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AGGREGATING HEAVY-TAILED RANDOM VECTORS: FROM FINITE SUMS TO LÉVY PROCESSES

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The tail behavior of aggregates of heavy-tailed random vectors is known to be determined by the so-called principle of "one large jump", be it for finite sums, random sums, or, Lévy processes. We establish that, in fact, a more general principle is at play. Assuming that the random vectors are multivariate regularly varying on various subcones of the positive quadrant, first we show that their aggregates are also multivariate regularly varying on these subcones. This allows us to approximate certain tail probabilities which were rendered asymptotically negligible under classical regular variation, despite the "one large jump" asymptotics. We also discover that depending on the structure of the tail event of concern, the tail behavior of the aggregates may be characterized by more than a single large jump. Eventually, we illustrate a similar phenomenon for multivariate regularly varying Lévy processes, establishing as well a relationship between multivariate regular variation of a Lévy process and multivariate regular variation of its Lévy measure on different subcones.

1. Introduction. In this paper we study the behavior of the asymptotic tail distribution of independent sums of heavy-tailed random vectors under the paradigm of multivariate regular variation [4, 39]. Assessment of such tail probabilities are of interest in risk management for many finance, insurance, queueing, and environmental applications [2, 16, 33]. Multidimensional tail events are often characterized by at least one variable exceeding a high threshold, and the asymptotic probability of such events follow the so-called "one large jump" principle, see [22].

Assume that all our random elements are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If $Z, Z^{(1)}, \dots, Z^{(n)}$ are independent and identically distributed (i.i.d.) random variables, then for fixed $x > 0$ we know that

$$\mathbb{P}(Z^{(1)} + \ldots + Z^{(n)} > tx) \sim n\,\mathbb{P}(Z > tx) \quad \text{as } t \to \infty, \tag{1.1}$$

if and only if Z is subexponential, i.e., $\mathbb{P}(Z^{(1)} + Z^{(2)} > t) \sim 2\,\mathbb{P}(Z > t)$ as $t \to \infty$; Chistyakov [6] proved this for non-negative random variables, later extended to $\mathbb{R}$ by Willekens [44]. This phenomenon of "one large jump" exhibited in (1.1) is so called because a high threshold-crossing of the sum of a set of random variables occurs with the same asymptotic probability as any one of them crossing the same threshold. Recall that a random variable Z has a regularly varying right tail if for $x > 0$, we have $\lim_{t\to\infty} \mathbb{P}(Z > tx)/\mathbb{P}(Z > t) = x^{-\alpha}$ for some $-\alpha < 0$, which is called the index of regular variation or tail index. We write $Z \in \mathcal{RV}_{-\alpha}$. Regularly varying distributions are subexponential as well (cf. [43]), i.e, and hence (1.1) holds when $Z \in \mathcal{RV}_{-\alpha}$.

In a multivariate context, an $\mathbb{R}_+^d$-valued random vector $\boldsymbol{Z}$ is multivariate regularly varying (MRV) on $\mathbb{R}_+^d \setminus \{\mathbf{0}\}$, if there exists a function $b(t) \to \infty$ as $t \to \infty$, and a non-null measure

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μ in the set of all Borel measures in $\mathbb{R}_+^d \setminus \{\mathbf{0}\}$ which are finite on sets bounded away from $\mathbf{0}$ such that

$$t\,\mathbb{P}(\boldsymbol{Z}/b(t) \in A) \to \mu(A) \quad \text{as } t \to \infty$$

for Borel sets A bounded away from $\mathbf{0}$ with $\mu(\partial A) = 0$; see Section 2 for details. In particular, we have $\mu(tA) = t^{-\alpha}\mu(A)$ for some $\alpha > 0$ and write $\boldsymbol{Z} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\})$. Now, for i.i.d. non-negative random vectors $\boldsymbol{Z}, \boldsymbol{Z}^{(1)}, \ldots, \boldsymbol{Z}^{(n)}$ with $\boldsymbol{Z} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\})$, we can deduce that

$$\boldsymbol{Z}^{(1)} + \ldots + \boldsymbol{Z}^{(n)} \in \mathcal{MRV}(\alpha, b, n\mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\}) \tag{1.2}$$

(cf. [38, Proposition 4.1], [39, Section 7.3] and [25, Lemma 3.11]). Hence, for Borel sets A bounded away from $\mathbf{0}$ with $\mu(\partial A) = 0$ and fixed $n \geq 1$ we may approximate

$$\mathbb{P}(\boldsymbol{Z}^{(1)} + \ldots + \boldsymbol{Z}^{(n)} \in tA) \sim \frac{n}{b^{\leftarrow}(t)}\,\mu(A) \sim n\,\mathbb{P}(\boldsymbol{Z} \in tA) \quad \text{as } t \to \infty, \tag{1.3}$$

if $\mu(A) > 0$. Thus, (1.3) extends (1.1) to higher dimensions and the principle of "one large jump" appears to hold. Curiously though, one often encounters examples where for a large class of sets A, the value of $\mu(A)$ is equal to zero and in this case,

$$\lim_{t\to\infty} b^{\leftarrow}(t)\,\mathbb{P}(\boldsymbol{Z}^{(1)} + \ldots + \boldsymbol{Z}^{(n)} \in tA) = 0 = \lim_{t\to\infty} b^{\leftarrow}(t)\,\mathbb{P}(\boldsymbol{Z} \in tA). \tag{1.4}$$

This makes (1.3) of limited practical use. For example, if the elements of $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ are themselves i.i.d. (with regularly varying marginal tail distributions) and we consider sets

$$A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \;\forall\, j \in S\}, \tag{1.5}$$

for indices $S \subseteq \mathbb{I} := \{1, \ldots, d\}$ with $|S| \geq 2$, $x_j > 0$ for $j \in S$, then for $\boldsymbol{Z} \in tA$ to hold, at least two components of $\boldsymbol{Z}$ need to be large together and under classical MRV assumptions we have $\mu(A) = 0$ verifying (1.4) (cf. Example 2.13). In fact, the components of $\boldsymbol{Z}$ need not to be independent at all, a Gaussian dependence among variables with tail equivalent regularly varying marginal distributions and pairwise correlations less than one will ensure $\mu(A) = 0$; see Section 2.4 for further examples.

In such a case, we do observe that a notion subtler than the well-known "one large jump" phenomenon holds depending on the joint dependence of the underlying random vector and on the type of tail set A considered. If multivariate regular variation holds on relevant subcones of $\mathbb{R}_+^d$ we realize that in many scenarios,

$$\mathbb{P}(\boldsymbol{Z}^{(1)} + \ldots + \boldsymbol{Z}^{(n)} \in tA) \sim \frac{n}{\widetilde{b}^{\leftarrow}(t)}\,\widetilde{\mu}(A) \sim n\,\mathbb{P}(\boldsymbol{Z} \in tA) \quad \text{as } t \to \infty, \tag{1.6}$$

where $0 < \widetilde{\mu}(A) < \infty$ and $\widetilde{b}^{\leftarrow}(t)$ is a function satisfying $\lim_{t\to\infty} b^{\leftarrow}(t)/\widetilde{b}^{\leftarrow}(t) = 0$. This means that the left and the right hand side of (1.3) are asymptotically equivalent, even though the term in the middle contains $\mu(A) = 0$. The relevant results and examples where we observe (1.6) are detailed in Section 3.1. In particular, the results hold for more general sets than the ones given in (1.5), and we can specify $\widetilde{\mu}(A)$ and $\widetilde{b}^{\leftarrow}(t)$. Formally, we show that $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)}$ is multivariate regularly varying on subcones of $\mathbb{R}_+^d$, obeying (1.2) but with α, b, μ replaced by $\widetilde{\alpha}, \widetilde{b}, \widetilde{\mu}$ and $\mathbb{R}_+^d \setminus \{\mathbf{0}\}$ replaced by an appropriate subcone. Note that (1.6) hints at connections to the notion of multivariate subexponentiality (cf. [7, 36, 40]).

Another phenomenon investigated for sets A as defined in (1.5) is that the tail event $\boldsymbol{Z}^{(1)} + \ldots + \boldsymbol{Z}^{(n)} \in tA$ may be determined by threshold crossings in different co-ordinates of S, by different variables $\boldsymbol{Z}^{(k)}$. Thus, an aggregation of random vectors leads to a tail event with a "few large jumps" and these jumps occur either together in one random vector $\boldsymbol{Z}^{(k)}$, or separately in a few different vectors; and for A as defined in (1.5) with $|S|=i$, we get

$$(b^{\leftarrow}(t))^i\,\mathbb{P}(\boldsymbol{Z}^{(1)}+\ldots+\boldsymbol{Z}^{(n)}\in tA)\sim C_{n,i}\prod_{j\in S}x_j^{-\alpha}\sim C_{n,i}\prod_{j\in S}b^{\leftarrow}(t)\,\mathbb{P}(Z_j>tx_j)$$

as $t\to\infty$ (cf. Remark 3.13). Thus,

$$\mathbb{P}(\boldsymbol{Z}^{(1)}+\ldots+\boldsymbol{Z}^{(n)}\in tA)\sim C_{n,i}\prod_{j\in S}\mathbb{P}(Z_j\in tx_j)\quad \text{as } t\to\infty, \tag{1.7}$$

where $C_{n,i}>0$ is not necessarily equal to n, and depends on the number of summands n, the index $i=|S|$ which represents the type of set A, as well as the distribution of $\boldsymbol{Z}$; the associated results are discussed in Proposition 3.3 (Section 3.1) and Section 3.2. We also notice that this phenomenon is often observed under the more general assumption of adapted-MRV (cf. Definition 2.10) for the underlying random vectors (cf. Section 3.2). Since $C_{n,i}\neq n$ in this example, it is no surprise that the limit measure observed in aggregating adapted-MRV is not linear in n anymore, in contrast to (1.2).

One of our primary interests is to characterize the asymptotic behavior of multi-dimensional regularly varying Lévy processes which have inherent applications to stochastic storage processes including insurance claims, inventory management, and more (cf. [2, 37]). This also happens to be a natural progression from computing tail probabilities of finite aggregation of random vectors. A Lévy process $\boldsymbol{L}=(\boldsymbol{L}(s))_{s\geq 0}$, is a stochastic process with $\boldsymbol{L}(0)=\mathbf{0}$ $\mathbb{P}$-almost surely, has stationary and independent increments, and has càdlàg sample paths (cf. [42]). Consequently, $\boldsymbol{L}(s)$ is infinitely divisible and has the same distribution as sums of i.i.d. random vectors; following the basic premise of this paper. A Lévy process $\boldsymbol{L}$ is characterized by its Lévy measure $\Pi(A)$, which measures the expected number of jumps of $\boldsymbol{L}$ in $[0,1]$ whose jump sizes are in A (cf. Section 5). The principle of one large jump is illustrated for multivariate regularly varying Lévy processes by Hult and Lindskog [21, 22] and the asymptotic rates of further hidden jumps have been characterized by Lindskog, Resnick and Roy [32] (for the univariate case). Our work addresses the case where the results of [22] hold, including and specifically addressing cases with negligible probability approximation for the tail event. In particular, a conclusion of [22, Proposition 3.1] is that $\boldsymbol{L}(1)\in\mathcal{MRV}(\alpha,b,\mu,\mathbb{R}_+^d\setminus\{\mathbf{0}\})$ if and only if $\Pi\in\mathcal{MRV}(\alpha,b,\mu,\mathbb{R}_+^d\setminus\{\mathbf{0}\})$ and then, for any Borel set A bounded away from $\mathbf{0}$ with $\mu(\partial A)=0$ we have

$$\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim\frac{s}{b^{\leftarrow}(t)}\,\mu(A)\sim s\,\Pi(tA)\quad \text{as } t\to\infty, \tag{1.8}$$

if $\mu(A)>0$ (see [15] for the one dimensional case). Naturally, if $\mu(A)=0$ then (1.8) gives a zero estimate. For example, this happens if we consider $\boldsymbol{L}$ to be comprised of d i.i.d. one-dimensional regularly varying Lévy processes and A is defined as in (1.5). Under quite general conditions, if the Lévy measure is multivariate regular varying on relevant subcones of $\mathbb{R}_+^d$, we show $\boldsymbol{L}(s)$ is multivariate regularly varying on that subcone as well. As a consequence under these conditions, we observe that,

$$\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim s\,\mathbb{P}(\boldsymbol{L}(1)\in tA)\sim\frac{s}{\widetilde{b}^{\leftarrow}(t)}\,\widetilde{\mu}(A)\sim s\,\Pi(tA)\quad \text{as } t\to\infty, \tag{1.9}$$

for sets A in appropriate subcones; here the function $\widetilde{b}^{\leftarrow}(t)$ satisfies $\lim_{t\to\infty}b^{\leftarrow}(t)/\widetilde{b}^{\leftarrow}(t)=0$ and in contrast to (1.8) where $\mu(A)=0$, here we have $0<\widetilde{\mu}(A)<\infty$. However, we also find cases where the asymptotics are different and we observe

$$\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim\phi(s)\,\Pi(tA)\quad \text{as } t\to\infty, \tag{1.10}$$

where ϕ is a function for which $\phi(s) = s$ may not hold, defying the linearity property often observed for Lévy processes, cf. (1.9). These results on the tail probabilities of heavy-tailed Lévy processes have obvious implications on risk and ruin problems, especially in the context of insurance and finance, and is addressed in an associated article [10].

This paper is organized as follows. In Section 2, we provide necessary preliminary results and background for our work; first we discuss the basic framework in terms of multivariate regular variation on subcones of $\mathbb{R}_+^d$ using $\mathbb{M}$-convergence. We also discuss copulas and survival copulas used to model dependence in the said random vectors. Our main result, Theorem 3.1, appears in Section 3 which is used to obtain results on "one or few large jumps" of the form (1.6) and (1.7) under a variety of assumptions. In particular, we show how multivariate regular variation of two independent random vectors on a subcone of $\mathbb{R}_+^d$ is used to obtain multivariate regular variation of their n-convolution. Our results are complemented with examples where they can be applied, especially for the convolution of finitely many i.i.d. random vectors. In Section 4, the results on finite convolutions are extended to random sums and finally applied to assess the tail behavior in regularly varying Lévy processes in Section 5. The proofs in the paper are relegated to the appendices.

2. Preliminaries. In this section, we discuss the framework for assessing probabilities of tail events where joint thresholds may be crossed; we briefly recall the theory of $\mathbb{M}$-convergence used to define multivariate regular variation on subcones of $\mathbb{R}_+^d = [0,\infty)^d$ (cf. [9, 32]). Furthermore, we characterize convergence for different types of tail sets in Proposition 2.7. The notion of multivariate regular variation on different cones of $\mathbb{R}_+^d$ is extended in Definition 2.10 to allow for a broader class of models and examples. This allows for a framework where the sum of two random vectors can be MRV on a specific subcone even if neither of the two summands is MRV on that subcone. In Section 2.4, we provide examples of joint distributions where our framework is useful, we use copulas to represent joint dependence. This allows us to illustrate our results in a variety of examples.

Unless otherwise specified, all random vectors are assumed to lie on the positive quadrant $\mathbb{E}_d := \mathbb{R}_+^d$. Notationally, vector operations are understood component-wise, e.g., for vectors $\boldsymbol{z} = (z_1, \ldots, z_d)$ and $\boldsymbol{x} = (x_1, \ldots, x_d)$, $\boldsymbol{x} \le \boldsymbol{z}$ means $x_i \le z_i$ for all i. Moreover, for a constant $t > 0$ and a set $A \subseteq \mathbb{R}_+^d$, we denote by $tA := \{t\boldsymbol{z} : \boldsymbol{z} \in A\}$.

2.1. *Regular variation.* The theory of regular variation provides a systematic framework to discuss heavy-tailed distributions; see Bingham, Goldie and Teugels [4], Resnick [39] for details. Here we briefly discuss regular varying functions and multivariate regular variation of measures and random vectors on Euclidean cones with $\mathbb{M}$-convergence.

A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying* at infinity if for all $x > 0$, we have $\lim_{t\to\infty} f(tx)/f(t) = x^\beta$ for some fixed $\beta \in \mathbb{R}$. We write $f \in \mathcal{RV}_\beta$; and if $\beta = 0$, then the function f is called *slowly varying*. A real-valued random variable Z with distribution function F, denoted by $Z \sim F$, is regularly varying (at $+\infty$) if $\overline{F} := 1 - F \in \mathcal{RV}_{-\alpha}$ for some $\alpha > 0$. Equivalently, $Z \sim F$ is regularly varying with index $-\alpha < 0$ if there exists a measurable function $b : \mathbb{R}_+ \to \mathbb{R}_+$ with $b(t) \to \infty$ as $t \to \infty$ such that

$$t\,\mathbb{P}(Z > b(t)x) = t\,\overline{F}(b(t)x) \xrightarrow{t\to\infty} x^{-\alpha} \quad \forall\, x > 0.$$

We write $\overline{F} \in \mathcal{RV}_{-\alpha}(b)$. As a consequence $b(t) \in \mathcal{RV}_{1/\alpha}$ and a canonical choice for b is $b(t) = F^{\leftarrow}(1 - 1/t) = \overline{F}^{\leftarrow}(1/t)$ where $F^{\leftarrow}(x) = \inf\{y \in \mathbb{R} : F(y) \ge x\}$.

Measures μ and ν defined on $\mathcal{B}(\mathbb{R}_+)$ are *(right) tail equivalent* if

$$\lim_{t\to\infty} \frac{\mu((t,\infty))}{\nu((t,\infty))} = c, \tag{2.1}$$

for some $c > 0$. Naturally, the same holds for probability measures and hence, distribution functions F and G are *(right) tail equivalent* if

$$\lim_{t\to\infty} \frac{\overline{F}(t)}{\overline{G}(t)} = \lim_{t\to\infty} \frac{1-F(t)}{1-G(t)} = c, \tag{2.2}$$

for some $c > 0$. We call measures μ and ν (respectively distributions F and G) *completely* tail equivalent if (2.1) (respectively (2.2)) holds with $c = 1$. We often assume that components of the random vectors considered in this paper are tail equivalent (if not identically distributed, or completely tail equivalent).

We discuss (multivariate) regular variation on Euclidean subspaces of $\mathbb{E}_d = \mathbb{R}_+^d$ using $\mathbb{M}$-convergence of measures which differs from vague convergence, the traditional notion used for multivariate regular variation. See [9, 23, 32] for further details and the preference for this notion over vague convergence; moreover see [3] for a broader notion of vague convergence.

Consider the space $\mathbb{E}_d$ endowed with the sup-norm metric $d(\boldsymbol{x},\boldsymbol{y}) = ||\boldsymbol{x}-\boldsymbol{y}||_\infty$. A *cone* $\mathbb{C} \subset \mathbb{E}_d$ is a set which is closed under scalar multiplication: if $\boldsymbol{z} \in \mathbb{C}$ then $t\boldsymbol{z} \in \mathbb{C}$ for $t > 0$; a *closed cone* is a cone which is a closed set in $\mathbb{E}_d$. Regular variation is defined using $\mathbb{M}$-convergence on a closed cone $\mathbb{C} \subset \mathbb{E}_d$ with a closed cone $\mathbb{C}_0 \subset \mathbb{C}$ deleted. We say that a subset $A \subset \mathbb{C}\setminus\mathbb{C}_0$ is *bounded away from* $\mathbb{C}_0$ if $d(A,\mathbb{C}_0) = \inf\{d(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{x} \in A, \boldsymbol{y} \in \mathbb{C}_0\} > 0$. Denote by $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$, the class of Borel measures on $\mathbb{C}\setminus\mathbb{C}_0$ assigning finite measures to all Borel sets $A \subset \mathbb{C}\setminus\mathbb{C}_0$, which are bounded away from $\mathbb{C}_0$. We often refer to a subspace of the closed cone $\mathbb{E}_d$ which is a cone, as a *subcone*.

Let $\mathbb{C}_0 \subset \mathbb{C} \subset \mathbb{R}_+^d$ be closed cones containing $\mathbf{0}$. We define $\mathbb{M}$-convergence first and subsequently use it to define regular variation on $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$.

DEFINITION 2.1 ($\mathbb{M}$-convergence). Let $\mu, \mu_n, n \geq 1$ be Borel measures on $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$. Suppose $\int f\,d\mu_n \to \int f\,d\mu$ as $n\to\infty$ for any bounded, continuous, real-valued function f whose support is bounded away from $\mathbb{C}_0$, then we say μ_n *converges to* μ *in* $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$, and write $\mu_n \to \mu$ in $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$.

Next we define regular variation of measures on $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$ which is an extension of the definition found in [23] for measures in $\mathbb{M}(\mathbb{R}_+^d \setminus \{\mathbf{0}\})$.

DEFINITION 2.2 (Regular variation of measures). A Borel measure Π on $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$ is regularly varying on $\mathbb{C}\setminus\mathbb{C}_0$ if there exists a regularly varying function $b(t) \in \mathcal{RV}_{1/\alpha}$, $\alpha > 0$, called the *scaling function* and a non-null (Borel) measure $\mu(\cdot) \in \mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$ called the *limit or tail measure* such that as $t\to\infty$,

$$t\,\Pi(b(t)\,\cdot\,) \to \mu(\cdot), \tag{2.3}$$

in $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$. We write $\Pi \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{C}\setminus\mathbb{C}_0)$.

The next definition is a modification of Definition 2.2 for multivariate probability measures and hence, random vectors in $\mathbb{R}_+^d$.

DEFINITION 2.3 (Multivariate regular variation). A random vector $\boldsymbol{Z} \in \mathbb{R}_+^d$ is multivariate regularly varying on $\mathbb{C}\setminus\mathbb{C}_0$ if there exists a regularly varying function $b(t) \in \mathcal{RV}_{1/\alpha}$, $\alpha > 0$, and a non-null (Borel) measure $\mu(\cdot) \in \mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$ such that as $t\to\infty$,

$$t\,\mathbb{P}\left(\boldsymbol{Z}/b(t) \in \,\cdot\,\right) \to \mu(\cdot), \tag{2.4}$$

in $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$. We write $\boldsymbol{Z} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{C}\setminus\mathbb{C}_0)$ and one or more parameters are often dropped according to convenience.

REMARK 2.4. In both Definitions 2.2 and 2.3, since $b(t) \in \mathcal{RV}_{1/\alpha}$, we observe that the limit measure $\mu(\cdot)$ has the scaling property $\mu(t\,\cdot\,) = t^{-\alpha}\mu(\,\cdot\,)$ for $t > 0$. Hence, if the measure or the random vector is $\mathcal{MRV}(\alpha, b, \mu, \mathbb{C}\setminus\mathbb{C}_0)$, we often refer to $-\alpha < 0$ as its tail index (in the subspace $\mathbb{C}\setminus\mathbb{C}_0$).

2.2. *Regular variation on co-ordinate subcones of the positive quadrant.* Equipped with the notion of $\mathbb{M}$-convergence and regular variation, we proceed to discuss regular variation on a particular set of subcones of $\mathbb{R}_+^d$ and also provide equivalent conditions for the same (cf. [34], [9, Section 2]). For $\boldsymbol{z} \in \mathbb{R}_+^d$ write $\boldsymbol{z} = (z_1, \ldots, z_d)$ and denote the (decreasing) order statistics of $\boldsymbol{z}$ by $z_{(1)} \geq z_{(2)} \geq \ldots \geq z_{(d)}$. For $0 \leq i \leq d-1$ let

$$\mathbb{CA}_d^{(i)} := \bigcup_{1\leq j_1 < \ldots < j_{d-i} \leq d} \{\boldsymbol{z} \in \mathbb{R}_+^d : z_{j_1} = 0, \ldots, z_{j_{d-i}} = 0\} = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_{(i+1)} = 0\},$$

and $\mathbb{CA}_d^{(d)} := \{\boldsymbol{z} \in \mathbb{R}_+^d : z_{(d)} > 0\}$. For any $i = 1, \ldots, d$, the closed cone $\mathbb{CA}_d^{(i)}$ represents the union of all i-dimensional co-ordinate hyperplanes in $\mathbb{R}_+^d$. Define the following sequence of subcones of $\mathbb{R}_+^d$ where we investigate regular variation (when it exists):

$$\mathbb{E}_d^{(i)} := \mathbb{R}_+^d \setminus \mathbb{CA}_d^{(i-1)} = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_{(i)} > 0\}, \qquad 1 \leq i \leq d. \tag{2.5}$$

We call the subsets $\mathbb{E}_d^{(i)}$ *co-ordinate subcones* since they are cones obtained from $\mathbb{R}_+^d$ by removing particular co-ordinate hyperplanes. Here $\mathbb{E}_d^{(1)}$ is the positive quadrant with $\{\mathbf{0}\} = \mathbb{CA}_d^{(0)}$ removed, $\mathbb{E}_d^{(2)}$ is the positive quadrant with all one-dimensional co-ordinate axes removed, $\mathbb{E}_d^{(3)}$ is the positive quadrant with all two-dimensional co-ordinate hyperplanes removed, and so on. Clearly,

$$\mathbb{E}_d^{(1)} \supset \mathbb{E}_d^{(2)} \supset \ldots \supset \mathbb{E}_d^{(d)} = \mathbb{CA}_d^{(d)}.$$

REMARK 2.5 (Asymptotic tail independence and hidden regular variation). Suppose a random vector $\boldsymbol{Z} \in \mathbb{R}_+^d$ admits regular variation on $\mathbb{E}_d^{(i)}$ with $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ and $\mu_i(\mathbb{E}_d^{(i+1)}) = 0$. We interpret this to be *asymptotic tail independence at level* i, meaning the probability with which $i+1$ or more components are simultaneously large, is negligible in comparison to the probability with which i (or fewer) components are simultaneously large. Suppose for some $j > i \geq 1$ we have $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_j, b_j, \mu_j, \mathbb{E}_d^{(j)})$ with $\lim_{t\to\infty}(b_i(t)/b_j(t)) = \infty$ then we say $\boldsymbol{Z}$ exhibits *hidden regular variation* (HRV) on $\mathbb{E}_d^{(j)}$ with respect to MRV on $\mathbb{E}_d^{(i)}$.

We investigate regular variation on cones of the form $\mathbb{E}_d^{(i)}$, since joint exceedances often occur in tail subsets of such cones. A recipe for seeking (hidden) regular variation on such subsets have been discussed in Section 2.1 of [9] and we do not repeat the steps here. In the rest of this section, we characterize a particular family of sets $\mathcal{R}^{(i)}$ (defined in (2.7) below), proving that it is an $\mathbb{M}$-convergence determining class on $\mathbb{E}_d^{(i)}$. The particular tail sets appear in multivariate risk and reliability problems where the quantity of interest is a finite or a random sum of identically distributed vectors.

Let $\mathcal{B} := \mathcal{B}(\mathbb{R}_+^d)$ denote the Borel σ-algebra on $\mathbb{R}_+^d$. For any $i \in \{1, \ldots, d\} =: \mathbb{I}$, $\mathbb{E}_d^{(i)}$ is a subspace of $\mathbb{R}_+^d$ and we denote its induced σ-algebra by

$$\mathcal{B}^{(i)} := \mathcal{B}(\mathbb{E}_d^{(i)}) = \{A \in \mathcal{B} : A \subseteq \mathbb{E}_d^{(i)}\}. \tag{2.6}$$

A *rectangular set* in $\mathbb{E}_d^{(i)}$ is defined as any set $A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \,\forall\, j \in S\}$ where $S \subseteq \mathbb{I}$, $|S| \geq i$ and $x_j > 0 \,\forall\, j \in S$. Let us denote the collection

$$\mathcal{R}^{(i)} := \{A \in \mathcal{B}^{(i)} : A \text{ is a rectangular set in } \mathbb{E}_d^{(i)}\}. \tag{2.7}$$

LEMMA 2.6. *$\mathcal{R}^{(i)}$ is a π-system and $\sigma(\mathcal{R}^{(i)}) = \mathcal{B}^{(i)}$.*

The proofs of Lemma 2.6 as well as other subsequent results in this section are given in Appendix A. The following proposition shows that for verifying convergence of measures in $\mathbb{E}_d^{(i)}$, we can restrict to testing convergence in sets belonging to $\mathcal{R}^{(i)}$. The result and the proof are in the spirit of [39, Lemma 6.1].

PROPOSITION 2.7. *Let $\mu, \mu_t \in \mathbb{M}(\mathbb{E}_d^{(i)})$ for all $t > 0$ and some fixed $i \in \mathbb{I}$. Then as $t \to \infty$,*

$$\mu_t \to \mu \quad \textit{in} \quad \mathbb{M}(\mathbb{E}_d^{(i)}) \tag{2.8}$$

if and only if

$$\lim_{t\to\infty} \mu_t(A) \to \mu(A), \quad \forall\, A \in \mathcal{R}^{(i)} \tag{2.9}$$

with $\mu(\partial A) = 0$ (μ-continuity set), where $\mathcal{R}^{(i)}$ is the collection of sets as defined in (2.7).

REMARK 2.8. In light of Proposition 2.7, when we seek regular variation (or any measure convergence) in the space $\mathbb{E}_d^{(i)}$ using $\mathbb{M}$-convergence (as in Definition 2.3), we can equivalently show this only for rectangular sets in $\mathcal{R}^{(i)}$ (which are also continuity sets with respect to the limit measure).

We wrap this section up with an extension of the so-called “heavier tail wins" phenomenon, in the context of multivariate regular variation on a subcone of $\mathbb{R}_+^d$. It is useful for many of the proofs in this paper.

LEMMA 2.9. *Suppose $\boldsymbol{X} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for some fixed $i \in \mathbb{I}$, and $\boldsymbol{X}$ is independent of the $\mathbb{R}^d$-valued random vector $\boldsymbol{Y}$ with $\mathbb{E}\|\boldsymbol{Y}\|^{\alpha_i+\gamma} < \infty$ for some $\gamma > 0$. Then*

$$\boldsymbol{X} + \boldsymbol{Y} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}).$$

2.3. *A joint multivariate regular variation condition.* Since our interest is in the tail behavior of aggregates over independent regularly varying vectors, when considering joint exceedances of sums of such vectors, regular variation on various combinations of subcones become important. Incidentally, in certain scenarios we encounter a sequence of random vectors, none of which possess MRV on a particular subcone, yet their sum admits MRV on the same subcone. The following definition provides conditions for such vectors to be tractable under aggregation in the multivariate regular variation framework.

DEFINITION 2.10 (Adapted Multivariate Regular Variation). Suppose $\boldsymbol{Z} \in \mathbb{R}_+^d$ is a random vector such that the following holds:

1. $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, \Delta$ for some $\Delta \leq d$.

2. If $\Delta < d$, then additionally assume that there exists a $\gamma \in (0, \alpha_1/d)$, such that for $i = \Delta+1, \ldots, d$ and any $A \in \mathcal{R}^{(i)}$, we have

$$\lim_{t\to\infty} t\,\mathbb{P}\left(\frac{\boldsymbol{Z}}{b_i(t)} \in A\right) = 0 \tag{2.10}$$

where $b_i(t) := t^{1/(i(\alpha_1+\gamma))}$, i.e., with the rate b_i, we have convergence to zero. We refer to (2.10) as *null convergence* and write $\boldsymbol{Z} \in \mathcal{NC}(b_i(t), \mathbb{E}_d^{(i)})$.

Then we say $\boldsymbol{Z}$ is *adapted multivariate regular varying* (or, *adapted-MRV*) on $\mathbb{R}_+^d$ and write $\{\boldsymbol{Z} \in \mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}); i = 1, \ldots, d; \Delta\}$ where $\alpha_i = \infty, \mu_i \equiv 0$ for $i = \Delta+1, \ldots, d$.

Adapted-MRV is defined jointly for all cones $\mathbb{E}_d^{(i)}, i = 1, \ldots, d$. Note the following.

i) For $i = 1, \ldots, \Delta$, $\mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ and $\mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ are equivalent.
ii) For $i = \Delta+1, \ldots, d$, the notation $\mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ means that for some $0 < \gamma < \alpha_1/d$, we have $\boldsymbol{Z} \in \mathcal{NC}(b_i(t) = t^{1/(i(\alpha_1+\gamma))}, \mathbb{E}_d^{(i)})$ and $\alpha_i = \infty, \mu_i \equiv 0$. The constant γ is chosen to be the same for all $i = \Delta+1, \ldots, d$ and the value of $\Delta = \arg\max_i\{\alpha_i : \alpha_i < \infty\}$ is implicit.

EXAMPLE 2.11. Let $X^{(1)}, X^{(2)} \sim F$ be i.i.d random variables with $\mathbb{P}(X^{(1)} > x) = x^{-\alpha}, x > 1$ for some $\alpha > 0$. Let $B^{(1)}, B^{(2)}$ be i.i.d random variables with $\mathbb{P}(B^{(1)} = 1) = 0.5 = \mathbb{P}(B^{(1)} = 0)$. Define for $k = 1, 2$,

$$\boldsymbol{Z}^{(k)} := B^{(k)}(X^{(k)}, 0) + (1 - B^{(k)})(0, X^{(k)}).$$

Then $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}$ are i.i.d. with $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha, b(t) = t^{1/\alpha}, \mu_1, \mathbb{E}_2^{(1)})$ where

$$\mu_1(([0, x_1] \times [0, x_2])^c) = 0.5x_1^{-\alpha} + 0.5x_2^{-\alpha}, \quad x_1, x_2 > 0.$$

Clearly, $\boldsymbol{Z}^{(1)} \in \mathcal{NC}(b^*(t) = t^{1/2(\alpha+\gamma)}, \mathbb{E}_2^{(2)})$ for any $\gamma > 0$. Hence, $\boldsymbol{Z}^{(k)}$ is adapted-MRV with $\Delta = 1$. But we can check that $\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in \mathcal{MRV}(\alpha, b(t) = t^{1/\alpha}, 2\mu_1, \mathbb{E}_2^{(1)})$ and $\boldsymbol{Z}_1 + \boldsymbol{Z}_2 \in \mathcal{MRV}(2\alpha, b(t) = t^{1/(2\alpha)}, \mu_2, \mathbb{E}_2^{(2)})$ where

$$\mu_2((x_1, \infty) \times (x_2, \infty)) = 0.5(x_1x_2)^{-\alpha}, \quad x_1, x_2 > 0.$$

REMARK 2.12. Many examples of regularly varying vectors on $\mathbb{E}_d^{(1)}$ are in fact adapted-MRV, including all examples mentioned in Section 2.4 and more. A few comments on how (2.10) enriches our class of models follows.

(a) If $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$, then clearly $\{\boldsymbol{Z} \in \mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}); i = 1, \ldots, d; \Delta = d\}$. Thus, condition (2.10) provides us a little more flexibility in case we fail to have MRV on subcone $\mathbb{E}_d^{(j)}$ for $j > \Delta$ for some $\Delta = 2, \ldots, d$.
(b) Condition (2.10) is satisfied if $\mathbb{E}(Z_{(i)})^{i(\alpha_1+\widetilde{\gamma})} < \infty$ for some $\widetilde{\gamma}$ with $\gamma < \widetilde{\gamma} < \alpha_1/d$; here $Z_{(1)} \geq \ldots \geq Z_{(d)}$ are the order statistics of the elements of $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$. In particular, one such example is when $Z_{(j)} = 0$ for $j > \Delta$, see Section 3.2. For further examples of multivariate heavy-tailed distributions exhibiting such a property see [8, 12].
(c) If $\Delta < d$, then (2.10) still allows for MRV to hold on $\mathbb{E}_d^{(i)}, i = \Delta+1, \ldots, d$ albeit with a lighter regularly varying tail rate than $i(\alpha_1+\gamma)$.

2.4. *Tail distributions, survival copulas and asymptotic tail independence.* In this section we discuss dependence structures for d-dimensional random vectors, which are used to model risks, claim sizes, or increments in general. Following common practice we model the marginal distributions separately from the dependence structures and hence, resort to using *copulas* (cf. [26, 35]). A key feature of most of the copulas we discuss is the presence of *asymptotic tail independence*, implying that the joint exceedance of a threshold by i components of the random vector occur at a rate negligible compared to joint exceedance of $(i-1)$ components for some or all $i=2,\ldots,d$.

Furthermore, we will also elaborate on multivariate regular variation properties under these copulas. To this end, for all examples in this section, we consider random vectors $\boldsymbol{Z}=(Z_1,\ldots,Z_d)$ with identically distributed continuous marginal components with distribution function F_α where $\overline{F}_\alpha \in \mathcal{RV}_{-\alpha}$ with $\alpha>1$, and the dependence is given by the particular (survival) copula. Moreover, we fix $b_\alpha(t)=\overline{F}_\alpha^{\leftarrow}(1/t), t>1$. Note that assuming tail equivalent marginals would lead to similar conclusions but notations become cumbersome.

Our interest is in tail sets, hence we will often use *survival copulas* along with copulas which we recall briefly here. For a random vector $\boldsymbol{Z}=(Z_1,\ldots,Z_d)\sim F$ with continuous marginal distributions $F_1,\ldots,F_d$, the copula $C:[0,1]^d\to[0,1]$ and the survival copula $\widehat{C}:[0,1]^d\to[0,1]$ are distribution functions such that:

$$F(x_1,\ldots,x_d):=\mathbb{P}(Z_1\le x_1,\ldots,Z_d\le x_d)=C(F_1(x_1),\ldots,F_d(x_d)),\ (x_1,\ldots,x_d)\in\mathbb{R}^d,$$

$$\overline{F}(x_1,\ldots,x_d):=\mathbb{P}(Z_1> x_1,\ldots,Z_d> x_d)=\widehat{C}(\overline{F}_1(x_1),\ldots,\overline{F}_d(x_d)),\ (x_1,\ldots,x_d)\in\mathbb{R}^d,$$

where $\overline{F}_j=1-F_j\ \forall\, j\in\mathbb{I}$.

EXAMPLE 2.13 (Independence copula). A widely used copula to exhibit asymptotic tail independence and hidden regular variation is the independence copula. The independence copula $C_\perp$ and survival copula $\widehat{C}_\perp$ are given by

$$C_\perp(u_1,\ldots,u_d)=\widehat{C}_\perp(u_1,\ldots,u_d)=u_1u_2\cdots u_d,\quad 0<u_i<1. \tag{2.11}$$

Let $\boldsymbol{Z}\sim F$ with identical (continuous) marginal F_α as defined above and dependence given by $C_\perp$ (or $\widehat{C}_\perp$). Then

$$\boldsymbol{Z}\in\mathcal{MRV}(i\alpha, b_\alpha^{1/i},\mu_i,\mathbb{E}_d^{(i)}) \tag{2.12}$$

where

$$\mu_i\left(\{\boldsymbol{z}\in\mathbb{E}_d^{(i)}: z_j>x_j\ \forall\, j\in S\}\right)=\prod_{j\in S}x_j^{-\alpha} \tag{2.13}$$

for $S\subseteq\mathbb{I}$ with $|S|=i, i=1,\ldots,d$ (cf. [9]) and $\mu_i(\mathbb{E}_d^{(i+1)})=0$. Clearly $\boldsymbol{Z}$ exhibits hidden regular variation on all cones $\mathbb{E}_d^{(i)}, i=2,\ldots,d$.

EXAMPLE 2.14 (Marshall-Olkin copula). In reliability theory, the Marshall-Olkin distribution provides an elegant mechanism to capture the dependence between the failure of subsystems in an entire system. We focus on a particular structure of the Marshall-Olkin survival copula as given in [30, eq. (2.4), page 58]. Assume that for all $\emptyset\neq S\subseteq\mathbb{I}$ there exists a parameter $\lambda_S>0$. Consider then the generalized Marshall-Olkin survival copula given by

$$\widehat{C}_{\mathrm{MO}}(u_1,\ldots,u_d)=\prod_{i=1}^{d}\prod_{|S|=i}\bigwedge_{j\in S}u_j^{\eta_j^S},\quad 0<u_j<1, \tag{2.14}$$

where

$$\eta_j^S = \frac{\lambda_S}{\sum\limits_{J \supseteq \{j\}} \lambda_J}, \quad j \in S \subseteq \mathbb{I}. \tag{2.15}$$

A typographical error in the formula for η_j^S in [30, eq. (2.4), page 58] is corrected in (2.15). We consider two particular choices of the parameters λ_S for our examples.

(a) *Equal parameter for all sets:* Let $\lambda_S = \lambda > 0$ for all $\emptyset \neq S \subseteq \mathbb{I}$; hence, from (2.15) we have

$$\eta_j^S = 2^{-(d-1)} =: \beta. \tag{2.16}$$

Therefore

$$\begin{aligned}
\mathbb{P}(Z_1 > x_1, \ldots, Z_d > x_d) &= \widehat{C}(\overline{F}_\alpha(x_1), \ldots, \overline{F}_\alpha(x_d)) \\
&= \prod_{i=1}^d \prod_{|S|=i} \bigwedge_{j \in S} (\overline{F}_\alpha(x_j))^\beta \\
&= \prod_{j=1}^d (\overline{F}_\alpha\left(x_{(j)}\right))^{2^{d-j}\beta} = \prod_{j=1}^d (\overline{F}_\alpha\left(x_{(j)}\right))^{2^{-(j-1)}},
\end{aligned}$$

where $x_{(1)} \geq \ldots \geq x_{(d)}$ denote the decreasing order statistics of $x_1, \ldots, x_d$. Now, we can check that for $i = 1, \ldots, d$,

$$\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$$

where

$$\begin{aligned}
\alpha_i &= (2 - 2^{-(i-1)})\alpha, \\
b_i(t) &= (b_\alpha(t))^{\alpha/\alpha_i} = (b_\alpha(t))^{1/(2-2^{-(i-1)})}, \\
\mu_i\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j \; \forall j \in S\}\right) &= \prod_{j=1}^i \left(x_{(j)}\right)^{-\alpha 2^{-(j-1)}},
\end{aligned} \tag{2.17}$$

where $S \subseteq \mathbb{I}$, $|S| = i$ with $x_j > 0$ for $j \in S$, and $x_{(1)} \geq \ldots \geq x_{(i)}$ denote the decreasing order statistics of $(x_j)_{j \in S}$ and $\mu_i(\mathbb{E}_d^{(i+1)}) = 0$.

(b) *Parameters proportional to cardinality of the sets:* Let $\lambda_S = |S|\,\lambda$ where $\lambda > 0$ for all $\emptyset \neq S \subseteq \mathbb{I}$. From (2.15) we have $\eta_j^S = |S|(d+1)2^{-(d-1)} = |S|(d+1)\beta$ using the definition in (2.16). Following a similar logic as in part (a), we obtain in this case

$$\mathbb{P}(Z_1 > x_1, \ldots, Z_d > x_d) = \prod_{j=1}^d \left(\overline{F}_\alpha(x_{(j)})\right)^{\left(1-\frac{j-1}{d+1}\right)2^{-(j-1)}}.$$

Again we can check that for $i = 1, \ldots, d$,

$$\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i^*, b_i^*, \mu_i^*, \mathbb{E}_d^{(i)})$$

where,

$$\alpha_i^* = \sum_{j=1}^{i}\Big(1-\frac{j-1}{d+1}\Big)\frac{\alpha}{2^{j-1}} = \alpha_i\frac{d}{d+1}+\frac{i\alpha}{(d+1)2^{i-1}},$$

$$b_i^*(t) = (b_\alpha(t))^{\alpha/\alpha_i^*}, \tag{2.18}$$

$$\mu_i^*\left(\{\boldsymbol{z}\in\mathbb{E}_d^{(i)} : z_j > x_j \;\forall\; j\in S\}\right) = \prod_{j=1}^{i}\left(x_{(j)}\right)^{-\alpha\left(1-\frac{j-1}{d+1}\right)2^{-(j-1)}},$$

where $|S| = i$ with $x_j > 0$ for $j \in S$ and $\mu_i^*(\mathbb{E}_d^{(i+1)}) = 0$.

In both examples of the Marshall-Olkin copula dependence (with identical regularly varying margins), $\boldsymbol{Z}$ exhibits hidden regular variation on all cones $\mathbb{E}_d^{(i)}, i = 2,\dots,d$.

EXAMPLE 2.15 (Archimedean copula (ACIG)). This Archimedean copula example based on the Laplace transform of the inverse gamma distribution, called ACIG copula in short, appears in [19] with its hidden regular variation discussed in [20, Example 4.4]. Suppose $\boldsymbol{Z} = (Z_1,\dots,Z_d)$ has an ACIG copula with dependence parameter $1 < \beta < 2$ and identical margins $\overline{F}_\alpha \in \mathcal{RV}_{-\alpha},\; i = 1,\dots,d$. Then $\boldsymbol{Z} \in \mathcal{MRV}(\alpha, b_\alpha, \mu_1, \mathbb{E}_d^{(1)})$ and $\boldsymbol{Z} \in \mathcal{MRV}(\alpha\beta, b_\alpha^{1/\beta}, \mu_2, \mathbb{E}_d^{(i)})$ for $i = 2,\dots,d$. In this particular example $\boldsymbol{Z}$ exhibits hidden regular variation on $\mathbb{E}_d^{(2)}$ but no further HRV at any subsequent co-ordinate subcone $\mathbb{E}_d^{(i)}, i \geq 3$.

EXAMPLE 2.16 (Asymptotically tail dependent copula). In the previous examples we observed distributions with regularly varying marginals and copulas exhibiting *asymptotic tail independence* leading to MRV with different indices on different spaces. But there are distributions which exhibit so-called *asymptotic tail dependence* which would lead to MRV with the same index, rate function and limit measure on all subcones $\mathbb{E}_d^{(i)}$; see [18] for general examples in dimension $d = 2$ and [5] for higher dimensional Archimedean copulas exhibiting asymptotic tail dependence. We illustrate this with one example. Let $\boldsymbol{Z} \sim F$ such that for $\alpha > 1$,

$$F(\boldsymbol{x}) = 1 - \left(1+\sum_{j=1}^{d} x_j^{\alpha}\right)^{-1}, \quad \boldsymbol{x}\in\mathbb{R}_+^d.$$

We can check that the marginals are identically Pareto distributed with index $-\alpha$ and hence, the tails are $\mathcal{RV}_{-\alpha}$. Moreover, $\boldsymbol{Z} \in \mathcal{MRV}(\alpha, b(t) = t^{1/\alpha}, \mu, \mathbb{E}_d^{(i)})$ for $i = 1,\dots,d$ where

$$\mu\left(\{\boldsymbol{z}\in\mathbb{E}_d : z_j > x_j \,\forall\, j\in S\}\right) = \sum_{j=1}^{|S|}(-1)^{j+1}\sum_{\substack{k_1<\dots<k_j\\ k_1,\dots,k_j\in S}}\left(\sum_{l=1}^{j} x_{k_l}^{\alpha}\right)^{-1} \tag{2.19}$$

for $S \subseteq \mathbb{I}$.

REMARK 2.17 (Gaussian copula). Gaussian copulas have been widely considered as a key example of asymptotic tail independence, for which coefficients of tail dependence, tail order and hidden regular variation have been studied in this context; see [17, 20, 29]. Surprisingly, the hidden regular variation properties of Gaussian copulas (with regularly varying marginal distributions) are not particularly well understood, especially in dimensions $d \geq 3$.

For example, the often used Gaussian copula defined by an equi-correlation correlation matrix does not seem to admit hidden regular variation in general; see [11] for details. Thus, we have refrained from using particular examples of Gaussian copulas here.

3. Aggregating regularly varying random vectors. In Section 1, we discussed the principle of one large jump determining the behavior of aggregates of multivariate regularly varying random vectors in the classical framework; here we extend the idea for a more general class of tail events. We start by assuming that individual random vectors have tail equivalent margins and they admit adapted multivariate regular variation on cones $\mathbb{E}_d^{(i)}, i = 1, \ldots, d$ (see Definition 2.10). In our first result, Theorem 3.1, we consider two independent random vectors which are not necessarily identically distributed and assess the tail behavior of the sum for various tail sets. This theorem forms the basis of many subsequent results where we do assume the underlying vectors to be identically distributed as well.

THEOREM 3.1. *Let $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)} \in \mathbb{R}_+^d$ be independent random vectors, each with tail equivalent marginal distributions and $\{\boldsymbol{Z}^{(k)} \in \mathcal{MRV}^*(\alpha_i^{(k)}, b_i^{(k)}, \mu_i^{(k)}, \mathbb{E}_d^{(i)}), i = 1, \ldots, d; \Delta_k\}$ for $k = 1, 2$, i.e., they are adapted-MRV on $\mathbb{R}_+^d$. Define $\alpha_0^{(k)} = 0$, $b_0^{(k)\leftarrow}(t) \equiv 1$, $\mu_0^{(k)} \equiv 1$ and*

$$I(i) := \operatorname{argmax}_{j \in \{0,\ldots,i\}} \{\bar{c}_j^{(i)} : \bar{c}_j^{(i)} < \infty\}$$

where

$$\bar{c}_j^{(i)} := \max_{0 \le m \le i} \left\{ \limsup_{t \to \infty} \frac{b_j^{(1)\leftarrow}(t) b_{i-j}^{(2)\leftarrow}(t)}{b_m^{(1)\leftarrow}(t) b_{i-m}^{(2)\leftarrow}(t)} \right\}.$$

Define for $i = 1, \ldots, d$, and $m = 0, \ldots, i$:

$$c_m^{I(i)} := \lim_{t \to \infty} \frac{b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)}{b_m^{(1)\leftarrow}(t) b_{i-m}^{(2)\leftarrow}(t)}.$$

Suppose that for $k = 1, 2$, and each $m = 1, \ldots, i-1$, either $c_m^{I(i)} = 0$ or $\lim_{t\to\infty} b_m^{(k)\leftarrow}(t)/b_{m+1}^{(k)\leftarrow}(t) = 0$. Then

$$\{\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in \mathcal{MRV}^*(\alpha_i, b_i, \mu_i^{\oplus}, \mathbb{E}_d^{(i)}), i = 1, \ldots, d; \Delta^{\oplus}\}$$

with $\alpha_i = \alpha_{I(i)}^{(1)} + \alpha_{i-I(i)}^{(2)}$, $b_i^{\leftarrow}(t) = b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)$, and

$$\mu_i^{\oplus}(A) = \sum_{m=0}^{i} c_m^{I(i)} \mu_{m,i}^*(A) \qquad \text{for } A \in \mathcal{B}(\mathbb{E}_d^{(i)}),$$

where $\mu_{m,i}^*$ is the measure which is uniquely defined on $\mathcal{R}^{(i)}$ as follows: for $A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \,\forall\, j \in S\} \in \mathcal{R}^{(i)}$ with $S \subseteq \mathbb{I}$, $|S| \ge i$ and $x_j > 0 \;\forall j \in S$ we have

$$\mu_{m,i}^*(A)$$
$$= \sum_{\substack{J \subseteq S \cup \{\emptyset\} \\ |J| = m}} \mu_m^{(1)}\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(m)} : z_j > x_j \,\forall\, j \in J\}\right) \mu_{i-m}^{(2)}\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i-m)} : z_j > x_j \,\forall\, j \in S \backslash J\}\right).$$

Moreover, $\Delta^{\oplus} \in \{\max(\Delta_1 + 1, \Delta_2 + 1), \ldots, \min(\Delta_1 + \Delta_2, d)\}$.

The proof of Theorem 3.1 is given in Appendix B.

REMARK 3.2. Since the output of argmax may contain multiple elements, $I(i)$ is not defined uniquely; hence, a value for $I(i)$ is often chosen from these outputs according to convenience.

We have refrained from stating a general result akin to Theorem 3.1 for adding n random vectors since the parameters of the limit model become notationally cumbersome without providing additional insight; on the other hand, for a variety of joint dependence behavior, we often observe nicer structures appearing. In the rest of the section, we discuss consequences of Theorem 3.1 on the finite sum of i.i.d. random vectors under various assumptions on their dependence structures.

3.1. *All subcones exhibit regular variation.* First, we investigate the case where we add i.i.d. random vectors which are multivariate regularly varying on all relevant cones. The results as we will see are direct consequences of Theorem 3.1. We begin with the well-known model where all components of each vector are i.i.d. random variables as well; Example 2.13 gives the structure of the limit measure in this case. The following proposition provides a slightly general version of this case. Proofs of the results of this subsection are available in Appendix C.

PROPOSITION 3.3 (Nearly independent case). *Let $\boldsymbol{Z}^{(1)},\dots,\boldsymbol{Z}^{(n)}$ be i.i.d. random vectors in $\mathbb{R}_+^d$ with tail equivalent marginal distributions and $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i=1,\dots,d$ where $b_i(t)=(b_1(t))^{1/i}$ and $b_1(t)\in\mathcal{RV}_{1/\alpha}$. Then $\alpha_i = i\alpha$ and*

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, \mathbb{E}_d^{(i)}) \quad \textit{for} \quad i=1,\dots,d. \tag{3.1}$$

Now if for some $\kappa_j>0$, $j=1,\dots,d$,

$$\mu_i\left(\{\boldsymbol{z}\in\mathbb{E}_d^{(i)} : z_j > x_j \,\forall\, j\in S\}\right) = \prod_{j\in S}\kappa_j x_j^{-\alpha} \tag{3.2}$$

for $S\subseteq\mathbb{I}$ with $|S|=i$, $x_j>0$ and $\mu_i(\mathbb{E}_d^{(i+1)})=0$, $i=1,\dots,d$, then

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(i\alpha, b_1^{1/i}, n^i\mu_i, \mathbb{E}_d^{(i)}) \quad \textit{for} \quad i=1,\dots,d.$$

REMARK 3.4. If all components of the random vectors $\boldsymbol{Z}^{(1)},\dots,\boldsymbol{Z}^{(n)}$ are completely tail equivalent then $\kappa_1=\kappa_2=\ldots=\kappa_d$.

Although condition (3.2) is rather restrictive, the result obtained in (3.1), i.e., if $\boldsymbol{Z}^{(1)}\in\mathcal{MRV}(\alpha_i,b_i,\mathbb{E}_d^{(i)})$ then $\sum_{k=1}^n \boldsymbol{Z}^{(k)}\in\mathcal{MRV}(\alpha_i,b_i,\mathbb{E}_d^{(i)})$, holds under much weaker assumptions. The particular assumption (3.2) helps only to calculate the exact form of the limit distribution. The following result provides a further case and helps in creating many examples.

PROPOSITION 3.5. *Let $\boldsymbol{Z}^{(1)},\dots,\boldsymbol{Z}^{(n)}$ be i.i.d. random vectors in $\mathbb{R}_+^d$ with tail equivalent marginal distributions and $\boldsymbol{Z}^{(1)}\in\mathcal{MRV}(\alpha_i,b_i,\mu_i,\mathbb{E}_d^{(i)})$ for $i=1,\dots,d$. Moreover, assume that $\alpha_i<\alpha_m+\alpha_{i-m}$ for all $m=1,\dots,i-1$ and $i=2,\dots,d$. Then*

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, n\mu_i, \mathbb{E}_d^{(i)}) \quad \textit{for} \quad i=1,\dots,d.$$

REMARK 3.6. Clearly, a sufficient condition for Proposition 3.5 to hold would be to assume that $c_m^{I(i)} = 0, m = 1, \ldots, i-1$ and $i = 2, \ldots, d$ instead of $\alpha_i < \alpha_m + \alpha_{i-m}$ for all $m = 1, \ldots, i-1$ and $i = 2, \ldots, d$. This requires the notation of Theorem 3.1, and we prefer the latter in lieu of interpretability.

REMARK 3.7. Both in Propositions 3.3 and 3.5, we observe that while adding finitely many random vectors $\boldsymbol{Z}^{(1)}, \ldots, \boldsymbol{Z}^{(n)}$, we obtain $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_{i,n}^{\oplus}, \mathbb{E}_d^{(i)})$. The indices of regular variation α_i and the scaling parameter b_i remain the same no matter how many vectors we add although the measure $\mu_{i,n}^{\oplus}$ are quite different for different values of n. Note the following.

i) Under the assumptions of Proposition 3.3, we have, $\alpha_i = \alpha_m + \alpha_{i-m}$ for $m = 0, \ldots, i$, and $\mu_{i,n}^{\oplus} = n^i \mu_i$. Interestingly, $\alpha_i = \alpha_m + \alpha_{i-m}$ for $m = 0, \ldots, i$ does not necessarily imply that $\mu_{i,n}^{\oplus} = n^i \mu_i$.
ii) Under the assumptions of Proposition 3.5, we have, $\alpha_i < \alpha_m + \alpha_{i-m}$ for $m = 1, \ldots, i-1$, which turns out to be a sufficient condition for $\mu_{i,n}^{\oplus} = n\mu_i$.

EXAMPLE 3.8 (Marshall-Olkin dependence). For this example let $\boldsymbol{Z}, \boldsymbol{Z}^{(1)}, \ldots, \boldsymbol{Z}^{(n)}$ be i.i.d. random vectors in $\mathbb{R}_+^d$ with all marginal distributions following F_α where $\overline{F}_\alpha \in \mathcal{RV}_{-\alpha}$ and $b_\alpha(t) = \overline{F}^{\leftarrow}(1/t)$ for some $\alpha > 0$. We consider two different Marshall-Olkin dependence parameters for $\boldsymbol{Z}$ which has copula $\widehat{C}_{\text{MO}}$ given by (2.14); see Example 2.14.

(a) *Equal parameter:* Suppose the parameters defining the Marshall-Olkin copula are given by (2.16). Then from Example 2.14(a) we have $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ where α_i, b_i, μ_i are given by (2.17). Now observe that for fixed $i = 1, \ldots, d$, and $m = 1, \ldots, i-1$,

$$\begin{aligned} \alpha_m + \alpha_{i-m} &= (2 - 2^{-(m-1)})\alpha + (2 - 2^{-(i-m-1)})\alpha \\ &\geq 2\alpha \\ &> (2 - 2^{-(i-1)})\alpha = \alpha_i. \end{aligned} \tag{3.3}$$

Hence, by Proposition 3.5, we have

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, n\mu_i, \mathbb{E}_d^{(i)}).$$

(b) *Proportional parameter:* We know from Example 2.14(b) that $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i^*, b_i^*, \mu_i^*, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ where $\alpha_i^*, b_i^*, \mu_i^*$ are given by (2.18). Again note that for fixed $i = 1, \ldots, d$, $m = 1, \ldots, i-1$, and α_i as in (2.17),

$$\begin{aligned} \alpha_m^* + \alpha_{i-m}^* &= \alpha_m \frac{d}{d+1} + \frac{m\alpha}{(d+1)2^{m-1}} + \alpha_{i-m} \frac{d}{d+1} + \frac{(i-m)\alpha}{(d+1)2^{i-m-1}} \\ &= (\alpha_m + \alpha_{i-m}) \frac{d}{d+1} + \frac{\alpha}{(d+1)} \left(\frac{m}{2^{m-1}} + \frac{i-m}{2^{i-m-1}} \right) \\ &> \alpha_i \frac{d}{d+1} + \frac{\alpha}{(d+1)} \left(\frac{m}{2^{i-1}} + \frac{i-m}{2^{i-1}} \right) \quad \text{(using (3.3))} \\ &= \alpha_i^*. \end{aligned}$$

Hence, by Proposition 3.5, we have

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i^*, b_i^*, n\mu_i^*, \mathbb{E}_d^{(i)}).$$

EXAMPLE 3.9 (Archimedean copula). Referring to Example 2.15, suppose we have i.i.d. random vectors $\boldsymbol{Z}^{(1)}, \dots, \boldsymbol{Z}^{(n)}$ with identical margins F_α so that $\overline{F}_\alpha \in \mathcal{RV}_{-\alpha}, b_\alpha = \overline{F}^{\leftarrow}(1/t)$ and they admit an ACIG copula as described with dependence parameter $1 < \beta < 2$. Then we have $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha, b_\alpha, \mu_1, \mathbb{E}_d^{(1)})$ and $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha\beta, b_\alpha^{1/\beta}, \mu_2, \mathbb{E}_d^{(i)})$ for $i = 2, \dots, d$. Now, clearly the conditions for Proposition 3.5 are satisfied and hence, we have $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha, b_\alpha, n\mu_1, \mathbb{E}_d^{(1)})$ and $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha\beta, b_\alpha^{1/\beta}, n\mu_2, \mathbb{E}_d^{(i)})$ for $i = 2, \dots, d$.

The two extreme cases of dependence considered in general are the case of fully independent components for $\boldsymbol{Z}^{(1)}$, which is covered in Proposition 3.3, and the case where the components of $\boldsymbol{Z}^{(1)}$ are dependent such that $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{E}_d^{(i)})$ for $i = 1, \dots, d$. The following corollary addresses the latter case.

COROLLARY 3.10 (Dependent case, corollary to Proposition 3.5). *Let $\boldsymbol{Z}^{(1)}, \dots, \boldsymbol{Z}^{(n)}$ be i.i.d. random vectors in $\mathbb{R}_+^d$ with tail equivalent marginal distributions and $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \dots, d$. Moreover, $(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}) = (\alpha, b, \mu, \mathbb{E}_d^{(i)})$ for $i = 1, \dots, i^*$ for some $i^* \le d$. Then*

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha, b, n\mu, \mathbb{E}_d^{(i)}) \quad \textit{for} \quad i = 1, \dots, i^*.$$

EXAMPLE 3.11. Example 2.16 exhibiting asymptotic tail dependence admits the property $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{E}_d^{(i)})$ for $i = 1, \dots, d$ and hence, we can compute tail asymptotics of its convolution using Corollary 3.10.

In all the propositions, corollaries and hence examples of this section, we observe that if $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$, then their finite sum $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, C_{n,i}\mu_i, \mathbb{E}_d^{(i)})$ for some constant $C_{n,i} > 0$. The "one large jump" phenomenon is observed here, in the sense of equivalence of the first and last expressions in (1.3) or (1.6) if $C_{n,i} = n$; cf. Proposition 3.5. But $C_{n,i}$ is not necessarily n, as for example in Proposition 3.3, and this is a case which we think of as a phenomenon of "more than one large jump" or "a few large jumps". Note that in both cases mentioned here, we did assume $\boldsymbol{Z}^{(1)}$ to have MRV on all cones, but not be strictly adapted-MRV. In Section 3.2, we illustrate that a similar principle holds, even under the assumption of adapted-MRV, although the characterizing jumps are now of the form (1.7) which relates to "a few large jumps" phenomenon.

3.2. *Not all subcones necessarily exhibit regular variation.* In certain contexts, we may be interested in adding random vectors which are not necessarily MRV on all relevant cones. For example, we may have a sequence of i.i.d. random vectors for which not all components are non-zero in each realization. An extension of such aggregation to random sums lead to general compound Poisson or Lévy processes, see Section 4 for details. In this section, we concentrate on a few such examples. The general structure for the limit measures in such problems are often not quite apparent.

PROPOSITION 3.12. *Let $\boldsymbol{Z}^{(1)},\dots,\boldsymbol{Z}^{(n)} \in \mathbb{R}_+^d$ be i.i.d. random vectors with tail equivalent marginal distributions which are $\mathcal{RV}_{-\alpha}$ and let $\{\boldsymbol{Z}^{(1)} \in \mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}), i = 1,\dots,d; \Delta = 1\}$ with $\alpha_1 = \alpha$. Then*

$$\left\{\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}^*(\alpha_{i,n}, b_{i,n}, \mu_{i,n}^{\oplus}, \mathbb{E}_d^{(i)}), i = 1,\dots,d; \Delta = \min\{d,n\}\right\}.$$

Specifically, for $i = 1,\dots,d$ we have the following:

(a) *For $n \geq i$,*

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}\left(\alpha_{i,n} = i\alpha, b_{i,n} = b_1^{1/i}, \mu_{i,n}^{\oplus}, \mathbb{E}_d^{(i)}\right), \tag{3.4}$$

where

$$\mu_{i,n}^{\oplus}\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j \,\forall\, j \in S\}\right) = \frac{n!}{(n-i)!} \prod_{j\in S} \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right), \tag{3.5}$$

for $S \subseteq \mathbb{I}$ with $|S| = i$, $x_j > 0$ for $j \in S$ and $\mu_{i,n}^{\oplus}(\mathbb{E}_d^{(i+1)}) = 0$.

(b) *For $1 \leq n < i$,*

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(k)} \in \mathcal{NC}\left(b_{i,n}(t) = t^{1/i(\alpha+\gamma)}, \mathbb{E}_d^{(i)}\right). \tag{3.6}$$

The proof of Proposition 3.12 is in Appendix D.

REMARK 3.13. In Proposition 3.12, if the marginal distributions are completely tail equivalent with distribution functions $F_j, j = 1,\dots,d$ and $b_1(t) = \overline{F}_1^{\leftarrow}(1/t)$, then $\mu_{i,n}^{\oplus}$ in (3.5) is given by

$$\mu_{i,n}^{\oplus}\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j \,\forall\, j \in S\}\right) = \frac{n!}{(n-i)!} \prod_{j\in S} x_j^{-\alpha}.$$

REMARK 3.14. For the conclusion of Proposition 3.12 to hold, the random variables $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \dots$ need not be identically distributed as long as they are independent and are all adapted-MRV with the same sets of parameters. The proof follows by similar arguments as the proof of Proposition 3.12 and is skipped.

The phenomenon of a “few large jumps" holds here too, as illustrated next. Assume that in Proposition 3.12, the marginal distributions are completely tail equivalent as in Remark 3.13; and $\boldsymbol{Z} = (Z_1,\dots,Z_d) \sim \boldsymbol{Z}^{(1)}$. Without loss of generality let tA be the tail event of interest where $A = \{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j \,\forall\, j \in \{1,\dots,i\}\}$. Note that from (3.4) and (3.5), we can infer that in fact as $t \to \infty$,

$$\begin{aligned}
\mathbb{P}(\boldsymbol{Z}^{(1)} + \dots + \boldsymbol{Z}^{(n)} \in tA) &\sim \frac{1}{(b_1^{\leftarrow}(t)^i)} \frac{n!}{(n-i)!} \prod_{j=1}^{i} x_j^{-\alpha} \\
&= C_{n,i} \prod_{j=1}^{i} \frac{x_j^{-\alpha}}{b_1^{\leftarrow}(t)} \\
&\sim C_{n,i} \prod_{j=1}^{i} \mathbb{P}(Z_j > tx_i),
\end{aligned}$$

where $C_{n,i} = (n!)/((n-i)!)$. Hence $\boldsymbol{Z}^{(1)} + \ldots + \boldsymbol{Z}^{(n)} \in tA$ occurs at the same rate with which i independent univariate marginals cross their respective thresholds, indicating i many large jumps. The constant $C_{n,i}$ gives the number of possible choices of independent jumps, here the marginals jumps counted are all from different variables $\boldsymbol{Z}^{(k)}$.

In the rest of the section we provide examples exhibiting Proposition 3.12 and its possible generalisation.

EXAMPLE 3.15. Let $(X^{(k)})_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with distribution function F and $\overline{F} \in \mathcal{RV}_{-\alpha}, \alpha > 0$. Let $(\boldsymbol{B}^{(k)})_{k\in\mathbb{N}}$ be i.i.d. random vectors taking values in $\{e_1, \ldots, e_d\}$ with $\mathbb{P}(\boldsymbol{B}^{(1)} = e_l) = p_l \geq 0, \sum_l p_l = 1$, and $e_l = (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0,1\}^d$ where the only non-zero entry 1 is at the l-th place. Define $\boldsymbol{Y}^{(k)} := X^{(k)}\boldsymbol{B}^{(k)}, k \in \mathbb{N}$. Moreover, let $(\boldsymbol{\varepsilon}^{(k)})_{k\in\mathbb{N}}$ be an i.i.d. sequence of random vectors with $\mathbb{E}\|\boldsymbol{\varepsilon}^{(k)}\|^{d(\alpha+\theta)} < \infty$ for some $\theta > 0$. Finally, also assume that $X^{(1)}, X^{(2)}, \ldots, \boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \ldots, \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}, \ldots$ are independent. Then $\boldsymbol{Z}^{(k)} := \boldsymbol{Y}^{(k)} + \boldsymbol{\varepsilon}^{(k)}$, $k \in \mathbb{N}$, are a sequence of i.i.d. adapted-MRV random vectors with $\Delta = 1$ (cf. Lemma 2.9) and hence, Proposition 3.12 (along with Remark 3.14) provides the tail asymptotic behavior of $\sum_{k=1}^n \boldsymbol{Z}^{(k)}$ for any $n \geq 1$.

The neat expressions for limit measures and tail indices as obtained by Proposition 3.12 in aggregating i.i.d adapted-MRV random vectors with $\Delta = 1$ does not extend as nicely for $\Delta > 1$. Nevertheless, we may still be able to find a pattern in certain cases and our next example with $\Delta = 2$ elaborates on this.

EXAMPLE 3.16. The setting is similar to Example 3.15. Let $X^{(1)}, X^{(2)}, \ldots, \widetilde{X}^{(1)}, \widetilde{X}^{(2)}, \ldots \sim F$ be i.i.d. random variables with $\overline{F} \in \mathcal{RV}_{-\alpha}, \alpha > 0$. Let $\boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \ldots, \widetilde{\boldsymbol{B}}^{(1)}, \widetilde{\boldsymbol{B}}^{(2)}, \ldots$ be i.i.d. random vectors taking values in $\{e_1, \ldots, e_d\}$ as defined in Example 3.15. Also assume that $X^{(1)}, X^{(2)}, \ldots, \widetilde{X}^{(1)}, \widetilde{X}^{(2)}, \ldots, \boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \ldots, \widetilde{\boldsymbol{B}}^{(1)}, \widetilde{\boldsymbol{B}}^{(2)}, \ldots$ are mutually independent. Then $\boldsymbol{Y}^{(k)} := 2^{-1/\alpha} X^{(k)} \boldsymbol{B}^{(k)} + 2^{-1/\alpha} \widetilde{X}^{(k)} \widetilde{\boldsymbol{B}}^{(k)}, k \in \mathbb{N}$, are i.i.d. adapted-MRV random vectors with $\Delta = 2$. Specifically for any $k \geq 1$:

i) $\boldsymbol{Y}^{(k)} \in \mathcal{MRV}(\alpha, b_1, \mu_1, \mathbb{E}_d^{(1)})$ where $b_1(t) = \overline{F}^{\leftarrow}(1/t)$ and with $A_1 = \{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\} \in \mathcal{R}^{(1)}$ for some $j \in \mathbb{I}$, $\mu_1(A_1) = p_j x_j^{-\alpha}$.
ii) $\boldsymbol{Y}^{(k)} \in \mathcal{MRV}(2\alpha, b_1^{1/2}, \mu_2, \mathbb{E}_d^{(2)})$ where with $A_2 = \{\boldsymbol{z} \in \mathbb{E}_d^{(2)} : z_j > x_j,\, z_\ell > x_\ell\} \in \mathcal{R}^{(2)}$ for some $j, \ell \in \mathbb{I}$, $j \neq \ell$ we have $\mu_2(A_2) = \frac{1}{2} p_j p_\ell (x_j x_\ell)^{-\alpha}$.
iii) For $i = 3, \ldots, d$ and some $0 < \gamma < \alpha/d$, we have $\boldsymbol{Y}^{(k)} \in \mathcal{NC}(t^{1/i(\alpha+\gamma)}, \mathbb{E}_d^{(i)})$.

Applying Theorem 3.1, and following the proof of Proposition 3.12, we can show that for $n \geq i$,

$$\sum_{k=1}^n \boldsymbol{Y}^{(k)} \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mu_{i,n}^{*\oplus}, \mathbb{E}_d^{(i)}),$$

where

$$\begin{aligned} \mu_{i,n}^{*\oplus}\Big(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j \,\forall j \in S\}\Big) &= f_i(n) \prod_{j\in S} \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right) \\ &= f_i(n) \prod_{j\in S} p_j x_j^{-\alpha}, \end{aligned} \tag{3.7}$$

for $S \subseteq \mathbb{I}$ with $|S| = i$, $x_j > 0$ for $j \in S$ and some function $f_i : \mathbb{N} \to \mathbb{R}_+$ where

$$\frac{n!}{(n-i)!} \leq f_i(n) \leq n^i, \quad n \geq i,\; i \in \mathbb{I}.$$

Furthermore, $\mu^{*\oplus}_{i,n}(\mathbb{E}_d^{(i+1)}) = 0$. In particular, we can check that

$$f_1(n) = n, \; n \geq 1, \qquad f_2(n) = n(n-1/2), \; n \geq 1,$$
$$f_3(n) = n(n-1/2)(n-1), \; n \geq 2, \qquad f_4(n) = n(n-1/2)(n-1)(n-3/2), \; n \geq 2.$$

A pattern in the value of f emerges for this example, but it depends on the limit measures of the underlying variables $\boldsymbol{Y}^{(k)}$. Examples in the same spirit can be computed for $\Delta \geq 3$ involving some careful combinatorial accounting.

REMARK 3.17. It is easy to extend Example 3.16 in the spirit of Example 3.15. Suppose $\boldsymbol{Y}^{(k)}$, $k \in \mathbb{N}$, are the same random vectors as in Example 3.16 and $(\boldsymbol{\varepsilon}^{(k)})_{k\in\mathbb{N}}$ are i.i.d. random vectors with $\mathbb{E}\|\boldsymbol{\varepsilon}^{(k)}\|^{d(\alpha+\theta)} < \infty$ for some $\theta > 0$, which are also independent of the sequence $(\boldsymbol{Y}^{(k)})_{k\in\mathbb{N}}$. Then $\boldsymbol{Z}^{(k)} := \boldsymbol{Y}^{(k)} + \boldsymbol{\varepsilon}^{(k)}$, $k \in \mathbb{N}$, is an adapted-MRV sequence of random vectors with $\Delta = 2$. All the conclusions for $(\boldsymbol{Y}^{(k)})_{k\in\mathbb{N}}$, and $\sum_{k=1}^n \boldsymbol{Y}^{(k)}, n \geq i$, in Example 3.16 also hold for $(\boldsymbol{Z}^{(k)})_{k\in\mathbb{N}}$, and $\sum_{k=1}^n \boldsymbol{Z}^{(k)}, n \geq i$, by an application of Lemma 2.9.

4. Random sums of regularly varying random vectors.

A natural extension from aggregating finitely many random vectors is to aggregate randomly many random vectors, which we discuss in this section, finally leading towards an extension to Lévy processes in Section 5. We observed in Section 3 that the behavior of the finite sum may take various forms even when they are multivariate regularly varying. Hence, for convenience, for the rest of the paper, we assume that the following is satisfied.

ASSUMPTION A. *Let $\boldsymbol{Z}, (\boldsymbol{Z}^{(k)})_{k\in\mathbb{N}}$ be a sequence of i.i.d. random vectors in $\mathbb{R}_+^d$. Assume that for all $i = 1, \ldots, d$ there exists a measurable function $f_i : \mathbb{N} \to \mathbb{R}_+$ and a non-null measure $\mu_i \in \mathbb{M}(\mathbb{E}_d^{(i)})$ such that for any $n \in \mathbb{N}$,*

$$\left\{ \sum_{k=1}^n \boldsymbol{Z}^{(k)} \in \mathcal{MRV}^*(\alpha_{i,n}, b_{i,n}, \mu^{\oplus}_{i,n} = f_i(n)\mu_i, \mathbb{E}_d^{(i)}); i = 1, \ldots, d; \Delta_n \right\}$$

and $\Delta_n = d$ for $n \geq d$. Furthermore, assume that for all $i = 1, \ldots, d$, there exist a finite constant $\alpha_i > 0$ and a regularly varying function $b_i(t) \in \mathcal{RV}_{1/\alpha_i}$ such that if $f_i(n) \neq 0$ we have $\alpha_{i,n} = \alpha_i$ and $b_{i,n} = b_i$.

REMARK 4.1.

(a) In general, the structure of the function f_i can be quite complex and often requires an involved combinatorial accounting procedure, see Example 3.16; nevertheless in several examples we do observe that $f_i(n) = n$ and in all our examples $0 \leq f_i(n) \leq n^i$. Assumption A allows us the flexibility to not get involved in the computation of f_i.

(b) For $\Delta_n < d$ we have $f_i(n) = 0$ and $\alpha_{i,n} = \infty$ for $i = \Delta_n + 1, \ldots, d$, and hence, we have null convergence. On the other hand, for $i = 1, \ldots, \Delta_n$ we have $0 < f_i(n) < \infty$, $\alpha_{i,n} = \alpha_i$ and $b_{i,n} = b_i$. For the examples considered in Section 3.2 this happens to be the case.

(c) Suppose $\Delta_1 = d$, then Assumption A implies that $I(i)$ as defined in Theorem 3.1 can be chosen to be i and hence, $\alpha_m + \alpha_{i-m} \geq \alpha_i$ for every $m = 0, \ldots, i$ and $i = 1, \ldots, d$. On the other hand, $\alpha_m + \alpha_{i-m} > \alpha_i$ for every $m = 1, \ldots, i-1$ is a sufficient condition for $I(i) = i$.

REMARK 4.2. Under Assumption A, define $\boldsymbol{Z}^{(\oplus,k)} := \sum_{l=(k-1)d+1}^{kd} \boldsymbol{Z}^{(l)}$, $k \in \mathbb{N}$. Also let $\boldsymbol{Z}^{\oplus} := (Z_1^{\oplus}, \ldots, Z_d^{\oplus}) \sim \boldsymbol{Z}^{(\oplus,1)}$ and $Z_{(1)}^{\oplus} \geq \ldots Z_{(d)}^{\oplus}$ be the order statistics of $Z_1^{\oplus}, \ldots, Z_d^{\oplus}$.

(a) From Assumption A we have

$$\sum_{k=1}^{n} \boldsymbol{Z}^{(\oplus,k)} \in \mathcal{MRV}(\alpha_i, b_i, f_i(dn)\mu_i, \mathbb{E}_d^{(i)})$$

with $0 < f_i(dn) < \infty$ for $i = 1, \ldots, d$ and $n \in \mathbb{N}$. Now, a consequence of Remark 4.1 (c) is that $I(i)$ (as defined in Theorem 3.1) for the random vector $\boldsymbol{Z}^{(\oplus,1)}$ (or equivalently $\boldsymbol{Z}^{\oplus}$) is equal to i and hence, $\alpha_m + \alpha_{i-m} \geq \alpha_i$ for every $m = 0, \ldots, i$ and $i = 1, \ldots, d$. Now, $I(i) = i$ implies as well that there exists a finite constant $C^* > 0$ such that

$$0 \leq \sup_{t>0} \sum_{i=1}^{d} \sum_{m=0}^{i} \frac{\mathbb{P}\left(Z_{(m)}^{\oplus} > t\right) \mathbb{P}\left(Z_{(i-m)}^{\oplus} > t\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} \leq C^*. \tag{4.1}$$

(b) Note that the function $g_i(t) := \mathbb{P}\left(Z_{(i)}^{\oplus} > t\right) \in \mathcal{RV}_{-\alpha_i}$ for any $i \in \mathbb{I}$. Hence, using Potter's bound [13, Proposition B.1.19 (5)] there exists a finite constant $C^{**} > 0$ such that

$$\sup_{i \in \mathbb{I}} \sup_{t>0} \frac{\mathbb{P}\left(Z_{(i)}^{\oplus} > t/2\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} \leq C^{**}. \tag{4.2}$$

THEOREM 4.3. *Let Assumption A hold and let the i.i.d. sequence $(\boldsymbol{Z}^{(k)})_{k \in \mathbb{N}}$ be independent of the $\mathbb{N}_0$-valued random variable τ with $\mathbb{E}(\kappa^{\tau}) < \infty$ for any $\kappa > 0$. Then for $i = 1, \ldots, d$,*

$$\sum_{k=1}^{\tau} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, \mathbb{E}(f_i(\tau))\mu_i, \mathbb{E}_d^{(i)}).$$

The proof is in Appendix E. Note that the examples considered in both Section 3.1 and Section 3.2, all satisfy Assumption A (as well as (4.1)). Hence, for any $\mathbb{N}_0$-valued random variable τ whose moment generating function exists on the positive real line, we can compute the tail probability of a random sum of τ many i.i.d. MRV random vectors using Theorem 4.3.

5. Regular variation in multivariate Lévy processes. A particular example of a random sum of i.i.d. random vectors as indicated in Theorem 4.3 is the compound Poisson process at a fixed time point where the number of summands τ is Poisson distributed, which in turn is an example of a Lévy process. In this section, we investigate multivariate regular variation of Lévy processes $\boldsymbol{L} = (\boldsymbol{L}(s))_{s \geq 0}$ on different subcones $\mathbb{E}_d^{(i)}$, $i = 1, \ldots, d$, and relate it to multivariate regular variation of the Lévy measure Π on those subcones. A Lévy process is characterized by its *Lévy-Khinchine representation* $\mathbb{E}(\mathrm{e}^{i\langle \Theta, \boldsymbol{L}(s) \rangle}) = \exp(-s\,\Psi(\Theta))$ for $\Theta \in \mathbb{R}^d$, where

$$\Psi(\Theta) = -i\langle \gamma, \Theta \rangle + \frac{1}{2}\langle \Theta, \Sigma\, \Theta \rangle + \int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{i\langle \Theta, \boldsymbol{x} \rangle} + i\langle \boldsymbol{x}, \Theta \rangle\right) \Pi(d\boldsymbol{x})$$

with $\gamma \in \mathbb{R}^d$, Σ a non-negative definite matrix in $\mathbb{R}^{d \times d}$ and a Borel measure Π on $\mathbb{R}^d$, called the Lévy measure which satisfies $\int_{\mathbb{R}^d} \min\{\|\boldsymbol{x}\|^2, 1\} \Pi(d\boldsymbol{x}) < \infty$ and $\Pi(\boldsymbol{0}) = 0$. Moreover, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$. The Lévy measure $\Pi(A)$ measures the expected number of jumps of the Lévy process in the interval $[0,1]$ which lies in the set A. We denote by Π_j for $j=1,\dots,d$ the marginal Lévy measures. In this paper, we restrict to Lévy processes in $\mathbb{R}_+^d$, i.e., the marginal Lévy processes are *subordinators*, which are increasing Lévy processes. For more details on Lévy processes see [1, 41].

Regular variation in multivariate Lévy processes, especially characterizing complex tail events, including but not restricted to (1.5), can happen in a variety of ways. We may observe regular variation for the Lévy process itself, or the Lévy measure, and they may have different implications depending on the dependence structure of the Lévy process. In the following three subsections we investigate this in detail; the proofs of the associated results are provided in Appendix F.

5.1. *The Lévy measure admits regular variation on all subcones.* In the first subsection, we assume that the Lévy measure is multivariate regularly varying on all subcones $\mathbb{E}_d^{(i)}$, $i=1,\dots,d$ and show that the same is true for the Lévy process, in fact, they are *tail equivalent* (in a multivariate sense) as we exhibit next. We understand (multivariate) tail equivalence as an extension of (2.1) to appropriate sets $A\in\mathcal{B}^{(i)}$. The result can be seen as an extension of (1.8) to subcones (cf. [22]).

PROPOSITION 5.1 (Extending Proposition 3.5). *Let $(\boldsymbol{L}(s))_{s\geq 0}$ be a Lévy process in $\mathbb{R}_+^d$ with Lévy measure $\Pi\in\mathcal{MRV}(\alpha_i,b_i,\mu_i,\mathbb{E}_d^{(i)})$ for $i=1,\dots,d$ whose univariate marginal Lévy measures are tail equivalent. Moreover assume that $\alpha_i<\alpha_m+\alpha_{i-m}$ for all $m=1,\dots,i-1$ and $i=2,\dots,d$. For $s>0$ we have then*

$$\boldsymbol{L}(s)\in\mathcal{MRV}(\alpha_i,b_i,s\mu_i,\mathbb{E}_d^{(i)})\quad \textit{for}\quad i=1,\dots,d.$$

A direct consequence of Proposition 5.1 is the tail equivalence of the Lévy measure of the set tA and the probability measure of the Lévy process belonging to tA, for Borel sets $A\in\mathcal{B}^{(i)}$ bounded away from $\mathbb{CA}_d^{(i-1)}$ with $\mu_i(A)>0$ and $\mu_i(\partial A)=0$ such that

$$\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim s\,\mathbb{P}(\boldsymbol{L}(1)\in tA)\sim s\,\Pi(tA)\sim \frac{s}{b_i^{\leftarrow}(t)}\,\mu_i(A)\qquad \text{as } t\to\infty.$$

Although the tail equivalence of the Lévy process and the Lévy measure holds for a variety of sets, the tail rate differs depending on which subcone $\mathbb{E}_d^{(i)}$ the set A belongs to. A similar conclusion was shown in [22], but only for sets A with $\mu_1(A)>0$. However, in many situations this is not the case as we see in the following examples.

EXAMPLE 5.2. Let $\boldsymbol{L}$ be a compound Poisson process of the form $\boldsymbol{L}(s)=\sum_{k=1}^{N(s)}\boldsymbol{Z}^{(k)}$ where the jump sizes $(\boldsymbol{Z}^{(k)})_{k\in\mathbb{N}}$ are i.i.d. and independent of the Poisson process $(N(s))_{s\geq 0}$ with intensity $\lambda>0$. Suppose the jump size $\boldsymbol{Z}^{(1)}$ has identical marginals which have $\mathcal{RV}_{-\alpha}$ tail distributions with tail index $-\alpha<0$.

(a) Let the dependence structure of $\boldsymbol{Z}^{(1)}$ be modelled by a Marshall-Olkin copula with equal parameters as in Example 3.8(a), then $\boldsymbol{L}(s)\in\mathcal{MRV}(\alpha_i,b_i,s\lambda\mu_i,\mathbb{E}_d^{(i)})$ with parameters α_i,b_i,μ_i given in (2.17).
(b) Let the dependence structure of $\boldsymbol{Z}^{(1)}$ be modelled by a Marshall-Olkin copula with proportional parameters as in Example 3.8(b), then $\boldsymbol{L}(s)\in\mathcal{MRV}(\alpha_i^*,b_i^*,s\lambda\mu_i^*,\mathbb{E}_d^{(i)})$ with parameters α_i^*,b_i^*,μ_i^* given in (2.18).
(c) Let the dependence structure of $\boldsymbol{Z}^{(1)}$ be modelled by an ACIG copula as in Example 3.9, then $\boldsymbol{L}(s)\in\mathcal{MRV}(\alpha,b_\alpha,s\lambda\mu_1,\mathbb{E}_d^{(1)})$ and $\boldsymbol{L}(s)\in\mathcal{MRV}(\alpha\beta,b_\alpha^{1/\beta},s\lambda\mu_2,\mathbb{E}_d^{(i)})$ for $i=2,\dots,d$ with parameters given in Example 2.15.

EXAMPLE 5.3. Suppose $L_\alpha^{(j)}$, $j=1,2,3$ are i.i.d. Lévy processes in $\mathbb{R}_+$ with Lévy measure $\Pi_\alpha \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0,\infty))$, $L_\beta^{(j)}$, $j=1,2$ are i.i.d. Lévy processes in $\mathbb{R}_+$ with Lévy measure $\Pi_\beta \in \mathcal{MRV}(\beta, b_\beta, \mu_\beta, (0,\infty))$ and L_γ is a Lévy process in $\mathbb{R}_+$ with Lévy measure $\Pi_\gamma \in \mathcal{MRV}(\gamma, b_\gamma, \mu_\gamma, (0,\infty))$. A typical example for L_α is an α-stable Lévy process with Lévy measure $\Pi_\alpha(\mathrm{d}x) = \alpha x^{-\alpha-1}\mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x$. Furthermore, assume all processes are independent and $\alpha < \beta < \gamma < 2\alpha$. Then the 3-dimensional Lévy process

$$\boldsymbol{L}(s) = (L_\alpha^{(1)}(s) + L_\beta^{(1)}(s) + L_\gamma(s), L_\alpha^{(2)}(s) + L_\beta^{(1)}(s) + L_\gamma(s), L_\alpha^{(3)}(s) + L_\beta^{(2)}(s) + L_\gamma(s)),$$

has Lévy measure

$$\Pi(A) = \sum_{j=1}^{3} \Pi_\alpha(A_j) + \Pi_\beta(A_1 \cap A_2) + \Pi_\beta(A_3) + \Pi_\gamma(A_1 \cap A_2 \cap A_3),$$

where $A_1 = \{z \in \mathbb{R}_+ : (z,0,0) \in A\}$, $A_2 = \{z \in \mathbb{R}_+ : (0,z,0) \in A\}$ and $A_3 = \{z \in \mathbb{R}_+ : (0,0,z) \in A\}$. Of course, Π satisfies the assumptions of Proposition 5.1 with $(\alpha_1,\alpha_2,\alpha_3) = (\alpha,\beta,\gamma)$, $(b_1,b_2,b_3) = (b_\alpha, b_\beta, b_\gamma)$ and

$$\mu_1(A) = \sum_{j=1}^{3} \mu_\alpha(A_j), \quad \mu_2(A) = \mu_\beta(A_1 \cap A_2) \quad \text{and} \quad \mu_3(A) = \mu_\gamma(A_1 \cap A_2 \cap A_3).$$

Finally, $\boldsymbol{L}(s) \in \mathcal{MRV}(\alpha_i, b_i, s\mu_i, \mathbb{E}_d^{(i)})$ for $i = 1,2,3$.

In each case for Examples 5.2 and 5.3, $\boldsymbol{L}(s)$ is MRV on $\mathbb{E}_d^{(1)}$ and $\mathbb{E}_d^{(2)}$ but with different indices and hence, $\mu_1(\mathbb{E}_d^{(2)}) = 0$. This implies that the components of $\boldsymbol{L}(s)$ are asymptotically tail independent. In the special case where the components of $\boldsymbol{L}(s)$ are (strongly) dependent, the next result follows directly from Proposition 5.1.

COROLLARY 5.4 (Extending Corollary 3.10: Dependent case). *Let $(\boldsymbol{L}(s))_{s\geq 0}$ be a Lévy process in $\mathbb{R}_+^d$ with Lévy measure $\Pi \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1,\dots,d$ whose univariate marginal Lévy measures are tail equivalent. Moreover, $(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}) = (\alpha, b, \mu, \mathbb{E}_d^{(i)})$ for $i = 1,\dots,i^*$ for some $i^* \leq d$. Then for $s > 0$ we have*

$$\boldsymbol{L}(s) \in \mathcal{MRV}(\alpha, b, s\mu, \mathbb{E}_d^{(i)}) \quad \textit{for} \quad i = 1,\dots,i^*.$$

EXAMPLE 5.5.

(a) Completely dependent case: Let $L_1 = (L_1(s))_{s\geq 0}$ be a Lévy process in $\mathbb{R}_+$ with univariate marginal Lévy measure $\Pi_\alpha \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0,\infty))$ and $\boldsymbol{L} = (L_1,\dots,L_1)$. Then the Lévy measure of $\boldsymbol{L}$ is given by

$$\Pi(A) = \min_{j\in S} \Pi_\alpha((x_j,\infty))$$

for a rectangular set $A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \ \forall\, j \in S\}$ with $S \subseteq \mathbb{I}$ and $x_j > 0$ for $j \in S$. In this case, we are in the setting of Corollary 3.10 with $i^* = d$ and $\mu(A) = \min_{j\in S} x_j^{-\alpha}$ for a rectangular set A as above.

More generally, if the marginal tail Lévy measures are not necessarily identical but are completely tail equivalent satisfying $\Pi_j \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0,\infty))$ for $j = 1,\dots,d$ and

$$\Pi(A) = \min_{j\in S} \Pi_j(x_j,\infty)$$

then $\Pi \in \mathcal{MRV}(\alpha, b_\alpha, \mu, \mathbb{E}_d^{(i)})$ for $i = 1,\dots,d$ as well and the assumptions of Corollary 5.4 are satisfied. Indeed, this is a Lévy measure, it is constructed by the complete dependence Lévy copula (cf. [27]).

(b) Suppose $L_j,\ j=1,\dots,d$ are Lévy processes in $\mathbb{R}_+$ with univariate completely tail equivalent marginal Lévy measures $\Pi_j \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0,\infty))$ and $\boldsymbol{L}(s) = (L_1(s),\dots,L_d(s))$ is a d-dimensional Lévy process with Lévy measure

$$\Pi(A) = \left(\sum_{j\in S}(\Pi_j((x_j,\infty)))^{-\theta}\right)^{\frac{1}{\theta}}$$

for some $\theta>0$, where $A=\{\boldsymbol{z}\in\mathbb{R}_+^d : z_j > x_j\ \forall\, j\in S\}$ is a rectangular set with $S\subseteq\mathbb{I}$ and $x_j>0$ for $j\in S$. This Lévy measure is constructed using the Clayton Lévy copula (cf. [27]). Let the measure μ on $\mathcal{B}^{(1)}$ be defined as

$$\mu(A) = \left(\sum_{j\in S} x_j^{-\alpha\theta}\right)^{-\frac{1}{\theta}}$$

for a rectangular set A as above. Then $\Pi\in\mathcal{MRV}(\alpha,b_\alpha,\mu,\mathbb{E}_d^{(i)})$ for $i=1,\dots,d$ and hence, due to Corollary 5.4, $\boldsymbol{L}(s)\in\mathcal{MRV}(\alpha,b_\alpha,s\mu,\mathbb{E}_d^{(i)})$ for $i=1,\dots,d$ as well.

(c) Another (dependent) example of a Lévy process can be constructed by a compound Poisson process where the jumps sizes have the distribution F as in Example 2.16.

REMARK 5.6. Regular variation of the Lévy measure on different subcones $\mathbb{E}_d^{(i)}$ can be related to regular variation of the Lévy copula and Pareto Lévy copula, respectively on these different subcones; cf. [14, 28] for classical regular variation of such Lévy measure on $\mathbb{E}_d^{(1)}$. This work is under investigation by the authors.

5.2. *The Lévy process is asymptotically tail independent.* In Proposition 5.1 and subsequently Corollary 5.4, the underlying Lévy measure admits regular variation on all relevant subcones; but this may not necessarily be the case in general. The next result includes the cases where the Lévy measure is adapted multivariate regularly varying; i.e., MRV need not exist in all the relevant subcones.

PROPOSITION 5.7 (Extending Proposition 3.12). *Let $(\boldsymbol{L}(s))_{s\geq 0}$ be a Lévy process in $\mathbb{R}_+^d$ with Lévy measure Π such that $\{\Pi\in\mathcal{MRV}^*(\alpha_i,b_i,\mu_i,\mathbb{E}_d^{(i)}), i=1,\dots,d;\Delta=1\}$ and Π has tail equivalent univariate marginal Lévy measures in $\mathcal{RV}_{-\alpha}$. Then for $s>0$ we have*

$$\boldsymbol{L}(s)\in\mathcal{MRV}(i\alpha_1, b_1^{1/i}, s^i\mu_i^L,\mathbb{E}_d^{(i)}) \quad \textit{for} \quad i=1,\dots,d.$$

with

$$\mu_i^L\Big(\bigcap_{j\in S}\{\boldsymbol{z}\in\mathbb{E}_d^{(i)} : z_j > x_j\}\Big) = \prod_{j\in S}\mu_1\left(\{\boldsymbol{z}\in\mathbb{E}_d^{(i)} : z_j>x_j\}\right). \tag{5.1}$$

for $S\subseteq\mathbb{I}$ with $|S|=i$, $x_j>0$ for $j\in S$ and $\mu_i^L(\mathbb{E}_d^{(i+1)})=0$.

Interestingly, for rectangular sets $A\in\mathcal{R}^{(i)}$ as in (1.5) with $|S|=i$, now we observe that

$$\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim s^i\,\mathbb{P}(\boldsymbol{L}(1)\in tA) \qquad \text{as } t\to\infty. \tag{5.2}$$

Hence, the linearity property of $\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim s\,\mathbb{P}(\boldsymbol{L}(1)\in tA)$ as $t\to\infty$, which we had noticed in the dependent cases of Proposition 5.1 and Corollary 5.4 respectively, vanishes here making this an unusual phenomenon for Lévy processes. Moreover, although Π is MRV on $\mathbb{E}_d^{(1)}$, for sets $A\in\mathcal{R}^{(2)}$ the tail measures $\Pi(tA)$ and $\mathbb{P}(\boldsymbol{L}(1)\in tA)$ are not tail equivalent anymore, in contrast to the common wisdom for regular variation of Lévy processes on $\mathbb{E}_d^{(1)}$.

EXAMPLE 5.8.

(a) Suppose the marginal Lévy processes $L_1, \ldots, L_d$ of $\boldsymbol{L} = (L_1, \ldots, L_d)$ are independent with tail equivalent univariate marginal Lévy measures Π_j which are regularly varying with tail index $-\alpha < 0$. Then the Lévy measure of $\boldsymbol{L}$ is

$$\Pi(A) = \sum_{j=1}^{d} \Pi_j(A_j)$$

for $A_j = \{z \in \mathbb{R}_+ : (0, \ldots, 0, z, 0, \ldots, 0) \in A\}$, where z appears in the j-th coordinate. This Lévy measure has mass only on the co-ordinate axes. Hence, the assumptions of Proposition 5.7 are satisfied and it can be applied to show MRV of $\boldsymbol{L}$ on various subcones. In particular, it satisfies (5.2).

(b) A compound Poisson process as defined in Example 5.2 with jumps sizes $(\boldsymbol{Z}^{(k)})_{k\in\mathbb{N}}$ as in Example 3.15 would also satisfy the assumptions of Proposition 5.7, providing an example for the same.

5.3. *The Lévy measure is asymptotically tail independent.* The next proposition covers the case of a compound Poisson process, as a special Lévy process, where the marginal distribution of the jump sizes are independent as well. The observed phenomena is again different from Proposition 5.7, which covers a compound Poisson process with independent marginal Lévy processes.

PROPOSITION 5.9 (Extending Proposition 3.3). *Let $(\boldsymbol{L}(s))_{s\geq 0}$ be a Lévy process in $\mathbb{R}_+^d$ with Lévy measure $\Pi \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ where $b_i(t) = (b_1(t))^{1/i}$, $b_1(t) \in \mathcal{RV}_{1/\alpha}$ and*

$$\mu_i\left(\bigcap_{j\in S}\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right) = \prod_{j\in S} \kappa_j x_j^{\alpha}$$

for $S \subseteq \mathbb{I}$ with $|S| = i$, $x_j > 0$ for $j \in S$ and $\mu_i(\mathbb{E}_d^{(i+1)}) = 0$, $i = 1, \ldots, d$, and Π has tail equivalent univariate marginal Lévy measures. Let $(N^*(s))_{s\geq 0}$ denote a Poisson process with intensity 1. Then for $s > 0$ we have

$$\boldsymbol{L}(s) \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mathbb{E}(N^*(s)^i)\mu_i, \mathbb{E}_d^{(i)}) \quad \textit{for} \quad i = 1, \ldots, d.$$

As a consequence of Proposition 5.9, for any rectangular set $A \in \mathcal{R}^{(i)}$ as in (1.5) with $|S| = i$ we obtain

$$\mathbb{P}(\boldsymbol{L}(s) \in tA) \sim \frac{\mathbb{E}(N^*(s)^i)}{\mathbb{E}(N^*(1)^i)} \mathbb{P}(\boldsymbol{L}(1) \in tA) \quad \text{as } t \to \infty,$$

where $\mathbb{E}(N^*(s)^i)$ is a polynomial of order i in s, and $s^i \leq \mathbb{E}(N^*(s)^i)$.

REMARK 5.10.

(a) We can verify that indeed the result in Proposition 5.9 is in accordance with Theorem 3.1. From Proposition 5.9 we get that for the i.i.d. random vectors

$$\boldsymbol{L}(1) \text{ and } \boldsymbol{L}(2) - \boldsymbol{L}(1) \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mathbb{E}(N^*(1)^i)\mu_i, \mathbb{E}_d^{(i)}) \quad \text{for} \quad i = 1, \ldots, d.$$

Applying Theorem 3.1 gives

$$\boldsymbol{L}(2) = \boldsymbol{L}(1) + [\boldsymbol{L}(2) - \boldsymbol{L}(1)] \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mu_i^{\oplus}, \mathbb{E}_d^{(i)}) \quad \text{for} \quad i = 1, \ldots, d$$

with

$$\mu_i^{\oplus} = \sum_{m=0}^{i} \binom{i}{m} \mathbb{E}(N^*(1)^m)\mathbb{E}([N^*(2)-N^*(1)]^{i-m})\mu_i$$
$$= \mathbb{E}\left(\sum_{m=0}^{i} \binom{i}{m} N^*(1)^m [N^*(2)-N^*(1)]^{i-m}\right)\mu_i$$
$$= \mathbb{E}(N^*(1)+[N^*(2)-N^*(1)])^i \mu_i = \mathbb{E}(N^*(2)^i)\mu_i$$

which is also a consequence of Proposition 5.9.

(b) Suppose $(\boldsymbol{L}(s))_{s\geq 0}$ is a compound Poisson process with Lévy measure $\Pi \in \mathcal{MRV}(\alpha_i, b_i, K\mu_i, \mathbb{E}_d^{(i)})$ for $i=1,\ldots,d$ with α_i, b_i, and μ_i as in Proposition 5.9 and $K>0$ is some positive constant. Furthermore, suppose the marginal Lévy measures of Π are tail-equivalent. Let $(\widetilde{\boldsymbol{L}}(s))_{s\geq}$ be another compound Poisson process with Lévy measure $\widetilde{\Pi} = \Pi/K$. Then $\widetilde{\Pi} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i=1,\ldots,d$ and due to Proposition 3.3, we have

$$\widetilde{\boldsymbol{L}}(s) \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mathbb{E}(N^*(s)^i)\mu_i, \mathbb{E}_d^{(i)}) \quad \text{for} \quad i=1,\ldots,d.$$

But $\boldsymbol{L}(s) \stackrel{\mathrm{d}}{=} \widetilde{\boldsymbol{L}}(Ks)$ and hence, we have

$$\boldsymbol{L}(s) \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mathbb{E}(N^*(Ks)^i)\mu_i, \mathbb{E}_d^{(i)}) \quad \text{for} \quad i=1,\ldots,d.$$

EXAMPLE 5.11. Consider the compound Poisson process

$$\boldsymbol{L}(s) = \sum_{k=1}^{N^*(s)} (Z_1^{(k)}, \ldots, Z_d^{(k)})$$

where $(N^*(s))_{s\geq 0}$ is a Poisson process with intensity 1, which is independent of the i.i.d. sequence of jump sizes $(Z_1^{(k)}, \ldots, Z_d^{(k)})_{k\in\mathbb{N}}$. Suppose $(Z_1^{(k)})_{k\in\mathbb{N}}, \ldots, (Z_d^{(k)})_{k\in\mathbb{N}}$ are as well independent of each other with tail equivalent marginal distributions F_j and $\overline{F}_j \in \mathcal{RV}_{-\alpha}$. Then for a rectangular set $A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \;\forall\, j \in S\}$ with $S \subseteq \mathbb{I}$ and $x_j > 0$ for $j \in S$ we have

$$\Pi(A) = \prod_{j\in S}^{d} \overline{F}_j(x_j).$$

Thus, the assumptions of Proposition 5.9 are again satisfied and hence, can be applied here.

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APPENDIX A: PROOFS OF THE RESULTS IN SECTION 2

PROOF OF LEMMA 2.6. The set $(1,\infty)^d \in \mathcal{R}^{(i)}$, and hence, it is non-empty. Now, let A and B be two arbitrary sets in $\mathcal{R}^{(i)}$. Then for some $m,n \geq i$, with $x_j > 0, j \in \{k_1,\dots,k_m\} =: S_1 \subseteq \mathbb{I}$ and $y_j > 0, j \in \{\ell_1,\dots,\ell_n\} =: S_2 \subseteq \mathbb{I}$ we have

$$A = \left\{ \boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \,\forall\, j \in S_1 \right\}, \quad B = \left\{ \boldsymbol{z} \in \mathbb{R}_+^d : z_j > y_j \,\forall\, j \in S_2 \right\}.$$

For $j \in S^* := S_1 \cup S_2$, define

$$w_j = \begin{cases} x_j, & \text{if } j \in S_1 \cap {S_2}^c, \\ y_j & \text{if } j \in {S_1}^c \cap S_2, \\ \max\{x_j, y_j\} & \text{if } j \in S_1 \cap S_2. \end{cases}$$

Thus, $A \cap B = \left\{ \boldsymbol{z} \in \mathbb{R}_+^d : z_j > w_j \,\forall\, j \in S^* \right\}$ where $|S^*| \geq \max(m,n) \geq i$ and $w_j > 0 \,\forall\, j \in S^*$. Hence, $A \cap B \in \mathcal{R}^{(i)}$ and $\mathcal{R}^{(i)}$ is a π-system. It can also be checked that $\sigma(\mathcal{R}^{(i)}) = \mathcal{B}^{(i)}$. □

PROOF OF PROPOSITION 2.7. (2.8) ⇒ (2.9): Using [32, Theorem 2.1], if (2.8) holds, then any set of the form $A \in \mathcal{R}^{(i)}$ with $\mu(\partial A) = 0$ is clearly bounded away from $\mathbb{CA}_i$ and belongs to the σ-algebra $\mathcal{B}^{(i)}$ as defined in (2.6). Hence, (2.9) holds as $t \to \infty$.
(2.9) ⇒ (2.8): Now assume (2.9) holds for all μ-continuity sets $A \in \mathcal{R}^{(i)}$. Denote by M, the collection

$$M = \{\mu_t : t > 0\} \subset \mathbb{M}(\mathbb{E}_d^{(i)}).$$

For any $r > 0$, let $\nu^{(r)}$ be the restriction of $\nu \in \mathbb{M}(\mathbb{E}_d^{(i)})$ to $\mathbb{E}_d^{(i)} \setminus \mathbb{CA}_i^{(r)}$ where $\mathbb{CA}_i^{(r)} = \{\boldsymbol{z} \in \mathbb{R}_+^d : d(\boldsymbol{z}, \mathbb{CA}_i) < r\}$. Let $M^{(r)} := \{\nu^{(r)} : \nu \in M\}$ and let $\{r_\ell\}$ be a sequence $r_\ell \downarrow 0$ as $\ell \to \infty$. Note that $M^{(r)} \subset \mathbb{M}(\mathbb{E}_d^{(i)} \setminus \mathbb{CA}_i^{(r)})$ which is a class of finite Borel measures.

Denote by $\mathcal{C}_i^{(r)}$ = all real-valued, bounded continuous functions f on $\mathbb{E}_d^{(i)}$ which vanishes on $\mathbb{CA}_i^{(r)}$. Fix $\ell \geq 1$ and pick any $f \in \mathcal{C}_i^{(r_\ell)}$ which is uniformly continuous. Then by definition, the support of f lies on a finite union of rectangular sets given by

$$A = \bigcup_{\substack{S \subset \mathbb{I} \\ |S| = i}} A_S,$$

where $A_S = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > r^* \,\forall\, j \in S\} \in \mathcal{R}^{(i)}$ for some $0 < r^* < r_\ell$. W.l.o.g. the sets A_S can be assumed to be μ-continuity sets using [32, Lemma 2.5]. Now, using (2.9) we have convergence on the sets A_S and therefore,

$$\sup_{\nu\in M^{(r_\ell)}} \nu(f) = \sup_t \mu_t^{(r_\ell)}(f) \le \sup_{\boldsymbol{z}\in\mathbb{R}_+^d} f(\boldsymbol{z}) \sup_t \mu_t(A) \le \sup_{\boldsymbol{z}\in\mathbb{R}_+^d} f(\boldsymbol{z}) \sum_{\substack{S\subset\mathbb{I}\\|S|=i}} \sup_t \mu_t^{(r_\ell)}(A_S) < \infty.$$

Hence, for any sequence of measures $\{\nu_n\}_{n\ge1}\in M^{(r_\ell)}$, the sequence $\nu_n(f)$ has a convergent subsequence. Since this is true for any uniformly continuous $f\in\mathcal{C}_i^{(r_\ell)}$, it is true for any $f\in\mathcal{C}_{i,K}^{(r_\ell)}$, which are compactly supported functions in $\mathcal{C}_i^{(r_\ell)}$. Since $\mathcal{C}_{i,K}^{(r_\ell)}$ is separable, using a countable dense collection $\{f_j\}_{j\ge1}\in\mathcal{C}_{i,K}^{(r_\ell)}$, and a diagonal argument we can show that any sequence of measures $\{\nu_n\}_{n\ge1}\in M^{(r_\ell)}$ has a subsequence $\{\nu_{n_k}\}$ such that $\lim_{n_k\to\infty}\nu_{n_k}(g)=\nu(g)$ for any $g\in\mathcal{C}_{i,K}^{(r_\ell)}$ and hence for all uniformly continuous functions $f\in\mathcal{C}_i^{(r_\ell)}$ (using a sequence $g_n\to f$ where $g_n\in\mathcal{C}_{i,K}^{(r_\ell)}$). Thus, $M^{(r_\ell)}=\{\mu_t^{(r_\ell)}:t>0\}$ is relatively compact; cf. [39, (3.16), p. 51)]; and this holds for a sequence $\{r_\ell\}$ where $r_\ell\downarrow 0$. Also $M^{(r_\ell)}\subset M$. Hence, by [32, Theorem 2.4] we have M is relatively compact.

Suppose as $t\to\infty$, μ_t has two different sequential limits μ_1 and μ_2, then by assumption they clearly agree on all sets $A\in\mathcal{R}^{(i)}$. By Lemma 2.6 such rectangular sets form a π-system generating the σ-algebra $\mathcal{B}^{(i)}$. Hence, $\mu_1=\mu_2=\mu$ on $\mathbb{E}_d^{(i)}$. □

PROOF OF LEMMA 2.9. Let $A=\{\boldsymbol{z}\in\mathbb{R}_+^d : z_j>x_j\ \forall j\in S\}$ where $S\subseteq\mathbb{I}$, $|S|\ge i$, $x_j>0$ for $j\in S$ and $\mu_i(\partial A)=0$. Furthermore, let $0<\varepsilon<\min_{j\in S}x_j$ and define the sets

$$A_\varepsilon^+ := \{\boldsymbol{z}\in\mathbb{R}_+^d : z_j > x_j-\varepsilon\ \forall j\in S\},$$
$$A_\varepsilon^- := \{\boldsymbol{z}\in\mathbb{R}_+^d : z_j > x_j+\varepsilon\ \forall j\in S\},$$
$$N_\varepsilon := \{\boldsymbol{z}\in\mathbb{R}_+^d : |z_j| \le \varepsilon\ \forall j\in S\}.$$

Suppose w.l.o.g. $\mu_i(\partial A_\varepsilon^+)=\mu_i(\partial A_\varepsilon^-)=0$ (otherwise choose ε appropriate). On the one hand,

$$\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) \le \mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^+) + \mathbb{P}(\boldsymbol{Y}\in tN_\varepsilon^c) \le \mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^+) + \mathbb{P}(\|\boldsymbol{Y}\|_\infty>\varepsilon t).$$

Hence,

$$\begin{aligned}\limsup_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) &\le \limsup_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^+) + \limsup_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\|\boldsymbol{Y}\|_\infty>\varepsilon t)\\ &\le \mu_i(A_\varepsilon^+) + \limsup_{t\to\infty} b_i^{\leftarrow}(t)(\gamma t)^{-(\alpha_i+\gamma)}\mathbb{E}\|\boldsymbol{Y}\|^{\alpha_i+\gamma}\\ &= \mu_i(A_\varepsilon^+)\downarrow\mu_i(A) \quad \text{as } \varepsilon\downarrow 0, \end{aligned}\tag{A.1}$$

since $\mu_i(\partial A)=0$. On the other hand,

$$\begin{aligned}\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) &\ge \mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA, \boldsymbol{Y}\in tN_\varepsilon)\\ &\ge \mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^-, \boldsymbol{Y}\in tN_\varepsilon)\\ &= \mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^-)\mathbb{P}(\boldsymbol{Y}\in tN_\varepsilon).\end{aligned}$$

Therefore,

$$\begin{aligned}\liminf_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) &\geq \limsup_{t\to\infty} b_i^{\leftarrow}(t)\,\mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^-)\mathbb{P}(\boldsymbol{Y}\in tN_\varepsilon)\\ &= \mu_i(A_\varepsilon^-)\uparrow\mu_i(A) \quad \text{ as } \varepsilon\downarrow 0, \qquad\qquad (A.2)\end{aligned}$$

since $\mu_i(\partial A)=0$. Thus (A.1) and (A.2) imply that

$$\lim_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA)=\mu_i(A),$$

and using Proposition 2.7 we can conclude the statement. □

APPENDIX B: PROOF OF THEOREM 3.1

The following auxiliary lemmas are used to prove Theorem 3.1.

LEMMA B.1. *Let the assumptions of Theorem 3.1 hold. Then for any $m=0,\dots,i-1$:*

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\,\mathbb{P}(Z_{(m+1)}^{(1)}>t)\,\mathbb{P}(Z_{(i-m)}^{(2)}>t)=0 \qquad (B.1)$$

and

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\,\mathbb{P}(Z_{(m+1)}^{(2)}>t)\,\mathbb{P}(Z_{(i-m)}^{(1)}>t)=0. \qquad (B.2)$$

PROOF. Let $\Gamma^{(k)}:=\arg\max_i\{\alpha_i^{(k)}<\infty\}$, $k=1,2$. By definition, for $i\leq\Gamma^{(k)}$, we have $\mathbb{P}(Z_{(i)}^{(k)}>t)=O(1/b_i^{(k)\leftarrow}(t))$; and for $i>\Gamma^{(k)},\mathbb{P}(Z_{(i)}^{(k)}>t)=o(1/b_i^{(k)\leftarrow}(t))$. Hence, to prove (B.1), we need only to show that

$$a_m:=\limsup_{t\to\infty}\frac{b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)}{b_{m+1}^{(1)\leftarrow}(t)b_{i-m}^{(2)\leftarrow}(t)}=0, \quad m=0,\dots,i-1.$$

For $m=0$ we have

$$a_0=c_0^{I(i)}\lim_{t\to\infty}1/b_1^{(1)\leftarrow}(t)=0.$$

Let $m\in\{1,\dots,i-1\}$. Note that

$$a_m=c_m^{I(i)}\limsup_{t\to\infty}\frac{b_m^{(1)\leftarrow}(t)}{b_{m+1}^{(1)\leftarrow}(t)}.$$

Since $\limsup_{t\to\infty}b_m^{(1)\leftarrow}(t)/b_{m+1}^{(1)\leftarrow}(t)<\infty$, the last equality implies that $a_m=0$ is only possible if either $c_m^{I(i)}=0$ or $\limsup_{t\to\infty}b_m^{(1)\leftarrow}(t)/b_{m+1}^{(1)\leftarrow}(t)=0$, which holds true by the assumptions in Theorem 3.1. The proof of (B.2) is analogous. □

LEMMA B.2. *Let the assumptions of Theorem 3.1 hold and $A=\{\boldsymbol{z}\in\mathbb{R}_+^d: z_j>x_j\ \forall\, j\in S\}$ be a rectangular set with $S\subseteq\mathbb{I}$, $|S|=i$, $x_j>0$ for $j\in S$ and $\mu_i^{\oplus}(\partial A)=0$. Then*

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\mathbb{P}\left(\boldsymbol{Z}^{(1)}+\boldsymbol{Z}^{(2)}\in tA\right)=\mu_i^{\oplus}(A). \qquad (B.3)$$

PROOF.
Step 1. First, we derive an upper bound for the left hand side of (B.3). Let $0<\epsilon<1$ such that $\epsilon x_j<(1-\epsilon)x_l$ for all $j,l\in S$. Then

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{Z}^{(1)}+\boldsymbol{Z}^{(2)}\in tA\right)=\Bigg[\sum_{\substack{J_2\subseteq S\cup\{\emptyset\}\\ J_1=S\setminus J_2}}+\sum_{\substack{J_2\subseteq S\cup\{\emptyset\}\\ J_1\subsetneq S\setminus J_2\cup\{\emptyset\}}}\Bigg]\mathbb{P}\Bigg(&\bigcap_{j\in S}\left\{Z_j^{(1)}+Z_j^{(2)}>tx_j\right\}\\ &\cap\bigcap_{j\in J_1}\left\{Z_j^{(1)}\le t\epsilon x_j\right\}\cap\bigcap_{j\in S\setminus J_1}\left\{Z_j^{(1)}>t(1-\epsilon)x_j\right\}\\ &\cap\bigcap_{j\in J_2}\left\{Z_j^{(2)}\le t\epsilon x_j\right\}\cap\bigcap_{j\in S\setminus J_2}\left\{Z_j^{(2)}>t(1-\epsilon)x_j\right\}\Bigg)\\ =:\ & M_1(t,\epsilon)+M_2(t,\epsilon).\end{aligned}\tag{B.4}$$

Note that in case $J_1\cap J_2\neq\emptyset$ these probabilities are zero since it results in computing probabilities of empty sets. Next, we find an upper bound for $M_1(t,\epsilon)$. Note that

$$\begin{aligned}M_1(t,\epsilon)&\le\sum_{\substack{J_2\subseteq S\cup\{\emptyset\}\\ J_1=S\setminus J_2}}\mathbb{P}\left(\bigcap_{j\in S\setminus J_1}\left\{Z_j^{(1)}>t(1-\epsilon)x_j\right\}\cap\bigcap_{j\in S\setminus J_2}\left\{Z_j^{(2)}>t(1-\epsilon)x_j\right\}\right)\\ &=\sum_{J_2\subseteq S\cup\{\emptyset\}}\mathbb{P}\left(\bigcap_{j\in J_2}\left\{Z_j^{(1)}>t(1-\epsilon)x_j\right\}\right)\mathbb{P}\left(\bigcap_{j\in S\setminus J_2}\left\{Z_j^{(2)}>t(1-\epsilon)x_j\right\}\right).\end{aligned}$$

Let $A_\epsilon:=\{\boldsymbol{z}\in\mathbb{R}_+^d: z_j>(1-\epsilon)x_j\,\forall\, j\in S\}$ and choose $\epsilon>0$ such that $\mu_i^{\oplus}(\partial A_\epsilon)=0$. Then

$$\limsup_{t\to\infty}b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)M_1(t,\epsilon)\le\mu_i^{\oplus}(A_\epsilon).\tag{B.5}$$

Define $x^*:=\min_{j\in S}x_j$. Following a similar argument for $M_2(t,\epsilon)$ we get the upper bound

$$\begin{aligned}M_2(t,\epsilon)&\le\sum_{\substack{J_2\subseteq S\cup\{\emptyset\}\\ J_1\subsetneq S\setminus J_2\cup\{\emptyset\}}}\mathbb{P}\Big(\bigcap_{j\in J_2\cup S\setminus J_1}\left\{Z_j^{(1)}>t(1-\epsilon)x_j\right\}\Big)\mathbb{P}\Big(\bigcap_{j\in J_1\cup S\setminus J_2}\left\{Z_j^{(2)}>t(1-\epsilon)x_j\right\}\Big)\\ &\le\sum_{\substack{J_2\subseteq S\cup\{\emptyset\}\\ J_1\subsetneq S\setminus J_2\cup\{\emptyset\}}}\mathbb{P}\left(Z^{(1)}_{(|J_2\cup S\setminus(J_1\cup J_2)|)}>t(1-\epsilon)x^*\right)\mathbb{P}\left(Z^{(2)}_{(|S\setminus J_2|)}>t(1-\epsilon)x^*\right)\\ &\le\sum_{m=0}^{i-1}\sum_{l=1}^{i-m}\sum_{\substack{J_2\subseteq S\cup\{\emptyset\}\\ |J_2|=m}}\sum_{\substack{J_1\subsetneq S\setminus J_2\cup\{\emptyset\}\\ |S\setminus(J_1\cup J_2)|=l}}\mathbb{P}\left(Z^{(1)}_{(m+l)}>t(1-\epsilon)x^*\right)\mathbb{P}\left(Z^{(2)}_{(i-m)}>t(1-\epsilon)x^*\right).\end{aligned}$$

Finally, an application of Lemma B.1 yields

$$\begin{aligned}&\limsup_{t\to\infty}b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)M_2(t,\epsilon)\\ &\le\sum_{m=0}^{i-1}\sum_{l=1}^{i-m}\sum_{\substack{J_2\subseteq S\cup\{\emptyset\}\\ |J_2|=m}}\sum_{\substack{J_1\subsetneq S\setminus J_2\cup\{\emptyset\}\\ |S\setminus(J_1\cup J_2)|=l}}\limsup_{t\to\infty}b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\\ &\qquad\mathbb{P}\left(Z^{(1)}_{(m+1)}>t(1-\epsilon)x^*\right)\mathbb{P}\left(Z^{(2)}_{(i-m)}>t(1-\epsilon)x^*\right)=0.\end{aligned}\tag{B.6}$$

Now from (B.4), (B.5) and (B.6) we have

$$\limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)\, \mathbb{P}\left(\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in tA\right) \leq \mu_i^{\oplus}(A_\epsilon) \downarrow \mu_i^{\oplus}(A) \quad \text{as } \epsilon \downarrow 0,$$

where in the last step we use the fact that $\mu_i^{\oplus}(\partial A) = 0$.

Step 2. Next, we derive a lower bound for the asymptotic limit. There are a total of $2^{|S|}$ subsets of S which we order as $J(1), \dots, J(2^{|S|})$ (in any way). Now, define the sets

$$C_l := \bigcap_{j\in J(l)} \left\{ Z_j^{(1)} > tx_j \right\} \cap \bigcap_{j\in S\setminus J(l)} \left\{ Z_j^{(2)} > tx_j \right\}, \quad l = 1, \dots, 2^{|S|}.$$

Then,

$$\left\{ \boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in tA \right\} \supseteq \bigcup_{l=1}^{2^{|S|}} C_l,$$

and using the inclusion-exclusion principle we have

$$\mathbb{P}\left(\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in tA\right) \geq \mathbb{P}\left(\bigcup_{l=1}^{2^{|S|}} C_l\right) \geq \sum_{l=1}^{2^{|S|}} \mathbb{P}(C_l) - \sum_{1\leq l_1<l_2\leq 2^{|S|}} \mathbb{P}(C_{l_1}\cap C_{l_2}). \quad \text{(B.7)}$$

Now on one hand,

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \sum_{l=1}^{2^{|S|}} \mathbb{P}(C_l) = \mu_i^{\oplus}(A), \quad \text{(B.8)}$$

and on the other hand, for any $1 \leq l_1 < l_2 \leq 2^{|S|}$ the inequality

$$0 \leq \mathbb{P}(C_{l_1}\cap C_{l_2}) \leq \mathbb{P}(Z^{(1)}_{(|J(l_1)\cup J(l_2)|)} > tx^*)\mathbb{P}(Z^{(2)}_{(|S\setminus(J(l_1)\cap J(l_2))|)} > tx^*)$$

holds. Define $m := |J(l_1) \cap J(l_2)|$. Since $J(l_1) \neq J(l_2)$ we have $|J(l_1) \cup J(l_2)| \geq |J(l_1) \cap J(l_2)| + 1 = m + 1$. Hence, a conclusion of Lemma B.1 is that

$$\begin{aligned} &\limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)\, \mathbb{P}(C_{l_1}\cap C_{l_2}) && \text{(B.9)} \\ &\leq \limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}(Z^{(1)}_{(m+1)} > tx^*)\mathbb{P}(Z^{(2)}_{(i-m)} > tx^*) = 0 \end{aligned}$$

for any $1 \leq l_1 < l_2 \leq 2^{|S|}$. Then (B.7), (B.8) and (B.9) result in the lower bound

$$\liminf_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)\, \mathbb{P}\left(\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in tA\right) \geq \mu_i^{\oplus}(A).$$

and together with the upper bound in Step 1 the lemma is proven. □

LEMMA B.3. *Let the assumptions of Theorem 3.1 hold and* $A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \ \forall\, j \in S\}$ *be a rectangular set with* $S \subseteq \mathbb{I}$, $|S| > i$ *and* $x_j > 0$ *for* $j \in S$ *where* $\mu_i^{\oplus}(\partial A) = 0$. *Then*

$$\mu_i^{\oplus}(A) = c_i^{I(i)} \mu_i^{(1)}(A) + c_0^{I(i)} \mu_i^{(2)}(A).$$

PROOF. Since

$$\mu_i^{\oplus}(A) = \sum_{m=0}^{i} c_m^{I(i)} \mu^*_{m,i}(A)$$

we have to show that $\sum_{m=1}^{i-1} c_m^{I(i)} \mu_{m,i}^*(A) = 0$. But

$$\begin{aligned}
0 &\le \sum_{m=1}^{i-1} c_m^{I(i)} \mu_{m,i}^*(A) \\
&= \lim_{t\to\infty} \sum_{m=1}^{i-1} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \sum_{\substack{J\subseteq S\cup\{\emptyset\} \\ |J|=m}} \mathbb{P}\left(\bigcap_{j\in J}\{Z_j^{(1)} > tx_j\}\right) \mathbb{P}\left(\bigcap_{j\in S\setminus J}\{Z_j^{(2)} > tx_j\}\right) \\
&\le 2^i \lim_{t\to\infty} \sum_{m=1}^{i-1} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}\left(Z_{(m)}^{(1)} > t \min_{j\in S} x_j\right) \mathbb{P}\left(Z_{|S|-m}^{(2)} > t \min_{j\in S} x_j\right) \\
&\le 2^i \sum_{m=1}^{i-1} \lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}\left(Z_{((m-1)+1)}^{(1)} > t \min_{j\in S} x_j\right) \mathbb{P}\left(Z_{i-(m-1)}^{(2)} > t \min_{j\in S} x_j\right).
\end{aligned}$$

The right hand side is equal to zero due to Lemma B.1. □

PROOF OF THEOREM 3.1. Due to Lemma 2.6 it is sufficient to study the convergence on the rectangular sets $A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \ \forall\, j \in S\}$ where $S \subseteq \mathbb{I}$, $|S| \ge i$ and $x_j > 0$ for $j \in S$ with $\mu_i^{\oplus}(\partial A) = 0$. If $|S| = i$, a consequence of Lemma 2.6 is that

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}\left(\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in tA\right) = \mu_i^{\oplus}(A).$$

If $|S| > i$ then $i \le d-1$. Thus, using Lemma B.1 and similar elaborate calculations as in the proof of Lemma B.2 (cf. proof of Lemma B.3) we can show that

$$\begin{aligned}
&\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}\left(\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in tA\right) \\
&\quad = \lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}\left(\boldsymbol{Z}^{(1)} \in tA\right) + \lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}\left(\boldsymbol{Z}^{(2)} \in tA\right) \\
&\quad = c_i^{I(i)} \mu_{|S|}^{(1)}(A) \lim_{t\to\infty} \frac{b_i^{(1)\leftarrow}(t)}{b_{|S|}^{(1)\leftarrow}(t)} + c_0^{I(i)} \mu_{|S|}^{(2)}(A) \lim_{t\to\infty} \frac{b_i^{(2)\leftarrow}(t)}{b_{|S|}^{(2)\leftarrow}(t)}.
\end{aligned}$$

In case $\lim_{t\to\infty} \frac{b_i^{(1)\leftarrow}(t)}{b_{|S|}^{(1)\leftarrow}(t)} = 0$ we have $\mu_i^{(1)}(A) = 0$. Otherwise, $\mu_{|S|}^{(1)}(A) \lim_{t\to\infty} \frac{b_i^{(1)\leftarrow}(t)}{b_{|S|}^{(1)\leftarrow}(t)} = \mu_i^{(1)}(A)$. In summary,

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}\left(\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in tA\right) = c_i^{I(i)} \mu_i^{(1)}(A) + c_0^{I(i)} \mu_i^{(2)}(A) = \mu_i^{\oplus}(A)$$

where the final equality is due to Lemma B.3. □

APPENDIX C: PROOFS OF THE RESULTS IN SECTION 3.1

PROOF OF PROPOSITION 3.3. Note that $\alpha_i = i\alpha$ is immediate from $b_i(t) = (b_1(t))^{1/i} \in \mathcal{RV}_{1/(i\alpha)}$. Using Theorem 3.1, it is sufficient to prove the statements for $n = 2$ and the rest follows by induction (which are direct and not shown here). Using the notation of Theorem 3.1, for any $i = 1, \ldots, d$, we have $I(i) = i$ and $c_m^{I(i)} = 1$ for $m = 0, \ldots, i$ for any $i = 1, \ldots, d$ and $\lim_{t\to\infty} b_m^{(1)\leftarrow}(t)/b_{m+1}^{(1)\leftarrow}(t) = 0$. Thus,

$$\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mathbb{E}_d^{(i)})$$

and (3.1) follows by induction. Now if (3.2) is satisfied then (for $n=2$), we have $\mu^*_{m,i}(\cdot) = \binom{i}{m}\mu_i(\cdot)$ and hence, for any $A \in \mathcal{B}(\mathbb{E}_d^{(i)})$ with $\mu_i(\partial A) = 0$ we get

$$\mu_i^{\oplus}(A) = \sum_{m=0}^{i} \binom{i}{m} \mu_i(A) = 2^i \mu_i(A)$$

implying that

$$\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in \mathcal{MRV}(i\alpha, b_1^{1/i}, 2^i\mu_i, \mathbb{E}_d^{(i)}).$$

Now (3.2) follows by induction using Theorem 3.1. □

PROOF OF PROPOSITION 3.5. Consider $n=2$ and the notation of Theorem 3.1. Fix some $i \in \{1,\dots,d\}$. We have $I(i) = i$, $c_m^{I(i)} = 0, m = 1,\dots,i-1$ and $c_0^{I(i)} = c_i^{I(i)} = 1$. Clearly, $\mu^*_{0,i} = \mu^*_{i,i} = \mu_i$ and hence, $\mu_i^{\oplus} = \mu^*_{0,i} + \mu^*_{i,i} = 2\mu_i$. Now by Theorem 3.1, we get

$$\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in \mathcal{MRV}(\alpha_i, b_i, 2\mu_i, \mathbb{E}_d^{(i)}).$$

The final result can now be derived using induction (which we skip here). □

PROOF OF COROLLARY 3.10. Since $\alpha_i = \alpha < \alpha + \alpha = \alpha_m + \alpha_{i-m}$, this holds as a direct consequence of Proposition 3.5. □

APPENDIX D: PROOF OF PROPOSITION 3.12

PROOF. By Definition 2.10, $\Delta = 1$, $b_1(t) \in \mathcal{RV}_{1/\alpha}$ and for some $0 < \gamma < \alpha/d$, $b_i(t) = t^{1/(i(\alpha+\gamma))}, i = 2,\dots,d$. Then $b_1^{\leftarrow}(t) \in \mathcal{RV}_\alpha$ and let $b_1^{\leftarrow}(t) = t^\alpha \ell(t)$ where $\ell(t)$ is some slowly varying function. Also define $\boldsymbol{S}^{(n)} := \sum_{k=1}^n \boldsymbol{Z}^{(k)}$. We prove the statement by induction.

(1) Consider $i=1$. By definition, (3.4) holds for $n=i=1$. Using classical MRV results [31, Theorem 1.30], [24, Example 3.2], we obtain $\boldsymbol{S}^{(n)} \in \mathcal{MRV}(\alpha, b_1, n\mu_1, \mathbb{E}_d^{(1)})$. Thus (3.4) holds for all $n \ge i = 1$; the form of the limit measure is (3.5) with $|S| = i = 1$.

(2) Fix $i_0 \in \{2,\dots,d\}$. By way of induction, assume that for any $i \in \{1,\dots,i_0-1\}$, (3.4) holds for all $n \ge i$ and (3.6) holds for $n < i$. First, in part (i) we show that for $i = i_0$, (3.6) holds for $n < i = i_0$. Then, we show (3.4) holds for $n = i = i_0$ in part (ii) and for all $n > i = i_0$ in part (iii). Note that the induction base case holds for $i_0 = 2$. Moreover, for $n = 1$, (3.6) holds for $i = 2,\dots,d$.

(i) Additionally assume that for $i = i_0$, (3.6) holds for all $n = 1,\dots,n_0-1$ where $n_0 < i_0$. Here we show (3.6) holds for $i = i_0$ and $n = n_0$. If $i_0 = 2$, then the only choice of n is $n = n_0 = 1$ and (3.6) holds since $\boldsymbol{S}^{(n_0)} = \boldsymbol{Z}^{(1)} \in \mathcal{NC}(t^{2(\alpha+\gamma)}, \mathbb{E}_d^{(2)})$. So assume $i_0 \ge 3$. Note that $\boldsymbol{S}^{(n_0)} = \boldsymbol{S}^{(n_0-1)} + \boldsymbol{Z}^{(n_0)}$. Using the notation of Theorem 3.1, for $j = 0$,

$$\bar{c}_{0,n_0}^{(i_0)} = \bar{c}_{j,n_0}^{(i_0)} := \max_{0\le m\le i_0} \left\{ \limsup_{t\to\infty} \frac{b_j^{(n_0-1)\leftarrow}(t) b_{i_0-j}^{\leftarrow}(t)}{b_m^{(n_0-1)\leftarrow}(t) b_{i_0-m}^{\leftarrow}(t)} \right\}$$

$$\ge \limsup_{t\to\infty} \frac{b_0^{(n_0-1)\leftarrow}(t) b_{i_0}^{\leftarrow}(t)}{b_1^{(n_0-1)\leftarrow}(t) b_{i_0-1}^{\leftarrow}(t)} = \limsup_{t\to\infty} \frac{1 \cdot t^{i_0(\alpha+\gamma)}}{t^\alpha \ell(t) \cdot t^{(i_0-1)(\alpha+\gamma)}} = \infty,$$

since $b_1^{(n_0-1)}(t) = b_1(t)$ from (1) and $b_1^{\leftarrow}(t) = t^\alpha \ell(t)$. Similarly for $j = 1$,

$$\bar{c}_{1,n_0}^{(i_0)} = \max_{0\le m\le i_0} \left\{ \limsup_{t\to\infty} \frac{b_1^{(n_0-1)\leftarrow}(t) b_{i_0-1}^{\leftarrow}(t)}{b_m^{(n_0-1)\leftarrow}(t) b_{i_0-m}^{\leftarrow}(t)} \right\}$$

$$= \max\left\{0, 1, \max_{2\le m\le i_0-1} \limsup_{t\to\infty} \frac{t^{\alpha}\ell(t)\cdot t^{(i_0-1)(\alpha+\gamma)}}{b_m^{(n_0-1)\leftarrow}(t)\cdot t^{(i_0-m)(\alpha+\gamma)}}, \limsup_{t\to\infty} \frac{t^{\alpha}\ell(t)\cdot t^{(i_0-1)(\alpha+\gamma)}}{b_{i_0}^{(n_0-1)\leftarrow}(t)b_0^{(1)\leftarrow}(t)}\right\}$$

$$= \max\left\{0, 1, \limsup_{t\to\infty} \frac{t^{i_0\alpha+(i_0-1)\gamma}\ell(t)}{t^{i_0(\alpha+\gamma)}}, \limsup_{t\to\infty} \frac{t^{i_0\alpha+(i_0-1)\gamma}\ell(t)}{t^{i_0(\alpha+\gamma)}}\right\} = 1.$$

The third equality results from $b_m^{(n_0-1)\leftarrow}(t) = t^{m(\alpha+\gamma)}$ for $m \in \{2,\dots,i_0-1\}$ where $n_0 - 1 < i_0 - 1$ and $b_{i_0}^{(n_0-1)\leftarrow}(t) = t^{i_0(\alpha+\gamma)}$ (by induction assumption). Similarly, for $j = i_0 - 1$, we can check that

$$c_{i_0-1,n_0} = 1.$$

Finally, for $j \in \{2,\dots,i_0-2, i_0\}$,

$$\bar{c}_{j,n_0}^{(i_0)} = \max_{0\le m\le i_0}\left\{\limsup_{t\to\infty} \frac{b_j^{(n_0-1)\leftarrow}(t)b_{i_0-j}^{\leftarrow}(t)}{b_m^{(n_0-1)\leftarrow}(t)b_{i_0-m}^{\leftarrow}(t)}\right\}$$

$$\ge \limsup_{t\to\infty} \frac{b_j^{(n_0-1)\leftarrow}(t)b_{i_0-j}^{\leftarrow}(t)}{b_1^{(n_0-1)\leftarrow}(t)b_{i_0-1}^{\leftarrow}(t)} = \limsup_{t\to\infty} \frac{t^{j(\alpha+\gamma)}\cdot t^{(i_0-j)(\alpha+\gamma)}}{t^{\alpha}\ell(t)\cdot t^{(i_0-1)(\alpha+\gamma)}} = \infty.$$

Hence, $I(i_0) = 1$ with

$$c_{m,n_0}^{I(i_0)} = \begin{cases} 1, & m \in \{1, i_0-1\},\\ 0, & \text{otherwise.}\end{cases}$$

Therefore by Theorem 3.1, $\boldsymbol{S}^{(n_0)} \in \mathcal{MRV}^*(\alpha_{i_0,n_0}, \widetilde{b}_{i_0,n_0}, \mu_{i_0,n_0}^{\oplus}, \mathbb{E}_d^{(i_0)})$ where $\alpha_{i_0,n_0} = \alpha_{1,n_0-1} + \alpha_{i_0-1} = 1 + \infty = \infty$ (recall $i_0 \ge 3$), $\widetilde{b}_{i_0,n_0}(t) = t^{i_0\alpha+(i_0-1)\gamma}\ell(t)$ and $\mu_{i_0,n_0}^{\oplus} \equiv 0$. Defining $b_{i_0,n_0}(t) = t^{i_0(\alpha+\gamma)} = b_{i_0}(t)$, since $\widetilde{b}_{i_0}(t) = o(b_{i_0}(t))$, we have $\boldsymbol{S}^{(n_0)} \in \mathcal{NC}(b_{i_0}(t), \mathbb{E}_d^{(i_0)})$. Therefore, by induction, (3.6) holds for all $n < i_0$.

(ii) Now, we show (3.4) holds for $n = n_0 = i_0$. For $j = 0,\dots,i_0-2$,

$$\bar{c}_{j,n_0}^{(i_0)} = \bar{c}_{j,i_0}^{(i_0)} := \max_{0\le m\le i_0}\left\{\limsup_{t\to\infty} \frac{b_j^{(i_0-1)\leftarrow}(t)b_{i_0-j}^{\leftarrow}(t)}{b_m^{(i_0-1)\leftarrow}(t)b_{i_0-m}^{\leftarrow}(t)}\right\}$$

$$\ge \limsup_{t\to\infty} \frac{b_j^{(i_0-1)\leftarrow}(t)b_{i_0-j}^{\leftarrow}(t)}{b_{i_0-1}^{(i_0-1)\leftarrow}(t)b_1^{\leftarrow}(t)}$$

$$= \limsup_{t\to\infty} \frac{(t^{\alpha}\ell(t))^{j}\cdot t^{(i_0-j)(\alpha+\gamma)}}{(t^{\alpha}\ell(t))^{(i_0-1)}\cdot t^{\alpha}\ell(t)} = \infty,$$

since by induction assumption $b_j^{(n_0-1)}(t) = b_j^{(i_0-1)}(t) = (b_1(t))^{1/j}$ for all $j \le n_0 - 1$. Similarly for $j = i_0$, we have

$$\bar{c}_{i_0,n_0} = \bar{c}_{i_0,i_0} = \infty$$

since we have $b_{i_0}^{(i_0-1)}(t) = t^{i_0(\alpha+\gamma)}$ from part 2(i) of the proof. For $j = i_0 - 1$,

$$\bar{c}_{i_0-1,n_0}^{(i_0)} = \bar{c}_{i_0-1,i_0}^{(i_0)} := \max_{0\le m\le i_0}\left\{\limsup_{t\to\infty} \frac{b_{i_0-1}^{(i_0-1)\leftarrow}(t)b_1^{\leftarrow}(t)}{b_m^{(i_0-1)\leftarrow}(t)b_{i_0-m}^{\leftarrow}(t)}\right\}$$

$$= \max_{0\le m\le i_0}\left\{\limsup_{t\to\infty} \frac{(t^{\alpha}\ell(t))^{(i_0-1)}\cdot t^{\alpha}\ell(t)}{b_m^{(i_0-1)\leftarrow}(t)b_{i_0-m}^{\leftarrow}(t)}\right\} = 1.$$

Finally, for $j = i_0$,

$$\bar{c}^{(i_0)}_{i_0,n_0} = \bar{c}^{(i_0)}_{i_0,i_0} := \max_{0\le m\le i_0} \left\{ \limsup_{t\to\infty} \frac{b_{i_0}^{(i_0-1)\leftarrow}(t) b_0^{\leftarrow}(t)}{b_m^{(i_0-1)\leftarrow}(t) b_{i_0-m}^{\leftarrow}(t)} \right\}$$
$$\ge \limsup_{t\to\infty} \frac{t^{i_0(\alpha+\gamma)}}{b_{i_0-1}^{(i_0-1)\leftarrow}(t) b_1^{\leftarrow}(t)} = \infty.$$

Hence, $I(i_0) = i_0 - 1$ with

$$c^{I(i_0)}_{m,n_0} = c^{I(i_0)}_{m,i_0} = \begin{cases} 1, & m = i_0 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, by Theorem 3.1, $\boldsymbol{S}^{(n_0)} = \boldsymbol{S}^{(i_0)} \in \mathcal{MRV}(\alpha_{i_0,n_0}, b_{i_0,n_0}, \mu^{\oplus}_{i_0,n_0}, \mathbb{E}_d^{(i_0)})$ where

$$\alpha_{i_0,n_0} = \alpha_{i_0,i_0} = \alpha_{i_0-1,i_0-1} + \alpha_1 = (i_0 - 1)\alpha + \alpha = i_0\alpha,$$

$b^{\leftarrow}_{i_0,n_0}(t) = b_{i_0}^{(i_0-1)\leftarrow}(t) b_1^{\leftarrow}(t) = (t^\alpha \ell(t))^{i_0}$ and

$$\mu^{\oplus}_{i_0,n_0} = \mu^{\oplus}_{i_0,i_0} = \sum_{m=0}^{i_0} c^{I(i_0)}_{m,i_0} \mu^{*}_{m,i_0,i_0} = \mu_{i_0-1,i_0,i_0},$$

where for $A = \{\boldsymbol{z} \in \mathbb{R}^d_+ : z_j > x_j \; \forall\, j \in S\} \in \mathcal{R}^{(i_0)}$ with $|S| = i_0$, $x_j > 0$ for $j \in S$

$$\begin{aligned}
&\mu_{i_0-1,i_0,i_0}(A) \\
&= \sum_{\substack{J\subseteq S \\ |J|=i_0-1}} \mu^{\oplus}_{i_0-1,i_0-1}\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i_0-1)} : z_j > x_j \,\forall\, j \in J\}\right) \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(1)} : z_j > x_j \,\forall\, j \in S \setminus J\}\right) \\
&= \sum_{\substack{J\subseteq S \\ |J|=i_0-1}} \left[(i_0-1)! \prod_{j\in J} \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right) \right] \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j \,\forall\, j \in S \setminus J\}\right) \\
&= \sum_{j\in S} (i_0-1)! \left[\prod_{k\in S\setminus\{j\}} \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_k > x_k\}\right) \right] \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right) \\
&= i_0! \prod_{j\in S} \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right).
\end{aligned}$$

Hence, (3.4) holds for $n_0 = i_0$.

(iii) Here we show (3.4) holds for all $n \ge i_0$. By way of induction (additionally) assume that for $i = i_0$, (3.4) holds for all $n \in \{i_0, i_0 + 1, \dots, n_0\}$. We will show that then it also holds for $n = n_0 + 1$. By part 2(ii), we know that it holds for $n_0 = i_0$. For $j = 0, \dots, i_0 - 2$,

$$\bar{c}^{(i_0)}_{j,n_0+1} := \max_{0\le m\le i_0} \left\{ \limsup_{t\to\infty} \frac{b_j^{(n_0)\leftarrow}(t) b_{i_0-j}^{\leftarrow}(t)}{b_m^{(n_0)\leftarrow}(t) b_{i_0-m}^{\leftarrow}(t)} \right\}$$
$$\ge \limsup_{t\to\infty} \frac{b_j^{(n_0)\leftarrow}(t) b_{i_0-j}^{\leftarrow}(t)}{b_{i_0}^{(n_0)\leftarrow}(t) b_0^{\leftarrow}(t)}$$
$$= \limsup_{t\to\infty} \frac{(t^\alpha \ell(t))^j \cdot t^{(i_0-j)(\alpha+\gamma)}}{(t^\alpha \ell(t))^{i_0} \cdot 1} = \limsup_{t\to\infty} t^{(i_0-j)\gamma} (\ell(t))^{(j-i_0)} = \infty,$$

since by induction assumption $b_j^{(n_0)}(t) = (b_1(t))^{1/j}$ for all $j \leq n_0$. By similar arguments we have for

$$\bar{c}^{(i_0)}_{i_0-1,n_0+1} = \bar{c}^{(i_0)}_{i_0,n_0+1} = 1.$$

Hence, $I(i_0) = i_0 - 1$, and

$$c^{I(i_0)}_{m,n_0+1} = \begin{cases} 1, & m \in \{i_0-1, i_0\}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore by Theorem 3.1, $\boldsymbol{S}^{(n_0+1)} \in \mathcal{MRV}(\alpha_{i_0,n_0+1}, b_{i_0,n_0+1}, \mu^{\oplus}_{i_0,n_0+1}, \mathbb{E}_d^{(i_0)})$ where

$$\alpha_{i_0,n_0+1} = \alpha_{i_0-1,n_0} + \alpha_1 = (i_0-1)\alpha + \alpha = i_0\alpha,$$

$$b^{\leftarrow}_{i_0,n_0+1}(t) = b^{(n_0)\leftarrow}_{i_0-1}(t) b_1^{\leftarrow}(t) = (t^{\alpha}\ell(t))^{i_0}, \text{ and,}$$

$$\mu^{\oplus}_{i_0,n_0+1} = \sum_{m=0}^{i_0} c^{I(i_0)}_{m,n_0+1}\mu^{*}_{m,i_0,n_0+1} = \mu_{i_0-1,i_0,n_0+1} + \mu_{i_0,i_0,n_0+1},$$

where for $A = \{\boldsymbol{z} \in \mathbb{R}^d_+ : z_j > x_j \; \forall\, j \in S\} \in \mathcal{R}^{(i_0)}$ with $|S| = i_0$, $x_j > 0$ for $j \in S$

$$\begin{aligned}
&\mu_{i_0-1,i_0,n_0+1}(A) \\
&= \sum_{\substack{J \subseteq S \\ |J| = i_0-1}} \mu^{\oplus}_{i_0-1,n_0}\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i_0-1)} : z_j > x_j \; \forall\, j \in J\}\right)\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(1)} : z_j > x_j \; \forall\, j \in S \setminus J\}\right) \\
&= \sum_{\substack{J \subseteq S \\ |J| = i_0-1}} \left[\frac{n_0!}{(n_0-i_0+1)!}\prod_{j\in J}\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right)\right]\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j, j \in S \setminus J\}\right) \\
&= \sum_{j \in S} \frac{n_0!}{(n_0-i_0+1)!}\left[\prod_{k \in S\setminus\{j\}} \mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_k > x_k\}\right)\right]\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right) \\
&= i_0 \cdot \frac{n_0!}{(n_0-i_0+1)!}\prod_{j\in S}\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right),
\end{aligned}$$

and,

$$\begin{aligned}
\mu_{i_0,i_0,n_0+1}(A) &= \sum_{\substack{J \subseteq S \\ |J| = i_0}} \mu^{\oplus}_{i_0,n_0}\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i_0-1)} : z_j > x_j \; \forall\, j \in J\}\right) \\
&= \frac{n_0!}{(n_0-i_0)!}\prod_{j \in J}\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i_0)} : z_j > x_j\}\right).
\end{aligned}$$

The measures $\mu^{\oplus}_{i_0-1,n_0}, \mu^{\oplus}_{i_0,n_0}, \mu_1$ are obtained from our assumptions and induction hypothesis. Now,

$$\begin{aligned}
\mu^{\oplus}_{i_0,n_0+1}(A) &= \mu_{i_0-1,i_0,n_0+1}(A) + \mu_{i_0,i_0,n_0+1}(A) \\
&= \left[i_0 \cdot \frac{n_0!}{(n_0-i_0+1)!} + \frac{n_0!}{(n_0-i_0)!}\right]\prod_{j\in S}\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right) \\
&= \frac{(n_0+1)!}{(n_0+1-i_0)!}\prod_{j\in S}\mu_1\left(\{\boldsymbol{z} \in \mathbb{E}_d^{(i)} : z_j > x_j\}\right).
\end{aligned}$$

Hence, (3.4) holds for $i=i_0$ and $n=n_0+1$, thus by induction it holds for all $n\geq i_0$.

□

APPENDIX E: PROOF OF THEOREM 4.3

For the proof we require some auxiliary results.

LEMMA E.1. *Let the assumptions of Theorem 4.3 hold. Define* $\boldsymbol{Z}^{\oplus}:=(Z_1^{\oplus},\dots,Z_d^{\oplus}):=\sum_{k=1}^d \boldsymbol{Z}^{(k)}$ *and denote by* $Z_{(1)}^{\oplus}\geq\ldots\geq Z_{(d)}^{\oplus}$ *the order statistics of* $Z_1^{\oplus},\dots,Z_d^{\oplus}$. *Also let* $Z_{(1)}^{(k)}\geq\ldots Z_{(d)}^{(k)})$ *be the order statistics of the elements of* $\boldsymbol{Z}^{(k)}=(Z_1^{(k)},\dots,Z_d^{(k)})$ *for any* $k\geq 1$. *Furthermore, for* $n\in\mathbb{N}$ *and* $i=1,\dots,d$ *define*

$$\alpha_{i,n}:=\sup_{S\subseteq\mathbb{I},\,|S|\leq i}\sup_{t>0}\frac{\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^n Z_j^{(k)}>t\right\}\right)}{\mathbb{P}\left(Z_{(|S|)}^{\oplus}>t\right)}.$$

Then there exists a finite constant $K_i>0$ *such that for any* $n\in\mathbb{N}$:

$$\alpha_{i,n+1}\leq K_i^n.$$

For one-dimensional random variables with $i=d=1$, a stronger result holds: for any $\epsilon>0$ there exists a constant $K>0$ such that the left hand side is bounded by $K(1+\epsilon)^n$ (cf. [16, Lemma 1.3.5]).

PROOF. First, we show recursively that for any $i\in\mathbb{I}$ there exists a constant $K_i>0$ such that $\alpha_{i,n+1}\leq K_i\alpha_{i,n}$ for any $n\geq d$. Let $S\subseteq\mathbb{I}$ with $|S|\leq i$ and $n\geq d$. Then

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n+1}Z_j^{(k)}>t\right\}\right)\\
&=\sum_{\substack{J\subseteq S\\J\neq\emptyset}}\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n+1}Z_j^{(k)}>t\right\}\cap\bigcap_{j\in J}\left\{Z_j^{(n+1)}>t/2\right\}\cap\bigcap_{j\in S\setminus J}\left\{Z_j^{(n+1)}\leq t/2\right\}\right)\\
&\quad+\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n+1}Z_j^{(k)}>t\right\}\cap\bigcap_{j\in S}\left\{Z_j^{(n+1)}\leq t/2\right\}\right)\\
&=:J_{n,1}(t,S)+J_{n,2}(t,S).
\end{aligned}\tag{E.1}$$

We investigate the two terms separately. First,

$$\begin{aligned}
&J_{n,1}(t,S)\\
&\leq\sum_{\substack{J\subseteq S\\J\neq\emptyset}}\mathbb{P}\left(\bigcap_{j\in S\setminus J}\left\{\sum_{k=1}^{n+1}Z_j^{(k)}>t\right\}\cap\bigcap_{j\in J}\left\{Z_j^{(n+1)}>t/2\right\}\right)\\
&\leq\sum_{\substack{J\subseteq S\\J\neq\emptyset}}\sum_{K\subseteq S\setminus J\cup\{\emptyset\}}\mathbb{P}\left(\bigcap_{j\in S\setminus(J\cup K)}\left\{\sum_{k=1}^{n}Z_j^{(k)}>t/2\right\}\cap\bigcap_{j\in J}\left\{Z_j^{(n+1)}>t/2\right\}\cap\bigcap_{j\in K}\left\{Z_j^{(n+1)}>t/2\right\}\right)\\
&\leq\sum_{\substack{J\subseteq S\\J\neq\emptyset}}\sum_{K\subseteq S\setminus J\cup\{\emptyset\}}\mathbb{P}\left(\bigcap_{j\in S\setminus(J\cup K)}\left\{\sum_{k=1}^{n}Z_j^{(k)}>t/2\right\}\right)\mathbb{P}\left(\bigcap_{j\in J\cup K}\left\{Z_j^{(n+1)}>t/2\right\}\right).
\end{aligned}$$

Since the set $S\backslash J \cup K$ has at most $i-1$ elements and by definition $\alpha_{i-1,n} \le \alpha_{i,n}$, we have that

$$J_{n,1}(t,S) \le \alpha_{i,n} \sum_{\substack{J\subseteq S \\ J\neq\emptyset}} \sum_{K\subseteq S\backslash J\cup\{\emptyset\}} \mathbb{P}\left(Z^{\oplus}_{(|S\backslash(J\cup K)|)} > t/2\right) \mathbb{P}\left(Z^{(n+1)}_{(|J\cup K|)} > t/2\right).$$

Now applying (4.1) and (4.2) we have

$$\sup_{\substack{S\subseteq\mathbb{I} \\ |S|\le i}} \sup_{t>0} \frac{J_{n,1}(t,S)}{\mathbb{P}\left(Z^{\oplus}_{(|S|)} > t\right)} \le \alpha_{i,n} \sup_{\substack{S\subseteq\mathbb{I} \\ |S|\le i}} \sum_{\substack{J\subseteq S \\ J\neq\emptyset}} \sum_{K\subseteq S\backslash J\cup\{\emptyset\}} C^* C^{**}$$
$$\le \alpha_{i,n} 2^{2i} C^* C^{**} = \alpha_{i,n} \widetilde{K}_i \tag{E.2}$$

with $\widetilde{K}_i := 2^{2i} C^* C^{**}$. Next, for the second term in (E.1) and $S = \{j_1, \ldots, j_i\}$ we have

$$\sup_{\substack{S\subseteq\mathbb{I} \\ |S|\le i}} \sup_{t>0} \frac{J_{n,2}(t,S)}{\mathbb{P}\left(Z^{\oplus}_{|S|} > t\right)}$$
$$= \sup_{\substack{S\subseteq\mathbb{I} \\ |S|\le i}} \sup_{t>0} \int_0^{t/2} \cdots \int_0^{t/2} \frac{\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^n Z_j^{(k)} > t - y_j\right\}\right)}{\mathbb{P}\left(Z^{\oplus}_{|S|} > t/2\right)}$$
$$\cdot \frac{\mathbb{P}\left(Z^{\oplus}_{|S|} > t/2\right)}{\mathbb{P}\left(Z^{\oplus}_{|S|} > t\right)} F_{Z_{j_1},\ldots,Z_{j_i}}(dy_1,\ldots,dy_i)$$
$$\le C^{**} \sup_{\substack{S\subseteq\mathbb{I} \\ |S|\le i}} \sup_{t>0} \int_0^{t/2} \cdots \int_0^{t/2} \frac{\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^n Z_j^{(k)} > t/2\right\}\right)}{\mathbb{P}\left(Z^{\oplus}_{(|S|)} > t/2\right)} F_{Z_{j_1},\ldots,Z_{j_i}}(dy_1,\ldots,dy_i)$$

where we applied (4.2) once more. Now the last term above is bounded by $C^{**}\alpha_{i,n}$ and hence we have

$$\sup_{\substack{S\subseteq\mathbb{I} \\ |S|\le i}} \sup_{t>0} \frac{J_{n,2}(t,S)}{\mathbb{P}\left(Z^{\oplus}_{|S|} > t\right)} \le C^{**}\alpha_{i,n}. \tag{E.3}$$

Now from (E.1), (E.2) and (E.3) we get

$$\alpha_{i,n+1} \le \alpha_{i,n}\widetilde{K}_i + \alpha_{i,n}C^{**} = (\widetilde{K}_i + C^{**})\alpha_{i,n}. \tag{E.4}$$

Note that $\alpha_{i,d} \le 1$ for $i = 1, \ldots, d$. Thus applying (E.4) recursively we obtain $\alpha_{i,n+1} \le (\widetilde{K}_i + C^{**})^{n-d}$ for $n \ge d$, $i = 1, \ldots, d$. But for $n \le d$ we have of course $\alpha_{i,n} \le 1$ for $i = 1, \ldots, d$. Thus, with $K_i = \max(1, \widetilde{K}_i + C^{**})$ the statement of the lemma is satisfied. □

PROOF OF THEOREM 4.3. Define $\boldsymbol{Z}^{\oplus} := (Z_1^{\oplus}, \ldots, Z_d^{\oplus}) := \sum_{k=1}^d \boldsymbol{Z}^{(k)}$ and denote by $Z^{\oplus}_{(1)} \ge \ldots \ge Z^{\oplus}_{(d)}$ the order statistics of $Z_1^{\oplus}, \ldots, Z_d^{\oplus}$. Let $A = \{\boldsymbol{z} \in \mathbb{R}_+^d : z_j > x_j \,\forall\, j \in S\}$ be a rectangular set in $\mathbb{E}_d^{(i)}$ where $S \subseteq \mathbb{I}$, $|S| \ge i$ and $x_j > 0$, $\forall\, j \in S$ with $\mu_i(\partial A) = 0$. Suppose $\widetilde{S} \subseteq S$ with $|\widetilde{S}| = i$. Then

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(\sum_{k=1}^{\tau} \boldsymbol{Z}^{(k)} \in tA\right)}{\mathbb{P}\left(Z^{\oplus}_{(i)} > t\right)}$$

$$= \lim_{t\to\infty} \sum_{n=0}^{\infty} \mathbb{P}(\tau = n) \frac{\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n} Z_j^{(k)} > tx_j\right\}\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)}. \tag{E.5}$$

But for any $n \in \mathbb{N}$ we have

$$\begin{aligned}
0 &\leq \sup_{t>0} \frac{\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n} Z_j^{(k)} > tx_j\right\}\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} \\
&\leq \sup_{t>0} \frac{\mathbb{P}\left(\bigcap_{j\in \widetilde{S}}\left\{\sum_{k=1}^{n} Z_j^{(k)} > t\min_{j\in S} x_j\right\}\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\min_{j\in S} x_j\right)} \frac{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\min_{j\in S} x_j\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} \\
&\leq \alpha_{i,n} \sup_{t>0} \frac{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\min_{j\in S} x_j\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)}.
\end{aligned} \tag{E.6}$$

Since $\boldsymbol{Z}^{(\oplus)} \in \mathcal{MRV}(\alpha_i, b_i, f_i(d)\mu_i, \mathbb{E}_d^{(i)})$ and $f_i(d)\mu_i\left(\{\boldsymbol{z} \in \mathbb{R}_+^d : z_{(i)} > 1\}\right) > 0$, we have

$$1 \leq \frac{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\min_{j\in S} x_j\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} \overset{t\to\infty}{\to} \left(\min_{j\in S} x_j\right)^{-\alpha_i} < \infty.$$

Hence, there exists a finite constant $C > 0$ such that

$$\sup_{t>0} \frac{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\min_{j\in S} x_j\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} \leq C. \tag{E.7}$$

Then an application of Lemma E.1 and (E.6), (E.7) yield

$$0 \leq \sup_{t>0} \frac{\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n} Z_j^{(k)} > tx_j\right\}\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} \leq CK_i^n, \quad n \in \mathbb{N}.$$

Thus, there exists a uniform finite upper bound of the right hand side of (E.5) such that due to Pratt's Theorem we are allowed to exchange the limit and the sum. A conclusion of Assumption A is then

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(\sum_{k=1}^{\tau} \boldsymbol{Z}^{(k)} \in tA\right)}{\mathbb{P}\left(Z_{(i)}^{\oplus} > t\right)} = \sum_{n=0}^{\infty} \mathbb{P}(\tau = n) f_i(n) \frac{\mu_i(A)}{f_i(d)\mu_i\left(\{\boldsymbol{z} \in \mathbb{R}_+^d : z_{(i)} > 1\}\right)}.$$

Then Proposition 2.7 and $\boldsymbol{Z}^{\oplus} \in \mathcal{MRV}(\alpha_i, b_i, f_i(d)\mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ result in $\sum_{k=1}^{\tau} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, \mathbb{E}(f_i(\tau))\mu_i, \mathbb{E}_d^{(i)})$. □

APPENDIX F: PROOFS OF THE RESULTS IN SECTION 5

PROOF OF PROPOSITION 5.1.
Step 1. To begin with, let $(\boldsymbol{L}(s))_{s\geq 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\mathbb{P}_{\boldsymbol{Z}} = \Pi/\lambda$, which is a proper probability measure on $\mathbb{R}^d_+$. Let us also assume that $(N(s))_{s\geq 0}$ is a Poisson process with intensity λ and $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \dots$ are i.i.d. with distribution $\mathbb{P}_{\boldsymbol{Z}}$. Then $\mathbb{P}_{\boldsymbol{Z}} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i/\lambda, \mathbb{E}_d^{(i)})$. Since $\mathbb{E}(N(s)) = \lambda s$, using Proposition 3.5 and Theorem 4.3 we have

$$\boldsymbol{L}(s) \overset{\mathrm{d}}{=} \sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, s\mu_i, \mathbb{E}_d^{(i)}) \quad \text{for} \quad i = 1, \dots, d.$$

Step 2. Now let $(\boldsymbol{L}(s))_{s\geq 0}$ be a general Lévy process. Define $D_{a,\infty} := \{\boldsymbol{z} \in \mathbb{R}^d : a < \|\boldsymbol{z}\| < \infty\}$ for any $a > 0$. Due to the Lévy-Itô decomposition (see [41, Theorem 19.2 and Theorem 19.3]) we can decompose $\boldsymbol{L}$ into two independent Lévy processes $\boldsymbol{L}_1 = (\boldsymbol{L}_1(s))_{s\geq 0}$ and $\boldsymbol{L}_2 = (\boldsymbol{L}_2(s))_{s\geq 0}$ such that

$$\boldsymbol{L}(s) = \boldsymbol{L}_1(s) + \boldsymbol{L}_2(s), \quad s \geq 0,$$

where $\boldsymbol{L}_1$ is a compound Poisson process with Lévy measure $\Pi(\cdot \cap D_{a,\infty})/\Pi(D_{a,\infty})$ and Poisson intensity $\Pi(D_{a,\infty})$, whereas $\boldsymbol{L}_2$ satisfies $\mathbb{E}\|\boldsymbol{L}_2(s)\|^\theta < \infty$ for any $\theta > 0$ (see [31, Lemma 2.2 and proof of Theorem 2.3]). Thus, the Lévy measure of $\boldsymbol{L}_1$ is $\Pi(\cdot \cap D_{a,\infty}) \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \dots, d$ and by step 1 we have

$$\boldsymbol{L}_1(s) \in \mathcal{MRV}(\alpha_i, b_i, s\mu_i, \mathbb{E}_d^{(i)}) \quad \text{for} \quad i = 1, \dots, d.$$

Then an application of Lemma 2.9 and $\boldsymbol{L}(s) = \boldsymbol{L}_1(s) + \boldsymbol{L}_2(s)$ gives us the result. □

PROOF OF PROPOSITION 5.7. As in Proposition 5.1 it is sufficient to investigate compound Poisson processes $(\boldsymbol{L}(s))_{s\geq 0} = (\sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)})_{s\geq 0}$ with intensity $\lambda > 0$ and jumps size distribution $\mathbb{P}_{\boldsymbol{Z}} = \Pi/\lambda$. Then for the jump size distribution we have $\mathbb{P}_{\boldsymbol{Z}} = \Pi/\lambda \in \mathcal{MRV}(\alpha_i, b_i, \mu_i/\lambda, \mathbb{E}_d^{(i)})$ for $i = 1, \dots, d$. Since $\mathbb{E}(N(s)(N(s)-1)(N(s)-i+1)) = (\lambda s)^i$, $f_i(n) = 0$ for $n < i$, $f_i(n) = n!/(n-i)!$ for $n \geq i$, Proposition 3.12 and Theorem 4.3 result in

$$\boldsymbol{L}(s) = \sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(i\alpha_1, b_1^{1/i}, s^i \mu_i^L, \mathbb{E}_d^{(i)}) \quad \text{for} \quad i = 1, \dots, d,$$

which is the statement. □

PROOF OF PROPOSITION 5.9. Suppose $(\boldsymbol{L}(s))_{s\geq 0}$ is a compound Poisson process with $\Pi(\mathbb{R}^d_+) \leq 1$. Let $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \dots$ be a sequence of i.i.d. random vectors with distribution $\mathbb{P}(\boldsymbol{Z}^{(1)} = \boldsymbol{0}) = 1 - \Pi(\mathbb{R}^d_+)$ and $\mathbb{P}(\boldsymbol{Z}^{(1)} \in A \backslash \{\boldsymbol{0}\}) = \Pi(A \backslash \{\boldsymbol{0}\})$ for all sets $A \in \mathcal{B}(\mathbb{R}^d_+)$. Then $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \dots, d$. Due to Proposition 3.3 and Theorem 4.3 we receive

$$\boldsymbol{L}(s) \overset{\mathrm{d}}{=} \sum_{k=1}^{N^*(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}(i\alpha, b_1^{1/i}, \mathbb{E}(N^*(s)^i)\mu_i, \mathbb{E}_d^{(i)}).$$

We extend this result to general Lévy measures and Lévy processes as in Proposition 5.1 by choosing a large enough so that $\Pi(D_{a,\infty}) \leq 1$. □