}{\partial f_l}(\underline{u}, t), \underline{u}(t) \right\rangle = 0\,.$$

Here $f_1, \dots, f_{q-1}, t$ is an orthonormal basis of $\mathbb{R}^q$. Thus

$$\begin{aligned} JT(\underline{u}, t) &= \left| \bigwedge_{j=1}^{q} \bigwedge_{i \in I_j^c} \left(\frac{t_j a_i(j)}{|\tilde{u}(t)|} + \lambda_i(j) \underline{u}(t) \right) \wedge \bigwedge_{i=1}^{q-1} \left(\frac{\tilde{u}(f_i)}{|\tilde{u}(t)|} + \lambda_i \underline{u}(t) \right) \right| \\ &= \left| \bigwedge_{j=1}^{q} \bigwedge_{i \in I_j^c} \left(\frac{t_j a_i(j)}{|\tilde{u}(t)|} \right) \wedge \bigwedge_{i=1}^{q-1} \left(\frac{\tilde{u}(f_i)}{|\tilde{u}(t)|} \right) \wedge \underline{u}(t) \right| \\ &= |\tilde{u}(t)|^{-(d-k)} \left| \bigwedge_{j=1}^{q} \bigwedge_{i \in I_j^c} (t_j a_i(j)) \wedge \bigwedge_{i=1}^{q-1} \tilde{u}(f_i) \wedge \tilde{u}(t) \right|\,, \end{aligned}$$

from which the formula for the Jacobian immediately follows.

References

[1] A. D. Aleksandrov. Smoothness of the convex surface of bounded Gaussian curvature. *Doklady Akad. Nauk SSSR*, 36:195–199, 1942.

[2] A. D. Aleksandrov. *Die innere Geometrie der konvexen Flächen.* Akademie-Verlag, Berlin, 1955.

[3] S. Alesker. Description of continuous isometry covariant valuations on convex sets. *Geom. Dedicata*, 74:241–248, 1999.

[4] S. Alesker. Continuous rotation invariant valuations on convex sets. *Ann. of Math.*, 149:977–1005, 1999.

[5] S. Alesker, S. Dar, and V. Milman. A remarkable measure preserving diffeomorphism between two convex bodies in $\mathbb{R}^n$. *Geom. Dedicata*, 74:201–212, 1999.

[6] R. Arnold. On the Aleksandrov-Fenchel inequality and the stability of the sphere. *Monatsh. Math.*, 115:1–11, 1993.

[7] A. J. Baddeley. Absolute curvatures in integral geometry. *Math. Proc. Camb. Philos. Soc.*, 886:45–58, 1980.

[8] A.J. Baddeley. Spatial sampling and censoring, Stochastic Geometry: Likelihood and Computation, ed. by O. Barndorff-Nielsen, W. Kendall, and M.N.M. van Lieshout, Chapman & Hall/CRC Boca Raton, 37–78, 1999.

[9] A.J. Baddeley and R.D. Gill. The empty space hazard of a spatial pattern, Preprint 845, University Utrecht, Department of Mathematics, 1994.

[10] A.J. Baddeley and R.D. Gill. Kaplan-Meier estimators of distance distributions for spatial point processes, *Ann. Statist.*, 25:263–292, 1997.

[11] V. Bangert. Analytische Eigenschaften konvexer Funktionen auf Riemannschen Mannigfaltigkeiten. *J. reine angew. Math.*, 307/308:309–324, 1979.

[12] V. Bangert. Convex hypersurfaces with bounded first mean curvature measure. *Calc. Var. Partial Differential Equations*, 8:259–278, 1999.

[13] F. Barthe. On a reverse form of the Brascamp-Lieb inequality. *Invent. Math.*, 134:335–361, 1998.

[14] Ch. Bauer. Minimal and reduced pairs of convex bodies. *Geom. Dedicata*, 62:179–192, 1996.

[15] M. Berger. Riemannian geometry during the second half of the twentieth century. *Jahresber. Deutsch. Math.-Verein.*, 100:45–208, 1998.

[16] U. Betke. Mixed volumes of polytopes. *Arch. Math.*, 58:388–391, 1992.

[17] J. Bourgain and J. Lindenstrauss. Projection bodies. In J. Lindenstrauss and V. D. Milman, editors, *Geometric Aspects of Functional Analysis*, pages 250–270, Berlin, 1988. Lecture Notes in Math. 1317, Springer.

[18] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44:375–417, 1991.

[19] J. N. Brooks and J. B. Strantzen. Blaschke's rolling theorem in $\mathbb{R}^n$. *Mem. Amer. Math. Soc.*, 80, No. 405:101 pp., 1989.

[20] J. E. Brothers. Rectifiability and integral-geometric measures in homogeneous spaces. *Bull. Amer. Math. Soc.*, 75:387–390, 1969.

[21] J. E. Brothers. The (φ, k) rectifiable subsets of a homogeneous space. *Acta Math.*, 122:197–229, 1969.

[22] Yu. D. Burago and B. V. Kalinin. Smoothness and singularities of convex hypersurfaces. *St. Petersbg. Math. J.*, 8:399–411, 1997.

[23] H. Busemann. *Convex surfaces.* Interscience Publishers, New York, 1958.

[24] S. Campi. Recovering a centred convex body from the areas of its shadows: A stability estimate. *Ann. Mat. Pura Appl.*, 151:289–302, 1988.

[25] G. D. Chakerian and H. Groemer. Convex bodies of constant width. In P. M. Gruber and J. M. Wills, editors, *Convexity and its Applications*, pages 49–96, Basel, Boston, Stuttgart, 1983. Birkhäuser Verlag.

[26] S.N. Chiu and D. Stoyan. Estimators of distance distributions for spatial patterns. *Statist. Neerlandica*, 52:239–246, 1998.

[27] F. H. Clarke. *Optimization and Nonsmooth Analysis.* John Wiley & Sons, New York, 1983. 308 pp.

[28] D. L. Cohn. *Measure Theory.* Birkhäuser, Boston, 1980.

[29] A. Colesanti and D. Hug. Steiner type formulae and weighted measures of singularities for semi-convex functions. Preprint Nr. 35/1996, Universität Freiburg, pp. 30. To appear in *Trans. Amer. Math. Soc.*

[30] A. Colesanti and D. Hug. Hessian measures of semi-convex functions and applications to support measures of convex bodies. Preprint Nr. 20/1999, Universität Freiburg, pp. 32. To appear in *Manuscripta Math.*

[31] B. V. Dekster. On reduced convex bodies. *Israel J. Math.*, 56:247–256, 1986.

[32] V. I. Diskant. Stability of a sphere in the class of convex surfaces of bounded specific curvature. *Siberian Math. J.*, 9:610–615, 1968.

[33] V. I. Diskant. Bounds for convex surfaces with bounded curvature functions. *Siberian Math. J.*, 12:78–89, 1971.

[34] V. I. Diskant. Bounds for the discrepancy between convex bodies in terms of the isoperimetric difference. *Siberian Math. J.*, 13:529–532, 1972.

[35] V. I. Diskant. Stability of the solution of the Minkowski equation. *Siberian Math. J.*, 14:466–469, 1973.

[36] V. I. Diskant. On the question of the order of the stability function in Minkowski's problem (in Russian). *Ukrain. Geom. Sb.*, 22:45–47, 1979.

[37] V. I. Diskant. Refinements of an isoperimetric inequality and stability theorems in the theory of convex bodies (in Russian). *Trudy Inst. Mat. (Novosibirsk)*, 14:98–132, MR 91c:52009, 1989, Sovrem. Probl. Geom. Analiz.

[38] M. Dyer, P. Gritzmann, and A. Hufnagel. On the complexity of computing mixed volumes. *SIAM J. Comput.*, 27:356–400, 1998.

[39] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton, 1992.

[40] K. J. Falconer. *Techniques in Fractal Geometry.* Mathematical Foundations and Applications. Wiley, Chichester, 1997.

[41] H. Fallert. Quermaßdichten für Punktprozesse konvexer Körper und Boolesche Modelle. *Math. Nachr.*, 181:165–184, 1996.

[42] H. Federer. *Geometric Measure Theory.* Springer, Berlin, 1969.

[43] W. J. Firey. The integral-geometric densities of tangent flats to convex bodies. In J. E. Goodman, E. Lutwak, J. Malkevitch, and R. Pollack, editors, *Discrete Geometry and Convexity*, volume 440, pages 92–96, New York 1982, 1985. Ann. N. Y. Acad. Sci.

[44] R. J. Gardner. *Geometric Tomography.* Encyclopedia of Mathematics and its Applications 58, Cambridge University Press, Cambridge, 1995.

[45] R. Gariepy and W. D. Pepe. On the level sets of a distance function in a Minkowski space. *Proc. Amer. Math. Soc.*, 31:255–259, 1972.

[46] A. Giannopoulos and V. D. Milman. Extremal problems and isotropic positions of convex bodies. Preprint 1999.

[47] A. Giannopoulos and M. Papadimitrakis. Isotropic surface area measures. Preprint 1999.

[48] A. Giannopoulos, and I. Perissinaki, and A. Tsolomitis. John's theorem for an arbitrary pair of convex bodies. Preprint 1999.

[49] S. Glasauer. *Integralgeometrie konvexer Körper im sphärischen Raum.* Dissertation, Freiburg, 1995.

[50] S. Glasauer. A generalization of intersection formulae of integral geometry. *Geom. Dedicata*, 68:101–121, 1997.

[51] S. Glasauer. An Euler-type version of the local Steiner formula for convex bodies. *Bull. London Math. Soc.*, 30:618–622, 1998.

[52] S. Glasauer. Translative and kinematic integral formulae concerning the convex hull operation. *Math. Z.*, 229:493–518, 1998.

[53] S. Glasauer. Kinematic formulae for support measures of convex bodies. *Beiträge Algebra Geom.*, 40:113–124, 1999.

[54] P. Goodey and W. Weil. Distributions and valuations. *Proc. London Math. Soc. (3)*, 49:504–516, 1984.

[55] P. Goodey, R. Schneider, and W. Weil. Projection functions on higher rank Grassmannians, Geometric Aspects of Functional Analysis (Israel, 1992-1994). In *Oper. Theory Adv. Appl.*, pages 75–90, Basel, 1995. Birkhäuser.

[56] P. Gritzmann and V. Klee. On the complexity of some basic problems in computational convexity. II. Volume and mixed volume. In T. Bisztriczky, P. McMullen, R. Schneider, and A. I. Weiss, editors, *Proceedings of the NATO Advanced Study Institute on Polytopes: Abstract, Convex and Computational*, volume C 440, pages 373–466, Dordrecht, 1994. Kluwer Academic Publishers.

[57] H. Groemer. On an inequality of Minkowski for mixed volumes. *Geom. Dedicata*, 33:117–122, 1990.

[58] H. Groemer and R. Schneider. Stability estimates for some geometric inequalities. *Bull. Lond. Math. Soc.*, 23:67–74, 1991.

[59] K. Hann. The average number of normals through a point in a convex body and a related Euler-type identity. *Geom. Dedicata*, 48:27–55, 1993.

[60] K. Hann. What's the bound on the average number of normals. *Amer. Math. Monthly*, 103:897–900, 1996.

[61] K. Hann. Normals in a Minkowski plane. *Geom. Dedicata*, 64:355–364, 1997.

[62] K. Hann. Minkowski normals for polycircles. *Geom. Dedicata*, 75:57–65, 1999.

[63] M.B. Hansen, A.J. Baddeley, and R.D. Gill. First contact distributions for spatial patterns: regularity and estimation. *Adv. in Appl. Probab.*, 31:15-33, 1999.

[64] M.B. Hansen, R.D. Gill, and A.J. Baddeley. Kaplan-Meier type estimators for linear contact distributions, *Scand. J. Statist.*, 23:129–155, 1998.

[65] G.H. Hardy, J.E. Littlewood, and G. Polya. *Inequalities.* Cambridge University Press, Cambridge, 1967.

[66] M. Henk, J. Richter-Gebert, and G. M. Ziegler. Basic properties of convex polytopes. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, pages 243–270, Boca Raton, 1997. CRC Press.

[67] R. A. Horn and C. A. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.

[68] A. Hufnagel. Algorithmic problems in Brunn-Minkowski theory. Thesis, Trier, FB 4, 153 pages, 1995.

[69] D. Hug. On the mean number of normals through a point in the interior of a convex body. *Geom. Dedicata*, 55:319–340, 1995.

[70] D. Hug. Contributions to affine surface area. *Manuscr. Math.*, 91:283–301, 1996.

[71] D. Hug. Curvature relations and affine surface area for a general convex body and its polar. *Result. Math.*, 29:233–248, 1996.

[72] D. Hug. Absolute continuity for curvature measures of convex sets I. *Math. Nachr.*, 195:139–158, 1998.

[73] D. Hug. Generalized curvature measures and singularities of sets with positive reach. *Forum Math.*, 10:699–728, 1998.

[74] D. Hug. Absolute continuity for curvature measures of convex sets II. To appear in Math. Z, 49 pp.

[75] D. Hug. Contact distributions of Boolean models. Preprint Nr. 21/1999, Universität Freiburg, 35 pp. Submitted.

[76] D. Hug and G. Last. On support measures in Minkowski spaces and contact distributions in stochastic geometry. Preprint Nr. 21/1998, Universität Freiburg, 55 pp. To appear in *Ann. Probab.*.

[77] D. Hug and R. Schätzle. Intersections and translative integral formulas for boundaries of convex bodies. To appear in *Man. Math.*, 30 pp.

[78] B. Huppert. *Angewandte Lineare Algebra.* Walter de Gruyter, Berlin, 1990.

[79] O. Kallenberg. *Random Measures.* Akademie Verlag and Academic Press, London, 1983.

[80] M. Kiderlen and W. Weil. Measure-valued valuations and mixed curvature measures of convex bodies. *Geom. Dedicata*, 76:291–329, 1999.

[81] P. Kohlmann. Compact convex bodies with one curvature measure near the surface measure. *J. Reine Angew. Math.*, 458:201–217, 1995.

[82] P. Kohlmann. Splitting and Pinching for convex sets. *Geom. Dedicata*, 60:125–143, 1996.

[83] P. Kohlmann. Characterization of metric balls in standard space forms via curvature. *Math. Z.*, 227:729–746, 1998.

[84] P. Kohlmann. Characterizations via linear combinations of curvature measures. *Arch. Math.*, 70:250–256, 1998.

[85] P. Kohlmann. Curvature measures and stability. To appear in J. Geom.

[86] P. Kohlmann. Stability of metric spheres in space forms. Preprint.

[87] P. Kohlmann. The convex–concave principle and uniqueness for hypersurfaces in space forms. Preprint.

[88] P. Kohlmann. Weyl–identities for pairs of higher order mean curvatures and uniqueness results. Preprint.

[89] U. Lang. Diameter bounds for convex surfaces with pinched mean curvature. *Manuscripta Math.*, 86:15–22, 1995.

[90] M. Lassak. Approximation of convex bodies by centrally symmetric bodies. *Geom. Dedicata*, 72:63–68, 1998.

[91] G. Last and M. Holtmann. On the empty space function of some germ-grain models. To appear in *Pattern Recognition*.

[92] G. Last and R. Schassberger. On the distribution of the spherical contact vector of stationary germ-grain models. *Adv. in Appl. Probab.*, 30:36–52, 1998.

[93] G. Last and R. Schassberger. On the second derivative of the spherical contact distribution function of smooth grain models. Preprint.

[94] K. Leichtweiß. Zur Affinoberfläche konvexer Körper. *Manuscr. Math.*, 56:429–464, 1986.

[95] K. Leichtweiß. Convexity and differential geometry. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, pages 1045–1080, Amsterdam, 1993. North-Holland.

[96] K. Leichtweiß. Nearly umbilical ovaloids in the n-space are close to spheres. *Results Math.*, 36:102–109, 1999.

[97] Senchun Lin. Splitting and Pinching for complete convex surfaces in E^3. *Geom. Dedicata*, 69:83–90, 1998.

[98] Senchun Lin. A mean curvature pinching theorem for convex hypersurfaces in E^{n+1}. Preprint.

[99] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge studies in advanced mathematics, Cambridge, 1995.

[100] R. J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.*, 80:309–323, 1995.

[101] P. McMullen. Continuous translation invariant valuations on the space of compact convex sets. *Arch. Math.*, 34:377–384, 1980.

[102] P. McMullen. Monotone translation invariant valuations on convex bodies. *Arch. Math.*, 55:595–598, 1990.

[103] P. McMullen. Isometry covariant valuations on convex bodies. *Rend. Circ. Mat. Palermo (2) Suppl.*, 50:259–271, 1997.

[104] A. V. Pogorelov. *Extrinsic geometry of convex surfaces*, volume 35 of *Transl. Math. Monographs.* American Mathematical Society, Providence, 1973.

[105] A. V. Pogorelov. *The Minkowski Multidimensional Problem.* Winston & Sons, Washington DC, 1978.

[106] D. Preiss. Geometry of measures in $\mathbb{R}^n$: Distribution, rectifiability, and densities. *Ann. Math.*, 125:537–643, 1987.

[107] S. T. Rachev and L. Rüschendorf *Mass Transportation Problems.* Volume I: Theory, Volume II: Applications Springer, New York, 1998.

[108] J. Rataj. The iterated version of a translative integral formula for sets of positive reach. *Rend. Circ. Mat. Palermo (2) Suppl.*, 46:129–138, 1997.

[109] J. Rataj. Translative and kinematic formulae for curvature measures of flat sections. *Math. Nachr.*, 197:89–101, 1999.

[110] J. Rataj and M. Zähle. Mixed curvature measures for sets of positive reach and a translative integral formula. *Geom. Dedicata*, 57:259–283, 1995.

[111] M. Ross. *Federer's Theorem.* Centre for Mathematical Analysis, Australian National University, Canberra, 1984.

[112] S. Rosset. Normalized symmetric functions, Newton's inequalities and a new set of stronger inequalities. *Amer. Math. Monthly*, 96:815–819, 1989.

[113] W. Rother and M. Zähle. Absolute curvature measures, II. *Geom. Dedicata*, 41:229–240, 1992.

[114] J. R. Sangwine-Yager. A representation for volume involving interior reach. *Math. Ann.*, 298:1–5, 1994.

[115] J. R. Sangwine-Yager. A generalization of outer parallel sets of a convex set. *Proc. Amer. Math. Soc.*, 123:1559–1564, 1995.

[116] L. A. Santaló. Total curvatures of compact manifolds immersed in Euclidean space. In *Symposia Mathematica,* Istituto Nazionale di Alta Matematica, volume XIV, pages 363–390, London and New York, 1974. Academic Press.

[117] E. Schmidt. Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie. I. *Math. Nachr.*, 1:81–157, 1948.

[118] R. Schneider. Kinematische Berührmaße für konvexe Körper. *Abh. Math. Sem. Univ. Hamburg*, 44:12–23, 1975.

[119] R. Schneider. Curvature measures of convex bodies. *Ann. di Mat. Pura Appl.*, 116:101–134, 1978.

[120] R. Schneider. Bestimmung konvexer Körper durch Krümmungsmaße. *Comment. Math. Helv.*, 54:42–60, 1979.

[121] R. Schneider. Boundary structure and curvature of convex bodies. In J. Tölke and J. M. Wills, editors, *Contributions to Geometry*, pages 13–59, Basel, 1979. Proc. Geometry Symp., Siegen 1978, Birkhäuser.

[122] R. Schneider. A stability estimate for the Aleksandrov-Fenchel inequality, with an application to mean curvature. *Manuscr. Math.*, 69:291–300, 1990.

[123] R. Schneider. *Convex Bodies: the Brunn-Minkowski Theory.* Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 1993.

[124] R. Schneider. Convex surfaces, curvature and surface area measures. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, volume A, pages 273–299, Amsterdam, 1993. North-Holland.

[125] R. Schneider. An extension of the principal kinematic formula of integral geometry. *Rend. Circ. Mat. Palermo*, 35:275–290, 1994.

[126] R. Schneider. Polytopes and Brunn-Minkowski Theory. In T. Bisztriczky, P. McMullen, R. Schneider, and A. I. Weiss, editors, *Proceedings of the NATO Advanced Study Institute on Polytopes: Abstract, Convex and Computational*, volume C 440, pages 273–299, Dordrecht, 1994. Kluwer Academic Publishers.

[127] R. Schneider. Simple valuations on convex bodies. *Mathematika*, 43:32–39, 1996.

[128] R. Schneider. Measures in convex geometry. *Rend. Istit. Mat. Univ. Trieste*, 29:215–265, 1998.

[129] R. Schneider. Convex bodies in exceptional relative positions. Preprint, Universität Freiburg 29/97, 16 pp.

[130] R. Schneider. Mixed functionals of convex bodies. Preprint, Universität Freiburg 9/1999, 12 pp.

[131] R. Schneider and W. Weil. Translative and kinematic integral formulae for curvature measures. *Math. Nachr.*, 129:67–80, 1986.

[132] R. Schneider and W. Weil. *Integralgeometrie.* Teubner, Stuttgart, 1992.

[133] R. Schneider and W. Weil. *Stochastische Geometrie.* Teubner, Stuttgart, *in preparation*, 2000.

[134] R. Schneider and J. A. Wieacker. Random touching of convex bodies. In R. V. Ambartzumian and W. Weil, editors, *Stochastic Geometry, Geometric Statistics, Stereology*, Oberwolfach 1983, pages 154–169, Leipzig, 1984. Teubner.

[135] R. Schneider and J. A. Wieacker. Integral geometry. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, volume B, pages 1349–1390, Amsterdam, 1993. North-Holland.

[136] L. M. Simon. *Lectures on Geometric Measure Theory.* Centre for Mathematical Analysis, Australian National University, Canberra, 1984.

[137] M. Spivak. *A Comprehensive Introduction to Differential Geometry.* Publish or Perish, Berkeley, 1979.

[138] A. C. Thompson. *Minkowski Geometry.* Cambridge University Press, Cambridge, 1996.

[139] J. Verschelde, K. Gatermann, and R. Cools. Mixed-volume computation by dynamic lifting applied to polynomial system solving. *Discrete Comput. Geom.*, 16:69–112, 1996.

[140] Yu. A. Volkov. Stability of the solution of Minkowski's problem (in Russian). *Vestnik Leningrad. Univ., Ser. Mat. Meh. Astronom.*, 18:33–43, 1963.

[141] R. Walter. Some analytical properties of geodesically convex sets. *Abh. Math. Sem. Univ. Hamburg*, 45:263–282, 1976.

[142] R. Walter. Konvexität in riemannschen Mannigfaltigkeiten. *Jber. d. Dt. Math. Verein.*, 83:1–31, 1981.

[143] R. Walter. Compact hypersurfaces with a constant higher mean curvature function. *Math. Ann.*, 270:125–145, 1985.

[144] W. Weil. Ein Approximationssatz für konvexe Körper. *Manuscr. Math.*, 8:335–362, 1973.

[145] W. Weil. Über den Vektorraum der Differenzen von Stützfunktionen konvexer Körper. *Math. Nachr.*, 59:353–369, 1974.

[146] W. Weil. Zufällige Berührung konvexer Körper durch q-dimensionale Ebenen. *Result. Math.*, 4:84–101, 1981.

[147] W. Weil. Das gemischte Volumen als Distribution. *Manuscripta Math.*, 36:1–18, 1981/82.

[148] W. Weil. Inner contact probabilities for convex bodies. *Adv. Appl. Prob.*, 14:582–599, 1982.

[149] W. Weil. Collision probabilities for convex sets. *J. Appl. Probab.*, 26:649–654, 1989.

[150] W. Weil. Iterations of translative integral formulae and non-isotropic poisson processes of particles. *Math. Z.*, 205:531–549, 1990.

[151] W. Weil. Mixed measures and functionals of translative integral geometry. Preprint.

[152] W. Weil. Densities of mixed volumes for Boolean models. Preprint.

[153] W. Weil and J. A. Wieacker. A representation theorem for random sets. *Probab. Math. Statist.*, 9:147–151, 1988.

[154] B. White. A new proof of Federer's structure theorem for k-dimensional subsets of $\mathbb{R}^n$. *J. Amer. Math. Soc.*, 11:693–701, 1998.

[155] M. Zähle. Integral and current representation of Federer's curvature measures. *Arch. Math.*, 46:557–567, 1986.

[156] M. Zähle. Absolute curvature measures. *Math. Nachr.*, 140:83–90, 1989.

[157] M. Zähle. Approximation and characterization of generalized Lipschitz-Killing curvatures. *Ann. Global Anal. Geom.*, 8:249–260, 1990.

[158] M. Zähle. Nonosculating sets of positive reach. *Geom. Dedicata*, 76:183–187, 1999.

[159] V. A. Zalgaller. On k-dimensional directions singular for a convex body F in $\mathbb{R}^n$. *J. Soviet Math.*, 3:437–441, 1975.