Boolean models

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Abstract

This survey is a preliminary version of a chapter of the forthcoming book "Geometry and Physics of Spatial Random Systems" edited and partially written by Daniel Hug, Michael Klatt, Klaus Mecke, Gerd Schröder Turk and Wolfgang Weil.

The topic of this survey are geometric functionals of a Boolean model (in Euclidean space) governed by a stationary Poisson process of convex grains. The Boolean model is a fundamental benchmark of stochastic geometry and continuum percolation. Moreover, it is often used to model amorphous connected structures in physics, materials science and biology. Deeper insight into the geometric and probabilistic properties of Boolean models and the dependence on the underlying Poisson process can be gained by considering various geometric functionals of Boolean models. Important examples are the intrinsic volumes and Minkowski tensors. We survey here local and asymptotic density (mean value) formulas as well as second order properties and central limit theorems.

Keywords: Boolean model, geometric functionals, mean values, covariances, central limit theorem

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1 Introduction

Consider a finite or countably infinite number of points X_n scattered randomly in d-dimensional space $\mathbb{R}^d$. Then attach a random particle (a nonempty compact set) Z_n to each point X_n which yields the grain $Z_n + X_n$. The union

$$Z := \bigcup_n (Z_n + X_n)$$

is then a random set in $\mathbb{R}^d$ which can serve as a model for spatial structures in physics and other applied sciences (biology, geology, mineralogy etc.). An important benchmark case arises if the random points X_n come from a stationary Poisson process X and the sequence (Z_n) is independent of X and formed by independent and identically distributed random compact sets, mostly assumed to be convex. Then the *particle process*

$$Y := \{Z_n + X_n : n \in \mathbb{N}\}$$

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is a stationary Poisson process on the space $\mathscr{C}^{(d)}$ of nonempty compact subsets of $\mathbb{R}^d$. The union set Z is called a *stationary Boolean model* (with compact particles (grains)). Its statistical properties are completely determined by two quantities, the *intensity* γ of the Poisson process X, a number which we assume to be positive and finite, and the distribution $\mathbb{Q}$ *of the typical particle (grain)* Z_1, a probability measure on the class $\mathscr{C}_0^d$ of suitably centered particles in $\mathbb{R}^d$ (we may assume, for example, that the particles have their circumcenter at the origin). Fundamental properties of Boolean models are summarized in [22, 1, 14]. It is often assumed that the particles are convex, that is random elements of the space $\mathscr{K}^d$ of all compact and convex subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric. We denote $\mathscr{K}^{(d)} := \mathscr{K}^d \setminus \{\emptyset\}$.

Section 2 describes some basic properties of the Boolean model. Starting with Subsection 3.1, we assume that the distribution of the typical grain is concentrated on $\mathscr{K}^{(d)}$. We consider an additive (and measurable) functional φ on the convex ring $\mathscr{R}^d$, the system of all finite unions of compact convex sets and study the random variable $\varphi(Z \cap W)$, where $W \in \mathscr{K}^{(d)}$. Starting with Subsection 3.2 we assume that the Poisson process Y and hence also the Boolean model Z is stationary. If φ satisfies a *translative integralgeometric principle*, then Theorem 3.13 expresses the *local density* $\mathbb{E}\varphi(Z \cap W)$ in terms of expected mixed functionals of independent copies of the typical grain. A first result of this type was obtained by Weil [26], dealing with mixed measures associated with curvature measures; see also [22, Theorem 9.1.5]. Later the framework was extended to tensor valuations; see [4, Theorem 5.4] and [24]. As illustrated with many examples, our theorem generalizes and unifies these results. We also consider (asymptotic) geometric *densities* of Z defined by the limit

$$\overline{\varphi}[Z] := \lim_{r\to\infty} \frac{\mathbb{E}\varphi(Z \cap rW)}{V_d(rW)},$$

where $W \in \mathscr{K}^{(d)}$ has nonempty interior. In fact, Theorem 3.17 treats asymptotic densities in a very general situation. In Subsection 3.3 we assume the Boolean model not only to be stationary but also isotropic. If φ satisfies a *kinematic integralgeometric principle*, then Theorem 3.25 expresses the local and the asymptotic densities of φ in terms of the intensity and the mean intrinsic volumes of the typical grain. Special cases of these results were first discovered by Miles [17] and Davy [2]; see [22, Section 9.1] for an extensive discussion. Potentially, the theorem could be used to construct estimators of the intensity γ. In particular, the classical results show that in the stationary and isotropic case, the densities of the intrinsic volumes of the Boolean model determine the intensity of the underlying particle process completely. Theorem 3.27 (taken from [6]) generalizes these ideas and establishes a corresponding uniqueness result in a non-isotropic situation.

Section 4 summarises some of the results from [11, 10] on second order properties of geometric (i.e. additive, locally bounded and translation invariant) functionals φ of the Boolean model. Variances and covariances are much harder to analyse than mean values. Even in the isotropic case explicit formulas are rather rare. Still it is possible to exploit the Fock space representation from [14, Theorem 18.6] to derive with Theorem 4.1 a series representation for the asymptotic covariance

$$\sigma(\varphi,\psi) := \lim_{r\to\infty} \frac{\mathbb{C}\mathrm{ov}(\psi(Z \cap rW), \varphi(Z \cap rW))}{V_d(rW)}$$

between two geometric functionals ϕ and ψ. The formulas involve geometric functionals φ^* and ψ^*, where $\varphi^*(K) := \mathbb{E}\varphi(Z \cap K) - \varphi(K)$, $K \in \mathscr{K}^d$. Without isotropy, these functionals can be treated for volume and surface area; see Theorem 4.2. The formulas are not completely explicit, but involve local volume and surface covariances of the typical grain. Further progress can be made under an isotropy assumption. Then Theorem 4.3 provides a formula for the asymptotic covariances between the intrinsic volumes, involving another series representation. In the planar case this leads to rather explicit formulas for the asymptotic covariance structure of intrinsic volumes; see Theorem 4.5. Subsection 4.4 presents simple assumptions guaranteeing positivity of asymptotic variances. In the final Section 5 we present some central limit theorems from [11], derived via a combination of Stein's method with stochastic analysis tools for general Poisson processes; see [19, 14].

2 Basic definitions and facts

Particle processes and germ-grain models can be introduced via a marking procedure or by considering processes of compact sets and their unions sets. For stationary processes, the two approaches are essentially equivalent, but the latter viewpoint seems to be preferable if a natural centering of the particles is not available.

2.1 Particle processes and germ-grain models

In the introduction we have used a marking procedure to introduce a Boolean model. Instead we can directly start with a Poisson process Y on $\mathscr{C}^{(d)}$ (equipped with the Borel σ-field generated by the Hausdorff metric) with intensity measure $\Theta \neq 0$ and defined on some given probability space $(\Omega, \mathscr{A}, \mathbb{P})$. We assume throughout that Θ is *locally finite*, that is,

$$\Theta(\{K \in \mathscr{C}^{(d)} : K \cap C \neq \emptyset\}) < \infty, \quad C \in \mathscr{C}^d, \tag{2.1}$$

where $\mathscr{C}^d := \mathscr{C}^{(d)} \cup \{\emptyset\}$ denotes the set of all compact subsets of $\mathbb{R}^d$. We also assume that Y is *diffuse*, so that Y can indeed be identified with a random subset of $\mathscr{C}^{(d)}$; see, e.g., [14, Proposition 6.9]. The point process Y is said to be a *Poisson particle process*. The *Boolean model* Z based on Y is given by

$$Z := \bigcup_{K \in Y} K. \tag{2.2}$$

It can be easily shown that $\{Z \cap C \neq \emptyset\}$ is an event, that is, an element of $\mathscr{A}$ (see [14, 22]). Moreover, it follows from (2.1) that Z is almost surely closed. Therefore Z is a *random closed set* in the sense of [22, 18]. It follows directly from the defining properties of a Poisson process that

$$\mathbb{P}(Z \cap C = \emptyset) = \exp[-\Theta(\mathscr{C}_C)], \quad C \in \mathscr{C}^d, \tag{2.3}$$

where $\mathscr{C}_C := \{K \in \mathscr{C}^{(d)} : K \cap C \neq \emptyset\}$. Since the *capacity functional* $C \mapsto \mathbb{P}(Z \cap C \neq \emptyset)$ determines the distribution of Z (see [22, 18]), it follows that the distribution of Z is determined by the values $\Theta(\mathscr{C}_C)$ for $C \in \mathscr{C}^d$.

2.2 Stationarity and isotropy

The Poisson process Y is called *stationary* if $Y + x := \{K + x : K \in Y\} \stackrel{d}{=} Y$ for each $x \in \mathbb{R}^d$. (This definition applies to general particle processes.) Under the Poisson assumption, it follows from (2.3) that Y is stationary if and only if the intensity measure Θ of Y is translation invariant, that is, $\Theta(\{K : K + x \in \cdot\}) = \Theta$ for each $x \in \mathbb{R}^d$. By [22, Theorem 4.1.1] the intensity measure Θ has the representation

$$\Theta(\cdot) = \gamma \iint \mathbf{1}\{K + x \in \cdot\}\, dx\, \mathbb{Q}(dK), \tag{2.4}$$

where $\gamma \in (0, \infty)$ is an *intensity* parameter and $\mathbb{Q}$ is a probability measure on $\mathscr{C}^{(d)}$ (the *grain distribution*) such that

$$\int \lambda_d(K + C)\, \mathbb{Q}(dK) < \infty, \quad C \in \mathscr{C}^d. \tag{2.5}$$

The intensity γ is uniquely determined by Θ. Without loss of generality we can assume that $\mathbb{Q}$ is concentrated on the particles for which the center of the circumscribed ball is the origin. Then Θ determines $\mathbb{Q}$ as well. It is convenient to introduce a *typical grain*, that is a random closed set Z_0 with distribution $\mathbb{Q}$.

A Boolean model Z (or a general random closed set) is said to be *stationary*, if $Z + x \stackrel{d}{=} Z$ for all $x \in \mathbb{R}^d$ (where $\stackrel{d}{=}$ means equality in distribution). Essentially it follows from (2.3) that this is the case if and only if the intensity measure of the underlying Poisson particle process Y is translation invariant; see, e.g., [22, Theorem 3.6.4] (and its proof). A Boolean model Z (or a general random closed set) is said to be *isotropic*, if $\vartheta Z \stackrel{d}{=} Z$ for all proper rotations ϑ of $\mathbb{R}^d$. In this case the intensity measure of Y is rotation invariant in the obvious sense. If Z is stationary it again follows from (2.3) that Z is isotropic if and only if the grain distribution is isotropic (that is invariant under proper rotations), provided that the center of the circumscribed ball is chosen as the center function.

If Z is stationary, then (2.3) can be written as

$$\mathbb{P}(Z \cap C = \emptyset) = \exp[-\gamma \mathbb{E} V_d(Z_0 + C^*)], \quad C \in \mathscr{C}^d, \tag{2.6}$$

where $C^* := \{-x : x \in C\}$. In particular, we obtain that

$$p := \mathbb{P}(x \in Z) = 1 - \exp[-\gamma \mathbb{E} V_d(Z_0)], \quad x \in \mathbb{R}^d. \tag{2.7}$$

Fubini's theorem implies

$$\mathbb{E} \lambda_d(Z \cap B) = p \lambda_d(B), \quad B \in \mathscr{B}^d, \tag{2.8}$$

which justifies to call p the *volume fraction* of Z.

2.3 Non-stationary structures

For some applications, stationarity might be too strong an assumption. A measure Θ on $\mathscr{C}^{(d)}$ is said to be *translation regular* (see [22]) if there exist a measurable function $\eta : \mathbb{R}^d \times \mathscr{C}^{(d)} \to [0, \infty)$ and a probability measure $\mathbb{Q}$ on $\mathscr{C}^{(d)}$ such that

$$\Theta = \iint \mathbf{1}\{K + x \in \cdot\}\, \eta(K, x)\, dx\, \mathbb{Q}(dK). \tag{2.9}$$

The function η and the measure $\mathbb{Q}$ are not determined by Θ, nevertheless partial information may be determined, in particular if η depends only on the location x and not on K. In the following, we focus on the stationary framework, but mention one particular result in Corollary 3.5. Relations between mean values of functionals of Boolean models and mean values of the underlying particle process are described, for instance, in [22, Chapter 11], [24, Section 11.8], [6, Section 6] and [28, 29]. In the non-stationary setting, local densities are defined as Radon-Nikodym derivatives and not via a limit involving an expanding observation window. Hence, local densities are functions of the location in space where the particles are distributed.

3 Mean values

In this section we fix a Boolean model Z as in (2.2). We assume that the underlying Poisson particle process Y is concentrated on the system $\mathscr{K}^{(d)}$ of nonempty convex bodies. Under suitable moment assumptions, Boolean models with more general (polyconvex) grains can be treated essentially in the same way.

Recall that we write $\mathscr{K}^d$ for the family of all compact convex subsets (convex bodies) of $\mathbb{R}^d$. and that the *convex ring* $\mathscr{R}^d$ is the system of all finite (possibly empty) unions of convex bodies. Elements of $\mathscr{R}^d$ are called *polyconvex sets*. By the proof of [22, Theorem 14.4.4], $\mathscr{R}^d$ is a Borel subset of $\mathscr{C}^d$. A function $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ is said to be *additive* if $\varphi(\emptyset) = 0$ and

$$\varphi(K \cup L) = \varphi(K) + \varphi(L) - \varphi(K \cap L) \quad \text{for all } K, L \in \mathscr{R}^d. \tag{3.1}$$

A function $\varphi\colon \mathscr{K}^d \to \mathbb{R}$ is additive if (3.1) is true whenever $K, L, K \cup L \in \mathscr{K}^d$. In this case it is no restriction of generality to assume that $\varphi(\emptyset) = 0$. An additive function on convex bodies is also called a *valuation*. For an additive functional $\varphi\colon \mathscr{R}^d \to \mathbb{R}$, measurability of φ is equivalent to measurability of the restriction of φ to $\mathscr{K}^d$ (see again [22, Theorem 14.4.4]).

Remark 3.1. Assume that $\varphi\colon \mathscr{K}^d \to \mathbb{R}$ is additive and continuous. A fundamental result by Groemer (see [22, Theorem 14.4.2], and [9, Theorem 4.19] for a generalization) says that φ has a unique additive extension to $\mathscr{R}^d$.

Let $j \in \{0, \dots, d\}$, and let $\varphi\colon \mathscr{K}^d \to \mathbb{R}$ be a function. Then φ is said to be (positively) *j-homogeneous* (or homogeneous of degree j) if

$$\varphi(cK) = c^j \varphi(K), \quad c > 0, K \in \mathscr{K}^d.$$

If $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ is additive and the restriction of φ to $\mathscr{K}^d$ is j-homogeneous, then φ is j-homogeneous on $\mathscr{R}^d$ in the obvious sense.

The *intrinsic volumes* $V_0, \dots, V_d$ are important examples of continuous additive functionals. On $\mathscr{K}^d$ they can be defined by the *Steiner formula* (see [22, Equation 14.5])

$$V_d(K + B_r^d) = \sum_{i=0}^{d} \kappa_{d-i} r^{d-i} V_i(K), \quad r \geq 0, K \in \mathscr{K}^d, \tag{3.2}$$

where B^d is the closed unit ball centered at the origin, $B_r^d := \{rx : x \in B^d\}$, κ_i denotes the volume of the i-dimensional unit ball and $\kappa_0 := 1$. By Remark 3.1 the intrinsic volumes can be extended to $\mathscr{R}^d$ in an additive way. The number $V_d(K)$ is the volume (Lebesgue measure) of $K \in \mathscr{R}^d$, while $V_0(K)$ is the *Euler characteristic* of K. If K has nonempty interior, then $V_{d-1}(K)$ is half the surface area of K. The intrinsic volumes are rigid motion invariant (in particular translation invariant), monotone increasing (with respect to set inclusion) and V_j is j-homogeneous. These properties are characteristic for the intrinsic volumes on convex bodies. Due to their different degrees of homogeneity, the intrinsic volumes are linearly independent functionals on convex bodies. The values of the extension of V_i to the convex ring can be obtained by means of the inclusion-exclusion formula.

3.1 The basic equation for additive functionals

The following result is a slight generalization of Theorem 9.1.2 in [22]. For additive functions φ, it expresses the mean value $\mathbb{E}\varphi(Z \cap K_0)$ in terms of iterated integrals with respect to the intensity measure Θ of the particle process Y on $\mathscr{K}^{(d)}$. Translation invariance of φ is not needed. But we require an integrability property.

Let $\varphi\colon \mathscr{K}^d \to \mathbb{R}$ be a measurable function. Then φ is said to satisfy the integrability condition $\mathbf{I}(\Theta)$, if for each $K_0 \in \mathscr{K}^d$ there exists a constant $c(K_0) \ge 0$ such that

$$\int \cdots \int |\varphi(K_0 \cap K_1 \cap \cdots \cap K_k)| \,\Theta(dK_1) \cdots \Theta(dK_k) \le c(K_0)^k, \quad k \in \mathbb{N}. \qquad (\mathbf{I}(\Theta))$$

If φ is defined on $\mathscr{R}^d$ then the restriction to $\mathscr{K}^d$ is required to satisfy $\mathbf{I}(\theta)$.

Theorem 3.2. *Suppose that $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ is measurable, additive and satisfies $\mathbf{I}(\Theta)$. Let $K_0 \in \mathscr{K}^d$. Then $\mathbb{E}|\varphi(Z \cap K_0)| < \infty$ and*

$$\mathbb{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int \cdots \int \varphi(K_0 \cap K_1 \cap \cdots \cap K_k) \,\Theta(dK_1) \cdots \Theta(dK_k), \qquad (3.3)$$

where the series converges absolutely (also with φ replaced by $|\varphi|$).

Proof. Almost surely the window K_0 is hit by only finitely many grains $M_1, \dots, M_\nu$ of the underlying particle process Y (here ν is a random variable). Since φ is additive, the inclusion-exclusion formula implies

$$\begin{aligned}
\varphi(Z \cap K_0) &= \varphi\Big(\bigcup_{i=1}^{\nu} M_i \cap K_0\Big) \\
&= \sum_{k=1}^{\nu} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \nu} \varphi(K_0 \cap M_{i_1} \cap \cdots \cap M_{i_k}) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{(K_1,\dots,K_k) \in Y^k_{\neq}} \varphi(K_0 \cap K_1 \cap \cdots \cap K_k),
\end{aligned} \qquad (3.4)$$

where $Y^k_{\neq}$ is the product process of all k-tupels of pairwise distinct bodies in Y. Here we could extend the summation to infinity since $\varphi(\emptyset) = 0$.

After taking expectations of (3.4) we wish to swap expectation and summation. By Fubini's theorem this is allowed, provided that

$$\sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E} \sum_{(K_1,\dots,K_k)\in Y_{\neq}^k} |\varphi(K_0 \cap K_1 \cap \cdots \cap K_k)| < \infty.$$

An application of [14, Corollary 4.10], which is a simple consequence of the multivariate Mecke equation, shows that the latter series equals

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int \cdots \int |\varphi(K_0 \cap K_1 \cap \cdots \cap K_k)| \, \Theta(dK_1) \cdots \Theta(dK_k),$$

which is finite by assumption $\mathbf{I}(\Theta)$. In particular, we have $\mathbb{E}|\varphi(Z \cap K_0)| < \infty$. Taking the expectation of (3.4) and repeating the argument, we obtain (3.3). □

Remark 3.3. The preceding proof still works with a slightly weaker integrability condition on $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ than $\mathbf{I}(\Theta)$. In fact, it is sufficient to assume that for each $K_0 \in \mathscr{K}^d$ and $k \in \mathbb{N}$ there is a constant $c_k(K_0) \geq 0$ such that

$$\int \cdots \int |\varphi(K_0 \cap K_1 \cap \cdots \cap K_k)| \, \Theta(dK_1) \cdots \Theta(dK_k) \leq c_0(K_0) k! c_k(K_0)^k \tag{3.5}$$

and

$$\sum_{k \in \mathbb{N}} c_k(K_0)^k < \infty,$$

that is, $c_k(K_0)^k$, $k \in \mathbb{N}$, is summable.

Remark 3.4. Suppose that $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ is *locally bounded*, that is,

$$\sup\{|\varphi(K)| : K \in \mathscr{K}^d, K \subset W\} < \infty, \quad W \in \mathscr{K}^d. \tag{3.6}$$

Hence, given $K_0 \in \mathscr{K}^d$, there exists a constant $c_0(K_0)$ such that $|\varphi(M)| \leq c_0(K_0)$ for all $M \in \mathscr{K}^d$ contained in K_0. Therefore for each $k \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\int \cdots \int |\varphi(K_0 \cap K_1 \cap \cdots \cap K_k)| \, \Theta(dK_1) \cdots \Theta(dK_k) \\
&\qquad \leq c_0(K_0) \int \cdots \int \mathbf{1}\{K_0 \cap K_1 \neq \emptyset, \dots, K_0 \cap K_k \neq \emptyset\} \, \Theta(dK_1) \cdots \Theta(dK_k) \\
&\qquad = c_0(K_0) \Theta(\mathscr{C}_{K_0})^k,
\end{aligned}$$

which is finite by our basic assumption (2.1). Hence φ satisfies the condition $\mathbf{I}(\Theta)$.

Using the decomposition of the intensity measure Θ in the translation regular case, we get the following result. Here and later we use the notation $K^x := K + x$ for $K \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$.

Corollary 3.5. *Let the assumptions of Theorem 3.2 be satisfied and assume that Θ is translation regular as in* (2.9). *Then*

$$\mathbb{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int \cdots \int F(K_0, K_1, \dots, K_k) \, \mathbb{Q}(dK_1) \cdots \mathbb{Q}(dK_k) \tag{3.7}$$

with

$$F(K_0,K_1,\dots,K_k) := \int \varphi(K_0\cap K_1^{x_1}\cap\cdots\cap K_k^{x_k}) \prod_{i=1}^k \eta(K_i,x_i)\, d(x_1,\dots,x_k).$$

3.2 The stationary case

From now on we assume that Y and hence also the Boolean model Z is stationary. Therefore the intensity measure Θ can be written in the form (2.4). Then Corollary 3.5 takes the following form.

Corollary 3.6. *Let the assumptions of Theorem 3.2 be satisfied and assume that Z is stationary. Then*

$$\mathbb{E}\varphi(Z\cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\gamma^k \int\cdots\int \varphi(K_0\cap K_1^{x_1}\cap\cdots\cap K_k^{x_k}) \\ \times\, d(x_1,\dots,x_k)\,\mathbb{Q}(dK_1)\cdots\mathbb{Q}(dK_k). \tag{3.8}$$

It becomes apparent that a further investigation of the mean values $\mathbb{E}\varphi(Z\cap K_0)$ requires iterated translative integral formulas for additive functionals φ.

Let $j\in\{0,\dots,d\}$ and $k\in\mathbb{N}$. Then we denote by $\mathrm{mix}(j,k)$ the set of all multi-indices $\mathbf{m}=(m_1,\dots,m_k)\in\{j,\dots,d\}^k$ satisfying $m_1+\cdots+m_k=(k-1)d+j$. A measurable function $\varphi\colon \mathscr{K}^d\to\mathbb{R}$ is said to satisfy a *translative integralgeometric principle of order j* if for each $k\in\mathbb{N}$ and each $\mathbf{m}\in\mathrm{mix}(j,k)$ there exists a measurable function $\varphi_{\mathbf{m}}\colon (\mathscr{K}^d)^k\to\mathbb{R}$ such that $\varphi_{\mathbf{m}}(K_1,\cdot)$ is for each $K_1\in\mathscr{K}^d$ symmetric on $(\mathscr{K}^d)^{k-1}$ (for $k=1$ this is no assumption) and the following two properties are satisfied. For all $K_1,\dots,K_k\in\mathscr{K}^d$,

- $$\int \varphi(K_1\cap K_2^{x_2}\cap\cdots\cap K_k^{x_k})\, d(x_2,\dots,x_k) = \sum_{\mathbf{m}\in\mathrm{mix}(j,k)} \varphi_{\mathbf{m}}(K_1,\dots,K_k) \tag{3.9}$$

  holds for $k\geq 2$ (in particular, the iterated translative integral is defined) and

- the decomposability property

  $$\varphi_{(m_1,\dots,m_{k-1},d)}(K_1,\dots,K_k) = \varphi_{(m_1,\dots,m_{k-1})}(K_1,\dots,K_{k-1})V_d(K_k) \tag{3.10}$$

  holds whenever $(m_1,\dots,m_{k-1},d)\in\mathrm{mix}(j,k)$ and $k\geq 2$. We simply write φ_j instead of $\varphi_{(j)}$.

We say that the condition $\mathbf{IP}(\mathbb{Q})$ holds for the translative integralgeometric principle satisfied by φ if for each $K_0\in\mathscr{K}^d$

$$\int\cdots\int |\varphi_{(m,m_1,\dots,m_s)}(K_0,K_1,\dots,K_s)|\,\mathbb{Q}(dK_1)\cdots\mathbb{Q}(dK_s) < \infty \tag{3.11}$$

for $s\in\{1,\dots,m-j\}$, $m\in\{j+1,\dots,d\}$ and $m_1,\dots,m_s\in\{j,\dots,d-1\}$ such that the relation $m_1+\cdots+m_s = sd+j-m$ holds.

The following remarks relate the preceding definition to the integrability condition $\mathbf{I}(\Theta)$.

Remark 3.7. If the functionals $\varphi_{\mathbf{m}}$ are all nonnegative, then condition $\mathbf{IP}(\mathbb{Q})$ is clearly implied by the integrability condition $\mathbf{I}(\Theta)$ on φ.

Remark 3.8. In many cases of interest, the functionals $(K_1,\dots,K_k) \mapsto \varphi_{\mathbf{m}}(K_1,\dots,K_k)$ are separately positively homogeneous of degree m_i (say) with respect to K_i for $i = 1,\dots,k$. If we define $F : \mathscr{K}^k \to \mathbb{R}$ by

$$F(K_1,\dots,K_k) := \int \varphi(K_1 \cap K_2^{x_2} \cap \cdots \cap K_k^{x_k})\, d(x_2,\dots,x_k),$$

then the relations

$$F(K_1,\dots,K_k) = \sum_{\mathbf{m}\in \mathrm{mix}(j,k)} \varphi_{\mathbf{m}}(K_1,\dots,K_k)$$

can be inverted by considering r_iK_i for $i = 1,\dots,k$ and sufficiently many integers $r_i \geq 1$. More specifically this means that for each $\mathbf{m} \in \mathrm{mix}(j,k)$ there are a finite set $\mathscr{I} \subset \mathbb{N}^k$ and constants $\alpha_I(\mathbf{m}) \in \mathbb{R}$ for $I \in \mathscr{I}$, independent of $K_1,\dots,K_k$, such that

$$\varphi_{\mathbf{m}}(K_1,\dots,K_k) = \sum_{(r_1,\dots,r_k)\in\mathscr{I}} \alpha_{(r_1,\dots,r_k)}(\mathbf{m}) F(r_1K_1,\dots,r_kK_k).$$

In this situation it follows that condition (3.11) is implied by the condition $\mathbf{I}(\Theta)$ for φ. (The existence of such an inversion follows by iterating the usual Vandermonde inversion for polynomials in a single variable.)

Remark 3.9. Suppose that $\varphi\colon \mathscr{K}^d \to \mathbb{R}$ is additive (a valuation), translation invariant and continuous. By a result of McMullen [15], [21, Theorem 6.3.5], φ can be uniquely decomposed into a sum $\varphi = \sum_{j=0}^d \varphi_j$, where $\varphi_j\colon \mathscr{K}^d \to \mathbb{R}$, $j \in \{0,\dots,d\}$, is additive, translation invariant, continuous and homogeneous of degree j. Note that φ_0 is a constant and φ_d is a multiple of the volume.

Assume now that φ is a j-homogeneous, translation invariant, continuous valuation, for some $j \in \{0,\dots,d\}$. It was shown in [29] (see also [24, Theorem 11.3]) that φ satisfies a translative integralgeometric principle of order j. Since in this situation the functionals $\varphi_{\mathbf{m}}$ have the homogeneity property described in Remark 3.8, condition $\mathbf{IP}(\mathbb{Q})$ is implied by condition $\mathbf{I}(\Theta)$ for φ. Moreover, in this case the mixed functionals are completely symmetric with respect to all entries and $\varphi_{(j,d)}(K_1,K_2) = \varphi(K)V_d(M)$, that is $\varphi_j = \varphi$.

Example 3.10. Let $j \in \{0,\dots,d\}$. The intrinsic volume $\varphi = V_j$ is nonnegative and satisfies a translative integralgeometric principle of order j; see, e.g., [29] and Remark 3.9. Intrinsic volumes are monotone on $\mathscr{K}^d$ (with respect to set inclusion) and therefore locally bounded in the sense of Remark 3.4. Therefore the functionals $\varphi = V_j$ satisfy the condition $\mathbf{I}(\Theta)$. Remark 3.9 shows that condition $\mathbf{IP}(\mathbb{Q})$ holds for the integralgeometric principle satisfied by $\varphi = V_j$ as well.

Example 3.11. Important examples of translation invariant, continuous valuations with a fixed degree of homogeneity have been discussed in [24, Section 11.6]. We mention them only briefly and refer to [24] for further details:

(1) Mixed volumes of the form

$$\varphi(K) := V(K[j], M_{j+1},\dots,M_d), \quad K \in \mathscr{K}^{(d)},$$

where $j \in \{0, \ldots, d\}$ and $M_{j+1}, \ldots, M_d \in \mathscr{K}^{(d)}$ are fixed; see [9, Section 3.3] for an introduction to mixed volumes as d-fold Minkowski linear functionals $V : (\mathscr{K}^d)^d \to \mathbb{R}$.

(2) Centered support function values (evaluated at a fixed vector $u \in \mathbb{R}^d$)

$$\varphi(K) := h^*(K,u) := h_K(u) - \langle s(K), u \rangle, \quad K \in \mathscr{K}^{(d)},$$

where $j = 1$ and $s : \mathscr{K}^{(d)} \to \mathbb{R}^d$ is the Steiner point map (see Example 3.18 for further details).

(3) Integrals of continuous functions $f : S^{d-1} \to \mathbb{R}$ against an area measure

$$\varphi(K) := \int_{S^{d-1}} f(u)\, S_j(K, du), \quad K \in \mathscr{K}^{(d)},$$

where $j \in \{0, \ldots, d-1\}$. The area measure $S_j(K, \cdot)$ is a finite Borel measure on the unit sphere that can be considered as a local version of the intrinsic volume V_j and may be introduced by a local Steiner formula [9, Theorem 4.10].

(4) As a generalization of the preceding example, we briefly also mention integrals of flag measures (projection means of area measures). If $G(d, j)$ denotes the linear Grassmannian of j-dimensional linear subspaces of $\mathbb{R}^d$, then $F(d, j) := \{(u, L) \in S^{d-1} \times G(d, j) : u \in L\}$ is a flag manifold and

$$\psi_j(K, \cdot) := \int_{G(d,j+1)} \int_{S^{d-1} \cap L} \mathbf{1}\{(u, L^\perp \vee u) \in \cdot\}\, S_j^L(K|L, du)\, \nu_{j+1}(dL)$$

denotes the j-th flag measure of K, which is a j-homogeneous Borel measure on the flag space $F(d, d-j)$. Here $L^\perp \in G(d, d-j-1)$ is the linear subspace orthogonal to $L \in G(d, j+1)$, $L^\perp \vee u \in G(d, d-j)$ is the linear span of $L^\perp$ and $u \in L$, $K|L$ denotes the orthogonal projection of K to the subspace L and $S_j^L(K|L, \cdot)$ is the j-th area measure of $K|L$ with respect to L as the ambient space. By the projection formula for area measures, the image measure of $\Psi_j(K, \cdot)$ under projection onto the first component is proportional to $S_j(K, \cdot)$. Thus

$$\varphi(K) := \int_{F(d,d-j)} f(u,L)\, \psi_j(K, d(u,L)), \quad K \in \mathscr{K}^{(d)},$$

for a given continuous function $f : F(d, d-j) \to \mathbb{R}$, generalizes the preceding example.

Each of these functionals φ satisfies an integralgeometric principle of order j and the associated mixed functionals $\varphi_{\mathbf{m}}$ are uniquely determined by φ. In Example (2) we obtain translative integralgeometric formulas for a function-valued valuation (the support function) by considering both sides of the resulting equation for a fixed $u \in S^{d-1}$ as a function of u. Similarly, if we consider the integralgeometric formulas in Examples (3) and (4) for general continuous functions f and apply the Riesz representation theorem, then we get integralgeomtric relations for measure-valued valuations. In this way it can be seen that the resulting formulas can be applied to not necessarily continuous functions.

The preceding Examples 3.10 and 3.11 can be generalized in another direction, thereby also dropping the assumption of translation invariance. To do so we need some further tools from convex and integral geometry. First we ask the reader to recall from [22, 21] the definition of the support measure $\Lambda_j(K,\cdot)$, $K \in \mathscr{K}^d$, for $j \in \{0,\dots,d\}$. These are finite measures on $\mathbb{R}^d \times \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. The total mass is given by $\Lambda_j(K,\mathbb{R}^d \times \mathbb{S}^{d-1}) = V_j(K)$. In the following, we call a function $f\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}$ *locally bounded* if $|f|$ is bounded on sets of the form $B \times \mathbb{S}^{d-1}$, where $B \subset \mathbb{R}^d$ is bounded. We now use [3, Theorem 3.14] (see also [4, Theorem 3.1]). By this result, for each $k \geq 2$ and each measurable, locally bounded function $f\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}$, we have

$$\begin{aligned}\iint f(x_1,u)\,\Lambda_j(K_1 \cap K_2^{x_2} \cap \dots \cap K_k^{x_k}, d(x_1,u))\,d(x_2,\dots,x_k)\\ = \sum_{\mathbf{m}\in \operatorname{mix}(j,k)} \int f(x_1,u)\,\Lambda^{(j)}_{\mathbf{m}}(K_1,\dots,K_k,d(x_1,\dots,x_k,u)),\end{aligned} \tag{3.12}$$

where the $\Lambda^{(j)}_{\mathbf{m}}(K_1,\dots,K_k,\cdot)$, $\mathbf{m} \in \operatorname{mix}(j,k)$, are finite measures on $(\mathbb{R}^d)^k \times \mathbb{S}^{d-1}$, the *mixed support measures* associated with $K_1,\dots,K_k \in \mathscr{K}^d$. We do not list all the nice properties of these measures, but refer to the above sources.

Example 3.12. Let $f\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}$ be a measurable, locally bounded function. Let $j \in \{0,\dots,d-1\}$ and define $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ by

$$\varphi(K) := \int f(x,u)\,\Lambda_j(K,d(x,u)), \tag{3.13}$$

where we use the additive extension of $K \mapsto \Lambda_j(K,\cdot)$ from $\mathscr{K}^d$ to $\mathscr{R}^d$. If f is independent of x, then φ is translation invariant and the integration with respect to $\Lambda_j(K,\cdot)$ can be replaced by an integration with respect to the j-th area measure $S_j(K,\cdot)$ of K. If f is independent of u, we can include $j = d$ in the range of j and then $\Lambda_d(K,\cdot)$ is replaced by d-dimensional Lebesgue measure restricted to K. For this example we restrict the discussion to $j \leq d-1$ though. Thanks to (3.12) and the properties of the mixed support measures (see [3, Theorem 3.14]), φ satisfies a translative integralgeometric principle of order j with

$$\varphi_{\mathbf{m}}(K_1,\dots,K_k) = \int f(x_1,u)\,\Lambda^{(j)}_{\mathbf{m}}(K_1,\dots,K_k,d(x_1,\dots,x_k,u)) \tag{3.14}$$

for $\mathbf{m} \in \operatorname{mix}(j,k)$ and $k \in \mathbb{N}$. Even though $\Lambda^{(j)}_{\mathbf{m}}$ has a natural symmetry property, $\varphi_{\mathbf{m}}$ is not symmetric in all of its arguments, but only with respect to $K_2,\dots,K_k$. Note that φ is in general not j-homogeneous and not translation invariant. Instead we have

$$\varphi(\lambda K) = \int f(x,u)\,\Lambda_j(\lambda K,d(x,u)) = \lambda^j \int f(\lambda x,u)\,\Lambda_j(K,d(x,u)).$$

Since $\Lambda_j(K,\mathbb{R}^d \times \mathbb{S}^{d-1}) = V_j(K)$ it follows again from Remark 3.4 or directly from the monotonicity of V_j (and the triangle inequality) that φ satisfies the integrability condition $\mathbf{I}(\Theta)$. We now argue that condition $\mathbf{IP}(\mathbb{Q})$ holds for this integralgeometric principle satisfied by φ. By the integral representation (3.14) and the triangle inequality we can assume that $f \geq 0$. Then all the numbers $\varphi_{\mathbf{m}}(K_1,\dots,K_k)$ are nonnegative, so that 3.11 follows from (3.9) and the fact that φ satisfies $\mathbf{I}(\Theta)$.

This example includes the tensor valuations (see [4, 23]) as a special case.

We need to introduce some notation. For $r \in \mathbb{N}$ and a measurable function ψ on $(\mathscr{K}^d)^r$, whenever the following integrals exist we set

$$\overline{\psi}[K,Y] := \gamma^{r-1} \int \psi(K, K_1 \dots, K_{r-1})\, \mathbb{Q}^{r-1}(d(K_1.\dots, K_{r-1})), \quad K \in \mathscr{K}^d, r \geq 2,$$
$$\overline{\psi}[Y] := \gamma^r \int \psi(K_1, \dots, K_r)\, \mathbb{Q}^r(d(K_1, \dots, K_r)).$$

The next result extends of Corollary 6.3 in [29]; see also [22, p. 390]. By the forthcoming Corollary 3.15 and Example 3.20 it also generalizes [4, Theorem 5.4]. The existence of the mean values $\overline{\varphi}_{(m,\mathbf{m})}[K_0,Y]$ in the following theorem is ensured by the condition $\mathbf{IP}(\mathbb{Q})$.

Theorem 3.13. *Assume that Z is stationary. Let $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ be measurable and additive. Let $j \in \{0,\dots,d\}$. Assume that φ satisfies a translative integralgeometric principle of order j such that the conditions $\mathbf{I}(\Theta)$ and $\mathbf{IP}(\mathbb{Q})$ hold. Then, for any $K_0 \in \mathscr{K}^d$,*

$$\mathbb{E}\varphi(Z \cap K_0) = \varphi_j(K_0)\big(1 - \mathrm{e}^{-\overline{V}_d[Y]}\big) + \mathrm{e}^{-\overline{V}_d[Y]} \sum_{m=j+1}^{d} \sum_{s=1}^{m-j} \frac{(-1)^{s-1}}{s!} \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\dots+m_s = sd+j-m}}^{d-1} \overline{\varphi}_{(m,\mathbf{m})}[K_0,Y], \tag{3.15}$$

where $\mathbf{m} = (m_1,\dots,m_s)$ in the inner summation on the right-hand side.

Proof. Since φ satisfies $\mathbf{I}(\Theta)$, we can apply Corollary 3.6. Together with (3.9) this yields

$$\begin{aligned}
&\mathbb{E}\varphi(Z \cap K_0)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \int \cdots \int \varphi(K_0 \cap K_1^{x_1} \cap \dots \cap K_k^{x_k})\, d(x_1,\dots,x_k)\, \mathbb{Q}(dK_1) \cdots \mathbb{Q}(dK_k)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \int \cdots \int \sum_{\mathbf{m}' \in \mathrm{mix}(j,k+1)} \varphi_{\mathbf{m}'}(K_0,K_1,\dots,K_k)\, \mathbb{Q}(dK_1) \cdots \mathbb{Q}(dK_k)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \sum_{m=j}^{d} \int \cdots \int \sum_{\mathbf{m} \in \mathrm{mix}(d-m+j,k)} \varphi_{(m,\mathbf{m})}(K_0,K_1,\dots,K_k)\\
&\qquad \times \mathbb{Q}(dK_1) \cdots \mathbb{Q}(dK_k).
\end{aligned} \tag{3.16}$$

By 3.11 this series converges absolutely. In the case $j = d$, we also have $m = d$ and the set $\mathrm{mix}(d,k)$ contains the single element $(d,\dots,d)$, hence repeated application of (3.10) yields

$$\varphi_{(d,\dots,d)}(K_0,K_1,\dots,K_k) = \varphi_d(K_0) V_d(K_1) \cdots V_d(K_k).$$

Then (3.16), for $j = d$, reduces to

$$\mathbb{E}\varphi(Z \cap K_0) = \varphi_d(K_0) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} (\overline{V}_d[Y])^k,$$

that is

$$\mathbb{E}\varphi(Z \cap K_0) = \varphi_d(K_0)\big(1 - \mathrm{e}^{-\overline{V}_d[Y]}\big). \tag{3.17}$$

For $j < d$ and $m = j$, we can proceed similarly to obtain that

$$\begin{aligned}\mathbb{E}\varphi(Z \cap K_0) &= \varphi_j(K_0)\big(1 - \mathrm{e}^{-\overline{V}_d[Y]}\big)\\ &\quad + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\gamma^k \sum_{m=j+1}^{d} \int \cdots \int S_{k,m}(K_0,\dots,K_k)\,\mathbb{Q}(dK_1)\cdots\mathbb{Q}(dK_k),\end{aligned}$$

where again (3.10) was used and

$$S_{k,m}(K_0,\dots,K_k) := \sum_{\mathbf{m}\in \mathrm{mix}(d-m+j,k)} \varphi_{(m,\mathbf{m})}(K_0,K_1,\dots,K_k).$$

Here, we have $\mathbf{m} = (m_1,\dots,m_k)$ with $m_1+\cdots+m_k = kd+j-m$. Since $m > j$, there must be indices m_i which are smaller than d. The number s of these indices ranges from 1 to $m-j$, since

$$kd-(m-j) = m_1+\cdots+m_k = m_1+\cdots+m_s+(k-s)d \le s(d-1)+(k-s)d.$$

We rearrange the arguments of $S_{k,m}$ according to this number s and use the symmetry and the decomposability property (3.10) of $\varphi_{(m,\mathbf{m})}$. We obtain

$$\begin{aligned}&S_{k,m}(K_0,\dots,K_k)\\ &= \sum_{s=1}^{k\wedge(m-j)} \binom{k}{s} \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\cdots+m_s=sd+j-m}}^{d-1} \varphi_{(m,m_1,\dots,m_s)}(K_0,K_1,\dots,K_s)V_d(K_{s+1})\cdots V_d(K_k).\end{aligned}$$

Thus we get

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\gamma^k \sum_{m=j+1}^{d} \int \cdots \int S_{k,m}(K_0,\dots,K_k)\,\mathbb{Q}(dK_1)\cdots\mathbb{Q}(dK_k)\\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{m=j+1}^{d} \sum_{s=1}^{k\wedge(m-j)} \binom{k}{s} \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\cdots+m_s=sd+j-m}}^{d-1} \overline{\varphi}_{(m,\mathbf{m})}[K_0,Y]\overline{V}_d[Y]^{k-s}\\ &= \sum_{m=j+1}^{d} \sum_{s=1}^{m-j} \sum_{r=0}^{\infty} \frac{(-1)^{r+s-1}}{r!s!}\overline{V}_d[Y]^r \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\cdots+m_s=sd+j-m}}^{d-1} \overline{\varphi}_{(m,\mathbf{m})}[K_0,Y].\end{aligned}$$

This gives the assertion (3.15). □

In the special $j = d$ equation (3.15) simplifies to (3.17). The special case $j = d-1$ is also worth mentioning. It extends [4, Theorem 5.2] significantly.

Corollary 3.14. *Let the assumptions of Theorem 3.13 be satisfied in the case $j = d-1$. Then*

$$\mathbb{E}\varphi(Z \cap K_0) = \varphi_{d-1}(K_0)\big(1 - \mathrm{e}^{-\overline{V}_d[Y]}\big) + \mathrm{e}^{-\overline{V}_d[Y]}\overline{\varphi}_{d,d-1}[K_0,Y]. \tag{3.18}$$

Next we specialize Theorem 3.13 to the case discussed in Example 3.12. To this end we introduce for $i \in \mathbb{N}$ and $\mathbf{m} = (m_1, \ldots, m_i) \in \mathbb{N}_0^i$ the mean measures

$$\bar{\Lambda}_{\mathbf{m}}^{(j)}(K_0, Y, \cdot) := \gamma^{i-1} \int \cdots \int \Lambda_{\mathbf{m}}^{(j)}(K_0, K_1, \ldots, K_{i-1}, \cdot)\, \mathbb{Q}(dK_1) \cdots \mathbb{Q}(dK_{i-1}), \quad K_0 \in \mathscr{K}^d,$$

if $i \geq 2$ and

$$\bar{\Lambda}_{\mathbf{m}}^{(j)}(Y, \cdot) := \gamma^{i} \int \cdots \int \Lambda_{\mathbf{m}}^{(j)}(K_0, K_1, \ldots, K_{i-1}, \cdot)\, \mathbb{Q}(dK_0) \cdots \mathbb{Q}(dK_{i-1}).$$

The next corollary generalizes Theorem [4, Theorem 5.4].

Corollary 3.15. *Assume that φ is given as in Example 3.12. Then* (3.15) *holds with $\varphi_j = \varphi$,*

$$\overline{\varphi}_{(m,\mathbf{m})}[K_0, Y] = \int f(x_0, u) \bar{\Lambda}_{(m,\mathbf{m})}^{(j)}(K_0, Y, d(x_0, \ldots, x_s, u)),$$

for $\mathbf{m} \in \mathbb{N}_0^s$, $s \in \mathbb{N}$ and $m \in \{0, \ldots, d-1\}$, and

$$\overline{\varphi}_{(d,\mathbf{m})}[K_0, Y] = \iint \mathbf{1}\{x_0 \in K_0\} f(x_0, u)\, dx_0\, \bar{\Lambda}_{\mathbf{m}}^{(j)}(Y, d(x_1, \ldots, x_s, u)).$$

Proof. The first identity is a direct consequence of the definition while the second follows from the decomposability property of mixed support measures; see [3, Theorem 3.14]. ☐

Let $\varphi \colon \mathscr{R}^d \to \mathbb{R}$ be measurable and additive and let $W \in \mathscr{K}^d$ with $V_d(W) > 0$. Let $\alpha \in \mathbb{R}$ and define

$$\overline{\varphi}_{W,\alpha}[Z] := \lim_{r \to \infty} \frac{\mathbb{E}\varphi(Z \cap rW)}{V_d(rW)^\alpha} \tag{3.19}$$

whenever this limit exists. If the limit is positive (and finite), we call $\overline{\varphi}_{W,\alpha}[Z]$ the (asymptotic) *φ-density* of Z (w.r.t. W). Of particular importance is the case $\alpha = 1$. In this case we write $\overline{\varphi}_W[Z] := \overline{\varphi}_{W,1}[Z]$. If this is independent of W we shorten this further to $\overline{\varphi}[Z]$ Under additional assumptions on φ, Theorem 3.13 implies the existence of these densities along with some formulas. We start with a simple example.

Example 3.16. Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be measurable and locally bounded. Define $\varphi \colon \mathscr{R}^d \to \mathbb{R}$ by $\varphi(K) := \int_K f(x)\, dx$. Then we can apply (3.17) with $\varphi_d = \varphi$ to obtain

$$\mathbb{E}\varphi(Z \cap W) = \left(1 - \mathrm{e}^{-\overline{V}_d[Y]}\right) \int_W f(x)\, dx,$$

where W is as in (3.19). Of course this formula is very simple and makes sense for an arbitrary Borel set W. Assume that there exists $\beta \in \mathbb{R}$ such that $\{r^{-\beta} f(rx) : r \geq 1, x \in W\}$ is bounded and $\lim_{r\to\infty} r^{-\beta} f(rx) = g(x)$, $x \in W$, for some (bounded) $g \colon W \to \mathbb{R}$. Then

$$\overline{\varphi}_{W,\alpha}[Z] = \left(1 - \mathrm{e}^{-\overline{V}_d[Y]}\right) V_d(W)^{-\alpha} \int_W g(x)\, dx,$$

where $\alpha := (d + \beta)/d$.

We now turn to Example 3.12. In the following, we write $\mathrm{mix}^*(j,s)$ for the set of all $(m_1,\dots,m_s)\in \mathrm{mix}(j,s)$ for which $m_1,\dots,m_s \le d-1$. Note that in this situation we necessarily have $m_1,\dots,m_s \ge j+1$.

Theorem 3.17. *Assume that φ is given as in Example 3.12. Assume there exists $\beta\in\mathbb{R}$ such that $\{r^{-\beta}f(rx,u): r\ge 1, (x,u)\in W\times\mathbb{S}^{d-1}\}$ is bounded and $\lim_{r\to\infty} r^{-\beta}f(rx,u)=g(x,u)$, $(x,u)\in W\times\mathbb{S}^{d-1}$, for some $g\colon W\times\mathbb{S}^{d-1}\to\mathbb{R}$. Then*

$$\overline{\varphi}_{W,\alpha}[Z] = V_d(W)^{-\alpha} \mathrm{e}^{-\overline{V}_d[Y]} \sum_{s=1}^{d-j} \frac{(-1)^{s-1}}{s!} \times \sum_{\mathbf{m}\in \mathrm{mix}^*(j,s)} \iint \mathbf{1}\{x_0\in W\} g(x_0,u)\, dx_0\, \bar{\Lambda}^{(j)}_{\mathbf{m}}(Y, d(x_1,\dots,x_s,u)),$$

where $\alpha := (d+\beta)/d$.

Proof. We use Corollary 3.15. By the homogeneity properties of the mixed support measures we have

$$\overline{\varphi}_{(m,\mathbf{m})}[rW,Y] = r^m \int f(rx_0,u) \bar{\Lambda}^{(j)}_{(m,\mathbf{m})}(W,Y,d(x_0,\dots,x_s,u)).$$

A similar formula holds for $\varphi_j(rW)$. Inserting this into (3.15), using the special form of $\overline{\varphi}_{(d,\mathbf{m})}[W,Y]$, and taking the limit as $r\to\infty$, yields the result. □

Suppose that a given measurable function $\varphi\colon \mathscr{K}^d\to\mathbb{R}$ satisfies a translative integralgeometric principle of order j. For a given $\beta\in\mathbb{R}$, we say that the principle is *β-homogeneous* if for each $k\in\mathbb{N}$ and each $\mathbf{m}=(m_1,\dots,m_k)\in\mathrm{mix}(j,k)$, the function $\varphi_{\mathbf{m}}$ is $(m_1+\beta)$-homogeneous in the first argument.

Example 3.18. For $K\in\mathscr{K}^{(d)}$, let $h_K\colon \mathbb{S}^{d-1}\to\mathbb{R}$ be the *support function* of K defined by $h_K(u) := \max\{\langle x,u\rangle : x\in K\}$. The *Steiner point* $s(K)\in\mathbb{R}^d$ of K is defined by

$$\frac{1}{d\kappa_d}\int h_K(u) u\, \mathscr{H}^{d-1}(du),$$

where $\mathscr{H}^{d-1}$ denotes $(d-1)$-dimensional Hausdorff measure. Fix $u\in\mathbb{S}^{d-1}$. Since the map $K\mapsto s(K)$ is translation covariant and additive, the map $\varphi\colon\mathscr{K}^{(d)}\to\mathbb{R}$ given by

$$\varphi(K) := h_{K-s(K)}(u) = h_K(u) - \langle s(K),u\rangle$$

defines an additive and translation invariant measurable function, which can be extended to $\mathscr{R}^d$ in an additive way; see [21, 22]. Since φ is locally bounded it follows from Remark 3.4 that φ satisfies condition $\mathbf{I}(\Theta)$. It was shown in [27, 20] that φ satisfies a 0-homogeneous translative integralgeometric principle of order 1. It was also proved there that φ admits a measure-valued extension. It follows from Remark 3.8 and the homogeneity properties of the functionals $\varphi_{\mathbf{m}}$ in this case that condition $\mathbf{IP}(\mathbb{Q})$ holds for this integralgeometric principle.

The next result can be proved as Theorem 3.17.

Corollary 3.19. *Let the assumption of Theorem 3.13 be satisfied and assume moreover, that the restriction of φ to $\mathscr{K}^d$ is β-homogeneous for some $\beta \in \mathbb{R}$. Let $\alpha := (d+\beta)/d$. Then*

$$\overline{\varphi}[Z]_{W,d+\beta} = V_d(W)^{-\alpha}\varphi_d(W)\big(1-\mathrm{e}^{-\overline{V}_d[Y]}\big)$$

in the case $j = d$ and by

$$\overline{\varphi}[Z]_{W,\alpha} = V_d(W)^{-\alpha}\mathrm{e}^{-\overline{V}_d[Y]}\sum_{s=1}^{d-j}\frac{(-1)^{s-1}}{s!}\sum_{\mathbf{m}\in\mathrm{mix}^*(j,s)}\overline{\varphi}_{\mathbf{m}}[Y] \tag{3.20}$$

in the case $j \in \{0,\dots,d-1\}$. In particular, if $j = d-1$, then

$$\overline{\varphi}[Z]_{W,\alpha} = V_d(W)^{-\alpha}\mathrm{e}^{-\overline{V}_d[Y]}\overline{\varphi}_{d-1}[Y].$$

Example 3.20. Define $f\colon \mathbb{R}^d\times\mathbb{S}^{d-1}\to\mathbb{R}$ by

$$f(x,u) := x_1^{\alpha_1}\cdots x_d^{\alpha_d}h(u), \quad x=(x_1,\dots,x_d)\in\mathbb{R}^d,\, u\in\mathbb{S}^{d-1},$$

where $\alpha_1,\dots,\alpha_d\in[0,\infty)$ and $h\colon \mathbb{S}^{d-1}\to\mathbb{R}$ is measurable and bounded. Let $j\in\{0,\dots,d-1\}$ and define φ by (3.13). Then both, Corollary 3.15 and Corollary 3.19 apply with the choices $\beta := \alpha_1+\cdots+\alpha_d$ and $g = f$. The non-asymptotic formulas are provided by Theorem 3.17. In fact the formulas can be simplified a bit by using the product form of g. If $\alpha_1,\dots,\alpha_d\in\mathbb{N}_0$ and h is of a similar product form, then φ is a component of the Minkowski tensors studied in [4].

We finish this section by highlighting density relations for homogeneous functionals as discussed in Remarks 3.8 and 3.9 and the subsequent Examples 3.10 and 3.11.

Corollary 3.21. *Let Z be a stationary Boolean model with convex grains, and let $K_0\in\mathscr{K}^d$. Let $\varphi\colon \mathscr{R}^d\to\mathbb{R}$ be a continuous translation invariant valuation satisfying $\mathbf{I}(\Theta)$ which is j-homogeneous for some $j\in\{0,\dots,d\}$. Then*

$$\begin{aligned}\mathbb{E}\varphi(Z\cap K_0) &= \varphi(K_0)\left(1-\mathrm{e}^{-\overline{V}_d[Y]}\right)\\ &\quad + \mathrm{e}^{-\overline{V}_d[Y]}\sum_{m=j+1}^{d}\sum_{s=1}^{m-j}\frac{(-1)^{s-1}}{s!}\sum_{\substack{m_1,\dots,m_s=j\\ m_1+\cdots+m_s=sd+j-m}}^{d-1}\overline{\varphi}_{(m,\mathbf{m})}[K_0,Y].\end{aligned}$$

Proof. As noted in Remark 3.9, φ satisfies a translative integralgeometric principle of order j and $\mathbf{IP}(\mathbb{Q})$ holds. Therefore we can apply Theorem 3.13. Since the $\varphi_{\mathbf{m}}$ are symmetric in all arguments, we have $\varphi_j = \varphi$ and the result follows. □

For the asymptotic densitities, we thus obtain the following relations.

Corollary 3.22. *Let the assumption of Corollary 3.21 be satisfied and let $W\in\mathscr{K}^d$ have nonempty interior. Then we have for $j = d$ that*

$$\overline{\varphi}_W[Z] = \frac{\varphi_d(W)}{V_d(W)}\left(1-\mathrm{e}^{-\overline{V}_d[Y]}\right)$$

and for $j\in\{0,\dots,d-1\}$ that

$$\overline{\varphi}_W[Z] = \mathrm{e}^{-\overline{V}_d[Y]}\sum_{s=1}^{d-j}\frac{(-1)^{s-1}}{s!}\sum_{\mathbf{m}\in\mathrm{mix}^*(j,s)}\overline{\varphi}_{\mathbf{m}}[Y].$$

3.3 The stationary and isotropic case

In this section we assume that the Boolean model Z is isotropic, so that the grain distribution $\mathbb{Q}$ is invariant under proper rotations. First we note the following consequence of Corollary 3.6. As in [22] we denote by μ the (suitable normalized) Haar measure on the group G_d of rigid motions of $\mathbb{R}^d$.

Corollary 3.23. *Let the assumptions of Theorem 3.2 be satisfied and assume that Z is stationary and isotropic. Then*

$$\mathbb{E}\varphi(Z\cap K_0) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\gamma^k\int\cdots\int\varphi(K_0\cap g_1K_1\cap\cdots\cap g_kK_k) \\ \times\mathbb{Q}(dK_1)\cdots\mathbb{Q}(dK_k)\,\mu^k(d(g_1,\dots,g_k)). \tag{3.21}$$

We say that a measurable function $\varphi\colon\mathscr{K}^d\to\mathbb{R}$ satisfies a *kinematic integralgeometric principle* if there exist measurable functions $\varphi_1,\dots,\varphi_d$ on $\mathscr{K}^d$ and constants $c_0,\dots,c_d\in\mathbb{R}$ such that for all $k\in\mathbb{N}$ and all $K_0,\dots,K_k\in\mathscr{K}^d$

$$\int\varphi(K_0\cap g_1K_1\cap\cdots\cap g_kK_k)\,\mu^k(d(g_1,\dots,g_k)) = \sum_{\substack{r_0,\dots,r_k=0\\ r_0+\cdots+r_k=kd}}^{d}\varphi_{r_0}(K_0)\prod_{i=1}^{k}c_{r_i}V_{r_i}(K_i), \tag{3.22}$$

where $\varphi_0:=\varphi$. If φ is defined on $\mathscr{R}^d$, this definition applies to the restriction of φ to $\mathscr{K}^d$.

In the following remark and also later we use the constants

$$c_j^i := \frac{i!\kappa_i}{j!\kappa_j},\quad i,j\in\{0,\dots,d\}; \tag{3.23}$$

see [22, (5.4)].

Remark 3.24. Suppose that $\varphi\colon\mathscr{K}^d\to\mathbb{R}$ is additive and continuous on $\mathscr{K}^{(d)}$. By [22, Theorem 5.1.4], φ satisfies a kinematic integralgeometric principle with $c_i:=c_d^i$, $i\in\{0,\dots,d\}$. If $\varphi=V_j$ for some $j\in\{0,\dots,d\}$, then the same result shows that

$$\varphi_i = c_j^{i+j}V_{j+i},\quad i\le d-j,$$

and $\varphi_i=0$ for $i>d-j$.

The function φ from Example 3.18 is continuous. The functions $\varphi_1,\dots,\varphi_d$ and the constants $c_0,\dots,c_d$ can be expressed in terms of so-called *mean section bodies* of K; see [27] for more details.

The following result is Theorem 9.1.3 in [22] if φ is assumed to be continuous, which ensures via Hadwiger's general integralgeometric theorem that φ satisfies a kinematic integralgeometric principle. The proof is similar to that of Theorem 3.13.

Theorem 3.25. *Let $\varphi\colon\mathscr{R}^d\to\mathbb{R}$ be measurable and additive. Assume that φ satisfies condition $\mathbf{I}(\Theta)$ as well as a kinematic integralgeometric principle. Let $K_0\in\mathscr{K}^d$. Then*

$$\mathbb{E}\varphi(Z\cap K_0) = \varphi(K_0)\left(1-\mathrm{e}^{-\overline{V}_d[Y]}\right) \\ +\mathrm{e}^{-\overline{V}_d[Y]}\sum_{m=1}^{d}\varphi_m(K_0)\sum_{s=1}^{m}\frac{(-1)^{s-1}}{s!}\sum_{\substack{m_1,\dots,m_s=0\\ m_1+\cdots+m_s=sd-m}}^{d-1}\prod_{i=1}^{s}c_{m_i}\overline{V}_{m_i}[Y]. \tag{3.24}$$

Corollary 3.26. *Let the assumptions of Theorem 3.25 be satisfied. Assume that there exists $j \in \{0,\dots,d-1\}$ such that φ_m is $(m+j)$-homogeneous for $m \in \{0,\dots,d-j\}$, and $\varphi_m = 0$ for $m > d-j$. Let $K_0 \in \mathscr{K}^d$ be such that $V_d(K_0) > 0$. Then*

$$\lim_{r\to\infty} \frac{\mathbb{E}\varphi(Z\cap rK_0)}{V_d(rK_0)} = \frac{\varphi_{d-j}(K_0)}{V_d(K_0)} \mathrm{e}^{-\overline{V}_d[Y]} \sum_{s=1}^{d-j} \frac{(-1)^{s-1}}{s!} \sum_{\substack{m_1,\dots,m_s=0 \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \prod_{i=1}^{s} c_{m_i} \overline{V}_{m_i}[Y].$$

By Remark 3.24 we may take $\varphi = V_j$ in Corollary 3.26. Then $\varphi_{d-j} = d!\kappa_d/(j!\kappa_j)V_d$, so that Corollary 3.26 gives the classical Miles' formulas; see [17, 2] and [22, Theorem 9.1.4].

3.4 Determination and estimation

The mean value formulas obtained so far can be used to estimate certain characteristics of the underlying particle process Y from observations of functionals φ of the Boolean model Z. Naturally, this requires the choice of suitable functionals φ. In the simplest case, the challenge is to estimate the intensity γ in the stationary case. For such a task we need to know a class of functionals φ such that the mean values $\mathbb{E}\varphi(Z\cap W)$, respectively the densities $\overline{\varphi}[Z]$, determine γ.

For a stationary and isotropic Boolean model Z with convex grains, Corollary 3.26 shows that the intensity γ is uniquely determined by the densities $\overline{V}_j[Z], j = 0,\dots,d$. In fact, the formulas build a triangular array, which can be seen better, if we write them in explicit form,

$$\overline{V}_j[Z] = \mathrm{e}^{-\overline{V}_d[Y]} \left(\overline{V}_j[Y] - c_j^d \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1 \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} \overline{V}_{m_i}[Y] \right). \qquad (3.25)$$

By recursion, the densities $\overline{V}_d[Y]$ and $\overline{V}_{m_i}[Y]$ in (3.25) are determined by the corresponding equations for $\overline{V}_i[Z]$, $i = d,\dots,j+1$. Therefore, (3.25) determines $\overline{V}_j[Y]$. At the end of the recursion, we obtain $\overline{V}_0[Y] = \gamma$ (since $V_0 = 1$ on $\mathscr{K}^{(d)}$).

Solving the system (3.25) for $\overline{V}_0[Y]$, we obtain an equation

$$\gamma = f_{d0}(\overline{V}_0[Z],\dots,\overline{V}_d[Z])$$

with an explicitly given rational function f_{d0}. For $\overline{V}_j[Z]$, various estimators are described in the literature (see, for example, [22, Section 9.4]), among them also unbiased ones. We thus obtain estimators for γ (usually biased ones). For other estimators of γ, see [22, Section 9.5]. More generally, there is a rational function f_{dj} such that

$$\overline{V}_j[Y] = f_{dj}(\overline{V}_j[Z],\dots,\overline{V}_d[Z]), \quad j = 0,\dots,d.$$

If γ is already determined (estimated), we thus get also estimators for the mean particle characteristics

$$\mathbb{E}_{\mathbb{Q}}[V_j] = \int V_j(K)\,\mathbb{Q}(dK), \quad j = 1\dots,d.$$

Since the intrinsic volumes V_j are rotation invariant, it is clear that the densities $\overline{V}_j[Z]$ do no longer determine γ, if Z is not isotropic. In fact, the formulas (3.20) then involve mixed densities $\overline{V}_{\mathbf{m}}[Y]$, for which a corresponding recursion procedure is not apparent. It is thus natural to consider Corollary 3.22 with functionals φ which are direction dependent and hence distinguish between different rotation images of convex bodies. Such functionals are the mixed volumes $K \mapsto V(K[j], M[d-j])$, $j = 0, \ldots, d$, where M varies in $\mathscr{K}^d$; see Example 3.10. Justified by Corollary 3.22, we denote by $\overline{V}(Z[j], M[d-j])$ the density of the functional $K \mapsto V(K[j], M[d-j])$. The following general result was obtained in [6].

Theorem 3.27. *Let Z be a stationary Boolean model with convex grains which satisfies*

$$\int V_1(K)^{d-2}\, \mathbb{Q}(dK) < \infty. \tag{3.26}$$

If for $j = 0, \ldots, d$ and all $M \in \mathscr{K}^d$ the densities of the mixed volumes $\overline{V}(Z[j], M[d-j])$ are given, then the intensity γ is uniquely determined.

Proof. We sketch the main ideas of the proof and refer to [6], for details. We first mention, that the density formulas for mixed volumes have the form

$$\overline{V}_d[Z] = 1 - \mathrm{e}^{-\overline{V}_d[Y]},$$
$$\overline{V}[Z[d-1], M[1]] = \mathrm{e}^{-\overline{V}_d[Y]}\overline{V}[Y[d-1], M[1]],$$

and

$$\binom{d}{j}\overline{V}[Z[j], M[d-j]] = \mathrm{e}^{-\overline{V}_d[X]}\Bigg(\overline{V}[Y[j], M[d-j]] - \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1, \ldots, m_s = j+1 \\ m_1 + \cdots + m_s = (s-1)d + j}}^{d-1} \overline{V}_{m_1, \ldots, m_s, d-j}[Y, \ldots, Y, M^*]\Bigg) \tag{3.27}$$

for $j = 0, \ldots, d-2$, and all $M \in \mathscr{K}^d$, as follows from Corollary 3.22.

For the mixed functionals $V_{m_1, \ldots, m_s, d-j}(K_1, \ldots, K_s, M^*)$ an integral representation was established in [8] (based on [5]),

$$V_{m_1, \ldots, m_s, d-j}(K_1, \ldots, K_s, M^*) = \int \cdots \int f_{m_1, \ldots, m_s, d-j}(u_1, L_1, \ldots, u_s, L_s, u, L) \times \psi_{d-j}(M^*, d(u, L))\, \psi_{m_s}(K_s, d(u_s, L_s)) \cdots \psi_{m_1}(K_1, d(u_1, L_1))$$

which involves a universal function $f_{m_1, \ldots, m_s, d-j}$ and the flag measures (of corresponding degree) of $M^*, K_1, \ldots, K_s$. This representation carries over to the densities

$$\overline{V}_{m_1, \ldots, m_s, d-j}[Y, \ldots, Y, M^*] = \int \cdots \int f_{m_1, \ldots, m_s, d-j}(u_1, L_1, \ldots, u_s, L_s, u, L) \times \psi_{d-j}(M^*, d(u, L))\, \overline{\psi}_{m_s}[Y](d(u_s, L_s)) \cdots \overline{\psi}_{m_1}[Y](d(u_1, L_1)),$$

where $\overline{\psi}_i[Y](\cdot) := \gamma \int \psi_i(K, \cdot)\, \mathbb{Q}(dK)$. Since $f_{m_1, \ldots, m_s, d-j}$ is not continuous, an additional approximation procedure is necessary here. The idea is now to use a recursion procedure, as in the isotropic situation described above, in order to show that the equations in (3.27) with index $> j$ determine the density $\psi_j[Y](\cdot)$. Since the bottom line, for $j = 0$, is equivalent to

$$\overline{V}_0[Z] = \mathrm{e}^{-\overline{V}_d[Y]} \left(\overline{V}_0[Y] - \sum_{s=2}^{d} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\ldots,m_s=1 \\ m_1+\cdots+m_s=(s-1)d}}^{d-1} \overline{V}_{m_1,\ldots,m_s}[Y,\ldots,Y] \right),$$

we obtain $\overline{V}_0[Y] = \gamma$, if, by recursion, all measures $\psi_j[Y](\cdot)$, $j = 1,\ldots,d$, are determined.

A major problem here is that by (3.27) we do not get $\overline{\psi}_j[Y](\cdot)$ directly, but the density $\overline{V}[Y[j],M[d-j]]$ of the mixed volume $V(\cdot[j],M[d-j])$ (for all $M \in \mathscr{K}^d$). For these mixed volumes, a similar flag representation was proved in [7], which states that

$$V(K[j],M[d-j]) = \iint g_j(u,L,u',L')\, \psi_j(K,d(u,L))\, \psi_{d-j}(M,d(u',L')),$$

again with a universal function g_j. Since also this function is not continuous, it is not apparent that the corresponding density

$$\overline{V}[Y[j],M[d-j]] = \iint g_j(u,L,u',L')\, \overline{\psi}_j[Y](d(u,L))\, \psi_{d-j}(M,d(u',L'))$$

determines $\overline{\psi}_j[Y](\cdot)$, as M varies in $\mathscr{K}^d$. In fact, here a direct functional analytic argument was not available and instead a deep result of Alesker was used which showed that linear combinations of mixed volumes $V(\cdot[j],M[d-j])$, $M \in \mathscr{K}^d$, lie dense in the Banach space $\mathbf{Val}_j$ (of translation invariant, continuous, j-homogeneous valuations). Therefore, all densities $\overline{\varphi}[Y]$ with φ in $\mathbf{Val}_j$ are determined; thus, in particular,

$$\int g(u,L)\, \overline{\psi}_j[Y](d(u,L)),$$

is determined for any convex function g on the corresponding flag space. This information determines $\overline{\psi}_j[Y](\cdot)$. □

As in the isotropic case, we get some more shape information from Theorem 3.27, namely we obtain all expected flag measures $\mathbb{E}_{\mathbb{Q}}\psi_j$, $j = 1,\ldots,d-1$, (in fact, we get all expectations $\mathbb{E}_{\mathbb{Q}}\varphi$ of translation invariant, continuous valuations φ). For isotropic Z, these measures are proportional to the Haar measure on the space of j-flags and the proportionality constant is $\mathbb{E}_{\mathbb{Q}}V_j$, hence we do not get more shape information in this case. However, if the distribution $\mathbb{Q}$ is concentrated on one shape K_0 say, the whole distribution $\mathbb{Q}$ is determined.

Because of the use of Alesker's approximation result, Theorem 3.27 does not seem lead to an estimation procedure for γ in a direct way.

4 Variances and covariances

We adapt the setting of Section 3 and assume that the Boolean model Z is stationary. Since we are interested in second order properties of Z we assume that

$$\int V_i(K)^2\, \mathbb{Q}(dK) < \infty, \quad i = 0,\ldots,d. \tag{4.1}$$

In view of the Steiner formula (3.2) this is stronger than (2.5).

In this section we focus on *geometric functionals*. A measurable function $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ is said to be geometric if it is additive, locally bounded in the sense of (3.6) and *translation invariant*, that is, $\varphi(B+x) = \varphi(B)$, for any $B \in \mathscr{R}^d$ and any $x \in \mathbb{R}^d$. Examples can be found in Examples 3.12 and 3.18. Note that if $\varphi\colon \mathscr{R}^d \to \mathbb{R}$ is additive, then φ is translation invariant if the restriction of φ to convex bodies is translation invariant.

4.1 A fundamental formula

Let φ be a geometric functional. It follows from the local boundedness of φ that

$$\mathbb{E}\varphi(Z\cap W)^2 < \infty; \tag{4.2}$$

see [11, Lemma 3.3] and [14, Corollary 4.10]. Given a second local functional ψ it makes hence sense to study the covariance $\mathbb{C}\mathrm{ov}(\varphi(Z\cap rW), \psi(Z\cap rW))$ between $\varphi(Z\cap rW)$ and $\psi(Z\cap rW)$. If $V_d(W) > 0$ we define the *asymptotic covariance*

$$\sigma(\varphi,\psi) := \lim_{r\to\infty} \frac{\mathbb{C}\mathrm{ov}(\psi(Z\cap rW), \varphi(Z\cap rW))}{V_d(rW)} \tag{4.3}$$

whenever this limit exists and is independent of W.

For two geometric functionals φ, ψ, we define the inner product

$$\rho(\varphi,\psi) := \sum_{n=1}^{\infty} \frac{\gamma}{n!} \iint \varphi(K_1\cap\cdots\cap K_n) \\ \psi(K_1\cap\cdots\cap K_n)\,\Theta^{n-1}(d(K_2,\ldots,K_n))\,\mathbb{Q}(dK_1), \tag{4.4}$$

whenever this infinite series is well defined. The functional $\varphi^*\colon \mathscr{K}^d \to \mathbb{R}$ defined by

$$\varphi^*(K) = \mathbb{E}\varphi(Z\cap K) - \varphi(K), \quad K \in \mathscr{K}^d,$$

is again geometric and additive, see [11, (3.11)]. The importance of these operations for the covariance analysis of the Boolean model is due to the following result from [11].

Theorem 4.1. *Let φ and ψ be geometric functionals and let $W \in \mathscr{K}^d$ with $V_d(W) > 0$. Then the limit* (4.3) *exists and is given by*

$$\sigma(\varphi,\psi) = \rho(\varphi^*, \psi^*). \tag{4.5}$$

Proof. (Sketch) From the Fock space representation of Poisson functionals (see [14, Theorem 18.6]) and additivity of φ and ψ we obtain for each $K_0 \in \mathscr{K}^d$ that

$$\mathbb{C}\mathrm{ov}(\varphi(Z\cap K_0), \psi(Z\cap K_0)) \\ = \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iint \varphi^*(K_0\cap K_1^{x_1}\cap\cdots\cap K_n^{x_n})\,\psi^*(K_0\cap K_1^{x_1}\cap\cdots\cap K_n^{x_n}) \\ \times d(x_1,\ldots,x_n)\,\mathbb{Q}^n(d(K_1,\cdots,K_n)); \tag{4.6}$$

cf. [14, (22.34)] for the case $\varphi = \psi$. Replacing here K_0 by rW for some large $r > 0$ this equals approximately

$$\sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iint \varphi^*(K_1^{x_1} \cap \cdots \cap K_n^{x_n}) \psi^*(K_1^{x_1} \cap \cdots \cap K_n^{x_n}) \mathbf{1}\{K_1 + x_1 \subset rW\} \times d(x_1, \ldots, x_n)\, \mathbb{Q}^n(d(K_1, \cdots, K_n)).$$

By a change of variables and translation invariance of φ and ψ the above series comes to

$$\sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint \varphi^*(K_1 \cap K_2^{y_2} \cap \cdots \cap K_n^{y_n}) \psi^*(K_1 \cap K_2^{y_2} \cap \cdots \cap K_n^{y_n}) \mathbf{1}\{K_1 + y_1 \subset rW\} \times dy_1\, d(y_2, \ldots, y_n)\, \mathbb{Q}^n(d(K_1, \cdots, K_n)).$$

Making the change of variables $y := r^{-1} y_1$ and dividing by $V_d(rW)$ we see that

$$V_d(rW)^{-1}\, \mathbb{C}\mathrm{ov}(\varphi(Z \cap rW), \psi(Z \cap rW))$$

approximately equals

$$V_d(W)^{-1} \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint \varphi^*(K_1 \cap K_2^{y_2} \cap \cdots \cap K_n^{y_n}) \psi^*(K_1 \cap K_2^{y_2} \cap \cdots \cap K_n^{y_n}) \times \mathbf{1}\{r^{-1} K_1 + y \subset W\}\, dy\, d(y_2, \ldots, y_n)\, \mathbb{Q}^n(d(K_1, \cdots, K_n)).$$

As $r \to \infty$ this converges to the right-hand side of (4.5). $\square$

4.2 Asymptotic covariances for intrinsic volumes

As a warm-up we may consider the geometric functional V_d. In this case

$$V_d^*(K) = \mathbb{E} V_d(Z \cap K) - V_d(K) = -(1-p) V_d(K), \quad K \in \mathscr{K}^d, \tag{4.7}$$

where $p := \mathbb{P}(0 \in Z)$ is the volume fraction of Z; see (2.8). In order to apply (4.5) to compute the asymptotic variance $\sigma(V_d, V_d)$ of the volume we need to compute

$$\rho(V_d, V_d) = \sum_{n=1}^{\infty} \frac{\gamma}{n!} \iint V_d(K_1 \cap \cdots \cap K_n)^2\, \Theta^{n-1}(d(K_2, \ldots, K_n))\, \mathbb{Q}(dK_1).$$

By Fubini's theorem the above series can be written as

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \cdots \int \mathbf{1}\{y \in K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_n + x_n)\} \\
&\times \mathbf{1}\{z \in K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_n + x_n)\}\, dy\, dz\, dx_2 \ldots dx_n\, \mathbb{Q}^n(d(K_1, \ldots, K_n)) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint V_d((K_2 - y) \cap (K_2 - z)) \cdots V_d((K_n - y) \cap (K_n - z)) \\
&\qquad \times \mathbf{1}\{y \in K_1\} \mathbf{1}\{z \in K_1\}\, dy\, dz\, \mathbb{Q}^n(d(K_1, \ldots, K_n)) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint (\mathbb{E} V_d(Z_0 \cap (Z_0 + y - z)))^{n-1} \mathbf{1}\{y, z \in K_1\}\, dy\, dz\, \mathbb{Q}(dK_1) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint (\mathbb{E} V_d(Z_0 \cap (Z_0 + y)))^{n-1} \mathbf{1}\{y + z \in K_1\} \mathbf{1}\{z \in K_1\}\, dy\, dz\, \mathbb{Q}(dK_1).
\end{aligned}$$

Hence we obtain that

$$\rho(V_d, V_d) = \int \left(e^{C_d[Y](y)} - 1\right) dy, \tag{4.8}$$

where the expected *covariogram* $C_d[Y]$ of the typical grain is the function on $\mathbb{R}^d$ defined by

$$C_d[Y](y) := \gamma \mathbb{E} V_d(Z_0 \cap (Z_0 + y)), \quad y \in \mathbb{R}^d.$$

By (4.5), (4.7) and (4.8) the asymptotic variance of the volume is given by

$$\sigma(V_d, V_d) = (1-p)^2 \int \left(e^{C_d[Y](y)} - 1\right) dy, \tag{4.9}$$

Formula (4.9) can be derived in a simpler way, without using Theorem 4.1 and without convexity assumptions on the grain distribution; see [14, Proposition 22.1]. The above calculation, however, can be generalized to other geometric functionals.

To derive an expression for $\sigma(V_{d-1}, V_d)$ we need to compute $\rho(V_{d-1}, V_d)$. To do so we use that $V_{d-1}(K) = \Phi_{d-1}(K, \mathbb{R}^d)$ for each $K \in \mathscr{K}^d$, where $\Phi_i(K, \cdot) = \Lambda_i(K, \cdot \times \mathbb{S}^{d-1})$, for $i \in \{0, \dots, d-1\}$, is the i-th curvature measure of K; see, e.g., [22, Section 14.2] or [21, Section 4.2]. Let B° denote the interior of a set $B \subset \mathbb{R}^d$. If

$$\mathbb{P}(Z_0^\circ \neq \emptyset) = 1 \tag{4.10}$$

it can be shown that

$$\Phi_{d-1}(K_1 \cap \dots \cap K_n, \cdot) = \sum_{k=1}^n \int \mathbf{1}\{x \in \cdot \cap_{r \neq k} K_r^\circ\} \Phi_{d-1}(K_k, dx) \tag{4.11}$$

for $(\mathbb{Q} \otimes \Theta^{n-1})$-a.e. $(K_1, \dots, K_n)$ and all $n \in \mathbb{N}$; see [11, Lemma 5.3]. Using these facts it is not hard to see that

$$\begin{aligned}
&\iint V_{d-1}(K_1 \cap \dots \cap K_n) V_d(K_1 \cap \dots \cap K_n)\, \Theta^{n-1}(d(K_2, \dots, K_n))\, \mathbb{Q}(dK_1) \\
&= n \iiiint \mathbf{1}\{y \in (K_2^\circ + x_2) \cap \dots \cap (K_n^\circ + x_n), z \in K_1 \cap (K_2 + x_2) \cap \dots \cap (K_n + x_n)\} \\
&\qquad \times \Phi_{d-1}(K_1, dy)\, dz\, d(x_2, \dots, x_n)\, \mathbb{Q}^n(d(K_1, \dots, K_n)).
\end{aligned}$$

Therefore we obtain similarly as above that $\rho(V_{d-1}, V_d)$ equals

$$\sum_{n=1}^\infty \frac{\gamma^n}{(n-1)!} \iint \mathbf{1}\{z \in K\} (\mathbb{E} V_d((Z_0 - y) \cap (Z_0 - z)))^{n-1}\, \Phi_{d-1}(K, dy)\, dz\, \mathbb{Q}(dK).$$

It follows that

$$\rho(V_{d-1}, V_d) = \int e^{C_d[Y](y-z)}\, M_{d-1,d}[Y](d(y,z)), \tag{4.12}$$

where the measure $M_{d-1,d}[Y]$ is defined by

$$\begin{aligned}
M_{d-1,d}[Y] &:= \gamma \iint \mathbf{1}\{(y,z) \in \cdot\} \mathbf{1}\{z \in K\}\, \Phi_{d-1}(K, dy)\, dz\, \mathbb{Q}(dK) \\
&= \gamma \iint \mathbf{1}\{(y,z) \in \cdot\}\, \Phi_{d-1}(K, dy)\, \Phi_d(K, dz)\, \mathbb{Q}(dK),
\end{aligned}$$

where $\Phi_d(K,\cdot)$ equals d-dimensional Lebesgue measure restricted to K.

If (4.10) holds, we obtain similarly to the derivation of (4.12) that

$$\begin{aligned}\rho(V_{d-1},V_{d-1}) = \gamma \int e^{C_d[Y](y-z)} C_{d-1}[Y](y-z)\, M_{d-1,d}[Y](d(y,z)) \\ + \gamma \int e^{C_d[Y](y-z)}\, M_{d-1,d-1}[Y](d(y,z)),\end{aligned} \tag{4.13}$$

where the measure $M_{d-1,d-1}[Y]$ is defined by

$$M_{d-1,d-1}[Y] := \gamma \iint \mathbf{1}\{(y,z) \in \cdot\}\, \Phi_{d-1}(K,dy)\, \Phi_{d-1}(K,dz)\, \mathbb{Q}(dK)$$

and the function $C_{d-1}[Y]$ is defined by

$$C_{d-1}[Y](y) := \gamma \int \Phi_{d-1}(K, K^\circ + y)\, \mathbb{Q}(dK), \quad y \in \mathbb{R}^d.$$

We can now prove a result from [11]; see [10, Theorem 2] for a slight generalization.

Theorem 4.2. *It holds that*

$$\begin{aligned}\sigma(V_{d-1},V_d) = -(1-p)^2 \overline{V}_{d-1}[Y] \int \left(e^{C_d[Y](x)} - 1\right) dx \\ + (1-p)^2 \int e^{C_d[Y](x-y)}\, M_{d-1,d}[Y](d(x,y)).\end{aligned}$$

If, in addition, (4.10) *holds, then*

$$\begin{aligned}\sigma(V_{d-1},V_{d-1}) = (1-p)^2 (\overline{V}_{d-1}[Y])^2 \int \left(e^{C_d[Y](x)} - 1\right) dx \\ + (1-p)^2 \int e^{C_d[Y](x-y)} C_{d-1}[Y](x-y)\, M_{d-1,d}[Y](d(y,x)) \\ - 2(1-p)^2 \overline{V}_{d-1}[Y] \int e^{C_d[Y](x-y)}\, M_{d-1,d}[Y](d(y,x)) \\ + (1-p)^2 \int e^{C_d[Y](x-y)}\, M_{d-1,d-1}[Y](d(x,y)).\end{aligned}$$

Proof. By Corollary 3.14 and (2.7),

$$V^*_{d-1}(K) = -(1-p)V_{d-1}(K) + (1-p)\overline{V}_{d-1}[Y] V_d(K), \quad K \in \mathscr{K}^d. \tag{4.14}$$

Combining (4.14) and (4.7) with (4.12), (4.13) and (4.5), we obtain the assertion. □

Formula (4.11) can be formulated for all curvature measures. Therefore it possible to derive integral representations for all asymptotic covariances $\rho(V_i,V_j)$. The resulting formulas, however, are less explicit than those in Theorem 4.2. We refer to [11] for the details. The special case of a planar Boolean model of aligned rectangles has been treated in [10]. Even though this model is not isotropic, the formulas become surprisingly explicit in this case.

4.3 The isotropic case

In this subsection we assume the Boolean model Z to be stationary and isotropic. For all $j \in \{0,\dots,d-1\}$ and $k \in \{j,\dots,d\}$ we define a polynomial $P_{j,k}$ on $\mathbb{R}^{d-j}$ of degree $k-j$ by

$$P_{j,k}(t_j,\dots,t_{d-1}) := \mathbf{1}\{k=j\} + c_j^k \sum_{s=1}^{k-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\ldots+m_s=sd+j-k}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} t_{m_i}, \tag{4.15}$$

where the constants c_j^i are given by (3.23). We also define $P_{d,d} := 1$. The following result is taken from [11]; see also [16].

Theorem 4.3. *Let $i,j \in \{0,\dots,d\}$. Then*

$$\sigma(V_i,V_j) = (1-p)^2 \sum_{k=i}^{d}\sum_{l=j}^{d} P_{i,k}(\overline{V}_i[Y],\dots,\overline{V}_{d-1}[Y]) P_{j,l}(\overline{V}_j[Y],\dots,\overline{V}_{d-1}[Y]) \rho(V_k,V_l),$$

Proof. By Theorem 3.25 and Remark 3.24 we have for all $j \in \{0,\dots,d\}$ and $K \in \mathscr{K}^d$ that

$$V_j^*(K) = -(1-p)\sum_{k=j}^{d} V_k(K) P_{j,k}(\overline{V}_j[Y],\dots,\overline{V}_{d-1}[Y]). \tag{4.16}$$

Using this formula in (4.5) we obtain the assertion. □

Theorem 4.3 requires the numbers $\rho(V_i,V_j)$. Thanks to our isotropy assumption we can complement (4.8), (4.12) and (4.13) as follows.

Proposition 4.4. *We have that*

$$\rho(V_0,V_d) = e^{\overline{V}_d[Y]} - 1 \tag{4.17}$$

and, for $i \in \{0,\dots,d-1\}$,

$$\rho(V_0,V_i) = e^{\overline{V}_d[Y]} c_i^d \sum_{l=1}^{d-i} \frac{1}{l!} \sum_{\substack{m_1,\dots,m_l=i \\ m_1+\cdots+m_l=(l-1)d+i}}^{d-1} \prod_{j=1}^{l} c_d^{m_j} \overline{V}_{m_j}[Y]. \tag{4.18}$$

The formulas for $\rho(V_0,V_d)$ and $\rho(V_{d-1},V_d)$ are true without isotropy assumption.

Proof. The proof of (4.17) is easy and left to the reader.

Let $i \in \{0,\dots,d-1\}$. Since $\mathbb{Q}$ is concentrated on nonempty grains we have that

$$\rho(V_0,V_i) = \sum_{n=1}^{\infty} \frac{\gamma}{n!} \iint V_i(K_1\cap\cdots\cap K_n)\, \Theta^{n-1}(d(K_2,\dots,K_n))\, \mathbb{Q}(dK_1).$$

By isotropy, this equals

$$\sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iint V_i(K_1 \cap g_2K_2 \cap\cdots\cap g_nK_n)\, \mu^{n-1}(d(g_2,\dots,g_n))\, \mathbb{Q}^n(d(K_1,\dots,K_n)).$$

By the principal kinematic formula (3.22) and Remark 3.24 (see also [22, Theorem 5.1.5]) this equals

$$\overline{V}_i[Y] + \sum_{n=2}^{\infty} \frac{\gamma^n}{n!} \int c_i^d \sum_{\substack{m_1,\dots,m_n=i \\ m_1+\dots+m_n=(n-1)d+i}}^{d} \prod_{j=1}^{n} c_d^{m_j} V_{m_j}(K_j)\, \mathbb{Q}^n(d(K_1,\dots,K_n)).$$

We now argue as in the proof of Theorem 3.15. If $m_1 + \dots + m_n = (n-1)d + i$ we can consider the indices m_k which are smaller than d. The number s of those indices ranges from 1 to $d-i$. Since $\binom{n}{s} = 0$ if $s > n$ and $c_d^d = 1$, we thus obtain that

$$\begin{aligned}\rho(V_0,V_i) &= \overline{V}_i[Y] + \sum_{n=2}^{\infty} \frac{1}{n!} c_i^d \sum_{s=1}^{d-i} \binom{n}{s} \overline{V}_d[Y]^{n-s} \sum_{\substack{m_1,\dots,m_s=i \\ m_1+\dots+m_s=(s-1)d+i}}^{d-1} \prod_{j=1}^{s} c_d^{m_j} \overline{V}_{m_j}[Y] \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} c_i^d \sum_{s=1}^{d-i} \binom{n}{s} \overline{V}_d[Y]^{n-s} \sum_{\substack{m_1,\dots,m_s=i \\ m_1+\dots+m_s=(s-1)d+i}}^{d-1} \prod_{j=1}^{s} c_d^{m_j} \overline{V}_{m_j}[Y].\end{aligned}$$

Therefore,

$$\rho(V_0,V_i) = \sum_{s=1}^{d-i} \sum_{n=s}^{\infty} \frac{1}{n!} c_i^d \frac{1}{s!(n-s)!} \overline{V}_d[Y]^{n-s} \sum_{\substack{m_1,\dots,m_s=i \\ m_1+\dots+m_s=(s-1)d+i}}^{d-1} \prod_{j=1}^{s} c_d^{m_j} \overline{V}_{m_j}[Y],$$

and the result (4.18) follows. □

When combined with (4.8), (4.12) and (4.13), Proposition 4.4 provides in the case $d = 2$ explicit formulas for the asymptotic covariances $\sigma(V_0,V_i)$, $i \in \{0,1,2\}$. Together with Theorem 4.2 this yields all asymptotic covariances for the intrinsic volumes. For instance we have:

Theorem 4.5. *For a planar stationary and isotropic Boolean model the asymptotic covariance between Euler characteristic and surface is given by*

$$\begin{aligned}\sigma(V_0,V_2) = p(1-p) - (1-p)^2\Big(\gamma - \frac{\overline{V}_1[Y]^2}{\pi}\Big) \int \big(e^{C_2[Y](x)} - 1\big)\, dx \\ - (1-p)^2 \frac{2\overline{V}_1[Y]}{\pi} \int e^{C_2[Y](x-y)} M_{1,2}[Y](d(x,y)).\end{aligned}$$

The formulas for $\sigma(V_0,V_1)$ and $\sigma(V_0,V_0)$ (again in the planar isotropic case) require the assumption (4.10) and are more lengthy. In view of (4.12) and (4.13) they not only involve the measure $M_{1,2}[Y]$ and the function $C_2[Y]$, but also the measure $M_{1,1}[Y]$ and the function $C_1[Y]$.

For the time being it is not clear whether the asymptotic covariances of a three-dimensional isotropic Boolean model can be made as explicit as in the two-dimensional case. More details on this point can be found in the extended preprint version of [11].

4.4 Positivity of asymptotic variances

It is of some interest to know whether the asymptotic variance $\sigma(\varphi,\varphi)$ of a geometric functional is positive. The following result (see [14, Exercise 22.4], where a typo needs to be corrected) shows that this is true in great generality.

Theorem 4.6. *Let φ be a geometric functional satisfying*

$$\iint |\varphi(K\cap(K_1+x_1)\cap\cdots\cap(K_n+x_n))|\,d(x_1,\ldots,x_n)\,\mathbb{Q}^n(d(K_1,\ldots,K_n))\,\mathbb{Q}(dK)>0 \quad (4.19)$$

for some $n\in\mathbb{N}_0$. Then $\sigma(\varphi,\varphi)>0$.

Proof. We proceed by contradiction and assume that $\sigma(\varphi,\varphi)=0$.

First, we assume that (4.19) holds for $n=0$, that is,

$$\int |\varphi(K)|\,\mathbb{Q}(dK)>0. \quad (4.20)$$

By Theorem 4.1 we then obtain, for all $m\in\mathbb{N}$, $(\lambda_d\otimes\mathbb{Q})^m$-a.e. $((y_1,K_1),\ldots,(y_m,K_m))$ and for $\mathbb{Q}$-a.e. K, that $\varphi^*(K)=0$ and

$$\varphi^*(K\cap(K_1+y_1)\cap\cdots\cap(K_m+y_m))=0. \quad (4.21)$$

Hence we obtain from (4.6) that $\mathbb{V}\mathrm{ar}(\varphi(Z\cap K))=0$ for $\mathbb{Q}$-a.e. K, that is

$$\varphi(Z\cap K)=\mathbb{E}\varphi(Z\cap K),\quad \mathbb{P}\text{-a.s., }\mathbb{Q}\text{-a.e. }K.$$

Since $\varphi^*(K)=0$ for $\mathbb{Q}$-a.e. K, we have $\mathbb{E}\varphi(Z\cap K)=\varphi(K)$ for $\mathbb{Q}$-a.e. K, and thus

$$\varphi(Z\cap K)=\varphi(K),\quad \mathbb{P}\text{-a.s, }\mathbb{Q}\text{-a.e. }K.$$

Moreover, by (2.6) and our basic integrability assumption (2.5), $\mathbb{P}(Z\cap K=\emptyset)>0$ for each $K\in\mathscr{K}^{(d)}$. Therefore $\varphi(K)=\varphi(\emptyset)=0$ for $\mathbb{Q}$-a.e. K, contradicting (4.20).

If (4.19) holds for $n=1$, we proceed in a similar way. Observe that it follows from (4.21) that $\mathbb{V}\mathrm{ar}(\varphi(Z\cap K\cap(K_1+y_1)))=0$ for $\mathbb{Q}$-a.e. K and $\lambda_d\otimes\mathbb{Q}$-a.e. (y_1,K_1), hence

$$\varphi(Z\cap K\cap(K_1+y_1))=\mathbb{E}\varphi(Z\cap K\cap(K_1+y_1))$$

holds $\mathbb{P}$-a.s., for $\mathbb{Q}$-a.e. K and for $\lambda_d\otimes\mathbb{Q}$-a.e. (y_1,K_1). Since $\varphi^*(K\cap(K_1+y_1))=0$ for $\mathbb{Q}$-a.e. K and $\lambda_d\otimes\mathbb{Q}$-a.e. (y_1,K_1), we conclude that

$$\varphi(Z\cap K\cap(K_1+y_1))=\varphi(K\cap(K_1+y_1))$$

holds $\mathbb{P}$-a.s., for $\mathbb{Q}$-a.e. K and for $\lambda_d\otimes\mathbb{Q}$-a.e. (y_1,K_1). Since $\mathbb{P}(Z\cap K\cap(K_1+y_1)=\emptyset)>0$ whenever $K\cap(K_1+y_1)\neq\emptyset$, we arrive at a contradiction as before.

The argument for arbitrary $n\in\mathbb{N}$ is the same, but the notation is more involved. □

In the case where $\varphi=V_i$, the preceding Theorem 4.6 is equivalent to the following corollary, since the intrinsic volumes are monotone and nonnegative.

Corollary 4.7. *Let $i\in\{0,\ldots,d\}$ be such that $\int V_i(K)\,\mathbb{Q}(dK)>0$. Then $\sigma(V_i,V_i)>0$.*

The next result is a special case of Theorem 4.1 in [11], dealing with more general geometric functionals. Thanks to Theorem 4.6 we can give here a much shorter proof.

Theorem 4.8. *Assume that* $\mathbb{P}(Z_0^\circ \neq \emptyset) > 0$. *Then the matrix* $(\sigma(V_i,V_j))_{i,j=0,\dots,d}$ *is positive definite.*

Proof. Let $a_0,\dots,a_d \in \mathbb{R}$ satisfy $\sum_{i=0}^d |a_i| > 0$. We need to show that

$$\sum_{i,j=0}^{d} a_i a_j \sigma(V_i,V_j) > 0,$$

that is $\sigma(\varphi,\varphi) > 0$, where $\varphi := \sum_{i=0}^d a_i V_i$. By Corollary 4.7 we can assume that $I^+ \neq \emptyset$ and $I^- \neq \emptyset$, where $I^+ := \{i \in \{0,\dots,n\} : a_i > 0\}$ and I^- is defined similarly. We check condition (4.19) for $n = d+1$. Assume it fails. Then there exist $K_1,\dots,K_{d+1} \in \mathscr{K}^d$ with nonempty interior such that

$$\varphi(L(x_2,\dots,x_{d+1})) = 0, \quad \lambda_d^d\text{-a.e. } (x_2,\dots,x_{d+1}),$$

where $L(x_2,\dots,x_{d+1}) := K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_{d+1} + x_{d+1})$. Assume without loss of generality that $\min I^+ > \min I^-$. Take $c_0 > 0$ to be specified later. Let $R(K)$ (resp. $r(K)$) denote the circumradius (resp. inradius) of $K \in \mathscr{K}^{(d)}$. By the monotonicity, translation invariance and homogeneity of intrinsic volumes we obtain for λ_d^d-a.e. $(x_2,\dots,x_{d+1})$ with $R(L) \le c_0 r(L)$ (abbreviating $L := L(x_2,\dots,x_{d+1})$) that

$$\begin{aligned}\sum_{i\in I^+} a_i R(L)^i V_i(B^d) \ge \sum_{i\in I^+} a_i V_i(L) &= \sum_{i\in I^-} (-a_i) V_i(L) \\ &\ge \sum_{i\in I^-} (-a_i) r(L)^i V_i(B^d) \ge \sum_{i\in I^-} (-a_i) c_0^{-1} R(L)^i V_i(B^d).\end{aligned}$$

Since $\min I^+ > \min I^-$ the above inequality fails, whenever $R(L) \le r$ for a sufficiently small r. However, choosing c_0 as in [11, Lemma 4.2], the inequality must hold on a set of points $(x_2,\dots,x_{d+1})$ with positive λ_d^d-measure. Hence (4.19) cannot fail for $n = d+1$, so that Theorem 4.6 implies $\sigma(\varphi,\varphi) > 0$. □

5 Central limit theorems

Again we adapt the setting of Section 3 and assume that (4.1) is satisfied. Throughout this section we fix some $W \in \mathscr{K}^d$ with $V_d(W) > 0$. Let φ be a geometric functional and define

$$\sigma_r(\varphi) := \mathbb{V}\mathrm{ar}(\varphi(Z \cap rW))^{1/2}, \quad r > 0. \tag{5.1}$$

Whenever $\sigma_r(\varphi) > 0$ we define

$$\hat{\varphi}_r := \sigma_r(\varphi)^{-1}(\varphi(Z \cap rW) - \mathbb{E}\varphi(Z \cap rW)) \tag{5.2}$$

and note that $\hat{\varphi}_r$ is a random variable (depending on Z) with mean zero and variance one. In this section we study the *asymptotic normality* of $\hat{\varphi}_r$ as $r \to \infty$.

5.1 Stein's method

Recall from the textbooks (from [12] for instance) that a sequence (X_n) of real-valued random variable is said to *converge in distribution* to a random variable X if $\lim_{n\to\infty}\mathbb{E}f(X_n)=\mathbb{E}f(X)$ for every bounded continuous function $f\colon \mathbb{R}\to\mathbb{R}$. One writes $X_n \stackrel{d}{\to} X$ as $n\to\infty$. (This definition extends to random vectors in the obvious way.) A possible way to quantify this convergence is the *Wasserstein distance* between random variables X_0, X. This distance is defined by

$$d_1(X_0,X)=\sup_{h\in\mathbf{Lip}(1)}|\mathbb{E}[h(X_0)]-\mathbb{E}[h(X)]|, \tag{5.3}$$

where $\mathbf{Lip}(1)$ denotes the space of all Lipschitz functions $h\colon\mathbb{R}\to\mathbb{R}$ with a Lipschitz constant less than or equal to one. If a sequence (X_n) of random variables satisfies $\lim_{n\to\infty}d_1(X_n,X)=0$, then it is not hard to see that X_n converges to X in distribution. Here we are interested in the *central limit theorem*, that is in the case where X has a standard normal distribution.

Let $\mathbf{AC}_{1,2}$ be the set of all differentiable functions $g\colon\mathbb{R}\to\mathbb{R}$ such that the derivative g' is absolutely continuous and satisfies $\sup\{|g'(x)| : x\in\mathbb{R}\}\le 1$ and $\sup\{|g''(x)| : x\in\mathbb{R}\}\le 2$, for some version g'' of the Radon–Nikodým derivative of g'. Throughout we let N denote a standard normal random variable. Let X be an integrable random variable. Stein [25] discovered that

$$d_1(X,N)\le\sup_{g\in\mathbf{AC}_{1,2}}|\mathbb{E}[g'(X)-Xg(X)]|. \tag{5.4}$$

To convert (5.4) into a bound for functions X of our Poisson process Y we need to introduce some notation. Let $X=f(Y)$ be a measurable function of Y. Then we define

$$D_KX := f(Y\cup\{K\})-f(Y),\quad K\in\mathscr{K}^d,$$

while DX denotes the mapping (the *difference operator*) $K\mapsto D_KF$ from $K\in\mathscr{K}^d$ to the space of random varables. Of course there exist other measurable function $\tilde f$ such that $X=\tilde f(Y)$ $\mathbb{P}$-a.s. However, it is not hard to see that the definition of D_KX is $\mathbb{P}$-almost surely and for Θ-almost every K independent of the specific choice of the *representative* f.

We can write $Y=\{Z_n : n\in\mathbb{N}\}$, where the Z_n are random convex bodies, depending measurably on Y. Let $B_1,B_2,\dots$ be a sequence of independent Bernoulli random variables with success probability $t\in[0,1]$, independent of Y. Define $\eta_t:=\{Z_n : B_n=1\}$ as a *t-thinning* of Y. By [14, Corollary 5.9], Y_t and $Y-Y_t$ are independent Poisson processes with intensity measures $t\Theta$ and $(1-t)\Theta$, respectively. Given $X=f(Y)$ and $t\in[0,1]$ we define a random variable P_t via the conditional expectation

$$P_tX=\mathbb{E}[f(Y_t+Y'_{1-t})\mid Y], \tag{5.5}$$

where Y'_{1-t} is a Poisson process with intensity measure $(1-t)\Theta$, independent of the pair (Y,Y_t).

Now we are in the position to state a seminal result from [19]. We take the version from [14, Chapter 21], specialized to Poisson particle processes.

Theorem 5.1. *Let X be a square integrable function of the Poisson process Y such that $\mathbb{E}\int (D_K X)^2\,\Theta(dK) < \infty$ and $\mathbb{E}X = 0$. Then*

$$d_1(X,N) \le \mathbb{E}\Big|1 - \iint_0^1 (P_t D_K X)(D_K X)\,dt\,\Theta(dK)\Big| \\ + \mathbb{E}\iint_0^1 |P_t D_K X|(D_K X)^2\,dt\,\Theta(dK). \tag{5.6}$$

The proof of this theorem is based on the Stein bound (5.4) and a covariance identity for Poisson functionals. The original proof in [19] uses Malliavin calculus.

5.2 Quantitative results

In this section we strengthen the assumption (4.1) and assume that

$$\int V_i(K)^3\,\mathbb{Q}(dK) < \infty, \quad i = 0,\dots,d. \tag{5.7}$$

Let B^d denote the unit ball in $\mathbb{R}^d$, and define a measurable function $\bar{V}: \mathscr{K}^d \to \mathbb{R}$ by

$$\overline{V}(K) := V_d(K \oplus B^d), \quad K \in \mathscr{K}^d, \tag{5.8}$$

where $A \oplus B := \{x + y : x \in A, y \in B\}$ is the *Minkowski addition* of two sets $A, B \subset \mathbb{R}^d$. The function $\bar{V}$ is translation invariant and additive.

Essentially, the following result is Theorem 9.3 in [11]. Recall the notation (5.1) and (5.2).

Theorem 5.2. *Suppose that φ is a geometric functional and that* (5.7) *holds. Then there exist constants $c_1, c_2 > 0$ such that*

$$d_1(\hat{\varphi}_r, N) \le c_1 \sigma_r^{-2}(\varphi)\overline{V}(W)^{1/2} + c_2 \sigma_r^{-3}(\varphi)\overline{V}(W), \tag{5.9}$$

whenever $\sigma_r(\varphi) > 0$.

The proof of Theorem 5.2 is beyond the scope of this chapter. The original argument in [11] rests on Theorem 5.1 and a tedious analysis of the *chaos expansion* of $\varphi(Z \cap rW)$. Our formulation is taken from [14, Chapter 22], where the proof is based on the *second order Poincaré inequality* from [13]. The latter result involves second order difference operators and is derived from Theorem 5.1 and the *Poincaré inequality*.

If the asymptotic variance $\sigma(\varphi,\varphi)$ is positive, Theorem 5.2 has the following immediate consequence.

Theorem 5.3. *Suppose that the assumptions of Theorem 5.2 hold and, in addition, that $\sigma(\varphi,\varphi) > 0$. Then there exist $\bar{c} > 0$ and $r_0 > 0$ such that*

$$d_1(\hat{\varphi}_r, N) \le \bar{c} r^{-1/2}, \quad r \ge r_0.$$

By [14, Proposition 21.6] the rate of convergence in Theorem 5.3 is presumably optimal.

5.3 Central limit theorems

In this subsection we return to the (minimal) integrability assumption (4.1). The following result stems from [11].

Theorem 5.4. *Suppose that φ is a geometric functional and that* (4.1) *holds. Assume also that* $\sigma(\varphi,\varphi)>0$. *Then $\hat{\varphi}_r \xrightarrow{d} N$ as $r\to\infty$.*

Proof. (Sketch) Under the stronger assumption (4.1) the result follows from Theorem 5.3. In the general case we define for $r>0$ the set

$$M_r = \left\{K \in \mathscr{K}^{(d)} : \bar{V}(K) \le (V_d(rW))^{1/2}\right\}.$$

The restriction of Y to M_r is a stationary Poisson process, generating a stationary Boolean model Z_r. By the triangle inequality for the Wasserstein distance,

$$d_1(\hat{\varphi}_r, N) \le d_1(\hat{\varphi}_r, \varphi_r^*) + d_1(\varphi_r^*, N), \tag{5.10}$$

where

$$\varphi_r^* := \mathbb{V}\mathrm{ar}(\varphi(Z_r \cap rW))^{-1/2}(\varphi(Z_r \cap rW) - \mathbb{E}\varphi(Z_r \cap rW)).$$

We have that $d_1(\hat{\varphi}_r, \varphi_r^*) \le (\mathbb{E}(\hat{\varphi} - \varphi_r^*))^{1/2}$, which can be shown to converge to 0 as $r\to\infty$; see [11, Lemma 9.6]. The distance $d_1(\varphi_r^*, N)$ can be treated with the non-asymptotic result of Theorem 5.3, applied to the Boolean model Z_r. The constant $\bar{c} \equiv \bar{c}_r$ depends on r. However, it can be seen from the explicit bounds in the proof of Theorem 22.7 in [14] (the third moment condition is required only in the last step of the proof) that $r^{-1/2}\bar{c}_r \to 0$ as $r\to\infty$. Therefore we obtain that $d_1(\hat{\varphi}_r, N) \to \infty$ as $r\to\infty$ and hence the assertion. □

We finish this section with a *multivariate central limit theorem* for the intrinsic volumes. Given a positive semi-definite matrix Σ, we let N_Σ denote a centered Gaussian random vector with covariance matrix Σ.

Theorem 5.5. *Assume that* (4.1) *and* $\mathbb{P}(Z_0^\circ \neq \emptyset) > 0$ *hold. Define the asymptotic covariance matrix* $\Sigma := (\sigma(V_i,V_j))_{i,j=0,\dots,d}$. *Then*

$$V_d(rW)^{-1/2}(V_0(Z\cap rW) - \mathbb{E}V_0(Z\cap rW), \dots, V_d(Z\cap rW) - \mathbb{E}V_d(Z\cap rW)) \xrightarrow{d} N_\Sigma,$$

as $r\to\infty$.

Proof. If φ is a geometric functional with $\sigma(\varphi,\varphi)>0$ then it follows from Theorem 5.4 and Slutzky's theorem that

$$V_d(rW)^{-1/2}(\varphi(Z\cap rW) - \varphi(Z\cap rW)) \xrightarrow{d} N_{\sigma(\varphi,\varphi)} \quad \text{as } r\to\infty. \tag{5.11}$$

We use the Cramér Wold theorem (see, e.g., [12]). Given a non-zero vector $(a_0,\dots,a_d) \in \mathbb{R}^{d+1}$ we need to show that (5.11) holds for the geometric functional $\varphi := \sum_{i=}^d a_i\varphi$. Note that the asymptotic variance of φ is given by

$$\sigma(\varphi,\varphi) = \sum_{i,j=0}^d a_i a_j \sigma(V_i,V_j),$$

which is also the variance of the centered Gaussian random variable $\sum_{i=0}^{d} a_i N_i$, where N_i is the i-th component of N_Σ. By Theorem 4.8, $\sigma(\varphi, \varphi) > 0$, so that (5.11) holds. This concludes the proof. □

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