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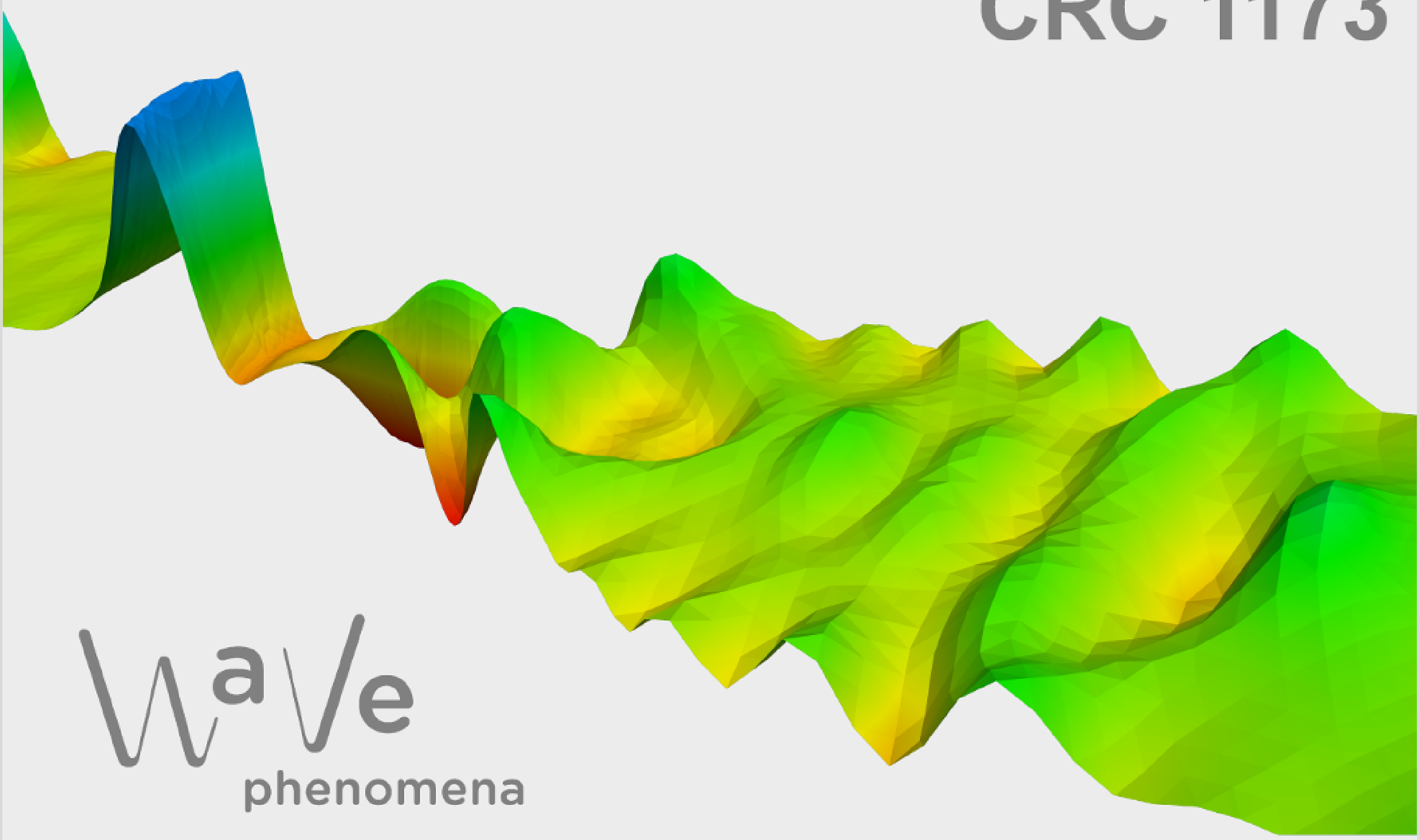

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO OPEN PERIODIC WAVEGUIDES

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Abstract. In this paper we consider the propagation of waves in an open waveguide in the half space $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$ under Dirichlet- or Neumann boundary condition for $x_2 = 0$. The index of refraction $n = n(x)$ is periodic along the axis of the waveguide (which we choose to be the x_1-axis) and equal to one for $x_2 > h_0$ for some $h_0 > 0$. Based on a Limiting Absorption Principle, derived in a former paper, we formulate a radiation condition, prove existence and uniqueness of a solution and describe explicitely the asymptotic behavior along the axis of the waveguide. This behavior depends crucially on the existence of non-evanescent (normal to the axis of the waveguide) modes.

MSC: 35J05

Key words: Helmholtz equation, open waveguide, propagating modes, radiation condition

1. Introduction

In this paper we study the boundary value problem

$$\Delta u + k^2 n\, u \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad u = 0 \quad \text{or} \quad \partial_{x_2} u = 0 \quad \text{for } x_2 = 0\,, \tag{1.1}$$

where $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 : x_2 > 0\}$. We assume that the (real valued) index of refraction $n \in L^\infty(\mathbb{R}^2_+)$ is $2\pi-$periodic with respect to x_1 and equals to 1 for $x_2 > h_0$ for some $h_0 > 0$ and $n(x) \geq n_0$ in $\mathbb{R}^2_+$ for some $n_0 > 0$. Furthermore, $k > 0$ denotes the (real) wave number which is fixed throughout the paper and $f \in L^2(\mathbb{R}^2_+)$ with compact support in W^{h_0} where $W^h := \mathbb{R} \times (0, h) \subset \mathbb{R}^2_+$ denotes the layer of height $h > 0$.

The boundary value problem (1.1) has to be complemented by a radiating condition. In contrast to the case of the scattering by bounded objects the present situation is more complicated due to the presense of guided waves (or propagating modes). The investigation of radiation problems for periodic structures (which include layered mediua as a special case) has a long history, and it is impossible to list all of the relevant literature. Instead, we refer to [BL22, Pet80] for a comprehensive introduction of electromagnetic scattering theory for diffraction gratings. The radiation condition plays a central role in our investigation. If guided modes are excluded, prominent examples are the upwards propagation radiation condition (see, e.g. [CWZ98]) or the angular spectrum representation condition, see e.g. [Hoh20, CM05, CE10, HLR21]. For layered or periodic media guided modes exist in general, see Example 1.1 below, and the radiation condition has to be modified. We believe that a proper radiation condition should be justified by a limiting absorption principle as we have done it for the case of (1.1) in [KL18, Kir19] or [Kir25]. In [Kir25] we included the important case that the cut-off values are critical (see Definition 2.1 below).

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However, if a radiation condition has been formulated, the question of existence and uniqueness should be answered without the help of the limiting absorption principle. This is the aim of the present work.

The formulation of the radiation condition requires a detailed description of the guided modes of this equation. Before we study modes we begin with a couple of examples which show one of the aims of this paper.

Example 1.1. *(a) We consider the simplest case of a homogeneous half plane* $\mathbb{R}^2_+$ *with Dirichet- or Neumann boundary conditions at* $x_2 = 0$*. Let* $f \in L^2(\mathbb{R}^2_+)$ *have compact support. Then the unique solutions* u_D *and* u_N *of the problems* $\Delta u + k^2 u = -f$ *in* $\mathbb{R}^2_+$ *for* $u = u_D$ *and* $u = u_N$*, and* $u_D = 0$ *and* $\partial_{x_2} u_N = 0$ *for* $x_2 = 0$*, respectively, satisfying the Sommerfeld radiation condition, is given by*

$$u_D(x) = \frac{i}{4} \int\limits_{\mathbb{R}^2_+} \left[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|) \right] f(y)\, dy\,, \quad x \in \mathbb{R}^2_+\,,$$

$$u_N(x) = \frac{i}{4} \int\limits_{\mathbb{R}^2_+} \left[H_0^{(1)}(k|x-y|) + H_0^{(1)}(k|x-y^*|) \right] f(y)\, dy\,, \quad x \in \mathbb{R}^2_+\,,$$

respectively, where $y^* = (y_1, -y_2)^\top$ *and* $H_0^{(1)}$ *denotes the Hankel function of order zero and type one. From the asymptotics*

$$\frac{i}{4} H_0^{(1)}(k|x-y|) = \frac{e^{i\pi/4}}{2\sqrt{2\pi k}} \frac{e^{ik|x|}}{\sqrt{|x|}} e^{-ik\frac{x}{|x|}\cdot y} + \mathcal{O}(|x|^{-3/2}) \text{ as } |x| \to \infty$$

(see, e.g. [Col04, Leb65]*) we obtain for fixed* $x_2 > 0$ *that*

$$u_D(x_1, x_2) = \mathcal{O}(|x_1|^{-3/2}) \quad \text{and} \quad u_N(x_1, x_2) = \mathcal{O}(|x_1|^{-1/2}) \quad \text{as } |x_1| \to \infty\,.$$

One might guess that the reason for the different rate of decay along the axis of the waveguide lies in the different kind of boundary condition. However, this is not the true reason as the following case of a one-layer medium shows.

(b) Let $n(x) = 1$ *for* $x_2 > 1$ *and* $n(x) = n_0$ *for* $0 < x_2 < 1$ *and any constant* n_0 *with* $0 < n_0 < 1$*, and let* f *have compact support in* $\mathbb{R} \times (0,1)$*. Then the unique radiating solution* u_N *of* $\Delta u_N + k^2 n u_N = -f$ *in* $\mathbb{R}^2_+$*,* $\partial_{x_2} u_N = 0$ *for* $x_2 = 0$*, decays as* $1/|x_1|^{3/2}$ *as* $|x_1| \to \infty$*.*

(c) If, on the other hand, $n(x) = 1$ *for* $x_2 > 1$ *and* $n(x) = 1 + \frac{\pi^2}{4k^2}$ *for* $0 < x_2 < 1$*, then the solution* u_D *of* $\Delta u_D + k^2 n u_D = -f$ *in* $\mathbb{R}^2_+$*,* $u_D = 0$ *for* $x_2 = 0$*, decays as* $1/|x_1|^{1/2}$ *as* $|x_1| \to \infty$*.*

One can show these statements of parts (b) and (c) by investigating the smoothness of the Fourier transform of the solution with respect to x_1*. It also follows from Theorem 2.10 below for general periodic refractive indices* $n = n(x)$ *(by using Example 2.3).*

It is one of the aims of this paper to give a precise characterization of the cases when the solution decays as $1/|x_1|^{1/2}$ or $1/|x_1|^{3/2}$ as $|x_1| \to \infty$.

Besides the layer W^h we define the sets $Q^h = (0, 2\pi) \times (0, h)$ and $Q^\infty = (0, 2\pi) \times (0, \infty)$ and $\Gamma_h = (0, 2\pi) \times \{h\}$ and set $Q = Q^{h_0}$ to simplify the notation.

2. Modes, Radiation Condition, and Formulation of the Main Result

We recall that a function $f : \mathbb{R} \to \mathbb{C}$ is called $\alpha-$quasi-periodic for some $\alpha \in \mathbb{R}$ if $f(t+2\pi) = e^{i\alpha 2\pi} f(t)$ for all $t \in \mathbb{R}$. We define the spaces

$$H^1_\alpha(Q^h) \;:=\; \big\{u|_{Q^h} : u \in H^1(\mathbb{R}^2),\ u(\cdot,x_2) \text{ is } \alpha-\text{quasi-periodic for all } x_2\big\}\,,$$
$$H^1_{\alpha,*}(Q^\infty) \;:=\; \big\{u|_{Q^\infty} : u \in H^1_{loc}(\mathbb{R}^2),\ u|_{Q^h} \in H^1_\alpha(Q^h) \text{ for all } h>0\big\}\,.$$

In addition, we include the boundary condition $u = 0$ for $x_2 = 0$ into the spaces in the case of a Dirichlet boundary condition without indicating this in the notation.
For $\alpha = 0$, i.e. for the periodic case, we denote the spaces by $H^1_{per}(Q^h)$ and $H^1_{per,*}(Q^\infty)$, respectively.

Definition 2.1. *(a) $\alpha \in \mathbb{R}$ is called **cut-off value** if there exists $\ell \in \mathbb{Z}$ with $|\alpha+\ell| = k$.*
*(b) $\alpha \in \mathbb{R}$ is called **critical value** if there exists a non-trivial solution $\phi \in H^1_{\alpha,*}(Q^\infty)$ of $\Delta\phi + k^2 n\phi = 0$ in Q^∞ satisfying the boundary condition at $x_2 = 0$ and the Rayleigh expansion*

$$\phi(x) \;=\; \sum_{\ell\in\mathbb{Z}} \phi_\ell\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)+i(\ell+\alpha)x_1}\,, \quad x_2 > h_0\,, \tag{2.1}$$

for some $\phi_\ell \in \mathbb{C}$ where the convergence is uniform for $x_2 \ge h$ for all $h > h_0$.
The set of critical values $\alpha \in \mathbb{R}$ is denoted by $\mathcal{A}$. For $\alpha \in \mathcal{A}$ we define the space $\mathcal{M}(\alpha)$ of modes by

$$\mathcal{M}(\alpha) \;:=\; \big\{\phi \in H^1_{\alpha,*}(Q^\infty) : \phi \text{ satisfies } \Delta\phi + k^2 n\phi = 0 \text{ in } Q^\infty, \text{ the b.c., and (2.1)}\big\}\,.$$

We note that α is critical or a cut-off value if, and only if, $\alpha + \ell$ is critical or a cut-off value, respectively, for every $\ell \in \mathbb{Z}$. From the definition we observe directly that the set of all cut-off values is given by $\{+k,-k\} + \mathbb{Z} = \{\sigma k + \ell : \sigma \in \{+1,-1\},\ \ell \in \mathbb{Z}\}$. It is obvious that α is a critical value with mode ϕ if, and only if, $-\alpha$ is a critical value with mode $\bar\phi$. Finally, we note that $\mathcal{M}(\alpha) = \mathcal{M}(\alpha+\ell)$ for all $\ell \in \mathbb{Z}$.

The following hermetian sesqui-linear form plays an important role.

$$E(u,v) \;:=\; i\int\limits_{Q^\infty} \big[u\,\partial_{x_1}\bar v - \bar v\,\partial_{x_1} u\big]\,dx \quad \text{for } u,v \in H^1(Q^\infty)\,. \tag{2.2}$$

We note that $E(u,v)$ is also well-defined if only $u\,\partial_{x_1}\bar v - \bar v\,\partial_{x_1} u \in L^1(Q^\infty)$. In [Kir25], the following properties have been shown.

Lemma 2.2. *Let $\alpha \in \mathcal{A}$ and $\phi \in \mathcal{M}(\alpha)$.*

(a) *Then the coefficients ϕ_ℓ in the Rayleigh expansion* (2.1) *vanish for all $\ell \in \mathbb{Z}$ with $|\ell+\alpha| < k$.*
(b) *Let $\phi_\ell = 0$ for all $\ell \in \mathbb{Z}$ with $|\ell+\alpha| \le k$. Then ϕ is evanescent, i.e. for every $h > h_0$ there exist $c,\eta > 0$ with $|\phi(x)| \le c\,e^{-\eta x_2}$ for $x_2 \ge h$.*
(c) *The mode spaces $\mathcal{M}(\alpha)$ are finite dimensional.*
(d) *Let $k \notin \frac{1}{2}\mathbb{N}$ and $\alpha \in \mathcal{A}$. Define the subspace $\mathcal{M}_{evan}(\alpha)$ of $\mathcal{M}(\alpha)$ for $\alpha \in \mathcal{A}$ by*

$$\mathcal{M}_{evan}(\alpha) \;:=\; \big\{\phi \in \mathcal{M}(\alpha) : \phi \text{ is evanescent}\big\}$$
$$= \big\{\phi \in \mathcal{M}(\alpha) : \phi_\ell = 0 \text{ for } |\ell+\alpha| \le k\big\}\,.$$

Then the codimension of $\mathcal{M}_{evan}(\alpha)$ in $\mathcal{M}(\alpha)$ is zero or one.

(e) *Let $u \in \mathcal{M}(\alpha)$ and $v \in \mathcal{M}(\beta)$ for $\alpha, \beta \in \mathcal{A}$ and at least one of them is evanescent. Then $E(u,v)$ exists and*

$$E(u,v) \;=\; 2\pi i \int\limits_{C_b} \big[u\, \partial_{x_1}\bar{v} - \bar{v}\, \partial_{x_1} u\big]\, ds \quad \textit{for any } b \in \mathbb{R} \tag{2.3}$$

where $C_b = \{b\} \times (0,\infty)$. Furthermore, $E(u,v) = 0$ if $\alpha - \beta \notin \mathbb{Z}$.

With respect to Example 1.1 we consider two simple examples for which the cut-off value k is critical.

Example 2.3. *(a) The function $\phi(x) = e^{ikx_1}$ is a mode ($k-$quasi-periodic) for the homogeneous half plane $\mathbb{R}^2_+$ with Neumann boundary condition, i.e. $\{0\} = \mathcal{M}_{evan}(k) \subsetneqq \mathcal{M}(k)$. This corresponds to part (a) of Example 1.1.*

(b) Let n be defined as $n(x) = 1$ for $x_2 > 1$ and $n(x) = 1 + \frac{\pi^2}{4k^2}$ for $0 < x_2 < 1$. Then

$$\phi(x_1,x_2) \;=\; e^{ikx_1} \cdot \begin{cases} 1\,, & x_2 > 1\,, \\ \sin\big(x_2\pi/2\big)\,, & 0 < x_2 < 1\,, \end{cases}$$

is in $C^1(\mathbb{R}^2_+)$ and satisfies (1.1) in $\mathbb{R}^2_+ \setminus \{x_2 = 1\}$ and $\phi = 0$ for $x_2 = 0$. Therefore, ϕ is a guided mode in $\mathcal{M}(k)$ which is not evanescent. It is easily seen that there exist $k > 0$ with $\mathcal{M}_{evan}(\pm k) = \{0\}$. This example corresponds to part (c) of Example 1.1.

A further example for the case $\{0\} \subsetneqq \mathcal{M}_{evan}(k) \subsetneqq \mathcal{M}(k)$ is given in [Kir25], Example 3.5.

We make the following assumption.

Assumption 2.4. *Let $k \notin \frac{1}{2}\mathbb{N}$ and let k (and thus also $-k$) be critical values, i.e. $\pm k \in \mathcal{A}$.*

The assumption that the cut-off values are critical decribes the new situation in this paper. If the set $\{+k, -k\} + \mathbb{Z}$ of cut-off values is disjoint from the set $\mathcal{A}$ of critical values we refer to [Kir22] for a complete discussion. If the assumption $k \notin \frac{1}{2}\mathbb{N}$ is violated, i.e. $k \in \frac{1}{2}\mathbb{N}$ then the space $\mathcal{M}_{evan}(k)$ can have co-dimension 2. This situation requires a different discussion. From Lemma 2.2 we conclude that every $\phi \in \mathcal{M}(\alpha)$ is evanescent if $\alpha \in \mathcal{A}$ is not a cut-off value. However, if $\alpha \in \mathcal{A}$ is a cut-off value, i.e. $|\hat{\ell} + \alpha| = k$ for some $\hat{\ell} \in \mathbb{Z}$, and $\phi \in \mathcal{M}(\alpha)$ then ϕ is evanescent if, and only if, $\phi_{\hat{\ell}} = 0$.

The following assumption is standard.

Assumption 2.5. *For every $\alpha \in \mathcal{A}$ and every $v \in \mathcal{M}_{evan}(\alpha)$ with $v \neq 0$ let the linear form $E(\cdot, v)$ be not trivial on $\mathcal{M}_{evan}(\alpha)$.*

Under this assumption it has been shown in [Kir25] (Theorem 4.1) that only finitely many critical values in the interval $(-1/2, 1/2]$ exist, i.e. $\mathcal{A} \cap (-1/2, 1/2]$ is finite. We number them by $\hat{\alpha}_j$, $j \in J$, where $J \subset \mathbb{Z}$ is a finite set, i.e. $\mathcal{A} \cap (-1/2, 1/2] = \{\hat{\alpha}_j : j \in J\}$.
By Assumption 2.4 the cut-off values are critical. If we decompose k into $k = \tilde{\ell} + \kappa$ with $\tilde{\ell} \in \mathbb{Z}_{\geq 0}$ and $\kappa \in (-1/2, 1/2]$ then also $\pm\kappa$ are critical and thus included in $\mathcal{A} \cap (-1/2, 1/2]$. Therefore, $|J| \geq 2$, and we denote these particular critical values by $\hat{\alpha}_1 = \kappa$ and $\hat{\alpha}_{-1} = -\kappa$.

With the form E from (2.2) we construct a basis of $\mathcal{M}(\hat{\alpha}_j)$ for every $j \in J$.

Let $j \in J$ and let $\mathcal{M}_{evan}(\hat{\alpha}_j) \neq \{0\}$. Let $\langle \cdot, \cdot \rangle_j$ be any inner product in the finite dimensional space $\mathcal{M}_{evan}(\hat{\alpha}_j)$. With the hermetian sesqui-linear form E from (2.2) we consider the self-adjoint eigenvalue problem to determine $\lambda_{\ell,j} \in \mathbb{R}$ and non-trivial $\phi^{\ell,j} \in \mathcal{M}_{evan}(\hat{\alpha}_j)$, $\ell = 1, \ldots, m_j := \dim \mathcal{M}_{evan}(\hat{\alpha}_j)$, with

$$E(\phi^{\ell,j}, \psi) \;=\; \lambda_{\ell,j} \, \langle \phi^{\ell,j} \, \psi \rangle_j \quad \text{for all } \psi \in \mathcal{M}_{evan}(\hat{\alpha}_j)\,. \tag{2.4}$$

The eigenfunctions $\phi^{\ell,j}$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_j$. We normalize the eigenfunctions such that $\langle \phi^{\ell,j}, \phi^{\ell',j} \rangle_j = \delta_{\ell,\ell'}$. Then $\{\phi^{\ell,j} : \ell = 1, \ldots, m_j\}$ is an orthonormal basis of $\mathcal{M}_{evan}(\hat{\alpha}_j)$ for every $j \in J$. Furthermore, by part (e) of Lemma 2.2 and Assumption 2.5,

$$E(\phi^{\ell,j}, \phi^{\ell',j'}) \;=\; \delta_{\ell,\ell'} \, \delta_{j,j'} \, \lambda_{\ell,j} \neq 0 \quad \text{for all } \ell = 1, \ldots, m_j \text{ and } \ell' = 1, \ldots, m_{j'}\,.$$

Note that the eigenfunctions depend on the chosen inner product. We have shown in [Kir25] that the Limiting Absorption Principle leads to inner products of the form $\langle u, v \rangle_j = k^2 \int_Q q \, u \, \bar{v} \, dx$ for positive functions $q \in L^\infty(W^{h_0})$ which are periodic with respect to x_1.

In the case $\mathcal{M}(\pm\kappa) \supsetneqq \mathcal{M}_{evan}(\pm\kappa)$ we extend this basis to a basis of $\mathcal{M}(\hat{\alpha}_{\pm 1}) = \mathcal{M}(\pm\kappa)$.

Lemma 2.6. *Let Assumptions 2.4 and 2.5 hold and let $\mathcal{M}(\hat{\alpha}_{\pm 1}) = \mathcal{M}(\pm\kappa) \supsetneqq \mathcal{M}_{evan}(\pm\kappa)$. Then there exist unique $\hat{\phi}^\pm \in \mathcal{M}(\pm k) = \mathcal{M}(\pm\kappa)$ with $E(\hat{\phi}^\pm, \psi) = 0$ for all $\psi \in \mathcal{M}_{evan}(\pm\kappa)$ and $\hat{\phi}^\pm$ has the form*

$$\hat{\phi}^\pm(x) \;=\; e^{\pm i k x_1} \;+\; \hat{\phi}^\pm_{evan}(x)\,, \quad x_2 > h_0\,,$$

where $\hat{\phi}^\pm_{evan}$ is evanescent. Therefore, if $\big\{\phi^{\ell,\pm 1} : \ell = 1, \ldots, m_1\big\}$ is a basis of $\mathcal{M}_{evan}(\pm k) = \mathcal{M}_{evan}(\hat{\alpha}_{\pm 1})$ determined by the eigenvalue problem (2.4) then $\big\{\phi^{\ell,\pm 1} : \ell = 1, \ldots, m_1\big\} \cup \{\hat{\phi}^\pm\}$ is a basis of $\mathcal{M}(\pm k)$.

For the proof we refer again to [Kir25].

The term $E(\hat{\phi}^\pm, \hat{\phi}^\pm)$ is not defined because $\hat{\phi}^\pm$ does not decay as $x_2 \to \infty$. However, we have the following analog of (2.3):

Lemma 2.7. *For $\sigma \in \{+, -\}$ set*

$$\begin{aligned}
\psi^\sigma(x) \;&:=\; \partial_{x_1} \hat{\phi}^\sigma(x) \, \overline{\hat{\phi}^\sigma(x)} - \partial_{x_1} \overline{\hat{\phi}^\sigma(x)} \, \hat{\phi}^\sigma(x) \;-\; 2\sigma i k \\
&=\; 2i \, \mathrm{Im}\big[\partial_{x_1} \hat{\phi}^\sigma(x) \, \overline{\hat{\phi}^\sigma(x)}\big] \;-\; 2\sigma i k\,, \\
\psi(x) \;&:=\; \partial_{x_1} \hat{\phi}^+(x) \, \overline{\hat{\phi}^-(x)} - \partial_{x_1} \overline{\hat{\phi}^-(x)} \, \hat{\phi}^+(x) \quad \text{for } x \in \mathbb{R}^2_+\,.
\end{aligned}$$

Then $\int_0^H \psi^\sigma(x_1, x_2) \, dx_2$ is uniformly bounded with respect to H and x_1 and converges to $\int_0^\infty \psi^\sigma(x_1, x_2) \, dx_2$ as $H \to \infty$, and this limit is constant with respect to x_1.
The same holds for ψ replacing ψ^σ.

Proof. The decomposition (for $x_2 > h_0$) as $\hat{\phi}^\sigma(x_1, x_2) = e^{\sigma i k x_1} + \hat{\phi}^\sigma_{evan}(x_1, x_2)$ and $\partial_{x_1} e^{\sigma i k x_1} \overline{e^{\sigma i k x_1}} = \sigma i k$ yields

$$\psi^\sigma(x) \;=\; 2i \, \mathrm{Im}\big[\partial_{x_1} \hat{\phi}^\sigma_{evan}(x) \, \overline{\hat{\phi}^\sigma_{evan}(x)} \;+\; \partial_{x_1} \hat{\phi}^\sigma_{evan}(x) \, \overline{e^{\sigma i k x_1}} \;+\; \hat{\phi}^\sigma_{evan}(x) \, \partial_{x_1} \overline{e^{\sigma i k x_1}}\big]\,.$$

Uniform (with respect to H and x_1) boundedness and existence of the limit follow from the quasi-periodicity and exponential decay of $\hat{\phi}^\sigma_{evan}(x)$ as $x_2 \to \infty$.

We fix $a < b$. For $H > h_0$ we use Green's theorem in the rectangle $(a,b)\times(0,H)$ to obtain

$$\begin{aligned}
&\int_0^H \psi^\sigma(b,x_2)\,dx_2 - \int_0^H \psi^\sigma(a,x_2)\,dx_2 \\
&= -\int_a^b \big[\partial_{x_2}\hat\phi^\sigma(x_1,H)\,\overline{\hat\phi^\sigma(x_1,H)} - \partial_{x_2}\overline{\hat\phi^\sigma(x_1,H)}\,\hat\phi^\sigma(x_1,H)\big]\,dx_1 \\
&= -\int_a^b \big[\partial_{x_2}\hat\phi^\sigma_{evan}(x_1,H)\,\overline{\hat\phi^\sigma(x_1,H)} - \partial_{x_2}\overline{\hat\phi^\sigma_{evan}(x_1,H)}\,\hat\phi^\sigma(x_1,H)\big]\,dx_1\,,
\end{aligned}$$

and this converges to zero as H tends to infinity.
For ψ we use the same arguments (note that $\partial_{x_1}e^{ikx_1}\,\overline{e^{-ikx_1}} - \partial_{x_1}\overline{e^{-ikx_1}}\,e^{ikx_1} = 0$). □

Analogously to $H^1_{\alpha,*}(Q^\infty)$ we define

$$H^1_*(\mathbb{R}^2_+) \;=\; \big\{u|_{\mathbb{R}^2_+} : u \in H^1_{loc}(\mathbb{R}^2),\ u|_{W^h} \in H^1(W^h) \text{ for all } h>0\big\}\,.$$

Now we are able to formulate the radiation condition.

Definition 2.8. *Let Assumptions 2.4 and 2.5 hold. Fix $R_0 > 1$, and let $\xi^\pm \in C^\infty(\mathbb{R})$ with $\xi^\pm(x_1) = 1$ for $\pm x_1 \ge R_0$ and $\xi^\pm(x_1) = 0$ for $\pm x_1 \le R_0 - 1$. A solution $u \in H^1_{loc}(\mathbb{R}^2_+)$ of (1.1); that is, of*

$$\Delta u + k^2 n\,u \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad u=0 \text{ or } \partial_{x_2}u = 0 \text{ for } x_2 = 0\,, \tag{2.5}$$

satisfies the ***open waveguide radiation condition*** *if u has a decomposition in the form $u = u^{prop} + u^{rad}$ where*

(a) *u^{prop} and u^{rad} have the forms*

$$u^{prop}(x) = \sum_{\sigma\in\{+,-\}} \xi^\sigma(x_1) \sum_{j\in J}\sum_{\ell:\sigma\lambda_{\ell,j}>0} a_{\ell,j}\,\phi^{\ell,j}(x)\,, \tag{2.6}$$

$$u^{rad}(x) = \; u^{rad}_*(x) \;+ \frac{1}{\sqrt{|x_1|}}\sum_{\sigma\in\{+,-\}} b^\sigma \xi^\sigma(x_1)\,\hat\phi^\sigma(x)\,, \quad x\in\mathbb{R}^2_+\,, \tag{2.7}$$

respectively, for some $a_{\ell,j}, b^\pm \in \mathbb{C}$ and $u^{rad}_* \in H^1_*(\mathbb{R}^2_+)$. If $\mathcal{M}_{evan}(\pm k) = \mathcal{M}(\pm k)$ then $b^\pm = 0$.

(b) *The Fourier transform $\mathcal{F}u^{rad} = (\mathcal{F}u^{rad})(\omega,x_2)$ of u^{rad} with respect to x_1 satisfies*

$$\lim_{x_2\to\infty}\big[\partial_{x_2}(\mathcal{F}u^{rad})(\omega,x_2) - i\sqrt{k^2-\omega^2}\,(\mathcal{F}u^{rad})(\omega,x_2)\big] \;=\; 0 \tag{2.8}$$

for almost all $\omega\in\mathbb{R}$.

Remark 2.9. *If $b^\pm \ne 0$ then $x_1 \mapsto u^{rad}(x)$ decays as $\mathcal{O}(1/\sqrt{|x_1|})$, and its Fourier transform is only defined in the distributional sense. However, this distribution is regular and in $L^q(\mathbb{R})$ for every $q\in(1,2)$ by Lemma B.2 of the Appendix. Therefore, (2.8) is well-defined.*

We can now formulate our main result.

Theorem 2.10. *Let Assumptions 2.4 and 2.5 hold and $f \in L^2(\mathbb{R}^2_+)$ have compact support in $W^{h_0} = \mathbb{R} \times (0, h_0)$.*
(a) Then there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^2_+)$ of (2.5) satisfying the radiation condition of Definition 2.8.
(b) Furthermore, the part $u^{rad}_* \in H^1_*(\mathbb{R}^2_+)$ decays as $\mathcal{O}(|x_1|^{-3/2})$ along the axis of the waveguide in the sense that for every $h > h_0$ there exists $c_h > 0$ with

$$\|u^{rad}_*\|_{H^1(Q^h_m)} \ \leq \ \frac{c_h}{|m|^{3/2}+1} \quad \text{for all } m \in \mathbb{Z} \tag{2.9}$$

where $Q^h_m := (m, m+2\pi) \times (0, h)$.
(c) Finally, for every $a > 0$ there exists $c = c(a) > 0$ such that $|u(x)| \leq c/\sqrt{x_2}$ for all $|x_1| \leq a$ and $x_2 \geq h_0$.

We note that this theorem extends Theorems 5.2 and 6.1 (for the unperturbed case $q = 0$) to the case where the cut-off values are critical. If the corresponding mode spaces $\mathcal{M}(\pm k)$ satisfy $\mathcal{M}_{evan}(\pm k) \subsetneqq \mathcal{M}(\pm k)$ then the radiating part u^{rad} decays as $\mathcal{O}(1/\sqrt{|x_1|})$ rather than $\mathcal{O}(1/|x_1|^{3/2})$ as $x_1 \to \pm\infty$.

The remaining part of this paper is devoted to the proof of this theorem.

3. The Floquet-Bloch Transformed Problems

3.1. **The Floquet-Bloch Transform.** The Floquet-Bloch transform $F\phi$ of a function $\phi \in C^\infty_0(\mathbb{R})$ is defined by

$$(F\phi)(t,\alpha) \ := \ \sum_{\ell\in\mathbb{Z}} \phi(t+2\pi\ell)\, e^{-2\pi\ell\alpha i}\,, \quad t, \alpha \in \mathbb{R}\,.$$

This formula directly shows that for smooth functions ϕ and fixed α the transformed function $t \mapsto (F\phi)(t,\alpha)$ is $\alpha-$quasi-periodic while for fixed t the function $\alpha \mapsto (F\phi)(t,\alpha)$ is periodic with period 1. It is hence sufficient to consider $(t,\alpha) \in (0,2\pi)\times(-1/2,1/2)$. Then F has an extension to a bounded isomorphism from $L^2(\mathbb{R})$ onto $L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ and

$$\|F\phi\|^2_{L^2((0,2\pi)\times(-1/2,1/2))} \ = \int_{-1/2}^{1/2}\int_0^{2\pi} |(F\phi)(t,\alpha)|^2\, dt\, d\alpha \ = \int_{-\infty}^{\infty} |\phi(t)|^2\, dt \ = \ \|\phi\|^2_{L^2(\mathbb{R})}\,.$$

The inverse transform is given by

$$(F^{-1}g)(t+2\pi\ell) \ = \int_{-1/2}^{1/2} g(t,\alpha)\, e^{2\pi\ell\alpha i} d\alpha\,, \quad t \in (0,2\pi)\,,\ \ell \in \mathbb{Z}\,.$$

If $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$ is $\alpha-$quasi-peridic for some $\alpha \in \mathbb{R}$ then

$$\begin{aligned} F(\varphi f)(x,\beta) \ &= \ \sum_{\ell\in\mathbb{Z}} \varphi(x+2\pi\ell)\, f(x+2\pi\ell)\, e^{-2\pi\ell\beta i} \\ &= \ \sum_{\ell\in\mathbb{Z}} \varphi(x+2\pi\ell)\, f(x)\, e^{-2\pi\ell(\beta-\alpha)i} \ = \ (F\varphi)(x,\beta-\alpha)\, f(x) \end{aligned} \tag{3.1}$$

and

$$\int_0^{2\pi} (F\varphi)(x,\alpha)\,\overline{f(x)}\,dx \;=\; \sum_{\ell\in\mathbb{Z}} \int_0^{2\pi} \varphi(x+2\pi\ell)\,\overline{f(x)e^{2\pi\ell\alpha i}}dx \;=\; \int_{-\infty}^{\infty} \varphi(x)\,\overline{f(x)}\,dx \tag{3.2}$$

because $f(x)e^{2\pi\ell\alpha i} = f(x+2\pi\ell)$. From this formula we note that the Fourier coefficients $a_\ell(\alpha) = \frac{1}{2\pi}\int_0^{2\pi}(F\varphi)(x,\alpha)\,e^{-i(\ell+\alpha)x}dx$ of the $\alpha-$quasi-periodic function $(F\varphi)(\cdot,\alpha)$ are given by

$$a_\ell(\alpha) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x)\,e^{-i(\ell+\alpha)x}dx \;=\; (\mathcal{F}\varphi)(\ell+\alpha) \tag{3.3}$$

where

$$(\mathcal{F}\varphi)(\omega) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x)\,e^{-i\omega x}dx\,, \quad \omega\in\mathbb{R}\,,$$

denotes the Fourier transform of φ.

In view of our scattering problem, we apply the Floquet-Bloch transform to the variable x_1 and consider x_2 as a parameter. Recalling that $W^h := \mathbb{R}\times(0,h)$ and $Q^h := (0,2\pi)\times(0,h)$ and $I = (-1/2,1/2)$ it has been shown (see, e.g. [LZ17, Section 6]) that F is an isomorphism from $H^1(W^h)$ onto

$$L^2\big((-1/2,1/2),H^1_\alpha(Q^h)\big) \;:=\; \left\{u\in L^2\big((-1/2,1/2),H^1(Q^h)\big) : \begin{array}{l} x_1\mapsto u(x,\alpha) \text{ is} \\ \alpha-\text{quasi-periodic}\end{array}\right\},$$

equipped with the norm of $L^2\big((-1/2,1/2),H^1(Q^h)\big)$.

3.2. **Formulation of the Quasi-Periodic Problems and Reduction.** We set $\varphi^\pm(x_1) := \xi^\pm(x_1)/\sqrt{|x_1|}$ to simplify the notation and note that the part $v := u_*^{rad}\in H^1_*(\mathbb{R}^2_+)$ satisfies

$$(\Delta+k^2n)v = -g \quad \text{in } \mathbb{R}^2_+\,, \quad v=0 \text{ or } \partial_{x_2}v=0 \text{ for } x_2=0\,,$$

where g is given by

$$\begin{aligned} g(x) \;:=&\; f(x) \;+\; \sum_{\sigma\in\{+,-\}} b^\sigma\,(\Delta+k^2n)\big[\varphi^\sigma(x_1)\,\hat\phi^\sigma(x)\big] \\ &+\; \sum_{\sigma\in\{+,-\}}\sum_{j\in J}\sum_{\sigma\lambda_{\ell,j}>0} a_{\ell,j}(\Delta+k^2n)\big[\xi^\sigma(x_1)\phi^{\ell,j}(x)\big] \\ =&\; f(x) \;+\; \sum_{\sigma\in\{+,-\}} b^\sigma\,\big[(\varphi^\sigma)'(x_1)\,\partial_{x_1}\hat\phi^\sigma(x) + \partial_{x_1}\big((\varphi^\sigma)'(x_1)\hat\phi^\sigma(x)\big)\big] \\ &+\; \sum_{\sigma\in\{+,-\}}\sum_{j\in J}\sum_{\sigma\lambda_{\ell,j}>0} a_{\ell,j}\big[(\xi^\sigma)'(x_1)\,\partial_{x_1}\phi^{\ell,j}(x) + \partial_{x_1}\big((\xi^\sigma)'(x_1)\phi^{\ell,j}(x)\big)\big] \end{aligned} \tag{3.4}$$

for $x\in Q^\infty$. The radiation condition (2.8) for u^{rad} turns into the following radiation condition for v.

$$\lim_{x_2\to\infty}\big[\partial_{x_2}(\mathcal{F}v)(\omega,x_2) - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v(\omega,x_2)\big] = i\sqrt{k^2-\omega^2}\sum_{\sigma\in\{+,-\}} b^\sigma\,\mathcal{F}\big(\varphi^\sigma\,e^{\sigma ik\cdot}\big)(\omega) \tag{3.5}$$

for almost all $\omega \in \mathbb{R}$ because $\sum_{\sigma\in\{+,-\}} b^\sigma\, \mathcal{F}\big(\varphi^\sigma\, \hat{\phi}^\sigma_{evan}(\cdot,x_2)\big)(\omega)$ and its derivative with respect to x_2 converge to zero as $x_2 \to \infty$. Here we note that $\mathcal{F}\big(\varphi^\sigma\, e^{\sigma ik\cdot}\big) \in L^q(\mathbb{R})$ for all $q \in (1,2)$ by Lemma B.2.

We now consider the quasi-periodic problems. For every fixed α we determine $\hat{v}(\cdot,\alpha) \in H^1_{\alpha,*}(Q^\infty)$ such that

$$(\Delta + k^2 n)\hat{v}(\cdot,\alpha) = -(Fg)(\cdot,\alpha) \quad \text{in } Q^\infty\,, \quad \hat{v} = 0 \text{ or } \partial_{x_2}\hat{v} = 0 \text{ for } x_2 = 0\,, \tag{3.6}$$

subject to the radiation condition, that

$$\lim_{x_2\to\infty} \Big[\partial_{x_2}\hat{v}_\ell(x_2,\alpha) - i\sqrt{k^2-(\ell+\alpha)^2}\,\hat{v}_\ell(x_2,\alpha)\Big] \quad \text{exist for all } \ell \in \mathbb{Z}\,. \tag{3.7}$$

where $\hat{v}_\ell(x_2,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \hat{v}_\ell(x_1,x_2,\alpha)e^{-i(\ell+\alpha)x_1}dx_1$ are the Fourier coefficients of $\hat{v}(\cdot,x_2,\alpha)$.

In order to reduce the problem to the bounded region $Q = Q^{h_0}$ we introduce a special solution $w = w(\cdot,\alpha) \in H^1_{\alpha,*}(Q^\infty \setminus Q)$ of

$$\Delta w(x,\alpha) + k^2 w(x,\alpha) = -(Fg)(x,\alpha) \text{ for } x_2 > h_0\,, \quad w(x,\alpha) = 0 \text{ for } x_2 = h_0\,. \tag{3.8}$$

Theorem 3.1. *For every $\alpha \in [-1/2,1/2]$ there exists a solution $w(\cdot,\alpha) \in H^1_{\alpha,*}(Q^\infty \setminus Q)$ of (3.8) such that $w_\ell(x_2,\alpha)$ satisfies the radiation condition (3.7).*
If $\ell+\alpha \neq \pm k$ the solution is unique and

$$\begin{aligned}&\lim_{x_2\to\infty} \Big[\partial_{x_2} w_\ell(x_2,\alpha) - i\sqrt{k^2-(\ell+\alpha)^2}\, w_\ell(x_2,\alpha)\Big] \\ &= \frac{i}{\sqrt{k^2-(\ell+\alpha)^2}} \sum_{\sigma\in\{+,-\}} b^\sigma \left[2\sigma ik\mathcal{F}(\varphi^{\sigma\prime} e^{\sigma ikx_1})(\ell+\alpha) + \mathcal{F}(\varphi^{\sigma\prime\prime} e^{\sigma ikx_1})(\ell+\alpha)\right].\end{aligned} \tag{3.9}$$

If $\ell+\alpha = \pm k$, i.e. $\ell = \pm\tilde{\ell}$ and $\alpha = \pm\kappa$, the solution is unique if one requires that (3.7) holds in the form

$$\lim_{x_2\to\infty} \partial_{x_2} w_{\pm\tilde{\ell}}(x_2,\pm\kappa) \;=\; -\sqrt{\frac{k}{2\pi}}\, b^\pm\, e^{-i\pi/4}\,. \tag{3.10}$$

The solution $w(\cdot,\alpha)$ has a decomposition in the form

$$\begin{aligned} w(x,\alpha) \;&= W_0(x,\alpha) \;+\; \sqrt{\kappa-\alpha}\, W_+(x,\alpha) \\ &\quad +\; \sqrt{\kappa+\alpha}\, W_-(x_2,\alpha) \;+\; \sqrt{\kappa-\alpha}\,\sqrt{\kappa+\alpha}\, \tilde{W}(x,\alpha)\end{aligned} \tag{3.11}$$

where $W_0, W_\pm, \tilde{W} \in C^\infty\big(Q^\infty \times [-1/2,1/2]\big)$.
Furthermore, there exists $c > 0$ with $\|w(\cdot,\alpha)\|_{L^2(\Gamma_H)} \le cH$ and $\|\partial_{x_2} w(\cdot,\alpha)\|_{L^2(\Gamma_H)} \le c$ for all $H > h_0$ and $\alpha \in [-1/2,1/2]$.

We will prove this theorem and further properties of the solution in Appendix A.

We recall the $\alpha-$quasi-periodic Dirichlet-to-Neumann operator $\Lambda_\alpha : H^{1/2}_\alpha(\Gamma) \to H^{-1/2}_\alpha(\Gamma)$ (where $\Gamma = \Gamma_{h_0} = (0,2\pi)\times\{h_0\}$), given by

$$(\Lambda_\alpha\phi)(x_1,h_0) \;:=\; i\sum_{\ell\in\mathbb{Z}} \sqrt{k^2-(\ell+\alpha)^2}\,\phi_\ell\, e^{i(\ell+\alpha)x_1}\,, \quad x_1 \in (0,2\pi)\,,$$

where $\phi_\ell = \frac{1}{2\pi}\int_0^{2\pi} \phi(x_1,h_0)\, e^{-i(\ell+\alpha)x_1}\, dx_1$ are the Fourier coefficients of $\phi(\cdot,h_0)$.

With the (unique) solution $w(\cdot,\alpha)$ of (3.8) we have:

Theorem 3.2. *Fix* $\alpha \in [-1/2, 1/2]$.
(a) Let $\hat v \in H^1_{\alpha,*}(Q^\infty)$ *satisfy* (3.6) *and the radiation condition* (3.7). *Then* $\hat v|_Q \in H^1_\alpha(Q)$ *satisfies*

$$\int\limits_Q [\nabla \hat v \cdot \nabla \overline{\psi} - k^2 n\, \hat v\, \overline{\psi}]\, dx - \int\limits_{\Gamma_{h_0}} (\Lambda_\alpha \hat v)\, \overline{\psi}\, ds = \int\limits_Q (Fg)(\cdot,\alpha)\, \overline{\psi}\, dx + \int\limits_{\Gamma_{h_0}} \partial_{x_2} w\, \overline{\psi}\, ds \tag{3.12}$$

for all $\psi \in H^1_\alpha(Q)$ *with* $w = w(\cdot,\alpha)$ *from Theorem 3.1.*
(b) Let $\hat v \in H^1_\alpha(Q)$ *satisfy* (3.12). *Extend* $\hat v$ *by*

$$\hat v(x) \;=\; w(x,\alpha) \;+\; \sum_{\ell\in\mathbb{Z}} \hat v_\ell(h_0,\alpha)\, e^{i(\ell+\alpha)x_1}\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)}\,, \quad x_2 > h_0\,,$$

with Fourier coefficients $\hat v_\ell(h_0,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \hat v(x_1,h_0,\alpha)\, e^{-i(\ell+\alpha)x_1}\, dx_1$. *Then* $\hat v \in H^1_{\alpha,*}(Q^\infty)$ *satisfies* (3.6) *and the radiation condition* (3.7).

Proof. (a) In $Q^\infty \setminus Q$ the difference $d := \hat v - w \in H^1_{\alpha,*}(Q^\infty \setminus Q)$ satifies $\Delta d + k^2 d = 0$, and their Fourier coefficients $d_\ell(x_2)$ satisfy $d_\ell''(x_2) + (k^2 - (\ell+\alpha)^2) d_\ell(x_2) = 0$ for $x_2 > h_0$ and $d_\ell(h_0) = \hat v_\ell(h_0)$ and the radiation condition (3.7). Since the general solution of the differential equation is $d_\ell(x_2) = a_\ell e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)} + b_\ell e^{-i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)}$ the radiation condition yields $b_\ell = 0$, i.e. $\hat v_\ell(x_2) - w_\ell(x_2) = \hat v_\ell(h_0)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)}$ and thus $\partial_{x_2}\hat v(x_1,h_0) = \partial_{x_2} w(x_1,h_0) + (\Lambda_\alpha \hat v)(x_1,h_0)$. Green's theorem, applied to $\hat v$ and $\bar\psi$ in Q, yields the form (3.12).
(b) The extended $\hat v$ satisfies the differential equation, the radiation condition (3.7), and the continuity conditions $\hat v|_+ = \hat v|_-$ on Γ_{h_0} and $\partial_{x_2}\hat v|_+ = \partial_{x_2} w + \Lambda_\alpha \hat v$ on Γ_{h_0}. Green's theorem, applied to $\hat v$ and $\bar\psi$ in $Q^\infty \setminus Q$, for some $\psi \in H^1_\alpha(Q^\infty)$ which vanishes for $x_2 > H$ (for some $H > h_0$) yields the variational form of (3.6). ☐

3.3. **Solvability of the Quasi-Periodic Problems.** After these preparations we want to show solvability of (3.12) for every $\alpha \in [-1/2, 1/2]$. We write (3.12) as an operator equation in the form $L_\alpha \hat v = r_\alpha$ where $L_\alpha : H^1_\alpha(Q) \to H^1_\alpha(Q)$ and $r_\alpha \in H^1_\alpha(Q)$ are given by

$$\begin{aligned}
\langle L_\alpha v, \psi\rangle_{H^1(Q)} &= \int\limits_Q [\nabla v \cdot \nabla \overline{\psi} - k^2 n\, v\, \overline{\psi}]\, dx - \int\limits_{\Gamma_{h_0}} (\Lambda_\alpha v)\, \overline{\psi}\, ds \\
&= \int\limits_Q [\nabla v \cdot \nabla \overline{\psi} - k^2 n\, v\, \overline{\psi}]\, dx - 2\pi i \sum_{\ell\in\mathbb{Z}} \sqrt{k^2-(\ell+\alpha)^2}\, v_\ell(\alpha)\, \overline{\psi_\ell(\alpha)}\,, \\
\langle r_\alpha, \psi\rangle_{H^1(Q)} &= \int\limits_Q (Fg)(\cdot,\alpha)\, \overline{\psi}\, dx + \int\limits_{\Gamma_{h_0}} \partial_{x_2} w(\cdot,\alpha)\, \overline{\psi}\, ds \\
&= \int\limits_Q (Fg)(\cdot,\alpha)\, \overline{\psi}\, dx + 2\pi \sum_{\ell\in\mathbb{Z}} \partial_{x_2} w_\ell(h_0,\alpha)\, \overline{\psi_\ell(\alpha)}
\end{aligned}$$

where $v_\ell(\alpha)$, $\psi_\ell(\alpha)$, and $w_\ell(x_2,\alpha)$ are the Fourier coefficients of $v(\cdot,h_0,\alpha)$, $\psi(\cdot,h_0,\alpha)$, and $w(\cdot,x_2,\alpha)$, respectively.

Since L_α is Fredholm with index zero (see, e.g., [Kir25], Lemma 3.7) a necessary and sufficient condition is that the right hand side r_α is orthogonal to the nullspace of the adjoint of L_α – which coincides with the null space of L_α itself (see again proof of Lemma 3.7 of [Kir25]). Therefore, we have to show that for every $j \in J$ the following holds

$$\int_Q (Fg)(\cdot,\hat\alpha_j)\,\overline{\psi}\,dx + \int_{\Gamma_{h_0}} \partial_{x_2} w(\cdot,\hat\alpha_j)\,\overline{\psi}\,ds \;=\; 0 \quad \text{for all } \psi\in\mathcal{M}(\hat\alpha_j)\,. \tag{3.13}$$

Theorem 3.3. *Let Assumptions 2.4 and 2.5 hold and let $a_{\ell,j}\in\mathbb{C}$ and $b^\pm\in\mathbb{C}$ (in the case that $\mathcal{M}_{evan}(\pm k)\subsetneqq\mathcal{M}(\pm k)$) be given by*

$$a_{\ell,j} \;=\; \frac{2\pi i}{|\lambda_{\ell,j}|}\int_{\mathbb{R}^2_+} f(x)\,\overline{\phi^{\ell,j}(x)}\,dx\,,\quad b^\pm \;=\; \frac{1}{\sqrt{2\pi k}}\,e^{i\pi/4}\int_{\mathbb{R}^2_+} f(x)\,\overline{\hat\phi^{\pm}(x)}\,dx\,. \tag{3.14}$$

Then (3.13) *holds, and for every $\alpha\in[-1/2,1/2]$ there exists a solution $\hat v\in H^1_\alpha(Q)$ of* (3.12) *and also a solution $\hat v\in H^1_{\alpha,*}(Q^\infty)$ of* (3.6) *satisfying the radiation condition* (3.7).

Proof. From the form (3.4) of g and (3.1) we obtain

$$\begin{aligned}
(Fg)(x,\alpha) \;&= (Ff)(x,\alpha) \;+ \sum_{s\in\{+,-\}} b^s\,F\big[2(\varphi^s)'\,\partial_{x_1}\hat\phi^s(\cdot,x_2) + (\varphi^s)''\hat\phi^s(\cdot,x_2)\big](x_1,\alpha)\\
&\quad + \sum_{s\in\{+,-\}}\sum_{j'\in J}\sum_{s\lambda_{\ell',j'}>0} a_{\ell',j'}F\big[2(\xi^s)'\,\partial_{x_1}\phi^{\ell',j'}(\cdot,x_2) + (\xi^s)''\phi^{\ell',j'}(\cdot,x_2)\big](x_1,\alpha)\\
&= (Ff)(x,\alpha) \qquad\qquad (3.15)\\
&\quad + \sum_{s\in\{+,-\}} b^s\,\big[2(F\varphi^{s\prime})(x_1,\alpha-s\kappa)\,\partial_{x_1}\hat\phi^s(x) + F(\varphi^{s\prime\prime})(x_1,\alpha-s\kappa)\hat\phi^s(x)\big]\\
&\quad + \sum_{s\in\{+,-\}}\sum_{j'\in J}\sum_{s\lambda_{\ell',j'}>0} a_{\ell',j'}\big[2(F\xi^{s\prime})(x_1,\alpha-\hat\alpha_{j'})\,\partial_{x_1}\phi^{\ell',j'}(x)\\
&\qquad\quad + F(\xi^{s\prime\prime})(x_1,\alpha-\hat\alpha_{j'})\,\phi^{\ell',j'}(x)\big].
\end{aligned}$$

(Here we denoted the summation indices by $s\in\{+,-\}$, ℓ' and j' instead of σ, ℓ and j, respectively.)

Let $\psi=\phi^{\ell,j}$ or $\psi=\hat\phi^\sigma$ (if $j\in\{1,-1\}$). For $H>h_0$ we use Green's theorem in $(0,2\pi)\times(h_0,H)$ for the $\hat\alpha_j-$quasi-periodic functions $w(\cdot,\hat\alpha_j),\psi\in H^1_{\hat\alpha_j,*}(Q^\infty\setminus Q)$ and obtain

$$\begin{aligned}
&\int\limits_{Q} (Fg)(\cdot,\hat{\alpha}_j)\,\overline{\psi}\,dx \;+\int\limits_{\Gamma_{h_0}} \partial_{x_2} w(\cdot,\hat{\alpha}_j)\,\overline{\psi}\,ds \\
&= \int\limits_{Q} (Fg)(\cdot,\hat{\alpha}_j)\,\overline{\psi}\,dx + \int\limits_{\Gamma_{h_0}} \big[\partial_{x_2} w(\cdot,\hat{\alpha}_j)\,\overline{\psi} - w(\cdot,\hat{\alpha}_j)\,\partial_{x_2}\overline{\psi}\big]\,ds \\
&= \int\limits_{Q} (Fg)(\cdot,\hat{\alpha}_j)\,\overline{\psi}\,dx \;+\; \int\limits_{\Gamma_H} \big[\partial_{x_2} w(\cdot,\hat{\alpha}_j)\,\overline{\psi} - w(\cdot,\hat{\alpha}_j)\,\partial_{x_2}\overline{\psi}\big]\,ds \\
&\quad - \int\limits_{Q^H\setminus Q} \big[\Delta w(\cdot,\hat{\alpha}_j)\,\overline{\psi} - \Delta\overline{\psi}\,w(\cdot,\hat{\alpha}_j)\big]\,dx \\
&= \int\limits_{Q^H} (Fg)(\cdot,\hat{\alpha}_j)\,\overline{\psi}\,dx \;+\; \int\limits_{\Gamma_H} \big[\partial_{x_2} w(\cdot,\hat{\alpha}_j)\,\overline{\psi} - w(\cdot,\hat{\alpha}_j)\,\partial_{x_2}\overline{\psi}\big]\,ds\,. \qquad (3.16)
\end{aligned}$$

In the following we discuss the line integral and the volume integral for the different choices of ψ.

Case (A): $\psi = \phi^{\ell,j}$.

The line integral tends to zero as $H \to \infty$ because of the boundedness of $\frac{1}{H}\|w\|_{L^2(\Gamma_H)}$ and $\|\partial_{x_2} w\|_{L^2(\Gamma_H)}$ and the exponential decay of $\phi^{\ell,j}$.

To consider the volume integral in (3.16) for the different terms in the form (3.15) of Fg let the pair $(\varphi(x_1),\phi(x))$ be given by $(\varphi,\phi) = (\xi^s,\phi^{\ell',j'})$ or $(\varphi,\phi) = (\xi^s/\sqrt{|\cdot|},\hat{\phi}^s)$. Then

$$\begin{aligned}
&\int\limits_{Q^\infty} F\big[(\Delta + k^2 n)(\varphi\,\phi)\big](x,\hat{\alpha}_j)\,\overline{\phi^{\ell,j}(x)}\,dx \;=\; \int\limits_{Q^\infty} F\big[2\varphi'\partial_{x_1}\phi + \varphi''\phi\big](x,\hat{\alpha}_j)\,\overline{\phi^{\ell,j}(x)}\,dx \\
&= \int\limits_{\mathbb{R}^2_+} \big[2\varphi'(x_1)\partial_{x_1}\phi(x) + \varphi''(x_1)\phi(x)\big]\,\overline{\phi^{\ell,j}(x)}\,dx \qquad (3.17) \\
&= \int\limits_{\mathbb{R}^2_+} \big[\varphi'(x_1)\partial_{x_1}\phi(x) + \partial_{x_1}\big(\varphi'(x_1)\phi(x)\big)\big]\,\overline{\phi^{\ell,j}(x)}\,dx \\
&= \int\limits_{\mathbb{R}^2_+} \varphi'(x_1)\,\big[\partial_{x_1}\phi(x)\,\overline{\phi^{\ell,j}(x)} - \phi(x)\,\partial_{x_1}\overline{\phi^{\ell,j}(x)}\big]\,dx \;=\; -\frac{1}{2\pi i}\,E(\phi,\phi^{\ell,j}) \int\limits_{-\infty}^{\infty} \varphi'(x_1)\,dx_1
\end{aligned}$$

where we used (2.3) in the last step.

Now we distinguish between two cases.

Case (A1): $\phi \neq \phi^{\ell,j}$. Then $E(\phi,\phi^{\ell,j}) = 0$, and the volume integral in (3.16) vanishes.

Case (A2): $\phi = \phi^{\ell,j}$. Then $\varphi = \xi^\sigma$ and $E(\phi^{\ell,j},\phi^{\ell,j}) = \lambda_{\ell,j}$. Furthermore, $\int_{-\infty}^{\infty}(\xi^\sigma)'(x_1)\,dx_1 = \sigma$ since $\xi^\pm(x_1) = 1$ for $\pm x_1 > R_0$ and $\xi^\pm(x_1) = 0$ for $\pm x_1 \leq R_0 - 1$.

Altogether we have

$$\int\limits_{Q^\infty} (Fg)(x,\hat\alpha_j)\,\overline{\phi^{\ell,j}(x)}\,dx \;=\; \int\limits_{Q^\infty} (Ff)(x,\hat\alpha_j)\,\overline{\phi^{\ell,j}(x)}\,dx \;-\; a_{\ell,j}\,\frac{|\lambda_{\ell,j}|}{2\pi i}$$

$$=\int\limits_{\mathbb{R}^2_+} f(x)\,\overline{\phi^{\ell,j}(x)}\,dx \;-\; a_{\ell,j}\,\frac{|\lambda_{\ell,j}|}{2\pi i} \;=\; 0$$

by the choice of $a_{\ell,j}$ from (3.14).

Case (B) $\hat\alpha_j = \sigma\kappa$ and $\psi = \hat\phi^\sigma$. We go back to (3.16) and discuss again the line integral and the volume integral. To discuss the line integral we decompose $\hat\phi^\sigma$ into $\hat\phi^\sigma(x) = e^{\sigma ikx_1} + \hat\phi^\sigma_{evan}(x)$. The part $\int_{\Gamma_H}\big[\partial_{x_2}w(\cdot,\hat\alpha_j)\,\overline{\hat\phi^\sigma_{evan}} - w(\cdot,\hat\alpha_j)\,\partial_{x_2}\overline{\hat\phi^\sigma_{evan}}\big]\,ds$ converges again to zero. By (3.10) of Theorem 3.1 we have

$$\lim_{H\to\infty}\int\limits_0^{2\pi}\big[\partial_{x_2}w(x_1,H,\sigma\kappa)\,\overline{e^{\sigma ikx_1}} - w(x_1,H,\sigma\kappa)\,\partial_{x_2}\overline{e^{\sigma ikx_1}}\big]\,ds \;=\; 2\pi\lim_{H\to\infty}\partial_{x_2}w_{\sigma\hat\ell}(H,\sigma\kappa)$$

$$=\; -\,b^\sigma\sqrt{2k\pi}\,e^{-i\pi/4}\,.$$

Next, we consider the volume integrals appearing in (3.16). We show that they converge to 0 as $H\to\infty$. Let again $(\varphi,\phi) = (\xi^s/\sqrt{|\cdot|},\hat\phi^s)$ or $(\varphi,\phi) = (\xi^s,\phi^{\ell',j'})$. Then, by the same arguments as in the derivation of (3.17),

$$\int\limits_{Q^H} F\big[(\Delta+k^2n)(\varphi\,\phi)\big](\cdot,\sigma\kappa)\,\overline{\hat\phi^\sigma}\,dx = \int\limits_{-\infty}^{\infty}\varphi'(x_1)\int\limits_0^H\big[\partial_{x_1}\phi(x)\,\overline{\hat\phi^\sigma(x)} - \phi(x)\,\partial_{x_1}\overline{\hat\phi^\sigma(x)}\big]\,dx_2\,dx_1\,.$$

If $\phi = \phi^{\ell',j'}$ and $\varphi = \xi^s$ then $\int_0^H\big[\partial_{x_1}\phi^{\ell',j'}\,\overline{\hat\phi^\sigma} - \phi^{\ell',j'}\,\partial_{x_1}\overline{\hat\phi^\sigma}\big]\,dx_2$ is bounded with respect to H and x_1 and converges to $\int_0^\infty\big[\partial_{x_1}\phi^{\ell',j'}\,\overline{\hat\phi^\sigma} - \phi^{\ell',j'}\,\partial_{x_1}\overline{\hat\phi^\sigma}\big]\,dx_2 = -\frac{1}{2\pi i}\,E(\phi^{\ell',j'},\hat\phi^\sigma) = 0$. Therefore,

$$\lim_{H\to\infty}\int\limits_{Q^H} F\big[(\Delta+k^2n)(\xi^s\,\phi^{\ell',j'})\big](x,\sigma\kappa)\,\overline{\hat\phi^\sigma(x)}\,dx \;=\; 0\,.$$

Let now $\phi = \hat\phi^s$ and $\varphi(x_1) = \varphi^s(x_1) := \xi^s(x_1)/\sqrt{|x_1|}$. Then

$$\int\limits_{Q^H} F\big[(\Delta+k^2n)(\varphi^s\,\hat\phi^s)\big](x,\sigma\kappa)\,\overline{\hat\phi^\sigma(x)}\,dx$$

$$=\int\limits_{-\infty}^{\infty}\varphi^{s\prime}(x_1)\int\limits_0^H\big[\partial_{x_1}\hat\phi^s(x)\,\overline{\hat\phi^\sigma(x)} - \hat\phi^s(x)\,\partial_{x_1}\overline{\hat\phi^\sigma(x)}\big]\,dx_2\,dx_1 \tag{3.18}$$

$$=\int\limits_{-\infty}^{\infty}\varphi^{s\prime}(x_1)\int\limits_0^H\big[\partial_{x_1}\hat\phi^s(x)\,\overline{\hat\phi^\sigma(x)} - \hat\phi^s(x)\,\partial_{x_1}\overline{\hat\phi^\sigma(x)} - 2\sigma ik\big]\,dx_2\,dx_1 \tag{3.19}$$

since $\varphi^s(x_1)$ vanishes for $x_1 \to \pm\infty$. In (3.18) and (3.19) we have the terms $\partial_{x_1}\hat\phi^\sigma\,\overline{\hat\phi^\sigma} - \hat\phi^\sigma\,\partial_{x_1}\overline{\hat\phi^\sigma}$ (if $s=\sigma$) and $\partial_{x_1}\hat\phi^{-\sigma}\,\overline{\hat\phi^\sigma} - \hat\phi^{-\sigma}\,\partial_{x_1}\overline{\hat\phi^\sigma}$ (if $s=-\sigma$). In the first case we apply Lemma 2.7 to the form (3.19) which yields that $\int_0^H \big[\partial_{x_1}\hat\phi^\sigma(x)\,\overline{\hat\phi^\sigma(x)} - \hat\phi^\sigma(x)\,\partial_{x_1}\overline{\hat\phi^\sigma(x)} - 2\sigma ik\big]\,dx_2$ is uniformly bounded with respect to H and x_1 and converges (as $H\to\infty$) to some $c(x_1)=c$ which is constant with respect to x_1. Therefore, application of Lebesgue's theorem on dominated convergence to (3.18) yields that

$$\lim_{H\to\infty}\int\limits_{Q^H} F\big[(\Delta+k^2n)(\varphi^\sigma\,\hat\phi^\sigma)\big](x,\sigma\kappa)\,\overline{\hat\phi^\sigma(x)}\,dx \;=\; 0\,.$$

For the second case we apply Lemma 2.7 to the form (3.18) and argue in the same way. Now we substitute these terms into (3.16) and obtain

$$\int\limits_Q (Fg)(x,\sigma\kappa)\,\overline{\hat\phi^\sigma(x)}\,dx + \int\limits_{\Gamma_{h_0}} \partial_{x_2}w(x,\sigma\kappa)\,\overline{\hat\phi^\sigma(x)}\,ds$$

$$= \int\limits_{Q^\infty} (Ff)(x,\sigma\kappa)\,\overline{\hat\phi^\sigma(x)}\,dx \;-\; b^\sigma\sqrt{2k\pi}\,e^{-i\pi/4} \;=\; 0$$

by the form (3.14) of $b^\pm$. □

3.4. **The Dependence on the Floquet-Parameter.** Next we study the dependence on α which is necessary for the application of the inverse Floquet-Bloch transform. It is convenient to transform the equation into the space $H^1_{per}(Q)$ for the function $\tilde v(x,\alpha) = e^{-i\alpha x_1}\hat v(x,\alpha)$. To simplify the notation we define the operator $J_\alpha : H^1_{\alpha,*}(Q^\infty) \to H^1_{per,*}(Q^\infty)$ as $(J_\alpha u)(x) = e^{-i\alpha x_1}u(x)$. Then the equation $L_\alpha \hat v = r_\alpha$ in $H^1_\alpha(Q)$ is equivalent to

$$J_\alpha L_\alpha J_\alpha^{-1}\tilde v(\cdot,\alpha) \;=\; J_\alpha r_\alpha \quad \text{in } H^1_{per}(Q)\,, \tag{3.20}$$

i.e.,

$$\int\limits_Q [\nabla(\tilde v\,e^{i\alpha x_1})\cdot\nabla(\overline{\psi\,e^{i\alpha x_1}}) - k^2 n\,\tilde v\,\overline{\psi}]\,dx \;-\; 2\pi i\sum_{\ell\in\mathbb{Z}}\sqrt{k^2-(\ell+\alpha)^2}\,\tilde v_\ell\,\overline{\psi_\ell}$$

$$= \langle J_\alpha r_\alpha,\psi\rangle_{H^1(Q)} \;=\; \int\limits_Q (Fg)(\cdot,\alpha)\,\overline{\psi\,e^{i\alpha x_1}}\,dx \;+\; \int\limits_{\Gamma_{h_0}} \partial_{x_2}w(\cdot,\alpha)\,\overline{\psi\,e^{i\alpha x_1}}\,ds$$

for all $\psi\in H^1_{per}(Q)$ where $\tilde v_\ell = \frac{1}{2\pi}\int_0^{2\pi}\tilde v(x_1,h_0)\,e^{-i\ell x_1}dx_1$ and ψ_ℓ are the Fourier coefficients of the $2\pi-$periodic functions $\tilde v(\cdot,h_0)$ and $\psi(\cdot,h_0)$, respectively. We observe that the operator $J_\alpha L_\alpha J_\alpha^{-1}$ is not differentiable at the cut-off values $\alpha=\pm\kappa$ – and also not invertible because $\pm\kappa$ are critical values by Assumption 2.4. This situation is new and has not treated before.

We consider first the dependence on α of the right hand side. The term "smooth" means C^∞ in the following.

Theorem 3.4. *(a) For $|j|\geq 2$ there exists a neighborhood of $\hat\alpha_j$ in which the transformed right hand side $J_\alpha r_\alpha \in H^1_{per}(Q)$ depends smoothly on α.*
For $j=\pm1$, i.e. $\hat\alpha_j=\pm\kappa$, it has the form $J_\alpha r_\alpha = r_1^\pm(\alpha)\sqrt{\kappa\mp\alpha} + r_2^\pm(\alpha)$ where $r_m^\pm(\alpha)\in H^1_{per}(Q)$ depend smoothly on α in a neighborhood of $\pm\kappa$.

(b) Furthermore, we have in a neighborhood of $\pm\kappa$:

$$\langle J_\alpha r_\alpha, J_{\pm\kappa}\phi^{\ell,\pm 1}\rangle_{H^1(Q)} = (\alpha\mp\kappa)\left[\eta^\pm_{1,\ell}(\alpha)+\sqrt{\kappa\mp\alpha}\,\eta^\pm_{2,\ell}(\alpha)\right],\ \ \ell=1,\ldots,m_1\,, \tag{3.21}$$

$$\langle J_\alpha r_\alpha, J_{\pm\kappa}\hat{\phi}^{\pm}\rangle_{H^1(Q)} = \sqrt{\kappa\mp\alpha}\,\rho^\pm_1(\alpha)\ +\ (\alpha\mp\kappa)\,\rho^\pm_2(\alpha)\,, \tag{3.22}$$

where $\eta^\pm_{j,\ell},\rho^\pm_j$ depend smoothly on α for $j=1,2$ and $\ell=1,\ldots,m_1$.

Proof. (a) We recall the form (3.15) of Fg. Since f and $(\xi^\sigma)'$ have compact support with respect to x_1 their Floquet Bloch transforms are smooth with respect to α. Furthermore, from Lemma B.1 of the Appendix we have $F(\varphi^{\sigma\prime})(x_1,\alpha-\sigma\kappa)=\sigma\frac{e^{-i\pi/4}}{2\sqrt{\pi}}\sqrt{\kappa-\sigma\alpha}\,e^{i(\alpha-\sigma\kappa)x_1}+A^\sigma(x_1,\alpha)$ with smooth A^σ. This proves the form of $J_\alpha r_\alpha$.

(b) Formula (3.22) follows directly from parts (a) and Theorem 3.3. Indeed, from (a) we obtain $\langle J_\alpha r_\alpha, J_{\pm\kappa}\hat{\phi}^\pm\rangle_{H^1(Q)}=\sqrt{\kappa\mp\alpha}\,\langle r^\pm_1(\alpha),J_{\pm\kappa}\hat{\phi}^\pm\rangle_{H^1(Q)}+\langle r^\pm_2(\alpha),J_{\pm\kappa}\hat{\phi}^\pm\rangle_{H^1(Q)}$, and from Theorem 3.3 we conclude that $\langle r^\pm_2(\pm\kappa),J_{\pm\kappa}\hat{\phi}^\pm\rangle_{H^1(Q)}=0$ and thus $\langle r^\pm_2(\alpha),J_{\pm\kappa}\hat{\phi}^\pm\rangle_{H^1(Q)}=(\alpha\mp\kappa)\,\rho^\pm_2(\alpha)$ for some smooth function $\rho^\pm_2$.

It remains to show (3.21). We note that $(\Delta+k^2)w=-Fg$ and $(\Delta+k^2)(e^{i(\pm\kappa-\alpha)x_1}\phi^{\ell,\pm1})=e^{i(\pm\kappa-\alpha)x_1}\big[2i(\pm\kappa-\alpha)\partial_{x_1}\phi^{\ell,\pm1}-(\pm\kappa-\alpha)^2\phi^{\ell,\pm1}\big]$. Exactly as for (3.16) we obtain by Green's theorem in $(0,2\pi)\times(h_0,H)$ for the $\alpha-$quasi-periodic functions $w(\cdot,\alpha)$ and $e^{-i(\pm\kappa-\alpha)x_1}\phi^{\ell,\pm1}\in H^1_{\alpha,*}(Q^\infty\setminus Q)$ that

$$\begin{aligned}
\langle J_\alpha r_\alpha, J_{\pm\kappa}\phi^{\ell,\pm1}\rangle_{H^1(Q)} &= \langle r_\alpha, e^{i(\alpha\mp\kappa)x_1}\phi^{\ell,\pm1}\rangle_{H^1(Q)}\\
=\int\limits_Q (Fg)(x,\alpha)\,\overline{e^{i(\alpha\mp\kappa)x_1}\phi^{\ell,\pm1}}(x)\,dx\ &+\int\limits_{\Gamma_{h_0}}\partial_{x_2}w(x,\alpha)\,\overline{\phi^{\ell,\pm1}(x)\,e^{i(\alpha\mp\kappa)x_1}}\,ds\\
=\int\limits_{Q^H}(Fg)(x,\alpha)\,&\overline{\phi^{\ell,\pm1}(x)\,e^{i(\alpha\mp\kappa)x_1}}\,dx\\
+\int\limits_{\Gamma_H}\Big[\partial_{x_2}w(x,\alpha)\,&\overline{\phi^{\ell,\pm1}(x)}-w(x,\alpha)\,\partial_{x_2}\overline{\phi^{\ell,\pm1}(x)}\Big]\,e^{-i(\alpha\mp\kappa)x_1}ds\\
+\int\limits_{Q^H\setminus Q}\Big[2i(\alpha\mp\kappa)\,&\partial_{x_1}\overline{\phi^{\ell,\pm1}(x)}+(\alpha\mp\kappa)^2\overline{\phi^{\ell,\pm1}(x)}\Big]\,w(x,\alpha)\,dx\,.
\end{aligned} \tag{3.23}$$

Since $\phi^{\ell,\pm1}$ decays exponentially and $\frac{1}{H}\|w(\cdot,\alpha)\|_{L^2(\Gamma_H)}$ and $\|\partial_{x_2}w(\cdot,\alpha)\|_{L^2(\Gamma_H)}$ are bounded by Theorem 3.1 we obtain again that the line integral converges to zero and thus

$$\begin{aligned}
\langle J_\alpha r_\alpha, J_{\pm\kappa}\phi^{\ell,\pm1}\rangle_{H^1(Q)} &= \int\limits_{Q^\infty}(Fg)(x,\alpha)\,\overline{\phi^{\ell,\pm1}(x)\,e^{i(\alpha\mp\kappa)x_1}}\,dx\\
&+\int\limits_{Q^\infty\setminus Q}\Big[2i(\alpha\mp\kappa)\,\partial_{x_1}\overline{\phi^{\ell,\pm1}(x)}+(\alpha\mp\kappa)^2\overline{\phi^{\ell,\pm1}(x)}\Big]\,w(x,\alpha)\,dx\,.
\end{aligned}$$

The second integral ist of the form $(\alpha\mp\kappa)\,[A_1(\alpha)+\sqrt{\kappa\mp\alpha}\,A_2(\alpha)]$ with smooth functions A_1, A_2 by Theorem 3.1. For the first integral we consider the different parts of Fg and use the representation (3.15). The mappings $\alpha\mapsto(Ff)(x,\alpha)$ and $\alpha\mapsto\big[2(F\xi^{s\prime})(x_1,\alpha-$

$\hat{\alpha}_{j'})\,\partial_{x_1}\phi^{\ell',j'}(x) + (F\xi^{s\prime\prime})(x_1,\alpha-\hat{\alpha}_{j'})\,\phi^{\ell',j'}(x)\big]$ are smooth because f and $\xi^{s\prime}$ have compact supports. Next, we consider for $s\in\{+,-\}$

$$\begin{aligned}
&\int\limits_{Q^\infty}\big[2(F\varphi^{s\prime})(x_1,\alpha-s\kappa)\,\partial_{x_1}\hat{\phi}^s(x)+(F\varphi^{s\prime\prime})(x_1,\alpha-s\kappa)\hat{\phi}^s(x)\big]\overline{\phi^{\ell,\pm1}(x)}\,e^{i(\pm\kappa-\alpha)x_1}\,dx\\
&=\int\limits_{Q^\infty}\big[(F\varphi^{s\prime})(x_1,\alpha-s\kappa)\,\partial_{x_1}\hat{\phi}^s+\partial_{x_1}\big((F\varphi^{s\prime})(x_1,\alpha-s\kappa)\hat{\phi}^s\big)\big]\,\overline{\phi^{\ell,\pm1}(x)}\,e^{i(\pm\kappa-\alpha)x_1}\,dx\\
&=\int\limits_{Q^\infty}(F\varphi^{s\prime})(x_1,\alpha-s\kappa)\big[\partial_{x_1}\hat{\phi}^s(x)\,\overline{(\phi^{\ell,\pm1}(x)\,e^{i(\alpha\mp\kappa)x_1})}-\hat{\phi}^s(x)\,\partial_{x_1}\overline{(\phi^{\ell,\pm1}(x)e^{i(\alpha\mp\kappa)x_1})}\big]\,dx\\
&=\int\limits_0^{2\pi}(F\varphi^{s\prime})(x_1,\alpha-s\kappa)e^{-i(\alpha\mp\kappa)x_1}\int\limits_0^\infty\big[\partial_{x_1}\hat{\phi}^s(x)\,\overline{\phi^{\ell,\pm1}(x)}-\hat{\phi}^s(x)\,\partial_{x_1}\overline{\phi^{\ell,\pm1}(x)}\big]\,dx_2\,dx_1\\
&\qquad+\;i(\alpha\mp\kappa)\int\limits_{Q^\infty}(F\varphi^{s\prime})(x_1,\alpha-s\kappa)\,\hat{\phi}^s(x)\,\overline{\phi^{\ell,\pm1}(x)}\,e^{-i(\alpha\mp\kappa)x_1}\,dx\\
&=i(\alpha\mp\kappa)\int\limits_{Q^\infty}(F\varphi^{s\prime})(x_1,\alpha-s\kappa)\,\hat{\phi}^s(x)\,\overline{\phi^{\ell,\pm1}(x)}\,e^{-i(\alpha\mp\kappa)x_1}\,dx
\end{aligned}$$

because $\int_0^\infty\big[\partial_{x_1}\hat{\phi}^s(x)\,\overline{\phi^{\ell,\pm1}(x)}-\hat{\phi}^s(x)\,\partial_{x_1}\overline{\phi^{\ell,\pm1}(x)}\big]\,dx_2=\frac{1}{2\pi i}\,E(\hat{\phi}^s,\phi^{\ell,\pm1})=0$. Now we use Lemma B.1 for $(F\varphi^{s\prime})(x_1,\alpha-s\kappa)$ to obtain the desired form (3.21). □

The dependence of the operator $J_\alpha L_\alpha J_\alpha^{-1}$ on α has been studied in, e.g. [Kir25] (Subsection 5.1) or [Kir22] (Theorem 4.3, part (c)), and we repeat the arguments. The operator is certainly smooth for α not being a cut-off value. We consider the case of the cut-off value $\alpha=\kappa$ and translate the value to $\alpha=0$, i.e. define $L(\alpha):=J_{\alpha+\kappa}L_{\alpha+\kappa}J_{\alpha+\kappa}^{-1}$ for α in a neighborhood of 0. Recalling that $k=\tilde{\ell}+\kappa$ with $\tilde{\ell}\in\mathbb{Z}_{\geq0}$ we decompose $L(\alpha)$ into $L(\alpha)=\tilde{L}(\alpha)+\rho(\alpha)B$ with

$$\begin{aligned}
\langle\tilde{L}(\alpha)u,v\rangle_{H^1(Q)}\;&:=\;\int\limits_Q[\nabla(u\,e^{i(\alpha+\kappa)x_1})\cdot\nabla(\overline{\psi\,e^{i(\alpha+\kappa)x_1}})-k^2n\,u\,\overline{\psi}]\,dx\\
&\qquad-\;2\pi i\sum_{\ell\neq\tilde{\ell}}u_\ell\,\overline{v_\ell}\,\sqrt{k^2-(\ell+\alpha+\kappa)^2}\,,\\
\rho(\alpha)\;&:=2\pi i\sqrt{k^2-(\tilde{\ell}+\alpha+\kappa)^2}\;=\;2\pi i\,\sqrt{2k+\alpha}\,\sqrt{-\alpha}\,,\\
\langle Bu,v\rangle_{H^1(Q)}\;&:=u_{\tilde{\ell}}\,\overline{v_{\tilde{\ell}}}
\end{aligned}$$

for $u,v\in H^1_{per}(Q)$ where $u_\ell=\frac{1}{2\pi}\int_0^{2\pi}u(x_1,h_0)\,e^{-\ell x_1}dx_1$ are the Fourier coefficients of the $2\pi-$periodic function $u(\cdot,h_0)$. We note that $Bu=\langle u,b\rangle_*b$ where $b\in H^1_{per}(Q)$ is defined by $\langle u,b\rangle_*=u_{\tilde{\ell}}=\frac{1}{2\pi}\int_0^{2\pi}u(x_1,h_0)e^{-i\tilde{\ell}x_1}dx_1$ for $u\in H^1_{per}(Q)$. Then $\tilde{L}(\alpha)$ is infinitely often differentiable in a neighborhood of 0 and $\rho(\alpha)^2$ is smooth with $\frac{d}{d\alpha}[\rho(\alpha)^2]|_{\alpha=0}=8\pi^2k\neq0$. We can now apply a functional analytic result presented in the following subsection.

3.5. **An Abstract Singular Perturbation Result.** In the following two theorems let H be a (complex) Hilbert space, $K(\alpha) : H \to H$ and $r(\alpha) \in H$ for $\alpha \in (-\delta_0, \delta_0) \subset \mathbb{R}$ be families of compact operators and elements, respectively, which depend continuously on $\alpha \in (-\delta_0, \delta_0)$. Set $L(\alpha) = I - K(\alpha)$ and assume that 1 is a semi-simple eigenvalue of $K(0)$; that is, $\mathcal{N}\big(L(0)^2\big) = \mathcal{N}\big(L(0)\big)$ where $\mathcal{N}(L) = \{x \in H : Lx = 0\}$ denotes the nullspace of an operator L. Furthermore, let $P : H \to \mathcal{N} \subset H$ be the projection onto the finite dimensional space $\mathcal{N} := \mathcal{N}\big(L(0)\big)$ with respect to the direct decomposition $H = \mathcal{N} \oplus \mathcal{R}\big(L(0)\big)$. We assume that this decomposition is orthogonal.
The case that $K(\alpha)$ and $r(\alpha)$ depend smoothly on α had been considered before, see, e.g., [KS24]. We consider the case that $K(\alpha)$ and $r(\alpha)$ are merely continuous with a square-root type continuity at zero.

Theorem 3.5. *Let U be neighborhood of 0 and let the compact operators $K(\alpha) : H \to H$ for $\alpha \in U \subset (-1/2, 1/2)$ have the forms $K(\alpha) = \tilde{K}(\alpha) + \rho(\alpha)\, B$ where $\tilde{K}(\alpha)$ is smooth, i.e. infinitely often differentiable, in U and $\rho(\alpha) : U \to \mathbb{C}$ is continuous with $\rho(0) = 0$ such that $\rho(\alpha) \neq 0$ for $\alpha \neq 0$ and ρ^2 is smooth with $\frac{d}{d\alpha}[\rho(\alpha)^2]|_{\alpha=0} \neq 0$. Furthermore, $B : H \to H$ is the one-dimensional operator, given by $Bv = \langle v, b\rangle\, b$ for $v \in H$ and some $b \in H$.*
Let $\mathcal{N}_0$ be the orthogonal complement of Pb in $\mathcal{N}$, i.e. $\mathcal{N}_0 := \{\phi \in \mathcal{N} : \langle \phi, Pb\rangle = 0\}$. (If $Pb = 0$ then $\mathcal{N}_0 = \mathcal{N}$.) Let $P_0 : \mathcal{N} \to \mathcal{N}_0$ be the orthogonal projection, given by $P_0 v = v - \langle v, Pb\rangle Pb/\|Pb\|^2$. (The operator P_0 is the identity if $Pb = 0$.) Furthermore, let $P_0 P \tilde{K}'(0)\big|_{\mathcal{N}_0} : \mathcal{N}_0 \to \mathcal{N}_0$ be one-to-one.

(a) *Then $L(\alpha)$ is invertible for $\alpha \neq 0$ sufficiently close to 0.*
(b) *Let, in addition, the right hand sides $r(\alpha)$ and their projections $Pr(\alpha)$ and $P_0 Pr(\alpha)$ have the forms*

$$\begin{aligned} r(\alpha) &= r_1(\alpha) \;+\; \rho(\alpha)\, r_2(\alpha)\,, \\ Pr(\alpha) &= \rho(\alpha)\, p_1(\alpha) \;+\; \alpha\, p_2(\alpha)\,, \\ P_0 Pr(\alpha) &= \alpha\big[q_1(\alpha) \;+\; \rho(\alpha)\, q_2(\alpha)\big]\,, \end{aligned} \tag{3.24}$$

with smooth functions $r_m(\alpha), p_m(\alpha), q_m(\alpha) \in H$. (In particular, $Pr(0) = 0$.) Then the unique solution $u(\alpha) = L(\alpha)^{-1} r(\alpha)$ for $\alpha \neq 0$ has a decomposition in the form

$$u(\alpha) \;=\; u_1(\alpha) \;+\; \rho(\alpha)\, u_2(\alpha)$$

where $u_j(\alpha)$ are smooth in a neighborhood of zero.

Proof. We use the splitting $H = \mathcal{N} \oplus \mathcal{R}$ where again $\mathcal{N} := \mathcal{N}\big(L(0)\big)$ and $\mathcal{R} := \mathcal{R}\big(L(0)\big)$ with corresponding projections P and $Q := I - P$, respectively. Furthermore, we set $\tilde{L}(\alpha) = I - \tilde{K}(\alpha)$. Then the equation $L(\alpha)u(\alpha) = r(\alpha)$ is quivalent to the set of equations

$$\begin{aligned} [P\tilde{L}(\alpha) - \rho(\alpha)PB]u^{\mathcal{N}}(\alpha) \;&+\; [P\tilde{L}(\alpha) - \rho(\alpha)PB]u^{\mathcal{R}}(\alpha) \;=\; Pr(\alpha)\,, \\ QL(\alpha)u^{\mathcal{N}}(\alpha) \;&+\; QL(\alpha)u^{\mathcal{R}}(\alpha) \;=\; Qr(\alpha) \end{aligned}$$

for $u(\alpha) = u^{\mathcal{N}}(\alpha) + u^{\mathcal{R}}(\alpha)$ with $\big(u^{\mathcal{N}}(\alpha), u^{\mathcal{R}}(\alpha)\big) \in \mathcal{N} \times \mathcal{R}$.
We consider first the case $Pb \neq 0$ and remark on the (simpler) case $Pb = 0$ below. We decompose $\mathcal{N} = \mathcal{N}\big(L(0)\big)$ into $\mathcal{N} = \mathcal{N}_0 \oplus \mathrm{span}\{Pb\}$, i.e. $\mathcal{N}_0 = \mathrm{span}\{Pb\}^\perp$, and recall that $P_0 : \mathcal{N} \to \mathcal{N}_0$ is the corresponding orthogonal projection. Then we have

$P_0PBv = \langle v,b\rangle P_0Pb = 0$ and $\langle PBv, Pb\rangle = \langle v,b\rangle \|Pb\|^2$. Applying P_0 to the first quation and multiplying the first equation by Pb results in the equivalent system

$$P_0P\tilde{L}(\alpha)u^{\mathcal{N}}(\alpha) \;+\; P_0P\tilde{L}(\alpha)u^{\mathcal{R}}(\alpha) = P_0Pr(\alpha)\,, \tag{3.25}$$

$$QL(\alpha)u^{\mathcal{N}}(\alpha) \;+\; QL(\alpha)u^{\mathcal{R}}(\alpha) \quad = Qr(\alpha)\,, \tag{3.26}$$

$$\langle P\tilde{L}u^{\mathcal{N}}, Pb\rangle - \rho\langle u^{\mathcal{N}}, b\rangle\|Pb\|^2 + \langle P\tilde{L}u^{\mathcal{R}}, Pb\rangle - \rho\langle u^{\mathcal{R}}, b\rangle\|Pb\|^2 \;=\; \langle Pr, Pb\rangle \tag{3.27}$$

where we dropped the argument α in the previous equation. Analogously, we decompose $u^{\mathcal{N}}(\alpha)$ into $u^{\mathcal{N}}(\alpha) = u_0^{\mathcal{N}}(\alpha) + s(\alpha)Pb$ with $u_0^{\mathcal{N}}(\alpha) \in \mathcal{N}_0$ and $s(\alpha) \in \mathbb{C}$. Then (3.25) – (3.27) is written as

$$P_0P\tilde{L}(\alpha)u_0^{\mathcal{N}} \;+\; P_0P\tilde{L}(\alpha)u^{\mathcal{R}} \;+\; s(\alpha)\,P_0P\tilde{L}(\alpha)Pb \;=\; P_0Pr(\alpha)\,, \tag{3.28}$$

$$QL(\alpha)u_0^{\mathcal{N}} \;+\; QL(\alpha)u^{\mathcal{R}} \;+\; s(\alpha)\,QL(\alpha)Pb \;=\; Qr(\alpha)\,, \tag{3.29}$$

$$\begin{aligned}&\langle P\tilde{L}u_0^{\mathcal{N}}, Pb\rangle + s\langle P\tilde{L}Pb, Pb\rangle - \rho\langle u_0^{\mathcal{N}}, b\rangle\|Pb\|^2 - s\rho\langle Pb, b\rangle\|Pb\|^2\\ &+ \langle P\tilde{L}u^{\mathcal{R}}, Pb\rangle - \rho\langle u^{\mathcal{R}}, b\rangle\|Pb\|^2 \;=\; \langle Pr, Pb\rangle\,.\end{aligned} \tag{3.30}$$

First we note that $\langle u_0^{\mathcal{N}}, b\rangle = \langle u_0^{\mathcal{N}}, Pb\rangle = 0$ for $u_0^{\mathcal{N}} \in \mathcal{N}_0$. We devide (3.28) by α and (3.30) by $\rho(\alpha)$ and write the resulting system in matrix form $A(\alpha)v(\alpha) = \hat{r}(\alpha)$ for $v(\alpha) = [u_0^{\mathcal{N}}(\alpha), s(\alpha), u^{\mathcal{R}}(\alpha)]^\top \in \mathcal{N}_0 \times \mathbb{C} \times \mathcal{R}$ where

$$A(\alpha) \;=\; \begin{bmatrix} \frac{1}{\alpha}P_0P\tilde{L}(\alpha)|_{\mathcal{N}_0} & \frac{1}{\alpha}P_0P\tilde{L}(\alpha)Pb & \frac{1}{\alpha}P_0P\tilde{L}(\alpha)|_{\mathcal{R}} \\ \frac{1}{\rho}\langle P\tilde{L}\cdot, Pb\rangle & \frac{1}{\rho}\langle P\tilde{L}Pb, Pb\rangle - \|Pb\|^4 & \frac{1}{\rho}\langle P\tilde{L}\cdot, Pb\rangle - \langle\cdot, b\rangle\|Pb\|^2 \\ QL(\alpha)|_{\mathcal{N}_0} & QL(\alpha)Pb & QL(\alpha)|_{\mathcal{R}} \end{bmatrix}$$

(where we dropped the argument α in the second row) and

$$\begin{aligned}\hat{r}(\alpha) \;&=\; \begin{bmatrix} \frac{1}{\alpha}P_0Pr(\alpha) \\ \frac{1}{\rho(\alpha)}\langle Pr(\alpha), Pb\rangle \\ Qr(\alpha)\end{bmatrix} \;=\; \begin{bmatrix} q_1(\alpha) \\ \langle p_1(\alpha), Pb\rangle \\ Qr_1(\alpha)\end{bmatrix} + \rho(\alpha)\begin{bmatrix} q_2(\alpha) \\ \frac{\alpha}{\rho(\alpha)^2}\langle p_2(\alpha), Pb\rangle \\ Qr_2(\alpha)\end{bmatrix}\\ &=\; \hat{r}_1(\alpha) \;+\; \rho(\alpha)\,\hat{r}_2(\alpha)\end{aligned}$$

where we used the assumptions on $r(\alpha)$, $Pr(\alpha)$, and $P_0Pr(\alpha)$. Since $\alpha \mapsto \tilde{L}(\alpha)$ is smooth there exists a smooth operator valued function $\alpha \mapsto M(\alpha)$ with

$$\tilde{L}(\alpha) \;=\; L(0) \;+\; \alpha\tilde{L}'(0) \;+\; \alpha^2M(\alpha)\,.$$

Substituting this into the form of $A(\alpha)$ and using $\tilde{L}(0)|_{\mathcal{N}} = L(0)|_{\mathcal{N}} \equiv 0$ and $P\tilde{L}(0)|_{\mathcal{R}} = PL(0)|_{\mathcal{R}} \equiv 0$ yields a decomposition in the form

$$A(\alpha) \;=\; A(0) \;+\; \rho(\alpha)\,A_1(\alpha) \;+\; \alpha\,A_2(\alpha)$$

where

$$
\begin{aligned}
A(0) &= \begin{bmatrix} P_0 P\tilde{L}'(0)|_{\mathcal{N}_0} & P_0 P\tilde{L}'(0)Pb & P_0 P\tilde{L}'(0)|_{\mathcal{R}} \\ 0 & -\|Pb\|^4 & -\langle \cdot, b\rangle \|Pb\|^2 \\ 0 & 0 & QL(0)|_{\mathcal{R}} \end{bmatrix}, \\
A_1(\alpha) &= \frac{\alpha}{\rho(\alpha)^2} \begin{bmatrix} 0 & 0 & 0 \\ \langle P\tilde{L}'(0)\cdot, Pb\rangle & \langle P\tilde{L}'(0)Pb, Pb\rangle & \langle P\tilde{L}'(0)\cdot, Pb\rangle \\ 0 & 0 & 0 \end{bmatrix}, \\
&+ \frac{\alpha^2}{\rho(\alpha)^2} \begin{bmatrix} 0 & 0 & 0 \\ \langle PM(\alpha)\cdot, Pb\rangle & \langle PM(\alpha)Pb, Pb\rangle & PM(\alpha)\cdot, Pb\rangle \\ 0 & 0 & 0 \end{bmatrix}, \\
A_2(\alpha) &= \begin{bmatrix} P_0 PM(\alpha)_{\mathcal{N}_0} & P_0 PM(\alpha)Pb & P_0 PM(\alpha)|_{\mathcal{R}} \\ 0 & 0 & 0 \\ Q\tilde{L}'(0)|_{\mathcal{N}_0} & Q\tilde{L}'(0)Pb & Q\tilde{L}'(0)|_{\mathcal{R}} \end{bmatrix} \\
&+ \alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ QM(\alpha)|_{\mathcal{N}_0} & QM(\alpha)Pb & QM(\alpha)|_{\mathcal{R}} \end{bmatrix}
\end{aligned}
$$

depend smoothly on α. As a trianguar matrix with invertible diagonal operators $P_0 P\tilde{L}'(0)|_{\mathcal{N}_0} : \mathcal{N}_0 \to \mathcal{N}_0$, $-\|Pb\|^4 \neq 0$, and $QL(0)|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ the matrix $A(0)$ is invertible. By a simple argument using the Neumann series and collecting terms with even and odd powers of $\rho(\alpha)$ (and noting that $\rho(\alpha)^2$ is smooth) we obtain that the solution $v(\alpha) \in \mathcal{N}_0 \times \mathbb{C} \times \mathcal{R}$ of $A(\alpha)v(\alpha) = \hat{r}(\alpha)$ has the form $v(\alpha) = v_0(\alpha) + \rho(\alpha)v_1(\alpha)$ where $v_j(\alpha)$ are smooth. Therefore, the solution $u(\alpha) \in H$ of $L(\alpha)u(\alpha) = r(\alpha)$ which is the sum of the three components of $v(\alpha)$ has the required form.

Finally, we look at the changes for the case $Pb = 0$. In this case the system (3.25) – (3.27) reduces to the system (3.25), (3.26) for $R = I$, and we obtain a $2 \times 2-$ matrix system for the unknowns $u^{\mathcal{D}} \in \mathcal{N}$ and $u^{\mathcal{R}} \in \mathcal{R}$. In the same way as before we obtain invertibility of $L(\alpha)$ for $\alpha \neq 0$ close to 0 and smoothness of $u^{\mathcal{D}}(\alpha)$ and $u^{\mathcal{R}}(\alpha)$ if $Pr(0) = 0$. □

3.6. **Application of the Abstract Theorem.** We go back to the discussion of equation (3.20) and consider α in a neighborhood of the critical cut-off value $\hat{\alpha}_1 = \kappa$. (The case $\hat{\alpha}_{-1} = -\kappa$ is treated analogously.) Therefore, we set $H = H^1_{per}(Q)$ and $L(\alpha) = J_{\alpha+\kappa} L_{\alpha+\kappa} J^{-1}_{\alpha+\kappa}$ and $r(\alpha) = J_{\alpha+\kappa} r_{\alpha+\kappa}$. We have to check the assumptions of Theorem 3.5. First we note that $\mathcal{N} = \mathcal{N}(L(0)) = \{J_\kappa \phi|_Q : \phi \in \mathcal{M}(\kappa)\}$ and $\mathcal{N}_0 = \{J_\kappa \phi|_Q : \phi \in \mathcal{M}_{evan}(\kappa)\}$. The decomposition $L(\alpha) = \tilde{L}(\alpha) + \rho(\alpha)\, B$ has been discussed at the end of Subsection 3.5. In [Kir25] (Lemma 5.3, part (b)) the formula

$$
\big\langle \tilde{L}'(0) J_\kappa w, J_\kappa \phi \big\rangle_{H^1(Q)} \;=\; -2i \int\limits_{Q^\infty} \bar{\phi}\, \partial_{x_1} w \, dx \;=\; E(w, \phi)
$$

for all $w \in \mathcal{M}(\kappa)$ and $\phi \in \mathcal{M}_{evan}(\kappa)$ has been shown. The Assumption 2.4 on E implies that $P_0 P\tilde{L}'(0)|_{\mathcal{N}_0}$ is one-to-one on $\mathcal{N}_0$. Therefore, all of the assumptions on $L(\alpha)$ are satisfied.

For the assumption with respect to the right hand side $r(\alpha)$ we apply Theorem 3.4. The conditions (3.24) follows from part (a) of Theorem 3.4 and (3.21) and (3.22) (replace $\kappa-\alpha$ by $\kappa-(\kappa+\alpha)=-\alpha$).

Therefore, all of the assumptions of Theorem 3.5 are satisfied. As mentioned above the same arguments hold also for the critical value $\hat\alpha_{-1}=-\kappa$. Furthermore, the critical values $\hat\alpha_j$ for $|j|\geq 2$ are different from the cut-off values $\pm\kappa$. In this case $L(\alpha)$ and $r(\alpha)$ are smooth near $\hat\alpha_j$, and we can apply the analogon of Theorem 3.5 (see, e.g., Theorem 3.2 of [KS24]) and obtain smoothness of $\tilde v$ near $\hat\alpha_j$.

Summarizing, we have shown the following representation of $\hat v(x,\alpha)=J_\alpha\hat u(x,\alpha)$.

Theorem 3.6. *Let Assumptions 2.5 and* (2.4) *hold and let $a_{\ell,j}\in\mathbb{C}$ and $b^\pm\in\mathbb{C}$ be given by* (3.14)*. Then, for every $\alpha\in[-1/2,1/2]$, there exists a solution $\hat v(\cdot,\alpha)\in H^1_\alpha(Q)$ of* (3.12) *and also a solution $\hat v(\cdot,\alpha)\in H^1_{\alpha,*}(Q^\infty)$ of* (3.6) *satisfying the radiation condition* (3.7)*.*

Furthermore, the solutions can be chosen to be smooth with respect to α in a neighborhood of the critical values $\hat\alpha_j$ for $|j|\geq 2$ and continuous in a neighborhood of $\hat\alpha_{\pm1}=\pm\kappa$ and have the form

$$\hat v(\cdot,\alpha) \;=\; \hat v_{1,\pm}(\cdot,\alpha) \;+\; \sqrt{\kappa\mp\alpha}\,\hat v_{2,\pm}(\cdot,\alpha) \tag{3.31}$$

in a neighborhood of $\hat\alpha_{\pm1}=\pm\kappa$ where $\hat v_{1,\pm}(\cdot,\alpha)$ and $\hat v_{2,\pm}(\cdot,\alpha)$ depend smoothly on α.

3.7. **Proof of Theorem 2.10.** (a) Uniqueness has been shown in [Kir25], Theorem 4.8.

Existence of a solution is constructed with the the solutions $\hat v(\cdot,\alpha)\in H^1_\alpha(Q)$ of (3.6). We define u_*^{rad} as the inverse Floquet-Bloch transform of $\hat v$, i.e.

$$u_*^{rad}(x_1+2\pi\ell,x_2) \;:=\; \int_{-1/2}^{1/2}\hat v(x,\alpha)\,e^{i2\pi\ell\alpha}\,d\alpha\,,\quad x\in Q^\infty\,,\ \ell\in\mathbb{Z}\,,$$

and $u:=u^{rad}+u^{prop}$ with u^{prop} from (2.6) and

$$u^{rad}(x) \;:=\; u_*^{rad}(x) \;+\; \frac{1}{\sqrt{|x_1|}}\sum_{\sigma\in\{+,-\}} b^\sigma\xi^\sigma(x_1)\,\hat\phi^\sigma(x)\,,\quad x\in\mathbb{R}^2_+\,.$$

Then u satisfies $\Delta u+k^2nu=-f$ in $\mathbb{R}^2_+$ and the boundary condition. To show the radiation condition (2.8), we observe that $(\mathcal{F}u^{rad})(\omega,x_2)$ has the form

$$(\mathcal{F}u^{rad})(\omega,x_2)=(\mathcal{F}u_*^{rad})(\omega,x_2)+\sum_{\sigma\in\{+,-\}}b^\sigma\,\mathcal{F}(\varphi^\sigma e^{\sigma ikx_1})(\omega)+\sum_{\sigma\in\{+,-\}}b^\sigma\,\mathcal{F}(\varphi^\sigma\hat\phi^\sigma_{evan})(\omega,x_2)\,.$$

For $\omega = \ell + \alpha$ and $x_2 > h_0$ we use Theorem 3.2 and obtain

$$\begin{aligned}
(\mathcal{F}u^{rad})(\omega, x_2) \;&=\; (Fu^{rad})_\ell(\alpha, x_2) \;=\; \hat{v}_\ell(x_2,\alpha) \;+\; \sum_{\sigma\in\{+,-\}} b^\sigma\,\mathcal{F}(\varphi^\sigma e^{\sigma i k x_1})(\ell+\alpha) \\
&\quad + \sum_{\sigma\in\{+,-\}} b^\sigma\,\mathcal{F}(\varphi^\sigma \hat{\phi}^\sigma_{evan})(\ell+\alpha, x_2) \\
&= w_\ell(x_2,\alpha) \;+\; \hat{v}_\ell(h_0,\alpha)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)} \;+\; \sum_{\sigma\in\{+,-\}} b^\sigma\,\mathcal{F}(\varphi^\sigma e^{\sigma i k x_1})(\ell+\alpha) \\
&\quad + \sum_{\sigma\in\{+,-\}} b^\sigma\,\mathcal{F}(\varphi^\sigma \hat{\phi}^\sigma_{evan})(\ell+\alpha, x_2)\,.
\end{aligned}$$

The last term tends to zero exponentially as $x_2 \to \infty$. With (3.9) we obtain for $\ell+\alpha \neq \pm k$

$$\lim_{x_2\to\infty}\left[\partial_{x_2}(\mathcal{F}u^{rad})(\ell+\alpha,x_2) - i\,\sqrt{k^2-(\ell+\alpha)^2}\,(\mathcal{F}u^{rad})(\ell+\alpha,x_2)\right] \;=$$

$$\begin{aligned}
&= \frac{i}{\sqrt{k^2-(\ell+\alpha)^2}} \sum_{\sigma\in\{+,-\}} b^\sigma\left[2\sigma i k\mathcal{F}(\varphi^{\sigma\prime} e^{\sigma i k x_1})(\ell+\alpha) + \mathcal{F}(\varphi^{\sigma\prime\prime} e^{\sigma i k x_1})(\ell+\alpha)\right] \\
&\quad -\; i\,\sqrt{k^2-(\ell+\alpha)^2} \sum_{\sigma\in\{+,-\}} b^\sigma\,\mathcal{F}(\varphi^\sigma\, e^{\sigma i k x_1})(\ell+\alpha) \\
&= \frac{i}{\sqrt{k^2-(\ell+\alpha)^2}} \sum_{\sigma\in\{+,-\}} b^\sigma\Bigg[\mathcal{F}\bigg(\Big(\frac{d^2}{dx_1^2}+k^2\Big)(\varphi^\sigma e^{\sigma i k x_1})\bigg)(\ell+\alpha) \\
&\quad -\; [k^2-(\ell+\alpha)^2]\,\mathcal{F}(\varphi^\sigma\, e^{\sigma i k x_1})(\ell+\alpha)\Bigg] \;=\; 0
\end{aligned}$$

for all $\ell \in \mathbb{Z}$ and $\alpha \in [-1/2, 1/2]$ where we used Lemma B.2. This proves the radiation condition (2.8).

(b) We show the decay property (2.9). Let $\psi_j \in C^\infty(\mathbb{R})$ for $j = -1, 0, 1$ be a partition of unity of the open covering $(-\kappa-\delta, -\kappa+\delta)\cup(\kappa-\delta,\kappa+\delta)\cup\big((-1/2-\delta, 1/2+\delta)\setminus\{\kappa,-\kappa\}\big)$ of $[-1/2.1/2]$, i.e. $\operatorname{supp}\psi_{\pm1} \subset (\pm\kappa-\delta, \pm\kappa+\delta)$ and $\operatorname{supp}\psi_0 \subset \big((-1/2-\delta, 1/2+\delta)\setminus\{\kappa,-\kappa\}\big)$ and $\sum_{j=-1}^{1}\psi_j(\alpha) = 1$ for $\alpha \in [-1/2, 1/2]$. Then, using (3.31)

$$\begin{aligned}
u_*^{rad}(x_1+2\pi\ell, x_2) \;&=\; \sum_{j=-1}^{1}\int_{-1/2}^{1/2}\psi_j(\alpha)\,\hat{v}(x,\alpha)\,e^{i2\pi\ell\alpha}\,d\alpha \;=\; \int_{-1/2}^{1/2} v_0(x,\alpha)\,e^{i2\pi\ell\alpha}\,d\alpha \\
&\quad + \int_{-1/2}^{1/2}\sqrt{\kappa-\alpha}\,v_+(x,\alpha)\,e^{i2\pi\ell\alpha}\,d\alpha \;+\; \int_{-1/2}^{1/2}\sqrt{\kappa+\alpha}\,v_-(x,\alpha)\,e^{i2\pi\ell\alpha}\,d\alpha
\end{aligned}$$

where $v_0(\cdot,\alpha) = \psi_0(\alpha)\hat{v}(\cdot,\alpha)+\psi_{+1}(\alpha)\hat{v}_{1,+}(\cdot,\alpha)+\psi_{-1}(\alpha)\hat{v}_{1,-}(\cdot,\alpha)$ and $v_\pm(\cdot,\alpha) = \psi_{\pm1}(\alpha)\hat{v}_{2,\pm}(\cdot,\alpha)$ depend smoothly on α. We note that $\operatorname{supp} v_\pm \subset (-1/2, 1/2)$ and $v_0 = \hat{v}$ near $\pm1/2$. Partial integration of the integral containing v_0 yields $\int_{-1/2}^{1/2} v_0(x,\alpha)\,e^{i2\pi\ell\alpha}\,d\alpha = \mathcal{O}(1/|\ell|^j)$ for any $j \in \mathbb{N}$ since v_0 is $1-$periodic with respect to α. For the integral containing $v_\pm$ we can also perform partial integration one times and obtain (for v_+)

$$\int\limits_{-1/2}^{1/2} \sqrt{\kappa-\alpha}\, v_+(x,\alpha)\, e^{i2\pi\ell\alpha}\, d\alpha \;=\; -\frac{1}{i2\pi\ell} \int\limits_{-1/2}^{1/2} \frac{d}{d\alpha}\big[\sqrt{\kappa-\alpha}\, v_+(x,\alpha)\big]\, e^{i2\pi\ell\alpha}\, d\alpha$$

$$= \frac{1}{i4\pi\ell} \int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}}\, v_+(x,\alpha)\, e^{i2\pi\ell\alpha}\, d\alpha \;-\; \frac{1}{i2\pi\ell} \int\limits_{-1/2}^{1/2} \sqrt{\kappa-\alpha}\, \partial_\alpha v_+(x,\alpha)\, e^{i2\pi\ell\alpha}\, d\alpha\,.$$

The second integral behaves as $\mathcal{O}(1/|\ell|)$ (partial integration again). The first integral behaves as $\mathcal{O}(1/\sqrt{|\ell|}$. Indeed, we write

$$\int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}}\, v_+(x,\alpha)\, e^{i2\pi\ell\alpha}\, d\alpha \;=$$

$$= \int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}} \big[v_+(x,\alpha) - v_+(x,\kappa)\big]\, e^{i2\pi\ell\alpha}\, d\alpha \;+\; v_+(x,\kappa) \int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}}\, e^{i2\pi\ell\alpha}\, d\alpha$$

$$= -\int\limits_{\kappa-\delta}^{\kappa+\delta} \sqrt{\kappa-\alpha}\, \tilde{v}(\alpha)\, e^{i2\pi\ell\alpha}\, d\alpha \;+\; v_+(x,\kappa) \int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}}\, e^{i2\pi\ell\alpha}\, d\alpha$$

since $v_+(x,\alpha) - v_+(x,\kappa) = \int_0^1 \frac{d}{dt} v_+\big(x, t\alpha + (1-t)\kappa\big)\, dt = (\alpha-\kappa) \int_0^1 \partial_\alpha v_+\big(x, t\alpha + (1-t)\kappa\big)\, dt = (\alpha-\kappa)\, \tilde{v}(\alpha)$. The first integral behaves again as $\mathcal{O}(1/|\ell|)$ by partial integration. For the second integral we write

$$\int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}}\, e^{i2\pi\ell\alpha}\, d\alpha \;=\; e^{i2\pi\ell\kappa} \int\limits_{-\delta}^{\delta} \frac{1}{\sqrt{\beta}}\, e^{-i2\pi\ell\beta}\, d\beta \;=\; \frac{e^{i2\pi\ell\kappa}}{\sqrt{2\pi|\ell|}} \int\limits_{-2\pi|\ell|\delta}^{2\pi|\ell|\delta} \frac{1}{\sqrt{t}}\, e^{-it\,\mathrm{sign}\,\ell}\, dt\,.$$

Since the sequence $a_\ell = \int_{-2\pi|\ell|\delta}^{2\pi|\ell|\delta} \frac{1}{\sqrt{t}}\, e^{-it\,\mathrm{sign}\,\ell}\, dt$ is bounded, this integral behaves as $\mathcal{O}(1/\sqrt{|\ell|}$.

(c) Since $u^{prop}(x)$ decays exponentially it is sufficient to study u^{rad}. First, we show a representation of u^{rad} in the half plane $x_2 > h_0$. First we note that u^{rad} satisfies $(\Delta + k^2 n)u^{rad} = -\tilde{g}$ for $x_2 > h_0$ where $\tilde{g}$ is given by

$$\tilde{g}(x) \;:=\; \sum_{\sigma\in\{+,-\}} \sum_{j\in J} \sum_{\sigma\lambda_{\ell,j}>0} a_{\ell,j} (\Delta + k^2 n)\big[\xi^\sigma(x_1)\phi^{\ell,j}(x)\big]\,.$$

We recall that $\tilde{g}(x)$ vanishes for $|x_1| > R$ and decays exponentially as $x_2 \to \infty$. We claim that u^{rad} has the representation

$$\begin{aligned} u^{rad}(x) &= \frac{i}{4} \int\limits_{\mathbb{R}^2_{x_2>h_0}} \tilde{g}(y) \left[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\right] dy \\ &\quad + \frac{i}{2} \int\limits_{\gamma_{h_0}} u^{rad}(y)\, \partial_{y_2} H_0^{(1)}(k|x-y|)\, ds(y) \\ &= \frac{i}{4} \int\limits_{\mathbb{R}^2_{x_2>h_0}} \tilde{g}(y) \left[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\right] dy \\ &\quad + \frac{i}{2} \int\limits_{\gamma_{h_0}} u_*^{rad}(y)\, \partial_{y_2} H_0^{(1)}(k|x-y|)\, ds(y) \\ &\quad + \sum_{\sigma\in\{+,-\}} b^\sigma \frac{i}{2} \int\limits_{\gamma_{h_0}} \varphi^\sigma(y_1)\hat{\phi}^\sigma(y)\, \partial_{y_2} H_0^{(1)}(k|x-y|)\, ds(y) \end{aligned} \tag{3.32}$$

where $y^* = (y_1, 2h_0 - y_2)^\top$ and $\gamma_{h_0} = \mathbb{R} \times \{h_0\}$. Indeed, it is well known (see, e.g. Lemma 7.1 of [Kir22] for $h = h_0$) that the volume potential satisfies the differential equation and the radiation condition (2.8) and vanishes on γ_h. The line integrals are of convolution type with the $L^1(\mathbb{R})-$function $x_1 \mapsto \Psi(x_1, x_2) := \frac{i}{2}\partial_{y_2} H_0^{(1)}(k\sqrt{x_1^2 + (x_2-y_2)^2})|_{y_2=h_0}$. The Fourier transform of Ψ is given by $(\mathcal{F}\Psi)(\omega, x_2) = e^{i\sqrt{k^2-\omega^2 1}(x_2-h_0)}$ (see formulas 3. and 4. in [GR07], Section 6.677), and satisfies (2.8) trivially. We can now use the convolution theorem for the Fourier transform (for $\varphi^\sigma\hat{\phi}^\sigma$ we refer to Lemma B.2 of Appendix B) which shows that the double potetial with density u^{rad} satisfies the homogeneous Helmholtz equation and the radiation condition (2.8). The jump condition yields that the line integral coincides with u^{rad} on γ_{h_0}. Therefore, the right hand side of (3.32) satisfies the same differential equation as u^{rad} and the radiation condition and coincides with u^{rad} on γ_{h_0}. Therefore, it has to coincide with u^{rad} everywhere, and (3.32) is shown.

For the volume integral and the line integral with density u_*^{rad} (which decays as $\mathcal{O}(1/|x_1|^{3/2})$ by part (b)) we can use the results of [Kir22] (see Lemma 7.1 and proof of Theorem 6.2). These two terms decay as $1/\sqrt{x_2}$ as $x_2 \to \infty$. The line integral with the density $\varphi^\sigma\hat{\phi}^\sigma$ which decays only of order $1/\sqrt{|x_1|}$ along γ_{h_0} is investigated in Lemma B.3 of Appendix B (replace $x_2 - h_0$ by x_2). This ends the proof of Theorem 2.10.

Appendix A. Proof of Theorem 3.1

A.1. **Preparations.** We recall that $\varphi^\sigma(x_1) = \xi^\sigma(x_1)/\sqrt{|x_1|}$. From (3.4) we observe that for $x_2 > h_0$ the right hand side g has a decomposition in the form $g(x) = g^{(1)}(x_1) + g^{(2)}(x)$

where

$$
\begin{aligned}
g^{(1)}(x_1) &:= \sum_{\sigma\in\{+,-\}} b^\sigma \left(\frac{d^2}{dx_1^2}+k^2\right)\big[\varphi^\sigma(x_1)\, e^{\sigma ikx_1}\big] \\
&= \sum_{\sigma\in\{+,-\}} b^\sigma \left[2\sigma ik\,\varphi^{\sigma\prime}(x_1)+\varphi^{\sigma\prime\prime}(x_1)\right] e^{\sigma ikx_1}\,, \\
g^{(2)}(x) &:= \sum_{\sigma\in\{+,-\}} b^\sigma\,(\Delta+k^2n)\big[\varphi^\sigma(x_1)\,\hat\phi^\sigma_{evan}(x)\big] \\
&\quad + \sum_{\sigma\in\{+,-\}}\sum_{j\in J}\sum_{\sigma\lambda_{\ell,j}>0} a_{\ell,j}(\Delta+k^2n)\big[\xi^\sigma(x_1)\phi^{\ell,j}(x)\big] \\
&= \sum_{\sigma\in\{+,-\}} b^\sigma\,\big[(\varphi^\sigma)'(x_1)\,\partial_{x_1}\hat\phi^\sigma(x)+\partial_{x_1}\big((\varphi^\sigma)'(x_1)\hat\phi^\sigma(x)\big)\big] \\
&\quad + \sum_{\sigma\in\{+,-\}}\sum_{j\in J}\sum_{\sigma\lambda_{\ell,j}>0} a_{\ell,j}\big[(\xi^\sigma)'(x_1)\,\partial_{x_1}\phi^{\ell,j}(x)+\partial_{x_1}\big((\xi^\sigma)'(x_1)\phi^{\ell,j}(x)\big)\big]\,.
\end{aligned}
$$

For the Floquet-Bloch transform we use (3.1) and the representation of Lemma B.1 (part (b)) to obtain

$$
\begin{aligned}
(Fg^{(1)})(x_1,\alpha) &= \frac{e^{i\pi/4}k}{\sqrt{\pi}}\sum_{\sigma\in\{+,-\}} b^\sigma\,\sqrt{\kappa-\sigma\alpha}\,\Big[1+\frac{\sigma}{2k}\,(\alpha-\sigma\kappa)\Big]\,e^{i(\alpha+\sigma\tilde\ell)x_1} \\
&\quad + \sum_{\sigma\in\{+,-\}} b^\sigma\,\big[2\,\sigma ik\,A^\sigma(x_1,\alpha-\sigma\kappa)+\partial_{x_1}A^\sigma(x_1,\alpha-\sigma\kappa)\big]e^{\sigma ikx_1}
\end{aligned}
$$

and

$$
\begin{aligned}
(Fg^{(2)})(x,\alpha) &= \frac{e^{-i\pi/4}}{2\sqrt{\pi}}\sum_{\sigma\in\{+,-\}} b^\sigma\,\sigma\,\sqrt{\kappa-\sigma\alpha}\,e^{i(\alpha-\sigma\kappa)x_1}\big[2\,\partial_{x_1}\hat\phi^\sigma_{evan}(x)+i(\alpha-\sigma\kappa)\,\hat\phi^\sigma_{evan}(x)\big] \\
&\quad + \sum_{\sigma\in\{+,-\}} b^\sigma\,\big[2\,A^\sigma(x_1,\alpha-\sigma\kappa)\,\partial_{x_1}\hat\phi^\sigma_{evan}(x)+\partial_{x_1}A^\sigma(x_1,\alpha-\sigma\kappa)\,\hat\phi^\sigma_{evan}(x)\big] \\
&\quad + \sum_{\sigma\in\{+,-\}}\sum_{j\in J}\sum_{\sigma\lambda_{\ell,j}>0} a_{\ell,j}\big[2(F\xi^{\sigma\prime})(x_1,\alpha-\hat\alpha_j)\,\partial_{x_1}\phi^{\ell,j}(x) \\
&\qquad + \; F(\xi^{\sigma\prime\prime})(x_1,\alpha-\hat\alpha_j)\,\phi^{\ell,j}(x)\big]
\end{aligned}
$$

with smooth functions $A^\pm$. We solve (3.8) by expanding Fg and w into Fourier series of the form $(Fg)(x,\alpha)=\sum_{\ell\in\mathbb{Z}}\hat g_\ell(x_2,\alpha)\,e^{i(\ell+\alpha)x_1}$ and $w(x,\alpha)=\sum_{\ell\in\mathbb{Z}}w_\ell(x_2,\alpha)\,e^{i(\ell+\alpha)x_1}$, respectively. The splitting $Fg=Fg^{(1)}+Fg^{(2)}$ implies a splitting of the Fourier coefficients

$\hat{g}_\ell(x_2,\alpha)$ into $\hat{g}_\ell(x_2,\alpha) = \hat{g}_\ell^{(1)}(\alpha) + \hat{g}_\ell^{(2)}(x_2,\alpha)$ where

$$\begin{aligned} \hat{g}^{(1)}_{\pm\tilde{\ell}}(\alpha) \;&=\; \frac{e^{i\pi/4}k}{\sqrt{\pi}}\, b^\pm \sqrt{\kappa \mp \alpha}\,\big[1 \pm \frac{1}{2k}\,(\alpha\mp\kappa)\big] \\ &\quad + \frac{1}{2\pi} \sum_{\sigma\in\{+,-\}} b^\sigma \int\limits_0^{2\pi} \big[2\sigma ik\, A^\sigma(x_1,\alpha-\sigma\kappa) + \partial_{x_1} A^\sigma(x_1,\alpha-\sigma\kappa)\big]\, e^{\sigma ikx_1} e^{-i(\alpha\pm\tilde{\ell})x_1}\, dx_1\,, \\ \hat{g}^{(1)}_{\ell}(\alpha) \;&=\; \frac{1}{2\pi} \sum_{\sigma\in\{+,-\}} b^\sigma \int\limits_0^{2\pi} \big[2\,\sigma ik\, A^\sigma(x_1,\alpha-\sigma\kappa) + \partial_{x_1} A^\sigma(x_1,\alpha-\sigma\kappa)\big]\, e^{\sigma ikx_1} e^{-i(\alpha+\ell)x_1}\, dx_1 \end{aligned}$$

for $\ell \neq \pm\tilde{\ell}$, and the analogous form for $\hat{g}_\ell^{(2)}(x_2,\alpha)$ which we decompose as

$$\begin{aligned} \hat{g}^{(2)}_{\ell}(x_2,\alpha) \;&=\; \frac{e^{-i\pi/4}}{4\pi\sqrt{\pi}} \sum_{\sigma\in\{+,-\}} b^\sigma\, \sigma\, \sqrt{\kappa-\sigma\alpha} \int\limits_0^{2\pi} \big[2\,\partial_{x_1}\hat{\phi}^\sigma_{evan}(x) + i(\alpha-\sigma\kappa)\,\hat{\phi}^\sigma_{evan}(x)\big]\cdot \\ &\qquad \cdot\; e^{-i(\sigma\kappa+\ell)x_1}\, dx_1 \;+\; G^{(2)}_\ell(x_2,\alpha) \end{aligned} \tag{A.1}$$

where $G_\ell^{(2)} \in C^\infty\big((h_0,\infty)\times[-1/2,1/2]\big)$ because of the smoothness of $A^\pm$ and $F\xi^\pm$. We note that $\hat{g}_\ell^{(1)}(\alpha)$ is independent of x_2 and all terms of $\hat{g}_\ell^{(2)}(x_2,\alpha)$ decay to zero exponentially as $x_2\to\infty$. Furthermore, we note that the Fourier coefficients tend to zero as $|\ell|\to\infty$ because of the smoothness of $(Fg)(\cdot,\alpha)$.

Lemma A.1. *(a) $\hat{g}_\ell^{(1)} \in C^\infty[-1/2,1/2]$ for $\ell\neq\pm\tilde{\ell}$. For every $m\in\mathbb{N}$ there exists $c_m>0$ with $|\hat{g}_\ell^{(1)}(\alpha)| \le \frac{c_m}{|\ell|^m+1}$ for all $\ell\in\mathbb{Z}$ and $\alpha\in[-1/2,1/2]$.*

(b) $\hat{g}^{(1)}_{\pm\tilde{\ell}}(\alpha)$ and $\hat{g}_\ell^{(2)}(x_2,\alpha)$ have the forms

$$\hat{g}^{(1)}_{\pm\tilde{\ell}}(\alpha) \;=\; \frac{e^{i\pi/4}k}{\sqrt{\pi}}\, b^\pm \sqrt{\kappa\mp\alpha}\,\big[1\pm\frac{1}{2k}\,(\alpha\mp\kappa)\big] \;+\; (\alpha\mp\kappa)\, G^{(1)}_\pm(\alpha)\,, \tag{A.2}$$

$$\hat{g}^{(2)}_{\ell}(x_2,\alpha) \;=\; \sqrt{\kappa-\alpha}\, G^{(2,+)}_\ell(x_2,\alpha) \;+\; \sqrt{\kappa+\alpha}\, G^{(2,-)}_\ell(x_2,\alpha) \;+\; G^{(2)}_\ell(x_2,\alpha)\,, \tag{A.3}$$

respectively, where $G_\pm^{(1)} \in C^\infty[-1/2,1/2]$ and $G_\ell^{(2)}, G_\ell^{(2,\pm)} \in C^\infty\big((h_0,\infty)\times[-1/2,1/2]\big)$. Furthermore, there exists $\eta>0$ and for every $m\in\mathbb{N}$ some $c_m>0$ with $|\hat{g}_\ell^{(2)}(x_2,\alpha)| \le \frac{c_m}{|\ell|^m+1} e^{-\eta x_2}$ for all $\ell\in\mathbb{Z}$ and $(x_2,\alpha)\in(h_0,\infty)\times[-1/2,1/2]$.

Proof. (a) This follows from the smoothness of $A^\pm$.

(b) We write $\hat{g}^{(1)}_{+\tilde{\ell}}(\alpha)$ as

$$\begin{aligned}\hat{g}^{(1)}_{\tilde{\ell}}(\alpha) &= \frac{e^{i\pi/4}k}{\sqrt{\pi}}\, b^+ \sqrt{\kappa-\alpha}\,\Big[1+\frac{1}{2k}\,(\alpha-\kappa)\Big] \\ &+ \frac{1}{2\pi} b^+ \int_0^{2\pi} \big[2ik\, A^+(x_1,\alpha-\kappa) + \partial_{x_1} A^+(x_1,\alpha-\kappa)\big]\, e^{i(\kappa\mp\alpha)x_1}\, dx_1 \\ &+ \frac{1}{2\pi} b^- \int_0^{2\pi} \big[\partial_{x_1}\big(A^-(x_1,\alpha+\kappa)\, e^{-2ikx_1}\big)\, e^{i(\kappa-\alpha)x_1}\, dx_1 \,.\end{aligned}$$

The integrals are smooth with respect to α since $A^\pm$ depends smoothly on x_1 and α. Furthermore, since $\int_0^{2\pi} A^+(x_1,0)dx_1 = 0$ and $A^+(\cdot,0)$ is $2\pi-$periodic and also $A^-(x_1,2\kappa)\, e^{-2ikx_1}$ is $2\pi-$periodic we conclude that both integrals vanish for $\alpha=\kappa$, i.e. $\hat{g}^{(1)}_{\tilde{\ell}}(\alpha)$ has the form (A.2). For $\hat{g}^{(1)}_{-\tilde{\ell}}(\alpha)$ one argues in the same way.

The representation (A.3) and the estimates for $|\hat{g}^{(2)}_{\ell}(x_2,\alpha)|$ follow directly from (A.1), the smoothness of $\hat{\phi}^\pm$ and its exponential decay. □

The coefficients $w_\ell(x_2,\alpha)$ have to solve

$$w''_\ell(x_2,\alpha) + [k^2-(\ell+\alpha)^2]\, w_\ell(x_2,\alpha) = -\hat{g}_\ell(x_2,\alpha) \text{ for } x_2>h_0\,, \quad w_\ell(h_0,\alpha)=0\,. \tag{A.4}$$

Therefore, if $\ell+\alpha \neq \pm k$ the solution $w_\ell(x_2,\alpha)$ of (A.4) is given by $w_\ell(x_2,\alpha) = w^{(1)}_\ell(x_2,\alpha) + w^{(2)}_\ell(x_2,\alpha)$ where

$$\begin{aligned} w^{(1)}_\ell(x_2,\alpha) &= -\frac{\hat{g}^{(1)}_\ell(\alpha)}{k^2-(\ell+\alpha)^2}\Big[1-e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)}\Big] \\ &= \frac{\hat{g}^{(1)}_\ell(\alpha)}{\sqrt{k^2-(\ell+\alpha)^2}}\,\frac{e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)}-1}{\sqrt{k^2-(\ell+\alpha)^2}}\,, \quad x_2>h_0\,, \qquad (A.5)\\ w^{(2)}_\ell(x_2,\alpha) &= \frac{i}{2}\int_{h_0}^{\infty} \hat{g}^{(2)}_\ell(y_2,\alpha)\, \frac{e^{i\sqrt{k^2-(\ell+\alpha)^2}|x_2-y_2|} - e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2+y_2-2h_0)}}{\sqrt{k^2-(\ell+\alpha)^2}}\, dy_2,\ \ x_2>h_0\,.\end{aligned}$$

They depend smoothly on x_2. We investigate the dependence on α and show

Lemma A.2. *(a)* $w^{(1)}_\ell \in C^\infty\big((h_0,\infty)\times[-1/2,1/2]\big)$ *for all* $\ell \neq \pm\tilde{\ell}$.
(b) For $\ell = \pm\tilde{\ell}$ *we have*

$$\begin{aligned} w^{(1)}_{\pm\tilde{\ell}}(x_2,\alpha) &= w^{(1)}_{\pm\tilde{\ell}}(x_2,\pm\kappa) \\ &+ (x_2-h_0)\big[\sqrt{\kappa\mp\alpha}\, W^{(1,1)}_\pm(x_2,\alpha) + (\kappa\mp\alpha)\, W^{(1,2)}_\pm(x_2,\alpha)\big]\end{aligned} \tag{A.6}$$

where

$$w^{(1)}_{\pm\tilde{\ell}}(x_2,\pm\kappa) := -\sqrt{\frac{k}{2\pi}}\, b^\pm\, e^{-i\pi/4}\,(x_2-h_0) \tag{A.7}$$

and $W^{(1,j)}_\pm \in C^\infty\big((h_0,\infty)\times[-1/2,1/2]\big)$ *are bounded.*

(c) For $\ell \in \mathbb{Z}$ we have

$$\begin{aligned} w_\ell^{(2)}(x_2,\alpha) \ &= W_\ell^{(2)}(x_2,\alpha) \ + \ \sqrt{\kappa-\alpha}\, W_\ell^{(2,+)}(x_2,\alpha) \\ &\quad + \ \sqrt{\kappa+\alpha}\, W_\ell^{(2,-)}(x_2,\alpha) \ + \ \sqrt{\kappa-\alpha}\,\sqrt{\kappa+\alpha}\, W_\ell^{(2,0)}(x_2,\alpha) \end{aligned} \tag{A.8}$$

where $W_\ell^{(2)}, W_\ell^{(2,\pm)}, W_\ell^{(2,0)} \in C^\infty\big((h_0,\infty)\times[-1/2,1/2]\big)$.

(d) $w_\ell(x_2,\alpha) = w_\ell^{(1)}(x_2,\alpha) + w_\ell^{(2)}(x_2,\alpha)$ satisfies the radiation conditions (3.9), (3.10) for all $\ell \in \mathbb{Z}$ and $\alpha \in [-1/2,1/2]$.

(e) For every $m \in \mathbb{N}$ there exists $c_m > 0$ with $|w_\ell(x_2,\alpha)| + |\partial_{x_2} w_\ell(x_2,\alpha)| \le \frac{c_m}{|\ell|^m}$ for all $|\ell| > \tilde\ell$ and $(x_2,\alpha) \in (h_0,\infty)\times[-1/2,1/2]$.

Proof. (a) This is obvious.

(b) If $\ell = \pm\tilde\ell$ we use the representation (A.2) and obtain

$$\begin{aligned} \frac{\hat g^{(1)}_{\pm\tilde\ell}(\alpha)}{\sqrt{k^2-(\pm\tilde\ell+\alpha)^2}} \ &= \frac{\hat g^{(1)}_{\pm\tilde\ell}(\alpha)}{\sqrt{\kappa\mp\alpha}\sqrt{k+\tilde\ell\pm\alpha}} \\ &= \frac{e^{i\pi/4}k}{\sqrt{\pi}\,\sqrt{k+\tilde\ell\pm\alpha}}\, b^\pm\big[1 \pm \frac{1}{2k}\,(\alpha\mp\kappa)\big] \ - \ \sqrt{\kappa\mp\alpha}\, G^{(1)}_\pm(\alpha) \end{aligned}$$

and $\lim_{\alpha\to\pm\kappa} \frac{\hat g^{(1)}_{\pm\tilde\ell}(\alpha)}{\sqrt{k^2-(\pm\tilde\ell+\alpha)^2}} = \sqrt{\frac{k}{2\pi}}\, e^{i\pi/4}\, b^\pm$. Since $(e^{iz}-1)/z$ is analytic in $\mathbb{C}$ we obtain that the second term in (A.5) has the form

$$\frac{e^{i\sqrt{k^2-(\pm\tilde\ell+\alpha)^2}(x_2-h_0)}-1}{\sqrt{k^2-(\pm\tilde\ell+\alpha)^2}} \ = \ i(x_2-h_0)\,\big[1 + \ \sqrt{\kappa\mp\alpha}\, B^{(1)}_\pm(x_2,\alpha) \ + \ (\kappa\mp\alpha)\, B^{(2)}_\pm(x_2,\alpha)\big]$$

for some smooth (even analytic) functions $B^{(j)}_\pm$ which are bounded. Substituting this into the form (A.5) of $w^{(1)}_{\pm\tilde\ell}(x_2,\alpha)$ we obtain the representation (A.6).

(c) For $w^{(2)}_\ell(x_2,\alpha)$ we use (A.3). If $\ell \ne \pm\tilde\ell$ the term $\frac{e^{i\sqrt{k^2-(\ell+\alpha)^2}|x_2-y_2|}-e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2+y_2-2h_0)}}{\sqrt{k^2-(\ell+\alpha)^2}}$ is smooth (and bounded), and we have the decomposition into

$$w_\ell^{(2)}(x_2,\alpha) \ = \ \sqrt{\kappa-\alpha}\, W_\ell^{(2,+)}(x_2,\alpha) \ + \ \sqrt{\kappa+\alpha}\, W_\ell^{(2,-)}(x_2,\alpha) \ + \ W_\ell^{(2)}(x_2,\alpha)$$

where $W_\ell^{(2)}, W_\ell^{(2,\pm)} \in C^\infty\big((h_0,\infty)\times[-1/2,1/2]\big)$ are bounded.
If $\ell = \pm\tilde\ell$ we have, in addition, to use

$$\begin{aligned} \frac{e^{i\sqrt{k^2-(\pm\tilde\ell+\alpha)^2}|x_2-y_2|} - e^{i\sqrt{k^2-(\pm\tilde\ell+\alpha)^2}(x_2+y_2-2h_0)}}{\sqrt{k^2-(\pm\tilde\ell+\alpha)^2}} \ &= \ i|x_2-h_0| - i(x_2+y_2-2h_0) \\ + \ \sqrt{\kappa\mp\alpha}\, B^{(3)}_\pm(x_2,\alpha) \ + \ (\kappa\mp\alpha)\, B^{(4)}_\pm(x_2,\alpha) \end{aligned}$$

with smooth functions $B_\pm^{(3)}$ and $B_\pm^{(4)}$ and obtain (A.8). For $\ell+\alpha=\pm k$ we obtain from (A.5)

$$w^{(2)}_{\pm\tilde\ell}(x_2,\pm\kappa) \;=\; i\int\limits_{h_0}^{\infty} \hat g^{(2)}_{\pm\tilde\ell}(y_2,\pm\kappa)\,\big[|x_2-y_2|-(x_2+y_2-2h_0)\big]\,dy_2\,. \tag{A.9}$$

(d) Let $\ell+\alpha\neq\pm k$. We look at the form (A.5) for $w_\ell^{(1)}$ and $w_\ell^{(2)}$. For $w_\ell^{(1)}$ we obtain

$$\partial_{x_2}w_\ell^{(1)}(x_2,\alpha)-i\sqrt{k^2-(\ell+\alpha)^2}\,w_\ell^{(1)}(x_2,\alpha) \;=\; \frac{i\hat g_\ell^{(1)}(\alpha)}{\sqrt{k^2-(\ell+\alpha)^2}} \;=\; \frac{i(\mathcal{F}g^{(1)})(\ell+\alpha)}{\sqrt{k^2-(\ell+\alpha)^2}}$$

$$=\;\frac{i}{\sqrt{k^2-(\ell+\alpha)^2}}\sum_{\sigma\in\{+,-\}} b^\sigma\,\big[2\sigma ik\mathcal{F}(\varphi^{\sigma\prime}e^{\sigma ikx_1})(\ell+\alpha)+\mathcal{F}(\varphi^{\sigma\prime\prime}e^{\sigma ikx_1})(\ell+\alpha)\big]\,.$$

For $w_\ell^{(2)}$ the decomposition of the interval (h_0,∞) of integration into $(h_0,x_2)\cup(x_2,\infty)$ yields after a short computation

$$\partial_{x_2}w_\ell^{(2)}(x_2,\alpha)-i\sqrt{k^2-(\ell+\alpha)^2}w_\ell^{(2)}(x_2,\alpha) \;=\; \int\limits_{x_2}^{\infty}\hat g_\ell^{(2)}(y_2,\alpha)\,e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-x_2)}dy_2\,,$$

and this tends to zero as $x_2\to\infty$ by the exponential decay of $\hat\phi^\sigma_{evan}$ and $\phi^{\ell,j}$. Therefore, the radiation condition (3.9) is shown for $\ell+\alpha\neq\pm k$.

For $\ell+\alpha=\pm k$, i.e. $\ell=\pm\tilde\ell$ and $\alpha=\pm\kappa$, from (A.7) and (A.9) we obtain after the same computation

$$\partial_{x_2}w_{\pm\tilde\ell}(x_2,\pm\kappa) \;=\; -\sqrt{\frac{k}{2\pi}}\,b^\pm\,e^{-i\pi/4}\;+\int\limits_{x_2}^{\infty}\hat g^{(2)}_{\pm\tilde\ell}(y_2,\pm\kappa)\,dy_2\,, \tag{A.10}$$

and this converges to $-\sqrt{\frac{k}{2\pi}}\,b^\pm\,e^{-i\pi/4}$ as $x_2\to\infty$. Therefore, $w_{\pm\tilde\ell}(x_2,\pm\kappa)$ satisfies the radiation condition (3.10).

(e) This follows directly from (A.5) and the estimates for $|\hat g_\ell^{(1)}(\alpha)|$ and $|\hat g_\ell^{(2)}(y_2,\alpha)|$ from Lemma A.1. □

A.2. **Proof of Theorem 3.1.** After these preparations we turn to the proof of Theorem 3.1. The Fourier series with the coefficients $w_\ell^{(1)}(x_2,\alpha)$ and $w_\ell^{(2)}(x_2,\alpha)$ from Lemma A.2 yields the form (3.11) of $w(x,\alpha)$. The terms $W_0(x,\alpha)$, $W_\pm(x,\alpha)$, and $\tilde W(x,\alpha)$ consists of the series of the corresponding Fourier coefficients when we separate terms involving $\sqrt{\kappa-\alpha}$, $\sqrt{\kappa+\alpha}$, or $\sqrt{\kappa-\alpha}\sqrt{\kappa+\alpha}$, e.g.,

$$\begin{aligned}W_0(x,\alpha) \;=\; &\sum_{\ell\neq\pm\tilde\ell}\big[w_\ell^{(1)}(x_2,\alpha)+W_\ell^{(2)}(x_2,\alpha)\big]\,e^{i(\ell+\alpha)x_1}\\ &+\sum_{\sigma\in\{+,-\}}\big[w^{(1)}_{\sigma\tilde\ell}(x_2,\sigma\kappa)+W^{(2)}_{\sigma\tilde\ell}(x_2,\alpha)\big]\,e^{i(\sigma\tilde\ell+\alpha)x_1}\\ &+\;(\kappa-\alpha)\,W_+^{(1,2)}(x_2,\alpha)\,e^{i(\tilde\ell+\alpha)x_1}\;+\;(\kappa+\alpha)\,W_-^{(1,2)}(x_2,\alpha)\,e^{i(-\tilde\ell+\alpha)x_1}\,.\end{aligned}$$

All coefficients are smooth with respect to x and α. Also, by part (e) of Lemma A.2 the coefficients tend to zero of every order as $\ell\to\pm\infty$, uniformly with respect to $(x,\alpha)\in$

$[0,2\pi]\times[h_0,H]\times[-1/2,1/2]$ for every $H>h_0$. This shows that $W_0(x,\alpha)$, $W_\pm(x,\alpha)$, and $\tilde{W}(x,\alpha)$ are smooth.

We finally show that there exists $c>0$ with $\|w(\cdot,\alpha)\|_{L^2(\Gamma_H)}\le cH$ and $\|\partial_{x_2}w(\cdot,\alpha)\|_{L^2(\Gamma_H)}\le c$ for all $H>h_0$ and $\alpha\in[-1/2,1/2]$.

For $|\ell|>\tilde{\ell}$ we have $\big|w_\ell(x_2,\alpha)\big|+\big|\partial_{x_2}w_\ell(x_2,\alpha)\big|\le\frac{c}{\ell^2}$ by part (e) of Lemma A.2 while for $\ell=\pm\tilde{\ell}$ we have $\frac{1}{x_2}\big|w_{\pm\tilde{\ell}}(x_2,\alpha)\big|+\big|\partial_{x_2}w_{\pm\tilde{\ell}}(x_2,\alpha)\big|\le c$ by part (b) and (c) of Lemma A.2. Therefore,

$$\begin{aligned}\|w(\cdot,x_2,\alpha)\|^2_{L^2(0,2\pi)} &= 2\pi\sum_{\ell\in\mathbb{Z}}\big|w_\ell(x_2,\alpha)\big|^2\\ &\le 2\pi\sum_{|\ell|<\tilde{\ell}}\big|w_\ell(x_2,\alpha)\big|^2\;+\;c\sum_{|\ell|>\tilde{\ell}}\frac{1}{\ell^2}\;+\;c\,x_2^2\;\le\;c'\,x_2^2\,.\end{aligned}$$

For the derivative we argue in the same way. This ends the proof of Theorem 3.1.

Appendix B. The Fourier Transform of some Particular Functions

Lemma B.1. *Let $\xi^+\in C^\infty(\mathbb{R})$ with $\xi^+(x_1)=1$ for $x_1>R_0$ (for some $R_0>2\pi$) and $\xi(x_1)=0$ for $x_1\le R_0-1$. Set $\xi^-(x_1)=\xi^+(-x_1)$ for $x_1\in\mathbb{R}$. Set $\varphi^\sigma(x_1)=\xi^\sigma(x_1)/\sqrt{|x_1|}$ for $\sigma\in\{+,-\}$. Then we have:*

(a) $$\int_{-\infty}^{\infty}(\varphi^\sigma)'(x_1)e^{-i\beta x_1}\,dx_1\;=\;\sigma\sqrt{-\sigma\pi\beta}\,e^{-i\pi/4}\;+\;\beta\,R^\sigma(\beta)\quad\text{for }\beta\in\mathbb{R}\,,$$

where $R^\sigma\in C^\infty(\mathbb{R})$.

(b) Set $R:=\big\{(x_1,\beta)\in[0,2\pi]\times(-1,1):\beta\neq0\big\}$. Then $F(\varphi^{\sigma\prime})\in C^\infty(R)$ and

$$F(\varphi^{\sigma\prime})(x_1,\beta)\;=\;\sigma\frac{e^{-i\pi/4}}{2\sqrt{\pi}}\,\sqrt{-\sigma\beta}\;e^{i\beta x_1}\;+\;A^\sigma(x_1,\beta)\quad\text{for }(x_1,\beta)\in[0,2\pi]\times(-1,1)$$

where $A^\pm\in C^\infty(\mathbb{R}\times\mathbb{R})$ is $\beta-$quasi-periodic with respect to x_1 and satisfies $\int_0^{2\pi}A^\pm(x_1,0)\,dx_1=0$.

Proof. (a) We consider $\sigma=+$ and φ for φ^+. For any $R>R_0$ we have by partial integration

$$\begin{aligned}\int_{-\infty}^{R}\varphi'(x_1)\,e^{-i\beta x_1}\,dx_1 &= \varphi(R)\,e^{-i\beta R}\;+\;i\beta\int_{-\infty}^{R}\varphi(x_1)\,e^{-i\beta x_1}\,dx_1\\ &= \frac{1}{\sqrt{R}}\,e^{-i\beta R}\;-\;i\beta\int_0^{R_0}\frac{1}{\sqrt{x_1}}\,e^{-i\beta x_1}\,dx_1\;+\;i\beta\int_{R_0-1}^{R_0}\varphi(x_1)\,e^{-i\beta x_1}\,dx_1\\ &\quad+\;i\beta\int_0^{R}\frac{1}{\sqrt{x_1}}\,e^{-i\beta x_1}\,dx_1\,.\end{aligned}$$

In the last integral we substitute $t = |\beta| x_1$. Then, with $s = \operatorname{sign} \beta$,

$$i\beta \int_0^R \frac{1}{\sqrt{x_1}} e^{-i\beta x_1} dx_1 \;=\; \frac{i\beta}{\sqrt{|\beta|}} \int_0^{R|\beta|} \frac{1}{\sqrt{t}} e^{-ist} dt \;=\; is\sqrt{|\beta|} \int_0^{R|\beta|} \frac{1}{\sqrt{t}} e^{-ist} dt\,,$$

and this converges to

$$is\sqrt{|\beta|} \int_0^\infty \frac{1}{\sqrt{t}} e^{-ist} dt \;= is\sqrt{|\beta|} \int_0^\infty \frac{\cos t - is \sin t}{\sqrt{t}} dt \;=\; is\sqrt{\frac{\pi}{2}|\beta|}\,(1-is)$$

$$= \sqrt{\pi|\beta|}\, e^{is\pi/4} \;=\; \sqrt{-\pi\beta}\, e^{-i\pi/4}$$

as R tends to infinity. Here we used $\int_0^\infty \frac{\cos t}{\sqrt{t}} dt = \int_0^\infty \frac{\sin t}{\sqrt{t}} dt = \sqrt{\pi/2}$ and $\sqrt{-\beta} = i\sqrt{|\beta|}$ for $\beta > 0$. Therefore,

$$\int_{-\infty}^\infty \varphi'(x_1)\, e^{-i\beta x_1} dx_1 \;=\; \sqrt{-\pi\beta}\, e^{-i\pi/4} \;+\; \beta\, R^+(\beta)$$

where $R^+(\beta) = i\int_0^{R_0} \frac{1}{\sqrt{x_1}} e^{i\beta x_1} dx_1 - i\int_{R_0-1}^{R_0} \varphi(x_1)\, e^{i\beta x_1} dx_1$ is smooth.

The case $\sigma = -$ is treated in the same way or we use that $\frac{d}{dx_1}\varphi^-(x_1) = -\varphi'(-x_1)$ where $\varphi^-(x_1) = \xi^-(x_1)/\sqrt{|x_1|} = \varphi(-x_1)$. Then

$$\int_{-\infty}^\infty (\varphi^-)'(x_1)\, e^{-i\beta x_1} dx_1 \;=\; -\int_{-\infty}^\infty \varphi'(-x_1)\, e^{-i\beta x_1} dx_1 \;=\; -\int_{-\infty}^\infty \varphi'(x_1)\, e^{i\beta x_1} dx_1$$

$$= \;-\sqrt{\pi\beta}\, e^{-i\pi/4} \;+\; \beta\, R^+(-\beta)\,.$$

(b) Expanding $F(\varphi^{\sigma\prime})(x_1,\beta)$ into a Fourier series yields

$$(F(\varphi^{\sigma\prime}))(x_1,\beta) \;=\; \varphi_0^\sigma(\beta)\, e^{i\beta x_1} \;+\; \sum_{\ell\neq 0} \varphi_\ell^\sigma(\beta)\, e^{i(\ell+\beta)x_1}$$

where the Fourier coefficients are given by

$$\varphi_\ell^\sigma(\beta) \;=\; \frac{1}{2\pi}\int_0^{2\pi} F(\varphi^{\sigma\prime})(x_1,\beta)\, e^{-i(\ell+\beta)x_1} dx_1 \;=\; \frac{1}{2\pi}\int_{-\infty}^\infty \varphi^{\sigma\prime}(x_1)\, e^{-i(\ell+\beta)x_1} dx_1\,.$$

For $\ell \neq 0$ and $|\beta| < 1$ the formula $\frac{\partial^q}{\partial\beta^q} e^{-i(\ell+\beta)x_1} = \frac{(-ix_1)^q}{[-i(\ell+\beta)]^p} \frac{\partial^p}{\partial x_1^p} e^{-i(\ell+\beta)x_1}$ and partial integration (sufficiently often) yields that $\varphi_\ell^\sigma(\beta)$ depends smoothly on β for $\ell \neq 0$ and decays of order $1/|\ell|^p$ for all $p \in \mathbb{N}$. Therefore, $\sum_{\ell\neq 0} \varphi_\ell^\sigma(\beta)\, e^{i(\ell+\beta)x_1}$ is smooth and $\int_0^{2\pi} \sum_{\ell\neq 0} \varphi_\ell^\sigma(0)\, e^{i\ell x_1} dx_1 = 0$.

Finally, the coefficient $\varphi_0^\sigma(\beta)$ has been treated in part (a) and gives the desired result. □

Lemma B.2. *Let $\xi \in C^\infty(\mathbb{R})$ with $\xi(x) = 1$ for $x \geq R$ and $\xi(x) = 0$ for $x \leq R-1$ (for some $R > 1$), and let $\phi \in C^\infty(\mathbb{R})$ be $\alpha-$quasi-periodic. Set*

$$g(x) \;=\; \frac{\xi(x)}{\sqrt{x}}\, \phi(x)\,, \quad x \in \mathbb{R}\,.$$

(a) Then $g \in L^p(\mathbb{R})$ for all $p > 2$, and the Fourier transform $\hat{g} = \mathcal{F}g$ (in the sense of distributions) of g is regular and in $L^q(\mathbb{R})$ for all $q \in (1,2)$.

(b) The Fourier transforms $\mathcal{F}\big(g^{(m)}\big)$ of the derivatives $g^{(m)}$ of order $m = 1,2,\ldots$ (in the sense of distributions) are in $L^1(\mathbb{R})$ and satisfy $\mathcal{F}\big(g^{(m)}\big)(\omega) = (i\omega)^m\,\hat{g}(\omega)$.

(c) For any $f \in L^1(\mathbb{R})$ the Fourier transform of the convolution $g \circ f \in L^p(\mathbb{R})$ is regular and is given by $\mathcal{F}(g \circ f) = \hat{g}\,\hat{f} \in L^q(\mathbb{R})$.

Proof. For any $j \in \mathbb{N}$ let $g_j(x) = g(x)$ for $|x| \le j$ and $g_j(x) = 0$ for $|x| > j$. Then g_j converges to g in $L^p(\mathbb{R})$ for all $p > 2$ as easily seen.

(a) We expand $\phi(x)$ into the Fourier series $\phi(x) = \sum_{\ell\in\mathbb{Z}} \phi_\ell\, e^{i(\ell+\alpha)x}$ with coefficients $\phi_\ell \in \mathbb{C}$ which decay arbitrarily fast to zero as $\ell \to \pm\infty$ because ϕ is smooth. We compute the Fourier transform as

$$\hat{g}_j(\omega) \;=\; \int_R^j \frac{\xi(x)}{\sqrt{x}}\,\phi(x)\,e^{-i\omega x}\,dx \;=\; \sum_{\ell\in\mathbb{Z}} \phi_\ell \int_R^j \frac{\xi(x)}{\sqrt{x}}\,e^{i(\ell+\alpha-\omega)x}\,dx \;=\; \sum_{\ell\in\mathbb{Z}} \phi_\ell\, h_j(\ell+\alpha-\omega)$$

where

$$\begin{aligned} h_j(t) \;&:=\; \int_R^j \frac{\xi(x)}{\sqrt{x}}\,e^{itx}\,dx \;=\; \int_R^j \frac{1}{\sqrt{x}}\,e^{itx}\,dx \;+\; \int_R^j \frac{\xi(x)-1}{\sqrt{x}}\,e^{itx}\,dx \\ &=\; \frac{1}{\sqrt{|t|}} \int_{R|t|}^{j|t|} \frac{1}{\sqrt{y}}\,e^{iy\,\operatorname{sign} t}\,dy \;+\; \int_R^{R+1} \frac{\xi(x)-1}{\sqrt{x}}\,e^{itx}\,dx \\ &=\; \frac{1}{\sqrt{|t|}}\big[A(jt) - A(Rt)\big] \;+\; \int_R^{R+1} \frac{\xi(x)-1}{\sqrt{x}}\,e^{itx}\,dx\,, \end{aligned}$$

where we used the substitution $y = |t|x$ and set

$$A(t) \;:=\; \int_0^{|t|} \frac{1}{\sqrt{y}}\,e^{iy\,\operatorname{sign} t}\,dy\,, \quad t \in \mathbb{R}\,.$$

Now we use $A(t) = A_{\pm\infty} + \mathcal{O}(1/\sqrt{|t|})$ as $t \to \pm\infty$ with $A_{\pm\infty} = \sqrt{\frac{\pi}{2}}\,(1 \pm i)$. We define

$$h(t) \;:=\; \frac{1}{\sqrt{|t|}}\big[A(\pm\infty) - A(Rt)\big] \;+\; \int_R^{R+1} \frac{\xi(x)-1}{\sqrt{x}}\,e^{itx}\,dx \quad \text{for } t \gtrless 0\,.$$

Since the second term decays as $\mathcal{O}(1/|t|)$ as $t \to \pm\infty$ (partial integration!) we observe that $h(t)$ – and also $h_j(t)$ – decays as $\mathcal{O}(1/|t|)$ as $t \to \pm\infty$ and are bounded near $t = 0$.

Next we show convergence of h_j to h in $L^q(\mathbb{R})$ for every $q \in (1,2)$. Indeed, we compute

$$\|h_j - h\|_{L^q}^q \;=\; \int_{-\infty}^0 \frac{|A(jt) - A(-\infty)|^q}{|t|^{q/2}}\,dx \;+\; \int_0^\infty \frac{|A(jt) - A(+\infty)|^q}{t^{q/2}}\,dx\,.$$

We bound the integrands (uniformly with respect to j) by

$$\frac{|A(jt)-A(-\infty)|^q}{|t|^{q/2}} \;\le\; \begin{cases} c\,|t|^{-q/2}\,, & -1<t<0\,,\\ c\,|t|^{-q}\,, & t<-1\,,\end{cases}$$

$$\frac{|A(jt)-A(+\infty)|^q}{|t|^{q/2}} \;\le\; \begin{cases} c\,t^{-q/2}\,, & 0<t<1\,,\\ c\,t^{-q}\,, & t>1\,.\end{cases}$$

where we used the boundedness of $|A(jt)-A(\pm\infty)|$ in $[-1,1]$ and the estimates $|A(jt)-A(\pm\infty)| \le c/\sqrt{j|t|} \le c/\sqrt{|t|}$ for $\pm t \ge 1$ and $j \ge 1$. Also the integrands converge to zero pointwise for all $t \neq 0$. Lebesgue's theorem on dominated convergence yields convergence of h_j to h in $L^q(\mathbb{R})$.

This implies convergence of $\hat{g}_j$ to $\hat{g}$ in $L^q(\mathbb{R})$ where $\hat{g}(\omega) := \sum_{\ell\in\mathbb{Z}} \phi_\ell\, h(\ell+\alpha-\omega)$.

It remains to show that $\hat{g}$ generates the Fourier transform of g. To show this let $\psi \in \mathcal{S}$. Then $\int_{\mathbb{R}} \hat{g}_j\,\psi\,dx$ converges to $\int_{\mathbb{R}} \hat{g}\,\psi\,dx$ by the convergence of $\hat{g}_j$ to $\hat{g}$ in $L^q(\mathbb{R})$. On the other hand, $\int_{\mathbb{R}} \hat{g}_j\,\psi\,dx = \int_{\mathbb{R}} g_j\,\hat{\psi}\,dx$ converges to $\int_{\mathbb{R}} g\,\hat{\psi}\,dx$ by the convergence of g_j to g in $L^p(\mathbb{R})$. This shows $\int_{\mathbb{R}} \hat{g}\,\psi\,dx = \int_{\mathbb{R}} g\,\hat{\psi}\,dx$, i.e. the Fourier transform of g in the distributional sense is regular and generated by $\hat{g} \in L^q(\mathbb{R})$.

(b) For any $\psi \in \mathcal{S}$ we have

$$\begin{aligned}\int_{\mathbb{R}} \big(\mathcal{F}g^{(m)}\big)(\omega)\,\psi(\omega)\,d\omega \;&=\; \int_{\mathbb{R}} g^{(m)}(x)\,(\mathcal{F}\psi)(x)\,dx \;=\; (-1)^m \int_{\mathbb{R}} g(x)\,\frac{d^m}{dx^m}(\mathcal{F}\psi)(x)\,dx\\ &= (-1)^m \int_{\mathbb{R}} g(x)\,\mathcal{F}\big((-i\omega)^m\psi\big)(x)\,dx\\ &= (-1)^m \int_{\mathbb{R}} \hat{g}(\omega)\,(-i\omega)^m\psi(\omega)\,d\omega \;=\; \int_{\mathbb{R}} \hat{g}(\omega)\,(i\omega)^m\psi(\omega)\,d\omega\end{aligned}$$

which shows the claim because ψ is arbitrary.

(c) $\int_{\mathbb{R}} (f\circ g_j)\,\hat{\psi}\,dx$ converges to $\int_{\mathbb{R}} (f\circ g)\,\hat{\psi}\,dx$ for any $\psi\in\mathcal{S}$. On the other hand

$$\int_{\mathbb{R}} (f\circ g_j)\,\hat{\psi}\,dx \;=\; \int_{\mathbb{R}} \widehat{(f\circ g_j)}\,\psi\,dx \;=\; \int_{\mathbb{R}} \hat{f}\,\hat{g}_j\,\psi\,dx \;\to\; \int_{\mathbb{R}} \hat{f}\,\hat{g}\,\psi\,dx$$

by the convergence of $\hat{g}_j$ to $\hat{g}$ in $L^q(\mathbb{R})$. This implies

$$\int_{\mathbb{R}} (f\circ g)\,\hat{\psi}\,dx \;=\; \int_{\mathbb{R}} \hat{f}\,\hat{g}\,\psi\,dx \quad \text{for all } \psi\in\mathcal{S}\,,$$

i.e. the Fourier transform $\widehat{(f\circ g)}$ of $f\circ g$ in the sense of distributions coincides with the regular distribution generated by $\hat{f}\,\hat{g} \in L^q(\mathbb{R})$ for all $q\in(1,2)$. □

Lemma B.3. *Let $\phi\in C^\infty(\mathbb{R})$ be $\kappa-$quasi-periodic for some $\kappa\in(-1/2,1/2)$, $\kappa\neq 0$. Let $\xi\in C^\infty(\mathbb{R})$ be equal to 1 for $x_1\ge R$ (for some $R>1$) and equal to 0 for $x_1\le R-1$. For* *every $a > 1$ then these exists $c = c(a) > 0$ with*

$$\Bigg| \int_{R-1}^{\infty} \frac{\xi(y_1)}{\sqrt{y_1}} \, \phi(y_1) \, \partial_{y_2} H_0^{(1)} \big(k \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \big)|_{y_2 = 0} \, dy_1 \Bigg| \;\leq\; \frac{c}{\sqrt{x_2}}$$

for all $x_2 \geq 1$ and $|x_1| \leq a$.

Proof. Define

$$\begin{aligned} \Psi(x_1, x_2, y_1) \;&:=\; \frac{1}{\sqrt{y_1}} \, \partial_{y_2} H_0^{(1)} \big(k \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \big)|_{y_2 = 0} \\ &=\; \frac{k x_2 \, H_0^{(1)\prime} \big(k \sqrt{(y_1 - x_1)^2 + x_2^2} }{\sqrt{(y_1 - x_1)^2 + x_2^2} \, \sqrt{y_1}} \, . \end{aligned}$$

By the asymptotics for the Hankel function we have $\Psi(x_1, x_2, y_1) = \Psi_\infty(x_1, x_2, y_1) + \tilde{\Psi}(x_1, x_2, y_1)$ where

$$\Psi_\infty(x_1, x_2, y_1) \;=\; \gamma \, \frac{x_2}{[(y_1 - x_1)^2 + x_2^2]^{3/4} \, \sqrt{y_1}} \, e^{ik\sqrt{(y_1 - x_1)^2 + x_2^2}}$$

with $\gamma = -\sqrt{\frac{2k}{\pi}} \, e^{i\pi/4}$ and

$$\big| \tilde{\Psi}(x_1, x_2, y_1) \big| \;\leq\; c \, \frac{x_2}{[(y_1 - x_1)^2 + x_2^2]^{5/4} \, \sqrt{y_1}} \, .$$

We estimate since $x_2 \geq 1$

$$\begin{aligned} \int_{R-1}^{\infty} \big| \tilde{\Psi}(x_1, x_2, y_1) \big| \, |\phi(y_1)| \, dy_1 \;&\leq c \, x_2 \int_{R-1}^{\infty} \frac{1}{[(y_1 - x_1)^2 + x_2^2]^{5/4} \, \sqrt{y_1}} \, dy_1 \\ &\leq c \, \frac{x_2}{x_2^2} \int_{R-1}^{\infty} \frac{1}{[(y_1 - x_1)^2 + 1]^{1/4} \, \sqrt{y_1}} \, dy_1 \;\leq\; \frac{c_1}{x_2} \, . \end{aligned}$$

Therefore, it remains to estimate the part with Ψ_∞. We split

$$\int_{R-1}^{\infty} \Psi_\infty(x_1, x_2, y_1) \, \phi(y_1) \, dy_1 \;=\; \int_{R-1}^{2a} \Psi_\infty(x_1, x_2, y_1) \, \phi(y_1) \, dy_1 \;+ \int_{2a}^{\infty} \Psi_\infty(x_1, x_2, y_1) \, \phi(y_1) \, dy_1 \, .$$

The first term on the right hand side is estimated by $c/\sqrt{x_2}$ because $|y_1 - x_1$ is bounded. We consider the second term and substitute $y_1 = x_1 + t x_2$. Then

$$\int_{2a}^{\infty} \Psi_\infty(x_1, x_2, y_1) \, \phi(y_1) \, dy \;=\; \gamma \, k \int_{\frac{2a - x_1}{x_2}}^{\infty} \frac{1}{[t^2 + 1]^{3/4} \, \sqrt{\frac{x_1}{x_2} + t}} \, e^{ikx_2\sqrt{t^2+1}} \, dt \, .$$

From $\frac{x_1}{x_2} + t \geq \frac{1}{2} t$ for $t \geq \frac{2a - x_1}{x_2}$ and thus

$$\left| \frac{1}{\sqrt{t + x_1/x_2}} - \frac{1}{\sqrt{t}} \right| \;=\; \frac{|x_1|}{x_2} \, \frac{1}{\sqrt{t + x_1/x_2} \, \sqrt{t} \, [\sqrt{t + x_1/x_2} + \sqrt{t}]} \;\leq\; \frac{|x_1|}{x_2} \, \frac{\sqrt{2}}{t^{3/2}}$$

we obtain $\int_{\frac{2a-x_1}{x_2}}^{\infty} \Big| \frac{1}{\sqrt{t+x_1/x_2}} - \frac{1}{\sqrt{t}} \Big| \, dt \leq \frac{c}{\sqrt{x_2}}$. Therefore, we can replace $\sqrt{\frac{x_1}{x_2}+1}$ by $\sqrt{t}$ and have to estimate

$$\int\limits_{\frac{2a-x_1}{x_2}}^{\infty} \frac{e^{ix_2 k\sqrt{t^2+1}}}{(t^2+1)^{3/4}\sqrt{t}} \, \phi(x_1+tx_2)\, dt$$

$$= \int\limits_0^{\infty} \frac{e^{ix_2 k\sqrt{t^2+1}}}{(t^2+1)^{3/4}\sqrt{t}} \, \phi(x_1+tx_2)\, dt \;-\; \int\limits_0^{\frac{2a-x_1}{x_2}} \frac{e^{ix_2 k\sqrt{t^2+1}}}{(t^2+1)^{3/4}\sqrt{t}} \, \phi(x_1+tx_2)\, dt$$

$$= \sum_{\ell\in\mathbb{Z}} \phi_\ell \, e^{i(\ell+\kappa)x_1} \int\limits_0^{\infty} \frac{e^{ix_2[k\sqrt{t^2+1}+(\ell+\kappa)t]}}{(t^2+1)^{3/4}\sqrt{t}} \, dt \;-\; \int\limits_0^{\frac{2a-x_1}{x_2}} \frac{e^{ix_2 k\sqrt{t^2+1}}}{(t^2+1)^{3/4}\sqrt{t}} \, \phi(x_1+tx_2)\, dt$$

where we expanded ϕ into a Fourier series of the form $\phi(y_1) = \sum_{\ell\in\mathbb{Z}} \phi_\ell \, e^{i(\ell+\kappa)y_1}$. The second integral can certainly be estimated by $c/\sqrt{x_2}$. Since $\sum_{\ell\in\mathbb{Z}} |\phi_\ell|$ is finite it is sufficient to estimate the integrals appearing in the Fourier series uniformly with respect to ℓ. We split

$$\int\limits_0^{\infty} \frac{e^{ix_2[k\sqrt{t^2+1}+(\ell+\kappa)t]}}{(t^2+1)^{3/4}\sqrt{t}} \, dt \;=\; \int\limits_0^{T} \frac{e^{ix_2[k\sqrt{t^2+1}+(\ell+\kappa)t]}}{(t^2+1)^{3/4}\sqrt{t}} \, dt \;+\; \int\limits_T^{\infty} \frac{e^{ix_2[k\sqrt{t^2+1}+(\ell+\kappa)t]}}{(t^2+1)^{3/4}\sqrt{t}} \, dt \tag{B.1}$$

where T is chosen such that $\frac{kT}{\sqrt{T^2+1}} \leq \frac{1}{2}|\kappa|$. We study the first integral on the right hand side and make the substitution $s = s(t) = \frac{k}{\ell+\kappa}\left[\sqrt{t^2+1}-1\right] + t$. Then $s(0)=0$ and $s'(t) = \frac{kt}{\sqrt{t^2+1}(\ell+\kappa)} + 1$, thus $s'(0)=1$. From $|\ell+\kappa| \geq |\kappa|$ for all $\ell\in\mathbb{Z}$ and the property of T we conclude that $s'(t) \geq 1 - \frac{kt}{\sqrt{t^2+1}\,|\kappa|} \geq 1 - \frac{kT}{\sqrt{T^2+1}\,|\kappa|} \geq \frac{1}{2}$. Furthermore, $|s'(t)| \leq k/|\kappa|$ and $|s^{(j)}(t)| \leq c$ for all $t\in[0,T]$ and $j=2,3$. Since s is strictly monotonously increasing there exists the inverse function $t=t(s)$ which satisfies $t(0)=0$ and $t'(0)=1$. By differentiating $s\big(t(s)\big)=s$ for all s three times we easily obtain an estimate of the form $|t^{(j)}(s)| \leq c$ for all $s\in[0,s(T)]$ and $j=1,2,3$. We write $t(s)$ in the form $t(s)=s\,\tau(s)$ with $\tau(s) := \int_0^1 t'(\rho s)\,d\rho$ and observe that $\tau(0)=1$ and $\tau(\rho)\geq|\kappa|/k$. Making this substitution for the first integral of the right hand side of (B.1) we obtain

$$\int\limits_0^{T} \frac{e^{ix_2[k\sqrt{t^2+1}+(\ell+\kappa)t]}}{(t^2+1)^{3/4}\sqrt{t}} \, dt \;=\; e^{ikx_2} \int\limits_0^{s(T)} \frac{e^{ix_2(\ell+\kappa)s}\, t'(s)}{[t(s)^2+1]^{3/4}\,\sqrt{t(s)}} \, ds$$

$$= e^{ikx_2} \int\limits_0^{s(T)} \frac{\psi(s)}{\sqrt{s}} \, e^{ix_2(\ell+\kappa)s}\, ds \;=\; e^{ikx_2} \int\limits_0^{s(T)} \frac{\psi(s)-1}{\sqrt{s}} \, e^{ix_2(\ell+\kappa)s}\, ds$$

$$+\; e^{ikx_2} \int\limits_0^{s(T)} \frac{1}{\sqrt{s}} \, e^{ix_2(\ell+\kappa)s}\, ds \quad \text{with} \quad \psi(s) := \frac{t'(s)}{[t(s)^2+1]^{3/4}\,\sqrt{\tau(s)}}\,.$$

Since $\psi(0)=1$ we write $\psi(s)$ in the form $\psi(s)=1+s\tilde\psi(s)$ with $\tilde\psi(s):=\int_0^1\psi'(\rho s)\,d\rho$. The function $\tilde\psi$ is continuously differentiable as seen from the smoothness of $t(s)$ and its derivative up to order 3. Therefore,

$$\int\limits_0^{s(T)}\frac{\psi(s)-1}{\sqrt{s}}\,e^{ix_2(\ell+\kappa)s}\,ds \;=\; \int\limits_0^{s(T)}\tilde\psi(s)\,\sqrt{s}\,e^{ix_2(\ell+\kappa)s}\,ds\,,$$

and we can apply partial integration to see that this integral can be estimated by $\frac{c}{x_2|\ell+\kappa|}$. We note that c is independent of ℓ and x_2.

Furthermore, the substitution $\sigma=x_2|\ell+\kappa|s$ and the boundedness of $\int_0^z\frac{1}{\sqrt{\sigma}}\,e^{\pm i\sigma}\,d\sigma$ yields that the integral $\int_0^{s(T)}\frac{1}{\sqrt{s}}\,e^{ix_2(\ell+\kappa)s}\,ds$ behaves is $\mathcal{O}(1/\sqrt{x_2|\ell+\kappa|})$. This finishes the discussion of the first integral on the right hand side of (B.1).

For the second integral on the right hand side of (B.1) we consider first ℓ with $|\ell+\kappa|\geq 2k$. Then $\frac{kt}{\sqrt{t^2+1}|\ell+\kappa|}\leq\frac{k}{2k}=\frac{1}{2}$. We make again the substitution $s=s(t)=\frac{k}{\ell+\kappa}\sqrt{t^2+1}+t$ and observe that $s'(t)=\frac{kt}{\sqrt{t^2+1}(\ell+\kappa)}+1$ is again positive. Since $s\mapsto\frac{t'(s)}{[t(s)^2+1]^{3/4}\sqrt{t(s)}}$ is differentiable on $[s(T),\infty)$ we can apply partial integration and obtain an order of $\frac{c}{x_2|\ell+\kappa|}$. We note again that c is independent of ℓ and x_2.

Let finally ℓ with $|\ell+\kappa|<2k$. (There are only finitely many of these ℓ.) Since the function $t\mapsto\frac{1}{[t^2+1]^{3/4}\sqrt{t}}$ is smooth on $[T,\infty)$ and $\frac{d^2}{dt^2}\big[k\sqrt{t^2+1}+(\ell+\kappa)t\big]>0$ the method of stationary phase yields the order $\mathcal{O}(1/\sqrt{x_2})$. $\square$

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