On the Existence and Stability of Nonlinear Waves in Lugiato-Lefever Models and in Systems with Periodic Coefficients

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M.Sc. Lukas Bengel
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1. Referent:	Prof. Dr. Wolfgang Reichel
2. Referent:	Prof. Dr.-Ing. Christian Koos
3. Referentin:	Prof. Dr. Mariana Haragus

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Contents

1 Introduction

Waves in the form of pulses, fronts or periodic patterns are *nonlinear* phenomena arising across various scientific fields such as physics, biology, and ecology. For instance, optical pulses can be generated in ring-shaped cavities and are used for high-speed data transmission [92]. Furthermore, the population density of a species that is spreading into unpopulated areas can be described by invasion fronts [84], and periodic patterns arise in the spatial structure of vegetation in dryland ecosystems [119]. In these examples, it is possible to derive nonlinear partial differential equations (PDEs) that model the dynamics of the systems. Constructing and analyzing solutions to the PDEs that capture these wave-like dynamics can then advance our understanding of the complex phenomena.

At the beginning of the mathematical analysis, the most fundamental questions concern the *existence* and *stability* of wave solutions in the underlying system of differential equations. Given a specific model, this means that one searches for special solutions of the PDE that exhibit a desired waveform and persist over time. Standing waves are of particular interest as they often emerge as asymptotic end states in the long-time dynamics. At the same time, the existence problem for stationary solutions in one spatial dimension reduces to an ordinary differential equation (ODE) equipped with suitable boundary conditions. This opens up the possibility of combining tools from functional analysis and dynamical systems theory to analyze the ODE for the wave profile. Once the existence of a waveform is established, the question of its stability naturally arises. From a practical point of view, stability properties of the solutions are critical to determine whether or not the waves can be observed in the experiments and are therefore of great importance. For dissipative systems, nonlinear stability often follows under general spectral stability assumptions on the stationary waves [79]. Spectral stability, however, typically depends on the specific waveform profile and requires a detailed analysis of the associated eigenvalue problem. Tools such as exponential dichotomies, Evans functions, or perturbation theory can then be employed to locate eigenvalues of the linearized system in the complex plane.

In this thesis, we study the existence and stability of standing nonlinear waves in one spatial dimension for two types of systems: *Lugiato-Lefever models* and *semilinear systems with periodic coefficients*.

1.1 The Lugiato-Lefever equation

The Lugiato-Lefever equation (LLE)

$$\mathrm{i}u_t = -du_{xx} + (\zeta - \mathrm{i}\mu)u - |u|^2u + \mathrm{i}f, \qquad x, t \in \mathbb{R}, \tag{LLE}$$

is a nonlinear Schrödinger (NLS) equation with damping and forcing. It was derived by Lugiato and Lefever in 1987 [89] to describe the dynamics of an electric field in a passive nonlinear optical cavity filled by a Kerr medium that is driven by a continuous-wave (CW) laser beam and subject to energy losses. Here, $u(x,t) \in \mathbb{C}$ denotes the normalized envelope of the electric field, $d \neq 0$ is the dispersion parameter ($d > 0$ and $d < 0$ correspond to anomalous and normal dispersion, respectively), $\zeta = \omega_0 - \omega_{p_0} \in \mathbb{R}$ denotes the detuning between the laser frequency ω_{p_0} and the resonant cavity frequency ω_0, damping is given by $\mu > 0$, and $f > 0$ is the strength of the laser input. The presence of dispersion and nonlinearity on the one hand and damping and forcing on the other allows for the formation of dissipative solitons in the LLE, which are used for the description of *frequency combs* [73].

Frequency combs are optical signals consisting of many equidistantly spaced frequency lines. They are promising tools with numerous applications including, for example, frequency metrology [144] and optical communication [92]. The importance of frequency combs was widely recognized in 2005, when Theodor Hänsch and John Hall were awarded the Nobel Prize for their contribution in frequency spectroscopy employing frequency comb technology [68]. Utilizing frequency combs across these fields requires an efficient and stable method for generating them. More recently, experimental demonstrations have shown that compact chip-scale ring-shaped microresonators, when pumped by a strong CW laser, are reliable devices for stable frequency comb generation, enabling comb formation with a small footprint [38]. Within the resonators, the nonlinear Kerr effect, inherent to the resonator material, converts monochromatic light into a frequency comb with many evenly spaced frequency lines, see Figure 1.1. For the complex field amplitude $u(x,t)$

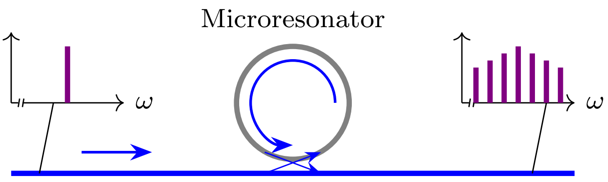

Figure 1.1: Schematic picture of the experimental set-up for frequency comb generation in a high-quality ring-resonator. Monochromatic light is injected into the resonator, which can store the light for many round-trips. Once a certain intensity threshold is attained, instabilities appear and subsequently the monochromatic light is converted into a signal with broad frequency spectrum through the nonlinear Kerr effect.

inside the resonator, the LLE equipped with periodic boundary conditions in x can then be derived, with the coordinate x corresponding to the angular position in the ring-resonator [66,73].

The formation of frequency combs in microresonators is a complex nonlinear dynamical process and strongly depends on the tuning of the control parameters ζ and f. Initially, the field inside the resonator is in a stable constant CW state, that is $u(x,0) = u_{CW} \in \mathbb{C}$. If we fix the forcing f at a sufficiently large value ≥ 1, the homogeneous CW states may become modulationally unstable as the detuning ζ is varied. At the onset of instability, a linear wave pattern emerges, which due to the Kerr nonlinearity subsequently evolves into a nonlinear wave. If the value of ζ is carefully chosen, the generated waveform converges to a periodic localized multi-soliton. Figure 1.2 shows the formation of a 2-soliton starting from a homogeneous CW state. We note that localized soliton states yield a broad frequency spectrum, which is desired for practical applications, see Figure 1.3.

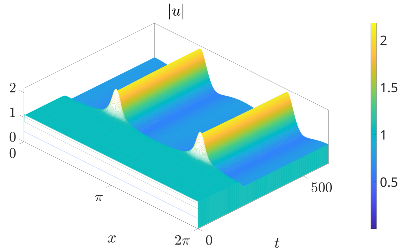

Figure 1.2: Time-integration of the LLE showing the formation of a 2-soliton starting from a CW state.

The mathematical literature for LLE primarily focuses on stationary states and their stability properties. In [37,54,61,67,91,94] the existence of stationary solutions has been proved using tools from bifurcation theory, center manifold reduction, or a spatial dynamics approach. Spectral and nonlinear stability is addressed in [36,37,62,67,69–71,137]. In Part I of this thesis, we add to the list of known results, the first rigorous construction and stability analysis of stationary (periodic) multi-solitons. This is conducted for anomalous dispersion $d > 0$.

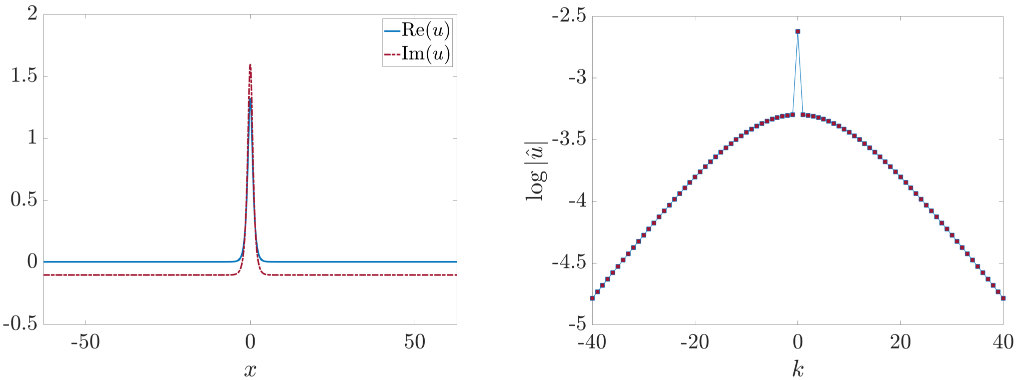

Figure 1.3: Numerical simulation of a stationary, localized, periodic 1-soliton solutions of LLE and its frequency spectrum.

Besides the LLE in its original formulation, we also consider two extensions that are both motivated from recent experiments in the field of frequency comb technologies [30, 46]. In the first extension, we consider the microresonator and the pump laser as a bidirectionally coupled dynamical system. This is different to previous set-ups, where only light from the laser is injected into the microresonator, but not the other way around. However, it has been demonstrated that the feedback of the resonator to the pump laser leads to synchronization of both devices and by this, tunes the laser frequency into the soliton existence range [22, 86, 117, 135, 153]. As a result, one observes automatic generation of frequency combs, bypassing any sweeping procedure of the frequency detuning ζ. We introduce and analyze the fully coupled system in Part II. In the second extension, we add a periodic potential multiplying a first order derivative to the LLE. This is motivated by a set-up, in which two microresonator is subject to a pumping scheme where two modes are excited, also known as dual laser pumping. In [46] it has been shown that the new pumping scheme can be used to lock the free spectral range of a frequency comb[1] to the external pump. The LLE with a periodic potential is studied in Part III and establishes the connection between the Lugiato-Lefever models and the general systems with spatially periodic coefficients.

1.2 Systems with periodic coefficients

We introduce the spatially periodic systems which are analyzed in Part III of this thesis. First, we consider the extended LLE (eLLE)

$$\mathrm{i}u_t = -du_{xx} + \mathrm{i}\varepsilon V(x)u_x + (\zeta - \mathrm{i}\mu)u - |u|^2u + \mathrm{i}f_0, \tag{eLLE}$$

with periodic potential $\varepsilon V(x)$. This equation is derived[2] in Section 5.7 to model microresonators which are pumped by a dual laser source with pump frequencies $\omega_{p_0}, \omega_{p_1}$ and pump strengths $|f_0| \gg |f_1|$, see Figure 1.4. The potential is given by $\varepsilon V(x) = \nu_1 - 2dk_1^2 f_0/f_1 \cos(x)$, where $k_1 \in \mathbb{N}$ is the number of the second pumped resonator mode with frequency ω_{k_1} ($k_0 = 0$ is the number of the first pumped mode with frequency ω_0) and $\nu_1 = \omega_0 - \omega_{p_0} + \omega_{p_1} - \omega_{k_1} + dk_1^2$. Despite its use to describe synchronization of frequency combs to external frequency oscillators [46], eLLE has not gained much attention in literature. The only mathematical result can be found in [57] where soliton solutions on $\mathbb{R}$ are constructed for the specific choice $V(x) = \alpha\cos(x)$ with $\alpha \in \mathbb{R}$. In this thesis, we investigate stationary periodic solutions for a large class of periodic potentials $V(x)$. From a mathematical point of view, the spatial heterogeneity breaks the continuous

[1] The free spectral range of a frequency comb equals the inverse of the repetition rate of the corresponding soliton in the ring resonator.

[2] The derivation was performed by Huanfa Peng, IPQ, KIT.

translation invariance in x. Thus, periodic soliton solutions of eLLE are expected to be pinned to specific points in the spatial domain, which turn out to be closely linked to zeros of $V(x)$. Physically, this pinning corresponds to a synchronization of frequency combs to the external pump.

Apart from nonlinear optics, systems with spatially periodic coefficients also occur in many other contexts. Examples include the Gross–Pitaevskii (GP) equation with a periodic trapping potential in the study of Bose–Einstein condensates [109], the Klausmeier model describing vegetation patterns in periodic topographies [13,14], and reaction-diffusion-systems with periodic diffusion coefficients in population dynamics [84, 136]. This motivates us to study a general semilinear system with spatially periodic coefficients, given by

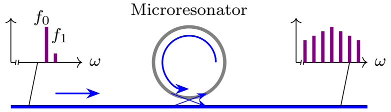

Figure 1.4: Microresonator pumped by a dual laser source with $|f_0| \gg |f_1| > 0$.

$$\partial_t u = \alpha_k(x)\partial_x^k u + \cdots + \alpha_1(x)\partial_x u + \mathcal{N}(x,u), \qquad u(x,t) \in \mathbb{F}^m,\ x \in \mathbb{R},\ t \geq 0. \qquad \text{(SYS)}$$

Here, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $m, k \in \mathbb{N}$, and $\alpha_i(x) \in \mathbb{F}^{m\times m}$ as well as $\mathcal{N}(x,u)$ are assumed to be continuous and T-periodic in x for all $i = 1, \ldots, k$. Moreover, we assume that $\det(\alpha_k(x)) \neq 0$ for all $x \in \mathbb{R}$ and that the nonlinearity $\mathcal{N}(x,u)$ is twice continuously differentiable with respect to u. We focus on the construction and stability analysis of stationary multifront and periodic pulse solutions of SYS. A multifront or M-front, where $M \in \mathbb{N}$, is a bounded solutions that connects $M+1$ periodic states through M front interfaces. In particular, it converges to periodic limit states $v_-(x)$ and $v_+(x)$ as $x \to \pm\infty$. Building blocks for the construction of M-front solutions are M nondegenerate 1-fronts. Here, nondegeneracy means the the linearized operator about the front is invertible. If the 1-fronts have matching periodic end states, we show that they are accompanied by a family of M-front solutions with well separated front interfaces. Moreover, we show that any given nondegenerate primary pulse solution of SYS, i.e., a solution that converge to a periodic background wave $v_0(x)$ as $|x| \to \infty$, is accompanied by a family of large wavelength periodic pulses. Our analysis extends previous results for autonomous dynamical systems [87,123,126,147] and appears to be the first systematic study of multiple front and periodic pulse solutions in general spatially heterogeneous systems.

We apply the abstract theory to a Klausmeier reaction-diffusion system with periodic coefficients f and g, that is

$$\begin{aligned}\partial_t w &= \partial_x^2 w + \varepsilon\big(f(x)\partial_x w + g(x)w\big) - w - wp^2 + a,\\ \partial_t p &= d^2\partial_x^2 p - mp + wp^2,\end{aligned}$$

and the GP equation

$$\mathrm{i}\partial_t u = -\partial_x^2 u + \mu V(x)u + \kappa|u|^2 u,$$

with periodic potential $\mu V(x)$, identifying novel stable solutions in both systems. In particular, we prove an orbital stability result for nontrivial time-harmonic spatially periodic solutions to the GP equation.

1.3 Outline of the thesis

This thesis is structured into three parts.

(I) **Multi-solitons in the Lugiato-Lefever equation**
Part I consists of the Chapters 2 and 3. Here we prove the existence and stability of periodic multi-soliton solutions of the LLE for anomalous disperison $d > 0$. Upon bifurcating from the bright soliton of the NLS equation, we establish the existence and stability of solitary wave solutions of the LLE in Chapter 2. In Chapter 3 we prove that the solitary waves are accompanied by periodic N-soliton solutions for every $N \in \mathbb{N}$. Moreover, we determine the stability of the N-solitons and establish their nonlinear stability against perturbations which are periodic or localized.

(II) **Solitons in a bi-directionally coupled laser-microresonator system**
Part II consists of Chapter 4. Inspired by recent experimental progress in frequency comb generation we model the microresonator and the CW laser as a bi-directionally coupled dynamical system. The resulting system couples the LLE to laser rate equations through a backward propagating resonator light field. We perform a numerical bifurcation analysis and time-integration simulations demonstrating the existence of stable soliton solutions for the fully coupled system. In comparison to the standard LLE model enhanced stability features of the solitons can now be observed. Moreover, we include sensitivity analysis unveiling the existence range for 1-solitons across a broad range of experimentally realistic parameter values.

(III) **Fronts and pulses in spatially periodic systems**
Part III consists of the Chapters 5, 6, and 7. In Chapter 5 we consider the eLLE with periodic potential $\varepsilon V(x)$. Depending on the shape of $V(x)$, we prove the existence and stability of periodic solutions which are pinned to zeros of an effective potential V_{eff}. In Chapter 6 we prove the existence and stability of multiple front and periodic pulse solutions in SYS, starting from finitely many nondegenerate fronts with matching end states and a nondegenerate primary pulse. We apply the abstract theory to a reaction-diffusion toy

model, the Klausmeier systems with periodic coefficients, as well as the Gross-Pitaevskii equation with a periodic potential. In Chapter 7 we employ the general toolbox of Chapter 6 to establish the existence and stability of periodic N-solitons to the eLLE.

In the following sections, we present a detailed overview of the main results of the Parts I, II, and III.

1.4 Publications, preprints, and code contained in this thesis

This thesis contains one single-author publication written by the author of this thesis, one co-authored publication and two co-authored preprints, which are submitted for publication. Chapter 4 is part of an ongoing project and at the time of submitting this thesis, the results have not been published or submitted elsewhere. Chapter 7 was written solely by the author of the thesis and has not been published elsewhere. We give a compact overview.

- Chapter 2 contains the article [15] written by the author of this thesis and published in *Zeitschrift für angewandte Mathematik und Physik.*
- Chapter 3 contains the article [16] written in collaboration with Björn de Rijk and submitted for publication.
- Chapter 4 is part of an ongoing project with Huanfa Peng, Björn de Rjk, Christian Koos, and Wolfgang Reichel.
- Chapter 5 contains the article [18] written in collaboration with Dmitry Pelinovsky and Wolfgang Reichel and published in *SIAM Journal of Mathematical Analysis*.
- Chapter 6 contains the article [17] written in collaboration with Björn de Rijk and submitted for publication.

The MATLAB code for reproducing the numerical simulations contained in this thesis can be downloaded using the following links.

- Code for the numerical simulations of Chapter 3:

  `https://www.waves.kit.edu/downloads/CRC1173_Preprint_2025-4_Codes.zip`

- Code for the numerical simulations of Chapter 4:

  `http://waves.kit.edu/downloads/CRC1173_Preprint_2025-preavailable_Codes.zip`

- Code for the numerical simulations of Chapter 5:

  `https://www.waves.kit.edu/downloads/CRC1173_Preprint_2023-6_supplement.zip`

- Code for the numerical simulations of Chapter 6:

  `https://www.waves.kit.edu/downloads/CRC1173_Preprint_2025-5_Codes.zip`

1.5 Main results of Part I: Multi-solitons in the Lugiato-Lefever equation

The aim of Part I is to establish the existence of stable periodic soliton solutions to the LLE in the anomalous dispersion regime $d > 0$. We recall that solitons are the physically relevant solutions to model optical frequency combs with broad frequency spectrum, cf. Figure 1.3. In [73] a close-to-perfect match was found between N-solitons $u(x)$ in the experiments and the analytical approximation formula

$$u(x) \approx \alpha_{CW} + \sum_{i=1}^{N} \phi_\theta(x - X_i). \tag{1.1}$$

Here, $\alpha_{CW} \in \mathbb{C}$ is the constant CW background, and

$$\phi_\theta(x) = \sqrt{2\zeta}\,\mathrm{sech}\left(\sqrt{\frac{\zeta}{d}}x\right)\mathrm{e}^{\mathrm{i}\theta}$$

is the bright soliton, which solves the nonlinear Schrödinger equation

$$\mathrm{i}\phi_t = -d\phi_{xx} + \zeta\phi - |\phi|^2\phi \tag{1.2}$$

for all $\theta \in \mathbb{R}$. The position of the solitons is denoted by X_i. We prove the existence of stationary solutions of the LLE, which satisfy the approximation formula (1.1). To this end, we introduce a small bifurcation parameter ε and study the LLE as a perturbed NLS,

$$\mathrm{i}u_t = -du_{xx} + \zeta u - |u|^2 u + \mathrm{i}\varepsilon\left(-u + f\right). \tag{1.3}$$

For $\varepsilon = \mu = 1$, we recover the original formulation of the LLE. The main result of Part I then reads as follows.

Theorem 1.1. *Let* $N \in \mathbb{N}$, *set* $n = \lfloor \frac{N}{2} \rfloor$ *and* $\alpha_0 = N \mod 2 \in \{0,1\}$. *Assume that* $d, \zeta, f > 0$ *and* $\theta_0 \in \mathbb{R}$ *is a simple zero of*

$$\theta \mapsto \pi f \cos(\theta) - 2\sqrt{2\zeta}$$

with $\sin(\theta_0) > 0$. *Then, there exist constants* $C_0, \varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *there exist distances* $T_1, \ldots, T_n \gg 1$ *and a period* $T \gg 1$ *satisfying*

$$2\sum_{i=1}^{n} T_i < T$$

such that equation (1.3) *admits a stationary, even, smooth, and* T*-periodic solution* $u_\varepsilon : \mathbb{R} \to \mathbb{C}$ *possessing the following properties:*

(i) **(Approximation).** *On a single periodicity interval, the solution* u_ε *is approximated by a superposition of* N *rotated bright solitons*

$$\left| u_\varepsilon(x) - \alpha_0 \phi_{\theta_0}(x) - \sum_{i=1}^{n} \left(\phi_{\theta_0}(x - T_1 - \ldots - T_i) + \phi_{\theta_0}(x + T_1 + \ldots + T_i) \right) \right| \leq C_0 \varepsilon$$

for $x \in [-\frac{1}{2}T, \frac{1}{2}T]$, $\varepsilon \in (0, \varepsilon_0)$.

(ii) **(Asymptotic orbital stability against periodic perturbations).** *Let* $\varepsilon \in (0, \varepsilon_0)$ *and* $M \in \mathbb{N}$. *There exist constants* $C, \delta, \eta > 0$ *such that for all* $v_0 \in H^1_{\mathrm{per}}(0, MT)$ *with* $\|v_0\|_{H^1_{\mathrm{per}}(0,MT)} < \delta$ *there exists a constant* $\gamma \in \mathbb{R}$ *and a global (mild) solution*

$$u \in C\big([0,\infty), H^1_{\mathrm{per}}(0, MT)\big)$$

of (1.3) *with initial condition* $u(0) = u_\varepsilon + v_0$, *satisfying*

$$\|u(\cdot, t) - u_\varepsilon(\cdot + \gamma)\|_{H^1_{\mathrm{per}}(0,MT)} \leq C\mathrm{e}^{-\eta t} \|v_0\|_{H^1_{\mathrm{per}}(0,MT)}$$

for $t \geq 0$.

(iii) **(Stability against localized perturbations).** *Let* $\varepsilon \in (0, \varepsilon_0)$. *There exist constants* $C, \delta > 0$ *such that for all* $v_0 \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$ *with* $\|v_0\|_{L^1 \cap H^4} < \delta$ *there exists a function*

$$v \in C\big([0,\infty), H^4(\mathbb{R})\big) \cap C^1\big([0,\infty), H^2(\mathbb{R})\big)$$

satisfying $v(0) = v_0$ *such that* $u = u_\varepsilon + v$ *is the unique global classical solution of* (1.3) *with* $u(0) = u_\varepsilon + v_0$. *Moreover, the estimates*

$$\|u(t) - u_\varepsilon\|_{L^2} \leq C(1+t)^{-\frac{1}{4}} \|v_0\|_{L^1 \cap H^4}$$

hold for $t \geq 0$.

For a more detailed version of Theorem 1.1 we refer to Theorem 3.2 in Chapter 3. By (i), the N-solitons consist of large amplitude pulses which are well separated in space and verify the approximation formula (1.1). They exist in the parameter regime $\pi^2 f^2 > 8\zeta$. Although the spatial period T is not controlled directly in our result, we can rescale dispersion and space according to $(d, u) \mapsto (d/T^2, u(T\cdot))$ to find 1-periodic solutions. Let us also comment on the stability results (ii) and (iii). The first states that the solitons are stable against subharmonic (i.e. MT-periodic) perturbations for every $M \in \mathbb{N}$. The second yields stability against perturbations which are integrable on the line. Both results are obtained by establishing spectral stability of the periodic solitons and subsequently applying the results from [70, 137].

We explain our contribution to the stability result in more detail. First, we write (1.3) as a system in the coordinates $\mathbf{u} = (\mathrm{Re}(u), \mathrm{Im}(u))^\top$,

$$\mathbf{u}_t = J\left(-d\mathbf{u}_{xx} + \zeta\mathbf{u} - |\mathbf{u}|^2\mathbf{u}\right) + \varepsilon(-\mathbf{u} + \mathbf{F}), \tag{1.4}$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Compared to (1.4), the advantage of the formulation (1.3) is that the nonlinearity is a polynomial and thus a Fréchet differentiable mapping from $H^1(\mathbb{R}) \to L^2(\mathbb{R})$. The linearization about a stationary solution $\underline{\mathbf{u}} \in L^\infty(\mathbb{R})$ of (1.4) is given by the matrix-valued operator $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon\colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with

$$\mathcal{L}(\underline{\mathbf{u}}) := JL(\underline{\mathbf{u}}), \qquad L(\underline{\mathbf{u}}) := -d\partial_x^2 + \zeta - \begin{pmatrix} 3\underline{\mathbf{u}}_1^2 + \underline{\mathbf{u}}_2^2 & 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 \\ 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 & \underline{\mathbf{u}}_1^2 + 3\underline{\mathbf{u}}_2^2 \end{pmatrix}.$$

The nonlinear stability results (ii) and (iii) follow from [70, 137] under a diffusive spectral stability assumption on the linearization $\mathcal{L}(\mathbf{u}_\varepsilon) - \varepsilon$ about the T-periodic soliton $\mathbf{u}_\varepsilon = (\mathrm{Re}(u_\varepsilon), \mathrm{Im}(u_\varepsilon))^\top$, which we establish in the proofs of Part I. Diffusive spectral stability is defined on the level of the Bloch operators $\mathcal{L}_\xi(\mathbf{u}_\varepsilon) - \varepsilon \colon H^2_{\text{per}}(0,T) \subset L^2_{\text{per}}(0,T) \to L^2_{\text{per}}(0,T)$, given by

$$\mathcal{L}_\xi(\mathbf{u}_\varepsilon) := JL_\xi(\mathbf{u}_\varepsilon), \qquad L_\xi(\underline{\mathbf{u}}) := -d(\partial_x + \mathrm{i}\xi T^{-1})^2 + \zeta - \begin{pmatrix} 3\underline{\mathbf{u}}_1^2 + \underline{\mathbf{u}}_2^2 & 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 \\ 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 & \underline{\mathbf{u}}_1^2 + 3\underline{\mathbf{u}}_2^2 \end{pmatrix},$$

which yield the spectral decomposition $\sigma(\mathcal{L}(\mathbf{u}_\varepsilon) - \varepsilon) = \bigcup_{\xi \in [-\pi,\pi)} \sigma(\mathcal{L}_\xi(\mathbf{u}_\varepsilon) - \varepsilon)$, cf. [52]. The T-periodic soliton is diffusively spectrally stable if the following assumptions are satisfied.

(i) We have $\sigma(\mathcal{L}(\mathbf{u}_\varepsilon) - \varepsilon) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\}$;

(ii) There exists $\vartheta > 0$ such that for all $\xi \in [-\pi, \pi)$ we have $\mathrm{Re}\left(\sigma\left(\mathcal{L}_\xi(\mathbf{u}_\varepsilon) - \varepsilon\right)\right) \leq -\vartheta\xi^2$;

(iii) 0 is a simple eigenvalue of the Bloch operator $\mathcal{L}_0(\mathbf{u}_\varepsilon) - \varepsilon$.

Note, that by the translational invariance of (1.4) we have $(\mathcal{L}_0(\mathbf{u}_\varepsilon) - \varepsilon)\partial_x \mathbf{u}_\varepsilon = 0$ and thus $0 \in \sigma(\mathcal{L}_0(\mathbf{u}_\varepsilon) - \varepsilon) \subset \sigma(\mathcal{L}(\mathbf{u}_\varepsilon) - \varepsilon)$. By (iii), this zero eigenvalue is simple and hence the implicit function theorem yields a curve consisting of simple eigenvalues $\lambda_0(\xi)$ to the Bloch operators $\mathcal{L}_\xi(\underline{\mathbf{u}}_\varepsilon) - \varepsilon$, which exists in a neighborhood of $\lambda_0(0) = 0$. This spectral curve is parametrized by the Bloch variable ξ. Condition (ii) ensures that it touches the origin quadratically and further yields that the remaining spectrum is confined to the open left-half plane with a spectral gap to the imaginary axis. Prior to Theorem 1.1, the only diffusive spectral stability result for periodic solutions of the LLE has been obtained in [37] for small amplitude Turing patterns.

We now sketch the proof of Theorem 1.1. It consists of two main steps carried out in Chapters 2 and 3, respectively.

In the first step, we prove the existence of 1-soliton solutions of (1.3) on the line $\mathbb{R}$ for small values of ε bifurcating from the NLS-soliton ϕ_{θ_0}. The rotational angle θ_0 has to satisfy the bifurcation equation $\pi f \cos(\theta) = 2\sqrt{2\zeta}$ with the transversality condition $\sin(\theta_0) \neq 0$. The construction relies on a Lyapunov–Schmidt reduction in parameter dependent subspaces exploiting symmetry breaking for $\varepsilon \neq 0$. We further prove that the 1-solitons are spectrally stable with a simple eigenvalue $\lambda = 0$, provided that $\varepsilon, \sin(\theta_0) > 0$ and that they are unstable for all other sign configurations of ε and $\sin(\theta_0)$. In the spectral analysis we use perturbation theory to locate small eigenvalues and exploit Krein index counting [31, 75, 76] for linear Hamiltonian systems to control all large eigenvalues. For the existence result we refer to Theorem 2.3 and the spectral stability result can be found in Theorem 2.6.

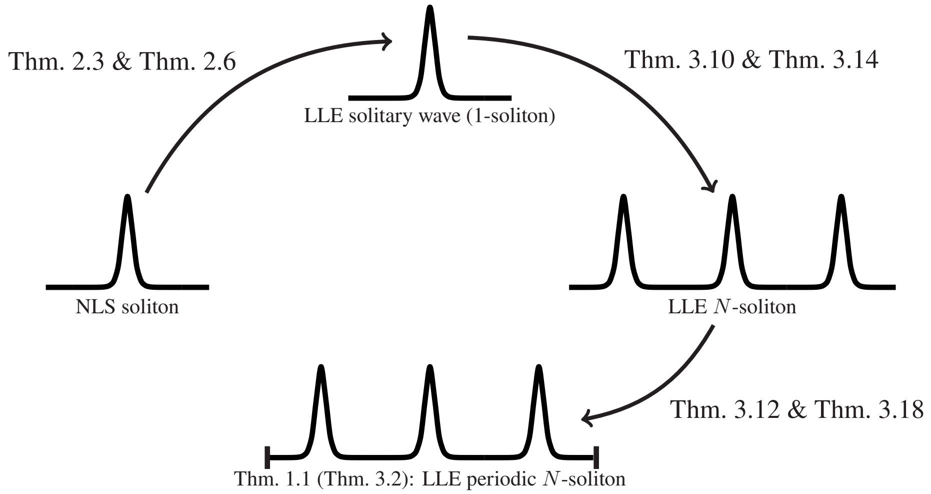

Figure 1.5: Overview of the main existence & stability results of Part I.

In the second step, we use the 1-solitons on $\mathbb{R}$ as starting points for the proof of the existence of periodic multi-solitons. We formulate the stationary problem (1.3) as a reversible dynamical system

$$U' = F(U), \qquad U = (\mathrm{Re}(u), \mathrm{Im}(u), \mathrm{Re}(u'), \mathrm{Im}(u'))^\top, \tag{1.5}$$

where $F\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a smooth nonlinear function. The 1-solitons then correspond to homoclinic solutions connecting a saddle focus equilibrium. This opens the way to apply results from homoclinic bifurcation theory for reversible systems [87, 123, 126, 127] that yield the existence of N-homoclinics for $N \in \mathbb{N}$ which bifurcate from the primary homoclinic solution, see Theorem 3.10. Further, we demonstrate that each N-homoclinic is accompanied by large wavelength periodic orbits, see Theorem 3.12. Returning to the second-order ODE formulation (1.3), we have established the existence of periodic multi-solitons. Proving diffusive spectral stability of the periodic N-solitons is achieved in Theorems 3.14 and 3.18. High-frequency resolvent estimates established in Theorem 3.13 allow us to reduce the spectral problem to a compact subset of $\mathbb{C}$. Within this set, we combine Evans functions techniques from Chapter 6, with Lin's method [126, 131], to close the proof. An overview of the results of Part I is given in Figure 1.5.

1.6 Main results of Part II: Solitons in a bi-directionally coupled laser-microresonator system

In Part II, we consider a microresonator and the CW laser as a bi-directionally coupled system for frequency comb generation. This set-up can be modeled by coupling the LLE to laser rate equations. This system differs from the one considered in Part I, where the laser injects light into the microresonator but an optical isolator prevents backcoupling from the resonator to the laser. Using numerical bifurcation analysis we demonstrate that the bi-directionally coupled system supports soliton solutions with enhanced stability properties. This means in particular, that for these soliton states, the pump laser is locked to the microresonator. This is confirmed by time-integration simulations. As a result, they are robust against perturbations in both the resonator and the laser fields.

The operational scheme of the coupled system works as follows. A CW single-mode semiconductor laser described by the carrier number $n(t)$ and the laser field $a_L(t)$ injects light into a Kerr microresonator. This excites the forward propagating resonator field $a_F(x,t)$, which in turn induces a backward propagating resonator field $a_B(t)$ through backscattering. In contrast to a_F, the backward field a_B is only represented by its central mode and hence does not depend on x. The microresonator is coupled to the CW laser via $a_B(t)$. Figure 1.6 summarizes the experimental set-up. Mathematically, it can be modeled by coupling the LLE describing the forward

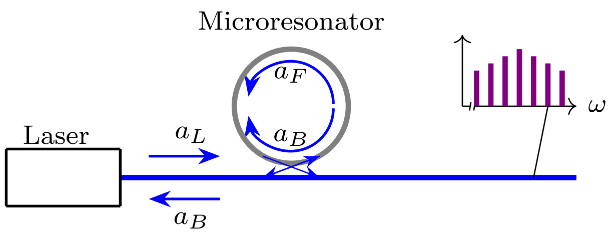

Figure 1.6: Experimental set-up of the bi-directionally coupled laser-microresonator system model by (1.6). The resonator field is represented by the sum of the forward field a_F and the backward field a_B. The forward field is driven by the laser field a_L and the backward field is coupled to the laser.

field dynamics, via an ODE for the backward field, to laser rate equations. The set of differential equations in normalized quantities is given by [86, 153]

$$\begin{aligned}
\dot{n}(t) =& \iota - \gamma n(t) - g(|a_L(t)|^2)(n(t)-1)|a_L(t)|^2, \\
\dot{a}_L(t) =& \Big[\frac{1-\mathrm{i}\alpha_H}{2}\big(g(|a_L(t)|^2)n_0(n(t)-1)-\kappa_d\big)+\mathrm{i}(\omega_m-\omega_0)\Big]a_L(t)-\kappa_{ext}a_B(t)\mathrm{e}^{\mathrm{i}\phi}, \\
\dot{a}_F(x,t) =& \Big[-1+\mathrm{i}d\frac{\partial^2}{\partial x^2}+\mathrm{i}(|a_F(x,t)|^2+2|a_B(t)|^2)\Big]a_F(x,t)+\mathrm{i}\kappa_{sc}a_B(t)-\kappa_{ext}a_L(t)\mathrm{e}^{\mathrm{i}\phi}, \\
\dot{a}_B(t) =& \left[-1+\mathrm{i}\big(|a_B(t)|^2+\frac{1}{\pi}\int_0^{2\pi}|a_F(x,t)|^2\mathrm{d}x\big)\right]a_B(t)+\mathrm{i}\overline{\kappa}_{sc}\frac{1}{2\pi}\int_0^{2\pi}a_F(x,t)\mathrm{d}x.
\end{aligned} \tag{1.6}$$

The dots represent time derivatives and the physical meanings of the parameters are explained in Table 4.1. Recent experimental and theoretical works have shown that soliton formation in the coupled system differs strongly from the more conventional soliton generation techniques without backcoupling. In fact, it was observed that, starting from a small perturbation of the zero state for all four fields, the instantaneous laser frequency $\omega_L(t) = \omega_m - \frac{d}{dt}\arg(a_L(t))$ automatically tunes into the soliton existence range. Subsequently, the formation of solitons can be observed [22, 30, 86, 117, 135, 153]. This mechanism is used to design compact devices that enable self-starting soliton formation [30].

Until now, theoretical investigations of the coupled system (1.6) have been limited mainly to time-integration simulations, see [86, 135, 153] and the references therein. In contrast, we perform a numerical bifurcation analysis with the MATLAB package `pde2path` [146] to systematically identify time-harmonic spatially localized soliton states for a large range of technically realistic parameter values. The main results, which have been obtained in the anomalous dispersion regime $d > 0$, can be summarized as follows.

- The bi-directionally coupled system (1.6) admits stable time-harmonic 1-soliton solutions of the form

$$\begin{aligned}
a_F(x,t) &= \underline{a}_F^0(x)\mathrm{e}^{\mathrm{i}\zeta_0 t}, & n(t) &= n^0, \\
a_L(t) &= \underline{a}_L^0\mathrm{e}^{\mathrm{i}\zeta_0 t}, & a_B(t) &= \underline{a}_B^0\mathrm{e}^{\mathrm{i}\zeta_0 t}.
\end{aligned}$$

  They bifurcate from trivial spatially homogeneous time-harmonic states and oscillate with the frequency $\zeta_0 = \zeta - \Delta\zeta$, where $\Delta\zeta = \frac{\kappa_{\mathrm{ext}}}{\underline{a}_L^0}\,\mathrm{Im}(\underline{a}_B^0\mathrm{e}^{\mathrm{i}\phi}(1-\mathrm{i}\alpha_{\mathrm{H}}))$. We use $\zeta := \omega_m - \omega_0$ as a bifurcation parameter, where ω_m and ω_0 denote the frequencies of the resonator mode and the laser cavity mode, respectively.

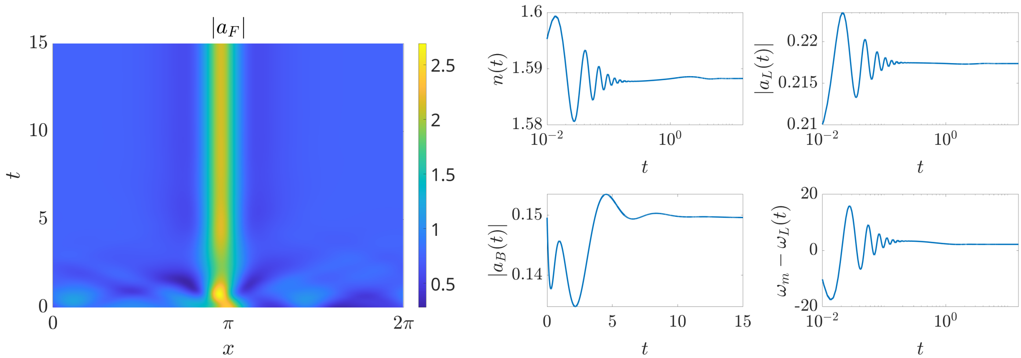

Figure 1.7: Time integration of (1.6) with a perturbed stable 1-soliton state as an initial condition. The simulation shows convergence of all four fields a_F, a_B, a_L, n and the instantaneous laser frequency ω_L towards the slightly spatially shifted soliton state as $t \to \infty$.

- If we add a small initial perturbation to a stable soliton state, the perturbed solution satisfies

$$a_F(x,t) \to \underline{a}_F^0(x+x_0)\mathrm{e}^{\mathrm{i}(\zeta_0 t+\phi_0)}, \qquad a_B(t) \to \underline{a}_B^0\mathrm{e}^{\mathrm{i}(\zeta_0 t+\phi_0)}$$
$$a_L(t) \to \underline{a}_L^0\mathrm{e}^{\mathrm{i}(\zeta_0 t+\phi_0)}, \qquad n(t) \to n^0,$$

exponentially fast as $t \to \infty$ for some $x_0, \phi_0 \in \mathbb{R}$. This convergence is illustrated in Figure 1.7 where we show a simulation of (1.6) with a perturbed 1-soliton state as an initial condition. In particular, not only the resonator fields a_F, a_B, but also the laser fields a_L, n self-correct towards the slightly spatially shifted soliton state. For the instantaneous laser frequency $\omega_L(t)$ we find that it converges to $\omega_0 + \Delta\zeta$ as $t \to \infty$. We emphasize that this self-correction of the laser fields is a new feature of the system (1.6), which cannot be observed in the standard LLE, where the laser is represented by a static forcing term.

1.7 Main results of Part III: Fronts and pulses in spatially periodic systems

Part III of this thesis is on systems with spatially periodic coefficients. We first study the eLLE with a small periodic potential and then move on to general semilinear systems of the form SYS. Finally, we return to specific model problems such as the Gross-Pitaevskii equation and the Klausmeier model.

Pinning in the extended LLE

In Chapter 5, we analyze the extended LLE

$$\mathrm{i}u_t = -du_{xx} + \mathrm{i}\varepsilon V(x)u_x + (\zeta - \mathrm{i}\mu)u - |u|^2u + \mathrm{i}f, \tag{1.7}$$

with a small 2π-periodic potential $\varepsilon V(x)$. The equation is derived in Section 5.6 from the physical model and we recall that it can be used to investigate synchronization of soliton repetition rates to external radio frequency oscillators [46]. We are interested in stationary solutions that solve

$$-du'' + \mathrm{i}\varepsilon V(x)u' + (\zeta - \mathrm{i}\mu)u - |u|^2u + \mathrm{i}f = 0, \tag{1.8}$$

equipped with 2π-periodic boundary conditions. Our goal is to establish the existence and stability of non-constant solutions to (1.8). For $\varepsilon = 0$, solutions exist due to [37, 67, 91] and Part I, and the idea is to prove the persistence of such solutions for $\varepsilon \neq 0$. We now explain the continuation argument in more detail. Let us fix a non-constant solution $u_0 \in H^2_{\text{per}}(-\pi,\pi)$ of (1.8) for $\varepsilon = 0$. We call u_0 nondegenerate if the kernel of the linearization $L_0\colon H^2_{\text{per}}(-\pi,\pi) \to L^2_{\text{per}}(-\pi,\pi)$ defined by

$$L_0 v := -dv'' + (\zeta - \mathrm{i}\mu)v - 2|u_0|^2 v - u_0^2\overline{v}$$

is only spanned by the derivative u_0'. Then, the adjoint operator L_0^* has a one-dimensional kernel as well and we fix $\phi_0^* \in H^2_{\text{per}}(-\pi,\pi)$ with $\ker(L_0^*) = \text{span}\{\phi_0^*\}$. This allows us to define the effective potential

$$V_{\text{eff}}(\sigma) := \mathrm{Re}\int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\overline{\phi_0^*}\,\mathrm{d}x.$$

The main result of Chapter 5 reads as follows.

Theorem 1.2. *Assume that $u_0 \in H^2_{per}(-\pi,\pi)$ is a nondegenerate solution of* (1.8) *for $\varepsilon = 0$ and that the effective potential V_{eff} has a simple zero σ_0. Then there exist $\varepsilon_0, C > 0$ such that for all $|\varepsilon| < \varepsilon_0$ there exists a solution $u_\varepsilon \in H^2_{per}(-\pi,\pi)$ of* (1.8) *with*

$$\|u_0(\cdot - \sigma_0) - u_\varepsilon\|_{H^2_{per}(-\pi,\pi)} \leq C|\varepsilon|.$$

If we further assume that u_0 is spectrally stable with a simple eigenvalue $\lambda = 0$, then the following holds.

(i) The solution u_ε is asymptotically stable as a solution to (1.7) *if $V_{\text{eff}}'(\sigma_0)\varepsilon > 0$.*

(ii) The solution u_ε is unstable as a solution to (1.7) *if $V_{\text{eff}}'(\sigma_0)\varepsilon < 0$.*

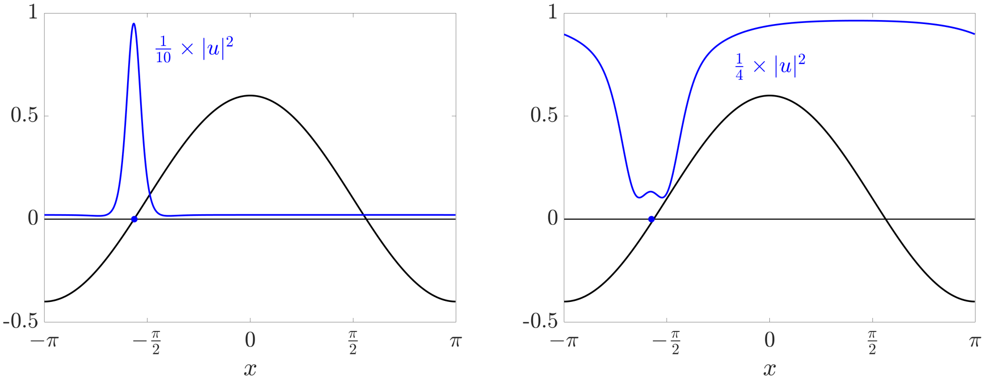

Figure 1.8: Left: bright soliton for anomalous dispersion $d > 0$ pinned to a zero (blue dot) of the effective potential V_{eff}. The black graph is the coefficient potential $V(x)$. Right: pinning of a dark soliton for normal dispersion $d < 0$. Zeros of V_{eff} almost coincide with zeros of V. The plots are from [18] which belongs to Part III of this thesis.

For the detailed statement we refer to the Theorems 5.3, 5.8, and 5.9 in Chapter 5. We note that the condition $V_{\text{eff}}(\sigma) = 0$ appears as the solvability condition for the linear inhomogeneous equation

$$L_0 v = \mathrm{i}V(x+\sigma)u_0'.$$

If V_{eff} has a simple zero σ_0 then the shifted version $u_0(\cdot - \sigma_0)$ can be continued as a solution for $\varepsilon \neq 0$. In this sense, these solutions are pinned to the zeros of V_{eff}, see Figure 1.8 for a numerical illustration of the pinning phenomena.

Remark 1.3. In Chapter 7 the above existence and stability result is extended to multi-solitons with arbitrary (but large) pulse distances, see Theorem 7.1.

An important property of the effective potential is that it approximates the coefficient potential V if u_0 is strongly localized around $x = 0$. Indeed, the strong localization allows us to approximate $\mathrm{Re}(\mathrm{i}u_0'\overline{\phi_0^*})$ by a multiple of the δ distribution and obtain

$$V_{\text{eff}}(\sigma) = \mathrm{Re}\int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\overline{\phi_0^*}\mathrm{d}x \approx \alpha\,\mathrm{Re}\int_{-\pi}^{\pi} V(x+\sigma)\delta(x)\mathrm{d}x = \alpha V(\sigma),$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$. In this case, solutions exist if V has a simple zero and they are pinned to zero of V.

Let us explain the physical meaning of the mathematical result. Since (1.7) is formulated in a co-moving frame moving with a speed determined by the external pump frequencies (see Section 5.6), pinned stationary solitons have a repetition rate that is locked to the laser pump. The condition $V(x) = \nu_1 - 2dk_1^2 f_0/f_1 \cos(x) = 0$ yields the locking range for solitons which is given by

$$|\nu_1| \leq 2dk_1^2 \frac{f_1}{f_0}.$$

Using the cold cavity dispersion relation $\omega_k = \omega_0 + k\omega_{\text{FSR}} + dk^2$ for the resonator frequencies ω_k with free spectral range (FSR) ω_{FSR}, the locking range can be rewritten as

$$|\Delta\omega - k_1\omega_{\text{FSR}}| \leq 2dk_1^2 \frac{f_1}{f_0}, \qquad \text{with} \qquad \Delta\omega := \omega_{p_1} - \omega_{p_0}.$$

Since $|f_1| \ll |f_0|$ we deduce that $\Delta\omega$ has to be tuned close to an integer multiple of the FSR to observe the pinning of solitons.

Multiple front and pulse solutions

In Chapter 6 we develop a mathematical toolbox for constructing stationary multifront and periodic pulse solutions to general semilinear equations with spatially periodic coefficients of the form SYS. In the autonomous case, this theory is well developed. The existence of N-homoclinics and periodic orbits bifurcating from a primary homoclinic solution in reversible or Hamiltonian dynamical systems of the form

$$U' = F(U), \qquad F\colon \mathbb{R}^n \to \mathbb{R}^n \text{ smooth}$$

has been proved in [28, 72, 74, 147]. With the help of the Lin's method [87], the eigenvalue problem

$$V' = (DF(U(x)) + \lambda B)V, \qquad \lambda \in \mathbb{C}, B \in \mathbb{R}^{n\times n}$$

associated to such a bifurcating solution U has been studied in [126, 127, 131]. This paved the way to determining stability properties of N-pulses and periodic patterns as stationary solutions for numerous nonlinear evolution equations.

Extending this theory to equations with spatially periodic coefficients of the form SYS is the main objective of Chapter 6. We develop the theory using an approach based on a combination of fixed point arguments, exponential dichotomies and Evans function techniques [4, 131]. Furthermore, we demonstrate the strength of the abstract approach by applying it to the Gross-Pitaevskii equation with a periodic potential and a Klausmeier reaction-diffusion model. For both models, we extending previously known results concerning the existence and stability of (multiple) pulses and fronts.

Let us explain the main results of Chapter 6. Setting the time derivative in SYS to zero, we obtain the ODE

$$\alpha_k(x)\partial_x^k u + \cdots + \alpha_1(x)\partial_x u + \mathcal{N}(x,u) = 0, \tag{1.9}$$

with T-periodic coefficients in x. For $\underline{u} \in L^\infty(\mathbb{R})$ we define the linearized operator

$$\mathcal{L}(\underline{u})u := \alpha_k(x)\partial_x^k u + \cdots + \alpha_1(x)\partial_x u + \partial_u \mathcal{N}(x,\underline{u})u.$$

Now let $Z_1(x),\dots,Z_M(x)$ be M front solutions to (1.9) converging to T-periodic end states $v_{1,\pm}(x),\dots,v_{M,\pm}(x)$ as $x \to \pm\infty$. We assume the matching condition $v_{j,+} = v_{j+1,-}$ for $j = 1,\dots,M-1$. Moreover, let $Z_0(x)$ be a stationary pulse solution of (1.9) converging to a T-periodic end state $v_0(x)$ as $x \to \pm\infty$. We define the formal concatenation and periodic extension

$$w_n(x) = \begin{cases} v_{1,-}(x), & x \le \frac{1}{2}nT, \\ Z_j(x - jnT), & x \in \left[(j-\frac{1}{2})nT, (j+\frac{1}{2})nT\right], \qquad j = 1,\dots,M, \\ v_{M,+}(x), & x \ge (M+\frac{1}{2})nT, \end{cases}$$

$$\tilde{w}_n(x) = Z_0(x - jnT), \qquad x \in \left[(j-\tfrac{1}{2})nT, (j+\tfrac{1}{2})nT\right], \qquad j \in \mathbb{Z}.$$

Then we have the following theorem.

Theorem 1.4. *Assume that Z_j is nondegenerate in the sense that the linearization $\mathcal{L}(Z_j)$ as an operator from $H^k(\mathbb{R})$ to $L^2(\mathbb{R})$ is invertible for $j = 0,\dots,M$. Then, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \ge N$ the following assertions hold.*

(i) There exists a stationary M-front solution $u_n(x)$ of (1.9)*, which converges uniformly to w_n as $n \to \infty$.*

(ii) There exists a stationary nT-periodic pulse solution $\tilde{u}_n(x)$ of (1.9)*, which converges uniformly to $\tilde{w}_n(x)$ as $n \to \infty$.*

Figure 1.9 gives an illustration of the solutions constructed in Theorem 1.4. For the detailed statements, we refer to Theorems 6.5 and 6.8. The proofs are based on contraction mapping arguments, which require to invert the linear operators $\mathcal{L}(w_n)$ and $\mathcal{L}(\tilde{w}_n)$ and to derive suitable bounds on the inverse. This is the main technical challenge, which is solved using exponential dichotomy techniques under the nondegeneracy conditions.

Next, we state a spectral stability result for the multifronts and periodic pulses. Again, we refer to corresponding Theorems 6.14 and 6.22 and Sections 6.6, 6.7 for the precise statements.

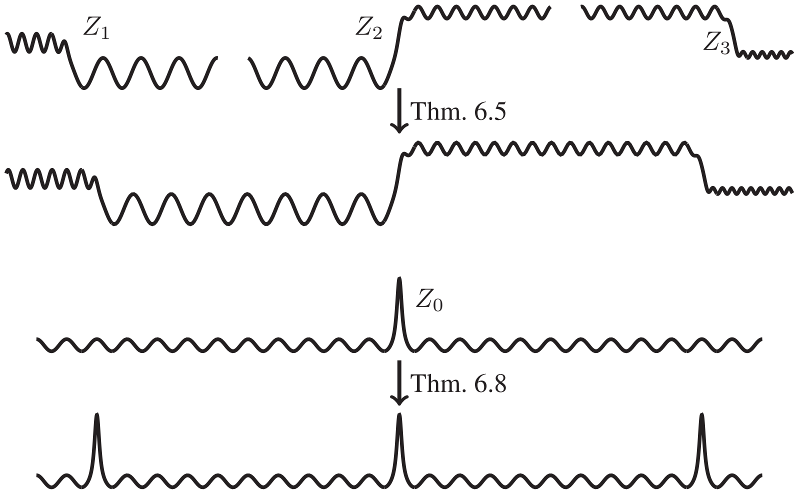

Figure 1.9: Illustration of the existence result in Theorem 1.4. Top: concatenation of fronts with matching end states. Bottom: periodic extension of a primary pulse.

Theorem 1.5. *Let $\mathcal{K} \subset \mathbb{C}$ be compact. Assume that the $L^2(\mathbb{R})$-spectrum in $\mathcal{K}$ of the linearization $\mathcal{L}(Z_j)$ consists of isolated eigenvalues of finite algebraic multiplicity only for $j = 0, \ldots, M$. Then, there exists $N_1 \in \mathbb{N}$ with $N_1 \geq N$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ the following holds.*

(i) The $L^2(\mathbb{R})$-spectrum of $\mathcal{L}(u_n)$ in the compact set $\mathcal{K}$ consists of isolated eigenvalues only and converges in Hausdorff distance to the union

$$\bigcup_{j=1}^{M} \sigma(\mathcal{L}(Z_j)) \cap \mathcal{K}$$

as $n \to \infty$. The total algebraic multiplicity of the eigenvalues of $\mathcal{L}(u_n)$ in $\mathcal{K}$ equals the sum of the total algebraic multiplicies of the eigenvalues of $\mathcal{L}(Z_1), \ldots, \mathcal{L}(Z_M)$ in $\mathcal{K}$.

(ii) The $L^2_{per}(0, nT)$-spectrum in $\mathcal{K}$ of the linearization $\mathcal{L}(\tilde{u}_n)$ consists of isolated eigenvalues only and converges in Hausdorff distance to

$$\sigma(\mathcal{L}(Z_0)) \cap \mathcal{K}$$

as $n \to \infty$. The total algebraic multiplicity of the eigenvalues of $\mathcal{L}(\tilde{u}_n)$ in $\mathcal{K}$ equals the total algebraic multiplicity of the eigenvalues of $\mathcal{L}(Z_0)$ in $\mathcal{K}$.

As a consequence of Theorem 1.5, the multifronts are spectrally unstable if at least one of the constituent primary fronts is spectrally unstable. The same holds for the periodic pulse solutions given the spectral instability of the primary pulse. Conversely, assume that all primary fronts as well as the primary pulse have only spectrum in the left-open half plane. Moreover, assume that the spectral stability problem for both, the multifronts and the periodic pulses can be reduced to a compact subset $\mathcal{K}$ via spectral a-priori bounds. Then they are spectrally stable by Theorem 1.5. The proof of Theorem 1.5 is based on Evans function techniques. The first statement results from a factorization of the Evans function for the multifront solution, which gives a product of Evans functions for the primary fronts up to a small error. The second statement extends the technique from [131], which relates the Evans function of the primary pulse to the Evans function of the periodic pulse for large periods.

As an application for our theory, we consider the Gross-Pitaevskii equation

$$\mathrm{i}\partial_t u = -\partial_x^2 u + \mu V(x)u + \kappa|u|^2 u, \tag{1.10}$$

with given T-periodic potential $\mu V(x)$ and $\kappa \in \{\pm 1\}$. We are interested in time-harmonic solutions of the form $u(x,t) = \mathrm{e}^{\mathrm{i}\omega t}\psi(x)$ with $\psi(x) \in \mathbb{R}$. Substituting this ansatz into (1.10), we obtain

$$-\psi'' + \mu V(x)\psi + \omega\psi + \kappa\psi^3 = 0, \tag{1.11}$$

which is of the form (1.9). We establish results for both the defocusing $\kappa = 1$ and focusing $\kappa = -1$ case.

In the defocusing case we establish the existence of multi-solitons bifurcating from concatenations of the black NLS soliton

$$\psi_0(x) = \sqrt{-\omega}\tanh\left(\sqrt{\frac{-\omega}{2}}x\right), \qquad x \in \mathbb{R},$$

for $\omega < 0$. For $M \in \mathbb{N}$ let us define the formal multi-soliton

$$\Psi_n(x) = \begin{cases} \psi_0(x - nT), & x \leq \frac{3}{2}nT, \\ (-1)^{j-1}\psi_0(x - jnT), & x \in \left[(j-\frac{1}{2})nT, (j+\frac{1}{2})nT\right], \quad j = 2,\ldots,M-1, \\ (-1)^{M-1}\psi_0(x - MnT), & x \geq (M-\frac{1}{2})nT. \end{cases}$$

This allows us to formulate the following theorem, cf. Theorem 6.33 and Corollary 6.35.

Theorem 1.6. *Let $\kappa = 1$, $\omega < 0$, $M \in \mathbb{N}$ and assume that*

$$\int_{\mathbb{R}} V'(x)\psi_0(x)\psi_0'(x)\mathrm{d}x \neq 0.$$

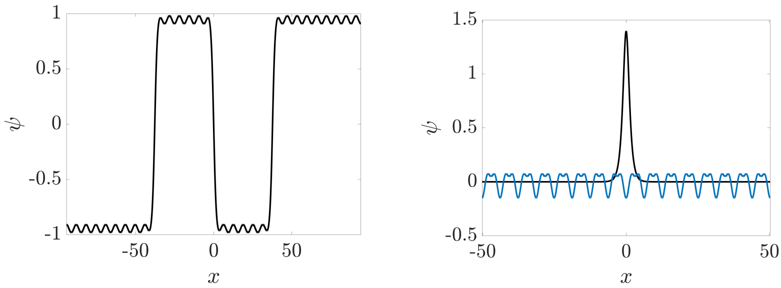

Figure 1.10: Left: black multi-soliton of (1.10) in the defocusing case $\kappa = 1$. Right: periodic 1-pulse in the focusing case $\kappa = -1$ and the potential $\mu V(x)$ (blue).

Then, there are constants $C, \mu_0 > 0$ such that for all $|\mu| \leq \mu_0$ there is $N \in \mathbb{N}$ such that if $n \geq N$ there exists a time-harmonic solution $u(x,t) = \mathrm{e}^{\mathrm{i}\omega t}\psi(x)$ of (1.10) with

$$\|\psi - \Psi_n\|_{L^\infty} \leq C|\mu|.$$

The stability of multi-solitons is challenging because the essential spectrum of the black NLS soliton covers the entire imaginary axis. For a spectral analysis, the Evans function would have to be extended across the essential spectrum using a "Gap Lemma", see [110]. This task is left open for future studies.

In the focusing case, we prove the existence of periodic pulse solutions of (1.11) in the semi-infinite gap $\inf \sigma(-\partial_x^2 + \mu V) > -\omega$. Here we assume that $\mu V(x)$ is a given periodic potential such that the existence of a primary pulse solution $\phi_0 \in H^2(\mathbb{R})$ of (1.11), which satisfies the spectral stability assumptions **(GP)** in Section 6.8, is guaranteed. The existence of such potentials has been verified, e.g. in [109]. We then have the following theorem, cf. Corollaries 6.44 and 6.21.

Theorem 1.7. *Assume that the primary pulse $\phi_0 \in H^2(\mathbb{R})$ satisfies the spectral assumptions* **(GP)** *with $\langle \partial_\omega \phi_0, \phi_0 \rangle_{L^2} > 0$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there exists a time-harmonic solution $u_n(x,t) = \mathrm{e}^{\mathrm{i}\omega t}\phi_n(x)$ of (1.10) with $\kappa = -1$ such that*

$$\phi_n \in H^2_{per}(0, nT), \qquad \|\phi_0 - \phi_n\|_{H^2_{per}(0,nT)} \to 0 \qquad \text{as } n \to \infty.$$

Moreover, u_n is orbitally stable.

To the best of our knowledge, Theorem 1.7 establishes the first orbital stability result of a nontrivial time-harmonic spatially periodic solution to (1.10). The proof relies on a delicate combination of Theorem 6.22, spectral bounds, Krein index theory [31, 75, 76], and the stability theorem of Grillakis, Shatah, and Strauss [63]. We end this section with numerical simulations of a black multi-soliton and a stable periodic bright soliton presented in Figure 1.10. The left panel depicts a black multi-soliton which is constructed by concatenating three front solutions. The right panel shows a periodic 1-pulse and the periodic potential $\mu V(x)$ in blue.

Part I

Multi-solitons in the Lugiato-Lefever equation

2 Stability of solitary wave solutions in the Lugiato-Lefever equation

This chapter is a reprint[1] of the published article [15] written by the author of the thesis. The article was adapted to fit the layout of this thesis.

Abstract

We analyze the spectral and dynamical stability of solitary wave solutions to the Lugiato-Lefever equation (LLE) on $\mathbb{R}$. Our interest lies in solutions that arise through bifurcations from the phase-shifted bright soliton of the nonlinear Schrödinger equation (NLS). These solutions are highly nonlinear, localized, far-from-equilibrium waves, and are the physical relevant solutions to model Kerr frequency combs. We show that bifurcating solitary waves are spectrally stable when the phase angle satisfies $\theta \in (0, \pi)$, while unstable waves are found for angles $\theta \in (\pi, 2\pi)$. Furthermore, we establish orbital asymptotical stability of spectrally stable solitary waves against localized perturbations. Our analysis exploits the Lyapunov-Schmidt reduction method, the instability index count developed for linear Hamiltonian systems, and resolvent estimates.

2.1 Introduction

Kerr frequency combs generated in an externally driven Kerr nonlinear microresonator are very promising devices in optical communications or frequency metrology, enabling, for instance, high-speed data transmission of up to 1.44 Tbit/s, cf. [112]. They are optical signals consisting of a multitude of equally spaced excited modes in frequency space and are modeled by stable highly localized stationary periodic solutions of the Lugiato-Lefever equation (LLE)

$$\mathrm{i}u_t = -du_{xx} + (\zeta - \mathrm{i})u - |u|^2 u + \mathrm{i}f, \qquad (x,t) \in \mathbb{R}^2. \tag{2.1}$$

[1] Reprint with permission of the journal

Here $u = u(x,t) \in \mathbb{C}$ is the field amplitude in the resonator, $d \neq 0$ is the dispersion, $\zeta \in \mathbb{R}$ is the offset between the external forcing frequency and the resonant frequency in the resonator called detuning, and $f \in \mathbb{R}$ describes the pump power inside the resonator of the external forcing. A physical derivation of (2.1) can be found in [89]. From a mathematical point of view, LLE is a damped and driven nonlinear Schrödinger equation (NLS). Motivated by the promising applications of Kerr frequency combs, the existence of stationary solutions of LLE has received considerable attention. A plethora of stationary solutions have been found in numerical simulations [9, 54, 97, 102, 104, 106] or have been constructed analytically [10, 55, 61, 81, 91, 94]. The naturally associated question of their spectral as well as their nonlinear stability with respect to different types of perturbations has gained interest recently [8, 36, 37, 47, 62, 67, 69–71, 111, 137, 141]. Whereas nonlinear stability can be obtained under general spectral stability assumptions, spectral stability analyses themselves rely on the specific structure of the solutions and typically employ similar methods as were used to construct them. So far, spectral stability has been obtained for periodic small amplitude solutions of (2.1) arising through a Turing bifurcation of a homogeneous rest state [36]. The only spectral stability result of far-from-equilibrium solutions of (2.1) that the author is aware of is that in [67], where solutions to (2.1) are constructed by bifurcation from the cnoidal wave solutions solving NLS equation on the torus. Stability results for explicitly available solitary wave solutions in the forced NLS equation without damping have been obtained in [7] and also recently in [47]. Here, we present the first spectral stability result of far-from-equilibrium *soliton solutions* of the damped and driven NLS (2.1). The solutions under consideration are highly nonlinear and arise by bifurcation from bright solitons in the NLS equation in the anomalous regime $d > 0$. They satisfy the approximation formula

$$u(x) \approx u_\infty + \sqrt{2\zeta}\,\mathrm{sech}\left(\sqrt{\frac{\zeta}{d}}x\right)\mathrm{e}^{\mathrm{i}\theta_0}, \qquad x \in \mathbb{R}, \tag{2.2}$$

where $u_\infty \in \mathbb{C}$ is a constant background due to the forcing in (2.1) and the angle θ_0 is found by solving the equation $\cos\theta_0 = 2\sqrt{2\zeta}/(\pi f)$. Although they are non-periodic solutions of (2.1), their exponential localization makes them a valid approximation of a frequency comb, cf. Figure 2.1. We emphasize that the approximation (2.2) is frequently used in the physics literature to approximate Kerr frequency combs [20, 73, 149], which facilitates (formal) computations.

Contributions of this paper are threefold. In Theorem 2.3, we prove the existence of solitary wave solutions of (2.1) that verify the approximation (2.2). Therefore we consider the LLE with dispersion rescaled to $d = 1$ as a perturbation of the focusing NLS in the following sense:

$$\mathrm{i}u_t = -u_{xx} + \zeta u - |u|^2 u + \varepsilon \mathrm{i}\Psi(u), \qquad (x,t) \in \mathbb{R}^2, \tag{2.3}$$

where

$$\Psi(u) = -u + f,$$

and ε is the bifurcation parameter. We show that solitary wave solutions $u = u(\varepsilon)$ bifurcate at $\varepsilon = 0$ from the rotated NLS soliton

$$\phi_{\theta_0}(x) := \sqrt{2\zeta}\,\mathrm{sech}(\sqrt{\zeta}x)\mathrm{e}^{\mathrm{i}\theta_0}.$$

In Theorem 2.6, we prove spectral stability of the solitary waves satisfying $\sin\theta_0, \varepsilon > 0$ and spectral instability for all other sign configurations of $\varepsilon, \sin\theta_0$. This result relies on a detailed analysis of the spectral problem, where the crucial point is to understand the behavior of small eigenvalues for ε small in the linearized problem.

Finally, in Theorem 2.9 we prove nonlinear asymptotic orbital stability of spectrally stable solitary waves. For the linear estimates, we establish high frequency resolvent estimates for families of operators in Hilbert spaces, see Theorem 2.22, which we then use to prove linear stability. Indeed, the resolvent estimates are needed to overcome the problem that a Spectral Mapping Theorem for the non-sectorial operator arising in the linearized equation is a-priori not available. Nonlinear stability then follows as a corollary of the linear stability result.

Remark 2.1. Existence of solitary waves bifurcating from the NLS soliton has already been proven in [57] using the Crandall-Rabinowitz Theorem of bifurcation from a simple eigenvalue. Here, we use a different approach based on a Lyapunov-Schmidt reduction in parameter-dependent spaces. This is advantageous because it directly yields an expansion of the solution needed in the spectral stability analysis. Further, we believe that our approach is flexible enough for possible extensions to bifurcation problems with higher dimensional kernels.

Remark 2.2. There is a long list of literature on persistence and stability of solitary solutions for other variants of the perturbed NLS, cf. [5, 11, 79, 115] and the references therein.

2.1.1 Main results

Solitary wave solutions of the perturbed NLS (2.3) are solutions of the stationary equation

$$-u'' + \zeta u - |u|^2 u + \mathrm{i}\varepsilon(-u + f) = 0, \quad x \in \mathbb{R}, \tag{2.4}$$

which decay to a limit state $u_\infty \in \mathbb{C}$ as $|x| \to \infty$.

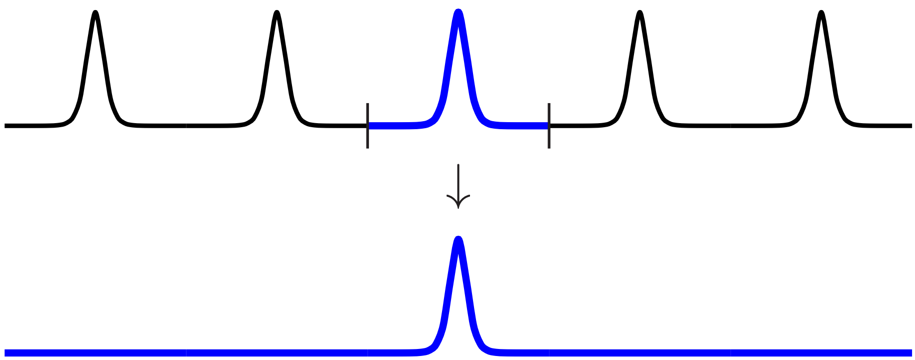

Figure 2.1: Approximation of a periodic solution on $\mathbb{R}/\mathbb{Z}$ by a solitary wave on $\mathbb{R}$.

The following theorem provides the first result of the paper on the existence of solitary waves for a suitable parameter region.

Theorem 2.3. *Let $\zeta, f > 0$ be fixed and suppose that $\theta_0 \in \mathbb{R}$ is a simple zero of the function*

$$\theta \mapsto \pi f \cos\theta - 2\sqrt{2\zeta}.$$

Then there exist $\varepsilon^* > 0$ and a branch $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto u(\varepsilon) \in \mathbb{C} + H^2(\mathbb{R})$ of solutions to the perturbed problem (2.4) *bifurcating from the rotated soliton*

$$\phi_{\theta_0}(x) = \sqrt{2\zeta}\,\mathrm{sech}\left(\sqrt{\zeta}x\right)\mathrm{e}^{\mathrm{i}\theta_0}.$$

More precisely, the branch is of the form

$$u(\varepsilon) = \phi_{\theta(\varepsilon)} + u_\infty(\varepsilon) + \varphi(\varepsilon), \quad u(0) = \phi_{\theta_0},$$

where

- *the map $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto \theta(\varepsilon) \in \mathbb{R}$ is real-analytic and describes the rotational angle of the soliton,*
- *the map $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto u_\infty(\varepsilon) \in \mathbb{C}$ is real-analytic and consists of the constant background of the solution at $\pm\infty$,*
- *the map $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto \varphi(\varepsilon) \in H^2(\mathbb{R})$ is real-analytic and describes a small correction term of order $\mathcal{O}(|\varepsilon|)$.*

Remark 2.4. In Theorem 2.3, a necessary and sufficient condition on the parameters ζ, f to find simple zeros is $\pi^2 f^2 > 8\zeta$, which yields an existence region in the ζ-f-plane, already obtained analytically or numerically in [8, 55, 57, 149]. Furthermore, it should be noted that solutions for negative forcing parameters $f < 0$ are obtained through the transformation $u \mapsto -u$.

Theorem 2.6 and Theorem 2.9 below provide the two main results on the spectral and nonlinear stability of solitary waves of Theorem 2.3 against localized perturbations in H^1.

Let $u = u(\varepsilon) \in \mathbb{C} + H^2(\mathbb{R})$ be a solution of the stationary LLE as in Theorem 2.3 for sufficiently small $\varepsilon \neq 0$. Expanding the solution as $\psi(x,t) = u(x) + v(x,t)$ results in the perturbation equation

$$\mathrm{i}v_t = -v_{xx} + \zeta v - 2|u|^2 v - u^2 \bar{v} - \mathrm{i}\varepsilon v - 2|v|^2 u - v^2 \bar{u} - |v|^2 v.$$

The evolution of the perturbation v is coupled with the evolution of the complex conjugate $\bar{v}$ so that we obtain the system:

$$\begin{cases} \mathrm{i}v_t &= -v_{xx} + \zeta v - 2|u|^2 v - u^2 \bar{v} - \mathrm{i}\varepsilon v - 2|v|^2 u - v^2 \bar{u} - |v|^2 v, \\ -\mathrm{i}\bar{v}_t &= -\bar{v}_{xx} + \zeta \bar{v} - 2|u|^2 \bar{v} - \bar{u}^2 v + \mathrm{i}\varepsilon \bar{v} - 2|v|^2 \bar{u} - \bar{v}^2 u - |v|^2 \bar{v}, \end{cases} \tag{2.5}$$

for which the mild formulation is locally well-posed[2] in $(H^1(\mathbb{R}))^2$. The linearized equation is then given by

$$V_t = (\mathcal{L} - \varepsilon)V, \quad V = (v_1, v_2)^T,$$

for the operator $\mathcal{L} := JL : (H^2(\mathbb{R}))^2 \to (L^2(\mathbb{R}))^2$, with

$$J := \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad L := \begin{pmatrix} -\partial_x^2 + \zeta - 2|u|^2 & -u^2 \\ -\bar{u}^2 & -\partial_x^2 + \zeta - 2|u|^2 \end{pmatrix}, \tag{2.6}$$

and the associated eigenvalue problem reads

$$\lambda V = (\mathcal{L} - \varepsilon)V, \quad V = (v_1, v_2)^T. \tag{2.7}$$

Spectral stability is now determined by the location of the spectrum of the linearized operator $\sigma(\mathcal{L} - \varepsilon) = \sigma(\mathcal{L}) - \varepsilon$ according to the following definition.

[2] This follows from standard semigroup theory.

Definition 2.5. A solution $u = u(\varepsilon) \in \mathbb{C} + H^2(\mathbb{R})$ of (2.4) is called *spectrally unstable*, if there exists $\lambda \in \sigma(\mathcal{L} - \varepsilon)$ such that $\mathrm{Re}(\lambda) > 0$. Otherwise the solution is called *spectrally stable*, i.e., if and only if $\sigma(\mathcal{L}) \subset \{z \in \mathbb{C} : \ \mathrm{Re}\, z \leq \varepsilon\}$.

The following theorem clarifies the spectral stability of the solitary wave solutions found in Theorem 2.3.

Theorem 2.6. *Suppose that $u = u(\varepsilon) \in \mathbb{C} + H^2(\mathbb{R})$ is a solution of (2.4) as in Theorem 2.3 for $\varepsilon \neq 0$ sufficiently small, that bifurcates from the NLS soliton*

$$\phi_{\theta_0}(x) = \sqrt{2\zeta}\,\mathrm{sech}\left(\sqrt{\zeta}x\right)\mathrm{e}^{\mathrm{i}\theta_0},$$

with $\sin\theta_0 \neq 0$. Then, the spectrum of $\mathcal{L} - \varepsilon$ is given by the disjoint union of essential and discrete spectrum $\sigma(\mathcal{L} - \varepsilon) = \sigma_{\mathrm{ess}}(\mathcal{L} - \varepsilon) \cup \sigma_d(\mathcal{L} - \varepsilon)$, where the essential spectrum is explicitly computable:

$$\sigma_{\mathrm{ess}}(\mathcal{L} - \varepsilon) = \{\mathrm{i}\omega \in \mathrm{i}\mathbb{R} : \ |\omega| \in [\zeta_\varepsilon, \infty)\} - \varepsilon, \quad \zeta_\varepsilon = \zeta + \mathcal{O}(\varepsilon^2).$$

Moreover,

(i) if $\varepsilon < 0$, then the solution u is spectrally unstable. The same is true if $\varepsilon, -\sin\theta_0 > 0$.

(ii) if $\varepsilon, \sin\theta_0 > 0$, then the solution $u(\varepsilon)$ is spectrally stable and the spectrum satisfies

$$\sigma(\mathcal{L} - \varepsilon) \subset \{-2\varepsilon\} \cup \{z \in \mathbb{C} : \ \mathrm{Re}\, z = -\varepsilon\} \cup \{0\}$$

with (algebraically) simple eigenvalues $\lambda = 0, -2\varepsilon$.

The different stability configurations of Theorem 2.6 are depicted in Figure 2.2.

Remark 2.7. In case (i) of Theorem 2.6, the instability is triggered by two different mechanism. If $\varepsilon < 0$, the essential spectrum of $\mathcal{L} - \varepsilon$ is unstable. If $\varepsilon > 0 > \sin\theta_0$, we find exactly one simple real unstable eigenvalue $\lambda_+ \in \sigma_d(\mathcal{L})$ of order $\mathcal{O}(\varepsilon^{1/2})$.

Remark 2.8. A similar spectral stability result for periodic solutions of (2.3) bifurcating from cnoidal wave solutions of the NLS has been obtained in [137]. Moreover, in [7, 47] stability and instability of purely imaginary soliton solutions of the forced NLS equation is proven. Here, the stable solutions have a strictly positive imaginary part, which is in agreement with our sign condition on $\sin\theta_0$.

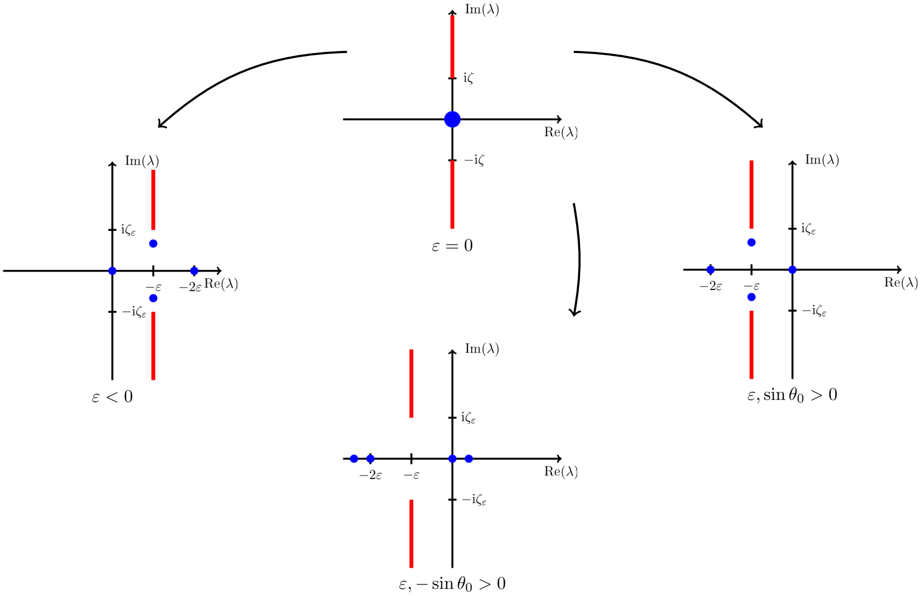

Figure 2.2: Stability configurations of Theorem 2.6. Blue dots = discrete spectrum, red lines = essential spectrum. Top: spectrum of the unperturbed stable NLS soliton. Left and bottom: spectrum of unstable solitary waves of LLE. Right: spectrum of a stable solitary wave of LLE.

The next theorem, provides the nonlinear stability result for spectrally stable solutions of Therorem 2.6.

Theorem 2.9. *Suppose that* $u = u(\varepsilon) \in \mathbb{C} + H^2(\mathbb{R})$ *is a spectrally stable solution as in Theorem 2.6. Then, the solution* $u(\varepsilon)$ *is asymptotically orbitally stable. More precisely, there exist constants* $\delta, \eta, C > 0$ *such that for all* $v_0 \in H^1(\mathbb{R})$ *satisfying* $\|v_0\|_{H^1} < \delta$ *there exists the unique global (mild) solution* $(v, \bar{v}) \in C([0, \infty), (H^1(\mathbb{R}))^2)$, $(v(0), \bar{v}(0)) = (v_0, \bar{v}_0)$ *of* (2.5) *and* $\sigma_\infty \in \mathbb{R}$ *such that with* $\psi = u + v$ *we have*

$$\|\psi(\cdot, t) - u(\cdot - \sigma_\infty)\|_{H^1} \leq C\delta \mathrm{e}^{-\eta t} \qquad \textit{for } t \geq 0.$$

Remark 2.10. In [137] asymptotic orbital stability of spectrally stable periodic solutions against co-periodic perturbations is proven. Theorem 2.9 can be seen as an analogous version of this result for stability of solitary wave solutions on $\mathbb{R}$ against localized perturbations.

Remark 2.11. If we restrict perturbations to the class of even functions the asymptotical orbital stability can be improved to asymptotic stability by exploiting a spectral gap in the linear stability problem.

2.1.2 Outline of the paper

In Section 2.2, we show that solitary waves for the LLE bifurcate from the NLS soliton as stated in Theorem 2.3. The spectral stability problem is analyzed in Section 2.3.1 and the proof of Theorem 2.6 is presented. The asymptotic orbital stability result of Theorem 2.9 is proven in Section 2.3.2. It relies upon uniform resolvent estimates for the linearized LLE, which we derive from high frequency resolvent estimates. These estimates are obtained in an abstract functional analytical set-up and can therefore also be applied to other NLS type equations.

2.2 Existence of solitary wave solutions

The goal of this section is to prove Theorem 2.3. We work in the Sobolev spaces $H^k(\mathbb{R}) = H^k(\mathbb{R}, \mathbb{C})$, $k \in \mathbb{N}_0$ of complex valued functions *over the field* $\mathbb{R}$. By this choice of function spaces, the map $H^2(\mathbb{R}) \ni u \mapsto |u|^2 u \in H^2(\mathbb{R})$ is Fréchet differentiable. Moreover, the L^2 scalar-product is defined by the $\mathbb{R}$-*valued map*

$$\forall f, g \in L^2(\mathbb{R}, \mathbb{C}) : \quad \langle f, g \rangle_{L^2} = \mathrm{Re} \int_{\mathbb{R}} f \bar{g} dx.$$

Let us fix the parameters $\zeta, f > 0$ such that $2\sqrt{2\zeta} < \pi f$ and let $\theta_0 \in \mathbb{R}$ be a simple zero of $\theta \mapsto \pi f \cos\theta - 2\sqrt{2\zeta}$. Recall that $\phi_0(x) = \sqrt{2\zeta}\,\mathrm{sech}\left(\sqrt{\zeta} x\right)$ denotes the soliton solution of the focusing cubic NLS equation

$$-\phi'' + \zeta\phi - |\phi|^2\phi = 0,$$

which decays exponentially to zero as $|x| \to \infty$. Let us introduce the manifold of rotated solitons

$$\mathcal{M} := \left\{ \phi_\theta = \phi_0 \mathrm{e}^{\mathrm{i}\theta} : \theta \in \mathbb{R} \right\}.$$

By the gauge invariance of NLS every $\phi \in \mathcal{M}$ is a solution of the NLS. In addition, NLS possesses a translational symmetry, i.e., $\phi(\cdot - \sigma)$ is a solution of NLS for all shifts $\sigma \in \mathbb{R}$. In the Lugiato-Lefever equation, the gauge symmetry is broken, and only the translational symmetry persists. Consequently, the continuation for $\varepsilon \neq 0$ is expected to be successful only for suitably rotated solitons $\phi \in \mathcal{M}$. To handle the translational symmetry, we fix the shift parameter $\sigma = 0$ and restrict our analysis to the spaces $H^2_{\text{ev}}(\mathbb{R}), L^2_{\text{ev}}(\mathbb{R})$ of even functions.

It is now important to note that the presence of the forcing term f in (2.4) prevents solutions from decaying to zero as $|x| \to \infty$. More precisely, the tails at $\pm\infty$ of every localized solution satisfy the algebraic equation

$$\zeta u_\infty - |u_\infty|^2 u_\infty + \mathrm{i}\varepsilon\big(- u_\infty + f\big) = 0, \quad u_\infty \in \mathbb{C}. \tag{2.8}$$

Thus, we adapt an ansatz of the form

$$u = \phi_\theta + u_\infty + \varphi,$$

where $\phi_\theta \in \mathcal{M}$ is a suitably rotated soliton, $u_\infty \in \mathbb{C}$ is the constant background solving the algebraic equation (2.8), and $\varphi \in H^2_{\text{ev}}(\mathbb{R})$ is a small correction. For the background we solve (2.8) to leading order by

$$u_\infty(\varepsilon) = -\frac{\mathrm{i}f}{\zeta}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{2.9}$$

which is the unique solution[3] close to 0. Inserting our ansatz into (2.4) yields the equation for the correction φ and rotational angle θ:

$$L_\theta \varphi = N(\varphi, \varepsilon, \theta) \tag{2.10}$$

where

$$L_\theta \varphi := -\varphi'' + \zeta\varphi - 2|\phi_\theta|^2\varphi - \phi_\theta^2\bar{\varphi}, \qquad \varphi \in H^2_{\text{ev}}(\mathbb{R}),$$

and

$$\begin{aligned} N(\varphi, \varepsilon, \theta) :=& 2|u_\infty(\varepsilon) + \varphi|^2\phi_\theta + (u_\infty(\varepsilon) + \varphi)^2\bar{\phi}_\theta + |u_\infty(\varepsilon) + \varphi|^2(u_\infty(\varepsilon) + \varphi) \\ & - |u_\infty(\varepsilon)|^2 u_\infty(\varepsilon) + 2|\phi_\theta|^2 u_\infty(\varepsilon) + \phi_\theta^2 \bar{u}_\infty(\varepsilon) + \mathrm{i}\varepsilon(\phi_\theta + \varphi). \end{aligned}$$

Note that we have $N(\varphi, \varepsilon, \theta) \in H^2_{\text{ev}}(\mathbb{R})$ for $\varphi \in H^2_{\text{ev}}(\mathbb{R})$. Our goal is to solve (2.10) by means of the Lyapunov-Schmidt reduction method. Therefore, let us collect relevant properties of the linearized operator L_θ.

3 There are two additional solutions to (2.8) given by $u_\infty(\varepsilon) = \pm\sqrt{\zeta} + \mathcal{O}(\varepsilon)$, which are not considered here.

Lemma 2.12. *For every $\theta \in \mathbb{R}$ the $\mathbb{R}$-linear operator $L_\theta : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint and Fredholm of index zero. Moreover we have*

$$\ker(L_\theta) = \operatorname{span}\{\mathrm{i}\phi_\theta, \phi_\theta'\}.$$

Proof. The fact that L_θ is Fredholm of index zero follows from $0 \notin \sigma_{\text{ess}}(L_\theta) = \sigma_{\text{ess}}(-\partial_x^2 + \zeta) = [\zeta, \infty)$ and the first equality is a consequence of Weyl's Theorem. Moreover the operator $-\partial_x^2 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint and hence the same holds for L_θ since it is a symmetric bounded perturbation of $-\partial_x^2 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$. Finally, the identity for the kernel follows from a similarity transformation and explicit formulas for the kernel of the linearized NLS operator, cf. [78]. □

Since we aim to solve (2.10) in the space of even functions, we note that the restricted operator $L_\theta|_{H^2_{\text{ev}}}$ has a one-dimensional kernel. Indeed, by Lemma 2.12 the kernel is explicitly given by $\ker(L_\theta|_{H^2_{\text{ev}}}) = \operatorname{span}\{\mathrm{i}\phi_\theta\}$. Lyapunov-Schmidt reduction now relies on the decomposition of L^2_{ev} with the orthogonal projection onto $\ker(L_\theta|_{H^2_{\text{ev}}})$ defined by

$$P_\theta\varphi := \frac{\langle \mathrm{i}\phi_\theta, \varphi\rangle_{L^2}}{\|\phi_0\|^2_{L^2}}\mathrm{i}\phi_\theta, \qquad \varphi \in H^2_{\text{ev}}(\mathbb{R}).$$

The operator P_θ, allows us to split the equation (2.10) into a singular and non-singular part:

$$(I - P_\theta)L_\theta(I - P_\theta)\varphi = (I - P_\theta)N(\varphi, \varepsilon, \theta) \tag{2.11}$$

$$P_\theta N(\varphi, \varepsilon, \theta) = 0 \tag{2.12}$$

where we additionally impose the phase-condition:

$$P_\theta\varphi = 0. \tag{2.13}$$

Note that the condition (2.13) depends on the free rotational parameter θ which means that our decomposition of L^2_{ev} is parameter dependent:

$$L^2_{\text{ev}}(\mathbb{R}) = \ker(P_\theta) \oplus \operatorname{ran}(P_\theta).$$

In the following lemma we solve the non-singular equation (2.11) subject to the phase-condition (2.13).

Lemma 2.13. *There exist open neighborhoods $U \subset \mathbb{R}^2$ of $(0, \theta_0)$, $V \subset H^2_{\text{ev}}(\mathbb{R})$ of 0 and an real-analytic map $U \ni (\varepsilon, \theta) \mapsto \varphi(\varepsilon, \theta) \in V$ such that $\varphi(\varepsilon, \theta)$ solves* (2.11) *subject to the phase condition* (2.13) *and $\varphi(0, \theta) = 0$ for all $(0, \theta) \in U$.*

Proof. Define the function $F : H^2_{\text{ev}}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to L^2_{\text{ev}}(\mathbb{R})$ given by

$$F(\varphi, \varepsilon, \theta) := (I - P_\theta)L_\theta(I - P_\theta)\varphi - (I - P_\theta)N(\varphi, \varepsilon, \theta) + P_\theta\varphi.$$

Then, F is real-analytic in $(\varphi, \varepsilon, \theta)$ as a composition of real-analytic functions (cf. [25] Theorem 4.5.7), $F(0, 0, \theta_0) = 0$ and

$$\partial_\varphi F(0, 0, \theta_0) = (I - P_{\theta_0})L_\theta(I - P_{\theta_0}) + P_{\theta_0} : H^2_{\text{ev}}(\mathbb{R}) \to L^2_{\text{ev}}(\mathbb{R})$$

is a homeomorphism by construction. The Implicit Function Theorem for analytic functions ([25] Theorem 4.5.4) yields the existence of open neighborhoods $U \subset \mathbb{R}^2$ of $(0, \theta_0)$, $V \subset H^2_{\text{ev}}(\mathbb{R})$ of 0 and a real-analytic map $U \ni (\varepsilon, \theta) \mapsto \varphi(\varepsilon, \theta) \in V$ such that the unique solution of $F(\varphi, \varepsilon, \theta) = 0$ in $V \times U$ is given by $(\varphi, \varepsilon, \theta) = (\varphi(\varepsilon, \theta), \varepsilon, \theta)$. Finally, since $F(0, 0, \theta) = 0$, we find $\varphi(0, \theta) = 0$ by the local uniqueness of the solution. □

Substitution of the solution obtained in Lemma 2.13 into (2.12) amounts to

$$P_\theta N(\varphi(\varepsilon, \theta), \varepsilon, \theta) = 0 \quad \Longleftrightarrow \quad f(\varepsilon, \theta) := \langle N(\varphi(\varepsilon, \theta), \varepsilon, \theta), \mathrm{i}\phi_\theta\rangle_{L^2} = 0,$$

where $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is again real-analytic as a composition of real-analytic functions and admits the expansion

$$f(\varepsilon, \theta) = \langle \varepsilon \mathrm{i}\phi_\theta + 2|\phi_\theta|^2 u_\infty(\varepsilon) + \phi_\theta^2 \bar{u}_\infty(\varepsilon), \mathrm{i}\phi_\theta\rangle_{L^2} + \mathcal{O}(\varepsilon^2).$$

Clearly, $f(0, \theta) = 0$ for all $(0, \theta) \in U$ and thus we find a real-analytic function $\tilde{f}$ with $f(\varepsilon, \theta) = \varepsilon\tilde{f}(\varepsilon, \theta)$. Nontrivial solutions of $f(\varepsilon, \theta) = 0$ then satisfy $\tilde{f}(\varepsilon, \theta) = 0$ and the equation is again solved by the Implicit Function Theorem. Indeed, in the subsequent Lemma 2.14 we show $\tilde{f}(0, \theta_0) = 0$ and $\partial_\theta\tilde{f}(0, \theta_0) \neq 0$ and thus there exist open intervals $(-\varepsilon^*, \varepsilon^*)$, $\Theta \subset \mathbb{R}$, $\theta_0 \in \Theta$ and a real-analytic branch $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto \theta(\varepsilon) \in \Theta$ such that $\tilde{f}(\varepsilon, \theta) = 0$ in $(-\varepsilon^*, \varepsilon^*) \times \Theta$ is uniquely solved by $(\varepsilon, \theta(\varepsilon)) = (\varepsilon, \theta)$. In summary we have constructed a solitary wave solution

$$u(\varepsilon) = \phi_{\theta(\varepsilon)} + u_\infty(\varepsilon) + \varphi(\varepsilon, \theta(\varepsilon)) \in \mathbb{C} + H^2(\mathbb{R}), \quad \varepsilon \in (-\varepsilon^*, \varepsilon^*)$$

of the Lugiato-Lefever equation (2.4). It remains to prove Lemma 2.14.

Lemma 2.14. *Let $\theta_0 \in \mathbb{R}$ be a simple zero of $\theta \mapsto \pi f \cos\theta - 2\sqrt{2\zeta}$. Then $\tilde{f}(0, \theta_0) = 0$ and $\partial_\theta\tilde{f}(0, \theta_0) \neq 0$.*

Proof. By definition of $\tilde{f}$, formula (2.9), and since $\pi f \cos\theta_0 = 2\sqrt{2\zeta}$, we have

$$\begin{aligned}
\tilde{f}(0,\theta_0) &= \langle \mathrm{i}\phi_{\theta_0} + 2|\phi_{\theta_0}|^2\partial_\varepsilon u_\infty(0) + \phi_{\theta_0}^2\partial_\varepsilon \bar{u}_\infty(0), \mathrm{i}\phi_{\theta_0}\rangle_{L^2} \\
&= \mathrm{Re}\int_{\mathbb{R}} |\phi_0|^2 - 2\mathrm{i}|\phi_0|^2\bar{\phi}_{\theta_0}\partial_\varepsilon u_\infty(0) - \mathrm{i}|\phi_0|^2\phi_{\theta_0}\partial_\varepsilon \bar{u}_\infty(0)dx \\
&= 4\sqrt{\zeta} - \frac{f}{\zeta}\,\mathrm{Re}\int_{\mathbb{R}} 2|\phi_0|^2\bar{\phi}_{\theta_0} - |\phi_0|^2\phi_{\theta_0}dx \\
&= 4\sqrt{\zeta} - \frac{f}{\zeta}\cos\theta_0\int_{\mathbb{R}}\phi_0^3 dx \\
&= 4\sqrt{\zeta} - \sqrt{2}\pi f\cos\theta_0 = 0
\end{aligned}$$

and similar computations lead to

$$\partial_\theta \tilde{f}(0,\theta_0) = \sqrt{2}\pi f \sin\theta_0 \neq 0,$$

where $\sin\theta_0 \neq 0$ by simplicity of the root θ_0, which proves the statement. □

Corollary 2.15. *The solutions $u(\varepsilon) \in \mathbb{C} + H^2(\mathbb{R})$ of Theorem 2.3 decay exponentially fast to their limit states $u_\infty(\varepsilon) \in \mathbb{C}$ as $|x| \to \infty$.*

Proof. Re-writing (2.4) in its dynamical system formulation

$$\partial_x U = \begin{pmatrix} U_3 \\ U_4 \\ \zeta U_1 - (U_1^2 + U_2^2)U_1 + \varepsilon U_2 \\ \zeta U_2 - (U_1^2 + U_2^2)U_2 + \varepsilon(-U_1 + f) \end{pmatrix}, \quad U = \begin{pmatrix} u_1(\varepsilon) \\ u_2(\varepsilon) \\ \partial_x u_1(\varepsilon) \\ \partial_x u_2(\varepsilon) \end{pmatrix}$$

for $u_1(\varepsilon) = \mathrm{Re}(u(\varepsilon)), u_2(\varepsilon) = \mathrm{Im}(u(\varepsilon))$ we easily see that U is homoclinic to a hyperbolic equilibrium and thus converges exponentially fast to its limit state $U_\infty = (\mathrm{Re}(u_\infty(\varepsilon)),$ $\mathrm{Im}(u_\infty(\varepsilon)), 0, 0)^T$. □

2.3 Stability analysis

In this section, we prove the stability results of Theorem 2.6 and Theorem 2.9. From now on $H^k(\mathbb{R}) = H^k(\mathbb{R}, \mathbb{C})$, $k \in \mathbb{N}_0$ denotes the Sobolev spaces *over the field* $\mathbb{C}$. In particular, the L^2-scalar product is now given by the $\mathbb{C}$-*valued map*

$$\forall f, g \in L^2(\mathbb{R}, \mathbb{C}) : \quad \langle f, g \rangle_{L^2} = \int_{\mathbb{R}} f \bar{g} dx.$$

Let us start with the proof of the spectral stability result of Theorem 2.6.

2.3.1 Proof of Theorem 2.6

Suppose that $u = u(\varepsilon) \in \mathbb{C} + H^2(\mathbb{R})$ is a solution of (2.4) as in Theorem 2.6 for $\varepsilon \neq 0$ sufficiently small. We determine the location of the spectrum of the operator $\mathcal{L} = JL$ defined in (2.6).

It is well-known [78] that we have a decomposition into essential and discrete spectrum

$$\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) \cup \sigma_d(\mathcal{L}). \tag{2.14}$$

Essential spectrum of $\mathcal{L}$

The essential spectrum can be computed explicitly.

Lemma 2.16. *Let ε be sufficiently small. The essential spectrum is given by*

$$\sigma_{ess}(\mathcal{L}) = \{\mathrm{i}\omega \in \mathrm{i}\mathbb{R} : \ |\omega| \in [\zeta_\varepsilon, \infty)\},$$

where $\zeta_\varepsilon = \zeta + \mathcal{O}(\varepsilon^2)$.

Proof. Since $u(x) \to u_\infty$ as $|x| \to \infty$ exponentially fast, cf. Corollary 2.15, we can use Weyl's Theorem [78] to find

$$\sigma_{\text{ess}}(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}^\infty)$$

where the asymptotic constant coefficient operator is given by

$$\mathcal{L}^\infty = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \begin{pmatrix} -\partial_x^2 + \zeta - 2|u_\infty|^2 & -u_\infty^2 \\ -\bar{u}_\infty^2 & -\partial_x^2 + \zeta - 2|u_\infty|^2 \end{pmatrix}.$$

We calculate the essential spectrum of $\mathcal{L}^\infty$. Therefore, we write the spectral problem for $\mathcal{L}^\infty$ in its first order reformulation $\partial_x V = A(\lambda)V$ with

$$A(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \zeta - 2|u_\infty|^2 - \mathrm{i}\lambda & u_\infty^2 & 0 & 0 \\ \bar{u}_\infty^2 & \zeta - 2|u_\infty|^2 + \mathrm{i}\lambda & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\ v_2 \\ \partial_x v_1 \\ \partial_x v_2 \end{pmatrix}.$$

Then, the essential spectrum is characterized as follows (see for instance [78]):

$$\lambda \in \sigma_{\text{ess}}(\mathcal{L}^\infty) \quad \Longleftrightarrow \quad \sigma(A(\lambda)) \cap \mathrm{i}\mathbb{R} \neq \emptyset.$$

Computing

$$p(k,\lambda) := \det(A(\lambda) - \mathrm{i}k) = k^4 + 2(\zeta - 2|u_\infty|^2)k^2 + (\zeta - 2|u_\infty|^2)^2 - |u_\infty|^4 + \lambda^2$$

we find that $A(\lambda)$ is nonhyperbolic if and only if

$$\exists k \in \mathbb{R}: \quad p(k,\lambda) = 0$$

which is equivalent to

$$\lambda \in \{\mathrm{i}\omega \in \mathrm{i}\mathbb{R} : \ |\omega| \in [\zeta_\varepsilon, \infty)\},$$

with $\zeta_\varepsilon = \zeta + \mathcal{O}(\varepsilon^2)$ since $u_\infty = \mathcal{O}(\varepsilon)$ and thus the claim follows. □

An immediate consequence of Lemma 2.16 is the spectral instability of the wave u if $\varepsilon < 0$. However, if $\varepsilon > 0$, the essential spectrum is stable and spectral stability is solely determined by the location of the discrete spectrum. From now on we focus on the case $\varepsilon > 0$.

Discrete spectrum of $\mathcal{L}$

Recall, that the discrete spectrum can be determined from the eigenvalue problem (2.7) which can be written as

$$\begin{cases} \mathrm{i}\lambda v_1 &= -v_{1xx} + \zeta v_1 - 2|u|^2 v_1 - u^2 v_2 - \mathrm{i}\varepsilon v_1, \\ -\mathrm{i}\lambda v_2 &= -v_{2xx} + \zeta v_2 - 2|u|^2 v_2 - \bar{u}^2 v_1 + \mathrm{i}\varepsilon v_2. \end{cases}$$

For $\varepsilon = 0$, we recover the spectral stability problem for the rotated soliton ϕ_{θ_0} of NLS and $\lambda = 0$ is an isolated eigenvalue of geometric multiplicity two and algebraic multiplicity four.

More precisely, we find two Jordan chains of length two and by Lemma 2.12 we have that the corresponding eigenspace is spanned by the vectors $(\mathrm{i}\phi_{\theta_0}, -\mathrm{i}\bar{\phi}_{\theta_0})$ and $(\phi'_{\theta_0}, \bar{\phi}'_{\theta_0})$. Consequently, it follows from standard perturbation theory [80], that for small values of ε the total multiplicity of all eigenvalues in a small neighborhood of zero is also four. We now focus on the bifurcations of these eigenvalues which in the end will determine the spectral stability.

Eigenvalues close to the origin

We compute expansions in ε of all eigenvalues of $\mathcal{L} = JL$ close to zero. Observe that we always find $0 \in \sigma(\mathcal{L} - \varepsilon)$ due to the translational invariance of (2.4). This yields $\varepsilon \in \sigma(JL)$ and the corresponding eigenfunction is given by $(\partial_x u, \partial_x \bar{u})$. Since the spectrum of JL is symmetric w.r.t. the imaginary axis we also find $-\varepsilon \in \sigma(JL)$. Note that the symmetry is an immediate consequence of the structure of JL, a composition of a skew-adjoint and self-adjoint operator. Thus, in the neighborhood of $\lambda = 0$ only two unknown eigenvalues remain and they correspond to the broken gauge-symmetry in the LLE.

To compute the expansions of the perturbed rotational eigenvalues, we restrict to spaces of even functions. Following [80], we expand both remaining eigenvalues along with their corresponding eigenfunctions in a Puiseux series:

$$\lambda = \sqrt{\varepsilon}\lambda_1 + \varepsilon\lambda_2 + \mathcal{O}(\varepsilon^{3/2}), \quad V = V_0 + \sqrt{\varepsilon}V_1 + \varepsilon V_2 + \mathcal{O}(\varepsilon^{3/2}), \quad V_0 = \begin{pmatrix} \mathrm{i}\phi_{\theta_0} \\ -\mathrm{i}\bar{\phi}_{\theta_0} \end{pmatrix},$$

and the expansions are in powers of $\varepsilon^{1/2}$ since we consider the splitting of a Jordan chain of length two. Moreover, we expand the operator L in powers of ε which relies on expansions of the solution u:

$$\begin{aligned}
L &= L_0 + \varepsilon L_1 + \mathcal{O}(\varepsilon^2) \\
L_0 &= \begin{pmatrix} -\partial_x^2 + \zeta - 2|u_0|^2 & -u_0^2 \\ -\bar{u}_0^2 & -\partial_x^2 + \zeta - 2|u_0|^2 \end{pmatrix}, \\
L_1 &= \begin{pmatrix} -2(u_0\bar{u}_1 + \bar{u}_0 u_1) & -2u_0u_1 \\ -2\bar{u}_0\bar{u}_1 & -2(u_0\bar{u}_1 + \bar{u}_0 u_1) \end{pmatrix}, \\
u &= u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2), \qquad u_0 = \phi_{\theta_0}, \qquad u_1 = \mathrm{i}\partial_\varepsilon\theta(0)\phi_{\theta_0} - \mathrm{i}\frac{f}{\zeta} + \varphi_1,
\end{aligned}$$

and the equation for φ_1 is found from differentiating (2.10) w.r.t. ε at $\varepsilon = 0$ (see also part two of the proof of Lemma 2.13),

$$L_{\theta_0}\varphi_1 = 2|\phi_{\theta_0}|^2\partial_\varepsilon u_\infty(0) + \phi_{\theta_0}^2\partial_\varepsilon \bar{u}_\infty(0) + \mathrm{i}\phi_{\theta_0}, \qquad \varphi_1 \perp \mathrm{i}\phi_{\theta_0}.$$

Substituting the expansions into the spectral problem $JLV = \lambda V$ yields equations at order $\varepsilon^{1/2}$ and ε,

$$L_0V_1 = \lambda_1 J^{-1}V_0, \quad L_0V_2 + L_1V_0 = \lambda_1 J^{-1}V_1 + \lambda_2 J^{-1}V_0.$$

Since $J^{-1}V_0 \perp \ker(L_0)$, we find $V_1 = \lambda_1 L_0^{-1}J^{-1}V_0 + \alpha V_0$, $\alpha \in \mathbb{C}$. Inserting this into the second equation amounts to

$$L_0V_2 = -L_1V_0 + \lambda_1^2 J^{-1}L_0^{-1}J^{-1}V_0 + \alpha\lambda_1 J^{-1}V_0 + \lambda_2 J^{-1}V_0 \tag{2.15}$$

and by the Fredholm alternative, (2.15) is solvable if and only if

$$-L_1V_0 + \lambda_1^2 J^{-1}L_0^{-1}J^{-1}V_0 + \alpha\lambda_1 J^{-1}V_0 + \lambda_2 J^{-1}V_0 \perp \ker(L_0).$$

Since $J^{-1}V_0 \perp \ker(L_0)$, this yields

$$\lambda_1^2\langle J^{-1}L_0^{-1}J^{-1}V_0, V_0\rangle_{L^2} = \langle L_1V_0, V_0\rangle_{L^2}. \tag{2.16}$$

Lemma 2.17. *Let $\varepsilon, \delta > 0$ be sufficiently small and $B_\delta(0) \subset \mathbb{C}$ be the ball of radius δ centered at $\lambda = 0$. Then $B_\delta(0) \cap \sigma_d(\mathcal{L})$ consists of four distinct simple eigenvalues, given by*

$$\pm\varepsilon, \pm\sqrt{\varepsilon}\sqrt{-\pi f\sqrt{2\zeta}\sin\theta_0} + \mathcal{O}(\varepsilon) \in \sigma_d(\mathcal{L}).$$

In particular, if $\sin\theta_0 > 0$, we have one unstable eigenvalue $\lambda = \varepsilon$ of $\mathcal{L}$, which corresponds to a simple zero eigenvalue of $\mathcal{L} - \varepsilon$, and two purely imaginary eigenvalues of $\mathcal{L}$:

$$\lambda_\pm = \pm\mathrm{i}\left(\sqrt{\varepsilon}\sqrt{\pi f\sqrt{2\zeta}\sin\theta_0} + \mathcal{O}(\varepsilon)\right).$$

If $\sin\theta_0 < 0$, we have two real unstable eigenvalues of $\mathcal{L}$:

$$\lambda = \varepsilon \quad \textit{and} \quad \lambda_+ = \sqrt{\varepsilon}\sqrt{\pi f\sqrt{2\zeta}|\sin\theta_0|} + \mathcal{O}(\varepsilon),$$

where λ_+ is also an unstable eigenvalue of the operator $\mathcal{L} - \varepsilon$ since it is of order $\mathcal{O}(\varepsilon^{1/2})$.

Proof. From the preceding discussion it remains to calculate the scalar products in (2.16). Let us start with $\langle J^{-1}L_0^{-1}J^{-1}V_0, V_0\rangle_{L^2}$. Consider the NLS

$$-\phi''_{\theta_0} + \zeta\phi_{\theta_0} - |\phi_{\theta_0}|^2\phi_{\theta_0} = 0.$$

Taking derivative w.r.t. ζ yields

$$L_0 \begin{pmatrix} \partial_\zeta \phi_{\theta_0} \\ \partial_\zeta \bar{\phi}_{\theta_0} \end{pmatrix} = - \begin{pmatrix} \phi_{\theta_0} \\ \bar{\phi}_{\theta_0} \end{pmatrix} = J^{-1}V_0.$$

The function $\partial_\zeta \phi_{\theta_0}$ is found from differentiating the formula of the NLS soliton and a straight forward calculation gives

$$\langle J^{-1}L_0^{-1}J^{-1}V_0, V_0\rangle_{L^2} = \int_{\mathbb{R}} \partial_\zeta \phi_{\theta_0}\bar{\phi}_{\theta_0} + \partial_\zeta \bar{\phi}_{\theta_0}\phi_{\theta_0}\, dx = 2\zeta^{-1/2}.$$

Next, we calculate the scalar product $\langle L_1 V_0, V_0\rangle_{L^2}$. We have

$$\langle L_1 V_0, V_0\rangle_{L^2} = 2\int_{\mathbb{R}} \begin{pmatrix} -\mathrm{i}u_0^2\bar{u}_1 \\ \mathrm{i}\bar{u}_0^2 u_1 \end{pmatrix} \cdot \overline{\begin{pmatrix} \mathrm{i}u_0 \\ -\mathrm{i}\bar{u}_0 \end{pmatrix}} dx = -4\,\mathrm{Re}\int_{\mathbb{R}} |u_0|^2 u_0 \bar{u}_1 dx.$$

Inserting the formula for u_1 into the integral yields

$$\begin{aligned} &\mathrm{Re}\int_{\mathbb{R}} |u_0|^2 u_0\bar{u}_1 dx = -\frac{f}{\zeta}\|\phi_0\|_{L^3}^3 \sin\theta_0 + \mathrm{Re}\int_{\mathbb{R}} |\phi_{\theta_0}|^2\phi_{\theta_0}\bar{\varphi}_1 dx \\ &= -\frac{f}{\zeta}\|\phi_0\|_{L^3}^3 \sin\theta_0 + \mathrm{Re}\int_{\mathbb{R}} |\phi_{\theta_0}|^2\phi_{\theta_0}\overline{L_{\theta_0}^{-1}(2|\phi_{\theta_0}|^2\partial_\varepsilon u_\infty(0) + \phi_{\theta_0}^2\partial_\varepsilon \bar{u}_\infty(0) + \mathrm{i}\phi_{\theta_0})}dx \\ &= -\frac{f}{\zeta}\|\phi_0\|_{L^3}^3 \sin\theta_0 + \mathrm{Re}\int_{\mathbb{R}} L_{\theta_0}^{-1}(|\phi_{\theta_0}|^2\phi_{\theta_0})\overline{(2|\phi_{\theta_0}|^2\partial_\varepsilon u_\infty(0) + \phi_{\theta_0}^2\partial_\varepsilon \bar{u}_\infty(0) + \mathrm{i}\phi_{\theta_0})}dx. \end{aligned}$$

Using that $L_{\theta_0}\phi_{\theta_0} = -2|\phi_{\theta_0}|^2\phi_{\theta_0}$ gives

$$\begin{aligned} &\mathrm{Re}\int_{\mathbb{R}} L_{\theta_0}^{-1}(|\phi_{\theta_0}|^2\phi_{\theta_0})\overline{(2|\phi_{\theta_0}|^2\partial_\varepsilon u_\infty(0) + \phi_{\theta_0}^2\partial_\varepsilon \bar{u}_\infty(0) + \mathrm{i}\phi_{\theta_0})}dx \\ &= -\frac{1}{2}\,\mathrm{Re}\int_{\mathbb{R}} \phi_{\theta_0}\overline{(2|\phi_{\theta_0}|^2\partial_\varepsilon u_\infty(0) + \phi_{\theta_0}^2\partial_\varepsilon \bar{u}_\infty(0) + \mathrm{i}\phi_{\theta_0})}dx \\ &= \frac{3f}{2\zeta}\|\phi_0\|_{L^3}^3 \sin\theta_0. \end{aligned}$$

Finally, since $\|\phi_0\|_{L^3}^3 = \sqrt{2}\zeta\pi$, we have

$$\langle L_1 V_0, V_0 \rangle_{L^2} = -2\frac{f}{\zeta}\|\phi_0\|_{L^3}^3 \sin\theta_0 = -2\sqrt{2} f\pi \sin\theta_0$$

and thus from (2.16) we obtain the desired formula for the simple eigenvalues

$$\lambda_\pm = \pm\sqrt{\varepsilon}\sqrt{-\pi f\sqrt{2\zeta}\sin\theta_0} + \mathcal{O}(\varepsilon),$$

where in the case $\sin\theta_0 > 0$ the $\mathcal{O}(\varepsilon)$ remainder is purely imaginary because of the Hamiltonian symmetry of the spectrum. □

Lemma 2.17 proves the spectral instability of the wave u if $\sin\theta_0 < 0$. However, if $\sin\theta_0 > 0$, no unstable eigenvalues of $\mathcal{L} - \varepsilon$ occur from the splitting of the zero eigenvalue. Instead, we find a pair of purely imaginary eigenvalues $\lambda_\pm \in \mathrm{i}\mathbb{R}$ of $\mathcal{L}$. Hence, we now focus on the case $\sin\theta_0 > 0$ and show that the only unstable eigenvalue of $\mathcal{L}$ is given by $\lambda = \varepsilon$, which then proves the spectral stability of the wave u. For this purpose, we employ the instablity index count developed in [75, 76] and also in [31]. To apply the instability count, we need to transform the spectral stability problem (2.7) into a problem with real-valued coefficients. Using similarity transformations with the matrizes

$$T_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \quad T_2 = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

such that $\tilde{J} = T_2 T_1 J T_1^{-1} T_2^{-1}$, $\tilde{L} = T_2 T_1 L T_1^{-1} T_2^{-1}$ we obtain the equivalent problem

$$\lambda\tilde{V} = (\tilde{J}\tilde{L} - \varepsilon)\tilde{V},$$

and the eigenfunctions are related by $\tilde{V} = T_2 T_1 V$. The real-valued operators $\tilde{J}, \tilde{L}$ are then of the form

$$\tilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{L} = \begin{pmatrix} -\partial_x^2 + \zeta - 3\phi_0^2 & 0 \\ 0 & -\partial_x^2 + \zeta - \phi_0^2 \end{pmatrix} + \mathcal{O}(\varepsilon).$$

We recall necessary notation from [78]:

- $n(A)$ denotes the *number of negative eigenvalues* (counting multiplicities) of a linear operator A.

- Let $\lambda \in \sigma_d(\tilde{J}\tilde{L}) \cap (\mathrm{i}\mathbb{R} \setminus \{0\})$ be an eigenvalue with algebraic multiplicity $m_a(\lambda)$ and $\ker(\bigcup_{n\in\mathbb{N}}(\tilde{J}\tilde{L}-\lambda)^n) = \mathrm{span}\{v_1, \dots, v_{m_a(\lambda)}\}$ be the generalized eigenspace. Then the *negative Krein index* of λ is defined by

$$k_i^-(\lambda) := n(H), \quad \text{with} \quad H_{ij} := \langle \tilde{L}v_i, v_j \rangle,$$

  and the total negative index is defined by $k_i^- := \sum_{\lambda \in \sigma_d(\tilde{J}\tilde{L}) \cap \mathrm{i}\mathbb{R}\setminus\{0\}} k_i^-(\lambda)$.

- The *real Krein index* is defined by $k_r := \sum_{\lambda \in \sigma_d(\tilde{J}\tilde{L}) \cap (0,\infty)} m_a(\lambda)$.
- The *complex Krein index* is defined by $k_c := \sum_{\lambda \in \sigma_d(\tilde{J}\tilde{L}), \mathrm{Re}(\lambda)>0, \mathrm{Im}(\lambda)\neq 0} m_a(\lambda)$.

Applying the instability index counting theory from [31,75,76] yields for $\varepsilon > 0$ sufficiently small the formula

$$k_r + k_i^- + k_c = n(\tilde{L}). \tag{2.17}$$

Remark 2.18. In [75,76] there appears an additional number $n(D)$ in the formula (2.17), which accounts for a nontrivial kernel generated by various symmetries of the problem. However, in Lemma 2.17 we proved that $\ker(\tilde{L}) = \{0\}$ provided $\varepsilon > 0$ is sufficiently small explaining the absence of this number in our situation (see also [31]).

In the next lemma, we use (2.17) to show that $\varepsilon \in \sigma(\mathcal{L})$ is the only unstable eigenvalue of $\mathcal{L}$ proving that $\sigma_d(\mathcal{L} - \varepsilon) \subset \{-2\varepsilon\} \cup \{\mathrm{Re} = -\varepsilon\} \cup \{0\}$ with simple eigenvalues $\lambda = 0, -2\varepsilon$.

Lemma 2.19. *Let $\varepsilon > 0$ be sufficiently small and $\sin\theta_0 > 0$. Then, $n(\tilde{L}) = 3$, $k_r = 1$, and $k_i^- = 2$.*

Proof. $k_r \geq 1$ is clear since $\varepsilon \in \sigma_d(\tilde{J}\tilde{L})$ due to the translational symmetry. Now consider the Sturm-Liouville operators on the line $\mathbb{R}$,

$$L_+ := -\partial_x^2 + \zeta - 3\phi_0^3, \quad L_- := -\partial_x^2 + \zeta - \phi_0^3.$$

We have $\ker(L_+) = \mathrm{span}\{\phi_0'\}$, $\ker(L_-) = \mathrm{span}\{\phi_0\}$ and since ϕ_0' has one zero and $\phi_0 > 0$ on $\mathbb{R}$ we find $n(L_+) = 1$, $n(L_-) = 0$ by the standard theory for Sturm-Liouville operators. In particular, we obtain

$$n\Big(\begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}\Big) = 1, \quad 0 \in \sigma_d\Big(\begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}\Big), \quad m_a(0) = 2,$$

and by means of perturbation theory for eigenvalues we conclude that there are at most three negative eigenvalues of $\tilde{L}$ for ε sufficiently small, which proves $n(\tilde{L}) \leq 3$. Moreover, by Lemma 2.17 we find purely imaginary simple eigenvalues $\lambda_\pm \in \mathrm{i}\mathbb{R}$ with $\bar{\lambda}_+ = \lambda_-$ by the symmetry of the spectrum and we show that $k_i^-(\lambda_\pm) = 1$. Indeed, from Lemma 2.17 and the discussion before, we have Puiseux expansions for the eigenvalue and corresponding eigenfunction

$$\lambda_+ = \sqrt{\varepsilon}\lambda_1 + \mathcal{O}(\varepsilon), \quad \tilde{V} = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} - \sqrt{\varepsilon}\left[\lambda_1 \begin{pmatrix} L_+^{-1}\phi_0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}\right] + \mathcal{O}(\varepsilon)$$

where the eigenfunction is found from the relation $\tilde{V} = T_2 T_1 V$. Hence direct calculations yield

$$\langle \tilde{L}\tilde{V}, \tilde{V}\rangle_{L^2} = -\lambda_1 \langle \tilde{J}\tilde{V}, \tilde{V}\rangle_{L^2} = \varepsilon 2|\lambda_1|^2 \langle L_+^{-1}\phi_0, \phi_0\rangle_{L^2} + \mathcal{O}(\varepsilon^{3/2}).$$

Since $\langle L_+^{-1}\phi_0, \phi_0\rangle_{L^2} = -\langle \partial_\zeta \phi_0, \phi_0\rangle_{L^2} < 0$, cf. the proof of Lemma 2.17, we find $k_i^-(\lambda_+) = 1$ and from $k_i^-(\lambda_+) = k_i^-(\bar{\lambda}_+)$ and $\bar{\lambda}_+ = \lambda_-$ it follows that $k_i^- \geq 2$. Thus, using (2.17) we infer $n(L) = 3$, $k_r = 1$, and $k_i^- = 2$, which finishes the proof. □

Combing our results on the discrete and essential spectrum we finally obtain

$$\sigma(\mathcal{L} - \varepsilon) \subset \{-2\varepsilon\} \cup \{z \in \mathbb{C} : \operatorname{Re} z = -\varepsilon\} \cup \{0\}$$

with simple eigenvalues $\lambda = -2\varepsilon, 0$. Thus, the wave is spectrally stable provided that $\varepsilon, \sin\theta_0 > 0$ as claimed.

2.3.2 Proof of Theorem 2.9

We prove the asymptotic orbital stability in H^1 of the spectrally stable solitary waves of Theorem 2.6. The strategy of the proof follows the work in [137] where asymptotic orbital stability is obtained for *periodic* spectrally stable solutions of LLE. Our method deviates from [137] when establishing uniform resolvent bounds. Indeed, high frequency resolvent estimates in [137] are only proven for the linearization operator with periodic coefficients. Using an abstract functional analytic approach, we extend this result to the case of localized perturbations.

Linearized stability

Let $u \in \mathbb{C} + H^2(\mathbb{R})$ be a spectrally stable solution of (2.4) for $\varepsilon > 0$ sufficiently small and denote by $\mathcal{L} - \varepsilon$ the linearization about u. In a first step we prove linearized stability. Note that decay of the semigroup $\left(e^{(\mathcal{L}-\varepsilon)t}\right)_{t\geq 0}$ cannot be concluded immediately from the spectral stability of $\mathcal{L}-\varepsilon$, since the spectrum of the operator is not confined to a sector of $\mathbb{C}$ and therefore the Spectral Mapping Theorem is a-priori not available. However, we can use the following characterization of exponential stability of semigroups in Hilbert spaces called the Prüss-Theorem, cf. [116] Corollary 4.

Theorem 2.20 ([116, Corollary 4]). *Let A be the generator of a C_0-semigroup $(e^{At})_{t\geq 0}$ in a Hilbert space H. Then $(e^{At})_{t\geq 0}$ is exponentially stable if and only if*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geq 0\} \subset \rho(A) \quad \textit{and} \quad \sup\left\{\|(A-\lambda)^{-1}\|_{H\to H} : \lambda \in \mathbb{C}, \operatorname{Re}(\lambda) \geq 0\right\} < \infty.$$

Recall, that by the presence of the translational symmetry, $0 \in \sigma(\mathcal{L}-\varepsilon)$, which violates the spectral condition in Prüss Theorem. To overcome this problem, we introduce the spectral projection P_0 onto $\ker(\mathcal{L}-\varepsilon)$ and show that the restricted operator $(\mathcal{L}-\varepsilon)|_E$ satisfies the conditions in Prüss Theorem, where $E := \ker(P_0)$. This then leads to decay of the semigroup restricted to the subspace E, which is enough to establish the orbital stability result (cf. Theorem 4.3.5 in [78]).

Recall the basic properties of the spectral projection: $(\mathcal{L}-\varepsilon)P_0 = P_0(\mathcal{L}-\varepsilon) = 0$ and

$$\sigma((\mathcal{L}-\varepsilon)|_E) = \sigma(\mathcal{L}-\varepsilon) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq -\varepsilon\}. \tag{2.18}$$

Lemma 2.21. *There exist constants $\eta > 0, C \geq 1$ such that*

$$\|e^{(\mathcal{L}-\varepsilon)t}|_E\|_{H^1\to H^1} \leq Ce^{-\eta t} \quad \textit{for } t \geq 0.$$

Proof. It follows from the Lumer-Phillips Theorem (Theorem II.3.15 in [44]) and the Bounded Perturbation Theorem (Theorem III.1.3 in [44]) that $\mathcal{L}-\varepsilon$ is the generator a C_0-semigroup on $(H^1(\mathbb{R}))^2$ and the same holds after restricting the operator to $E = \ker(P_0)$. According to the Prüss Theorem and (2.18) the claim of the lemma follows if we show the uniform resolvent bound

$$\exists C > 0 : \quad \sup_{\operatorname{Re}(\lambda)\geq 0} \|(\mathcal{L}-\varepsilon-\lambda)^{-1}(I-P_0)\|_{H^1\to H^1} \leq C.$$

This estimate is obtained in two steps.

Step 1 (Uniform bound in L^2): We show uniform resolvent estimates for the operator $(\mathcal{L}-\varepsilon-\lambda)^{-1}(I-P_0): L^2 \to L^2$. First note that the Hille-Yosida Theorem ([44] Theorem 3.8) ensures existence of constants $\gamma_1, C' > 0$ such that

$$\sup_{\mathrm{Re}(\lambda)\geq\gamma_1} \|(\mathcal{L}-\varepsilon-\lambda)^{-1}\|_{L^2\to L^2} \leq C'.$$

Moreover, using Theorem 2.22 and Remark 2.23 with $H = L^2(\mathbb{R}), A_\pm = -\partial_x^2+\zeta-2|u|^2, B = -u^2$ we find constants $\gamma_2, C'' > 0$ such that

$$\sup_{\mathrm{Re}(\lambda)\geq 0, |\,\mathrm{Im}(\lambda)|\geq\gamma_2} \|(\mathcal{L}-\varepsilon-\lambda)^{-1}\|_{L^2\to L^2} \leq C''.$$

Finally, observe that $\lambda \mapsto (\mathcal{L}-\varepsilon-\lambda)^{-1}(I-P_0)$ is an analytic function on $\{\lambda\in\mathbb{C}: \ \mathrm{Re}(\lambda)\geq 0\}$ and hence uniformly bounded on the compact set $\{\lambda\in\mathbb{C}: \ 0\leq \mathrm{Re}(\lambda)\leq\gamma_1, |\,\mathrm{Im}(\lambda)|\leq\gamma_2\}$ with a bound $C''' \geq 0$. Thus for $C := \max\{C', C'', C'''\}$ we have

$$\sup_{\mathrm{Re}(\lambda)\geq 0} \|(\mathcal{L}-\varepsilon-\lambda)^{-1}(I-P_0)\|_{L^2\to L^2} \leq C.$$

Step 2 (Uniform bound in H^1): First, we establish a uniform bound in H^2 and then use an interpolation argument to find the desired H^1 bound. Note that there exist constants $c, \gamma \gg 1$ such that

$$\forall V = (v_1, v_2)^T \in (H^2(\mathbb{R}))^2 : \quad \|V\|_{H^2} \leq c\|(L+J\varepsilon+\gamma)V\|_{L^2}, \quad \|(JL-\varepsilon)V\|_{L^2} \leq c\|V\|_{H^2},$$

where we recall the decomposition $\mathcal{L} = JL$ from (2.6). Hence we find for $\mathrm{Re}(\lambda)\geq 0, V\in H^2$,

$$\begin{aligned}
&\|(JL-\varepsilon-\lambda)^{-1}(I-P_0)V\|_{H^2} \\
&\quad\leq c\|(L+J\varepsilon+\gamma)(JL-\varepsilon-\lambda)^{-1}(I-P_0)V\|_{L^2} \\
&\quad\leq c\|(JL-\varepsilon)(JL-\varepsilon-\lambda)^{-1}(I-P_0)V\|_{L^2} \\
&\qquad + c\gamma\|(JL-\varepsilon-\lambda)^{-1}(I-P_0)V\|_{L^2} \\
&\quad\leq c\|(JL-\varepsilon-\lambda)^{-1}(I-P_0)(JL-\varepsilon)V\|_{L^2} + c\gamma C\|V\|_{H^2} \\
&\quad\leq cC(c+\gamma)\|V\|_{H^2},
\end{aligned}$$

by step 1, which implies the uniform bound

$$\sup_{\mathrm{Re}(\lambda)\geq 0} \|(JL-\varepsilon-\lambda)^{-1}(I-P_0)\|_{H^2\to H^2} \leq cC(c+\gamma).$$

Interpolation of both estimates according to [90] Theorem 2.6 yields

$$\sup_{\mathrm{Re}(\lambda)\geq 0} \|(JL-\varepsilon-\lambda)^{-1}(I-P_0)\|_{H^1\to H^1} \leq C$$

for some constant $C > 0$ and thus the claim follows. □

We still need to prove the uniform resolvent estimate for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$, $|\operatorname{Im}\lambda| \gg 1$ and conclude the nonlinear stability.

High frequency resolvent estimates

We establish uniform resolvent estimates for a family of operators in Hilbert spaces which generalizes the linearization operator $\mathcal{L} - \varepsilon$. Our proof relies on techniques from [51] Section 3 where uniform resolvent estimates for NLS are considered. Similar resolvent estimates can also be found in [18, 115, 137].

Let H be a complex Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. On $H \times H$ we consider the spectral problem

$$\left[\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\begin{pmatrix} A_+ & B \\ B^* & A_- \end{pmatrix} - \lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.19}$$

where $\lambda = \lambda_r + \mathrm{i}\lambda_i \in \mathbb{C}$ is a spectral parameter, $A_\pm : D \subset H \to H$ are closed self-adjoint linear operators with common domains $D = D(A_\pm)$ which are either both bounded from below or from above by the bound $\gamma \in \mathbb{R}$, $B : H \to H$ is a bounded linear operator, $I : H \to H$ is the identity, $\phi_1, \phi_2 \in D$ and $\psi_1, \psi_2 \in H$. Under these assumptions the following theorem on uniform high-frequency resolvent estimates holds.

Theorem 2.22. *There exists $\rho = \rho(\gamma, \|B\|_{H\to H}) > 0$ such that for all $\lambda = \lambda_r + \mathrm{i}\lambda_i \in \mathbb{C}$ with $|\lambda_i| \geq \rho$ and $\lambda_r \neq 0$ we have that for every given $(\psi_1, \psi_2) \in H \times H$ the spectral problem* (2.19) *has a unique solution $(\phi_1, \phi_2) \in D \times D$ such that*

$$\|\phi_1\| + \|\phi_2\| \lesssim |\lambda_r|^{-1}(\|\psi_1\| + \|\psi_2\|).$$

Remark 2.23. Theorem 2.22 gives a uniform resolvent estimate for the linear operator

$$\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\begin{pmatrix} A_+ & B \\ B^* & A_- \end{pmatrix} : D \times D \to H \times H$$

for high frequencies $\lambda \in \mathbb{C}$ such that $|\operatorname{Im}(\lambda)| \gg 1$, $\operatorname{Re}(\lambda) \neq 0$. For $\operatorname{Re}(\lambda) = 0$ we cannot expect such an estimate to be true, since the intersection of the spectrum and the imaginary axis is typically non-empty.

Remark 2.24. Theorem 2.22 can be applied to different variants of LLE. Indeed, consider the extended LLE

$$\mathrm{i}u_t = \sum_{k=1}^{2n} d_k \left(\mathrm{i}\partial_x\right)^k u + (\zeta(x) - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f(x), \quad (x,t) \in \Omega \times \mathbb{R},$$

where $\Omega \in \{\mathbb{R}, \mathbb{R}/\mathbb{Z}\}$, $n \in \mathbb{N}$, $d_{2n} \neq 0$, $d_{2n-1}, \ldots, d_1 \in \mathbb{R}$, $\zeta \in L^\infty(\Omega, \mathbb{R})$, $f \in H^2(\Omega, \mathbb{C})$, and $\mu \geq 0$. A similar equation is studied in [18, 58, 59]. Then, the linearization $\mathcal{L} : (H^{2n}(\Omega))^2 \to (L^2(\Omega))^2$ about a stationary solution $u = u(x)$ reads

$$\mathcal{L} := \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \begin{pmatrix} \sum_{k=1}^{2n} d_k \left(\mathrm{i}\partial_x\right)^k + \zeta(x) - 2|u|^2 & -u^2 \\ -\bar{u}^2 & \sum_{k=1}^{2n} d_k \left(-\mathrm{i}\partial_x\right)^k + \zeta(x) - 2|u|^2 \end{pmatrix} - \mu.$$

If we set $A_\pm = \sum_{k=1}^{2n} d_k \left(\pm\mathrm{i}\partial_x\right)^k + \zeta(x) - 2|u|^2$, $B = -u^2$, and replaced λ by $\lambda + \mu$, the associated spectral problem fits into the framework of Theorem 2.22. In particular, the uniform resolvent estimates can be used to study the dynamics of the extended LLE close to stationary waves.

We need the following property of self-adjoint operators.

Lemma 2.25 ([80, Chapter 5, Section 5]). *Let $A : D(A) \subset H \to H$ be a self-adjoint operator and $\lambda \in \rho(A)$ be in the resolvent set of A. Then*

$$\|(A - \lambda)^{-1}\|_{H \to H} = \frac{1}{\operatorname{dist}(\sigma(A), \lambda)}.$$

Proof of Theorem 2.22. We follow the strategy in [51] Section 3. Let us assume that $A_\pm$ are both bounded from below, i.e., $A_\pm \geq -\gamma$ for $\gamma > 0$. The proof for the case that $A_\pm$ is bounded from above is similar. We write the spectral problem (2.19) as the system

$$\begin{cases} (A_+ + \lambda_i - \mathrm{i}\lambda_r)\phi_1 + B\phi_2 = \psi_1, \\ (A_- - \lambda_i + \mathrm{i}\lambda_r)\phi_2 + B^*\phi_1 = \psi_2, \end{cases} \tag{2.20}$$

where we have replaced $\mathrm{i}\psi_1$ by ψ_1 and $-\mathrm{i}\psi_2$ by ψ_2. Now, consider the case $\lambda_i > 0$. By the previous lemma we infer that for all $\lambda_i \geq 2\gamma$ we find $-\lambda_i, -\lambda_i + \mathrm{i}\lambda_r \in \rho(A_+)$ and

$$\|(A_+ + \lambda_i)^{-1}\|_{H\to H}, \|(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}\|_{H\to H} \leq \frac{2}{\lambda_i}, \quad \|(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}\|_{H\to H} \leq \frac{1}{|\lambda_r|}.$$

In particular, we can solve the first equation in (2.20) for ϕ_1:

$$\phi_1 = -(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1} B\phi_2 + (A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}\psi_1$$

and substituting the expression for ϕ_1 into the second equation amounts to

$$(A_- - \lambda_i + \mathrm{i}\lambda_r)\phi_2 - B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}B\phi_2 = -B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}\psi_1 + \psi_2.$$

By the resolvent identity we find

$$(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1} - (A_+ + \lambda_i)^{-1} = \mathrm{i}\lambda_r(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}(A_+ + \lambda_i)^{-1}$$

and consequently

$$\begin{aligned}\big(A_- - B^*(A_+ + \lambda_i)^{-1}B - \lambda_i + \mathrm{i}\lambda_r\big)\phi_2 - \mathrm{i}\lambda_r B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}(A_+ + \lambda_i)^{-1}B\phi_2 \\ = -B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}\psi_1 + \psi_2.\end{aligned}$$

The operator $B^*(A_+ + \lambda_i)^{-1}B : H \to H$ is bounded and symmetric and thus

$$\mathcal{A} := A_- - B^*(A_+ + \lambda_i)^{-1}B : D \to H$$

is self-adjoint. From the previous lemma on resolvent bounds of self-adjoint operators we have $\|(\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1}\|_{H\to H} \leq |\lambda_r|^{-1}$. Hence, we infer

$$\begin{aligned}&\big[I - \mathrm{i}\lambda_r(\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1}B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}(A_+ + \lambda_i)^{-1}B\big]\phi_2 \\ &\quad = -(\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1}B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}\psi_1 + (\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1}\psi_2.\end{aligned}$$

Now observe that

$$\|\mathrm{i}\lambda_r(\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1}B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}(A_+ + \lambda_i)^{-1}B\|_{H\to H} \lesssim \lambda_i^{-2}.$$

Thus for $\lambda_i > 0$ sufficiently large the operator

$$I - \mathrm{i}\lambda_r(\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1}B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}(A_+ + \lambda_i)^{-1}B : H \to H$$

is invertible as a small perturbation of the identity and

$$\left\| \left[I - \mathrm{i}\lambda_r(\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1} B^*(A_+ + \lambda_i - \mathrm{i}\lambda_r)^{-1}(A_+ + \lambda_i)^{-1} B \right]^{-1} \right\|_{H \to H} \leq \frac{1}{2}$$

uniformly in $\lambda_i \gg 1$. For this reason and using $\|(\mathcal{A} - \lambda_i + \mathrm{i}\lambda_r)^{-1}\|_{H \to H} \leq |\lambda_r|^{-1}$ we find

$$\|\phi_2\| \lesssim |\lambda_r|^{-1}(\|\psi_1\| + \|\psi_2\|)$$

as well as

$$\|\phi_1\| \lesssim |\lambda_r|^{-1}(\|\psi_1\| + \|\psi_2\|)$$

and both estimates are independent of λ_i provided $\lambda_i \gg 1$. In summary for $\lambda = \lambda_r + \mathrm{i}\lambda_i \in \mathbb{C}$ with $\lambda_r \neq 0$ and $\lambda_i \gg 1$ the resolvent in (2.19) exists and is uniformly bounded in λ_i. In the same way one can show the existence and boundedness of the resolvent for $\lambda_i \ll -1$ and the claim follows. □

Asymptotical orbital stability

The proof of the nonlinear stability follows as a direct consequence of the exponential decay estimate in Lemma 2.21 and Theorem 4.3.5 in [78] applied to (2.5). Notice that the nonlinearity in (2.5) is locally Lipschitz, so that all assumptions of the theorem in [78] are satisfied. In conclusion, the solitary wave bifurcating from ϕ_{θ_0} with $\sin\theta_0 > 0$ is asymptotically orbitally stable against localized perturbations in $H^1(\mathbb{R})$ and the proof of Theorem 2.9 is completed.

3 Existence and stability of soliton-based frequency combs in the Lugiato-Lefever equation

This chapter is a reprint of the article [16] written by the author of the thesis in collaboration with Björn de Rijk and submitted for publication. The article was adapted to fit the layout of this thesis.

Abstract

Kerr frequency combs are optical signals consisting of a multitude of equally spaced excited modes in frequency space. They are generated in optical microresonators pumped by a continuouse-wave laser. It has been experimentally observed that the interplay of Kerr nonlinearity and dispersion in the microresonator can lead to a stable optical signal consisting of a periodic sequence of highly localized ultra-short pulses, resulting in broad frequency spectrum. The discovery that stable broadband frequency combs can be generated in microresonators has unlocked a wide range of promising applications, particularly in optical communications, spectroscopy and frequency metrology. In its simplest form, the physics in the microresonator is modeled by the Lugiato-Lefever equation, a damped nonlinear Schrödinger equation with forcing. In this paper, we rigorously demonstrate that the Lugiato-Lefever equation indeed supports arbitrarily broad Kerr frequency combs by proving the first existence and stability results of periodic solutions consisting of any number of well-separated, strongly localized and highly nonlinear pulses on a single periodicity interval. We realize these periodic multi-soliton solutions as concatenations of individual bright cavity solitons by phrasing the problem as a reversible dynamical system and employing results from homoclinic bifurcation theory. The spatial dynamics formulation enables us to harness general results, based on Evans-function techniques and Lin's method, to rigorously establish diffusive spectral stability. This, in turn, yields nonlinear stability of the periodic multi-soliton solutions against localized and subharmonic perturbations.

3.1 Introduction

In this paper we rigorously construct periodic multi-soliton solutions to the Lugiato-Lefever equation and determine their spectral and nonlinear stability. The Lugiato-Lefever equation (LLE) is a damped and forced nonlinear Schrödinger equation given by

$$\mathrm{i}u_t = -du_{xx} + \zeta u - |u|^2 u - \mathrm{i}u + \mathrm{i}f, \tag{3.1}$$

where $u\colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a complex-valued function, $d \neq 0$ denotes the dispersion, $\zeta \in \mathbb{R}$ is a detuning parameter and $f > 0$ represents the forcing. The LLE was derived from Maxwell's equations in [89] to describe the optical field in a dissipative and nonlinear cavity filled with a Kerr medium and subjected to a continuous-wave laser pump. As such it serves as a canonical model, see [29] and further references therein, for Kerr frequency comb generation in continuous-wave laser driven microresonators, which are microscopic ring- or disk-shaped cavities that confine light by circulating it in a closed path and enhance the interaction of light through resonance. Kerr frequency combs, which are optical signals whose frequency spectrum is carved into a series of regularly spaced δ-functions, arise due to four-wave mixing mediated by the nonlinear Kerr effect in the microresonator.

Over the past decades the experimental generation of combs with broad frequency spectrum has become the subject of intensive research, mainly due to the fact that such broad bandwidth frequency combs have revolutionized the precision and accuracy with which different optical transition frequencies can be measured, a discovery that was awarded with the Nobel prize in physics and has promising applications to optical communications [92], broadband gas sensing [134], spectroscopy [32, 113], and frequency metrology [144], to name but a few. The generation of broadband frequency combs in high-quality microresonators has sparked significant interest [38, 49], mainly due to the potential of chip-scale implementation, which facilitates the integration of frequency comb technology into applications outside the laboratory. As high sensitivity to noise is undesired for practical implementation, attention must be given to the stability of these combs.

A breakthrough addressing the stability issue is the experimental realization [20, 73] of frequency combs comprised of a multitude of well-separated bright (cavity) solitons, which are remarkably stable thanks to a double balance between anomalous dispersion and the Kerr nonlinearity (which defines their shape) and between gain and dissipation (which defines their amplitude). The individual solitons correspond to ultrashort pulses whose frequency spectrum is broad and smooth, which, together with their stability properties, makes soliton-based frequency combs highly attractive for applications, see e.g. [92, 150] and further references therein.

In experiments [20, 73] the number of solitons constituting the generated frequency comb turns out to be stochastic, but, once generated, the waveform is stable. That is, on a single periodicity interval $I \subset \mathbb{R}$, provided by the ring (or disk) shape of the microresonator, stable optical signals of any number $N \in \mathbb{N}$ of ultrashort, well-separated, pulses can be generated. In simulations in [73] of the microresonator system with parameters similar to the experimental setup a close-to-perfect match was found between the numerical solution and the formal approximate solution

$$u(x) = \alpha_{CW} + \sum_{i=1}^{N} \phi_\theta(x - X_i), \tag{3.2}$$

corresponding to $N \in \mathbb{N}$ solitons superposed on a background $\alpha_{CW} \in \mathbb{C}$. Here, $x \in I$ resembles the angular coordinate inside the resonator, $X_i \in I$ represents the position of the i-th soliton and

$$\phi_\theta(x) = \sqrt{2\zeta}\,\mathrm{sech}\left(\sqrt{\frac{\zeta}{d}}\,x\right)\mathrm{e}^{\mathrm{i}\theta} \tag{3.3}$$

is the well-known bright soliton with phase $\theta \in \mathbb{R}$ solving the focusing nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t = -du_{xx} + \zeta u - |u|^2 u \tag{3.4}$$

with dispersion and detuning parameters $d, \zeta > 0$.

Subsequent bifurcation analyses [54, 62, 91, 102] based on numerical continuation indicate that frequency-comb solutions consisting of a multitude of bright solitons on a periodicity interval can be found in the LLE (3.1) in the anomalous dispersion regime $d > 0$. More precisely, these numerical analyses suggest that, as a result of a snaking bifurcation, the LLE supports periodic soliton solutions comprised of any number of solitons on a single periodicity interval.

In this paper, we affirm the above experimental and numerical findings by providing the first rigorous proof that the LLE (3.1) supports stable soliton-based frequency combs, whose leading-order profiles are of the form (3.2) on a single periodicity interval. Specifically, we construct even stationary periodic N-soliton solutions to (3.1), allowing for arbitrary $N \in \mathbb{N}$, in which $\lfloor \frac{N}{2} \rfloor$ inter-soliton distances can be tuned independently, permitting unequally spaced soliton configurations. We establish their nonlinear stability with respect to both localized and subharmonic perturbations, yielding the strongest known stability results for frequency-comb solutions to the LLE. We refer to §3.1.2 for further details.

The periodic N-soliton solutions to (3.1) arise in the anomalous dispersion regime $d > 0$ with small damping and forcing. In this regime stable 1-soliton solutions to (3.1) on $\mathbb{R}$ were constructed in [15] by bifurcating from the rotated bright soliton solution (3.3) to the focusing NLS equation (3.4). Thus, we regard the LLE as a perturbed focusing NLS equation

$$\mathrm{i}u_t = -u_{xx} + \zeta u - |u|^2 u + \varepsilon \mathrm{i}(-u + f) \tag{3.5}$$

with parameters $\zeta, f > 0$ and $0 < \varepsilon \ll 1$, where we have set the dispersion to 1 by rescaling space. We note that, given a dispersion coefficient $d > 0$, solutions of (3.5) are in 1-to-1-correspondence with solutions of the original formulation (3.1) of the LLE, see Remark 3.3.

3.1.1 Embedding in the mathematical literature

Before stating our main result, we provide an overview of prior mathematically rigorous studies on periodic solutions to the LLE. The first mathematical works [36, 37, 61, 94] focus on proving the existence of small amplitude periodic solutions of (3.1) by bifurcating from spatially homogeneous steady states. These stationary periodic solutions are weakly nonlinear patterns, i.e., their leading-order profile is a (co)sine wave superposed on the homogeneous background state. In particular, they do not exhibit broad frequency spectrum.

To date, far-from-equilibrium periodic solutions have, to the authors' best knowledge, only been established rigorously in [67, 91]. In [91] global branches of stationary periodic solutions and bounds on their location in parameter space were obtained using global bifurcation theory. Yet, the results in [91] do not provide any rigorous mathematical control on the profile or size of the periodic solutions away from the branch of homogeneous background states.

In [67] stationary periodic solutions to (3.1) are constructed by bifurcating from the well-known one-parameter family of real-valued periodic dnoidal solutions of the focusing NLS equation (3.4). The homoclinic limit of the real dnoidal family is the bright soliton (3.3) with $\theta \in \{0, \pi\}$. As the dnoidal waves approach the homoclinic limit, their profile thus consists of one strongly localized bright soliton on each periodicity interval. Consequently, the bifurcating solutions to the LLE resemble periodic 1-soliton solutions, which exhibit broad frequency spectrum. These bifurcating periodic 1-soliton solutions are however different from the ones constructed in this paper, because they are unstable against localized and long-wavelength subharmonic perturbations, whereas the soliton trains constructed in this paper are stable against localized perturbations and against subharmonic perturbations of any wavelength. We refer to Remark 3.1 for further details.

As far as the authors are aware, the only class of periodic solutions to (3.1) whose stability against localized perturbations and against subharmonic perturbations of any wavelength has been rigorously established in the current literature [70, 137], are the small-amplitude, weakly nonlinear Turing patterns constructed in [36, 37]. This follows by the fact that they are diffusively spectrally stable, as proved in [36, 37]. On the other hand, spectral and nonlinear stability of small amplitude periodic solutions of (3.1) against co-periodic perturbations is obtained in [94, 95]. Finally, although the periodic solutions to (3.1) bifurcating from the family of dnoidal NLS-solutions are unstable against localized perturbations and large-wavelength subharmonic perturbations, their spectral and nonlinear stability against co-periodic perturbations is proven in [67, 137], thereby confirming the formal asymptotic analysis presented in [139].

Remark 3.1. The one-parameter family of real-valued periodic dnoidal solutions of the focusing NLS equation (3.4) can be extended to a two-parameter family of stationary periodic solutions through rotation. Indeed, if u is a stationary solution of (3.4), so is $\mathrm{e}^{\mathrm{i}\theta}u$ for any $\theta \in \mathbb{R}$. For each fixed rotation angle $\theta \in \mathbb{R}$ the homoclinic limit of the family is given by the rotated bright solition (3.3).

It is a classical result that any nonconstant real-valued stationary solution of period $T > 0$ of the focusing NLS equation (3.4) is long-wavelength (or sideband) unstable [122], i.e., it is spectrally unstable against MT-periodic perturbations for $M \in \mathbb{N}$ sufficiently large. In particular, any real-valued dnoidal solution of (3.4) is long-wavelength unstable. Since the spectrum is unaffected by the rotation $u \mapsto \mathrm{e}^{\mathrm{i}\theta}u$, it follows that the full two-parameter family of dnoidal waves is sideband unstable.

The long-wavelength instability is inherited by periodic solutions of the LLE bifurcating from any member of the two-parameter dnoidal family. The reason is as follows. The linearization of (3.1) or (3.4) about a stationary T-periodic solution posed on $L^2_{\mathrm{per}}(0, MT)$ has compact resolvent for any $M \in \mathbb{N}$ due to the compact embedding of its domain $H^2_{\mathrm{per}}(0, MT)$ into $L^2_{\mathrm{per}}(0, MT)$ by the Rellich–Kondrachov theorem. Consequently, its spectrum consists of isolated eigenvalues of finite multiplicity. It is well-known, cf. [80, Section 4.3.5], that a finite set of eigenvalues of finite multiplicity changes continuously under bounded perturbations. Therefore, stationary T-periodic solutions to (3.1) bifurcating from any T-periodic dnoidal solution of (3.4) are long-wavelength unstable. By Floquet-Bloch theory, cf. [52, 69], the spectrum of the linearization of (3.1) about a T-periodic stationary wave posed on $L^2(\mathbb{R})$ arises by taking the union over $M \in \mathbb{N}$ of all spectra of the linearizations of (3.1) about the wave posed on $L^2_{\mathrm{per}}(0, MT)$. Hence, any long-wavelength unstable periodic stationary solution is also spectrally unstable against localized perturbations. We conclude that the periodic stationary solutions of the LLE, which were established in [67] by bifurcating from the dnoidal waves, are (spectrally) unstable against localized perturbations and against MT-periodic perturbations for $M \in \mathbb{N}$ sufficiently large. This is confirmed by numerical simulations in [139]. Interestingly, these simulations also indicate that, outside of a neighborhood of the bifurcation point, the $L^2_{\mathrm{per}}(0, MT)$-spectrum might stabilize for given $M \in \mathbb{N}$.

However, the homoclinic limits, given by the rotated bright solitons (3.3), of the two-parameter dnoidal family are spectrally and orbitally stable [26, 151, 152] as solutions to the NLS equation (3.4) against localized perturbations. It has been shown in [15] that the spectral stability is inherited by some of the bifurcating soliton solutions of (3.5), see also Theorem 3.6. In this paper, we exploit the spectral stability of the soliton solutions to construct periodic multi-soliton solutions, which are spectrally and nonlinearly stable against localized perturbations and against subharmonic perturbations of any wavelength.

3.1.2 Main result

Our main result may now be formulated as follows.

Theorem 3.2. *Fix $N \in \mathbb{N}$. Set $n = \lfloor \frac{N}{2} \rfloor$ and $\alpha_0 = N \mod 2 \in \{0,1\}$. Assume that $\zeta, f > 0$ and $\theta_0 \in \mathbb{R}$ obey $8\zeta < \pi^2 f^2$, $\pi f \cos\theta_0 = 2\sqrt{2\zeta}$ and $\sin\theta_0 > 0$. Then, there exist constants $C_0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, $\mathbf{k} = (k_1,\ldots,k_n) \in \mathbb{N}^n$ and $m \in \mathbb{N}$ there exist distances $T_{1,\varepsilon}^{k_1},\ldots,T_{n,\varepsilon}^{k_n} > 0$ and periods $L_{\mathbf{k}}^m > 0$ satisfying*

$$2\sum_{i=1}^{n} T_{i,\varepsilon}^{k_i} < L_{\mathbf{k}}^m \tag{3.6}$$

such that equation (3.5) admits a stationary smooth solution $u_{m,\mathbf{k},\varepsilon} \colon \mathbb{R} \to \mathbb{C}$ enjoying the following properties:

(i) **(Symmetry).** *The solution $u_{m,\mathbf{k},\varepsilon}$ is even, i.e., it holds $u_{m,\mathbf{k},\varepsilon}(x) = u_{m,\mathbf{k},\varepsilon}(-x)$ for all $x \in \mathbb{R}$, $\varepsilon \in (0,\varepsilon_0)$, $\mathbf{k} \in \mathbb{N}^n$ and $m \in \mathbb{N}$.*

(ii) **(Periodicity).** *The solution $u_{m,\mathbf{k},\varepsilon}$ is $L_{\mathbf{k},\varepsilon}^m$-periodic for each $\varepsilon \in (0,\varepsilon_0)$, $\mathbf{k} \in \mathbb{N}^n$ and $m \in \mathbb{N}$. For each fixed $\varepsilon \in (0,\varepsilon_0)$ and $\mathbf{k} \in \mathbb{N}^n$ the sequence $\{L_{\mathbf{k},\varepsilon}^m\}_m$ of periods is monotonically increasing and tends to ∞ as $m \to \infty$.*

(iii) **(Approximation).** *On a single periodicity interval the solution $u_{m,\mathbf{k},\varepsilon}$ is approximated by a superposition of N rotated bright solitons of the form (3.3) as*

$$\Big| u_{m,\mathbf{k},\varepsilon}(x) - \alpha_0\phi_{\theta_0}(x) - \sum_{i=1}^{n}\left(\phi_{\theta_0}\left(x - T_{1,\varepsilon}^{k_1} - \ldots - T_{i,\varepsilon}^{k_i}\right) + \phi_{\theta_0}\left(x + T_{1,\varepsilon}^{k_1} + \ldots + T_{i,\varepsilon}^{k_i}\right)\right)\Big| \leq C_0\varepsilon$$

for $x \in [-\frac{1}{2}L_{\mathbf{k},\varepsilon}^m, \frac{1}{2}L_{\mathbf{k},\varepsilon}^m]$, $\varepsilon \in (0,\varepsilon_0)$, $\mathbf{k} \in \mathbb{N}^n$ and $m \in \mathbb{N}$.

(iv) **(Soliton separation).** *For all* $i = 1, \dots, n$ *and* $\varepsilon \in (0, \varepsilon_0)$ *the sequence* $\{T^k_{i,\varepsilon}\}_k$ *of soliton distances is monotonically increasing with* $T^k_{i,\varepsilon} \to \infty$ *as* $k \to \infty$.

(v) **(Asymptotic orbital stability against subharmonic perturbations).** *Let* $\varepsilon \in (0, \varepsilon_0)$, $\mathbf{k} \in \mathbb{N}^n$ *and* $m, M \in \mathbb{N}$. *There exist constants* $C, \delta, \eta > 0$ *such that for all* $v_0 \in H^1_{\text{per}}(0, ML^m_{\mathbf{k},\varepsilon})$ *with* $\|v_0\|_{H^1_{\text{per}}(0,ML^m_{\mathbf{k},\varepsilon})} < \delta$ *there exist a constant* $\gamma \in \mathbb{R}$ *and a global (mild) solution*

$$u \in C\big([0,\infty), H^1_{\text{per}}(0, ML^m_{\mathbf{k},\varepsilon})\big)$$

of (3.5) *with initial condition* $u(0) = u_{m,\mathbf{k},\varepsilon} + v_0$ *satisfying*

$$|\gamma| \leq C\|v_0\|_{H^1_{\text{per}}(0,ML^m_{\mathbf{k},\varepsilon})}, \qquad \|u(\cdot,t) - u_{m,\mathbf{k},\varepsilon}(\cdot+\gamma)\|_{H^1_{\text{per}}(0,ML^m_{\mathbf{k},\varepsilon})} \leq C\mathrm{e}^{-\eta t}\|v_0\|_{H^1_{\text{per}}}$$

for $t \geq 0$.

(vi) **(Diffusive stability against localized perturbations).** *Let* $\varepsilon \in (0, \varepsilon_0)$, $\mathbf{k} \in \mathbb{N}^n$ *and* $m \in \mathbb{N}$. *There exist constants* $C, \delta > 0$ *such that for all* $v_0 \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$ *with* $\|v_0\|_{L^1 \cap H^4} < \delta$ *there exist functions*

$$\gamma, v \in C\big([0,\infty), H^4(\mathbb{R})\big) \cap C^1\big([0,\infty), H^2(\mathbb{R})\big)$$

satisfying $\gamma(0) = 0$ *and* $v(0) = v_0$ *such that* $u = u_{m,\mathbf{k},\varepsilon} + v$ *is the unique global classical solution of* (3.5) *with* $u(0) = u_{m,\mathbf{k},\varepsilon} + v_0$. *Moreover, the estimates*

$$\begin{aligned}\|\gamma(t)\|_{L^2}, \|u(t) - u_{m,\mathbf{k},\varepsilon}\|_{L^2} &\leq C(1+t)^{-\frac{1}{4}}\|v_0\|_{L^1\cap H^4},\\ \|u(\cdot,t) - u_{m,\mathbf{k},\varepsilon}(\cdot + \gamma(\cdot,t))\|_{L^2} &\leq C(1+t)^{-\frac{3}{4}}\|v_0\|_{L^1\cap H^4}\end{aligned}$$

hold for $t \geq 0$.

Theorem 3.2 shows that for any $N \in \mathbb{N}$ there exist even periodic stationary solutions to (3.5) composed of a superposition of N bright solitons on a single periodicity interval. The period length, as well as the distance between the individual solitons, can be chosen arbitrarily large through the $n+1$ degrees of freedom $k_1, \dots, k_n$ and m in Theorem 3.2, where we set $n = \lfloor \frac{N}{2} \rfloor$. More precisely, after fixing $k_1, \dots, k_i \in \mathbb{N}$ so that the distances $T^{k_1}_{1,\varepsilon}, \dots, T^{k_i}_{i,\varepsilon}$ between the first $i \in \{1, \dots, n\}$ solitons and their symmetric counterparts are set, there remain $n+1-i$ degrees of freedom, namely $k_{i+1}, \dots, k_n, m \in \mathbb{N}$, which can still be adjusted to make the distances between the remaining $N - 2i$ solitons, as well as the period of the solution, arbitrarily large. In particular, for fixed parameters $\zeta, f > 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, Theorem 3.2 yields a $(n+1)$-parameter family of even periodic stationary N-soliton solutions of (3.5), whose soliton locations can be unequally spaced. Since any fixed spatial translate of a solution of (3.5) is again a solution, we find that Theorem 3.2 provides in fact a $(n+2)$-parameter family of periodic stationary solutions to (3.5).

Upon rescaling the spatial variable x, the optical field u, and the detuning and forcing parameters $\zeta, f > 0$ in (3.5) and upon reintroducing the dispersion parameter $d > 0$, we can adjust the period length to match the (fixed) circumference of the microresonator and we can normalize the damping coefficient, see Remark 3.3. In particular, one finds after rescaling that the periodic N-soliton solutions in Theorem 3.2 correspond to frequency-comb solutions of (3.1) in the regime of large detuning, strong forcing and high anomalous dispersion. These frequency combs are comprised of ultrashort, well-separated, large-amplitude bright solitons and were thus numerically and experimentally observed in [20, 54, 62, 73, 91, 102]. In fact, the amplitude of the frequency-comb solution to (3.1) can be made arbitrarily large by taking $\varepsilon > 0$ sufficiently small in Theorem 3.2, cf. Remark 3.3. On the other hand, by fixing $\varepsilon > 0$ and taking the parameters $k_1, \ldots, k_n$ and m in Theorem 3.2 sufficiently large, the individual solitons constituting the frequency comb can be arbitrarily localized, while maintaining the same amplitude. Therefore, their frequency spectrum can be made arbitrarily broad.

The periodic multi-soliton solutions, or *multi-soliton trains*, established in Theorem 3.2, exhibit the strongest stability properties attainable for (nonconstant) stationary periodic solutions of (3.5). Since any spatial translate corresponds to a co-periodic perturbation, $u_{m,\mathbf{k},\varepsilon}$ cannot be asymptotically stable against subharmonic perturbations. However, Theorem 3.2 shows that, except for a small spatial shift of the original periodic solution, the effect of subharmonic perturbations fades exponentially quickly in time. In addition, by Floquet-Bloch theory, cf. [52, 69], the linearization of (3.5) about $u_{m,\mathbf{k},\varepsilon}$ possesses, when posed on $L^2(\mathbb{R})$, continuous spectrum, which, in the most stable situation, touches the origin due to translational invariance in a single quadratic tangency. It is well-known that such *diffusively stable* spectrum, cf. Definition 3.17, leads to algebraic decay rates of perturbations, whose leading-order behavior is captured by a diffusively decaying spatio-temporal phase modulation satisfying a viscous Hamilton-Jacobi equation, cf. [41, 70, 125, 159]. That is, the algebraic decay rates stated in Theorem 3.2 are the best achievable in the case of localized perturbations.

In summary, Theorem 3.2 rigorously shows that the LLE admits for any number $N \in \mathbb{N}$ an $(\lfloor \frac{N}{2} \rfloor + 2)$-parameter family of frequency combs. These combs are periodic solutions of (3.1) consisting of N well-separated, generally unequally spaced bright solitons on a single periodicity interval. The amplitude and frequency spectra of these solitions can be made arbitrarily large and broad, respectively. Moreover, the frequency combs exhibit the best attainable stability properties. These features are, as outlined above, of key importance for applications relying on frequency-comb technology. Moreover, as explained in §3.1.1 below, Theorem 3.2 is the first rigorous mathematical result establishing periodic *multiple* soliton solutions of the LLE. In addition, the solutions in Theorem 3.2 are the first far-from-equilibrium periodic solutions of the LLE, whose stability against localized perturbations and against subharmonic perturbations of any wavelength has been rigorously proven.

Remark 3.3. Let $d > 0$. If $u(x,t)$ is a solution to (3.5), then the rescaled solution $\tilde{u}(x,t) = \varepsilon^{-1/2}u((d\varepsilon)^{-1/2}x, \varepsilon^{-1}t)$ solves

$$\mathrm{i}\tilde{u}_t = -d\tilde{u}_{xx} + \tilde{\zeta}\tilde{u} - |\tilde{u}|^2\tilde{u} - \mathrm{i}\tilde{u} + \mathrm{i}\tilde{f}$$

with $\tilde{\zeta} = \varepsilon^{-1}\zeta$ and $\tilde{f} = \varepsilon^{-1/2}f$. Hence, we find that solutions of (3.5) are in 1-to-1-correspondence with solutions of the original formulation (3.1) of the LLE with normalized damping coefficient and dispersion coefficient $d > 0$. In particular, the solutions $u_{m,\mathbf{k},\varepsilon}$, established in Theorem 3.2, correspond to stable soliton-train solutions of (3.1) of period $(d\varepsilon)^{1/2}L^m_{\mathbf{k},\varepsilon}$, whose amplitude can be made arbitrarily large by taking $\varepsilon > 0$ sufficiently small. Moreover, choosing $d = 1/(\varepsilon L^m_{\mathbf{k},\varepsilon})^{1/2}$ yields 1-periodic multi-soliton solutions to (3.1), whose individual solitons become highly localized and well-separated by taking $k_1, \ldots, k_n, m \in \mathbb{N}$ sufficiently large.

3.1.3 Dynamical systems approach

The results presented in this paper are the outcome of a dynamical systems approach to analyze the existence and spectral stability problems for stationary solutions to (3.5). These problems, being independent of time, may be written as first-order dynamical systems of ordinary differential equations in the spatial variable x. Due to the reflection symmetry $x \mapsto -x$ present in (3.5), these dynamical systems admit a reversible symmetry.

Our basic ingredients for the construction of periodic multi-soliton solution to (3.5) are the 1-soliton solutions bifurcating from the one-parameter family of rotated bright NLS solitons (3.3). These primary 1-soliton solutions to (3.5) were rigorously established in [15, 55, 57] and their spectral and nonlinear stability was analyzed in [15] using Krein index counting and analytic perturbation theory. Upon formulating the existence problem as a reversible dynamical system, the 1-solitons correspond to symmetric nondegenerate homoclinics to a saddle-focus equilibrium. Homoclinic bifurcations results [28, 72, 127] for reversible dynamical systems, which rely on Shil'nikov analysis or a Lyapunov-Schmidt reduction method called Lin's method [87, 123], allow us to concatenate any number $N \in \mathbb{N}$ of these nondegenerate homoclinics, yielding so-called N-*homoclinics*, which are again nondegenerate and symmetric. The N-homoclinics correspond to even stationary multiple soliton solutions of (3.5) comprised of N well-separated solitons, which can, in turn, be approximated by the rotated bright NLS solitons (3.3). The spectral stability of these N-soliton solutions follows by combining results from [17, 126, 127] with a-priori bounds on the spectrum. More specifically, in [126] general eigenvalue problems arising in the spectral stability analysis of N-pulses bifurcating from a formal concatenation of N primary pulses are studied using Lin's method, providing leading-order control over the N small eigenvalues bifurcating from the translational eigenvalue residing at the origin. An application of the theory of [126] to reversible systems can be found in [127], see also [124]. On the other hand, the Evans-function analysis in [17] yields that the absence of eigenvalues in compact regions of the spectral plane associated with the primary 1-soliton solutions is inherited by the bifurcating N-soliton solutions. We note that the spectral stability of the multi-soliton solutions implies their asymptotic orbital stability through standard arguments relying on high-frequency resolvent bounds established in [15]. We emphasize that these rigorous existence and stability results of multiple soliton solutions to the LLE are novel and interesting in their own right. We refer to §3.2.3 and §3.3.2 for the precise statements.

Having established a nondegenerate N-homoclinic in the dynamical systems formulation of the existence problem, we employ the homoclinic bifurcation results in [147], which again rely on Lin's method, to find nearby periodic orbits, which have large spatial periods $T > 0$ and are reversibly symmetric. The bifurcating periodic orbits correspond to a family of periodic N-soliton solutions to (3.5). We study the spectral stability of these T-periodic soliton solutions against localized perturbations and subharmonic perturbations using [17, 53, 131]. There are M eigenvalues of the linearization of (3.5) about the multi-soliton train posed on $L^2_{\mathrm{per}}(0, MT)$ bifurcating from each isolated eigenvalue associated with the underlying multi-soliton solution, cf. [52]. Since there is an eigenvalue associated with the underlying multi-soliton solution at the origin due to translational invariance, the multi-soliton train can be spectrally unstable even in the case of spectral stability of the underlying multi-soliton. In the case of localized perturbations each eigenvalue associated with the underlying multi-soliton solution yields a bifurcating spectral curve consisting of the union of eigenvalues of the linearizations posed on $L^2_{\mathrm{per}}(0, MT)$ for each $M \in \mathbb{N}$. Leading-order control on the bifurcating eigenvalues in the case of subharmonic perturbations and on the bifurcating spectral curve in the case of localized perturbations close to the origin is provided by results in [131], which rely on Lin's method and Floquet-Bloch theory. On the other hand, the Evans function analysis in [17, 53] yield that the absence of eigenvalues in compact regions of the spectral plane associated with the underlying multi-soliton solution is inherited by the bifurcating multi-soliton trains. Combining the latter with spectral a-priori bounds then leads to the desired diffusive spectral stability result for the periodic multi-soliton solution. Finally, diffusive spectral stability implies nonlinear stability against localized perturbations and against subharmonic perturbations of any wavelength, cf. [70, 71, 137].

3.1.4 Outline of paper

The remainder of this paper is structured as follows. In §3.2 we collect previous results on the existence and spectral stability of the 1-soliton solutions to the LLE bifurcating from the rotated bright NLS-solitons. Moreover, we formulate the existence problem as a dynamical system and establish multiple and periodic soliton solutions. The spectral and nonlinear stability analysis of the multiple and periodic soliton solutions, as well as the proof of our main result, Theorem 3.2, can be found in §3.3.

3.2 Soliton-based pulse solutions

In this section we establish multiple and periodic soliton solutions to (3.5). As outlined in §3.1.3, the fundamental building blocks of these soliton solutions are the stationary 1-soliton solutions of (3.5), constructed in [15, 55, 57] by bifurcating from the 1-parameter family of rotated bright NLS solitions (3.3). After we have formulated the existence problem as a dynamical system, we collect the relevant properties of these primary 1-soliton solutions from [15] and show that they correspond to nondegenerate symmetric homoclinics in the dynamical system. Then, we employ homoclinic bifurcation results from [123, 127, 147] to obtain the desired multiple and periodic soliton solutions.

3.2.1 Spatial dynamics formulation

Multiplying (3.5) by $-\mathrm{i}$ and decomposing u into its real and imaginary part yields the real two-component system

$$\mathbf{u}_t = J(-\mathbf{u}_{xx} + \zeta\mathbf{u} - |\mathbf{u}|^2\mathbf{u}) + \varepsilon(-\mathbf{u} + \mathbf{F}), \tag{3.7}$$

written in vector notation $\mathbf{u} = (\mathrm{Re}(u), \mathrm{Im}(u))^\top$, where we denote

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

and where $|\mathbf{u}| = \sqrt{\mathbf{u}_1^2 + \mathbf{u}_2^2}$ is the usual Euclidean norm. The advantage of the fomulation (3.7) over (3.5) is that the nonlinearity is now a (Fréchet) differentiable function of the vector $\mathbf{u}$. We note that any real-valued solution $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)^\top$ to (3.7) gives rise to a complex-valued solution to (3.5) by setting $u = \mathbf{u}_1 + \mathrm{i}\mathbf{u}_2$.

Stationary solutions of (3.7) solve the second-order system

$$J(-\mathbf{u}_{xx} + \zeta \mathbf{u} - (\mathbf{u}_1^2 + \mathbf{u}_2^2)\mathbf{u}) + \varepsilon(-\mathbf{u} + \mathbf{F}) = 0 \tag{3.8}$$

of ordinary differential equations. By introducing the variable

$$U = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1', \mathbf{u}_2')^\top$$

the existence problem (3.8) can be written as a dynamical system on $\mathbb{R}^4$ given by

$$U' = F(U; \varepsilon), \tag{3.9}$$

where $F \colon \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is the smooth nonlinear function defined by

$$F(U; \varepsilon) = \begin{pmatrix} U_3 \\ U_4 \\ \zeta U_1 + \varepsilon U_2 - (U_1^2 + U_2^2)U_1 \\ \zeta U_2 - \varepsilon U_1 - (U_1^2 + U_2^2)U_2 + \varepsilon f \end{pmatrix}.$$

The reflection symmetry $x \mapsto -x$ of (3.5) yields that the first-order system (3.9) of ordinary differential equations is *reversible* for the linear involution

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \qquad R^2 = I.$$

That is, the relation $RF(U; \varepsilon) = -F(RU; \varepsilon)$ holds true for all $U \in \mathbb{R}^4$ and $\varepsilon \in \mathbb{R}$. A solution U of (3.9) is called *symmetric* if we have $RU(-x) = U(x)$ for all $x \in \mathbb{R}$. We note that symmetric solutions of (3.9) give rise to even stationary solutions of (3.5). Since F is smooth, we find that all solutions of (3.9) are smooth by standard local existence theory for ODEs. Consequently, all stationary bounded solutions of (3.7) are smooth.

Taking advantage of the differentiability of the nonlinearity in (3.7), we can linearize system (3.7) about a bounded stationary solution $\underline{\mathbf{u}} = (\underline{\mathbf{u}}_1, \underline{\mathbf{u}}_2)^\top \colon \mathbb{R} \to \mathbb{R}^2$. The linearization of (3.7) about

$\underline{\mathbf{u}}$ equals $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$, where $L(\underline{\mathbf{u}}), \mathcal{L}(\underline{\mathbf{u}})\colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are the closed and densely defined operators given by

$$L(\underline{\mathbf{u}}) = -\partial_x^2 + \zeta - \begin{pmatrix} 3\underline{\mathbf{u}}_1^2 + \underline{\mathbf{u}}_2^2 & 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 \\ 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 & \underline{\mathbf{u}}_1^2 + 3\underline{\mathbf{u}}_2^2 \end{pmatrix}, \qquad \mathcal{L}(\underline{\mathbf{u}}) = JL(\underline{\mathbf{u}}).$$

3.2.2 Primary 1-soliton solutions

Stationary 1-soliton solutions of (3.7) were constructed in [15, 55, 57] by bifurcating from the rotated bright soliton solution (3.3) of the NLS equation (3.4), where the rotation parameter $\theta \in \mathbb{R}$ has to satisfy the bifurcation equation $\pi f \cos\theta = 2\sqrt{2\zeta}$. In the following we collect the relevant details from [15] on the existence and spectral stability of these 1-soliton solutions, which will serve as building blocks for the upcoming construction of stationary multi-soliton solutions to (3.7). In order to formulate the result from [15], we first state the definition of spectral stability for stationary pulse solutions of (3.7), as well as the definition of spectral instability.

Definition 3.4. Let $\mathbf{u}_\infty \in \mathbb{R}^2$ and let $\underline{\mathbf{u}}\colon \mathbb{R} \to \mathbb{R}^2$ be a smooth stationary solution to (3.7) such that $\underline{\mathbf{u}}(x)$ converges to $\mathbf{u}_\infty$ as $x \to \pm\infty$. The stationary pulse solution $\underline{\mathbf{u}}$ to (3.7) is *spectrally stable* if there exists $\tau > 0$ such that

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\tau\} \cup \{0\}$$

and 0 is an algebraically simple eigenvalue of $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$.

Definition 3.5. A smooth stationary bounded solution $\underline{\mathbf{u}}\colon \mathbb{R} \to \mathbb{R}^2$ to (3.7) is *spectrally unstable* if there exists $\lambda \in \sigma(\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon)$ with $\mathrm{Re}(\lambda) > 0$.

It has been proved in [15, Theorem 3] that spectral stability yields nonlinear stability of pulse solutions to (3.7) against L^2-localized perturbations, see also Theorem 3.16. We are now ready to summarize the existence and spectral stability results on 1-soliton solutions from [15].

Theorem 3.6 ([15, Theorem 1,2])**.** *Assume that* $\zeta, f > 0$ *and* $\theta_0 \in \mathbb{R}$ *obey* $f^2\pi^2 > 8\zeta$, $\pi f \cos\theta_0 = 2\sqrt{2\zeta}$ *and* $\sin\theta_0 \neq 0$. *Then, there exist* $C_0, \varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there exist an asymptotic state* $\mathbf{u}_{\infty,\varepsilon} \in \mathbb{R}^2$ *and an even smooth solution* $\underline{\mathbf{u}}_\varepsilon\colon \mathbb{R} \to \mathbb{R}^2$ *of* (3.8) *satisfying*

$$\|\underline{\mathbf{u}}_\varepsilon - \phi_{\theta_0}\|_{L^\infty}, |\mathbf{u}_{\infty,\varepsilon}| \leq C_0\varepsilon, \qquad \underline{\mathbf{u}}_\varepsilon - \mathbf{u}_{\infty,\varepsilon} \in H^2(\mathbb{R}), \tag{3.10}$$

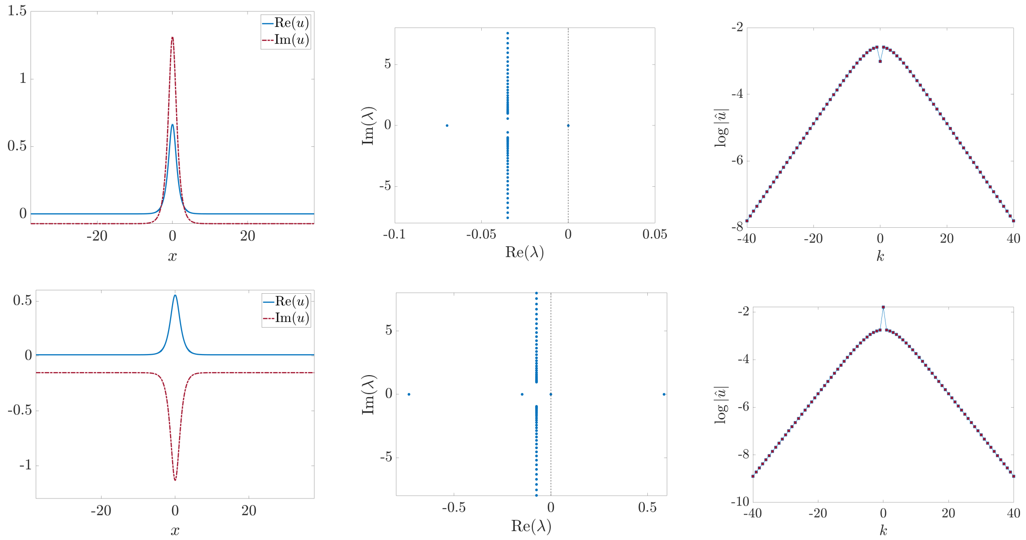

Figure 3.1: Periodic approximations of the primary 1-soliton solutions established in Theorem 3.6 (see also Theorem 3.12) with parameters $\zeta = 1, f = 2$. The solutions were computed with the MATLAB package `pde2path` [146]. The top row shows a stable 1-soliton bifurcating with $\sin\theta_0 > 0$, its spectrum against co-periodic perturbations, and the corresponding frequency comb obtained by plotting $\log|\hat{u}(k)|$ against the Fourier frequency variable k. The bottom row depicts an unstable 1-soliton bifurcating with $\sin\theta_0 < 0$, the associated spectrum, and its frequency comb.

where we denote $\boldsymbol{\phi}_\theta = \phi_0(\cos\theta, \sin\theta)^\top$ *and* ϕ_0 *is the bright soliton given by* (3.3). *The spectral stability of the solution* $\underline{\mathbf{u}}_\varepsilon$ *depends on the rotational angle* θ_0 *as follows.*

(i) If $\sin\theta_0 > 0$*, then* $\underline{\mathbf{u}}_\varepsilon$ *is spectrally stable as a stationary pulse solution to* (3.7).

(ii) If $\sin\theta_0 < 0$*, then* $\underline{\mathbf{u}}_\varepsilon$ *is spectrally unstable as a stationary solution to* (3.7).

In both cases $\ker(\mathcal{L}(\underline{\mathbf{u}}_\varepsilon) - \varepsilon)$ *is spanned by* $\underline{\mathbf{u}}_\varepsilon'$.

We prove that the 1-soliton solutions, established in Theorem 3.6, correspond to nondegenerate homoclinics connecting to a saddle-focus equilibrium in the dynamical system (3.9). The definition of nondegeneracy, cf. [147], reads as follows.

Definition 3.7. A homoclinic solution $\underline{U}$ of (3.9) is called *nondegenerate* if all bounded solutions $U\colon \mathbb{R} \to \mathbb{R}^4$ to the variational problem

$$U' = \partial_U F(\underline{U}(x); \varepsilon)U \qquad (3.11)$$

are given by scalar multiples of $\underline{U}'$.

We exploit the spectral properties of the linearization of (3.7) about the primary 1-soliton to show nondegeneracy of the corresponding homoclinic in (3.9). In particular, the nondegeneracy follows from the fact that 0 is a geometrically simple eigenvalue of the linearization.

Lemma 3.8. *Let* $\mathbf{u}_\infty \in \mathbb{R}^2$ *and let* $\underline{\mathbf{u}} \colon \mathbb{R} \to \mathbb{R}^2$ *be a smooth solution of* (3.8) *such that* $\underline{\mathbf{u}}(x) \to \mathbf{u}_\infty$ *and* $\underline{\mathbf{u}}'(x) \to 0$ *as* $x \to \pm\infty$*. Assume that* 0 *is a geometrically simple eigenvalue of* $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$ *with associated eigenfunction* $\underline{\mathbf{u}}'$*. Then, the corresponding homoclinic solution* $\underline{U} \colon \mathbb{R} \to \mathbb{R}^4$ *to* (3.9) *given by* $\underline{U} = (\underline{\mathbf{u}}, \underline{\mathbf{u}}')^\top$ *is nondegenerate if its asymptotic state* $U_\infty = (\mathbf{u}_\infty, 0)^\top$ *is hyperbolic.*

Proof. First, we note that, since the variational equation (3.11) has smooth coefficients, all of its solutions are smooth. Moreover, if $U = (U_1, U_2, U_3, U_4)^\top$ is an L^2-localized solution of (3.11), then it follows, by expressing derivatives through the equation, that U' and U'' are also L^2-localized. Hence, we infer that $\mathbf{u} = (U_1, U_2)^\top \in H^2(\mathbb{R})$ must lie in $\ker(\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon) = \mathrm{span}\{\underline{\mathbf{u}}'\}$. Consequently, U is a scalar multiple of $\underline{U}'$, implying that $\underline{U}$ is nondegenerate. Thus, in order to prove the result, it suffices to show that all bounded solutions of (3.11) are L^2-localized. This will be achieved with the aid of exponential dichotomies. We first look at the limiting system

$$U' = \partial_U F(U_\infty; \varepsilon) U. \tag{3.12}$$

Since the matrix $\partial_U F(U_\infty; \varepsilon)$ is hyperbolic by assumption, system (3.12) admits an exponential dichotomy on $\mathbb{R}$. Combining the latter with the fact that the coefficient matrix $\partial_U F(\underline{U}(x); \varepsilon)$ of (3.11) converges to $\partial_U F(U_\infty; \varepsilon)$ as $x \to \pm\infty$, we infer that system (3.11) possesses exponential dichotomies on both half-lines $(-\infty, 0]$ and $[0, \infty)$ by [99, Lemma 3.4]. Therefore, every bounded solution of (3.11) is exponentially localized and, thus, lies in $L^2(\mathbb{R})$. □

With the aid of Lemma 3.8, we establish that the primary 1-soliton solutions in Theorem 3.6 correspond to nondegenerate homoclinics in (3.9) connecting to a saddle-focus equilibrium.

Proposition 3.9. *Let* $\underline{\mathbf{u}}_\varepsilon$ *be a stationary* 1*-soliton solution, established in Theorem 3.6. Then, provided* $\varepsilon > 0$ *is sufficiently small,*

$$\underline{U}_\varepsilon = (\underline{\mathbf{u}}_{1,\varepsilon}, \underline{\mathbf{u}}_{2,\varepsilon}, \underline{\mathbf{u}}_{1,\varepsilon}', \underline{\mathbf{u}}_{2,\varepsilon}')^\top$$

is a nondegenerate homoclinic solution to (3.9)*, whose asymptotic state* $U_{\infty,\varepsilon} = \lim_{x \to \pm\infty} \underline{U}_\varepsilon(x)$ *is a saddle-focus equilibrium, i.e.,* $\partial_U F(U_{\infty,\varepsilon}; \varepsilon)$ *has the four eigenvalues* $\pm\alpha \pm \beta\mathrm{i}$ *with* $\alpha, \beta > 0$*.*

Proof. It holds $F(0; 0) = 0$ and $\det(\partial_U F(0; 0)) = \zeta^2 > 0$. So, with the aid of the implicit function theorem, we find a unique equilibrium $U_{\infty,\varepsilon} \in \mathbb{R}^4$ of (3.9) converging to $0 \in \mathbb{R}^4$ as

$\varepsilon \to 0$. On the other hand, since $\underline{\mathbf{u}}_\varepsilon - \mathbf{u}_{\infty,\varepsilon}$ lies in $H^2(\mathbb{R})$ by Theorem 3.6 and functions in $H^1(\mathbb{R})$, being square integrable and uniformly continuous, converge to 0 as $x \to \pm\infty$, it follows that $\underline{U}_\varepsilon$ is a homoclinic solution of (3.9) connecting to the equilibrium $(\mathbf{u}_{\infty,\varepsilon}, 0)^\top \in \mathbb{R}^4$. Since the equilibrium $(\mathbf{u}_{\infty,\varepsilon}, 0)^\top$ converges to 0 as $\varepsilon \to 0$ by Theorem 3.6, it must hold $U_{\infty,\varepsilon} = (\mathbf{u}_{\infty,\varepsilon}, 0)^\top$. One readily observes that the matrix $\partial_U F(U_{\infty,\varepsilon}; \varepsilon) \in \mathbb{R}^{4\times 4}$ possesses four eigenvalues $\pm\nu_{\pm,\varepsilon} \in \mathbb{C}$ satisfying

$$\nu_{\pm,\varepsilon}^2 = \zeta - 2|\mathbf{u}_{\infty,\varepsilon}|^2 \pm \mathrm{i}\sqrt{\varepsilon^2 - |\mathbf{u}_{\infty,\varepsilon}|^4}.$$

Since we have $|\mathbf{u}_{\infty,\varepsilon}| = O(\varepsilon)$ by Theorem 3.6, these four eigenvalues are, provided $\varepsilon > 0$ is sufficiently small, of the form $\pm\alpha \pm \mathrm{i}\beta$ with $\alpha, \beta > 0$, implying that $U_{\infty,\varepsilon}$ is a saddle focus. Finally, using that $U_{\infty,\varepsilon}$ is hyperbolic and 0 is a simple eigenvalue of $\mathcal{L}(\underline{\mathbf{u}}_\varepsilon) - \varepsilon$ by Theorem 3.6, we infer that the homoclinic $\underline{U}_\varepsilon$ is nondegenerate. □

3.2.3 N-soliton solutions

In this subsection we establish stationary multi-soliton solutions to (3.7) composed of any number $N \in \mathbb{N}$ of well-separated primary 1-solitons, which were obtained in Theorem 3.6. We allow for superpositions of 1-solitons bifurcating from bright solitons with different phase rotations. The constructed N-solitons are even and the individual distances between the first $\lfloor \frac{N}{2} \rfloor + 1$ solitons can be chosen arbitrarily large, independently of each other. The linearization of (3.7) about the N-solitons possesses N eigenvalues, which converge to 0 as the distances between the individual solitons tend to ∞. These N small eigenvalues are associated with the translational eigenvalues of the N individual 1-solitons constituting the multi-soliton. We employ [127, Theorem 3.6] and [126, Theorem 1], see also [123, Section 3], to establish for any number $\ell \in \{0, \ldots, N-1\}$ the existence of an N-soliton solution such that there are precisely ℓ stable eigenvalues and $N - 1 - \ell$ unstable eigenvalues among the N small eigenvalues. The N-th eigenvalue resides at the origin and is algebraically simple, implying that the associated N-homoclinic in the dynamical system (3.9) is nondegenerate, cf. Lemma 3.8. In summary, given a sequence of $\lceil \frac{N}{2} \rceil$ primary 1-solitons, an application of [123, Theorem 1] and [127, Theorem 3.6] yield an $(n+1)$-parameter family of associated even N-solitons, where the first $n = \lfloor \frac{N}{2} \rfloor$ parameters regulate the distances between the first $n + 1$ solitons and the last parameter corresponds to the number of stable small eigenvalues.

Theorem 3.10. *Let $N \in \mathbb{N}$ and $\ell \in \{0, \ldots, N-1\}$. Set $n = \lfloor \frac{N}{2} \rfloor$ and $\alpha_0 = N \mod 2 \in \{0, 1\}$. Let $\underline{\mathbf{u}}_1, \ldots, \underline{\mathbf{u}}_{n+\alpha_0}$ be a sequence of stationary 1-soliton solutions, established in Theorem 3.6 and converging to the asymptotic end state $\mathbf{u}_\infty \in \mathbb{R}^2$ as $x \to \pm\infty$. Then, there exist constants*

$\delta_0, C_0 > 0$ such that for each $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ there exist distances $T_{1,\ell}^{k_1}, \ldots, T_{n,\ell}^{k_n} > 0$ such that (3.8) *admits an even smooth solution $\underline{\mathbf{u}}_{\mathbf{k},\ell} \colon \mathbb{R} \to \mathbb{R}^2$ satisfying*

$$\begin{aligned}\Bigg|\underline{\mathbf{u}}_{\mathbf{k},\ell}(x) - \mathbf{u}_\infty - \alpha_0\left(\underline{\mathbf{u}}_{n+\alpha_0}(x) - \mathbf{u}_\infty\right) - \sum_{i=1}^{n}\Big(\underline{\mathbf{u}}_i\left(x - T_{1,\ell}^{k_1} - \ldots - T_{i,\ell}^{k_i}\right) \\ + \underline{\mathbf{u}}_i\left(x + T_{1,\ell}^{k_1} + \ldots + T_{i,\ell}^{k_i}\right) - 2\mathbf{u}_\infty\Big)\Bigg| \leq \frac{C_0}{\min\{k_1, \ldots, k_n\}},\end{aligned} \tag{3.13}$$

for $x \in \mathbb{R}$. The spectrum of $\mathcal{L}(\underline{\mathbf{u}}_{\mathbf{k},\ell}) - \varepsilon$ in the ball $B_{\delta_0}(0)$ consists of N eigenvalues of which ℓ have negative real part and $N-1-\ell$ have positive real part (all counted with algebraic multiplicities). Moreover, 0 is an algebraically simple eigenvalue. In addition, $\underline{U}_{\mathbf{k},\ell} = (\underline{\mathbf{u}}_{\mathbf{k},\ell}, \underline{\mathbf{u}}_{\mathbf{k},\ell}')^\top$ is a nondegenerate homoclinic solution to (3.9)*, whose asymptotic end state $U_\infty = \lim_{x\to\pm\infty} \underline{U}_{\mathbf{k},\ell}(x)$ is a saddle-focus equilibrium. Finally, the sequence $\{T_{i,\ell}^k\}_k$ of pulse distances is monotonically increasing with $T_{i,\ell}^k \to \infty$ as $k \to \infty$ for $i = 1, \ldots, n$.*

Proof. The proof follows by a direct application of the homoclinic bifurcation result [127, Theorem 3.6], with the exception that estimate (3.13) is a consequence of [126, Theorem 1, (2.6)(i)]. Both theorems apply in the setting of the reversible system (3.9) once [127, Hypothesis 3.1, 3.2, and 3.5] are verified. In the following, we will check these hypothesis step by step.

By Theorem 3.6 and Proposition 3.9 we find that $U_j = (\underline{\mathbf{u}}_j, \underline{\mathbf{u}}_j')^\top$ is a symmetric homoclinic solution to (3.9) connecting to the saddle-focus equilibrium $U_\infty = (\mathbf{u}_\infty, 0)^\top$ for $j = 1, \ldots, n + \alpha_0$, which establishes [127, Hypothesis 3.1].

Next, we check that the Melnikov integral $\mathcal{M}$ of [127, Hypothesis 3.5] is not vanishing. Let $j \in \{1, \ldots, n + \alpha_0\}$. By Theorem 3.6 we know that $\lambda = 0$ is an isolated, algebraically simple eigenvalue of the linearization $\mathcal{L}(\underline{\mathbf{u}}_j) - \varepsilon$. Combining this with Lemma 3.8, it follows that U_j is nondegenerate. Since 0 lies in the discrete spectrum, $\mathcal{L}(\underline{\mathbf{u}}_j) - \varepsilon$ is Fredholm of index 0 and 0 must be an isolated, algebraically simple eigenvalue of the adjoint $\mathcal{L}(\underline{\mathbf{u}}_j)^* - \varepsilon$. Consequently, the space of bounded solutions to the adjoint problem $\Psi' = -\partial_U F(U_j; \varepsilon)^* \Psi$ is also one-dimensional. Let Ψ_j^{ad} be a nontrivial element of this space. Direct computations yield that there exists $\boldsymbol{\psi}_j \in \ker(\mathcal{L}(\underline{\mathbf{u}}_j)^* - \varepsilon)$ such that $\Psi_j^{\mathrm{ad}} = (J^* \boldsymbol{\psi}_j', J\boldsymbol{\psi}_j)^\top$. It follows that the Melnikov integral in [127, Hypothesis 3.5] reduces to $\mathcal{M} = \langle \boldsymbol{\psi}_j, \underline{\mathbf{u}}_j' \rangle_{L^2}$, which is non-zero by the simplicity of the eigenvalue 0 of $\mathcal{L}(\underline{\mathbf{u}}_j) - \varepsilon$, verifying [127, Hypothesis 3.5].

Finally, we check [127, Hypothesis 3.2], that is $\lim_{x\to\infty} \mathrm{e}^{2\alpha x}\|U_j(x) - U_\infty\|\|\Psi_j^{\mathrm{ad}}(-x)\| \neq 0$, using a slight adaptation of the arguments in [107, Proposition 1.2]. Let $j \in \{1, \ldots, n+\alpha_0\}$. We observe that $V_j = U_j - U_\infty$ solves the ODE

$$V_j' = A_\infty V_j + R(V_j, \varepsilon)$$

with $A_\infty := \partial_U F(U_\infty, \varepsilon)$, $\sigma(A_\infty) = \{\pm\alpha \pm \mathrm{i}\beta\}$ and $R(V, \varepsilon) := F(V + U_\infty, \varepsilon) - A_\infty V = \mathcal{O}(\|V\|^2)$ as $\|V\| \to 0$. We define the stable space $E^s = \cup_{\bullet\in\{\pm\}} \ker(A_\infty + \alpha \bullet \mathrm{i}\beta)$ and for $a, x_0 \in \mathbb{R}$ the weighted function space

$$X_{a,x_0}^+ := \{V \in C([x_0,\infty), \mathbb{R}^4) : \|V\|_{X_{a,x_0}^+} := \sup_{x\in[x_0,\infty)} \|V(x)\|\mathrm{e}^{ax} < \infty\}.$$

Clearly, we have $\lim_{x\to\infty} V_j(x) = 0$ and by [74, Section 2] we further deduce $\|V_j\|_{X_{\alpha,0}^+} < \infty$. This means that there exist constants $C_{1,2} > 0$ such that in the variation of constant formula

$$V_j(x) = \mathrm{e}^{A_\infty(x-y)}V_j(y) + \int_y^x \mathrm{e}^{A_\infty(x-s)}R(V_j(s), \varepsilon)\mathrm{d}s, \tag{3.14}$$

we can bound

$$\int_x^y \left\|\mathrm{e}^{A_\infty(x-s)}R(V_j(s), \varepsilon)\right\| \mathrm{d}s \leq C_1 \int_x^y \mathrm{e}^{-\alpha(x-s)} \|V_j(s)\|^2 \,\mathrm{d}s \leq C_2\|V_j\|_{X_{\alpha,0}^+}^2 \mathrm{e}^{-2\alpha x}$$

for $y \geq x \geq 0$. Taking the limit $y \to \infty$, we infer that

$$\lim_{y\to\infty} \int_y^x \mathrm{e}^{A_\infty(x-s)}R(V_j(s), \varepsilon)\mathrm{d}s$$

exists for every $x \geq 0$. Thus, taking the limit $y \to \infty$ in (3.14), we find that the same holds for $p_j := \lim_{y\to\infty} \mathrm{e}^{-A_\infty y}V_j(y) \in \mathbb{R}^2$. Since $V_j \in X_{\alpha,0}^+$ we obtain $p_j \in E^s$. In particular, $V_j \in X_{\alpha,0}^+$ solves the integral equation

$$V(x) = \mathrm{e}^{A_\infty x}p + \int_\infty^x \mathrm{e}^{A_\infty(x-y)}R(V(y), \varepsilon)\mathrm{d}y \tag{3.15}$$

with $p = p_j$ and satisfies

$$V_j(x) = \mathrm{e}^{A_\infty x}p_j + \mathcal{O}(\mathrm{e}^{-2\alpha x}), \qquad \text{as } x \to \infty.$$

Conversely, a contraction mapping argument shows that (3.15) has for every $p \in E^s$ a unique solution $V(x)$ in X_{α,x_0}^+ provided $x_0 \geq 0$ is sufficiently large. Hence, p_j cannot be zero, as this would imply that $V_j \in X_{\alpha,x_0}^+$ is identically zero. Thus, we have $p_j \neq 0$ and deduce

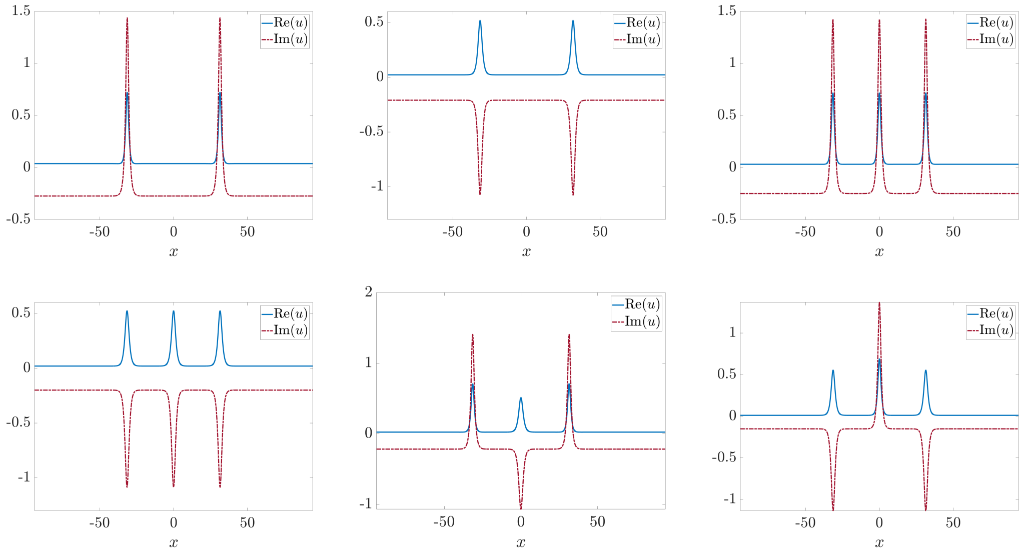

Figure 3.2: The figure shows periodic approximations of 2- and 3-solitons, which were established in Theorem 3.10, see also Theorem 3.12, computed with `pde2path` [146]. The parameters are $\zeta = 1$ and $f = 2$. Continuation was performed in the small parameter ε starting from superpositions of bright NLS-solitons.

$\lim_{x\to\infty} \mathrm{e}^{\alpha x}\|V_j(x)\| = \lim_{x\to\infty} \|\mathrm{e}^{(A_\infty+\alpha)x} p_j\| \neq 0$. Using very similar arguments, one finds $\lim_{x\to\infty} \mathrm{e}^{\alpha x}\|\Psi_j^{\text{ad}}(-x)\| \neq 0$, thereby establishing [127, Hypothesis 3.2].

We conclude that [127, Theorem 3.6] and [126, Theorem 1, Estimate (2.6)] apply. Finally, we point out that the nondegeneracy of $\underline{U}_{\mathbf{k},\ell}$ follows from Lemma 3.8. □

We emphasize that taking $\ell \neq N-1$ in Theorem 3.10 results in spectrally unstable N-solitons, even if all primary 1-pulses $\underline{\mathbf{u}}_1, \ldots, \underline{\mathbf{u}}_{n+\alpha_0}$ are spectrally stable.

Remark 3.11. In [126] Lin's method is employed to determine leading-order expressions of the N small eigenvalues bifurcating from 0. The leading-order expressions in [126, Theorem 2] imply that there exist constants $C, \eta > 0$ such that that any of the N eigenvalues $\lambda \in B_{\delta_0}(0)$ in Theorem 3.10 obeys the estimate $|\lambda| \leq C\exp(-\eta \min\{T_{1,\ell}^{k_1}, \ldots, T_{n,\ell}^{k_n}\})$. Particularly, any potential instabilities arising from the eigenvalues near zero can be interpreted as *long-time instabilities*, i.e. instabilities that are only observed on time scales which are exponentially long in terms of the soliton distances.

3.2.4 Periodic N-soliton solutions

We obtain periodic multi-solitons solutions to (3.8) by employing the dynamical systems formulation (3.9). Specifically, we use that any nondegenerate symmetric homoclinic in a reversible dynamical system connecting to a saddle-focus equilibrium is accompanied by a 1-parameter family of symmetric periodic orbits parameterized by the period T. The T-periodic orbits converge uniformly to the homoclinic on a single periodicity interval as $T \to \infty$. This result, which was obtained in [147, Theorem 5], follows from an application of Lin's method.

Theorem 3.12. *Let* $\underline{U}\colon \mathbb{R} \to \mathbb{R}^4$ *be a nondegenerate symmetric solution of* (3.9) *homoclinic to a saddle-focus equilibrium* $U_\infty \in \mathbb{R}^4$*, as established in Theorem 3.6 or Theorem 3.10. Then, there exists* $T_0 > 0$ *such that for every* $T \geq T_0$ *there exists a smooth* T*-periodic symmetric solution* $\underline{U}_T\colon \mathbb{R} \to \mathbb{R}^4$ *to* (3.9) *satisfying*

$$\lim_{T\to\infty} \sup_{x\in\left[-\frac{T}{2},\frac{T}{2}\right]} |\underline{U}_T(x) - \underline{U}(x)| = 0. \tag{3.16}$$

Proof. The result follows directly from [147, Theorem 5], provided that the symmetric homoclinic solution is elementary. This is ensured by [147, Lemma 4], where the statement (ii) holds by definition of nondegeneracy. □

Combining Theorem 3.12 with Theorem 3.10 and Proposition 3.9 we readily establish the existence of periodic N-soliton solutions to (3.8) for any $N \in \mathbb{N}$.

3.3 Stability analysis

In this section, we establish the spectral and nonlinear stability of the multiple and periodic soliton solutions to (3.7), which were obtained in Theorems 3.10 and 3.12. As outlined in §3.1.3, the spectral stability analysis of these soliton solutions hinges on a-priori bounds on the spectrum, a detailed assessment of the spectrum in a neighborhood of the origin relying on [127, 131], and Evans-function arguments from [17] to control the spectrum in a compact set away from the origin. After having obtained spectral stability, nonlinear stability of the multiple and periodic soliton solutions follows by invoking results from [15] and [70, 137], respectively.

3.3.1 Spectral a-priori bounds

Given a constant $\rho > 0$ and a smooth stationary solution $\underline{\mathbf{u}}$ of (3.7) obeying $\|\underline{\mathbf{u}}\|_{L^\infty} \leq \rho$, we establish a-priori bounds on the spectrum of the linearization $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$ of (3.7) about $\underline{\mathbf{u}}$. The bounds ensure that the spectrum of $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$ in the closed right-half plane is confined to a compact set, whose boundary depends on ρ and the detuning parameter ζ only.

Lemma 3.13. *Fix $\rho > 0$ and $\zeta \in \mathbb{R}$. There exist constants $\eta_1, \eta_2 > 0$ such that for all $\underline{\mathbf{u}} \in L^\infty(\mathbb{R})$ with $\|\underline{\mathbf{u}}\|_{L^\infty} \leq \rho$ the set*

$$\Omega = \{\lambda \in \mathbb{C} : |\operatorname{Re}(\lambda)| \geq \eta_1\} \cup \{\lambda \in \mathbb{C} : |\operatorname{Im}(\lambda)| \geq \eta_2, \operatorname{Re}(\lambda) \neq 0\}$$

belongs to the resolvent set of $\mathcal{L}(\underline{\mathbf{u}})$, i.e., we have $\sigma(\mathcal{L}(\underline{\mathbf{u}})) \cap \Omega = \emptyset$.

Proof. The densely defined skew-adjoint operator $-J\partial_x^2 \colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ generates a unitary group on the Hilbert space $L^2(\mathbb{R})$ by Stone's theorem, see [44, Theorem II.3.24]. In particular, [44, Theorem I.1.10] yields the resolvent bound

$$\left\|\left(-J\partial_x^2 - \lambda\right)^{-1}\right\|_{L^2 \to L^2} \leq \frac{1}{|\operatorname{Re}(\lambda)|} \tag{3.17}$$

for $\lambda \in \mathbb{C}$ with $|\operatorname{Re}(\lambda)| > 0$. The residual $\mathcal{L}(\underline{\mathbf{u}}) + J\partial_x^2$ can be bounded as

$$\left\|\left(\mathcal{L}(\underline{\mathbf{u}}) + J\partial_x^2\right)\mathbf{u}\right\|_{L^2} \leq \left(|\zeta| + 4\|\underline{\mathbf{u}}\|_{L^\infty}^2\right)\|\mathbf{u}\|_{L^2} \leq C_1\|\mathbf{u}\|_{L^2},$$

for all $\mathbf{u} \in H^2(\mathbb{R})$, where we set $C_1 = |\zeta| + 4\rho^2$. By estimate (3.17) there exists a constant $\eta_1 > 0$, depending on ρ and ζ only, such that for $\lambda \in \mathbb{C}$ with $|\operatorname{Re}(\lambda)| \geq \eta_1$ we have

$$C_1\left\|(-J\partial_x^2 - \lambda)^{-1}\right\|_{L^2 \to L^2} < 1.$$

Therefore, $\mathcal{L}(\underline{\mathbf{u}}) - \lambda = -J\partial_x^2 - \lambda + \left(\mathcal{L}(\underline{\mathbf{u}}) + J\partial_x^2\right)$ is by [80, Theorem IV.1.16] bounded invertible for each $\lambda \in \mathbb{C}$ with $|\operatorname{Re}(\lambda)| \geq \eta_1$. This yields

$$\{\lambda \in \mathbb{C} : |\operatorname{Re}(\lambda)| \geq \eta_1\} \cap \sigma(\mathcal{L}(\underline{\mathbf{u}})) = \emptyset.$$

Next, let $\lambda \in \mathbb{C}$ and $\mathbf{w} \in L^2(\mathbb{R})$. Consider the resolvent problem

$$(\mathcal{L}(\underline{\mathbf{u}}) - \lambda)\mathbf{u} = \mathbf{w}. \tag{3.18}$$

We show that there exists $\eta_2 > 0$, depending on ρ and ζ only, such that (3.18) has a unique solution $\mathbf{u} \in H^2(\mathbb{R})$ provided $|\operatorname{Im}(\lambda)| \geq \eta_2$ and $\operatorname{Re}(\lambda) \neq 0$. To this end, we observe that the resolvent problem (3.18) has a unique solution $\mathbf{u} \in H^2(\mathbb{R})$ if and only if the conjugate problem

$$(\widetilde{\mathcal{L}}(\underline{\mathbf{u}}) - \lambda)\widetilde{\mathbf{u}} = \widetilde{\mathbf{w}} \tag{3.19}$$

posses a unique solution $\widetilde{\mathbf{u}} \in H^2(\mathbb{R})$, where we denote $\widetilde{\mathcal{L}}(\underline{\mathbf{u}}) = \widetilde{J}\widetilde{L}(\underline{\mathbf{u}})$ with $\widetilde{J} = S^{-1}JS$, $\widetilde{L}(\underline{\mathbf{u}}) = S^{-1}L(\underline{\mathbf{u}})S$, $\widetilde{\mathbf{u}} = S^{-1}\mathbf{u}$, $\widetilde{\mathbf{w}} = S^{-1}\mathbf{w}$ and

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} \in \mathbb{C}^{2\times 2}.$$

We note that conjugation with the matrix S corresponds to a coordinate transform transferring the formulation (3.7) of (3.5) as a system in $(\operatorname{Re}(u), \operatorname{Im}(u))^\top$ into its formulation as a system in $(u, \overline{u})^\top$. One readily computes

$$\widetilde{J} = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \qquad \widetilde{L}(\underline{\mathbf{u}}) = \begin{pmatrix} -\partial_x^2 + \zeta - 2|\underline{\tilde{\mathbf{u}}}|^2 & -\underline{\tilde{\mathbf{u}}}^2 \\ -\overline{\underline{\tilde{\mathbf{u}}}}^2 & -\partial_x^2 + \zeta - 2|\underline{\tilde{\mathbf{u}}}|^2 \end{pmatrix},$$

where we denote $\underline{\tilde{\mathbf{u}}} = \underline{\mathbf{u}}_1 + \mathrm{i}\underline{\mathbf{u}}_2$. We define the operators $A\colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $B\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Au = -u'' + (\zeta - 2|\underline{\tilde{\mathbf{u}}}|^2)u$ and $Bu = -\underline{\tilde{\mathbf{u}}}^2 u$, respectively. Then, using integration by parts we infer that the self-adjoint operator A can be bounded from below as

$$\langle Au, u\rangle_{L^2} \geq \|u'\|_{L^2}^2 + (\zeta - 2\|\underline{\tilde{\mathbf{u}}}\|_{L^\infty}^2)\|u\|_{L^2}^2 \geq (\zeta - 2\rho^2)\|u\|_{L^2}^2$$

for all $u \in H^2(\mathbb{R})$. We note that the lower bound of A depends on ρ and ζ only. Clearly, B is a bounded operator with $\|B\|_{L^2\to L^2} \leq \rho^2$. Employing [15, Theorem 4], we find constants $C_2, \eta_2 > 0$, depending on ρ and ζ only, such that for every $\widetilde{\mathbf{w}} \in L^2(\mathbb{R})$ the problem (3.19) has a unique solution $\widetilde{\mathbf{u}} \in H^2(\mathbb{R})$ with

$$\|\widetilde{\mathbf{u}}\|_{L^2} \leq C_2 \frac{\|\widetilde{\mathbf{w}}\|_{L^2}}{|\operatorname{Re}(\lambda)|},$$

provided that $\operatorname{Re}(\lambda) \neq 0$ and $|\operatorname{Im}(\lambda)| \geq \eta_2$. Thus, we have proved

$$\{\lambda \in \mathbb{C} : |\operatorname{Im}(\lambda)| \geq \eta_2, \operatorname{Re}(\lambda) \neq 0\} \cap \sigma(\mathcal{L}(\underline{\mathbf{u}})) = \emptyset$$

and the claim follows. $\square$

3.3.2 Stability of N-soliton solutions

We determine the spectral stability of the N-soliton solutions established in Theorem 3.10. If one of the primary 1-soliton solutions constituting the N-soliton is spectrally unstable, spectral instability of the N-soliton follows from [17]. On the other hand, if the N-soliton is comprised of N spectrally stable 1-solitons, then we employ Lemma 3.13 and [17, Lemma 3.3], to confine the spectral stability problem to a small ball $B_{\delta_0}(0)$ centered at the origin. Spectral stability is then decided by the N small eigenvalues in $B_{\delta_0}(0)$, whose position with respect to the imaginary axis is described by Theorem 3.10. All in all, we arrive at the following result.

Theorem 3.14. *Let $N \in \mathbb{N}$ and $\ell \in \{0, \ldots, N-1\}$. Set $n = \lfloor \frac{N}{2} \rfloor$ and $\alpha_0 = N \mod 2 \in \{0, 1\}$. Let $\underline{\mathbf{u}}_1, \ldots, \underline{\mathbf{u}}_{n+\alpha_0}$ be a sequence of stationary 1-soliton solutions to* (3.7)*, established in Theorem 3.6. For each $\mathbf{k} \in \mathbb{N}^n$, we denote by $\underline{\mathbf{u}}_{\mathbf{k},\ell}$ the associated N-soliton solutions to* (3.7)*, established in Theorem 3.10.*

The following assertions hold.

(i) If $\underline{\mathbf{u}}_1, \ldots, \underline{\mathbf{u}}_{n+\alpha_0}$ are spectrally stable soliton solutions of (3.7)*, then there exists a constant $k_0 \in \mathbb{N}$ such that, whenever $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ satisfies $\min\{k_1, \ldots, k_n\} \geq k_0$, the N-soliton solution $\underline{\mathbf{u}}_{\mathbf{k},\ell}$ of* (3.7) *is spectrally stable if $\ell = N-1$ and spectrally unstable if $\ell \neq N-1$.*

(ii) Assume that there exists $i \in \{1, \ldots, n+\alpha_0\}$ such that $\underline{\mathbf{u}}_i$ is a spectrally unstable solution of (3.7)*, then there exists a constant $k_0 \in \mathbb{N}$ such that, whenever $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ satisfies $\min\{k_1, \ldots, k_n\} \geq k_0$, the N-soliton solution $\underline{\mathbf{u}}_{\mathbf{k},\ell}$ of* (3.7) *is spectrally unstable.*

Proof. We start with the proof of the first assertion. Let $\delta_0 > 0$ be as in Theorem 3.10. Using the bound (3.13) in Theorem 3.10 and applying Lemma 3.13, we find $\mathbf{k}$-independent constants $\eta_1, \eta_2 > 0$ such that

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}_{\mathbf{k},\ell}) - \varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\tfrac{\varepsilon}{2}\} \subset \{\lambda \in \mathbb{C} : -\tfrac{\varepsilon}{2} \leq \mathrm{Re}(\lambda) \leq \eta_1, |\mathrm{Im}(\lambda)| \leq \eta_2\}.$$

On the other hand, by the spectral stability of $\underline{\mathbf{u}}_i$, there exists a $\mathbf{k}$-independent constant $\tau > 0$ such that

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}_i) - \varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\tau\} = \{0\}$$

for $i = 1, \ldots, n + \alpha_0$. Hence, [17, Lemma 3.3] yields a constant $k_0 \in \mathbb{N}$ such that, provided $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ satisfies $\min\{k_1, \ldots, k_n\} \geq k_0$, the compact set

$$\{\lambda \in \mathbb{C} : -\tau \leq \mathrm{Re}(\lambda) \leq \eta_1, |\,\mathrm{Im}(\lambda)| \leq \eta_2\} \setminus B_{\delta_0}(0)$$

lies in the resolvent set of $\mathcal{L}(\underline{\mathbf{u}}_{\mathbf{k},\ell}) - \varepsilon$. We conclude that all spectrum of $\mathcal{L}(\underline{\mathbf{u}}_{\mathbf{k},\ell}) - \varepsilon$ in the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\tau\}$ must be confined to the ball $B_{\delta_0}(0)$. The spectrum of $\mathcal{L}(\underline{\mathbf{u}}_{\mathbf{k},\ell}) - \varepsilon$ in the ball $B_{\delta_0}(0)$ consists of precisely ℓ eigenvalues of negative real part, $N - 1 - \ell$ eigenvalues of positive real part and one algebraically simple eigenvalue 0 by Theorem 3.10 (all counted with algebraic multiplicities). This yields the first assertion.

For the second assertion, we observe that if $\underline{\mathbf{u}}_i$ is spectrally unstable, then there exists $\lambda_0 \in \sigma(\mathcal{L}(\underline{\mathbf{u}}_i) - \varepsilon)$ with $\mathrm{Re}(\lambda_0) > 0$. By [15, Lemma 4] the essential spectrum of $\mathcal{L}(\underline{\mathbf{u}}_j) - \varepsilon$ lies on the line $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) = -\varepsilon\}$ for $j = 1, \ldots, n + \alpha_0$. Combining the last two lines with [17, Theorem 6.2] yields a constant $k_0 \in \mathbb{N}$ such that, provided $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ satisfies $\min\{k_1, \ldots, k_n\} \geq k_0$, $\mathcal{L}(\underline{\mathbf{u}}_{\mathbf{k},\ell}) - \varepsilon$ possesses an eigenvalue λ of real part $\mathrm{Re}(\lambda) \geq \mathrm{Re}(\lambda_0)/2 > 0$, which proves the second assertion. □

Remark 3.15. As mentioned in Remark 3.11, the N-solitons, established in Theorem 3.10, can suffer from long-time instabilities triggered by the N eigenvalues near the origin, which are exponentially small with respect to the distances between solitons. On the other hand, if one of the primary 1-solitons constituting the N-soliton is spectrally unstable, the proof of Theorem 3.14 shows that we find an eigenvalue λ of the linearization of (3.7) about the N-soliton of real part $\mathrm{Re}(\lambda) \geq \mathrm{Re}(\lambda_0)/2 > 0$. That is, the real part of this eigenvalue obeys a positive lower bound, which is independent of the distances between the 1-solitons constituting the N-soliton. Therefore, the instability in Theorem 3.14.(ii) can be interpreted as a *short-time instability*.

The asymptotic orbital stability of spectrally stable N-pulses is now a direct consequence of the following statement, which was proved in [15].

Theorem 3.16 ([15, Theorem 3]). *Let $\mathbf{u}_\infty \in \mathbb{R}^2$ and let $\underline{\mathbf{u}} \colon \mathbb{R} \to \mathbb{R}^2$ be a smooth stationary solution to* (3.7) *such that $\underline{\mathbf{u}}(x)$ converges to $\mathbf{u}_\infty$ as $x \to \pm\infty$. If $\underline{\mathbf{u}}$ is spectrally stable, then there exist constants $C, \delta, \eta > 0$ such that for all $\mathbf{v}_0 \in H^1(\mathbb{R})$ with $\|\mathbf{v}_0\|_{H^1} \leq \delta$ there exist a constant $\gamma \in \mathbb{R}$ and a function $\mathbf{v} \in C\big([0, \infty), H^1(\mathbb{R})\big)$ with $\mathbf{v}(0) = \mathbf{v}_0$ such that $\mathbf{u} = \underline{\mathbf{u}} + \mathbf{v}$ is the unique global mild solution of* (3.7) *with $\mathbf{u}(0) = \underline{\mathbf{u}} + \mathbf{v}_0$ enjoying the estimate*

$$\|\mathbf{u}(t) - \underline{\mathbf{u}}(\cdot + \gamma)\|_{H^1} \leq C\mathrm{e}^{-\eta t}\delta$$

for all $t \geq 0$.

3.3.3 Stability of periodic N-soliton solutions

We now turn to the stability analysis of the periodic multi-soliton solutions to (3.7), whose existence, as outlined in §3.2.3, follows by combining Theorems 3.10 and 3.12. Our first step is to establish *diffusive spectral stability* of these solutions, which, in turn, yields their nonlinear stability against localized perturbations and against subharmonic perturbations of any wavelength.[1]

Diffusive spectral stability is defined in terms of the 1-parameter family of Bloch operators associated with the linearization $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$ of system (3.7) about a periodic smooth stationary solution $\underline{\mathbf{u}} = (\underline{\mathbf{u}}_1, \underline{\mathbf{u}}_2)^\top$ with period $T > 0$. These Bloch operators $\mathcal{L}_\xi(\underline{\mathbf{u}}) - \varepsilon \colon H^2_{\text{per}}(0,T) \subset L^2_{\text{per}}(0,T) \to L^2_{\text{per}}(0,T)$ are given by

$$L_\xi(\underline{\mathbf{u}}) = -(\partial_x + \mathrm{i}\xi T^{-1})^2 + \zeta - \begin{pmatrix} 3\mathbf{u}_1^2 + \mathbf{u}_2^2 & 2\mathbf{u}_1\mathbf{u}_2 \\ 2\mathbf{u}_1\mathbf{u}_2 & \mathbf{u}_1^2 + 3\mathbf{u}_2^2 \end{pmatrix}, \qquad \mathcal{L}_\xi(\underline{\mathbf{u}}) = JL_\xi(\underline{\mathbf{u}})$$

with $\xi \in [-\pi, \pi)$. We then have the well-known spectral decomposition

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon) = \bigcup_{\xi \in [-\pi,\pi)} \sigma\left(\mathcal{L}_\xi(\underline{\mathbf{u}}) - \varepsilon\right), \tag{3.20}$$

cf. [52]. The definition of diffusive spectral stability now reads as follows.

Definition 3.17. A smooth stationary T-periodic solution $\underline{\mathbf{u}} \colon \mathbb{R} \to \mathbb{R}^2$ of (3.7) is *diffusively spectrally stable* provided the following conditions hold:

(i) We have $\sigma(\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\}$;

(ii) There exists $\vartheta > 0$ such that for all $\xi \in [-\pi, \pi)$ we have $\mathrm{Re}\left(\sigma\left(\mathcal{L}_\xi(\underline{\mathbf{u}}) - \varepsilon\right)\right) \leq -\vartheta\xi^2$;

(iii) 0 is a simple eigenvalue of the Bloch operator $\mathcal{L}_0(\underline{\mathbf{u}}) - \varepsilon$.

Diffusive spectral stability of the periodic multi-soliton solutions to (3.7), studied in this paper, can be obtained if the underlying multi-soliton is spectrally stable. In this case, we can apply the a-priori bounds in Lemma 3.13 and the results in [17, 53], to preclude any unstable spectrum outside a small ball $B_\varrho(0)$ of radius $\varrho > 0$ centered at the origin. By [17, 53] the spectrum inside the ball $B_\varrho(0)$ is given by a single smooth curve $\{\lambda_0(\xi) : \xi \in [-\pi, \pi)\}$, which touches the origin by translational invariance. Leading-order control on this critical spectral curve, provided

[1] We note that diffusive spectral stability is a standard assumption in the nonlinear stability analysis of periodic traveling or steady waves in dissipative systems, see [70] and further references therein.

by [131], then yields constants $\widetilde{T}_0, \varpi > 0$ and a partitioning of the half line $[\widetilde{T}_0, \infty)$ in intervals $I_j = (\widetilde{T}_0 + j\varpi, \widetilde{T}_0 + (j+1)\varpi)$, $j \in \mathbb{N}_0$ such that diffusive spectral stability holds if the period T lies in I_j with j even, whereas spectral instability holds for $T \in I_j$ if j is odd. That is, the stability of the multi-soliton train alternates with the period. On the other hand, if the underlying multi-soliton solution is spectrally unstable, then the associated multi-soliton trains are spectrally unstable by [17, 53] for *all* periods $T > 0$ sufficiently large.

Theorem 3.18. *Let*

$$\underline{U} = (\underline{U}_1, \underline{U}_2, \underline{U}_3, \underline{U}_4)^\top : \mathbb{R} \to \mathbb{R}^4$$

be a nondegenerate symmetric homoclinic solution of (3.9) *connecting to a saddle-focus equilibrium $U_\infty \in \mathbb{R}^4$, as established in Theorem 3.10. Let $\{\underline{U}_T\}_{T \geq T_0}$ be the corresponding family of smooth T-periodic symmetric solutions*

$$\underline{U}_T = (\underline{U}_{T,1}, \underline{U}_{T,2}, \underline{U}_{T,3}, \underline{U}_{T,4})^\top : \mathbb{R} \to \mathbb{R}^4$$

of (3.9)*, established in Theorem 3.12. Denote by $\underline{\mathbf{u}} = (\underline{U}_1, \underline{U}_2)^\top, \underline{\mathbf{u}}_T = (\underline{U}_{T,1}, \underline{U}_{T,2})^\top : \mathbb{R} \to \mathbb{R}^2$ the associated stationary solutions of* (3.7).

The following assertions hold.

(i) If $\underline{\mathbf{u}}$ is a spectrally stable pulse solution of (3.7)*, then there exist constants $\widetilde{T}_0, C, \tau, \delta, a > 0$ and $b \in \mathbb{R}$, an open set $U \subset \mathbb{C}$ containing the real interval $[-\pi, \pi]$ and an analytic map $\lambda_0 : U \to \mathbb{C}$ such that for all $T \geq \widetilde{T}_0$ we have*

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}_T) - \varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\tau\} = \{\lambda_0(\xi) : \xi \in [-\pi, \pi)\}. \tag{3.21}$$

Moreover, $\lambda_0(\xi) = \lambda_0(-\xi)$ is a real-valued algebraically simple eigenvalue of the Bloch operator $\mathcal{L}_\xi(\underline{\mathbf{u}}_T) - \varepsilon$ for each $\xi \in [-\pi, \pi)$ and $T \geq \widetilde{T}_0$. Finally, the estimate

$$\left|\lambda_0(\xi) - a(\cos(\xi) - 1)\mathrm{e}^{-2\alpha T} \sin(2\beta T + b)\right| \leq C|\mathrm{e}^{\mathrm{i}\xi} - 1|\mathrm{e}^{-(2\alpha+\delta)T} \tag{3.22}$$

holds for all $\xi \in U$ and $T \geq \widetilde{T}_0$, where $\alpha, \beta > 0$ are as in Proposition 3.9.

If $T \geq \widetilde{T}_0$ is such that $\sin(2\beta T + b) > 0$*, then $\underline{\mathbf{u}}_T$ is diffusively spectrally stable as a periodic stationary solution to* (3.7)*. Moreover, if $T \geq \widetilde{T}_0$ is such that* $\sin(2\beta T + b) < 0$*, then $\underline{\mathbf{u}}_T$ is spectrally unstable as a stationary solution to* (3.7)*.*

(ii) *Assume that $\underline{\mathbf{u}}$ is spectrally unstable. Then, there exists a constant $\widetilde{T}_0 > 0$ such that for all $T \geq \widetilde{T}_0$ the stationary solution $\underline{\mathbf{u}}_T$ of* (3.7) *is spectrally unstable. In particular, $\mathcal{L}_\xi(\underline{\mathbf{u}}_T) - \varepsilon$ possesses spectrum in the open right-half plane for all $\xi \in [-\pi, \pi)$.*

Proof. We start with the proof of the first assertion. Since the stationary pulse solution $\underline{\mathbf{u}}$ of (3.7) is spectrally stable, there exists a constant $\tau_0 > 0$ such that

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\tau_0\} = \{0\}, \tag{3.23}$$

and 0 is an algebraically simple eigenvalue of $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$. Moreover, by the bound (3.16) in Theorem 3.12, the L^∞-norm of the solution $\underline{\mathbf{u}}_T$ can be bounded by a T-independent constant for $T \geq T_0$. Therefore, Lemma 3.13, yields a T-independent compact set $\mathcal{K} \subset \mathbb{C}$ such that

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}_T) - \varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\tfrac{\varepsilon}{2}\} \subset \mathcal{K}. \tag{3.24}$$

for all $T \geq T_0$. Take $\varrho \in (0, \min\{\frac{\varepsilon}{2}, \tau_0\})$ and set $\tilde{\mathcal{K}} := \{\lambda \in \mathcal{K} : \mathrm{Re}(\lambda) \geq -\tau_0\} \setminus B_\varrho(0)$. Using that $\tilde{\mathcal{K}}$ is compact and (3.23) holds, [17, Lemma 4.2] yields, provided $T \geq T_0$ is sufficiently large, that

$$\sigma(\mathcal{L}(\underline{\mathbf{u}}_T) - \varepsilon) \cap \tilde{\mathcal{K}} = \emptyset \tag{3.25}$$

and, moreover,

$$\sigma(\mathcal{L}_\xi(\underline{\mathbf{u}}_T) - \varepsilon) \cap B_\varrho(0) = \{\lambda_0(\xi)\}, \tag{3.26}$$

where $\lambda_0(\xi)$ is an algebraically simple eigenvalue of the Bloch operator $\mathcal{L}_\xi(\underline{\mathbf{u}}_T) - \varepsilon$ for each $\xi \in [-\pi, \pi)$. Hence, using that the Bloch operators depend analytically on ξ, it follows from standard analytic perturbation theory, see [80, Sections II.1 and VII.3], that there exists an open neighborhood $U \subset \mathbb{C}$ of the real interval $[-\pi, \pi]$ such that $\lambda_0(\xi)$ can be extended to an analytic map $\lambda_0 \colon U \to \mathbb{C}$. Provided $T > 0$ is sufficiently large, [131, Theorem 5.6] yields that $\lambda_0(\xi)$ is real-valued, we have $\lambda_0(\xi) = \lambda_0(-\xi)$ for all $\xi \in [-\pi, \pi)$ and the approximation (3.22) holds with $a > 0$ (after possibly shifting $b \mapsto b + \pi$). Finally, the identities (3.20), (3.24), (3.25) and (3.26) imply (3.21) with $\tau = \min\{\tau_0, \varepsilon/2\} > 0$.

If $T > 0$ is sufficiently large with $\sin(2\beta T + b) > 0$, then Cauchy's estimate in conjunction with the bound (3.22) yield $\lambda_0''(0) < 0$. Hence, combining the latter with (3.22) and $\lambda_0(0) = 0 = \lambda_0'(0)$, we infer that, provided $T > 0$ is sufficiently large, there exists $\vartheta > 0$ such that $\lambda_0(\xi) \leq -\vartheta\xi^2$ for all $\xi \in [-\xi, \xi]$. This, together with (3.21), implies diffusive spectral stability of $\underline{\mathbf{u}}_T$ as a periodic stationary solution to (3.7). On the other hand, if $T > 0$ is sufficiently large with $\sin(2\beta T + b) < 0$, then by estimate (3.22) there exists $\xi \in [-\pi, \pi) \setminus \{0\}$ such that $\lambda_0(\xi) > 0$.

Therefore, $\underline{\mathbf{u}}_T$ is spectrally unstable as a stationary solution to (3.7) upon recalling (3.21). This finishes the proof of the first assertion.

We proceed with proving the second assertion. Assume that $\underline{\mathbf{u}}$ is spectrally unstable. Then, there exists $\lambda_0 \in \sigma(\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon)$ with $\mathrm{Re}(\lambda_0) > 0$. By [15, Lemma 4] the essential spectrum of $\mathcal{L}(\underline{\mathbf{u}}) - \varepsilon$ is confined to the line $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) = -\varepsilon\}$. Therefore, we can apply [17, Theorem 7.2] to yield that, provided $T > 0$ is sufficiently large, $\sigma(\mathcal{L}_\xi(\underline{\mathbf{u}}_T) - \varepsilon)$ possesses for each $\xi \in [-\pi, \pi)$ an element $\lambda_*(\xi)$ of real part $\mathrm{Re}(\lambda_*(\xi)) \geq \mathrm{Re}(\lambda_0)/2 > 0$, which proves the second assertion. □

Remark 3.19. If the underlying multi-soliton solution is spectrally unstable, then Theorem 3.18.(ii) shows that each of the Bloch operators $\mathcal{L}_\xi(\underline{\mathbf{u}}_T) - \varepsilon$ with $\xi \in [-\pi, \pi)$ possesses spectrum in the open right-half plane. That is, the multi-soliton train $\underline{\mathbf{u}}_T$ is spectrally unstable against subharmonic perturbations of any wavelength. In particular, it is spectrally unstable against co-periodic perturbations. On the other hand, if the underlying multi-pulse solution is spectrally stable, then Theorem 3.18.(i) yields that the spectrum of the Bloch operator $\mathcal{L}_0(\underline{\mathbf{u}}_T) - \varepsilon$ lies in the open left-half plane except for the algebraically simple eigenvalue $\lambda_0(0) = 0$. That is, the multi-soliton train $\underline{\mathbf{u}}_T$ is spectrally stable against co-periodic perturbations. Moreover, if in addition $\sin(2\beta T + b) < 0$ holds, we find $\lambda_0''(0) > 0$ by applying Cauchy's estimate to the bound (3.22). Hence, using that $\lambda_0(0), \lambda_0'(0) = 0$ holds by Theorem 3.18, we infer that $\mathcal{L}_\xi(\underline{\mathbf{u}}_T) - \varepsilon$ possesses unstable spectrum for each $\xi \in [-\pi, \pi) \setminus \{0\}$ sufficiently small. We conclude that the multi-soliton train is sideband unstable, i.e., it is spectrally unstable against subharmonic perturbations of sufficiently large wavelength.

Remark 3.20. Similarly as in the case for multi-soliton solutions, cf. Remark 3.15, we can distinguish between short- and long-time instabilities of multi-soliton trains. On the one hand, the critical spectral curve $\lambda_0(\xi)$ in Theorem 3.18.(i) is exponentially small in terms of the period T by estimate (3.22). Thus, any instabilities arising from this curve can be interpreted as long-time instabilities of the multi-soliton train. On the other hand, if the underlying multi-soliton solution is spectrally unstable, then the proof of Theorem 3.18.(ii) shows that the linearization of (3.7) about the multi-soliton train admits unstable spectrum, whose real part can be bounded from below by a positive bound, independent of T. Thus, these instabilities can be interpreted as short-time instabilities.

The proof of our main result, Theorem 3.2, now follows by combining the diffusive spectral stability result in Theorem 3.18 with [70, 137].

Proof of Theorem 3.2. Since it holds $8\zeta < \pi^2 f^2, \pi f \cos\theta_0 = 2\sqrt{2\zeta}$ and $\sin\theta_0 > 0$, Theorem 3.6 provides constants $C_0, \varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exist an asymptotic state $\mathbf{u}_{\infty,\varepsilon}$ and an even spectrally stable smooth stationary 1-pulse solution $\underline{\mathbf{u}}_\varepsilon$ of (3.7) obeying (3.10).

Fix $\varepsilon \in (0, \varepsilon_0)$. We apply Theorems 3.10 and 3.14 with $\ell = N-1$ to yield constants $C_1, k_0 > 0$ such that for each $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $\min\{k_1, \ldots, k_n\} \geq k_0$ there exist distances $T_{1,\varepsilon}^{k_1}, \ldots, T_{n,\varepsilon}^{k_n}$ and an even spectrally stable smooth stationary N-pulse solution $\underline{\mathbf{u}}_{\mathbf{k},\varepsilon}$ of (3.7) enjoying the estimate

$$\left|\underline{\mathbf{u}}_{\mathbf{k},\varepsilon}(x) - \mathbf{u}_{\infty,\varepsilon} - \alpha_0\left(\underline{\mathbf{u}}_\varepsilon(x) - \mathbf{u}_{\infty,\varepsilon}\right) - \sum_{i=1}^n \left(\underline{\mathbf{u}}_\varepsilon\left(x - T_{1,\varepsilon}^{k_1} - \ldots - T_{i,\varepsilon}^{k_i}\right)\right.\right.$$
$$\left.\left. + \underline{\mathbf{u}}_\varepsilon\left(x + T_{1,\varepsilon}^{k_1} + \ldots + T_{i,\varepsilon}^{k_i}\right) - 2\mathbf{u}_{\infty,\varepsilon}\right)\right| \leq \frac{C_1}{\min\{k_1, \ldots, k_n\}}$$

for $x \in \mathbb{R}$. Here, the sequence $\{T_{i,\varepsilon}^k\}_k$ of pulse distances is monotonically increasing with $T_{i,\varepsilon}^k \to \infty$ as $k \to \infty$ for $i = 1, \ldots, n$. Combining the latter estimate with (3.10) implies

$$\begin{aligned}&\left|\underline{\mathbf{u}}_{\mathbf{k},\varepsilon}(x) - \alpha_0\boldsymbol{\phi}_{\theta_0}(x) - \sum_{i=1}^n \left(\boldsymbol{\phi}_{\theta_0}\left(x - T_{1,\varepsilon}^{k_1} - \ldots - T_{i,\varepsilon}^{k_i}\right) + \boldsymbol{\phi}_{\theta_0}\left(x + T_{1,\varepsilon}^{k_1} + \ldots + T_{i,\varepsilon}^{k_i}\right)\right)\right| \\ &\quad \leq 2C_0\varepsilon\end{aligned} \tag{3.27}$$

for each $x \in \mathbb{R}$ and $\mathbf{k} \in \mathbb{N}^n$ with $\min\{k_1, \ldots, k_n\} \geq k_0$, upon taking $k_0 > 0$ larger if necessary.

Now, fix $\mathbf{k} \in \mathbb{N}^n$ with $\min\{k_1, \ldots, k_n\} \geq k_0$. By Proposition 3.9 and Theorems 3.12 and 3.18 there exists a monotonically increasing sequence of periods $\{L_{\mathbf{k},\varepsilon}^m\}_m$ such that for each $m \in \mathbb{N}$ there exists an even diffusively spectrally stable smooth stationary periodic solution $\underline{\mathbf{u}}_{m,\mathbf{k},\varepsilon}(x)$ of (3.7) of period $L_{\mathbf{k},\varepsilon}^m$ satisfying the estimate

$$\sup_{x \in \left[-\frac{1}{2}L_{\mathbf{k},\varepsilon}^m, \frac{1}{2}L_{\mathbf{k},\varepsilon}^m\right]} \left|\underline{\mathbf{u}}_{m,\mathbf{k},\varepsilon}(x) - \underline{\mathbf{u}}_{\mathbf{k},\varepsilon}(x)\right| \leq C_0\varepsilon. \tag{3.28}$$

Here, the sequence $\{L_{\mathbf{k},\varepsilon}^m\}_m$ tends to ∞ as $m \to \infty$ and obeys (3.6) for each $m \in \mathbb{N}$. Thus, we have established assertions (i), (ii) and (iv). Moreover, assertion (iii) follows readily by combing (3.27) and (3.28). Assertion (v) is a direct consequence of the diffusive spectral stability of $\underline{\mathbf{u}}_{m,\mathbf{k},\varepsilon}$ in combination with [137, Theorem 1], see also [71, Theorem 1.2]. Similarly, assertion (vi) follows immediately from [70, Theorem 1.3]. □

Remark 3.21. Diffusive spectral stability of T-periodic stationary solutions to the LLE even yields a nonlinear stability result [71] against MT-periodic perturbations that is uniform in $M \in \mathbb{N}$, as well as nonlinear stability against nonlocalized phase modulations [159]. We refer to [71, 159] for further details.

Part II

Solitons in a bi-directionally coupled laser-microresonator system

4 Solitons in a bi-directionally coupled laser-microresonator system

This chapter is based on joint work in progress with Huanfa Peng, Björn de Rijk, Christian Koos, and Wolfgang Reichel. We thank Tobias Jahnke for providing us the MATLAB code for the time-integration of the system (4.7).

4.1 Introduction

We consider a Kerr-nonlinear microresonator and a continuous-wave single-mode semiconductor laser as a bi-directionally coupled system. The light emitted from the pump laser is described by the carrier number $N(t)$ inside the cavity and the laser field $A_L(t)$ which is coupled into the ring-shaped Kerr microresonator. There a counter-clockwise propagating forward field $A_F(x,t)$ and a clockwise propagating backscattered field $A_B(x,t)$ are generated, where $x \in [0, 2\pi]$ is the angular position and $t \geq 0$ is the time. Inside the resonator the fields A_F and A_B interact with each other and a part of A_B is coupled out of the resonator back into the laser where it interacts with the laser field A_L, see Figure 4.1. It has been demonstrated both experimentally and theoretically that such a bi-directional laser–resonator system, when operating in the right regime, features fully automatic soliton formation by simply switching on the laser pump [22, 86, 117, 135, 153]. Backscattering from the microresonator to the laser leads to self-injection locking [85, 98] and then to the emergence of stable cavity solitons. Due to the absence of photonic and electronic control circuitry, which are needless here, it is possible to fabricate such devices with a compact footprint enabling full integration into industrial packages [135, 153].

In this chapter we identify and investigate stable 1-soliton states of the bi-directionally coupled system using numerical bifurcation analysis in the anomalous dispersion regime. Unlike time-integration methods, which were used in previous studies [86, 135, 148, 153], our numerical bifurcation approach allows for a systematic exploration of soliton states for a large range of technically relevant parameter values. Moreover, our numerical analysis shows that, unlike

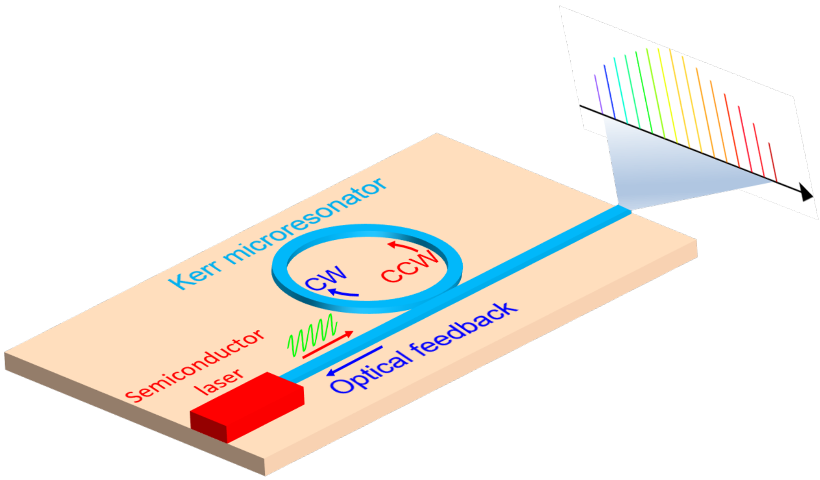

Figure 4.1: Schematic diagram depicting the bi-directional coupling of the semiconductor laser with a Kerr microresonator. The light emitted from the semiconductor laser is injected into the Kerr microresonator in counterclockwise (CCW) direction. Due to surface and internal inhomogeneities, the CCW field is backscattered clockwise (CW) and injected back into the semiconductor laser.

its one-directionally coupled counterpart, the bi-directionally coupled laser-resonator system can self-correct perturbations of the laser frequency such that stable 1-solitons persist. We supplement our study by a sensitivity analysis of the 1-soliton states with respect to parameter variations.

The chapter is structured as follows. In Section 4.2 we formulate the mathematical model for the bi-directionally coupled system. In Section 4.3 we present numerical simulations and compare our findings with those obtained in the standard uni-directionally coupled laser-microresonator model. In Appendix 4.5 we give details on the numerical bifurcation analysis as well as the algorithm for the parameter sensitivity analysis. Appendix 4.6 is devoted to the derivation of the normalized form of the bi-directionally coupled system from the system in physical quantities.

4.2 Coupled laser-resonator model equations

The standard set-up of a Kerr-nonlinear microresonator pumped by a cw-laser without backcoupling is described by the Lugiato-Lefever equation (LLE) [89]:

$$\dot{a}(x,t) = \left[-1 + \mathrm{i}d\frac{\partial^2}{\partial x^2} + \mathrm{i}|a(x,t)|^2\right]a(x,t) - \kappa_{ext} f \mathrm{e}^{\mathrm{i}(\zeta t+\phi)} \tag{4.1}$$

in normalized time t and angular position $x \in [0, 2\pi]$ (a superscripted dot denotes the time derivative). Here a is 2π-periodic in x and describes the optical field in the microresonator driven by the invariant laser field $a_L(t) = f\mathrm{e}^{\mathrm{i}\zeta t}$ where the detuning $\zeta = \omega_m - \omega_0$ is static and is given by the mismatch between the frequency ω_m of the pumped resonator mode and the frequency ω_0 of the laser cavity mode. Dispersion d and coupling κ_{ext} are as in Table 4.1 and f represents the strength of the input laser.

The set-up in this chapter differs from the one described by the LLE (4.1) in the sense that the backscattered field interacts with the laser. This leads to a bi-directionally coupled system, whose dynamics is governed by the equations [86, 153]:

$$\begin{aligned}
\dot{n}(t) =& \iota - \gamma n(t) - g(|a_L(t)|^2)(n(t)-1)|a_L(t)|^2, \\
\dot{a}_L(t) =& \Big[\frac{1-\mathrm{i}\alpha_H}{2}\big(g(|a_L(t)|^2)n_0(n(t)-1) - \kappa_d\big) + \mathrm{i}(\omega_m - \omega_0)\Big]a_L(t) \\
& - \kappa_{ext} a_B(t)\mathrm{e}^{\mathrm{i}\phi}, \\
\dot{a}_F(x,t) =& \Big[-1 + \mathrm{i}d\frac{\partial^2}{\partial x^2} + \mathrm{i}(|a_F(x,t)|^2 + 2|a_B(t)|^2)\Big]a_F(x,t) \\
& + \mathrm{i}\kappa_{sc} a_B(t) - \kappa_{ext} a_L(t)\mathrm{e}^{\mathrm{i}\phi}, \\
\dot{a}_B(t) =& \left[-1 + \mathrm{i}\Big(|a_B(t)|^2 + \frac{1}{\pi}\int_0^{2\pi}|a_F(x,t)|^2\mathrm{d}x\Big)\right]a_B(t) \\
& + \mathrm{i}\overline{\kappa}_{sc}\frac{1}{2\pi}\int_0^{2\pi} a_F(x,t)\mathrm{d}x,
\end{aligned} \tag{4.2}$$

where a_F, a_B, and a_L represent the normalized forward, backward, and laser fields, respectively, and n denotes the normalized carrier number. For simplicity, the backscattered field a_B is only represented by its central mode $a_B(t) := \frac{1}{2\pi}\int_0^{2\pi} a_B(x,t)\mathrm{d}x$ which is independent of the angular variable x. The derivation of the system (4.2) for the normalized unknowns a_F, a_L, a_B, n from the system (4.12) for the physical quantities A_F, A_B, A_L, N is detailed in Appendix 4.6. Again, $a_F(x,t)$ is 2π-periodic in x. The first two ordinary differential equations (ODEs) describe the laser dynamics for the carrier number n and laser field a_L with an additional coupling to the backscattered field a_B. Due to saturation, the gain $g = g(|a_L|^2)$ of the laser is modeled by a decreasing function of the power $|a_L|^2$ of the laser field and takes the following explicit form

$$g(p) = \frac{\mu^2\sigma}{g_0 + \varepsilon p}.$$

The third and fourth equation describe the microresonator dynamics by a coupled system of a partial differential equation (PDE) for the forward field a_F and an ODE for the backscattered field a_B with the laser field a_L as a source term for the forward field. All normalized constants are

quantity	normal. quantity	physical meaning
I/e	ι	bias current of laser/ elementary electronic charge
Γ	γ	carrier recombination rate
N_0	n_0	carrier number at transparency
μ	–	optical mode confinement factor
α_{H}	–	linewidth enhancement factor
K_d	κ_d	photon decay rate of laser cavity
Ω_0	ω_0	laser cavity mode frequency
K_{ext}	κ_{ext}	external coupling
K	–	photon decay rate of resonator
Ω_{m}	ω_{m}	frequency of resonator mode
D	d	group velocity dispersion
g_0	–	single photon induced Kerr frequency shift
K_{sc}	κ_{sc}	$\mathbb{C}$-valued coupling between forward and backward modes
ϕ	–	accumulative phase between laser and resonator
σ	–	small-signal gain
ϵ	ε	saturation power

Table 4.1: Description of the physical meaning of the constants and their normalized counterparts. A dash indicates that there is no normalized version of the corresponding quantity in the equations.

explained in Table 4.1. The values of the physical constants and their relation to the normalized constants can be found in Table 4.2 in Appendix 4.6.

4.3 Results and comparison between uni-directionally and bi-directionally coupled system

Here, we present the results of our numerical bifurcation analysis for the bi-directionally coupled model (4.2) and compare them with those for the uni-directionally coupled system (4.1). An important part of the model is the normalized instantaneous laser frequency $\omega_L(t)$, which is defined by $\omega_L(t) - \omega_m := -\frac{d}{dt}\arg(a_L(t))$. In the standard LLE-model (4.1) we have that $\omega_L(t) = \omega_0$ is fixed whereas in the extended model (4.2) it is in general time-dependent and part of the unknowns.

A typical scenario found in the standard LLE (4.1) is the following:

(i) Stable 1-soliton solutions of the form $a(x,t) = \underline{a}^0(x)\mathrm{e}^{\mathrm{i}(\zeta t+\phi)}$, $\zeta = \omega_m - \omega_0$ can be determined by a bifurcation analysis, [54, 102, 103, 106].

(ii) Stability means that, while keeping the laser in its prescribed invariant state, small perturbations of the resonator field a relax to a translate of the 1-soliton on an exponentially fast time scale [62, 137].

For the extended system (4.2), instead, we find the following scenario:

(iii) A bifurcation analysis reveals stable time-harmonic 1-soliton solutions

$$\begin{aligned} a_F(x,t) &= \underline{a}_F^0(x)\mathrm{e}^{\mathrm{i}\zeta_{\text{new}}t}, & n(t) &= n^0 \\ a_L(t) &= \underline{a}_L^0\mathrm{e}^{\mathrm{i}\zeta_{\text{new}}t}, & a_B(t) &= \underline{a}_B^0\mathrm{e}^{\mathrm{i}\zeta_{\text{new}}t}. \end{aligned} \tag{4.3}$$

These 1-solitons bifurcate from spatially constant time-harmonic states with $\zeta = \omega_m - \omega_0$ as the bifurcation parameter. They are qualitatively very similar to the ones found in (4.1) and the laser is still operating in a cw-mode with constant detuning $\zeta_{\text{new}} = \omega_m - \omega_L = \zeta - \Delta\zeta$ where $\Delta\zeta = \frac{\kappa_{\text{ext}}}{\underline{a}_L^0}\,\mathrm{Im}(\underline{a}_B^0\mathrm{e}^{\mathrm{i}\phi}(1 - \mathrm{i}\alpha_{\mathrm{H}}))$.

We illustrate this bifurcation phenomenon in Figure 4.2 by a numerically computed bifurcation diagram together with examples of the localized steady states. The y-axis plots the power

$$\|\underline{a}_F\|_2 = \left(\int_0^{2\pi} |\underline{a}_F(x)|^2 \mathrm{d}x\right)^{1/2}$$

of the forward-field component of the solution in the bifurcation diagram. The black line in the left panel of Figure 4.2 indicates the curve of homogeneous constants solutions with the blue circles corresponding to the bifurcation points (BP). Continuation at the BPs (blue curves) then gives rise to many localized states (1-, 2-, 3-, and 4-soliton solutions) which are depicted in the panels on the right. Most importantly, we find that the 1-soliton solutions lie on the bifurcation branch which bifurcates from the BP with the largest value of ζ. As we follow the curve of constant solutions starting from $\zeta = -20$, this corresponds to the last BP in the diagram. The observation, that the 1-solitons are found on the last bifurcation branch is consistent with the theory developed for the standard LLE without backward laser coupling [54].

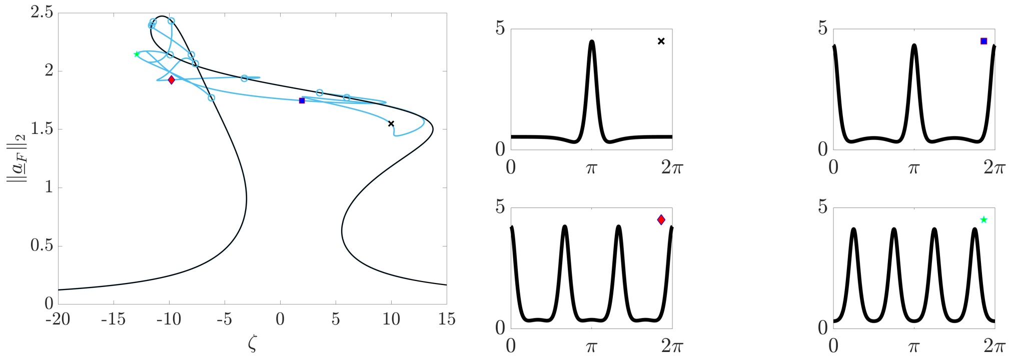

Figure 4.2: Left: Bifurcation diagram of stationary solutions to (4.2), with physical constants set as in Table 4.2 and $\phi = 0.64 \times \pi \approx 2.0106, \kappa_{ext} = 6.4550$. As bifurcation parameter we choose the frequency difference ζ. Right: Plots of the forward field component $\underline{a}_F$ at the markers, which are localized states and the y-axis shows $|\underline{a}_F|^2$.

An important difference to (4.1) is the stability behavior of the solitons.

(iv) We stress that stability of 1-solitons as solutions to the extended model (2) implies robustness against perturbations in all four unknowns n, a_L, a_F and a_B. In particular, applying such perturbations to the soliton solutions (4.3), we find solutions $a_F(x,t) = \underline{a}_F(x,t)\mathrm{e}^{\mathrm{i}\int_0^t (\omega_m - \omega_L(s))\, ds}$ with the property that

$$\omega_L(t) \to \omega_0 + \Delta\zeta, \tag{4.4}$$

$$a_F(x,t) \to \underline{a}_F^0(x + x_0)\mathrm{e}^{\mathrm{i}(\zeta_{\text{new}} t + \phi_0)} \tag{4.5}$$

with $\Delta\zeta = \frac{\kappa_{\text{ext}}}{\underline{a}_L^0} \operatorname{Im}(\underline{a}_B^0 \mathrm{e}^{\mathrm{i}\phi}(1 - \mathrm{i}\alpha_{\mathrm{H}}))$ exponentially fast as $t \to \infty$. Likewise

$$a_L(t) \to \underline{a}_L^0 \mathrm{e}^{\mathrm{i}(\zeta_{\text{new}} t + \phi_0)}, \quad n(t) \to n^0, \quad a_B(t) \to \underline{a}_B^0 \mathrm{e}^{\mathrm{i}(\zeta_{\text{new}} t + \phi_0)}$$

at an exponential rate. This means that due to the backcoupling the laser automatically adjusts and self-corrects towards a time-harmonic cw-mode with constant detuning. We emphasize that this self-correction of the laser found in the extended system (4.2) cannot be observed in (4.1) due to the static form of the laser field.

(v) Here $\phi_0 = \int_0^\infty (\omega_L(s) - \omega_L)\, ds$ is an accumulated phase and x_0 is a translational displacement of the asymptotic steady state a_F^0 induced by the perturbation, see Appendix 4.5 for more details. The translational shift is visible in the left panel of Figure 4.3. Its value is fully determined by the initial perturbation.

We use numerical time integration to illustrate the enhanced dynamical stability features of soliton solutions to (4.2) described in (iv). To this end, we employ a Lie-Trotter splitting to approximate the dynamics with the perturbed soliton state **x** of Figure 4.2 as an initial condition. The simulations in Figure 4.3 confirms that a perturbation of a stable 1-soliton solution causes all four unknowns to relax against a steady state. In particular, as $t \to \infty$ we see that $r(t) := |a_L(t)|$ and $\underline{a}_B(t)$ converge to $r^{\mathbf{x}}$ (magnitude of laser field) and $\underline{a}_B^{\mathbf{x}}$ (central mode of back-scattered field of the 1-soliton state), respectively. Moreover, the lower-right panel of Figure 4.3 shows the long-term stabilization of the laser field as in (4.4), where the frequency detuning is automatically adjusted to sustain a soliton solution.

Next, we systematically analyze the influence of the parameters ϕ and κ_{ext} on the existence range of 1-soliton solutions. This is of particular interest for practical applications as both parameters can be changed in the experimental set-up. The sensitivity analysis relies on an algorithm based on numerical bifurcation theory, which is explained in Appendix 4.5. The results are shown in Figure 4.4. Here, each point in the colored areas represents a pair (ζ, ϕ) (left panel) or (ζ, κ_{ext}) (right panel) for which 1-solitons can be found. The green areas indicate subsets of the existence range which consist of highly localized 1-solitons with a prescribed full width at half maximum (FWHM) value of at most 0.4. In addition to that, the blue line in the left panel shows the values of the parameters at which these solitons bifurcate from the homogeneous stead states.

For the accumulated phase ϕ, we observe that the existence range bends from negative values of ζ to positive values, and then back to negative, as ϕ increases from 0 to π. We note that due to the symmetry

$$(\underline{a}_F, \underline{a}_B, \underline{a}_L, n, \phi) \mapsto (\underline{a}_F, \underline{a}_B, -\underline{a}_L, n, \phi + \pi)$$

the existence range is π-periodic in ϕ and thus we can deduce from Figure 4.4 the existence range for all values of $\phi \in [0, 2\pi]$. The right panel of Figure 4.4 shows the sensitivity with respect to the coupling κ_{ext} where the remaining parameters are fixed to the physical values in Table 4.2. We see that the minimal value of $\kappa_{ext} \approx 6.0$ is needed in order to form bifurcating branches supporting 1-soliton solutions. As κ_{ext} increases beyond a second value, we find localized solitons with a FWHM of at most 0.4. This is explained by the fact that low values of κ_{ext} lead to a small excitation of the cavity field obstructing the existence of highly localized states, whereas high values allow energy transfer from the laser to the microresonator resulting in strong forcing.

4.4 Summary

We have investigated the bi-directionally coupled system (4.2) of a cw-laser and a Kerr-nonlinear microresonator where the backward propagating resonator field interacts with the pump laser.

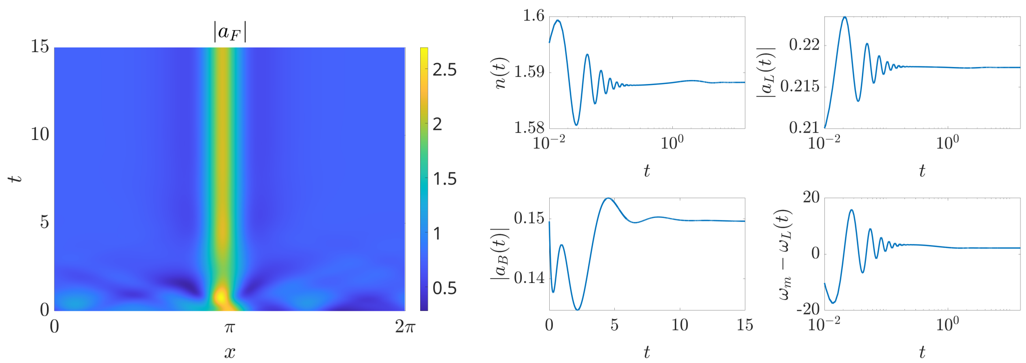

Figure 4.3: Time integration starting from a small perturbation of the stable 1-soliton solution with label **x** in Figure 4.2. Left: Space-time plot of the forward field $a_F(x,t)$, where the color code indicates the magnitude $|a_F(x,t)|$. Right: Simulations of the carrier density $n(t)$ (top left), magnitude of the laser field $|a_L(t)|$ (top right), the backward field $|a_B(t)|$ (bottom left), and the instantaneous laser frequency $\omega_m - \omega_L(t)$ (bottom right).

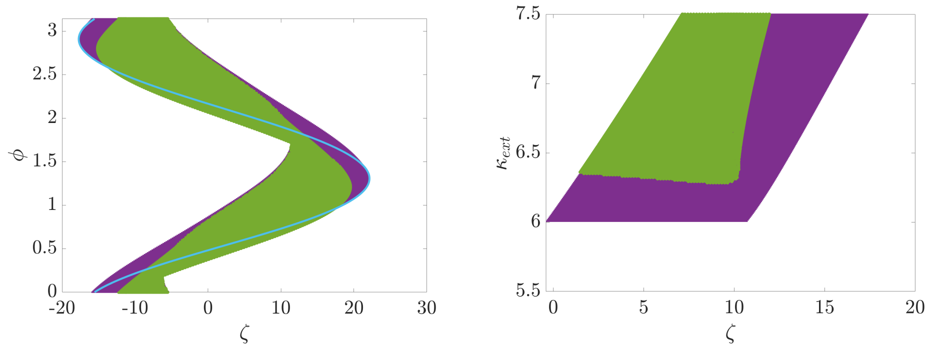

Figure 4.4: Left: Existence chart in the ζ–ϕ plane of stationary 1-soliton solutions to (4.7), bifurcating from flat homogeneous states. Colored area (purple) shows the 1-soliton existence range. The blue curve indicates the position of the BP from which these states emerge. Right: Existence range (purple) in the ζ–κ_{ext} plane. In both figures, the green subsets depicts the range of 1-solitons with FWHM of at most 0.4. The other parameters are fixed to the physical values in Tablet 4.2.

Using numerical bifurcation tools we have shown that the system supports stable time-harmonic 1-solitons states with stationary normalized instantaneous laser frequency $\omega_L = \omega_0 + \Delta\zeta$. These states have enhanced stability properties in the sense that under sufficiently small perturbations of the unknowns not only the laser and microresonator fields relax towards the 1-solitons but most importantly $\omega_L(t)$ self-corrects towards the value ω_L which sustains the 1-soliton state. Such a dynamic self-correction of the laser is impossible for the standard LLE where the laser field has a static frequency. Our approach also offers the advantage of systematically exploring existence ranges of 1-solitons with respect to parameter variations. We have investigated the sensitivity with respect to accumulated phase ϕ and external coupling strength κ_{ext}. It turns out that 1-solitons can only exist provided ϕ lies within a finite band of values and the external coupling strength κ_{ext} exceeds a minimal value.

4.5 A: Numerical bifurcation and sensitivity analysis

Here we explain the set-up of the numerical bifurcation analysis and we introduce the algorithm which we use for analyzing the sensitivity of the 1-soliton existence range with respect to parameter variations.

First note that if the unknowns a_L, a_F and a_B in (4.2) are all shifted by a common constant phase factor then they still solve the same system. This means that in contrast to the standard Lugiato-Lefever equation (4.1), which is only invariant under spatial shifts, the system (4.2) has an additional invariance under phase shifts of a_L, a_F, a_B. We eliminate the phase invariance by transitioning into a rotating frame in which the laser field a_L is real-valued. This is necessary for a robust convergence of the numerical bifurcation algorithm as it excludes a continuation of nontrivial states in the direction where all components are shifted by a constant phase factor. To this end, we write the laser field in polar coordinates

$$a_L(t) = r(t)\mathrm{e}^{-\mathrm{i}\int_0^t (\omega_L(s)-\omega_m)\mathrm{d}s}, \quad r(t) = |a_L(t)|,$$

where we recall that $\omega_L(t) - \omega_m = -\frac{d}{dt}\arg(a_L(t))$. Moreover, we introduce $\underline{a}_F, \underline{a}_B$ defined by

$$a_F(x,t) = \underline{a}_F(x,t)\mathrm{e}^{-\mathrm{i}\int_0^t (\omega_L(s)-\omega_m)\mathrm{d}s}, \quad a_B(t) = \underline{a}_B(t)\mathrm{e}^{-\mathrm{i}\int_0^t (\omega_L(s)-\omega_m)\mathrm{d}s}.$$

Substituting the formula for a_L into (4.2) we obtain an equation for the instantaneous laser frequency

$$\omega_L(t) = \omega_0 + \frac{1}{2}\alpha_H \left(g(r(t)^2)n_0(n(t)-1) - \kappa_d\right) + \frac{\kappa_{\text{ext}}}{r(t)} \operatorname{Im}\left(\underline{a}_B(t)\mathrm{e}^{\mathrm{i}\phi}\right). \tag{4.6}$$

With the shorthand

$$\Theta(r,z,n) := \frac{1}{2}\alpha_H \left(g(r^2)n_0(n-1) - \kappa_d\right) + \frac{\kappa_{\text{ext}}}{r} \operatorname{Im}\left(z\mathrm{e}^{\mathrm{i}\phi}\right)$$

and the frequency offset $\zeta = \omega_m - \omega_0$ the remaining system for $n, r, \underline{a}_F, \underline{a}_B$ becomes (we suppress their arguments):

$$\begin{aligned}
\dot{n} =& \iota - \gamma n - g(r^2)(n-1)r^2, \\
\dot{r} =& \frac{1}{2}\left(g(r^2)n_0(n-1) - \kappa_d\right) r - \kappa_{ext} \operatorname{Re}\left(\underline{a}_B \mathrm{e}^{\mathrm{i}\phi}\right), \\
\underline{\dot{a}}_F =& \Big[-1 - \mathrm{i}\left(\zeta - \Theta(r, \underline{a}_B, n)\right) + \mathrm{i}d\frac{\partial^2}{\partial x^2} + \mathrm{i}(|\underline{a}_F|^2 + 2|\underline{a}_B|^2)\Big]\underline{a}_F \\
&+ \mathrm{i}\kappa_{sc}\underline{a}_B - \kappa_{ext} r \mathrm{e}^{\mathrm{i}\phi}, \\
\underline{\dot{a}}_B =& \left[(-1 - \mathrm{i}(\zeta - \Theta(r, \underline{a}_B, n)) + \mathrm{i}\left(|\underline{a}_B|^2 + \frac{2}{2\pi}\int_0^{2\pi} |\underline{a}_F(x,\cdot)|^2 \mathrm{d}x\right)\right] a_B \\
&+ \mathrm{i}\overline{\kappa}_{sc}\frac{1}{2\pi}\int_0^{2\pi} \underline{a}_F(x,\cdot)\mathrm{d}x.
\end{aligned} \tag{4.7}$$

Observe that in this system the invariance under phase-shifts is no longer present since r is a real-valued quantity.

When searching for steady states of (4.7) we set the time-derivatives $\dot{n}, \dot{r}, \underline{\dot{a}}_F, \underline{\dot{a}}_B$ to zero. This results in a ODE for $\underline{a}_F$ coupled to three algebraic equations for the remaining fields which include nonlocal terms of $\underline{a}_F$. For the instantaneous laser frequency ω_L, equation (4.6) implies that it will also be constant. This means that stationary solutions of (4.7) correspond to time-harmonic solutions of (4.2).

Combing (4.6) and the equation for r in (4.7) we obtain

$$\omega_L = \omega_0 + \frac{\kappa_{ext}}{r} \operatorname{Im}\left(\underline{a}_B \mathrm{e}^{\mathrm{i}\phi}(1 - \mathrm{i}\alpha_H)\right). \tag{4.8}$$

The steady equation for n can be solved in terms of r, yielding the formula

$$n = n(r) = \frac{\iota + g(r^2)r^2}{\gamma + g(r^2)r^2}.$$

Upon substituting this into the system (4.7), we obtain an ODE for $\underline{a}_F$, coupled to two algebraic equations for r and $\underline{a}_B$, given by

$$
\begin{aligned}
0 =& \Big[-1 - \mathrm{i}\,(\zeta - \Theta(r, \underline{a}_B, n(r))) + \mathrm{i}d\frac{\partial^2}{\partial x^2} + \mathrm{i}(|\underline{a}_F|^2 + 2|\underline{a}_B|^2)\Big]\underline{a}_F \\
&+ \mathrm{i}\kappa_{sc}\underline{a}_B - \kappa_{ext} r \mathrm{e}^{\mathrm{i}\phi}, \\
0 =& \frac{1}{2}\left(g(r^2)n_0(n(r)-1) - \kappa_d\right) r - \kappa_{ext}\,\mathrm{Re}\left(\underline{a}_B \mathrm{e}^{\mathrm{i}\phi}\right), \\
0 =& \left[(-1 - \mathrm{i}(\zeta - \Theta(r, \underline{a}_B, n(r))) + \mathrm{i}\left(|\underline{a}_B|^2 + \frac{2}{2\pi}\int_0^{2\pi} |\underline{a}_F(x)|^2 \mathrm{d}x\right)\right]\underline{a}_B \\
&+ \mathrm{i}\overline{\kappa}_{sc}\frac{1}{2\pi}\int_0^{2\pi} \underline{a}_F(x)\mathrm{d}x.
\end{aligned}
\tag{4.9}
$$

We consider the ODE for $\underline{a}_F$ on the interval $x \in [0, \pi]$ equipped with homogeneous Neumann boundary conditions

$$\partial_x \underline{a}_F(0) = \partial_x \underline{a}_F(\pi) = 0. \tag{4.10}$$

This eliminates the spatial shift invariance of $\underline{a}_F$ in (4.7), which is necessary to avoid unwanted path continuation in the translational direction. Note that we recover periodic solutions on $[0, 2\pi]$ by reflection:

$$\underline{a}_{F,\mathrm{per}}(x) = \begin{cases} \underline{a}_F(x), & x \in [0, \pi], \\ \underline{a}_F(2\pi - x), & x \in [\pi, 2\pi]. \end{cases}$$

We then perform the numerical bifurcation analysis for (4.9)-(4.10) . The bifurcation analysis is conducted using the MATLAB package `pde2path` [146], which is designed for the numerical continuation and bifurcation analysis of systems of PDEs. We discretize the boundary value problem for $\underline{a}_F$ using linear finite elements, and add the equations for $\underline{a}_B$ and r as algebraic constraints. Throughout our bifurcation experiments, we choose the frequency offset ζ as the bifurcation parameter, since it can be varied in the physical experiments.

Stability of the bifurcating solutions can be determined by computing the eigenvalues of the Jacobian $\mathcal{J}$ associated with the discretized version of (4.9). For this purpose it is necessary to reintroduce the translational invariance by using periodic boundary conditions on $[0, 2\pi]$ for the field $\underline{a}_F$ as otherwise only stability with respect to even perturbations is determined. Thus, the solution is stable if all eigenvalues λ of $\mathcal{J}$ satisfy $\mathrm{Re}(\lambda) < 0$ with the exception of a zero eigenvalue present due to the continuous shift symmetry of the system. As a result, stable solutions exhibit a spectral gap (if we neglect to zero eigenvalue), which causes a relaxation of

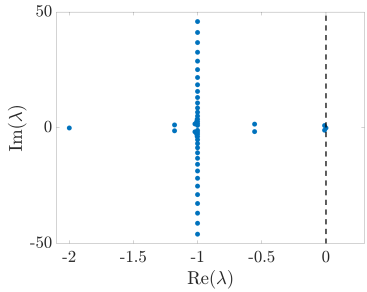

Figure 4.5: Eigenvalues of the Jacobian operator evaluated at the 1-soliton state with label **x** in Figure 4.2. We see that all eigenvalues except for the zero eigenvalue have strictly negative real part indicating dynamical stability and relaxation to a spatial translate of the 1-soliton solution at an exponential rate.

the perturbed solutions to a spatial translate of it on exponentially fast timescales, cf. Figure 4.3. Figure 4.5 shows the spectrum of the 1-soliton states **x** indicating its stability.

The bifurcation analysis is also used to algorithmically determine (a subset of) the 1-soliton existence range in the parameter space. To describe our algorithm for the sensitivity analysis, we need an auxiliary result from bifurcation theory. Suppose that $(\underline{a}_{F,0}, r, \underline{a}_{B,0}, \zeta_0) \in \mathbb{C}\times\mathbb{R}\times\mathbb{C}\times\mathbb{R}$ is a nondegenerate BP on the constant solution curve, where nondegenerate means that the associated Jacobian of (4.9) has a simple zero eigenvalue. Then the forward field component of the bifurcating solutions can be locally parametrized by $s \in (-s_0, s_0)$ with some $0 < s_0 \ll 1$, [83, Corollary I.4.2]. More precisely, we have the expansion

$$\underline{a}_F(x; s) = \underline{a}_{F,0} + \underline{a}_{F,1}\cos(kx)s + \mathcal{O}(s^2) \text{ as } s \to 0 \tag{4.11}$$

with $k \in \mathbb{N}_0$ and $\underline{a}_{F,1} \in \mathbb{C}$. Below, we will use this formula for $k = 1$ to determine those BPs at which 1-solitons branch away from flat states. Now let us fix a system parameter $\varsigma \neq \zeta$. We analyze the influence of ς on the existence range of the 1-solitons. Therefore, we chose a discretization $\varsigma_{\text{min}} = \varsigma_1 < \varsigma_2 < \cdots < \varsigma_m = \varsigma_{\text{max}}$ with $m \in \mathbb{N}$. Our algorithm consists of three steps.

Algorithm: For $j = 1, \ldots, m$ do:

1. Set $\varsigma = \varsigma_j$ and compute the constant solution curve with bifurcation parameter $\zeta \in [\zeta_{\text{min}}, \zeta_{\text{max}}]$.

2. Mark all nondegenerate BPs on this curve for which (4.11) holds with $k = 1$.

3. Compute and plot all bifurcation branches starting from the marked BPs of Step 2 in ζ as long as the forward field component $\underline{a}_F$ satisfies the following conditions:

$$\text{(i)} \quad \{x_{\max}\} = \text{argmax}_x |\underline{a}_F(x)|^2 \subset \{0, \pi\},$$
$$\text{(ii)} \quad \left| \frac{\int_{\pi/2}^{x_{\max}} |\underline{a}'_F(z)|^2 \mathrm{d}z}{\int_{\pi/2}^{\pi - x_{\max}} |\underline{a}'_F(z)|^2 \mathrm{d}z} \right| > 1.75.$$

We briefly comment on each step. Steps 1 and 2 are necessary to compute nontrivial solutions of (4.9) where Step 2 ensures that we only calculate bifurcation branches on which we expect to find 1-soliton states. Heuristically, we have observed that k_0-solitons emerge from BPs where (4.11) holds with $k = k_0$. This observation is consistent with simulations in [54] for the standard LLE (4.1). Once we are on a 1-soliton branch, we stop the continuation as soon as the solutions deform into 2-solitons states (or different wave forms). In system (4.9) we have observed that a 1-soliton branch generically connects to a 2-soliton branch. The stopping criteria are given by conditions (i) and (ii) in Step 3. The first ensures that the periodic extension of $\underline{a}_F(x)$ has a single maximum on the periodicity cell $[0, 2\pi]$, while the second ensures that the solutions are sufficiently localized around this maximum. Note that the value 1.75 determines the degree of localization; it can also be varied, with low values meaning that we look for a low degree of localization, while high values enforce a high degree of localization.

4.6 B: Derivation of the normalized system from the physical model

Here we show how the normalized coupled system (4.2) is derived from the system in physical units and we give the values of the physical and normalized parameters used in our simulations.

quantity	value	normal. quantity	value
I/e	$\frac{0.5}{1.6\times10^{-19}}$ A/C	ι	41.4466
Γ	1×10^{9} s^{-1}	γ	2.6526
N_0	2×10^{8}	n_0	1.0610
μ	0.5	–	–
α_{H}	2.5	–	–
K_d	$2\times10^{10}\times2\pi$ rad/s	κ_d	833.3333
K_{ext}	$[5.5,7]\times120\times10^{6}\times\pi$ rad/s	κ_{ext}	$[5.5,7]$
K	$120\times10^{6}\times2\pi$ rad/s	–	–
D	$7.5\times10^{6}\times2\pi$ rad/s	d	0.125
g_0	1.866 rad^{-1}	–	–
K_{sc}	$15\times10^{6}\times2\pi$ rad/s	κ_{sc}	0.25
ϕ	$[0,\pi]$	–	–
σ	1×10^{4} s^{-1}	–	–
ϵ	0	ε	0

Table 4.2: Values of the physical constants and their normalized counterparts. A dash indicates that there is no normalized version of the corresponding quantity in the equations.

The bi-directionally coupled laser-resonator system in physical units is given by [86, 153]

$$\begin{aligned}\dot{N}(t) =& I - \Gamma N(t) - \frac{\mu^2\sigma}{1+\epsilon|A_L(t)|^2}(N(t)-N_0)|A_L(t)|^2,\\ \dot{A}_L(t) =& \Big[\frac{1-\mathrm{i}\alpha_H}{2}\left(\frac{\mu^2\sigma}{1+\epsilon|A_L(t)|^2}(N(t)-N_0)-K_d\right)+\mathrm{i}(\Omega_m-\Omega_0)\Big]A_L(t)\\ &-K_{ext}A_B(t)\mathrm{e}^{\mathrm{i}\phi}\\ \dot{A}_F(x,t) =& \Big[-\frac{K}{2}+\mathrm{i}D\frac{\partial^2}{\partial x^2}+\mathrm{i}g_0(|A_F(x,t)|^2+2|A_B(t)|^2)\Big]A_F(x,t)\\ &+\mathrm{i}K_{sc}A_B(t)-K_{ext}A_L(t)\mathrm{e}^{\mathrm{i}\phi},\\ \dot{A}_B(t) =& \left[-\frac{K}{2}+\mathrm{i}g_0\big(|A_B(t)|^2+\frac{1}{\pi}\int_0^{2\pi}|A_F(x,t)|^2\mathrm{d}x\big)\right]A_B(t)\\ &+\mathrm{i}\overline{K}_{sc}\frac{1}{2\pi}\int_0^{2\pi}A_F(x,t)\mathrm{d}x.\end{aligned} \tag{4.12}$$

The physical meanings of the constants are explained in Table 4.1. The system (4.2) is then obtained by the transformation

$$a_F(x,t) = \sqrt{\frac{2g_0}{K}} A_F\left(x, \frac{2}{K}t\right), \qquad a_B(t) = \sqrt{\frac{2g_0}{K}} A_B\left(\frac{2}{K}t\right),$$
$$a_L(t) = \sqrt{\frac{2g_0}{K}} A_L\left(\frac{2}{K}t\right), \qquad n(t) = N_0^{-1} N\left(\frac{2}{K}t\right),$$

and the normalized constants are given by

$$\iota = 2\frac{I}{KN_0}, \quad \gamma = 2\frac{\Gamma}{K}, \quad \kappa_d = 2\frac{K_d}{K}, \quad \omega_m = 2\frac{\Omega_m}{K}, \quad \omega_0 = 2\frac{\Omega_0}{K},$$
$$\kappa_{\text{ext}} = 2\frac{K_{\text{ext}}}{K}, \quad d = 2\frac{D_2}{K}, \quad \kappa_{\text{sc}} = 2\frac{K_{\text{sc}}}{K}, \quad n_0 = 2\frac{N_0}{K\mu}, \quad \varepsilon = \frac{\epsilon K}{2}.$$

Part III

Fronts and pulses in spatially periodic systems

5 Pinning in the extended Lugiato-Lefever equation

This chapter is a reprint[1] of the published article [18] written by the author of the thesis in collaboration with Dmitry Pelinovsky[2] and Wolfgang Reichel. The article was adapted to fit the layout of this thesis. We thank Huanfa Peng for showing us how to derive our main equation (5.3) from the two-mode pumping variant (5.2) of the LLE.

Abstract

We consider a variant of the Lugiato-Lefever equation (LLE), which is a nonlinear Schrödinger equation on a one-dimensional torus with forcing and damping, to which we add a first-order derivative term with a potential $\varepsilon V(x)$. The potential breaks the translation invariance of LLE. Depending on the existence of zeroes of the effective potential V_{eff}, which is a suitably weighted and integrated version of V, we show that stationary solutions from $\varepsilon = 0$ can be continued locally into the range $\varepsilon \neq 0$. Moreover, the extremal points of the ε-continued solutions are located near zeros of V_{eff}. We therefore call this phenomenon *pinning* of stationary solutions. If we assume additionally that the starting stationary solution at $\varepsilon = 0$ is spectrally stable with the simple zero eigenvalue due to translation invariance being the only eigenvalue on the imaginary axis, we can prove asymptotic stability or instability of its ε-continuation depending on the sign of V_{eff}' at the zero of V_{eff} and the sign of ε. The variant of the LLE arises in the description of optical frequency combs in a Kerr nonlinear ring-shaped microresonator which is pumped by two different continuous monochromatic light sources of different frequencies and different powers. Our analytical findings are illustrated by numerical simulations.

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5.1 Introduction

The Lugiato-Lefever equation [89] is the most commonly used model to describe electromagnetic fields inside a resonant cavity that is pumped by a strong continuous laser source. Inside the cavity the electromagnetic field propagates and suffers losses due to curvature and/or material imperfections. Most importantly, the cavity consists of a Kerr-nonlinear material so that triggered by modulation instability the field may experience a nonlinear interaction of the pumped and resonantly enhanced modes of the cavity. Under appropriate driving conditions of the resonant cavity and the laser, a stable Kerr-frequency comb may form in the cavity, which is a spatially localized and spectrally broad waveform.

Since their discovery by the 2005 noble prize laureate Theodor Hänsch, frequency combs have seen an enormously wide field of applications, e.g., in high capacity optical communications [92], ultrafast optical ranging [143], optical frequency metrology [144], or spectroscopy [113, 156]. The Lugiato-Lefever equation (LLE) is an amplitude equation for the electromagnetic field inside the cavity derived by means of the slowly varying envelope approximation.

In the following we assume that the cavity is a ring-shaped microresonator with normalized perimeter 2π. Using dimensionless quantities and writing $u(x,t) = \sum_{k\in\mathbb{Z}} u_k(t)\mathrm{e}^{\mathrm{i}kx}$ for the slowly varying and 2π-periodic amplitude of the electromagnetic field, the LLE in its original form [89] reads as

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0, \qquad (x,t) \in \mathbb{T}\times\mathbb{R}, \tag{5.1}$$

where $\mathbb{T}$ is a circle of length 2π. Here $d > 0$ is the case of anomalous dispersion, whereas $d < 0$ refers to normal dispersion. The laser pump with frequency ω_{p_0} has the power $|f_0|^2$ and the detuning value ζ represents the off-set between the pump frequency ω_{p_0} and the closest resonance frequency ω_0 of the resonator. Finally, the value $\mu > 0$ quantifies the damping coefficient.

More recently, novel pumping schemes have been discussed [140], where instead of one monochromatic laser pump one uses a dual laser pump with two different frequencies as a source term. Using again dimensionless quantities the resulting equation is given by

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0 + \mathrm{i}f_1\mathrm{e}^{\mathrm{i}(k_1 x - \nu_1 t)}, \qquad (x,t) \in \mathbb{T}\times\mathbb{R}, \tag{5.2}$$

cf. [56, 58, 140] for a detailed derivation. In contrast to (5.1) there is now a second source term with pump strength f_1 and k_1 stands for the second pumped mode (the first pumped mode is again $k_0 = 0$). This gives rise to two detuning variables $\zeta = \frac{2}{\kappa}(\omega_0 - \omega_{p_0})$, $\zeta_1 = \frac{2}{\kappa}(\omega_{k_1} - \omega_{p_1})$ and they define $\nu_1 = \zeta - \zeta_1 + dk_1^2$. One of the main outcomes of [56] is that the stationary states of

(5.2) are far more localized than the stationary states of (5.1), and the best results can be achieved when $f_0 = f_1$ among all power distributions such that $f_0^2 + f_1^2$ is kept constant.

However, there are cases where a power distribution $|f_0| \gg |f_1|$ is more adequate in physical experiments. In this case, it is shown in Appendix 5.6 that one can derive from (5.2) the perturbed LLE in the form

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + \mathrm{i}\varepsilon V(x)\partial_x u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{5.3}$$

where in the physical context $V(x) = \omega_1 - 2dk_1^2 \frac{f_1}{f_0}\cos(x)$ and $\varepsilon = 1$. However, if ω_1 and $k_1^2 f_1/f_0$ are small, we will consider (5.3) as the perturbed LLE with $\varepsilon \in \mathbb{R}$ being small and $V \in C^1([-\pi,\pi],\mathbb{R})$ being a generic periodic potential. Recall that (5.3) is already set in a moving coordinate frame. In its stationary form the equation becomes

$$-du'' + \mathrm{i}\varepsilon V(x)u' + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0 = 0, \qquad x \in \mathbb{T}. \tag{5.4}$$

The main questions addressed in this paper are the existence and stability of the stationary solution of (5.3). Our main results, which are stated in detail in Section 5.2, can be summarized as follows:

- In Theorem 5.3 we prove existence of solutions of (5.4) for small ε provided the effective potential V_{eff} changes sign, where V_{eff} is a weighted integrated version of the coefficient function V.
- In Theorems 5.8 and 5.9 we prove stability/instability properties of the solution obtained from Theorem 5.3 with the time evolution of (5.3).
- In Section 5.3 we illustrate the findings of our theorems by numerical simulations. The numerical simulations show that the location of the intensity extremum of the ε-continued solutions does not change significantly for small ε. Therefore, we call this phenomenon *pinning of solutions at zeroes of the effective potential V_{eff}*.

Existence and bifurcation behavior of solutions of (5.1) have been studied quite well, cf. [54, 55, 61, 62, 91, 94, 102, 105, 106] and their stability properties have been investigated in [8, 36, 37, 67, 69, 70, 111, 137, 141]. Analytical and numerical investigations of (5.2) have recently been reported [9, 56, 58]. In contrast, we are not aware of any treatment of (5.3). However, a related problem, where instead of $\mathrm{i}\varepsilon V(x)u'$ a term of the form $\varepsilon V(x)u$ appears in the NLS equation, has been quite well studied, cf. [2, 50, 110]. In this case solutions are pinned near nondegenerate critical points of V_{eff} instead of the zeroes of V_{eff} as in our case. A different kind of pinning at the intensity maximum of an external forcing term occurs in [121].

5.2 Main results

In this section we present our main results regarding existence and stability of stationary solutions of (5.3). For $\varepsilon = 0$ there is a plethora of non-trivial (non-constant) stationary solutions, cf. [54,91]. We start with such a solution under the assumption of its non-degeneracy according to the following definition.

Definition 5.1. A non-constant solution $u \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ of (5.4) for $\varepsilon = 0$ is called non-degenerate if the kernel of the linearized operator

$$L_u\varphi := -d\varphi'' + (\zeta - \mathrm{i}\mu - 2|u|^2)\varphi - u^2\bar{\varphi}, \quad \varphi \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$$

consists only of $\operatorname{span}\{u'\}$.

Remark 5.2. Note that $L_u : H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \to L^2([-\pi,\pi],\mathbb{C})$ is a compact perturbation of the isomorphism $-d\partial_x^2 + \operatorname{sign}(d) : H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \to L^2([-\pi,\pi],\mathbb{C})$ and hence a Fredholm operator. Notice also that $\operatorname{span}\{u'\}$ always belongs to the kernel of L_u due to translation invariance in x for $\varepsilon = 0$. Non-degeneracy means that except for the obvious candidate u' (and its real multiples) there is no other element in the kernel of L_u.

One can ask the question whether non-constant non-degenerate solutions at $\varepsilon = 0$ in Definition 5.1 may be continued into the regime of $\varepsilon \neq 0$. In order to describe the continuation, we denote such a solution by u_0 and its spatial translations by $u_\sigma(x) := u_0(x-\sigma)$. The non-degeneracy assumption implies that $\ker L_{u_\sigma} = \operatorname{span}\{u'_\sigma\}$. Since the adjoint operator $L^*_{u_0}$ also has a one-dimensional kernel there exists $\phi^*_0 \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ such that $\ker L^*_{u_0} = \operatorname{span}\{\phi^*_0\}$. Notice that with $\phi^*_\sigma(x) = \phi^*_0(x-\sigma)$ we find $\ker L^*_{u_\sigma} = \operatorname{span}\{\phi^*_\sigma\}$.

Before stating our existence result, let us clarify the assumption on the potential V.

(A1) The potential $V : [-\pi,\pi] \to \mathbb{R}, x \mapsto V(x)$ is a 2π-periodic, continuously differentiable function.

The existence result is given by the following theorem.

Theorem 5.3. *Let $d \in \mathbb{R}\setminus\{0\}, f_0, \zeta \in \mathbb{R}$, $\mu > 0$ be fixed and assume that (A1) holds. Let furthermore $u_0 \in H^2_{per}([-\pi,\pi],\mathbb{C})$ be a non-constant, non-degenerate solution of* (5.4) *for $\varepsilon = 0$. If σ_0 is a simple zero of the function*

$$\sigma \mapsto V_{\text{eff}}(\sigma) := \operatorname{Re}\int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\bar{\phi}^*_0\,dx \tag{5.5}$$

then there exists a continuous curve $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \to u(\varepsilon) \in H^2_{per}([-\pi,\pi],\mathbb{C})$ *consisting of solutions of* (5.4) *with* $\|u(\varepsilon) - u_0(\cdot - \sigma_0)\|_{H^2} \le C|\varepsilon|$ *for some constant* $C > 0$.

Remark 5.4. The value of σ_0 is determined from the existence of a unique solution $v \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ of the linear inhomogeneous equation

$$L_{u_{\sigma_0}} v = -\mathrm{i}V(x)u'_{\sigma_0}$$

with the property that $v \perp_{L^2} u'_{\sigma_0}$. Fredholm's condition shows that σ_0 is a zero of V_{eff} if and only if this equation is uniquely solvable. Simplicity of the zero of V_{eff} yields the result of Theorem 5.3.

Remark 5.5. The observation, that V_{eff} having a zero is a necessary condition for continuability of solutions in the case where $V(x) \equiv V_0$ is constant occured in [12], where traveling solitons with speed V_0 were considered.

To investigate the stability of a stationary solution u we introduce the expansion

$$u(x) + v(x,t) = u_1(x) + \mathrm{i}u_2(x) + v_1(x,t) + \mathrm{i}v_2(x,t)$$

and substitute this into the perturbed LLE (5.3). After neglecting the quadratic and cubic terms in v and separating real and imaginary parts we obtain the linearized system for $\boldsymbol{v} = (v_1, v_2)$ which reads as

$$\partial_t \boldsymbol{v} = \widetilde{L}_{u,\varepsilon}\boldsymbol{v}$$

and the linearization has the form

$$\widetilde{L}_{u,\varepsilon} = JA_u - I(\mu - \varepsilon V(x)\partial_x) \tag{5.6}$$

with

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_u := \begin{pmatrix} -d\partial_x^2 + \zeta - (3u_1^2 + u_2^2) & -2u_1u_2 \\ -2u_1u_2 & -d\partial_x^2 + \zeta - (u_1^2 + 3u_2^2) \end{pmatrix}.$$

In the following we will often identify functions in $\mathbb{C}$ as vector-valued functions in $\mathbb{R} \times \mathbb{R}$ and use the notation

$$u = u_1 + \mathrm{i}u_2 \in \mathbb{C} \quad \leftrightarrow \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2.$$

We denote the spectrum of $\widetilde{L}_{u,\varepsilon}$ in $L^2([-\pi,\pi]) \times L^2([-\pi,\pi])$ by $\sigma(\widetilde{L}_{u,\varepsilon})$ and the resolvent set of $\widetilde{L}_{u,\varepsilon}$ by $\rho(\widetilde{L}_{u,\varepsilon})$.

For our stability results we require one additional spectral assumption on the non-degenerate solution u_0 regarding the spectrum of $\widetilde{L}_{u_0,0}$.

(A2) The eigenvalue $0 \in \sigma(\widetilde{L}_{u_0,0})$ is algebraically simple and there exists $\xi > 0$ such that

$$\sigma(\widetilde{L}_{u_0,0}) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -\xi\} \cup \{0\}.$$

Remark 5.6. By Fredholm theory, the assumption of simplicity of the zero eigenvalue of $\widetilde{L}_{u_0,0}$ is equivalent to $\boldsymbol{u}_0' \notin \operatorname{range} \widetilde{L}_{u_0,0} = \operatorname{span}\{J\boldsymbol{\phi}_0^*\}^\perp$. It will be convenient to use the normalization $\langle \boldsymbol{u}_0', J\boldsymbol{\phi}_0^* \rangle_{L^2} = \int_{-\pi}^{\pi} \boldsymbol{u}_0' \cdot J\boldsymbol{\phi}_0^* \, dx = 1$. We also note that

$$\int_{-\pi}^{\pi} \boldsymbol{u}_0' \cdot J\boldsymbol{\phi}_0^* \, dx = \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i} u_0' \bar{\phi}_0^* \, dx.$$

Before stating the stability results, let us clarify that u_0' and ϕ_0^* are linearly independent if $\mu \neq 0$ and the integrand of V_{eff} is generically nonzero. We also clarify the parity of eigenfunctions in $\ker L_u^*$ and $\ker L_u$ if u_0 is even in x. This is used for many practical computations.

Lemma 5.7. *Let $u_0 \in H^2_{per}([-\pi,\pi],\mathbb{C})$ be a non-constant, non-degenerate solution of* (5.4) *for $\varepsilon = 0$ and $\mu \neq 0$. Then the following holds:*

(i) u_0' and ϕ_0^* are linearly independent and $\operatorname{Re} \mathrm{i} u_0' \bar{\phi}_0^* \not\equiv 0$,

(ii) if u_0 is even then ϕ_0^* is odd.

Proof. Part (i): By using the decomposition (5.6) with $u = u_0$ and $\varepsilon = 0$, the eigenvalue problems $L_{u_0} u_0' = 0$ and $L_{u_0}^* \phi_0^* = 0$ are equivalent to

$$JA_{u_0} \begin{pmatrix} u_{01}' \\ u_{02}' \end{pmatrix} = \mu \begin{pmatrix} u_{01}' \\ u_{02}' \end{pmatrix}, \qquad JA_{u_0} \begin{pmatrix} \phi_{01}^* \\ \phi_{02}^* \end{pmatrix} = -\mu \begin{pmatrix} \phi_{01}^* \\ \phi_{02}^* \end{pmatrix}.$$

But since (u_{01}', u_{02}') and $(\phi_{01}^*, \phi_{02}^*)$ are eigenvectors to the different eigenvalues μ and $-\mu$ of JA_{u_0}, respectively, they are linearly independent. Moreover, the determinant of the matrix with columns $(u_{01}', u_{02}')^T$ and $(\phi_{01}^*, \phi_{02}^*)^T$ coincides with $\operatorname{Re} \mathrm{i} u_0' \bar{\phi}_0^*$, and is not identically zero.

Part (ii): By assumption we have that $\ker L_{u_0} = \operatorname{span}\{u_0'\}$ and u_0' is an odd function. Let us define the restriction of L_{u_0} onto the odd functions

$$L_{u_0}^{\#} : H^2_{\text{per,odd}} \to L^2_{\text{per,odd}},\ \varphi \mapsto L_{u_0}\varphi.$$

Then $L_{u_0}^{\#}$ is again an index 0 Fredholm operator with $\ker L_{u_0}^{\#} = \operatorname{span}\{u_0'\}$. Further we have $(L_{u_0}^{\#})^* = (L_{u_0}^*)^{\#}$ where

$$(L_{u_0}^*)^{\#} : H^2_{\text{per,odd}} \to L^2_{\text{per,odd}},\ \varphi \mapsto L_{u_0}^*\varphi$$

is the restriction of the adjoint onto the odd functions. But since $1 = \dim\ker(L_{u_0}^*)^{\#} = \dim\ker L_{u_0}^*$ it follows that $\ker(L_{u_0}^*)^{\#} = \ker L_{u_0}^*$ and hence $\phi_0^* \in H^2_{\text{per,odd}}$ as claimed. □

The stability results are given by the following two theorems. A stationary solution u of (5.4) is called spectrally stable if $\operatorname{Re}(\lambda) \leq 0$ for all eigenvalues λ of $\widetilde{L}_{u,\varepsilon}$. It is called spectrally unstable if there exists one eigenvalue λ with $\operatorname{Re}(\lambda) > 0$.

Theorem 5.8. *Let $d \in \mathbb{R} \setminus \{0\}, f_0, \zeta \in \mathbb{R}, \mu > 0$ be fixed, $u_0 \in H^2_{per}([-\pi,\pi],\mathbb{C})$ be as in Theorem 5.3, and assume that (A1) and (A2) hold. With σ_0 being a simple zero of V_{eff} as in Theorem 5.3, we have*

$$V_{\text{eff}}'(\sigma_0) = \operatorname{Re}\int_{-\pi}^{\pi} \mathrm{i}V'(x+\sigma_0)u_0'\bar{\phi}_0^*\,dx = \langle V'(\cdot+\sigma_0)\boldsymbol{u}_0', J\boldsymbol{\phi}_0^*\rangle_{L^2} \neq 0.$$

Then there exists $\varepsilon_0 \in (0,\varepsilon^*]$ such that on the solution branch $(-\varepsilon_0,\varepsilon_0) \ni \varepsilon \to u(\varepsilon) \in H^2_{per}([-\pi,\pi],\mathbb{C})$ of (5.4) with $u(0) = u_{\sigma_0}$ the solutions $u(\varepsilon)$ are spectrally stable for $V_{\text{eff}}'(\sigma_0)\cdot\varepsilon > 0$ and spectrally unstable for $V_{\text{eff}}'(\sigma_0)\cdot\varepsilon < 0$.

In the next theorem we will show that spectral stability leads to nonlinear asymptotic stability because $\varepsilon \neq 0$ breaks the translational symmetry. Thus, the zero eigenvalue of the linearisation disappears and the asymptotic orbital stability result from [137] can be adapted and leads to a slightly improved result.

Theorem 5.9. *Let $u(\varepsilon) \in H^2_{per}([-\pi,\pi],\mathbb{C})$ be a spectrally stable stationary solution of* (5.3) *for a small value of $\varepsilon \neq 0$ as in Theorem 5.8. Then $u(\varepsilon)$ is asymptotically stable, i.e., there exist $\eta, \delta, C > 0$ with the following properties. If $\varphi \in C([0,T), H^1_{per}([-\pi,\pi],\mathbb{C}))$ is a solution of* (5.3) *with maximal existence time T and*

$$\|\varphi(\cdot,0) - u(\varepsilon)\|_{H^1} < \delta$$

then $T = \infty$ *and*

$$\|\varphi(\cdot,t) - u(\varepsilon)\|_{H^1} \leq C\mathrm{e}^{-\eta t}\|\varphi(\cdot,0) - u(\varepsilon)\|_{H^1} \quad \textit{for all } t \geq 0.$$

Remark 5.10. Due to periodicity of V_{eff} on $\mathbb{T}$, simple zeros of V_{eff} comes in pairs. By Theorems 5.8 and 5.9 , one simple zero gives a solution branch consisting of asymptotically stable solutions for any sign of ε. Moreover, at the bifurcation point $\varepsilon = 0$ there is an exchange of stability, i.e., the zero eigenvalue crosses the imaginary axis with non zero speed.

Remark 5.11. In [37, 67] the authors constructed spectrally stable solutions u of (5.4) for $\varepsilon = 0$ in the case of anomalous dispersion $d > 0$. These solutions satisfy the spectral condition $\sigma(\widetilde{L}_{u,0}) \subset \{-2\mu\} \cup \{\operatorname{Re} z = -\mu\} \cup \{0\}$ and are therefore non-degenerate starting solutions for which our main results from Theorems 5.3, 5.8, and 5.9 hold. Moreover, in Section 5.3 we provide examples of numerically computed solutions for which we checked (A2) numerically.

Remark 5.12. If u is a solution of (5.4) then the relation

$$\int_{-\pi}^{\pi} (u'\bar{u} - \bar{u}'u)dx = 0$$

holds. This constraint is satisfied by every even function u. In fact, the only solutions of equation (5.4) for $\varepsilon = 0$ that we are aware of are even around $x = 0$ (up to a shift).

Remark 5.13. Using Fourier-series expansions of the potential V and the function $\operatorname{Re}(\mathrm{i}u_0'\bar{\phi}_0^*)$, we can derive the Fourier expansion of the effective potential V_{eff}. Assume that u_0 is a non-constant, non-degenerate, even solution of (5.4) for $\varepsilon = 0$ and $\mu \neq 0$. Then, according to Lemma 5.7, the function $f := \operatorname{Re}(\mathrm{i}u_0'\bar{\phi}_0^*)$ is a non-constant, even, real-valued, periodic function and we can write $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{\mathrm{i}kx}$ with Fourier-coefficients satisfying $\hat{f}_k = \hat{f}_{-k} = \overline{\hat{f}_k}$ for all $k \in \mathbb{Z}$. Expanding $V(x) = \sum_{k\in\mathbb{Z}} \hat{V}_k \mathrm{e}^{\mathrm{i}kx}$ a straightforward calculation shows

$$V_{\text{eff}}(\sigma) = \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\bar{\phi}_0^* \, dx = \sum_{k\in\mathbb{Z}} \hat{V}_k \hat{f}_k \mathrm{e}^{\mathrm{i}k\sigma}.$$

In particular, since not all $\hat{f}_k$, $|k| \geq 1$, vanish, we can choose V as an average-zero trigonometric polynomial such that V_{eff} is also an average-zero non-trivial trigonometric polynomial, and thus generally has simple zeroes.

Remark 5.14. In the limit where u_0 is highly localized around 0 (e.g. the limit $d \to 0\pm$) and the potential V is wide, the effective potential V_{eff} is well approximated by the actual potential V. More precisely we find the asymptotic

$$V_{\text{eff}}(\sigma) = \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0' \bar{\phi}_0^* \, dx \approx V(\sigma) \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i}u_0' \bar{\phi}_0^* \, dx = V(\sigma)$$

provided $\langle \mathrm{i}u_0', \phi_0^* \rangle_{L^2} = 1$. Thus, the asymptotically stable branch bifurcates from a simple zero σ_0 of V with $V'(\sigma_0)\varepsilon > 0$.

Remark 5.15. The criterion for stability of stationary solutions in Theorem 5.8 can be written in a more precise form for small μ in the case of solitary waves. This limit is considered in Appendix 5.7.

To summarize, our main results show that nondegenerate solutions of (5.4) for $\varepsilon = 0$ can be extended locally for small $\varepsilon \neq 0$ provided the effective potential V_{eff} has a sign-change. Depending on the derivative of V_{eff} at a simple zero we determined the stability properties of these solutions. It remains an open problem to give a criterion on V or V_{eff} for the existence/stability of stationary solutions which applies when $|\varepsilon|$ is large.

5.3 Numerical simulations

In the following we describe numerical simulations of solutions to (5.4). We choose $f_0 = 2$, $\mu = 1$, $V(x) = 0.1 + 0.5\cos(x)$ and $d = \pm 0.1$. All computations are done with help of the Matlab package `pde2path` (cf. [42, 146]) which has been designed to numerically treat continuation and bifurcation in boundary value problems for systems of PDEs.

We begin with the description of the stationary solutions of the LLE (5.1), which are the same as the solutions of (5.4) for $\varepsilon = 0$. The corresponding results are mainly taken from [54, 91]. There is a curve of trivial, spatially constant solutions, cf. black line in Figure 5.1, and this is the same curve for anomalous dispersion ($d = 0.1$) and normal dispersion ($d = -0.1$). Next one finds that there are finitely many bifurcation points on the curve of trivial solutions (blue dots). Depending on the sign of the dispersion parameter d one can find now the branches of the single solitons on the periodic domain $\mathbb{T}$. In the following descriptions we always follow the path of trivial solutions by starting from negative values of ζ.

For $d = 0.1$ (left panel in Figure 5.1) along the trivial branch there is a last bifurcation point which gives rise to a single bright soliton branch (red line). This branch has a turning point, at which the solutions change from unstable (dashed) to stable (solid), and after the turning point it tends back

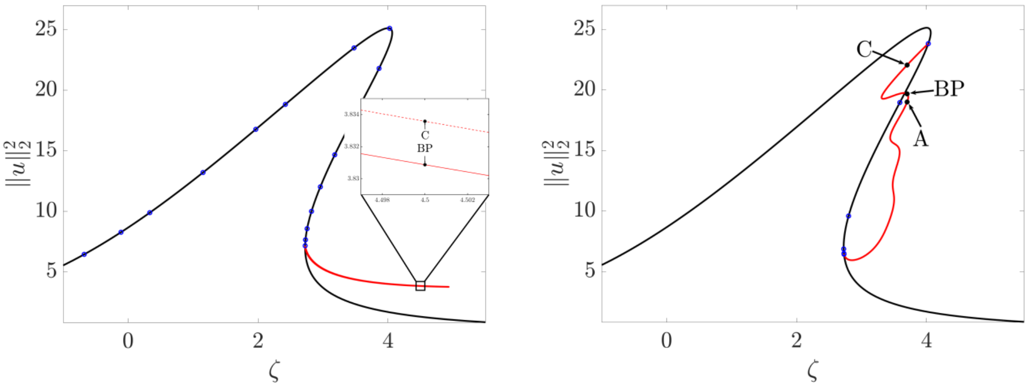

Figure 5.1: Bifurcation diagram for the case $\varepsilon = 0$. Blue dots indicate bifurcation points on the line of trivial solutions (black). The red curve denotes the single soliton solution branch. The point BP is chosen as a starting point for Theorem 5.3. Further solutions on the same branch for the same value of ζ are denoted by C (left panel) and A, C (right panel). Left panel for $d = 0.1$, right panel for $d = -0.1$.

towards the trivial branch. Thus, the red line in the left panel of Figure 5.1 represents two different but almost identical curves, which can be seen in the enlarged inset. We have chosen a solution at the point BP on the stable branch as a starting point for the illustration of Theorems 5.3 and 5.8.

In the case where $d = -0.1$ (right panel in Figure 5.1) along the trivial branch there is a first bifurcation point from which a single dark soliton branch (red line) bifurcates. Near the second turning point of this branch the most localized single solitons live and we have chosen a stable dark soliton solution at the point BP as a starting point for the illustration of Theorems 5.3 and 5.8.

Next we explain the global picture in Figure 5.2 of the continuation in ε of the chosen point BPs from the $\varepsilon = 0$ case in Figure 5.1. The local picture is covered by Theorem 5.3. First we note the following symmetry: since $V(x)$ is even around $x = 0$ we find that $(u(x), \varepsilon)$ solves (5.4) if and only if $(u(-x), -\varepsilon)$ satisfies (5.4). Since reflecting u does not affect the L^2-norm we see for $\varepsilon > 0$ an exact mirror image of the one for $\varepsilon < 0$.

Next we observe that continuation curves in ε appear to be unbounded for $d = 0.1$ (upper left panel of Figure 5.2) and closed and bounded for $d = -0.1$ (lower left panel of Figure 5.2). In our example the map $\sigma \mapsto V_{\text{eff}}(\sigma) := \mathrm{Re}\int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\bar{\phi}_0^*\,dx$ has two zeroes in the periodic domain $\mathbb{T}$ denoted by σ_0 and σ_1. Since moreover u_0 is even and consequently u_0', ϕ_0^* are odd we see that the effective potential V_{eff} is also even and hence $\sigma_0 = -\sigma_1$. Thus, continuation in ε works for the starting point $u_0(\cdot - \sigma_0)$ (blue curve) and $u_0(\cdot + \sigma_0)$ (green curve) with $\sigma_0 < 0$. As predicted from Theorem 5.8 locally on one side of $\varepsilon = 0$ we have stable and on the other side unstable solutions. On the top and bottom right panels of Figure 5.2 we see the graph of $|u|^2$ for several solutions on the continuation diagram. The top left panel and the bottom left panel

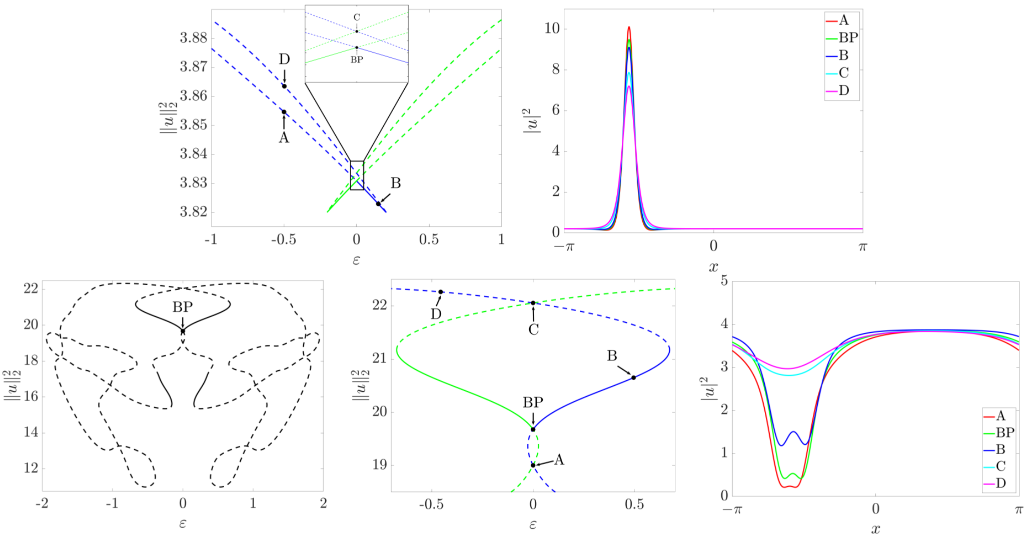

Figure 5.2: Continuation diagrams w.r.t ε with stability regions (solid = stable; dashed = unstable) and solutions at designated points. The two different zeroes of V_{eff} give rise to two different continuation curves (blue and green). Top panels: $d = 0.1$, $\zeta = 3.7$. Bottom panels: $d = -0.1$, $\zeta = 4.5$ with zoom (middle panel) of the continuation curve near the starting point.

indicate that the ε-continuation curves meet all other nontrivial points (C for $d = 0.1$ and A, C for $d = -0.1$) at $\varepsilon = 0$ from Figure 5.1.

In Figure 5.3 we show the starting solutions $u_0(x - \sigma_0)$ and $u_0(x - \sigma_1)$ together with the potential $V(x)$. Here the zeroes $\sigma_0 < 0 < \sigma_1$ of the effective potential V_{eff} are shown as blue and green dots and we already observed $\sigma_0 = -\sigma_1$ due to the evenness of both V and V_{eff}. Since u_0 is sufficiently strongly localized the zeroes of V_{eff} are well approximated by the zeroes of V and the starting solutions are thus centered near the zeroes of V. Therefore, by applying Remark 5.14, we see that slope of V at the center of the soliton being positive in the blue bifurcation point indicates that the ε-continuation will be stable for $\varepsilon > 0$ and unstable for $\varepsilon < 0$. The stability behavior is exactly opposite for the green bifurcation point. The stability considerations are valid both for $d = 0.1$ and $d = -0.1$.

Finally, let us illustrate the spectral stability properties of the ε-continuations in Figure 5.4. For $\varepsilon = 0$ we see in the left panel the spectrum of the linearization around u_0 with most of spectrum having real part -1 due to damping $\mu = 1$ and further spectrum in the left half plane together with the zero eigenvalue caused by shift-invariance. Now we consider how the critical eigenvalue behaves when ε varies. We do this for the case where the starting soliton sits at a zero of V_{eff} with positive slope, cf. blue bifurcation point in Figure 5.3. As predicted, the critical eigenvalue moves into the complex left half plane for $\varepsilon > 0$ rendering the ε-continuations stable. Since the starting

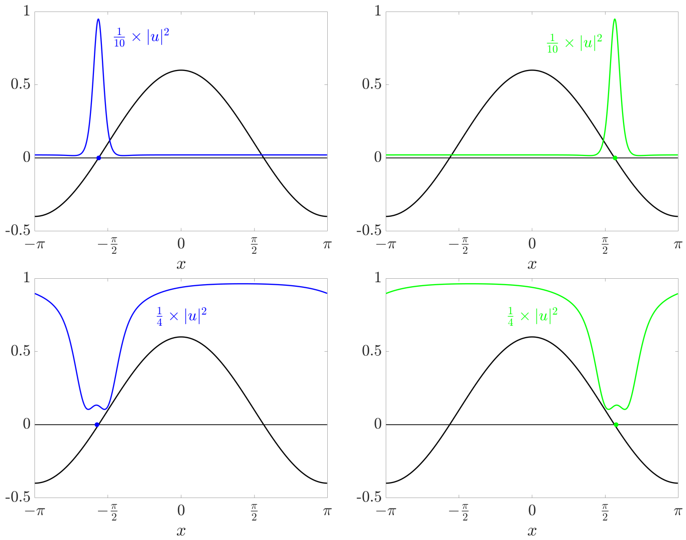

Figure 5.3: Top row: $d = 0.1$, bottom row: $d = -0.1$. Left panels: starting solutions $u_0(x - \sigma_0)$ together with $V(x)$ and negative zero σ_0 of V_{eff} (blue dot). Stability for $\varepsilon > 0$, instability for $\varepsilon < 0$. Right panels: starting solutions $u_0(x + \sigma_0)$ together with $V(x)$ and positive zero $\sigma_1 = -\sigma_0$ of V_{eff} (green dot). Stability for $\varepsilon < 0$, instability for $\varepsilon > 0$.

solitons are sufficiently localized $-V'(\sigma_0)$ predicts well the slope of the critical eigenvalue, cf. Lemma 5.18 and Remark 5.14.

5.4 Proof of the existence result

Theorem 5.3 will be proved via Lyapunov-Schmidt reduction and the Implicit Function Theorem. Fix the values of d, ζ, μ and f_0. Let $u_0 \in H^2_{\text{per}}([-\pi, \pi], \mathbb{C})$ be a non-degenerate solution of (5.4) for $\varepsilon = 0$ and recall that for $\sigma \in \mathbb{R}$ its shifted copy $u_\sigma(x) := u_0(x - \sigma)$ is also a solution of (5.4) for $\varepsilon = 0$.

Proof of Theorem 5.3: We seek solutions u of (5.4) of the form

$$u = u_\sigma + v, \quad \langle v, u'_\sigma \rangle_{L^2} = 0, \quad v \in H^2_{\text{per}}([-\pi, \pi], \mathbb{C}).$$

Inserting it into (5.4) we obtain the following equation for the correction term v:

$$L_{u_\sigma} v + \mathrm{i}\varepsilon V(u'_\sigma + v') - N(v, \sigma) = 0, \quad \langle v, u'_\sigma \rangle_{L^2} = 0, \tag{5.7}$$

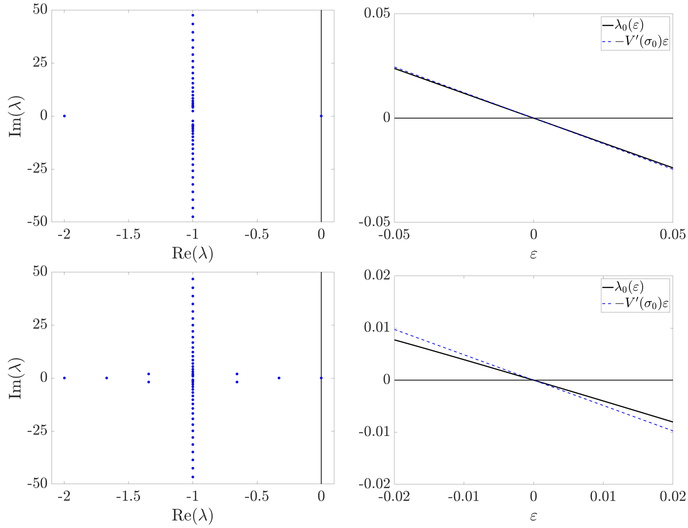

Figure 5.4: Top: $d = 0.1$, bottom: $d = -0.1$. Left: spectrum for $\varepsilon = 0$. Right: critical eigenvalue $\lambda_0(\varepsilon)$ together with $-V'(\sigma_0)\varepsilon$ as functions of ε.

with nonlinearity given by

$$N(v,\sigma) = \bar{u}_\sigma v^2 + 2u_\sigma|v|^2 + |v|^2 v.$$

The nonlinearity is a sum of quadratic and cubic terms in v. Since H^2_{per} is a Banach algebra, it is clear that for every $R > 0$, there exists $C_R > 0$ such that

$$\|N(v,\sigma)\|_{L^2} \leq C_R \|v\|^2_{H^2}, \quad \text{for every } \; v \in H^2_{\text{per}} : \;\; \|v\|_{H^2} \leq R. \tag{5.8}$$

Moreover, since $V \in L^\infty$ it follows that

$$\|\mathrm{i}\varepsilon V(u'_\sigma + v')\|_{L^2} \leq |\varepsilon| \|V\|_{L^\infty} \|u_\sigma + v\|_{H^2}.$$

Next we solve (5.7) according to the Lyapunov-Schmidt reduction method. Define the orthogonal projections

$$P_\sigma : L^2 \to \operatorname{span}\{u'_\sigma\} \subset L^2, \quad Q_\sigma : L^2 \to \operatorname{span}\{\phi^*_\sigma\}^\perp \subset L^2$$

onto $\ker L_{u_\sigma}$ and $(\ker L^*_{u_\sigma})^\perp = \operatorname{span}\{\phi^*_\sigma\}^\perp = \operatorname{range} L_{u_\sigma}$, respectively. Then (5.7) can be decomposed into a non-singular and singular equation

$$Q_\sigma \left(L_{u_\sigma}(I-P_\sigma)v + \mathrm{i}\varepsilon V(u'_\sigma + v') - N(v,\sigma)\right) = 0, \tag{5.9}$$

$$\langle \mathrm{i}\varepsilon V u'_\sigma, \phi^*_\sigma\rangle_{L^2} + \langle \mathrm{i}\varepsilon V v' - N(v,\sigma), \phi^*_\sigma\rangle_{L^2} = 0, \tag{5.10}$$

$$\langle v, u'_\sigma\rangle_{L^2} = 0. \tag{5.11}$$

Notice that the linear part $Q_\sigma L_{u_\sigma}(I-P_\sigma)$ in (5.9) is invertible between the σ-dependent subspaces $(\ker L_{u_\sigma})^\perp$ and $\operatorname{range} L_{u_\sigma}$. Therefore, the Implicit Function Theorem cannot be applied directly to solve (5.9). However, (5.9) together with the orthogonality condition (5.11) are equivalent to $F(v,\varepsilon,\sigma) = 0$ with

$$F(v,\varepsilon,\sigma) := Q_\sigma \left(L_{u_\sigma}(I-P_\sigma)v + \mathrm{i}\varepsilon V(u'_\sigma + v') - N(v,\sigma)\right) + \phi^*_\sigma\langle v, u'_\sigma\rangle_{L^2}$$

and $F : H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \times \mathbb{R} \times \mathbb{R} \to L^2([-\pi,\pi],\mathbb{C})$. Here the added term $\phi^*_\sigma\langle v, u'_\sigma\rangle_{L^2}$ enforces $v \perp u'_\sigma$. For any fixed $\sigma_0 \in \mathbb{R}$ we have $F(0,0,\sigma_0) = 0$. Since

$$D_v F(0,0,\sigma_0)\varphi = L_{u_{\sigma_0}}\varphi + \phi^*_{\sigma_0}\langle \varphi, u'_{\sigma_0}\rangle_{L^2}$$

is an isomorphism from H^2_{per} to L^2, we can apply the Implicit Function Theorem which gives the existence of a smooth function $v = v(\varepsilon,\sigma)$ solving the problem $F(v(\varepsilon,\sigma),\varepsilon,\sigma) = 0$ for (ε,σ) in a neighborhood of $(0,\sigma_0)$. Moreover from the definition of F we see that $F(0,0,\sigma) = 0$ so that $v(0,\sigma) = 0$ which implies the bound

$$\|v(\varepsilon,\sigma)\|_{H^2} \le C|\varepsilon|, \tag{5.12}$$

for (ε,σ) close to $(0,\sigma_0)$. As a consequence, $\|v'(\varepsilon,\sigma)\|_{L^2} \le C|\varepsilon|$, where $v'(\varepsilon,\sigma)$ denotes the derivative of v with respect to x. Inserting $v(\varepsilon,\sigma)$ into the singular equation (5.10) we end up with with the 2-dimensional problem

$$f(\varepsilon,\sigma) := \langle \mathrm{i}\varepsilon V u'_\sigma, \phi^*_\sigma\rangle_{L^2} + \langle \mathrm{i}\varepsilon V v'(\varepsilon,\sigma) - N(v(\varepsilon,\sigma),\sigma), \phi^*_\sigma\rangle_{L^2} = 0.$$

To find non-trivial solutions let us define the C^1-function

$$\tilde{f}(\varepsilon,\sigma) := \begin{cases} \varepsilon^{-1} f(\varepsilon,\sigma), & \varepsilon \neq 0, \\ \partial_\varepsilon f(0,\sigma), & \varepsilon = 0. \end{cases}$$

By (5.8), (5.12) and by our assumption on the effective potential V_{eff} there exists $\sigma_0 \in \mathbb{R}$ such that

$$\tilde{f}(0,\sigma_0) = \langle \mathrm{i}V u'_{\sigma_0}, \phi^*_{\sigma_0} \rangle_{L^2} = \mathrm{Re} \int_{-\pi}^{\pi} \mathrm{i}V(x) u'_{\sigma_0} \bar{\phi}^*_{\sigma_0} dx = V_{\text{eff}}(\sigma_0) = 0$$

and

$$\partial_\sigma \tilde{f}(0,\sigma_0) = \partial_\sigma \langle \mathrm{i}V u'_\sigma, \phi^*_\sigma \rangle_{L^2} |_{\sigma=\sigma_0} = \partial_\sigma \, \mathrm{Re} \int_{-\pi}^{\pi} \mathrm{i}V(x) u'_\sigma \bar{\phi}^*_\sigma dx \Big|_{\sigma=\sigma_0} = V'_{\text{eff}}(\sigma_0) \neq 0.$$

Hence the Implicit Function Theorem can be applied to the problem $\tilde{f}(\varepsilon,\sigma) = 0$ and yields a curve of unique non-trivial solutions $\sigma = \sigma(\varepsilon)$ to the singular equation $f(\varepsilon,\sigma) = 0$ such that $\sigma(0) = \sigma_0$. Finally we conclude that $u(\varepsilon) = u_0(\cdot - \sigma(\varepsilon)) + v(\varepsilon, \sigma(\varepsilon))$ solves (5.4) for small ε. □

5.5 Proof of the stability result

In this section we will find the condition when the stationary solutions obtained in Theorem 5.3 as a continuation of a stable solution u_0 of the LLE (5.1) are spectrally stable against co-periodic perturbations in the perturbed LLE (5.3). Moreover, we prove the nonlinear asymptotic stability of stationary spectrally stable solutions.

5.5.1 Preliminary notes

For our stability analysis we consider (5.3) as a 2 dimensional system by decomposing the function $u = u_1 + \mathrm{i}u_2$ into real and imaginary part. This leads us to the system of dynamical equations

$$\begin{cases} \partial_t u_1 = -d\partial_x^2 u_2 + \varepsilon V(x)\partial_x u_1 + \zeta u_2 - \mu u_1 - (u_1^2 + u_2^2)u_2 + f_0, \\ \partial_t u_2 = d\partial_x^2 u_1 + \varepsilon V(x)\partial_x u_2 - \zeta u_1 - \mu u_2 + (u_1^2 + u_2^2)u_1, \end{cases} \tag{5.13}$$

equipped with the 2π-periodic boundary condition on $\mathbb{R}$. The spectral problem associated to the nonlinear system (5.13) can be written as

$$\widetilde{L}_{u,\varepsilon} \boldsymbol{v} = \lambda \boldsymbol{v}, \quad \lambda \in \mathbb{C}, \quad \boldsymbol{v} \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \times H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$$

and the linearized operator $\widetilde{L}_{u,\varepsilon}$ is given by (5.6). Note that the operator A_u in the decomposition (5.6) is self-adjoint on $L^2([-\pi,\pi],\mathbb{C}) \times L^2([-\pi,\pi],\mathbb{C})$ and $\widetilde{L}_{u,\varepsilon}$ is an index 0 Fredholm operator.

Moreover we see that if u_0 is a non-degenerate solution of (5.4) for $\varepsilon = 0$ then the following relations for the linearized operators are true:

$$\ker \widetilde{L}_{u_0,0} = \operatorname{span}\{\boldsymbol{u}_0'\}, \quad \ker \widetilde{L}^*_{u_0,0} = \operatorname{span}\{J\boldsymbol{\phi}_0^*\},$$

where the vectors $\boldsymbol{u}_0' = (u_{01}', u_{02}')$ and $\boldsymbol{\phi}_0^* = (\phi_{01}^*, \phi_{02}^*)$ are obtained from $u_0' = u_{01}' + \mathrm{i}u_{02}'$ and $\phi_0^* = \phi_{01}^* + \mathrm{i}\phi_{02}^*$. We recall that $\langle \boldsymbol{u}_0', J\boldsymbol{\phi}_0^* \rangle_{L^2} = 1$ due to normalization, cf. Remark 5.6.

Finally we observe that since the embedding

$$H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \times H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \hookrightarrow L^2([-\pi,\pi],\mathbb{C}) \times L^2([-\pi,\pi],\mathbb{C})$$

is compact, the linearization has compact resolvents and thus the spectrum of $\widetilde{L}_{u,\varepsilon}$ consists of isolated eigenvalues with finite multiplicity where the only possible accumulation point is at ∞. In the following we will use the spaces

$$H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) =: X, \quad H^1_{\text{per}}([-\pi,\pi],\mathbb{C}) =: Y, \quad L^2([-\pi,\pi],\mathbb{C}) =: Z.$$

Both the proof of Theorem 5.8 and Theorem 5.9 rely on the next lemma for the linearized operator $\widetilde{L}_{u(\varepsilon),\varepsilon}$ where $u(\varepsilon)$ lies on the solution branch of Theorem 5.3 and $|\varepsilon|$ is small. The lemma gives spectral bounds for eigenvalues with large imaginary part together with a uniform resolvent estimate. The proof is presented in Section 5.5.4.

Lemma 5.16. *Denote* $\Lambda_{\lambda^*} := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geq 0, |\operatorname{Im}(\lambda)| \geq \lambda^*\}$. *Given* $\varepsilon_1 > 0$ *sufficiently small there exists* $\lambda^* > 0$ *such that we have the uniform resolvent bound*

$$\sup_{\lambda \in \Lambda_{\lambda^*}} \|(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}\|_{L^2 \to L^2} < \infty$$

for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$.

Remark 5.17. The uniformity of the resolvent estimate on the imaginary axis allows to sharpen the above result as follows. If we define S as the supremum from Lemma 5.16 and let $0 < \delta < 1/S$ then the estimate

$$\sup_{\lambda \in \Lambda_{\lambda^*} - \delta} \|(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}\|_{L^2 \to L^2} < \infty$$

holds. This follows from taking inverses in the identity

$$(\lambda - \delta - \widetilde{L}_{u(\varepsilon),\varepsilon}) = (\lambda - \widetilde{L}_{u(\varepsilon),\varepsilon})(I - \delta(\lambda - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}).$$

5.5.2 Proof of Theorem 5.8

For $\lambda \in \mathbb{C}$ we study the spectral problem

$$\widetilde{L}_{u,\varepsilon}\boldsymbol{v} = \lambda \boldsymbol{v}. \tag{5.14}$$

Since (5.4) has the translational symmetry in the case that $\varepsilon = 0$ we find

$$\widetilde{L}_{u,0}\boldsymbol{u}' = 0.$$

For $\varepsilon \neq 0$, this symmetry is broken, and the zero eigenvalue is expected to move either into the stable or unstable half-plane. In our stability analysis, it is therefore important to understand how the critical zero eigenvalue behaves along the bifurcating solution branch given by $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto u(\varepsilon) \in X$ with $u(0) = u_{\sigma_0}$, where σ_0 is a simple zero of V_{eff} as in Theorem 5.3. For the following calculations we will identify the $\mathbb{C}$-valued function $u(\varepsilon) : \mathbb{T} \to \mathbb{C}$ with the $\mathbb{R}^2$ vector-valued function $\boldsymbol{u}(\varepsilon) : \mathbb{T} \to \mathbb{R}^2$ and, understanding that the set of $\mathbb{R}^2$ valued functions is a subset of the set of $\mathbb{C}^2$ valued functions, write this as $\boldsymbol{u}(\varepsilon) \in X \times X$.

We start with the tracking of the simple critical zero eigenvalue and set up the equation for the perturbed eigenvalue $\lambda_0 = \lambda_0(\varepsilon)$ which reads

$$\widetilde{L}_{u(\varepsilon),\varepsilon}\boldsymbol{v}(\varepsilon) = \lambda_0(\varepsilon)\boldsymbol{v}(\varepsilon).$$

After a possible re-scaling we find that $\boldsymbol{v}(0) = \boldsymbol{u}'_{\sigma_0}$ and using regular perturbation theory for simple eigenvalues, cf. [83] Proposition I.7.2, the mapping $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto \lambda_0(\varepsilon) \in \mathbb{R}$ is continuously differentiable. Our first goal is to derive a formula for $\lambda_0'(0)$. If $\lambda_0'(0) > 0$ this means that the solutions $u(\varepsilon)$ for $\varepsilon > 0$ are spectrally unstable. In contrast, if $\lambda_0'(0) < 0$, the solutions $u(\varepsilon)$ for $\varepsilon > 0$ are spectrally stable.

Lemma 5.18. *Let $\varepsilon \mapsto \lambda_0(\varepsilon)$ be the C^1 parametrization of the perturbed zero eigenvalue. Then the following formula holds true:*

$$\lambda_0'(0) = -\int_{-\pi}^{\pi} V'(x)\boldsymbol{u}'_{\sigma_0} \cdot J\boldsymbol{\phi}^*_{\sigma_0}\,dx.$$

Proof. On the one hand, if we differentiate the equation

$$\widetilde{L}_{u(\varepsilon),\varepsilon}\boldsymbol{v}(\varepsilon) = \lambda_0(\varepsilon)\boldsymbol{v}(\varepsilon).$$

with respect to ε and evaluate at $\varepsilon = 0$ we find

$$\widetilde{L}_{u_{\sigma_0},0}\partial_\varepsilon \boldsymbol{v}(0) - JN_u \boldsymbol{u}'_{\sigma_0} + V(x)\boldsymbol{u}''_{\sigma_0} = \lambda_0'(0)\boldsymbol{u}'_{\sigma_0}, \tag{5.15}$$

where N_u is given by

$$N_u = 2\begin{pmatrix} 3u_{\sigma_0 1}\partial_\varepsilon u_1(0) + u_{\sigma_0 2}\partial_\varepsilon u_2(0) & u_{\sigma_0 1}\partial_\varepsilon u_2(0) + u_{\sigma_0 2}\partial_\varepsilon u_1(0) \\ u_{\sigma_0 1}\partial_\varepsilon u_2(0) + u_{\sigma_0 2}\partial_\varepsilon u_1(0) & u_{\sigma_0 1}\partial_\varepsilon u_1(0) + 3u_{\sigma_0 2}\partial_\varepsilon u_2(0) \end{pmatrix}.$$

On the other hand, if we differentiate (5.4) with respect to ε at $\varepsilon = 0$, then we obtain

$$\widetilde{L}_{u_{\sigma_0},0}\partial_\varepsilon \boldsymbol{u}(0) + V(x)\boldsymbol{u}'_{\sigma_0} = 0.$$

If we differentiate this equation with respect to x we find

$$\widetilde{L}_{u_{\sigma_0},0}\partial_\varepsilon \boldsymbol{u}'(0) + V(x)\boldsymbol{u}''_{\sigma_0} + V'(x)\boldsymbol{u}'_{\sigma_0} - JN_u\boldsymbol{u}'_{\sigma_0} = 0. \tag{5.16}$$

Combining (5.15) and (5.16) yields

$$\widetilde{L}_{u_{\sigma_0},0}[\partial_\varepsilon \boldsymbol{v}(0) - \partial_\varepsilon \boldsymbol{u}'(0)] - V'(x)\boldsymbol{u}'_{\sigma_0} = \lambda_0'(0)\boldsymbol{u}'_{\sigma_0}$$

and testing this equation with $J\boldsymbol{\phi}^*_{\sigma_0} \in \ker \widetilde{L}^*_{u_{\sigma_0},0}$ we obtain

$$-\int_{-\pi}^{\pi} V'(x)\boldsymbol{u}'_{\sigma_0} \cdot J\boldsymbol{\phi}^*_{\sigma_0} dx = -\langle V'(x)\boldsymbol{u}'_{\sigma_0}, J\boldsymbol{\phi}^*_{\sigma_0}\rangle_{L^2} = \lambda_0'(0)\langle \boldsymbol{u}'_{\sigma_0}, J\boldsymbol{\phi}^*_{\sigma_0}\rangle_{L^2} = \lambda_0'(0)$$

which finishes the proof. □

By Lemma 5.18 we can control the critical part of the spectrum close to the origin along the bifurcating solution branch. In fact, using standard perturbation theory, cf. [80], we know that all the eigenvalues of $\widetilde{L}_{u(\varepsilon),\varepsilon}$ depend continuously on the parameter ε. However, this dependence is in general not uniform w.r.t. all eigenvalues, so we have to make sure that no unstable spectrum occurs far from the origin. At this point, it is worth mentioning that we have an a-priori bound on the spectrum of the form

$$\exists \lambda_* = \lambda_*(u(\varepsilon), \varepsilon) > 0 : \quad \lambda \in \sigma(\widetilde{L}_{u(\varepsilon),\varepsilon}) \implies \mathrm{Re}(\lambda) \leq \lambda_*.$$

This bound follows from the Hille-Yoshida Theorem since $\widetilde{L}_{u(\varepsilon),\varepsilon}$ generates a C_0-semigroup on $Z \times Z$, cf. Lemma 5.19 below. It can also be shown directly by testing the eigenvalue problem with the corresponding eigenfunction and integration by parts. As a conclusion, spectral stability holds if we can prove that there exists $\lambda^* > 0$ such that

$$\{\lambda \in \mathbb{C} : 0 \leq \mathrm{Re}(\lambda) \leq \lambda_*, |\,\mathrm{Im}(\lambda)| \geq \lambda^*\} \subset \rho(\widetilde{L}_{u(\varepsilon),\varepsilon}).$$

This relation is shown as part of Lemma 5.16 and it is extended to the left of the origin by the subsequent Remark 5.17. Since in any rectangle $\{\lambda \in \mathbb{C} : -M \leq \mathrm{Re}(\lambda) \leq \lambda_*, |\,\mathrm{Im}(\lambda)| \leq \lambda^*\}$ there are only finitely many eigenvalues of $\widetilde{L}_{u(\varepsilon),\varepsilon}$ and they depend (uniformly) continuoulsy on ε, our assumption (A2) on $\widetilde{L}_{u_0,0}$ shows that none of these eigenvalues (except possibly the critical one) can move into the right half plane if $|\varepsilon|$ is small. Therefore, only the movement of the critical eigenvalue determines the spectral stability and therefore Theorem 5.8 is true.

5.5.3 Proof of Theorem 5.9

In order to prove nonlinear asymptotic stability of stationary solutions of (5.13) it is enough to show exponential stability of the semigroup of the linearization in $Y \times Y$, see e.g. [27]. For the proof of Theorem 5.9 we will show the following three steps:

(i) Prove that $\widetilde{L}_{u(\varepsilon),\varepsilon}$ is the generator of a C_0-semigroup on $Z \times Z$.

(ii) Show exponential decay of $(\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t})_{t\geq 0}$ in $Z \times Z$.

(iii) Show exponential decay of $(\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t})_{t\geq 0}$ in $Y \times Y$.

For step (i), we establish the generator properties of the linearization in $Z \times Z$.

Lemma 5.19. *The operator $\widetilde{L}_{u(\varepsilon),\varepsilon}$ generates a C_0-semigroup on $Z \times Z$.*

Proof. We split the operator into

$$\widetilde{L}_{u(\varepsilon),\varepsilon} = L_1 + L_2 + L_3,$$

where $L_1 : X \times X \to Z \times Z$, $L_2 : Y \times Y \to Z \times Z$, and $L_3 : Z \times Z \to Z \times Z$ are defined by

$$L_1 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} := \begin{pmatrix} -d\varphi_2'' - \mu\varphi_1 \\ d\varphi_1'' - \mu\varphi_2 \end{pmatrix},$$

$$L_2 \boldsymbol{\varphi} := \varepsilon V(x)\boldsymbol{\varphi}' - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\boldsymbol{\varphi},$$

and

$$L_3\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} := \begin{pmatrix}\frac{|\varepsilon|}{2}\|V'\|_{L^\infty} - 2u_1u_2 & \zeta - (u_1^2 + 3u_2^2)\\ -\zeta + 3u_1^2 + u_2^2 & \frac{|\varepsilon|}{2}\|V'\|_{L^\infty} + 2u_1u_2\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

We will show that

(i) L_1 generates a contraction semigroup.

(ii) L_2 is dissipative and bounded relative to L_1.

(iii) L_3 is a bounded operator on $Z \times Z$.

By using the semigroup theory, this will prove that the sum $L_1 + L_2 + L_3$ is the generator of a C_0-semigroup on $Z \times Z$.

Part (i): It follows that $\mathrm{Re}\langle L_1\boldsymbol{\varphi}, \boldsymbol{\varphi}\rangle_{L^2} = -\mu\|\boldsymbol{\varphi}\|_{L^2}^2 \le 0$ for every $\boldsymbol{\varphi} \in X \times X$, and $\lambda - L_1$ is invertible for every $\lambda > 0$ which can be seen using Fourier transform. By the Lumer-Phillips Theorem we find that L_1 generates a contraction semigroup on $Z \times Z$.

Part (ii): We have to show that

$$\forall \boldsymbol{\varphi} \in Y \times Y: \quad \mathrm{Re}\langle L_2\boldsymbol{\varphi}, \boldsymbol{\varphi}\rangle_{L^2} \le 0$$

and

$$\forall a > 0,\ \exists b > 0: \quad \|L_2\boldsymbol{\varphi}\|_{L^2} \le a\|L_1\boldsymbol{\varphi}\|_{L^2} + b\|\boldsymbol{\varphi}\|_{L^2} \quad \forall \boldsymbol{\varphi} \in X \times X.$$

Let $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in Y \times Y$ and observe that integration by parts yields

$$\begin{aligned}\mathrm{Re}\int_{-\pi}^{\pi} \varepsilon V(x)(\varphi_1'\bar{\varphi}_1 + \varphi_2'\bar{\varphi}_2) - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}|\boldsymbol{\varphi}|^2 dx\\ = \int_{-\pi}^{\pi} -\frac{\varepsilon}{2}V'(x)|\boldsymbol{\varphi}|^2 - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}|\boldsymbol{\varphi}|^2 dx \le 0\end{aligned}$$

which shows that L_2 is dissipative. Further, if $\boldsymbol{\varphi} \in X \times X$, then for every $a > 0$ we have

$$\begin{aligned}&\|\varepsilon V\boldsymbol{\varphi}' - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\boldsymbol{\varphi}\|_{L^2}\\ &\qquad \le |\varepsilon|\|V\|_{L^\infty}\|\boldsymbol{\varphi}'\|_{L^2} + \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\|\boldsymbol{\varphi}\|_{L^2}\\ &\qquad \le |\varepsilon|a\|V\|_{L^\infty}\|\boldsymbol{\varphi}''\|_{L^2} + \frac{|\varepsilon|}{4a}\|V\|_{L^\infty}\|\boldsymbol{\varphi}\|_{L^2} + \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\|\boldsymbol{\varphi}\|_{L^2}\\ &\qquad \le \frac{|\varepsilon|a}{|d|}\|V\|_{L^\infty}\|L_1\boldsymbol{\varphi}\|_{L^2} + |\varepsilon|\left(\left(\frac{a\mu}{|d|} + \frac{1}{4a}\right)\|V\|_{L^\infty} + \frac{1}{2}\|V'\|_{L^\infty}\right)\|\boldsymbol{\varphi}\|_{L^2}\end{aligned}$$

where we used the inequality

$$\forall \boldsymbol{\varphi} \in X \times X, \forall a > 0: \quad \|\boldsymbol{\varphi}'\|_{L^2} \le a\|\boldsymbol{\varphi}''\|_{L^2} + \frac{1}{4a}\|\boldsymbol{\varphi}\|_{L^2}.$$

Hence, by the dissipative perturbation theorem, cf. Chapter III, Theorem 2.7 in [44], for generators the operator $L_1 + L_2 : X \times X \to Z \times Z$ generates a contraction semigroup.

Part (iii): It follows that L_3 is bounded on $Z \times Z$. Then the bounded perturbation theorem for generators, cf. Chapter III, Theorem 1.3 in [44], yields that $\widetilde{L}_{u(\varepsilon),\varepsilon} = L_1 + L_2 + L_3$ generates a C_0-semigroup on $Z \times Z$ as desired. □

Remark 5.20. Using similar arguments, one can show that $\widetilde{L}_{u(\varepsilon),\varepsilon}$ is the generator of a C_0-semigroup on $Y \times Y$.

For step (ii), we use a characterization of exponential decay of semigroups in Hilbert spaces known as the Gearhart-Greiner-Prüss Theorem, cf. Chapter V, Theorem 1.11 in [44].

Theorem 5.21 (Gearhart-Greiner-Prüss Theorem)**.** *Let L be the generator of a C_0-semigroup $(\mathrm{e}^{Lt})_{t\ge 0}$ on a complex Hilbert space H. Then $(\mathrm{e}^{Lt})_{t\ge 0}$ is exponentially stable in H if and only if*

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\} \subset \rho(L) \quad \textit{and} \quad \sup_{\mathrm{Re}\,\lambda \ge 0} \|(\lambda I - L)^{-1}\|_{H \to H} < \infty.$$

By the assumption of Theorem 5.9, spectral stability of the solution $u(\varepsilon)$ is guaranteed and we are left with the proof of the uniform resolvent estimate on $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$. Using Lemma 5.16, we find $\lambda^* \gg 1$ such that $(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}$ is uniformly bounded on the set Λ_{λ^*} for sufficiently small ε. Moreover, since $\widetilde{L}_{u(\varepsilon),\varepsilon}$ is the generator of a C_0-semigroup on the state-space $Z \times Z$, the Hille-Yosida Theorem ensures a uniform bound of the resolvent on $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \lambda_*\}$ for some constant $\lambda_* > 0$. From the fact that $\lambda \mapsto (\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}$ is a meromorphic function with no poles in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$, the resolvent is uniformly bounded on compact subsets of $\mathbb{C}$ in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$. Thus, we can conclude that $\widetilde{L}_{u(\varepsilon),\varepsilon}$ satisfies the Gearhart-Greiner-Prüss resolvent bound and exponential stability in $Z \times Z$ follows.

Finally, for step (iii), we will interpolate the decay estimate between the spaces $Z \times Z$ and $X \times X$. To do so, we have to establish bounds in $X \times X$ which is done the next lemma. The interpolation argument is then in the spirit of Lemma 5 in [137] and will also lead to decay estimates in the more general interpolation spaces $H^s_{\mathrm{per}} \times H^s_{\mathrm{per}}$ for $s \in [0, 2]$.

Lemma 5.22. *For any* $s \in [0,2]$ *and sufficiently small* ε *the semigroup* $(\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t})_{t\geq 0}$ *has exponential decay in* $H^s_{per}([-\pi,\pi],\mathbb{C}) \times H^s_{per}([-\pi,\pi],\mathbb{C})$*, i.e., there exist* $C_s > 0$ *such that*

$$\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t}\|_{H^s \to H^s} \leq C_s \mathrm{e}^{-\eta t} \quad \textit{for } t \geq 0,$$

where $-\eta < 0$ *is the previously established growth bound of the semigroup in* $Z \times Z$.

Proof. We consider only the case $d > 0$, since the other case can be shown by rewriting JA_u as $-J(-A_u)$ and using the same arguments as presented below. If $d > 0$, the operator $A_{u(\varepsilon)} + \gamma I$ is positive and self-adjoint provided $\gamma > 0$ is sufficiently large. Hence, for $z \in \mathbb{C}$ we can define the complex powers by

$$(A_{u(\varepsilon)} + \gamma I)^z \boldsymbol{v} = \int_0^\infty \lambda^z dE_\lambda \boldsymbol{v}, \quad \text{for } \boldsymbol{v} \in \mathrm{dom}(A_u u(\varepsilon) + \gamma I)^z,$$

with domain given by

$$\mathrm{dom}(A_{u(\varepsilon)} + \gamma I)^z = \left\{ \boldsymbol{v} \in Z \times Z : \|(A_{u(\varepsilon)} + \gamma I)^z \boldsymbol{v}\|_{L^2}^2 = \int_0^\infty \lambda^{2\,\mathrm{Re}\, z} d\|E_\lambda \boldsymbol{v}\|_{L^2}^2 < \infty \right\}$$

and where E_λ for $\lambda \in \mathbb{R}$ is the family of self-adjoint spectral projections associated to $A_{u(\varepsilon)} + \gamma I$. Note that for $\theta \in [0,1]$ the relation

$$\mathrm{dom}(A_{u(\varepsilon)} + \gamma I)^\theta = H^{2\theta}_{\mathrm{per}}([-\pi,\pi],\mathbb{C}) \times H^{2\theta}_{\mathrm{per}}([-\pi,\pi],\mathbb{C})$$

is true, cf. [90] Theorem 4.36, and further for any $r \in \mathbb{R}$ the operator $(A_{u(\varepsilon)} + \gamma I)^{\mathrm{i}r}$ is unitary on $Z \times Z$. If $\theta = 0, 1$ we will show that there exists $C_\theta > 0$ such that

$$\forall r \in \mathbb{R},\, \forall t \geq 0,\, \forall \boldsymbol{v} \in X \times X : \quad \|(A_{u(\varepsilon)} + \gamma I)^{\theta + \mathrm{i}r} \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} \leq C_\theta \mathrm{e}^{-\eta t} \|\boldsymbol{v}\|_{H^{2\theta}},$$

which implies

$$\forall r \in \mathbb{R},\, \forall t \geq 0,\, \forall \theta \in (0,1),\, \forall \boldsymbol{v} \in X \times X :$$
$$\|(A_{u(\varepsilon)} + \gamma I)^{\theta + \mathrm{i}r} \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} \leq C_0^{1-\theta} C_1^\theta \mathrm{e}^{-\eta t} \|\boldsymbol{v}\|_{H^{2\theta}},$$

by complex interpolation, cf. [90] Theorem 2.7. In particular, we see that

$$\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t}\|_{H^s \to H^s} \leq C_0^{1-s} C_1^s \mathrm{e}^{-\eta t}$$

which is precisely our claim. The estimate for $\theta = 0$ has already been shown in the preceding discussion, so it remains to check the estimate for $\theta = 1$. Let $\boldsymbol{v} \in X \times X$ and observe that

$$\begin{aligned}
&\|(A_{u(\varepsilon)} + \gamma I)^{1+\mathrm{i}r} \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} \\
&\qquad = \|(A_{u(\varepsilon)} + \gamma I) \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} \\
&\qquad = \|(\widetilde{L}_{u(\varepsilon),\varepsilon} + J\gamma + I(\mu - \varepsilon V(x)\partial_x)) \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} \\
&\qquad \leq \|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \widetilde{L}_{u(\varepsilon),\varepsilon} \boldsymbol{v}\|_{L^2} + C\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} + |\varepsilon| \|V\|_{L^\infty} \|\partial_x \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} \\
&\qquad \leq C\mathrm{e}^{-\eta t} \|\widetilde{L}_{u(\varepsilon),\varepsilon} \boldsymbol{v}\|_{L^2} + C\mathrm{e}^{-\eta t} \|\boldsymbol{v}\|_{L^2} + |\varepsilon| \|V\|_{L^\infty} \|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{H^1} \\
&\qquad \leq C\mathrm{e}^{-\eta t} \|\boldsymbol{v}\|_{H^2} + |\varepsilon| C \|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{H^2},
\end{aligned}$$

which yields $\|(A_{u(\varepsilon)} + \gamma I)^{1+\mathrm{i}r} \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t} \boldsymbol{v}\|_{L^2} \leq C\mathrm{e}^{-\eta t} \|\boldsymbol{v}\|_{H^2}$ if ε is sufficiently small because of the norm equivalence $\|\boldsymbol{v}\|_{H^2} \sim \|(A_{u(\varepsilon)} + \gamma I)\boldsymbol{v}\|_{L^2}$. □

In particular Lemma 5.22 establishes exponential stability of the linearization in $Y \times Y$, thus we have proved Theorem 5.9.

5.5.4 Proof of Lemma 5.16

The uniform resolvent estimate is proved if we can find a constant $C > 0$ independent of $\lambda \in \Lambda_{\lambda^*}$ such that

$$\forall \boldsymbol{\varphi} \in X \times X : \quad \|(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})\boldsymbol{\varphi}\|_{L^2} \geq C\|\boldsymbol{\varphi}\|_{L^2}. \tag{5.17}$$

In order to simplify the situation, let us introduce the rotation on $Z \times Z$ as follows:

$$R \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

with spatially varying angular $\theta(x) = \frac{\varepsilon}{2d} \int_{-\pi}^{x} [V(y) - \hat{V}_0] dy$ where $\hat{V}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(y)\, dy$ is the mean of the potential V. Since R is an isometry on $Z \times Z$ the resolvent estimate (5.17) is equivalent to

$$\forall \boldsymbol{\varphi} \in X \times X : \quad \|(\lambda I - R\widetilde{L}_{u(\varepsilon),\varepsilon} R^{-1})\boldsymbol{\varphi}\|_{L^2} \geq C\|\boldsymbol{\varphi}\|_{L^2},$$

where we note that $\sigma(\widetilde{L}_{u(\varepsilon),\varepsilon}) = \sigma(R\widetilde{L}_{u(\varepsilon),\varepsilon} R^{-1})$. The advantage of considering the operator $R\widetilde{L}_{u(\varepsilon),\varepsilon} R^{-1}$ becomes clear if we calculate

$$R\widetilde{L}_{u(\varepsilon),\varepsilon} R^{-1} = J\tilde{A}_{u(\varepsilon),\varepsilon,V} - I(\mu - \varepsilon \hat{V}_0 \partial_x)$$

where the operator $\tilde{A}_{u(\varepsilon),\varepsilon,V}$ given by

$$\tilde{A}_{u(\varepsilon),\varepsilon,V} := \begin{pmatrix} -d\partial_x^2 + W_1 & W_2 + W_4 \\ W_2 - W_4 & -d\partial_x^2 + W_3 \end{pmatrix}$$

with potentials

$$\begin{aligned} W_1 &= \zeta + \cos^2\theta U_1 + 2\cos\theta\sin\theta U_2 + \sin^2\theta U_3 + d\theta'^2 - \varepsilon\theta' V, \\ W_2 &= (\cos^2\theta - \sin^2\theta)U_2 + \cos\theta\sin\theta(U_3 - U_1), \\ W_3 &= \zeta + \sin^2\theta U_1 - 2\cos\theta\sin\theta U_2 + \cos^2\theta U_3 + d\theta'^2 - \varepsilon\theta' V, \\ W_4 &= d\theta'', \end{aligned}$$

and functions

$$U_1 = -(3u_1^2(\varepsilon) + u_2^2(\varepsilon)),\ U_2 = -2u_1(\varepsilon)u_2(\varepsilon),\ U_3 = -(u_1^2(\varepsilon) + 3u_2^2(\varepsilon)).$$

Clearly, the first order derivative is now multiplied by a constant instead of a spatially varying potential which will be used in the following calculations. We also note that the functions $W_i \in X$, $i = 1, 2, 3$ depend upon the solution u and the potential V whereas $W_4 \in X$ only depends upon the potential V. For the proof of the resolvent estimate we use techniques presented in [137], where the authors construct resolvents for the unperturbed LLE (5.1).

We need the following proposition, which is Lemma 4 in [137].

Proposition 5.23. *Let $d \neq 0$ and $\mu > 0$. Then there exists $\lambda^* > 0$ depending on d and μ with the property that for all $\omega \geq \lambda^*$ there is at most one $k_0 = k_0(\omega, \mu) \in \mathbb{N}$ such that*

$$\omega \geq |d^2 k_0^4 + \mu^2 - \omega^2|.$$

For all other $k \in \mathbb{Z} \setminus \{\pm k_0(\omega, \mu)\}$ we have

$$|d^2 k^4 + \mu^2 - \omega^2| \geq \frac{1}{10} \max\{d^2 k^2, \omega\}^{3/2}.$$

Moreover, we find $k_0(\omega, \mu) = \mathcal{O}(\omega^{1/2})$ as $\omega \to \infty$.

Now we can start to construct and bound the resolvent. By the Hille-Yoshida Theorem, a uniform resolvent estimate holds whenever $\operatorname{Re}\lambda$ is sufficiently large. It therefore remains to consider $\lambda = \delta + i\omega \in \Lambda_{\lambda^*}$ for some $\lambda^* > 0$ and $\delta \geq 0$ on a compact set. Since δ replaces μ in $\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon}$ by $\mu + \delta$ and the estimates of Proposition 5.23 holds for any $\mu > 0$ on a compact set, it sufficies to prove the uniform estimates for $\delta = 0$. For now, we do not specify the value of λ^*, since this will be done a posteriori. More precisely, we will choose λ^* such that the conditions (5.20),(5.21),(5.22),(5.28), which we derive below, are all satisfied. We can restrict to the case $\omega \geq \lambda^*$, since the proof for $\omega \leq -\lambda^*$ follows from symmetries of the spectral problem under complex conjugation. For $\boldsymbol{v} \in X \times X$ we define

$$(\lambda I - R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1})\boldsymbol{v} =: \boldsymbol{\psi} \in Z \times Z \tag{5.18}$$

and show that there exist bounded operators T_1 and T_2 on $Z \times Z$ depending on λ with norms satisfying $\|T_1\|_{L^2\to L^2} = \mathcal{O}(\omega^{-1/2})$ and $\|T_2\|_{L^2\to L^2} = \mathcal{O}(1)$ as $\omega \to \infty$ such that (5.18) implies

$$(I + T_1)\boldsymbol{v} = T_2\boldsymbol{\psi}. \tag{5.19}$$

If λ^* is sufficiently large such that

$$\forall \omega \geq \lambda^* : \quad \|T_1\|_{L^2\to L^2} \leq \frac{1}{2} \tag{5.20}$$

we then deduce that $I + T_1$ is a small perturbation of the identity, and hence invertible with norm uniformly bounded in λ which is our claim. Therefore, it remains to show (5.19). We introduce the matrix-valued potential

$$W = \begin{pmatrix} W_1 & W_2 + W_4 \\ W_2 - W_4 & W_3 \end{pmatrix}$$

in order to write

$$\lambda I - R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1} = \mathrm{i}\omega I - J(-d\partial_x^2 + W) + I(\mu - \varepsilon\hat{V}_0\partial_x).$$

Now, let $A = \lambda I - R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1} + JW$ and observe that $A\boldsymbol{v}(x) = \sum_{k\in\mathbb{Z}} A_k\hat{\boldsymbol{v}}_k \mathrm{e}^{\mathrm{i}kx}$ with $\boldsymbol{v}(x) = \sum_{k\in\mathbb{Z}} \hat{\boldsymbol{v}}_k \mathrm{e}^{\mathrm{i}kx}$ and Fourier multiplier

$$A_k = A_k^1 + A_k^2 = \begin{pmatrix} \mathrm{i}\omega + \mu & -dk^2 \\ dk^2 & \mathrm{i}\omega + \mu \end{pmatrix} + \begin{pmatrix} -\mathrm{i}\varepsilon\hat{V}_0 k & 0 \\ 0 & -\mathrm{i}\varepsilon\hat{V}_0 k \end{pmatrix}.$$

The inverse of A_k^1 is given by

$$(A_k^1)^{-1} = \frac{1}{\det(A_k^1)} \begin{pmatrix} \mathrm{i}\omega + \mu & dk^2 \\ -dk^2 & \mathrm{i}\omega + \mu \end{pmatrix}$$

and by Proposition 5.23 there exists at most one $k_0 = k_0(\omega, \mu) \in \mathbb{N}$ such that

$$\omega \geq |d^2 k_0^4 + \mu^2 - \omega^2| \tag{5.21}$$

and

$$|\det(A_k^1)| \geq |d^2 k^4 + \mu^2 - \omega^2| \geq \frac{1}{10} \max\{d^2 k^2, \omega\}^{3/2} \text{ for all } k \neq \pm k_0 \tag{5.22}$$

provided that λ^* is sufficiently large. Thus A_k^1 is invertible with bound $\|(A_k^1)^{-1}\|_{\mathbb{C}^{2\times 2}} \leq C/\max\{\omega^{1/2}, |k|\}$ for all $k \neq \pm k_0$. Using again Proposition 5.23, we have the asymptotic $k_0 = k_0(\omega) = \mathcal{O}(\omega^{1/2})$ as $\omega \to \infty$. Consequently, if $|\varepsilon|$ is sufficiently small, then $A_k = A_k^1(I + (A_k^1)^{-1} A_k^2)$, $k \neq \pm k_0$, is also invertible with the bound $\|(A_k)^{-1}\|_{\mathbb{C}^{2\times 2}} = \mathcal{O}(\omega^{-1/2})$ as $\omega \to \infty$. Next, for the above $k_0 \in \mathbb{N}$, we introduce the orthogonal projections $P, Q, Q_1, Q_2 : Z \times Z \to Z \times Z$ as follows:

$$Q_1 \boldsymbol{v} = \hat{\boldsymbol{v}}_{k_0} \mathrm{e}^{\mathrm{i} k_0 (\cdot)}, \quad Q_2 \boldsymbol{v} = \hat{\boldsymbol{v}}_{-k_0} \mathrm{e}^{-\mathrm{i} k_0 (\cdot)}$$

and

$$Q = Q_1 + Q_2, \quad P = I - Q.$$

This allows us to decompose (5.18) as follows:

$$PAP\boldsymbol{v} - PJW\boldsymbol{v} = P\boldsymbol{\psi}, \tag{5.23}$$

$$QAQ\boldsymbol{v} - QJW\boldsymbol{v} = Q\boldsymbol{\psi}. \tag{5.24}$$

From the preceding arguments we find

$$\|(PAP)^{-1}\|_{PL^2 \to PL^2} = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty$$

which implies that (5.23) is equivalent to

$$P\boldsymbol{v} - (PAP)^{-1} PJW\boldsymbol{v} = (PAP)^{-1} P\boldsymbol{\psi} \tag{5.25}$$

with bound $\|(PAP)^{-1} PJW\|_{L^2 \to L^2} = \mathcal{O}(\omega^{-1/2})$ as $\omega \to \infty$.

Next we investigate (5.24) which we decompose a second time to find

$$Q_1 A Q_1 \boldsymbol{v} - Q_1 JW Q_1 \boldsymbol{v} - Q_1 JW Q_2 \boldsymbol{v} - Q_1 JW P\boldsymbol{v} = Q_1 \boldsymbol{\psi}, \tag{5.26}$$

$$Q_2 A Q_2 \boldsymbol{v} - Q_2 JW Q_1 \boldsymbol{v} - Q_2 JW Q_2 \boldsymbol{v} - Q_2 JW P\boldsymbol{v} = Q_2 \boldsymbol{\psi}. \tag{5.27}$$

Both equations can be handled similarly and thus we focus on the first one. Using (5.25) we can write (5.26) as

$$[Q_1AQ_1 - Q_1JWQ_1]\boldsymbol{v} - Q_1JWQ_2\boldsymbol{v} - Q_1JW(PAP)^{-1}PJW\boldsymbol{v} = Q_1JW(PAP)^{-1}\boldsymbol{\psi} + Q_1\boldsymbol{\psi}.$$

The operator $B := Q_1AQ_1 - Q_1JWQ_1$ acts like a Fourier-multiplier on $\operatorname{range} Q_1$ with matrix

$$B_{k_0} = \begin{pmatrix} \mathrm{i}(\omega - \varepsilon\hat{V}_0k_0) + \mu - (\hat{W}_2)_0 + (\hat{W}_4)_0 & -dk_0^2 - (\hat{W}_3)_0 \\ dk_0^2 + (\hat{W}_1)_0 & \mathrm{i}(\omega - \varepsilon\hat{V}_0k_0) + \mu + (\hat{W}_2)_0 + (\hat{W}_4)_0 \end{pmatrix}$$

and we observe that

$$|\det(B_{k_0})| \geq |\operatorname{Im}\det(B_{k_0})| = 2|\omega - \varepsilon\hat{V}_0k_0||\mu + (\hat{W}_4)_0| \sim \omega$$

since $k_0 = \mathcal{O}(\omega^{1/2})$. This means that we find $\lambda^* \gg 1$ such that

$$\forall\omega \geq \lambda^* : \quad B_{k_0} \text{ is invertible with } \|B_{k_0}^{-1}\|_{\mathbb{C}^{2\times 2}} = \mathcal{O}(1) \text{ as } \omega \to \infty, \tag{5.28}$$

and thus the same holds for the operator B. Inverting B yields

$$Q_1\boldsymbol{v} - B^{-1}[Q_1JWQ_2 + Q_1JW(PAP)^{-1}PJW]\boldsymbol{v} = B^{-1}Q_1JW(PAP)^{-1}P\boldsymbol{\psi} + B^{-1}Q_1\boldsymbol{\psi}$$

and since we have $W_i \in Y$ for $i = 1, 2, 3, 4$ we can exploit decay of the Fourier-coefficients

$$|(\hat{W}_i)_k| \leq \frac{C}{\sqrt{1+k^2}} \quad \text{for all } k \in \mathbb{Z}$$

to bound $Q_1JWQ_2\boldsymbol{v} = (J\hat{W})_{2k_0}\hat{\boldsymbol{v}}_{-k_0}\mathrm{e}^{\mathrm{i}k_0(\cdot)}$:

$$\|Q_1JWQ_2\|_{L^2\to L^2} = \mathcal{O}(k_0(\omega,\mu)^{-1}) = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty.$$

Finally from the bounds of the first part we infer that

$$\|Q_1JW(PAP)^{-1}PJW\|_{L^2\to L^2} = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty,$$
$$\|Q_1JW(PAP)^{-1}\|_{L^2\to L^2} = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty$$

and as a conclusion we arrive at (5.19) which is all we had to prove.

5.6 A: Derivation of the perturbed LLE

The following is a derivation of the perturbed LLE (5.3) from the dual laser pump equation (5.2). We start by taking a solution $u = u(x,t)$ of (5.2). Jumping in a moving coordinate system we set $\tilde{u}(x,t) = u(k_1 x - \nu_1 t, t)$ and find that $\tilde{u}$ satisfies

$$\mathrm{i}\partial_t \tilde{u} - \mathrm{i}\nu_1 \partial_\xi \tilde{u} = -dk_1^2 \partial_\xi^2 \tilde{u} + (-\mathrm{i}\mu + \zeta)\tilde{u} - |\tilde{u}|^2 \tilde{u} + \mathrm{i}f_0 + \mathrm{i}f_1 \mathrm{e}^{\mathrm{i}\xi}, \tag{5.29}$$

where $\xi := k_1 x - \nu_1 t$. Next, using the approximation $\arctan s \approx s$ for $|s|$ small, we find for $|f_0| \gg |f_1|$ that

$$f_0 + f_1 \mathrm{e}^{\mathrm{i}\xi} = \sqrt{f_0^2 + 2f_0 f_1 \cos\xi + f_1^2}\,\mathrm{e}^{\mathrm{i}\arctan\frac{f_1 \sin\xi}{f_0 + f_1\cos\xi}} \approx f_0 \mathrm{e}^{\mathrm{i}\frac{f_1}{f_0}\sin\xi}.$$

Inserting this into (5.29) we find that approximately the following equation holds for $\tilde{u}$

$$\mathrm{i}\partial_t \tilde{u} - \mathrm{i}\nu_1 \partial_\xi \tilde{u} = -dk_1^2 \partial_\xi^2 \tilde{u} + (-\mathrm{i}\mu + \zeta)\tilde{u} - |\tilde{u}|^2 \tilde{u} + \mathrm{i}f_0 \mathrm{e}^{\mathrm{i}\frac{f_1}{f_0}\sin\xi}. \tag{5.30}$$

This suggests to set $\tilde{u}(\xi,t) = w(\xi,t)\mathrm{e}^{\mathrm{i}\frac{f_1}{f_0}\sin\xi}$ so that w solves

$$\begin{aligned}\mathrm{i}\partial_t w = &- dk_1^2 \partial_\xi^2 w + \left(\mathrm{i}\nu_1 - \mathrm{i}2dk_1^2 \frac{f_1}{f_0}\cos\xi\right)\partial_\xi w \\ &+ \Big(-\mathrm{i}\mu + \zeta \underbrace{-\nu_1 \frac{f_1}{f_0}\cos\xi + dk_1^2 \frac{f_1^2}{f_0^2}\cos^2\xi + \mathrm{i}dk_1^2 \frac{f_1}{f_0}\sin\xi}_{=:\alpha(\xi)}\Big) w - |w|^2 w + \mathrm{i}f_0.\end{aligned}$$

Using $|f_1| \ll |f_0|$ we see that the term $\alpha(\xi)$ is much smaller than $-\mathrm{i}\mu + \zeta$ for physically relevant (normalized) values of $\mu = \mathcal{O}(1)$ and ζ between $\mathcal{O}(1)$ and $\mathcal{O}(10)$. Neglecting $\alpha(\xi)$ we arrive at

$$\mathrm{i}\partial_t w = -dk_1^2 \partial_\xi^2 w + \mathrm{i}\underbrace{\big(\nu_1 - 2dk_1^2 \frac{f_1}{f_0}\cos\xi\big)}_{=:V(\xi)}\partial_\xi w + (-\mathrm{i}\mu + \zeta)w - |w|^2 w + \mathrm{i}f_0$$

which is our target equation (5.3) in the case $\varepsilon = 1$ and with d replaced by dk_1^2.

5.7 B: Stability criterion for solitary waves in the limit of small μ

The stability criterion of Theorem 5.8 becomes more explicit in the limit $\mu \to 0+$ for solitary waves on $\mathbb{R}$ for the focusing case $d > 0$. We thus consider the stationary LLE in the form:

$$-du'' + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}\mu f_0 = 0, \qquad x \in \mathbb{R}. \tag{5.31}$$

Here both the pumping term $\mathrm{i}\mu f_0$ and the dissipative term $-\mathrm{i}\mu u$ are small and of equal order in μ. When μ is small, the solution can be expanded asymptotically as

$$u = u^{(0)} + \mu u^{(1)} + \mathcal{O}(\mu^2). \tag{5.32}$$

Here $u^{(0)}$ is the solitary wave of the nonlinear Schrödinger equation (NLSE) which exists if $d > 0$ and $u^{(1)}$ is found from the linear inhomogeneous equation

$$(-d\partial_x^2 + \zeta - 2|u^{(0)}|^2)u^{(1)} - (u^{(0)})^2 \bar{u}^{(1)} = \mathrm{i}u^{(0)} - \mathrm{i}f_0. \tag{5.33}$$

By using the vector form with $u = u_1 + \mathrm{i}u_2$ and the notation from (5.6), we can rewrite (5.33) in the form: $JA_{u^{(0)}}\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(0)} + f_0$. Recall that

$$\ker \widetilde{L}_u = \operatorname{span}\{\boldsymbol{u}'\}, \quad \ker \widetilde{L}_u^* = \operatorname{span}\{J\boldsymbol{\phi}^*\},$$

according to Assumption (A2), which implies that

$$JA_u\boldsymbol{u}' = \mu\boldsymbol{u}', \qquad JA_u\boldsymbol{\phi}^* = -\mu\boldsymbol{\phi}^*.$$

Inserting expansion (5.32) into the operator A_u and using expansions for the eigenfunctions $\boldsymbol{u}'$ and $\boldsymbol{\phi}^*$ in powers of μ up to the order of $\mathcal{O}(\mu)$ one can derive that

$$\begin{aligned} \boldsymbol{u}' &= (\boldsymbol{u}^{(0)})' + \mu(\boldsymbol{u}^{(1)})' + \mathcal{O}(\mu^2), \\ \boldsymbol{\phi}^* &= C\left[(\boldsymbol{u}^{(0)})' + \mu[(\boldsymbol{u}^{(1)})' + 2\boldsymbol{v}^{(1)}] + \mathcal{O}(\mu^2)\right], \end{aligned}$$

where $\boldsymbol{v}^{(1)}$ is a solution of the linear inhomogeneous equation $JA_{u^{(0)}}\boldsymbol{v}^{(1)} = -(\boldsymbol{u}^{(0)})'$ and the constant $C = C(\mu) \in \mathbb{C}$ is found from the normalization condition $\langle \boldsymbol{u}', J\boldsymbol{\phi}^* \rangle_{L^2} = 1$. The solution of $JA_{u^{(0)}}\boldsymbol{v}^{(1)} = -(\boldsymbol{u}^{(0)})'$ on the line $\mathbb{R}$ is available explicitly:

$$\boldsymbol{v}^{(1)} = -\frac{1}{2d} xJ\boldsymbol{u}^{(0)},$$

where $\boldsymbol{u}^{(0)}(x) \to 0$ as $|x| \to \infty$ exponentially fast in the case of solitary waves for $d > 0$. This allows us to compute by using integration by parts:

$$\begin{aligned}
\langle \boldsymbol{u}', J\boldsymbol{\phi}^* \rangle_{L^2} &= C \left[2\mu \int_{\mathbb{R}} [(u_1^{(0)})' v_2^{(1)} - (u_2^{(0)})' v_1^{(1)}] dx + \mathcal{O}(\mu^2) \right] \\
&= C \left[\mu d^{-1} \int_{\mathbb{R}} x [(u_1^{(0)})' u_1^{(0)} + (u_2^{(0)})' u_2^{(0)}] dx + \mathcal{O}(\mu^2) \right] \\
&= C \left[-\frac{\mu}{2d} \|u^{(0)}\|_{L^2}^2 + \mathcal{O}(\mu^2) \right].
\end{aligned}$$

Normalization $\langle \boldsymbol{u}', J\boldsymbol{\phi}^* \rangle_{L^2} = 1$ defines C asymptotically as follows:

$$C = -\frac{2d}{\mu \|u^{(0)}\|_{L^2}^2} \left[1 + \mathcal{O}(\mu) \right].$$

The stability condition of Theorem 5.8 is expressed in terms of the sign of $V_{\text{eff}}'(\sigma_0)$, where σ_0 is a simple root of V_{eff}. The effective potential can now be written more explicitly as

$$\begin{aligned}
V_{\text{eff}}(\sigma_0) &= \langle V(\cdot + \sigma_0) \boldsymbol{u}', J\boldsymbol{\phi}^* \rangle_{L^2} \\
&= C \left[\mu d^{-1} \int_{\mathbb{R}} x V(x + \sigma_0) [(u_1^{(0)})' u_1^{(0)} + (u_2^{(0)})' u_2^{(0)}] dx + \mathcal{O}(\mu^2) \right] \\
&= \frac{1}{\|u^{(0)}\|_{L^2}^2} \int_{\mathbb{R}} [x V'(x + \sigma_0) + V(x + \sigma_0)] |u^{(0)}|^2 dx + \mathcal{O}(\mu).
\end{aligned}$$

If $V_{\text{eff}}(\sigma_0) = 0$, then the solitary wave of the stationary LLE (5.31) with small $\mu \neq 0$ is uniquely continued in the perturbed equation for small ε and the unique continuation is spectrally stable if $V_{\text{eff}}'(\sigma_0) \cdot \varepsilon > 0$.

6 Multiple front and pulse solutions in spatially periodic systems

This chapter is a reprint of the article [17] written by the author of the thesis in collaboration with Björn de Rijk and submitted for publication. The article was adapted to fit the layout of this thesis.

Abstract

In this paper, we develop a comprehensive mathematical toolbox for the construction and spectral stability analysis of stationary multiple front and pulse solutions to general semilinear evolution problems on the real line with spatially periodic coefficients. Starting from a collection of N nondegenerate primary front solutions with matching periodic end states, we realize multifront solutions near a formal concatenation of these N primary fronts, provided the distances between the front interfaces is sufficiently large. Moreover, we prove that nondegenerate primary pulses are accompanied by periodic pulse solutions of large spatial period. We show that spectral (in)stability properties of the underlying primary fronts or pulses are inherited by the bifurcating multifronts or periodic pulse solutions. The existence and spectral analyses rely on contraction-mapping arguments and Evans-function techniques, leveraging exponential dichotomies to characterize invertibility and Fredholm properties. To demonstrate the applicability of our methods, we analyze the existence and stability of multifronts and periodic pulse solutions in some benchmark models, such as the Gross-Pitaevskii equation with periodic potential and a Klausmeier reaction-diffusion-advection system, thereby identifying novel classes of (stable) solutions. In particular, our methods yield the first spectral and orbital stability result of periodic waves in the Gross-Pitaevskii equation with periodic potential, as well as new instability criteria for multipulse solutions to this equation.

6.1 Introduction

Let $k, m \in \mathbb{N}, T > 0$, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. This paper focuses on stationary front and pulse solutions in general semilinear evolution systems on the real line of the form

$$\partial_t u = \alpha_k(x)\partial_x^k u + \ldots + \alpha_1(x)\partial_x u + \mathcal{N}(u, x), \qquad u(x,t) \in \mathbb{F}^m,\, x \in \mathbb{R},\, t \geq 0, \tag{6.1}$$

with continuous T-periodic coefficient functions $\alpha_i : \mathbb{R} \to \mathbb{F}^{m \times m}$, and continuous nonlinearity $\mathcal{N} : \mathbb{F}^m \times \mathbb{R} \to \mathbb{F}^m$ which is C^2 in its first argument and T-periodic in its second argument. We further assume that $\alpha_k(x)$ is invertible for each $x \in \mathbb{R}$.

Spatially periodic systems of the form (6.1) arise in a wide range of contexts. For instance, they appear as a mean-field approximations in the study of Bose-Einstein condensates in periodic trapping lattices [109], as models in nonlinear optics with periodic potential [18] or periodic forcing [56], as hydrodynamic bifurcation problems over oscillating domains [43, 133], as ecological models for the dynamics of vegetation patterns on periodic topographies [13], as equations describing the vertical infiltration of water through periodically layered unsaturated soils [48], or as reaction-diffusion systems in population biology in which the environment consists of favorable and unfavorable patches that are arranged periodically [84, 136]. The solutions of interest in these problems are typically pulse or front solutions: stationary gap solitons in Bose-Einstein condensates [109], optical signals composed of (a sequence of) standing, highly localized pulses in [18, 56], stationary patterns consisting of localized patches of vegetation in [13], so-called wetting fronts describing water infiltration in [48], and invasion fronts mediating population spreading in [84, 136].

Due to the significance to applications, mathematical studies on the existence, stability, and dynamics of front and pulse solutions have been extensively conducted across a wide variety of spatially periodic systems, see, for instance, the survey paper [154], the memoirs [157], and references therein. We note that *traveling* front and pulse solutions in such systems are typically *modulated*, i.e., they are time-periodic in the co-moving frame, see Remark 6.1.

In this paper, we focus on *stationary* front and pulse solutions to general spatially periodic systems of the form (6.1). We show that any collection of nondegenerate front (or pulse) solutions with matching asymptotic end states is accompanied by a family of multiple front (or pulse) solutions arising through concatenation or periodic extension. Moreover, we prove that their spectral (in)stability properties are determined by the comprising primary front (or pulse) solutions.

While stationary multifront and multipulse solutions have been studied in specific model problems, such as the Allen-Cahn equation with spatial inhomogeneity [14] and the Gross-Pitaevskii equation with periodic potential [1, 6, 109], a systematic framework for their construction and spectral stability analysis in general spatially periodic systems of the form (6.1) appears to be novel.[1] Furthermore, the spectral stability of large-wavelength *periodic* pulse solutions accompanying a primary pulse has, to the authors' best knowledge, not yet been rigorously addressed in any spatially periodic system prior to this paper.

Remark 6.1. The existence problem for *traveling-wave solutions* of the form $u(x,t) = \phi(x-ct)$ to the constant-coefficient system (6.2) with wavespeed $c \in \mathbb{R}$ is the same as (6.3) with the sole difference that the coefficient α_1 is replaced by $\alpha_1 + c$. However, this no longer holds when the coefficients are spatially periodic. In a frame $\xi = x - ct$ moving with speed $c \in \mathbb{R}$, the coefficients of (6.1) read $\alpha_j(\xi + ct)$. That is, they are periodic in time and space. Consequently, traveling wave solutions to (6.1) of the form $u(x,t) = \phi(x-ct)$ propagating with nonzero speed c while maintaining a fixed shape, cannot generally be expected. Instead, one typically finds modulated traveling waves of the form $u(x,t) = \phi(x-ct, x)$, where ϕ is T-periodic in its second component, cf. [39, 60, 65, 108, 145].

6.1.1 The case of constant coefficients

The construction of multiple front (or pulse) solutions through concatenation or periodic extension is well-documented in general semilinear problems with constant coefficients. If the coefficients α_i and nonlinearity $\mathcal{N}(u)$ do not depend on x, then system (6.1) reads

$$\partial_t u = \alpha_k \partial_x^k u + \ldots + \alpha_1 \partial_x u + \mathcal{N}(u). \tag{6.2}$$

Stationary solutions of (6.2) obey the autonomous ordinary differential equation

$$\alpha_k \partial_x^k u + \ldots + \alpha_1 \partial_x u + \mathcal{N}(u) = 0, \tag{6.3}$$

which can be written as a dynamical system $U' = F(U)$ in $U = (u, \partial_x u, \ldots, \partial_x^{k-1} u)$. Pulse and front solutions can be identified with homoclinic and heteroclinic connections in $U' = F(U)$. The existence of periodic, homoclinic or heteroclinic orbits near a nondegenerate homoclinic connection or a heteroclinic chain follows from homoclinic or heteroclinic bifurcation theory, relying on techniques such as Lin's method, Shil'nikov variables, and homoclinic center manifolds. The bifurcating orbits correspond to periodic pulse solutions with large spatial periods, as well as multifront (or multipulse) solutions with well-separated interfaces. An overview of homoclinic and heteroclinic bifurcation theory can be found in the survey paper [74].

[1] We note that the existence and spectral analysis of multifronts in [14] depend on the smallness rather than the periodicity of the inhomogeneity, in contrast to our results and those in [1, 6, 109].

Spectral stability of the bifurcating periodic pulse and multifront solutions to (6.2) has been studied in [4, 53, 126, 131] using Lin's method and Evans-function techniques. One finds that there are precisely M eigenvalues of the linearization about an M-pulse solution bifurcating from each simple isolated eigenvalue of the underlying primary pulse [4, 126]. Moreover, periodic pulse solutions have continua of eigenvalues in a neighborhood of each isolated eigenvalue of the primary pulse [53]. Since (6.2) is translational invariant, 0 must be an eigenvalue of each primary pulse or front. Hence, there are M eigenvalues of the linearization of (6.2) about an M-front or M-pulse converging to the 0 as the distance between interfaces tends to infinity. In addition, the linearization of (6.2) about a periodic pulse solution, posed on a space of localized perturbations, features a spectral curve converging to 0 as the period tends to infinity. Leading-order control on the spectrum in a neighborhood of the origin, established in [126, 131], shows that the bifurcating periodic pulse and multifront solutions to (6.2) can be unstable, even if all the underlying primary front or pulse solutions are spectrally stable.

6.1.2 Main results

Our existence and spectral stability analysis of stationary multiple front and pulse solutions in spatially periodic systems differs fundamentally from the constant-coefficient case. On the one hand, there seems to be no natural way to formulate the existence problem as an autonomous dynamical system that would facilitate the application of homoclinic or heteroclinic bifurcation theory. Furthermore, stationary pulse and front solutions to (6.1) generally converge to spatially periodic end states rather than to constant states. As a result, it appears that there is no obvious method for augmenting the eigenvalue problem and transforming it into an autonomous system that would enable the use of geometric dynamical systems techniques for analyzing spectral stability as in [4, 53].

On the other hand, the spatially periodic coefficients of (6.1) break the translational invariance, so that the linearization about a stationary front or pulse solution is generically invertible. Unlike the constant-coefficient case, we can (and do) leverage this property in the existence analysis of periodic pulse and multifront solutions. If system (6.1) is dissipative, then front and pulse solutions can be strongly spectrally stable, meaning that the spectrum of the linearization about the front or pulse is confined to the open left-half plane. This significantly reduces the complexity of the spectral analysis compared to the constant-coefficient case where an eigenvalue must reside at 0 due to translational invariance. However, we emphasize that our spectral techniques are also useful if system (6.1) is conservative, which naturally precludes strong spectral stability of solutions. We will illustrate this by establishing spectral stability and instability for periodic pulses and multipulses in the Gross-Pitaevskii equation with periodic potential.

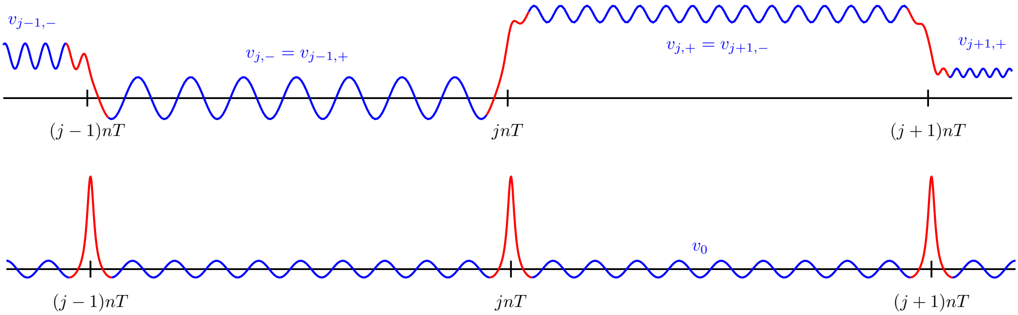

Figure 6.1: Illustration of the multifront solution $u_n(x)$ (top) and periodic pulse solution $\tilde{u}_n(x)$ (bottom) as established in Theorem 6.2. Both are depicted on the interval $[(j-1)nT,(j+1)nT]$ for some $j \in \{2,\ldots,M-1\}$. The multifront $u_n(x)$ transitions between the T-periodic states $v_{\ell,\pm}$ (in blue), with front interfaces (in red) located at $x = \ell nT$, $\ell = 1,\ldots,M$. The periodic solution $\tilde{u}_n(x)$ consists of a series of localized pulses (in red) centered at $x = jnT$, $j \in \mathbb{Z}$, superimposed on the T-periodic background state v_0 (in blue).

Our existence result may informally be stated as follows, see also Figure 6.1.

Theorem 6.2 (Informal existence result)**.** *Let $M \in \mathbb{N}$. Let $Z_1(x),\ldots,Z_M(x)$ be M stationary front solutions to system* (6.1) *converging to T-periodic end states $v_{1,\pm}(x),\ldots,v_{M,\pm}(x)$ as $x \to \pm\infty$. Assume that $v_{j,+} = v_{j+1,-}$ for $j = 1,\ldots,M-1$. Take a stationary pulse solution $Z_0(x)$ to* (6.1) *converging to a T-periodic end state $v_0(x)$ as $x \to \pm\infty$. Assume that Z_j is nondegenerate in the sense that the linearization of* (6.1) *about Z_j is invertible for $j = 0,\ldots,M$. Then, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ the following assertions hold.*

1. *There exists a stationary M-front solution $u_n(x)$ of* (6.1)*, which converges uniformly to the formal concatenation*

$$w_n(x) = \begin{cases} v_{1,-}(x), & x \leq \frac{1}{2}nT, \\ Z_j(x - jnT), & x \in \left[(j-\frac{1}{2})nT,(j+\frac{1}{2})nT\right], \qquad j = 1,\ldots,M, \\ v_{M,+}(x), & x \geq (M+\frac{1}{2})nT, \end{cases}$$

as $n \to \infty$.

2. *There exists a stationary nT-periodic pulse solution $\tilde{u}_n(x)$ of* (6.1)*, which converges uniformly to the formal periodic extension $\tilde{w}_n(x)$ given by*

$$\tilde{w}_n(x) = Z_0(x - jnT), \qquad x \in \left[(j-\tfrac{1}{2})nT,(j+\tfrac{1}{2})nT\right], \ j \in \mathbb{Z},$$

as $n \to \infty$.

For the precise statements, we refer to Theorems 6.5 and 6.8. The proof of the existence of the multifronts and periodic pulse solutions relies on a contraction-mapping argument in the function spaces $H^k(\mathbb{R})$ and $H^k_{\mathrm{per}}(0, nT)$, respectively. The key idea is to insert the formal multifront w_n or periodic pulse solution $\tilde{w}_n$ into system (6.1) and derive an equation for the resulting error. We then convert the error equation into a fix-point problem by showing that the linearization about the formal solution is invertible. Showing the invertibility is the main technical challenge, which follows from the nondegeneracy of the primary fronts or pulses with the aid of exponential dichotomies.

We emphasize that this procedure sharply contrasts with the constant-coefficient case, where the lack of invertibility of the linearization about the primary front or pulses necessitates a Lyapunov-Schmidt reduction argument. The reduced problem is typically solved by introducing an additional degree of freedom in the form of a bifurcation parameter or by exploiting additional structure such as reversible symmetry, see [74] and references therein.

We note that the distances between the interfaces of the multifront solutions in Theorem 6.2, as well as the wavelength of the periodic pulse solutions, are multiples of the period T, see Figure 6.1. In fact, the possible locations of front interfaces and pulse peaks are restricted by the spatial periodicity of (6.1). This phenomenon, known as *trapping* or *pinning*, cf. [14, 40] and references therein, precludes translational invariance and contributes to the enhanced stability properties compared to the constant-coefficient case, where the interfaces are typically free to occupy a continuum of positions. For example, if the constant-coefficient system (6.2) admits a reversible symmetry, then stationary periodic pulse solutions exist for any sufficiently large wavelength [147].

The main outcomes of our spectral analysis may informally be summarized as follows.

Theorem 6.3 (Informal spectral result)**.** *Let $\mathcal{K} \subset \mathbb{C}$ be compact. Let M, Z_j, N, u_n and $\tilde{u}_n$ be as in Theorem 6.2. Assume that the L^2-spectrum in $\mathcal{K}$ of the linearization $\mathcal{L}(Z_j)$ of* (6.1) *about Z_j consists of isolated eigenvalues of finite algebraic multiplicity only for $j = 0, \ldots, M$. Then, there exists $N_1 \in \mathbb{N}$ with $N \geq N_1$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ the following holds.*

1. *The L^2-spectrum of the linearization $\mathcal{L}(u_n)$ of* (6.1) *about u_n in the compact set $\mathcal{K}$ consists of isolated eigenvalues only and converges in Hausdorff distance to the union*

$$\bigcup_{j=1}^{M} \sigma(\mathcal{L}(Z_j)) \cap \mathcal{K}$$

as $n \to \infty$. The total algebraic multiplicity of the eigenvalues of $\mathcal{L}(u_n)$ in $\mathcal{K}$ equals the sum of the total algebraic multiplicies of the eigenvalues of $\mathcal{L}(Z_1), \ldots, \mathcal{L}(Z_M)$ in $\mathcal{K}$.

2. *The spectrum in $\mathcal{K}$ of the linearization $\mathcal{L}_{per}(\tilde{u}_n)$ of* (6.1) *about $\tilde{u}_n$ on $L^2_{\mathrm{per}}(0, nT)$ consists of isolated eigenvalues only and converges in Hausdorff distance to*

$$\sigma(\mathcal{L}(Z_0)) \cap \mathcal{K}$$

as $n \to \infty$. The total algebraic multiplicity of the eigenvalues of $\mathcal{L}_{per}(\tilde{u}_n)$ in $\mathcal{K}$ equals the total algebraic multiplicity of the eigenvalues of $\mathcal{L}(Z_0)$ in $\mathcal{K}$.

For the precise statements and further extensions of Theorem 6.3, we refer to Theorems 6.14 and 6.22 and Corollaries 6.13, 6.17 and 6.21.

The spectral analysis in this paper is divided into two parts. In the first part, we follow an approach similar to the one used in the existence analysis. Specifically, we utilize exponential dichotomies to characterize invertibility and demonstrate that, if the resolvent problem associated with the linearization about the primary fronts or pulses is uniquely solvable on a compact set in $\mathbb{C}$, then the same holds for the resolvent problem corresponding to the linearization about the multifront or periodic pulse solution.

In the second part, we identify eigenvalues with zeros of the analytic Evans function, see [78, 128] and references therein, to show that isolated eigenvalues of finite algebraic multiplicity of the linearization about the primary front or pulse perturb continuously into eigenvalues of the linearization about the multifront or periodic pulse posed on $L^2(\mathbb{R})$ or $L^2_{\mathrm{per}}(0, nT)$, respectively, thereby preserving the total algebraic multiplicity.

As mentioned earlier, similar results have been obtained for the constant-coefficient case, cf. [4, 53]. Unlike our existence analysis, which fails in the constant-coefficient setting due to the nondegeneracy condition, our spectral analysis does apply to multifronts and periodic pulse solutions to constant-coefficient systems. Yet, our method differs significantly from the geometric dynamical systems approach employed in [4, 53]. Instead, it is inspired by the Evans-function analyses in [35, 129]. It employs exponential weights and relies on roughness and analyticity properties of exponential dichotomies.

Theorem 6.3 shows that, if a point $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ is an eigenvalue of finite algebraic multiplicity of the linearizations about some of the primary fronts or pulses and lies in the resolvent set of the linearizations about the other primary fronts or pulses, then bifurcating multifronts and periodic pulse solutions are spectrally unstable. However, if spectral instability of the primary fronts is induced by unstable *essential spectrum*, then the associated multifront may still be spectrally stable. This phenomenon is well-documented in systems with constant coefficients [120, 130]. In §6.6, we present an extension of Theorem 6.3, providing control on the spectrum of the multifront outside the so-called *absolute spectrum*, cf. [129, 130]. This result can be employed to establish strong spectral stability of multifronts, even when the constituting primary fronts are spectrally unstable.

Many dissipative systems admit a-priori bounds that preclude spectrum with nonnegative real part and large modulus. By combining such a-priori bounds with Theorem 6.3, one finds that strong spectral stability of the primary fronts or pulses is carried over to the bifurcating multifronts and periodic pulse solutions. This contrasts sharply with the constant-coefficient case where multifronts or periodic pulse solutions can be spectrally unstable, even if the constituting primary fronts are all spectrally stable, cf. [126, 131]

6.1.3 Application to benchmark models

To illustrate the applicability of our methods, we construct multifronts and periodic pulse solutions in several prototypical models, analyze their spectral stability, and corroborate our findings with numerical simulations performed with the MATLAB package `pde2path` [146]. Specifically, we examine multifronts in a scalar reaction-diffusion toy model with a periodic potential and consider multipulses and periodic pulses in an extended Klausmeier model, which describes the dynamics of vegetation patterns on periodic topographies [13]. We demonstrate that the spectral (in)stability of the multifronts, multipulses and periodic pulses is inherited from the comprising primary fronts and pulses. In particular, our analysis shows that the Klausmeier model supports stable periodic (multi)pulses, which has not been identified in the previous work [13].

Additionally, we consider the Gross-Pitaevskii equation with a general periodic potential, which arises in the study of Bose-Einstein condensates in optical lattices [109]. Our methods lead to multifront, multipulse and periodic pulse solutions. By combining our spectral results with Krein index counting theory [75, 76, 78], we obtain novel spectral instability and stability results for the constructed multipulse and periodic pulse solutions. Due to the conservative nature of the Gross-Piteavskii equation, spectral stability entails that the spectrum of the linearization is confined to the imaginary axis. Notably, the preservation of algebraic multiplicities, as stated in Theorem 6.3, is instrumental for the effective application of Krein index counting theory. The spectral stability analysis of the periodic pulse solutions yields that they are orbitally stable. To the best of the authors' knowledge, this is the first orbital stability result of periodic waves in the Gross-Pitaevskii equation with periodic potential.

Outline of paper. In §6.2 we introduce the necessary notation and formulate the existence and (weighted) eigenvalue problems associated with stationary solutions to (6.1). The existence analysis of multifronts and periodic pulse solutions is presented in §6.3 and §6.4, respectively. In §6.5 we introduce the necessary concepts for the spectral analysis of fronts solutions with periodic tails. Sections 6.6 and §6.7 are devoted to the spectral analysis of multifronts and periodic pulse solutions, respectively. We demonstrate the applicability of our methods in several benchmark models in §6.8 and corroborate our findings with numerical simulations. Finally, the Appendices 6.9, 6.10 and 6.11 contain several auxiliary results on projections, exponential dichotomies, and multiplication operators, respectively.

6.2 Notation and set-up

Let $k, m \in \mathbb{N}, T > 0$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. This paper focuses on the existence and spectral analysis of stationary front and pulse solutions to the general semilinear evolution system (6.1) of m components on the real line. For notational convenience, we abbreviate the k-th order linear differential operator in (6.1) as

$$Au = \alpha_k(x)\partial_x^k u + \alpha_{k-1}(x)\partial_x^{k-1} u + \ldots + \alpha_1(x)\partial_x u.$$

We recall that A has T-periodic coefficients $\alpha_1, \ldots, \alpha_k \in C(\mathbb{R}, \mathbb{F}^{m\times m})$, where $\alpha_k(x)$ invertible for all $x \in \mathbb{R}$. Moreover, the nonlinearity $\mathcal{N} \in C\big(\mathbb{F}^m \times \mathbb{R}, \mathbb{F}^m\big)$ in (6.1) is twice continuously differentiable in its first argument and T-periodic in its second argument.

6.2.1 Formulation of the existence and eigenvalue problems

Stationary solutions to (6.1) obey

$$Au + \mathcal{N}(u, \cdot) = 0. \tag{6.4}$$

The ordinary differential equation (6.4) is the main object of study in the existence analysis of stationary front and pulse solutions to (6.1).

For $\underline{u} \in L^\infty(\mathbb{R})$ we define the linear differential operator $\mathcal{L}(\underline{u})\colon D(\mathcal{L}(\underline{u})) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathcal{L}(\underline{u})u = Au + \partial_u \mathcal{N}(\underline{u}, \cdot)u.$$

Clearly, $\mathcal{L}(\underline{u})$ is a closed operator and has dense domain $D(\mathcal{L}(\underline{u})) = H^k(\mathbb{R})$. If $\underline{u}$ is a solution of (6.4), then $\mathcal{L}(\underline{u})$ corresponds to the linearization of (6.1) about $\underline{u}$. The associated eigenvalue problem reads

$$\left(\mathcal{L}(\underline{u}) - \lambda\right) u = 0. \tag{6.5}$$

The linear ordinary differential equation (6.5) is the main object of study in the spectral analysis of the stationary front and pulse solutions to (6.1). We adopt the following notions of nondegeneracy and spectral stability.

Definition 6.4. Let $\underline{u} \in L^\infty(\mathbb{R})$.

(i) We call $\underline{u}$ *nondegenerate* if $\mathcal{L}(\underline{u})$ is invertible.

(ii) We say that $\underline{u}$ is *spectrally stable* if

$$\sigma(\mathcal{L}(\underline{u})) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\}.$$

(iii) We say that $\underline{u}$ is *spectrally stable with simple eigenvalue* $\lambda = 0$ if there exists $\varrho > 0$ such that

$$\sigma(\mathcal{L}(\underline{u})) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\varrho\} \cup \{0\},$$

and the algebraic multiplicity of $\lambda = 0$ is one.

(iv) We say that $\underline{u}$ is *strongly spectrally stable* if there exists $\varrho > 0$ such that

$$\sigma(\mathcal{L}(\underline{u})) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\varrho\}.$$

(v) We call $\underline{u}$ *spectrally unstable* it there exists $\lambda \in \sigma(\mathcal{L}(\underline{u}))$ with $\mathrm{Re}(\lambda) > 0$.

As explained in the introduction, the concept of nondegeneracy plays a key role in the construction of multiple front and pulse solutions.

Spectrally stable front or pulse solutions with a simple eigenvalue at $\lambda = 0$ arise in systems with translational invariance. Such solutions serve as a basis for bifurcation arguments in the models explored in the application section §6.8.

6.2.2 First-order formulation

Because the coefficient matrix $\alpha_k(x)$ is invertible for all $x \in \mathbb{R}$ and the nonlinearity $\mathcal{N}$ is twice continuously differentiable in its first argument, the eigenvalue problem (6.5) can be written as a first-order system

$$U' = \mathcal{A}(x, \underline{u}(x); \lambda)U, \tag{6.6}$$

by setting $U = (u, \partial_x u, \ldots, \partial_x^{k-1} u)^\top$, where the coefficient matrix $\mathcal{A}\colon \mathbb{R} \times \mathbb{F}^m \times \mathbb{C} \to \mathbb{C}^{km}$ is continuous and T-periodic in its first argument, continuously differentiable in its second argument, and analytic in its third argument. The formulation (6.6) of the eigenvalue problem as a linear nonautonomous first-order system is essential for applying the theory of exponential dichotomies.

6.2.3 Exponentially weighted linearization operator

Let $\underline{u} \in L^\infty(\mathbb{R})$ and $\eta_\pm \in \mathbb{R}$. For the spectral analysis of stationary front solutions to (6.1), it is convenient to consider the exponentially weighted linearization operator $\mathcal{L}_{\eta_-,\eta_+}(\underline{u})\colon D(\mathcal{L}_{\eta_-,\eta_+}(\underline{u})) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with dense domain $D(\mathcal{L}_{\eta_-,\eta_+}(\underline{u})) = H^k(\mathbb{R})$ given by

$$\mathcal{L}_{\eta_-,\eta_+}(\underline{u})u = \mathrm{e}^{-\omega_{\eta_-,\eta_+}} \mathcal{L}(\underline{u})\,[\mathrm{e}^{\omega_{\eta_-,\eta_+}} u]$$

where $\omega_{\eta_-,\eta_+}\colon \mathbb{R} \to \mathbb{R}$ is a smooth weight function whose derivative satisfies

$$\omega'_{\eta_-,\eta_+}(x) = \begin{cases} \eta_-, & x \leq -1, \\ \eta_+, & x \geq 1. \end{cases}$$

The associated eigenvalue problem

$$\big(\mathcal{L}_{\eta_-,\eta_+}(\underline{u}) - \lambda\big)\, u = 0$$

can be written as the first-order system

$$U' = \Big(\mathcal{A}(x, \underline{u}(x); \lambda) - \omega'_{\eta_-,\eta_+}(x)\Big)\, U.$$

In case $\eta_+ = \eta_- = \eta \in \mathbb{R}$, we take $\omega_{\eta_-,\eta_+}(x) = \eta x$ and adopt the notation $\mathcal{L}_{\eta_-,\eta_+}(\underline{u}) = \mathcal{L}_\eta(\underline{u})$. Consequently, it holds $\mathcal{L}_{0,0}(\underline{u}) = \mathcal{L}_0(\underline{u}) = \mathcal{L}(\underline{u})$.

6.2.4 Periodic differential operators

Let $n \in \mathbb{N}$ and let $\underline{u} \in C(\mathbb{R})$ be an nT-periodic function. Then, $\mathcal{L}(\underline{u})$ is a differential operator with nT-periodic coefficients. We collect some basic properties of periodic differential operators and their Bloch transforms, which are essential for our spectral analysis. We refer to [52, 78, 118, 132] for further background.

Since $\mathcal{L}(\underline{u})$ has nT-periodic coefficients, we can study its action on the space $L^2_{\text{per}}(0,nT)$, which is convenient for analyzing the spectral stability of $\underline{u}$ against co-periodic perturbations. Thus, we define the operator $\mathcal{L}_{\text{per}}(\underline{u})\colon D(\mathcal{L}_{\text{per}}(\underline{u})) \subset L^2_{\text{per}}(0,nT) \to L^2_{\text{per}}(0,nT)$ by

$$\mathcal{L}_{\text{per}}(\underline{u})u = Au + \partial_u \mathcal{N}(\underline{u},\cdot)u.$$

The operator $\mathcal{L}_{\text{per}}(\underline{u})$ is closed and has dense domain $D(\mathcal{L}_{\text{per}}(\underline{u})) = H^k_{\text{per}}(0,nT)$. Due to the compact embedding $H^k_{\text{per}}(0,nT) \hookrightarrow L^2_{\text{per}}(0,nT)$, it has compact resolvent and its spectrum consists of isolated eigenvalues of finite algebraic multiplicity only. Hence, a point $\lambda \in \mathbb{C}$ lies in the spectrum $\sigma(\mathcal{L}_{\text{per}}(\underline{u}))$ if and only if the first-order eigenvalue problem (6.6) admits a nontrivial nT-periodic solution.

On the other hand, the spectrum of the nT-periodic differential operator $\mathcal{L}(\underline{u})$ on $L^2(\mathbb{R})$ is purely essential. It is characterized by family of Bloch operators $\mathcal{L}_{\xi,\text{per}}(\underline{u})\colon D(\mathcal{L}_{\xi,\text{per}}(\underline{u})) \subset L^2_{\text{per}}(0,nT) \to L^2_{\text{per}}(0,nT)$ with dense domain $D(\mathcal{L}_{\xi,\text{per}}(\underline{u})) = H^k_{\text{per}}(0,nT)$ given by

$$\mathcal{L}_{\xi,\text{per}}(\underline{u})u = M_\xi^{-1}\mathcal{L}(\underline{u})M_\xi u, \qquad \xi \in \left[-\frac{\pi}{nT}, \frac{\pi}{nT}\right),$$

where $M_\xi\colon H^\ell(\mathbb{R}) \to H^\ell(\mathbb{R})$ is the invertible multiplication operator defined by $(M_\xi u)(x) = \mathrm{e}^{\mathrm{i}\xi x}u(x)$ for $\ell \in \mathbb{N}_0$. Since the Bloch operators $\mathcal{L}_{\xi,\text{per}}(\underline{u})$ have compact resolvent, their spectrum consists of isolated eigenvalues of finite algebraic multiplicity only. Therefore, a point $\lambda \in \mathbb{C}$ lies in the spectrum of $\mathcal{L}_{\xi,\text{per}}(\underline{u})$ if and only if the first-order system (6.6) admits a nontrivial solution $U(x)$ obeying the boundary condition $U(-\frac{nT}{2}) = \mathrm{e}^{\mathrm{i}\xi nT}U(\frac{nT}{2})$. The spectrum of $\mathcal{L}(\underline{u})$ is then given by the union

$$\sigma(\mathcal{L}(\underline{u})) = \bigcup_{\xi \in \left[-\frac{\pi}{nT}, \frac{\pi}{nT}\right)} \sigma(\mathcal{L}_{\xi,\text{per}}(u_n)),$$

which implies

$$\sigma(\mathcal{L}_{\text{per}}(\underline{u})) \subset \sigma(\mathcal{L}(\underline{u})). \tag{6.7}$$

Hence, $\lambda \in \mathbb{C}$ lies in the spectrum of $\mathcal{L}(\underline{u})$ if and only if there exist a point γ on the unit circle $S^1 \subset \mathbb{C}$ and a nontrivial solution $U(x)$ of (6.6) obeying $U(-\frac{nT}{2}) = \gamma U(\frac{nT}{2})$.

6.3 Existence of multifront solutions

Let $M \in \mathbb{N}_{\geq 2}$. In this section, we construct an M-front solution to (6.4) by concatenating M nondegenerate front solutions with matching periodic limit states. Specifically, we impose the following assumptions:

(H1) There exist M fronts $Z_1, \ldots, Z_M \in L^\infty(\mathbb{R})$ with associated end states $v_{1,\pm}, \ldots, v_{M,\pm} \in H^k_{\text{per}}(0,T)$. It holds $\chi_\pm (Z_j - v_{j,\pm}) \in H^k(\mathbb{R})$ for $j = 1, \ldots, M$, where $\chi_\pm \colon \mathbb{R} \to [0,1]$ is a smooth partition of unity such that χ_+ is supported on $(-1,\infty)$ and χ_- is supported on $(-\infty, 1)$.

(H2) The matching condition $v_{j,+} = v_{j+1,-}$ holds for $j = 1, \ldots, M-1$.

(H3) The front Z_j is a nondegenerate solution of (6.4) for $j = 1, \ldots, M$.

We realize the M-front close to the formal concatenation of the M primary fronts $Z_1, \ldots, Z_M$, see Figure 6.1. Thus, our ansatz for the multifront solution to (6.4) reads

$$u_n = a_n + w_n, \qquad w_n := \sum_{j=1}^{M} \chi_{j,n} Z_j(\cdot - jnT) \tag{6.8}$$

with $\chi_{j,n} \colon \mathbb{R} \to [0,1]$, $j = 1, \ldots, M$ a smooth partition of unity satisfying $\|\chi_{j,n}\|_{W^{k,\infty}} \leq 1$, where $\chi_{1,n}$ is supported on $(-\infty, \frac{3}{2}nT + 1)$, $\chi_{M,n}$ is supported on $((M - \frac{1}{2})nT - 1, \infty)$, and $\chi_{j,n}$ is supported on $((j-\frac{1}{2})nT - 1, (j+\frac{1}{2})nT + 1)$ for $j = 2, \ldots, M-1$ (only in case $M > 2$). Moreover, $a_n \in H^k(\mathbb{R})$ is an error term that accounts for the fact that the formal concatenation w_n of the M fronts is not an actual solution to (6.4). Our main result of this section confirms that there exists a small $a_n \in H^k(\mathbb{R})$ such that u_n is indeed a solution to (6.4), provided $n \in \mathbb{N}$ is sufficiently large.

Theorem 6.5. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Then, there exist $C > 0$ and $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N$ there exists an M-front solution to* (6.4) *given by* (6.8) *with $a_n \in H^k(\mathbb{R})$ satisfying*

$$\|a_n\|_{H^k} \leq C \sum_{j=1}^{M} \Big(\|\chi_-(Z_j - v_{j,-})\|_{H^k(\mathbb{R} \setminus [-\frac{n}{2}T+1, \frac{n}{2}T-1])} + \|\chi_+(Z_j - v_{j,+})\|_{H^k(\mathbb{R} \setminus [-\frac{n}{2}T+1, \frac{n}{2}T-1])} \Big).$$

In particular, $\|a_n\|_{H^k}$ converges to 0 as $n \to \infty$.

Remark 6.6. One could prove a more general result where the front interfaces are located on positions $n_1T, \dots, n_mT$ as long as the distances $n_{i+1} - n_i$ are sufficiently large for each $i = 1, \dots, M-1$. More precisely, there exist $C > 0$ and $N \in \mathbb{N}$ such that for each vector $n = (n_1, \dots, n_m) \in \mathbb{N}^M$ with $\kappa := \min\{n_{i+1} - n_i : i \in \{1, \dots, M-1\}\} \geq N$, there exists an M-front solution

$$u_n = a_n + \sum_{j=1}^{M} \tilde{\chi}_{j,n} Z_j(\cdot - jn_jT)$$

of (6.4), where $\tilde{\chi}_{j,n} \colon \mathbb{R} \to [0,1]$, $j = 1, \dots, M$ is a smooth partition of unity satisfying $\|\tilde{\chi}_{j,n}\|_{W^{k,\infty}} \leq 1$ with $\tilde{\chi}_{1,n}$ supported on $(-\infty, \frac{1}{2}(n_1+n_2)T+1)$, $\tilde{\chi}_{M,n}$ supported on $(\frac{1}{2}(n_{M-1}+n_M)T-1, \infty)$, and $\tilde{\chi}_{j,n}$ supported on $(\frac{1}{2}(n_{j-1}+n_j)T-1, \frac{1}{2}(n_j+n_{j+1})T+1)$ for $j = 2, \dots, M-1$ (only in case $M > 2$). Moreover, the error $a_n \in H^k(\mathbb{R})$ converges to 0 as $\kappa \to \infty$, The proof of this statement proceeds along the lines of Theorem 6.5, but with considerably more involved notation, and is therefore omitted.

The proof of Theorem 6.5 relies on a contraction-mapping argument. Inserting the ansatz (6.8) into (6.4), one arrives at an equation for the error a_n, whose linear part is given by $\mathcal{L}(w_n)a_n$. Here, $\mathcal{L}(w_n)$ represents the linearization of (6.1) about the formal concatenation w_n of the M fronts, with $\mathcal{L}(\cdot)$ defined in §6.2. To solve for the error, we recast the equation as a fixed-point problem in $H^k(\mathbb{R})$ by inverting the linear operator $\mathcal{L}(w_n)$.

The invertibility of $\mathcal{L}(w_n)$ is established by transferring the nondegeneracy of the primary fronts $Z_1, \dots, Z_M$ to the concatenation w_n. This is achieved by characterizing invertibility through exponential dichotomies [93]. Specifically, $\mathcal{L}(\underline{u}) - \lambda$ can be inverted if and only if the first-order formulation (6.6) of the eigenvalue problem admits an exponential dichotomy on $\mathbb{R}$. By applying pasting and roughness techniques to the exponential dichotomies arising through the nondegeneracy of the individual fronts, we construct an exponential dichotomy on $\mathbb{R}$ for the first-order formulation of the eigenvalue problem associated with $\mathcal{L}(w_n)$, thereby establishing its invertibility.

The invertibility result is formalized in the following lemma, which is stated in a slightly more general form. This generalization also plays a central role in the subsequent spectral analysis of the multifront.

Lemma 6.7. *Assume* **(H1)** *and* **(H2)**. *Let $\mathcal{K} \subset \mathbb{C}$ be a compact set. Moreover, let $\{a_n\}_n$ be a sequence in $H^k(\mathbb{R})$ with $\|a_n\|_{H^k} \to 0$ as $n \to \infty$.*

Suppose that the linear operator $\mathcal{L}(Z_j) - \lambda$ is invertible for each $\lambda \in \mathcal{K}$ and $j = 1, \ldots, M$. Then, there exist $C > 0$ and $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N$ the resolvent set of the operator

$$\mathcal{L}(w_n + a_n), \qquad w_n := \sum_{j=1}^{M} \chi_{j,n} Z_j(\cdot - jnT)$$

contains $\mathcal{K}$ and the resolvent obeys the bound

$$\left\| (\mathcal{L}(w_n + a_n) - \lambda)^{-1} \right\|_{L^2 \to H^k} \leq C \tag{6.9}$$

for $\lambda \in \mathcal{K}$.

Proof. Lemma 6.57 yields that the first-order system

$$U' = \mathcal{A}\left(x, Z_j(x); \lambda\right) U \tag{6.10}$$

admits an exponential dichotomy on $\mathbb{R}$ for each $\lambda \in \mathcal{K}$ and for $j = 1, \ldots, M$. By continuity of $\mathcal{A}$ and roughness of exponential dichotomies, cf. [33, Proposition 4.1], there exists for each $\lambda_0 \in \mathcal{K}$ an open disk $B_{\lambda_0} \subset \mathbb{C}$ with $\lambda_0 \in B_{\lambda_0}$ and constants $K_{\lambda_0}, \mu_{\lambda_0} > 0$ such that (6.10) has an exponential dichotomy on $\mathbb{R}$ for each $\lambda \in B_{\lambda_0}$ with constants $K_{\lambda_0}, \mu_{\lambda_0} > 0$. By compactness of $\mathcal{K}$ the open cover $\{B_{\lambda_0} : \lambda_0 \in \mathcal{K}\}$ has a finite subcover. It follows that (6.10) has for each $\lambda \in \mathcal{K}$ an exponential dichotomy on $\mathbb{R}$ with λ-independent constants.

Clearly, system

$$U' = \mathcal{A}\left(x, Z_j(x - jnT); \lambda\right) U \tag{6.11}$$

is for each $n \in \mathbb{N}$ and $j = 1, \ldots, M$ a jnT-translation of system (6.10). So, (6.11) possesses for each $n \in \mathbb{N}$, $j = 1, \ldots, M$ and $\lambda \in \mathcal{K}$ an exponential dichotomy on $\mathbb{R}$ with λ- and n-independent constants.

We use roughness techniques to transfer the exponential dichotomy of system (6.11) to an exponential dichotomy of system

$$U' = \mathcal{A}(x, w_n(x) + a_n(x); \lambda) U \tag{6.12}$$

on an interval I_j for $j = 1, \ldots, M$, where we denote $I_1 = (-\infty, \frac{5}{3}nT]$, $I_j = [(j - \frac{2}{3})nT, (j + \frac{2}{3})nT]$ for $j = 2, \ldots, M-1$ (only in case $M > 2$), and $I_M = [(M - \frac{2}{3})nT, \infty)$. Since $\partial_u \mathcal{A}$ is continuous, $\mathcal{K}$ is compact and we have $Z_1, \ldots, Z_M \in L^\infty(\mathbb{R})$ and $a_n \in H^k(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we obtain by the mean value theorem a λ- and n-independent constant $K_0 > 0$ such that

$$\begin{aligned} &\|\mathcal{A}(x, w_n(x) + a_n(x); \lambda) - \mathcal{A}(x, Z_j(x - jnT); \lambda)\| \\ &\quad \leq K_0 \Bigg(\|a_n\|_{L^\infty} + \sum_{\ell \in \{1,\ldots,M\}} \Big(\|\chi_-(Z_\ell - v_{\ell,-})\|_{L^\infty(\mathbb{R}\setminus(-\frac{1}{3}nT, \frac{1}{3}nT))} \\ &\qquad\qquad + \|\chi_+(Z_\ell - v_{\ell,+})\|_{L^\infty(\mathbb{R}\setminus(-\frac{1}{3}nT, \frac{1}{3}nT))} \Big) \Bigg) \end{aligned} \tag{6.13}$$

for $x \in I_j$, $n \in \mathbb{N}$, $\lambda \in \mathcal{K}$ and $j = 1, \ldots, M$.

It is readily seen by approximation with simple functions that for each $g \in L^2(\mathbb{R})$ it holds $\|g\|_{L^2(\mathbb{R}\setminus[-R,R])} \to 0$ as $R \to \infty$. Thus, $\|\chi_\pm(Z_j - v_{j,\pm})\|_{H^1(\mathbb{R}\setminus[-R,R])}$ converges to 0 as $R \to \infty$ for $j = 1, \ldots, M$. Hence, noting that $H^1(I)$ continuously embeds into $L^\infty(I)$ for each interval $I \subset \mathbb{R}$, the right-hand side of the estimate (6.13) converges to 0 uniformly on I_j as $n \to \infty$ for $j = 1, \ldots, M$. Therefore, using that (6.11) has an exponential dichotomy on $\mathbb{R}$ with λ- and n-independent constants, we establish, provided $n \in \mathbb{N}$ is sufficiently large, by [33, Proposition 4.1] an exponential dichotomy for (6.12) on I_j with λ-, j- and n-independent constants $K_1, \mu_1 > 0$ and projections $P_{j,n}(x;\lambda)$ for each $\lambda \in \mathcal{K}$ and $j = 1, \ldots, M$.

For $j = 1, \ldots, M-1$ we iteratively paste the exponential dichotomies for (6.12) on the intervals $(-\infty, (j+\frac{2}{3})nT]$ and I_{j+1} together at the point $x = (j+\frac{1}{2})nT$ to obtain an exponential dichotomy on $\mathbb{R}$. Let $j \in \{1, \ldots, M-1\}$. Given an exponential dichotomy for (6.12) on $(-\infty, (j+\frac{2}{3})nT]$ with λ- and n-independent constants $K_j, \mu_j > 0$ and projections $Q_{j,n}(x;\lambda)$, we employ Lemma 6.54 and arrive, provided $n \in \mathbb{N}$ is sufficiently large, at

$$\left\|Q_{j,n}\left((j+\tfrac{1}{2})nT;\lambda\right) - P_{j+1,n}\left((j+\tfrac{1}{2})nT;\lambda\right)\right\| \leq 2K_1K_j \mathrm{e}^{-(\mu_1+\mu_j)\frac{n}{6}T} < 1,$$

for each $\lambda \in \mathcal{K}$. Hence, the subspaces $\ker(Q_{j,n}((j+\frac{1}{2})nT;\lambda))$ and $\operatorname{ran}(P_{j+1,n}((j+\frac{1}{2})nT;\lambda))$ are complementary by Lemma 6.48 and the associated projection $P_{\circ,n}(\lambda)$ onto $\operatorname{ran}(P_{j+1,n}((j+\frac{1}{2})nT;\lambda))$ along $\ker(Q_{j,n}((j+\frac{1}{2})nT;\lambda))$ is well-defined and satisfies

$$\|P_{\circ,n}(\lambda)\| \leq \frac{K_j}{1 - 2K_1K_j\mathrm{e}^{-(\mu_1+\mu_j)\frac{n}{6}T}},$$

for each $\lambda \in \mathcal{K}$. Hence, provided $n \in \mathbb{N}$ is sufficiently large, Lemma 6.53 yields an exponential dichotomy for system (6.12) on $(-\infty, (j+\frac{2}{3})nT] \cup I_{j+1}$ with λ- and n-independent constants for each $\lambda \in \mathcal{K}$. Thus, iteratively repeating the above procedure for $j = 1, \ldots, M-1$, we establish, provided $n \in \mathbb{N}$ is sufficiently large, an exponential dichotomy of (6.12) on $(-\infty, (M - \frac{1}{3})nT] \cup I_M = \mathbb{R}$ with λ- and n-independent constants for each $\lambda \in \mathcal{K}$.

Using the compactness of $\mathcal{K}$ and the continuity of $\mathcal{A}$, it follows that $\|\mathcal{A}(x, w_n(x) + a_n(x); \lambda)\|_{L^\infty}$ can be bounded by a λ- and n-independent constant for each $\lambda \in \mathcal{K}$ and $n \in \mathbb{N}$. So, provided $n \in \mathbb{N}$ is sufficiently large, Lemma 6.56 yields a λ- and n-independent constant $C > 0$ such that for each $g \in H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ and $\lambda \in \mathcal{K}$ the inhomogeneous linear problem

$$U' = \mathcal{A}(x, w_n(x) + a_n(x); \lambda)U + \psi$$

with inhomogeneity $\psi = (0, \ldots, 0, g)^\top \in H^1(\mathbb{R})$ has a solution $U \in H^1(\mathbb{R})$ satisfying

$$\|U\|_{H^1} \leq C\|\psi\|_{L^2} = C\|g\|_{L^2}.$$

Using that we have $U_i' = U_{i+1} \in H^1(\mathbb{R})$ for $i = 1, \ldots, k-1$, we readily observe that $u = U_1 \in H^k(\mathbb{R})$ solves the resolvent problem

$$(\mathcal{L}(w_n + a_n) - \lambda)\, u = g, \tag{6.14}$$

and satisfies

$$\|u\|_{H^k} \leq \|U\|_{H^1} \leq C\|g\|_{L^2}. \tag{6.15}$$

Since the operator $\mathcal{L}(w_n + a_n)$ is closed, it follows by the density of $H^1(\mathbb{R})$ in $L^2(\mathbb{R})$ that, provided $n \in \mathbb{N}$ is sufficiently large, the resolvent problem (6.14) possesses for each $g \in L^2(\mathbb{R})$ and $\lambda \in \mathcal{K}$ a solution $u \in H^k(\mathbb{R})$ satisfying (6.15).

Finally, if $u \in H^k(\mathbb{R})$ lies in the kernel of $\mathcal{L}(w_n + a_n) - \lambda$, then $U = (u, \partial_x u, \ldots, \partial_x^{k-1} U)^\top \in H^1(\mathbb{R})$ is a localized solution of the first-order variational problem (6.12). Since (6.12) has an exponential dichotomy on $\mathbb{R}$, U must be the trivial solution and, thus, we find $u = 0$.

So, we have established that, provided $n \in \mathbb{N}$ is sufficiently large, $\mathcal{L}(w_n + a_n) - \lambda$ is bounded invertible and satisfies (6.9) for each $\lambda \in \mathcal{K}$. □

With the aid of Lemma 6.7, we now prove Theorem 6.5 using a contraction-mapping argument.

Proof of Theorem 6.5. First, Lemma 6.7 implies that the linear operator $\mathcal{L}(w_n)$ is invertible and there exists an n-independent constant $K > 0$ such that

$$\|\mathcal{L}(w_n)^{-1}\|_{L^2 \to H^k} \leq K. \tag{6.16}$$

Inserting the ansatz $u = w_n + a$ with correction term $a \in H^k(\mathbb{R})$ into (6.4) yields the equation

$$a = \widetilde{\mathcal{N}}(a) + R, \tag{6.17}$$

where the nonlinear map $\widetilde{\mathcal{N}} \colon H^k(\mathbb{R}) \to H^k(\mathbb{R})$ is given by

$$\widetilde{\mathcal{N}}(a) = \mathcal{L}(w_n)^{-1}\left(\mathcal{N}\left(w_n, \cdot\right) + \partial_u \mathcal{N}\left(w_n, \cdot\right) a - \mathcal{N}\left(w_n + a, \cdot\right)\right)$$

and the residual $R \in H^k(\mathbb{R})$ is given by

$$R = -\mathcal{L}(w_n)^{-1}\left(\mathcal{N}(w_n, \cdot) + A(w_n)\right).$$

Using the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the fact that the nonlinearity $\mathcal{N}$ is twice continuously differentiable in its first argument, it follows by Taylor's theorem and estimate (6.16) that $\widetilde{\mathcal{N}} \colon H^k(\mathbb{R}) \to H^k(\mathbb{R})$ is well-defined and for all $\rho > 0$ there exists an n-independent constant $C_1 > 0$ such that

$$\left\|\widetilde{\mathcal{N}}(a_0) - \widetilde{\mathcal{N}}(a_1)\right\|_{H^k} \leq C_1 \left(\|a_0\|_{H^k} + \|a_1\|_{H^k}\right) \|a_0 - a_1\|_{H^k}. \tag{6.18}$$

for $a_0, a_1 \in H^k(\mathbb{R})$ with $\|a_0\|_{L^\infty}, \|a_1\|_{L^\infty} \leq \rho$.

Next, the fact that $Z_j(\cdot - jnT)$ is a solution of (6.4) implies that

$$R = \mathcal{L}(w_n)^{-1}\left(\mathcal{N}\left(Z_j(\cdot - jnT), \cdot\right) - \mathcal{N}(w_n, \cdot) - A\left(w_n - Z_j(\cdot - jnT)\right)\right)$$

for $j = 1, \ldots, M$. We partition $\mathbb{R} = I_1 \cup \ldots \cup I_M$ with $I_1 = (-\infty, \frac{3}{2}nT]$, $I_j = ((j - \frac{1}{2})nT, (j + \frac{1}{2})nT]$ for $j = 2, \ldots, M-1$ (only in case $M > 2$), and $I_M = ((M - \frac{1}{2})nT, \infty)$. Since the nonlinearity $\mathcal{N}$ is continuously differentiable in its first argument and it holds $\|\chi_{j,n}\|_{W^{k,\infty}} \leq 1$ for $j = 1, \ldots, M$, the mean value theorem and estimate (6.16) yield an n-independent constant $C_0 > 0$ such that

$$\|R\|_{H^k} \leq K \sum_{j=1}^{M} \left\|\mathcal{N}\left(Z_j(\cdot - jnT), \cdot\right) - \mathcal{N}(w_n, \cdot) - A\left(w_n - Z_j(\cdot - jnT)\right)\right\|_{L^2(I_j)} \leq C_0 \delta_n, \tag{6.19}$$

where we denote

$$\delta_n = \sum_{j=1}^{M} \Big(\|\chi_-(Z_j - v_{j,-})\|_{H^k(\mathbb{R} \setminus [-\frac{n}{2}T+1, \frac{n}{2}T-1])} + \|\chi_+(Z_j - v_{j,+})\|_{H^k(\mathbb{R} \setminus [-\frac{n}{2}T+1, \frac{n}{2}T-1])} \Big).$$

We observe that δ_n converges to 0 as $n \to \infty$.

Motivated by the estimate (6.19), we introduce the rescaled variable

$$\mathsf{a} = \delta_n^{-1} a, \tag{6.20}$$

in which (6.17) reads

$$\mathsf{a} = \delta_n^{-1} \widetilde{\mathcal{N}}(\delta_n \mathsf{a}) + \delta_n^{-1} R. \tag{6.21}$$

We regard (6.21) as an abstract fixed point problem

$$\mathsf{a} = \mathcal{F}_n(\mathsf{a}) \tag{6.22}$$

and show that $\mathcal{F}_n \colon B_0(2C_0) \to B_0(2C_0)$ is a well-defined contracting mapping on the ball $B_0(2C_0)$ of radius $2C_0$ in $H^k(\mathbb{R})$ centered at the origin, where $C_0 > 0$ is the n-independent constant appearing in the bound (6.19) on R. Combining the estimates (6.18) and (6.19) and noting $\widetilde{\mathcal{N}}(0) = 0$, yields an n-independent constant $K_0 > 0$ such that

$$\begin{aligned} \|\mathcal{F}_n(g)\|_{H^k} &\leq C_0 + K_0 \delta_n, \\ \|\mathcal{F}_n(g) - \mathcal{F}_n(h)\|_{H^2} &\leq K_0 \delta_n \|g - h\|_{H^k}, \end{aligned}$$

for $g, h \in B_0(2C_0)$. Therefore, using that $\delta_n \to 0$ as $n \to \infty$, $\mathcal{F}_n$ is a well-defined contraction mapping on $B_0(2C_0)$, provided $n \in \mathbb{N}$ is sufficiently large. By the Banach fixed point theorem there exists a unique solution $\mathsf{a}_n \in B_0(2C_0)$ to (6.22). Undoing the rescaling (6.20), we found a solution $a_n \in H^k(\mathbb{R})$ of equation (6.17) with

$$\|a_n\|_{H^k} \leq 2C_0 \delta_n,$$

provided $n \in \mathbb{N}$ is sufficiently large, which yields the result. □

6.4 Existence of periodic pulse solutions

In this section, we construct a periodic solution to (6.4) by periodically extending a nondegenerate pulse solution, see Figure 6.1. Thus, we impose the following assumptions:

(H4) There exist $v \in H^k_{\text{per}}(0,T)$ and $z \in H^k(\mathbb{R})$ such that v and $z + v$ are solutions to (6.4).

(H5) The pulse $z + v$ is nondegenerate.

The main result of this section reads as follows.

Theorem 6.8. *Assume* **(H4)** *and* **(H5)**. *Then, there exist* $C > 0$ *and* $N \in \mathbb{N}$ *such that for each* $n \in \mathbb{N}$ *with* $n \geq N$ *there exists an* nT*-periodic pulse solution* $u_n \in H^k_{per}(0, nT)$ *of* (6.4) *given by*

$$u_n(x) = \chi_n(x)z(x) + v(x) + a_n(x), \qquad x \in \left[-\tfrac{n}{2}T, \tfrac{n}{2}T\right), \tag{6.23}$$

with $a_n \in H^k_{per}(0, nT)$ *satisfying*

$$\|a_n\|_{H^k_{per}(0,nT)} \leq C\|z\|_{H^k(\mathbb{R}\setminus(-\frac{n}{6}T,\frac{n}{6}T))},$$

and where $\chi_n\colon \mathbb{R} \to [0,1]$ *is a smooth* nT*-periodic cut-off function satisfying* $\|\chi_n\|_{W^{k,\infty}} \leq 1$, $\chi_n(x) = 1$ *for* $x \in [-\frac{n}{6}T, \frac{n}{6}T]$ *and* $\chi_n(x) = 0$ *for* $x \in [-\frac{n}{2}T, -\frac{n}{3}T] \cup [\frac{n}{3}T, \frac{n}{2}T]$. *In particular,* $\|a_n\|_{H^k_{per}(0,nT)}$ *converges to* 0 *as* $n \to \infty$.

We prove Theorem 6.8 using a similar strategy as for Theorem 6.5. Specifically, we arrive at an equation for the error a_n by substituting the ansatz $u_n = w_n + a_n$ into (6.4), where w_n is the nT-periodic extension of the primary pulse $\chi_n z + v$ on $[-\frac{n}{2}nT, \frac{n}{2}nT)$. We convert this equation into a fixed-point problem in $H^k_{\text{per}}(0, nT)$ by inverting its linear component, given by the linearization $\mathcal{L}_{\text{per}}(w_n)$ of (6.3) about the formal periodic extension w_n, with $\mathcal{L}_{\text{per}}(\cdot)$ defined in §6.2. The proof is then completed by applying a contraction-mapping argument.

The invertibility of $\mathcal{L}_{\text{per}}(w_n)$ is ensured by the following lemma, which serves as the periodic counterpart of Lemma 6.7 and also plays an essential role in the subsequent spectral analysis of the periodic pulse solution. Its proof proceeds along the lines of the proof of Lemma 6.7, now relying on a periodic extension result for exponential dichotomies [100].

Lemma 6.9. *Let* $\mathcal{K} \subset \mathbb{C}$ *be a compact. Let* $v \in H^k_{per}(0, T)$ *and* $z \in H^k(\mathbb{R})$. *Moreover, let* $\{a_n\}_n$ *be a sequence with* $a_n \in H^k_{per}(0, nT)$ *satisfying* $\|a_n\|_{H^k_{per}(0,nT)} \to 0$ *as* $n \to \infty$. *Finally, let* $z_n \in H^k_{per}(0, nT)$ *be the* nT*-periodic extension of the function* $\chi_n z$ *on* $[-\frac{n}{2}T, \frac{n}{2}T)$, *where* χ_n *is the cut-off function from Theorem 6.8.*

Suppose that the linear operator $\mathcal{L}(z + v) - \lambda$ *is invertible for each* $\lambda \in \mathcal{K}$. *Then, there exist* $C > 0$ *and* $N \in \mathbb{N}$ *such that for each* $\lambda \in \mathcal{K}$ *and each* $n \in \mathbb{N}$ *with* $n \geq N$ *the operators* $\mathcal{L}(z_n + v + a_n) - \lambda$ *and* $\mathcal{L}_{per}(z_n + v + a_n) - \lambda$ *are invertible with*

$$\left\|(\mathcal{L}(z_n + v + a_n) - \lambda)^{-1}\right\|_{L^2 \to H^k} \leq C, \tag{6.24}$$

and

$$\left\|(\mathcal{L}_{per}(z_n+v+a_n)-\lambda)^{-1}\right\|_{L^2_{per}(0,nT)\to H^k_{per}(0,nT)} \leq C. \tag{6.25}$$

Proof. As in the proof of Lemma 6.7, we obtain that for each $\lambda \in \mathcal{K}$ the first-order system

$$U' = \mathcal{A}\,(x, z(x)+v(x);\lambda)\,U \tag{6.26}$$

admits an exponential dichotomy on $\mathbb{R}$ with λ-independent constants and projections $P(x;\lambda)$. So, we find that the nT-translated system

$$U' = \mathcal{A}\,(x, z(x-nT)+v(x);\lambda)\,U \tag{6.27}$$

has for each $n \in \mathbb{N}$ and $\lambda \in \mathcal{K}$ also an exponential dichotomy on $\mathbb{R}$ with λ- and n-independent constants.

We apply roughness techniques to carry over the exponential dichotomies of (6.26) and (6.27) to the system

$$U' = \mathcal{A}\,(x, z_n(x)+v(x)+a_n(x);\lambda)\,U. \tag{6.28}$$

Since $\partial_u \mathcal{A}$ is continuous, $\mathcal{K}$ is compact, it holds $z \in H^1(\mathbb{R})$ and $v, a_n \in H^1_{\text{per}}(0,nT)$, and $H^1(\mathbb{R})$ and $H^1_{\text{per}}(0,nT)$ embed continuously into $L^\infty(\mathbb{R})$ with n-independent constant, we obtain by the mean value theorem a λ- and n-independent constant $K_0 > 0$ such that

$$\begin{aligned} &\|\mathcal{A}\,(x, z_n(x)+v(x)+a_n(x);\lambda) - \mathcal{A}\,(x, z(x)+v(x);\lambda)\| \\ &\quad \leq K_0 \begin{cases} (\|(1-\chi_n(x))z(x)\| + \|a_n(x)\|)\,, & x \in \left[-\frac{n}{2}T, \frac{n}{2}T\right], \\ (\|z(x)\| + \|z_n(x)\| + \|a_n(x)\|)\,, & x \in \left[\frac{n}{2}T, \frac{5n}{6}T\right], \end{cases} \end{aligned} \tag{6.29}$$

for $x \in [-\frac{n}{2}T, \frac{5n}{6}T]$ and $\lambda \in \mathcal{K}$, and

$$\begin{aligned} &\|\mathcal{A}\,(x, z_n(x)+v(x)+a_n(x);\lambda) - \mathcal{A}\,(x, z(x-nT)+v(x);\lambda)\| \\ &\quad \leq K_0\,(\|(1-\chi_n(x-nT))z(x-nT)\| + \|a_n(x)\|) \end{aligned} \tag{6.30}$$

for $x \in [\frac{n}{2}T, \frac{3n}{2}T]$ and $\lambda \in \mathcal{K}$. Since $H^1(I)$ embeds continuously into $L^\infty(I)$ for each interval $I \subset \mathbb{R}$ and $\|z\|_{H^1(\mathbb{R}\setminus[-R,R])}$ converges to 0 as $R \to \infty$, we find that the right-hand sides of the estimates (6.29) and (6.30) converge to 0 uniformly on $[-\frac{n}{2}T, \frac{5n}{6}T]$ and on $[\frac{n}{2}T, \frac{3n}{2}T]$, respectively, as $n \to \infty$. Combing the latter with the fact that (6.26) admits an exponential dichotomy on $\mathbb{R}$ with λ- and n-independent constants, we infer thanks to [33, Proposition 5.1] that, provided $n \in \mathbb{N}$ is sufficiently large, (6.28) has an exponential dichotomy on $[-\frac{n}{2}T, \frac{5n}{6}T]$ with λ- and n-independent constants $K_1, \mu_1 > 0$ and projections $P_{1,n}(x;\lambda)$ for each $\lambda \in \mathcal{K}$. Similarly, provided $n \in \mathbb{N}$ is sufficiently large, the exponential dichotomy of (6.27) on $\mathbb{R}$ yields an exponential dichotomy of (6.28) on $[\frac{n}{2}T, \frac{3n}{2}T]$ with λ- and n-independent constants $K_2, \mu_2 > 0$ and projections $P_{2,n}(x;\lambda)$ for each $\lambda \in \mathcal{K}$.

We glue the exponential dichotomies of (6.28) on $[-\frac{n}{2}T, \frac{5n}{6}T]$ and on $[\frac{n}{2}T, \frac{3n}{2}T]$ together at the point $x = \frac{2n}{3}T$. First, Lemma 6.54 yields, provided $n \in \mathbb{N}$ is sufficiently large, the bound

$$\left\|P_{1,n}\left(\tfrac{2n}{3}T;\lambda\right) - P_{2,n}\left(\tfrac{2n}{3}T;\lambda\right)\right\| \leq 2K_1K_2\mathrm{e}^{-(\mu_1+\mu_2)\frac{n}{6}T} < 1,$$

for each $\lambda \in \mathcal{K}$. So, the subspaces $\ker(P_{1,n}(\frac{2n}{3}T;\lambda))$ and $\operatorname{ran}(P_{2,n}(\frac{2n}{3}T;\lambda))$ are complementary by Lemma 6.48. We infer that the projection $P_{\circ,n}(\lambda)$ onto $\operatorname{ran}(P_{2,n}(\frac{2n}{3}T;\lambda))$ along $\ker(P_{1,n}(\frac{2n}{3}T;\lambda))$ is well-defined and satisfies

$$\|P_{\circ,n}(\lambda)\| \leq \frac{K_2}{1 - 2K_1K_2\mathrm{e}^{-(\mu_1+\mu_2)\frac{n}{6}T}},$$

for each $\lambda \in \mathcal{K}$. Therefore, provided $n \in \mathbb{N}$ is sufficiently large, Lemma 6.53 establishes an exponential dichotomy for system (6.28) on the interval $[-\frac{n}{2}T, \frac{3n}{2}T]$ of length $2nT$ with λ- and n-independent constants for each $\lambda \in \mathcal{K}$. Moreover, using the continuity of $\mathcal{A}$ and the compactness of $\mathcal{K}$, it follows that $\|\mathcal{A}(\cdot, z_n(\cdot) + v(\cdot) + a_n(\cdot);\lambda)\|_{L^\infty}$ can be bounded by a λ- and n-independent constant for each $\lambda \in \mathcal{K}$. Combining the last two sentences with the fact that system (6.28) has nT-periodic coefficients, [100, Theorem 1] yields, provided $n \in \mathbb{N}$ is sufficiently large, an exponential dichotomy of system (6.28) on $\mathbb{R}$ with λ- and n-independent constants $K_0, \mu_0 > 0$ for each $\lambda \in \mathcal{K}$.

In the following, we denote by $\mathcal{H}^l$ either the space $H^l(\mathbb{R})$ or the space $H^l_{\text{per}}(0, nT)$ and $l \in \mathbb{N}_0$. Since $\|\mathcal{A}(\cdot, z_n(\cdot) + v(\cdot) + a_n(\cdot);\lambda)\|_{L^\infty}$ can be bounded by a λ- and n-independent constant and the nT-periodic system (6.28) has an exponential dichotomy on $\mathbb{R}$ with λ- and n-independent constants, Lemma 6.56 yields, provided $n \in \mathbb{N}$ is sufficiently large, a λ- and n-independent constant $C > 0$ such that for each $g \in \mathcal{H}^1 \hookrightarrow C(\mathbb{R})$ and $\lambda \in \mathcal{K}$ the inhomogeneous problem

$$U' = \mathcal{A}(x, z_n(x) + v(x) + a_n(x);\lambda)\,U + \psi$$

with $\psi = (0, \ldots, 0, g)^\top \in \mathcal{H}^1$ has a solution $U \in \mathcal{H}^1$ satisfying

$$\|U\|_{\mathcal{H}^1} \leq C\|\psi\|_{\mathcal{H}^0} = C\|g\|_{\mathcal{H}^0}.$$

Using that we have $U_j' = U_{j+1} \in \mathcal{H}^1$ for $j = 1, \ldots, k-1$, we readily observe that $u = U_1 \in \mathcal{H}^k$ solves the resolvent problem

$$(\mathcal{L}(z_n + v + a_n) - \lambda)\, u = g, \tag{6.31}$$

in case $\mathcal{H}^l = H^l(\mathbb{R})$, and

$$(\mathcal{L}_{\text{per}}(z_n + v + a_n) - \lambda)\, u = g, \tag{6.32}$$

in case $\mathcal{H}^l = H^l_{\text{per}}(0, nT)$. Moreover, it obeys the estimate

$$\|u\|_{\mathcal{H}^k} \leq \|U\|_{\mathcal{H}^1} \leq C\|g\|_{\mathcal{H}^0}. \tag{6.33}$$

Since $\mathcal{L}(z_n + v + a_n)$ and $\mathcal{L}_{\text{per}}(z_n + v + a_n)$ are closed operators, it follows by the density of $\mathcal{H}^1$ in $\mathcal{H}^0$ that the resolvent problem (6.31), in case $\mathcal{H}^l = H^l(\mathbb{R})$, and the resolvent problem (6.32), in case $\mathcal{H}^l = H^l_{\text{per}}(0, nT)$, possesses for each $g \in \mathcal{H}^0$ and $\lambda \in \mathcal{K}$ a solution $u \in \mathcal{H}^k$ satisfying (6.33).

On the other hand, an element $u \in \ker(\mathcal{L}(z_n+v+a_n)-\lambda)$ or $u \in \ker(\mathcal{L}_{\text{per}}(z_n+v+a_n)-\lambda)$ yields a bounded solution $U = (u, \partial_x u, \ldots, \partial_x^{k-1} u)$ of system (6.28), which must be 0, because (6.28) has an exponential dichotomy on $\mathbb{R}$ for each $\lambda \in \mathcal{K}$.

We conclude that, provided $n \in \mathbb{N}$ is sufficiently large, $\mathcal{L}(z_n+v+a_n)-\lambda$ and $\mathcal{L}_{\text{per}}(z_n+v+a_n)-\lambda$ are bounded invertible and obey (6.24) and (6.25), respectively, for each $\lambda \in \mathcal{K}$. □

With the aid of Lemma 6.9, we are now able to establish the main result of this section using a contraction-mapping argument.

Proof of Theorem 6.8. Let $z_n \in H^k_{\text{per}}(0, nT)$ be the nT-periodic extension of the function $\chi_n z$ on $[-\frac{n}{2}T, \frac{n}{2}T)$. By Lemma 6.9 the linear operator $\mathcal{L}_\circ := \mathcal{L}_{\text{per}}(z_n + v)$ is invertible and there exists an n-independent constant $K > 0$ such that

$$\|\mathcal{L}_\circ^{-1}\|_{L^2_{\text{per}}(0,nT) \to H^k_{\text{per}}(0,nT)} \leq K. \tag{6.34}$$

We substitute the ansatz $u = \chi_n z + v + a$ with correction term $a \in H^k_{\text{per}}(0, nT)$ into (6.4) and arrive at the equation

$$a = \widetilde{\mathcal{N}}(a) + R, \tag{6.35}$$

with nonlinearity $\widetilde{\mathcal{N}} \colon H^k_{\mathrm{per}}(0,nT) \to H^k_{\mathrm{per}}(0,nT)$ given by

$$\widetilde{\mathcal{N}}(a) = \mathcal{L}_\circ^{-1}\left(\mathcal{N}(z_n+v,\cdot) + \partial_u\mathcal{N}(z_n+v,\cdot)a - \mathcal{N}(z_n+v+a,\cdot)\right)$$

and residual $R \in H^k_{\mathrm{per}}(0,nT)$ given by

$$R = -\mathcal{L}_\circ^{-1}\left(\mathcal{N}(z_n+v,\cdot) + A(z_n+v)\right).$$

Since $\mathcal{N}$ is twice continuously differentiable in its first argument and $H^1_{\mathrm{per}}(0,nT)$ embeds continuously into $L^\infty(\mathbb{R})$ with n-independent constant, Taylor's theorem and estimate (6.34) imply that $\widetilde{\mathcal{N}}$ is well-defined and for all $\rho > 0$ there exists an n-independent constant $C_1 > 0$ such that

$$\left\|\widetilde{\mathcal{N}}(a_0) - \widetilde{\mathcal{N}}(a_1)\right\|_{H^k_{\mathrm{per}}(0,nT)} \leq C_1\left(\|a_0\|_{H^k_{\mathrm{per}}(0,nT)} + \|a_1\|_{H^k_{\mathrm{per}}(0,nT)}\right)\|a_0 - a_1\|_{H^k_{\mathrm{per}}(0,nT)},$$

for $a_0, a_1 \in H^k_{\mathrm{per}}(0,nT)$ with $\|a_0\|_{L^\infty}, \|a_1\|_{L^\infty} \leq \rho$.

We proceed with bounding the residual. First, as $z+v$ and v are solutions of (6.4), we have

$$\mathcal{N}(z_n(x)+v(x),x) + Az_n(x) + Av(x) = 0,$$

for $x \in [-\frac{n}{6}T, \frac{n}{6}T]$ and

$$\mathcal{N}(z_n(x)+v(x),x) + Az_n(x) + Av(x) = \mathcal{N}(z_n(x)+v(x),x) - \mathcal{N}(v(x),x) + Az_n(x),$$

for $x \in [-\frac{n}{2}T, \frac{n}{2}T]$. Hence, since $H^1_{\mathrm{per}}(0,ntT)$ embeds continuously into $L^\infty(\mathbb{R})$ with n-independent constant, $\mathcal{N}$ is continuously differentiable and it holds $\|\chi_n\|_{W^{k,\infty}} \leq 1$, there exists by the mean value theorem and estimate (6.34) an n-independent constant $C_0 > 0$ such that

$$\|R\|_{H^k_{\mathrm{per}}(0,nT)} \leq C_0\delta_n, \tag{6.36}$$

where we denote

$$\delta_n := \|z\|_{H^k(\mathbb{R}\setminus(-\frac{n}{6}T,\frac{n}{6}T))}.$$

We observe that δ_n converges to 0 as $n \to \infty$.

We introduce the rescaled variable

$$\mathsf{a} = \delta_n^{-1}a, \tag{6.37}$$

in which (6.35) reads

$$\mathsf{a} = \delta_n^{-1}\widetilde{\mathcal{N}}(\delta_n \mathsf{a}) + \delta_n^{-1}R. \tag{6.38}$$

Regard (6.38) as an abstract fixed point problem

$$\mathsf{a} = \mathcal{F}_n(\mathsf{a}). \tag{6.39}$$

Analogous to the proof of Theorem 6.5, one establishes, provided $n \in \mathbb{N}$ is sufficiently large, that $\mathcal{F}_n \colon B_0(2C_0) \to B_0(2C_0)$ is a well-defined contracting mapping on the ball $B_0(2C_0)$ of radius $2C_0$ in $H^k_{\text{per}}(0,nT)$ centered at the origin, where $C_0 > 0$ is the n-independent constant appearing in the bound (6.36) on the residual R. Then, an application of the Banach fixed point theorem yields a unique solution $\mathsf{a}_n \in B_0(2C_0)$ to (6.39). Undoing the rescaling (6.37), we obtain a solution $a_n \in H^k_{\text{per}}(0,nT)$ of equation (6.35) with $\|a_n\|_{H^k_{\text{per}}(0,nT)} \leq 2C_0\delta_n$, provided $n \in \mathbb{N}$ is sufficiently large, which yields the result. □

6.5 Spectral analysis of front solutions with periodic tails

In this section, we collect some background material on the spectral stability of front solutions connecting periodic end states. Specifically, we impose the following assumption:

(H6) There exists a front $Z \in L^\infty(\mathbb{R})$ with associated end states $v_\pm \in H^k_{\text{per}}(0,T)$. We have $\chi_\pm(Z - v_\pm) \in H^k(\mathbb{R})$, where $\chi_\pm \colon \mathbb{R} \to [0,1]$ is a smooth partition of unity such that χ_+ is supported on $(-1,\infty)$ and χ_- is supported on $(-\infty,1)$.

We adopt the following distinction between essential and point spectrum, cf. [78, 128].

Definition 6.10. Let $L \colon D(L) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a linear operator with domain $D(L) = H^k(\mathbb{R})$. The *essential spectrum* of L is defined as the set of all $\lambda \in \mathbb{C}$ for which the operator $L - \lambda$ is not Fredholm of index zero. The *point spectrum* is defined as the complement $\sigma(L) \setminus \sigma_{\text{ess}}(L)$.

Assume **(H6)**. Let $\eta_\pm \in \mathbb{R}$. The Fredholm properties of the exponentially weighted linearization operator $\mathcal{L}_{\eta_-,\eta_+}(Z)$, see §6.2, are determined by the periodic end states $v_\pm$. By leveraging Lemma 6.59 and Weyl's theorem, cf. [80, Theorem VI.5.26], we infer that, for each $\lambda \in \mathbb{C}$, the operator $\mathcal{L}_{\eta_-,\eta_+}(\chi_- v_- + \chi_+ v_+) - \lambda$ is Fredholm of index $j \in \mathbb{Z}$ if and only if $\mathcal{L}_{\eta_-,\eta_+}(Z) - \lambda$ is Fredholm of index j. Consequently, the essential spectra of $\mathcal{L}_{\eta_-,\eta_+}(Z)$ and $\mathcal{L}_{\eta_-,\eta_+}(\chi_- v_- + \chi_+ v_+)$ coincide.

The results of Palmer, [99, Lemma 4.2], [100, Theorem 1] and [101, Theorem 1], imply that $\mathcal{L}_{\eta_-,\eta_+}(\chi_- v_- + \chi_+ v_+) - \lambda$ is Fredholm if and only if both of the associated first-order eigenvalue problems

$$U' = (\mathcal{A}(x, v_\pm(x); \lambda) - \eta_\pm)\, U \tag{6.40}$$

admit an exponential dichotomy on $\mathbb{R}$. Its Fredholm index is then given by

$$\mathrm{ind}(\mathcal{L}_{\eta_-,\eta_+}(Z) - \lambda) = \mathrm{ind}(\mathcal{L}_{\eta_-,\eta_+}(\chi_- v_- + \chi_+ v_+) - \lambda) = l_-(\lambda) - l_+(\lambda),$$

where the *Morse index* $l_\pm(\lambda)$ is the rank of the projection associated with the exponential dichotomy of (6.40) on $\mathbb{R}$. We note that (6.40) possesses an exponential dichotomy on $\mathbb{R}$ if and only if the operator $\mathcal{L}_{\eta_\pm}(v_\pm) - \lambda$ is invertible, cf. Lemmas 6.56 and 6.57. Furthermore, Floquet's theorem, [78, Theorem 2.1.27], yields that the T-periodic system (6.40) has an exponential dichotomy on $\mathbb{R}$ if and only if it possesses no Floquet exponents $\nu \in \mathbb{C}$ on the imaginary axis [2]. The Morse index is then equal to the number of Floquet exponents $\nu \in \mathbb{C}$ with negative real part (counted with algebraic multiplicity). We summarize these observations in the following proposition.

Proposition 6.11. *Assume* **(H6)**. *Let* $\eta_\pm \in \mathbb{R}$. *Then, the following assertions hold true.*

1. *A point* $\lambda \in \mathbb{C}$ *lies in the spectrum of* $\mathcal{L}_{\eta_\pm}(v_\pm)$ *if and only if the* T*-periodic system* (6.40) *possesses purely imaginary Floquet exponents.*
2. *We have*

$$\begin{aligned}\sigma_{\mathrm{ess}}\left(\mathcal{L}_{\eta_-,\eta_+}(Z)\right) &= \sigma_{\mathrm{ess}}\left(\mathcal{L}_{\eta_-,\eta_+}(\chi_- v_- + \chi_+ v_+)\right) \\ &= \{\lambda \in \mathbb{C} : l_-(\lambda) \neq l_+(\lambda)\} \cup \sigma(\mathcal{L}_{\eta_-}(v_-)) \cup \sigma(\mathcal{L}_{\eta_+}(v_+)),\end{aligned}$$

where $l_\pm(\lambda)$ *is the number of Floquet exponenents* $\nu \in \mathbb{C}$ *(counted with algebraic multiplicity) of* (6.40) *with negative real part.*

In the following proposition, we introduce the *Evans function*, a well-known tool to locate point spectrum in the stability analysis of nonlinear waves, see [4,45,78,107,128] and references therein. The Evans function is an analytic determinantal function measuring the alignment or mismatch between the subspace of solutions decaying as $x \to \infty$ and the subspace of solutions decaying as $x \to -\infty$. Consequently, its zeros correspond to eigenvalues, including their multiplicities.

[2] The Floquet exponents are the principal logarithms of the eigenvalues of the monodromy matrix $\mathcal{T}_\pm(T, 0; \lambda)$, where $\mathcal{T}_\pm(x, y; \lambda)$ is the evolution of system (6.40). We refer to [24, 78] for further background on Floquet theory.

Proposition 6.12. *Assume* **(H6)**. *Let $\eta_\pm \in \mathbb{R}$. Let Ω be a connected component of*

$$\mathbb{C} \setminus \big(\sigma_{\mathrm{ess}}(\mathcal{L}_{\eta_-}(v_-)) \cup \sigma_{\mathrm{ess}}(\mathcal{L}_{\eta_+}(v_+))\big).$$

Then, the following assertions hold true.

1. *The number $l_{0,\pm} \in \{0, \ldots, km\}$ of Floquet exponents $\nu \in \mathbb{C}$ (counted with algebraic multiplicity) of* (6.40) *with negative real part is constant for each $\lambda \in \Omega$.*
2. *System* (6.40) *has for each $\lambda \in \Omega$ an exponential dichotomy on $\mathbb{R}$ with projections $Q_\pm(x;\lambda)$ of rank $l_{0,\pm}$. Here, $Q_\pm(\cdot;\lambda)\colon \mathbb{R} \to \mathbb{C}^{km\times km}$ is T-periodic for each $\lambda \in \Omega$ and $Q_\pm(x;\cdot)\colon \Omega \to \mathbb{C}^{km\times km}$ is analytic for each $x \in \mathbb{R}$.*
3. *System*

$$U' = \Big(\mathcal{A}\,(x, Z(x);\lambda) - \omega'_{\eta_-,\eta_+}(x)\Big)\,U \tag{6.41}$$

possesses for each $\lambda \in \Omega$ exponential dichotomies on $\mathbb{R}_\pm$ with projections $P_\pm(\pm x;\lambda), x \geq 0$ of fixed rank $l_{0,\pm}$, where ω_{η_-,η_+} is the weight function defined in §6.2. Moreover, there exist analytic functions $B_{\mathrm{s}}\colon \Omega \to \mathbb{C}^{km\times l_{0,+}}$ and $B_{\mathrm{u}}\colon \Omega \to \mathbb{C}^{km\times(km-l_{0,-})}$ such that $B_{\mathrm{s}}(\lambda)$ is a basis of $\mathrm{ran}(P_+(0;\lambda))$ and $B_{\mathrm{u}}(\lambda)$ is a basis of $\ker(P_-(0;\lambda))$ for each $\lambda \in \Omega$.

4. *Assume further $l_{0,-} = l_{0,+}$. Then, there is no essential spectrum of $\mathcal{L}_{\eta_-,\eta_+}(Z)$ in Ω. Moreover, a point $\lambda_0 \in \Omega$ lies in the point spectrum of $\mathcal{L}_{\eta_-,\eta_+}(Z)$ if and only if λ_0 is a root of the analytic Evans function $\mathcal{E}\colon \Omega \to \mathbb{C}$ given by*

$$\mathcal{E}(\lambda) = \det(B_{\mathrm{u}}(\lambda) \mid B_{\mathrm{s}}(\lambda)).$$

The geometric multiplicity $m_g(\lambda_0)$ of an eigenvalue $\lambda_0 \in \Omega$ of the operator $\mathcal{L}_{\eta_-,\eta_+}(Z)$ is equal to $\dim(\ker(P_-(0;\lambda_0)) \cap \mathrm{ran}(P_+(0;\lambda_0)))$. Moreover, if $\mathcal{E}$ does not vanish identically on Ω, then the algebraic multiplicity $m_a(\lambda_0)$ of an eigenvalue $\lambda_0 \in \Omega$ of $\mathcal{L}_{\eta_-,\eta_+}(Z)$ is equal to the multiplicity of λ_0 as a root of $\mathcal{E}$. In particular, the roots of $\mathcal{E}$ and their multiplicities are independent of the choice of bases.

5. *Let $\mathcal{K} \subset \Omega$ be compact. Then, there exist constants $K_0, \mu_0, \tau_0 > 0$ such that system* (6.41) *admits for each $\lambda \in \mathcal{K}$ exponential dichotomies on $\mathbb{R}_\pm$ with constants $K_0, \mu_0 > 0$ and projections $\tilde{P}_\pm(\pm x;\lambda), x \geq 0$ satisfying*

$$\big\|\tilde{P}_\pm(\pm x;\lambda) - Q_\pm(\pm x;\lambda)\big\| \leq K_0 \left(\mathrm{e}^{-\mu_0 x} + \sup_{y\in[x,\infty)} \|Z(\pm y) - v_\pm(\pm y)\|\right), \tag{6.42}$$

for each $x \geq \tau_0$.

Proof. The first assertion is an immediate consequence of Proposition 6.11 and the fact that the Floquet exponents of (6.40) depend continuously on λ.

It follows from Floquet's theorem, cf. [78, Theorem 2.1.27], Proposition 6.11 and Lemma 6.55 that the T-periodic system (6.40) possesses for each $\lambda \in \Omega$ an exponential dichotomy on $\mathbb{R}$ with projections $Q_\pm(x;\lambda)$, which have rank $l_{0,\pm}$ and are T-periodic in their first component. Thanks to the uniqueness of exponential dichotomies on $\mathbb{R}$, cf. [33, p. 19], and the fact that (6.40) depends analytically on λ, it follows from [34, Lemma A.2] that $Q_\pm(x;\cdot)$ is analytic on Ω for each $x \in \mathbb{R}$. This proves the second assertion.

Next, we observe that [99, Lemma 3.4], **(H6)** and the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ yield for each $\lambda \in \Omega$ exponential dichotomies for system (6.41) on $\mathbb{R}_\pm$ with projections $P_\pm(\pm x;\lambda), x \geq 0$ of rank $l_{0,\pm}$. Using that system (6.41) depends analytically on λ and the subspaces $\ker(P_-(-x;\lambda))$ and $\operatorname{ran}(P_+(x;\lambda))$ are by [33, p. 19] uniquely determined for $x \geq 0$ and $\lambda \in \Omega$, [34, Lemma A.2] and [80, Section II.4.2] provide analytic functions $B_\mathrm{s} \colon \Omega \to \mathbb{C}^{km \times l_0}$ and $B_\mathrm{u} \colon \Omega \to \mathbb{C}^{km \times (km - l_0)}$ such that $B_\mathrm{s}(\lambda)$ is a basis of $\operatorname{ran}(P_+(0;\lambda))$ and $B_\mathrm{u}(\lambda)$ is a basis of $\ker(P_-(0;\lambda))$ for each $\lambda \in \Omega$. Thus, we have established the third assertion.

Assume $l_{0,+} = l_{0,-}$. Then, there is no essential spectrum of $\mathcal{L}_{\eta_-,\eta_+}(Z)$ in Ω by Proposition 6.11. Set $l_0 = l_{0,\pm}$ and take $\lambda_0 \in \Omega$. As $\mathcal{L}_{\eta_-,\eta_+}(Z) - \lambda_0$ is Fredholm of index 0, it is invertible if and only if λ_0 is not an eigenvalue of $\mathcal{L}_{\eta_-,\eta_+}(Z)$. Since (6.41) is the first-order formulation of the eigenvalue problem $\mathcal{L}_{\eta_-,\eta_+}(Z)u = \lambda u$, there is a one-to-one correspondence between elements $u_0 \in \ker(\mathcal{L}_{\eta_-,\eta_+}(Z) - \lambda_0)$ and H^1-solutions $U_0 = (u_0, \partial_x u_0, \ldots, \partial_x^{k-1} u_0) \in H^1(\mathbb{R})$ of (6.41) at $\lambda = \lambda_0$. The exponential dichotomies of (6.41) on $\mathbb{R}_\pm$ yield that U_0 is an H^1-solution of (6.41) at $\lambda = \lambda_0$ if and only if $U_0(0) \in \ker(P_-(0;\lambda_0)) \cap \operatorname{ran}(P_+(0;\lambda_0))$. Therefore, $\lambda_0 \in \Omega$ is an eigenvalue of $\mathcal{L}_{\eta_-,\eta_+}(Z)$ if and only if $\mathcal{E}(\lambda_0) = 0$. In addition, the geometric multiplicity $m_g(\lambda_0)$ of λ_0 equals $\dim(\ker(P_-(0;\lambda_0)) \cap \operatorname{ran}(P_+(0;\lambda_0)))$. Finally, [77, Theorem 2.9] asserts that, if $\mathcal{E}$ is not identically 0, then the algebraic multiplicity $m_a(\lambda_0)$ of λ_0 is equal to the multiplicity of λ_0 as a root of $\mathcal{E}$. This completes the proof of the fourth assertion.

Finally, we recall that system (6.40) possesses an exponential dichotomy on $\mathbb{R}$ for each λ in the compact set $\mathcal{K}$ with projections $Q_\pm(x;\lambda)$. Thus, as in the proof of Lemma 6.7, we find that (6.40) has for each $\lambda \in \mathcal{K}$ an exponential dichotomy on $\mathbb{R}$ with λ-independent constants and, by uniqueness, projections $Q_\pm(x;\lambda)$. Thus, the fifth assertion follows from [99, Lemma 3.4] and its proof. □

6.6 Spectral analysis of multifront solutions

This section is devoted to the spectral stability analysis of stationary multifront solutions to (6.1). We consider a multifront u_n of the form (6.8), where w_n is a formal concatenation of M primary fronts $Z_1, \ldots, Z_M$ with matching periodic end states, and a_n is an error term converging to 0 in $H^k(\mathbb{R})$ as $n \to \infty$. Our goal is to transfer spectral properties of the linearizations $\mathcal{L}(Z_1), \ldots, \mathcal{L}(Z_M)$ of (6.1) about the primary front solutions to the linearization $\mathcal{L}(u_n)$ about the multifront.

A first key observation, provided by Lemma 6.7, is that, if $\mathcal{L}(Z_1) - \lambda, \ldots, \mathcal{L}(Z_M) - \lambda$ are invertible for each λ in a compact set $\mathcal{K} \subset \mathbb{C}$, then so is $\mathcal{L}(u_n) - \lambda$ for $n \in \mathbb{N}$ sufficiently large. In other words, if $\mathcal{K}$ is a compact subset of the resolvent set $\rho(\mathcal{L}(Z_j))$ for each $j = 1, \ldots, M$, then $\mathcal{K}$ is also contained in $\rho(\mathcal{L}(u_n))$, provided $n \in \mathbb{N}$ is sufficiently large.

In dissipative systems, such as the reaction-diffusion models discussed in §6.8, the spectral stability analysis can often be reduced to an n-independent compact set with the aid of a-priori bounds that preclude spectrum with nonnegative real part and large modulus. As a result, Lemma 6.7 yields the following corollary, asserting that strong spectral stability of the M primary fronts $Z_1, \ldots, Z_M$ is inherited by the M-front u_n.

Corollary 6.13. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Suppose that the front solutions Z_j are strongly spectrally stable for $j = 1, \ldots, M$. Moreover, assume that there exist a compact set $\mathcal{K} \subset \mathbb{C}$ and $N \in \mathbb{N}$ such that*

$$\sigma(\mathcal{L}(u_n)) \cap \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\} \subset \mathcal{K}$$

for $n \geq N$, where u_n is the multifront solution established in Theorem 6.5. Then, there exists $N_1 \in \mathbb{N}$ with $N_1 \geq N$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ the multifront u_n is strongly spectrally stable.

In the remainder of this section, we study the spectra associated with stationary multifront solutions to (6.1) in more detail. The obtained results are particularly useful in scenarios where one of the primary fronts is either spectrally unstable or only marginally stable. We assume that **(H1)** and **(H2)** hold and consider a multifront u_n of the form (6.8), where a_n is an error term converging to 0 in $H^k(\mathbb{R})$ as $n \to \infty$.

By Proposition 6.11, the essential spectrum of $\mathcal{L}(u_n)$ is determined solely by its periodic end states $v_{1,-}$ and $v_{M,+}$, making it independent of n. Specifically, it is given by

$$\begin{aligned}\sigma_{\text{ess}}(\mathcal{L}(u_n)) &= \sigma_{\text{ess}}\left(\mathcal{L}(\chi_- v_{1,-} + \chi_+ v_{M,+})\right) \\ &= \{\lambda \in \mathbb{C} : l_-(\lambda) \neq l_+(\lambda)\} \cup \sigma(\mathcal{L}(v_{1,-})) \cup \sigma(\mathcal{L}(v_{M,+})),\end{aligned}$$

where $l_-(\lambda)$ and $l_+(\lambda)$ are the Morse indices, corresponding to the number of Floquet exponents $\nu \in \mathbb{C}$ of negative real part (counted with algebraic multiplicity), of the asymptotic systems

$$U' = \mathcal{A}(x, v_{1,-}(x); \lambda)U, \qquad U' = \mathcal{A}(x, v_{M,+}(x); \lambda)U, \tag{6.43}$$

respectively.

The main result of this section concerns the approximation of the point spectrum of $\mathcal{L}(u_n)$. For each connected component Ω of $\mathbb{C} \setminus \sigma_{\text{ess}}(\mathcal{L}(u_n))$, we establish such an approximation in the subset

$$\rho_{\text{abs},\Omega} := \left\{ \lambda \in \Omega : \begin{array}{c} \text{for each } j \in \{1, \ldots, M-1\} \\ \text{there exists } \eta_j \in \mathbb{R} \text{ such that } \ell_{\eta_j,j}(\lambda) = l_+(\lambda) \end{array} \right\},$$

where $\ell_{\eta,j}(\lambda) \in \mathbb{N}_0$ denotes the number of Floquet exponents $\nu \in \mathbb{C}$ with negative real part (counted with algebraic multiplicity) of the T-periodic system

$$U' = \left(\mathcal{A}(x, v_{j,+}(x); \lambda) - \eta\right) U.$$

In accordance with [130], we refer to the complement $\sigma_{\text{abs},\Omega} := \Omega \setminus \rho_{\text{abs},\Omega}$ as the *absolute spectrum of* $\mathcal{L}(u_n)$ *in* Ω, see also Remark 6.19. We observe that a point $\lambda \in \Omega$ lies in the absolute spectrum $\sigma_{\text{abs},\Omega}$ if and only if there is a $j \in \{1, \ldots, M-1\}$ such that there is no $\eta \in \mathbb{R}$ separating the Floquet exponents of the asymptotic system

$$U' = \mathcal{A}(x, v_{j,+}(x); \lambda)U, \tag{6.44}$$

into $l_+(\lambda)$ exponents in the half plane $\{\nu \in \mathbb{C} : \text{Re}(\nu) < \eta\}$ and $km - l_+(\lambda)$ exponents in its complement (counting algebraic multiplicities). So, ordering the Floquet exponents $\nu_{1,j}(\lambda), \ldots, \nu_{km,j}(\lambda)$ of the system (6.44) by their real parts,

$$\text{Re}(\nu_{1,j}(\lambda)) \leq \ldots \leq \text{Re}(\nu_{km,j}(\lambda)),$$

a point $\lambda \in \Omega$ lies in the absolute spectrum $\sigma_{\text{abs},\Omega}$ if and only if we have $\text{Re}(\nu_{l_+(\lambda),j}(\lambda)) = \text{Re}(\nu_{l_+(\lambda)+1,j}(\lambda))$ for some $j \in \{1, \ldots, M-1\}$. Since the Floquet exponents $\nu_{i,j}(\lambda)$ depend continuously on λ, and Ω is open, we infer that $\rho_{\mathrm{abs},\Omega}$ is also open. We note that the imaginary axis enforces the desired splitting of Floquet exponents if $\lambda \in \Omega$ lies outside the essential spectra of the linearizations about the primary fronts, see Proposition 6.11. So, we have the inclusion

$$\sigma_{\mathrm{abs},\Omega} \subset \bigcup_{j=1}^{M} \sigma_{\mathrm{ess}}(\mathcal{L}(Z_j)).$$

Let Ω be a connected component of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{L}(u_n))$ and let $\lambda_0 \in \rho_{\mathrm{abs},\Omega}$. Set $\eta_0 = 0$ and $\eta_M = 0$. Then, by continuous dependence of the Floquet exponents on λ, there exist an open neighborhood $\mathcal{U} \subset \rho_{\mathrm{abs},\Omega}$ of λ_0 and $\eta_1, \ldots, \eta_{M-1} \in \mathbb{R}$ such that the T-periodic asymptotic system

$$U' = (\mathcal{A}(x, v_{j,+}(x); \lambda) - \eta_j)\, U \tag{6.45}$$

has $l_+(\lambda_0)$ Floquet exponents $\nu \in \mathbb{C}$ in the open left-half plane and $km - l_+(\lambda_0)$ Floquet exponents in the open right-half plane for all $\lambda \in \mathcal{U}$ and $j = 1, \ldots, M$ (counting algebraic multiplicities), cf. Figure 6.2(top). The main result of this section establishes that the point spectrum in $\mathcal{U}$ of the linearization $\mathcal{L}(u_n)$ about the multifront converges, as $n \to \infty$, to the union of the point spectra in $\mathcal{U}$ of the (exponentially weighted) linearizations $\mathcal{L}_{\eta_0,\eta_1}(Z_1), \ldots, \mathcal{L}_{\eta_{M-1},\eta_M}(Z_M)$ about the primary fronts, thereby preserving the total algebraic multiplicity of eigenvalues.

Theorem 6.14. *Let $M \in \mathbb{N}_{\geq 2}$. Assume* **(H1)** *and* **(H2)**. *Suppose that there exists $N \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$ with $n \geq N$, there exists an M-front u_n of the form* (6.8)*, where $\{a_n\}_n$ is a sequence in $H^k(\mathbb{R})$ converging to* 0.

Let Ω be a connected component of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}\left(\mathcal{L}(\chi_- v_{1,-} + \chi_+ v_{M,+})\right)$. Let $\lambda_0 \in \rho_{\mathrm{abs},\Omega}$. Take an open neighborhood $\mathcal{U} \subset \rho_{\mathrm{abs},\Omega}$ of λ_0 and real numbers $\eta_j \in \mathbb{R}$ such that for each $j = 1, \ldots, M-1$ and all $\lambda \in \mathcal{U}$ systems (6.43) *and* (6.45) *have the same number of Floquet exponents in both the open left-half plane and the open right-half plane (counting algebraic multiplicities). Set $\eta_0 = 0$ and $\eta_M = 0$. Let $\mathcal{E}_j \colon \mathcal{U} \to \mathbb{C}$ be the Evans function of the first-order system*

$$U' = \left(\mathcal{A}\left(x, Z_j(x); \lambda\right) - \omega'_{\eta_{j-1},\eta_j}(x)\right) U \tag{6.46}$$

for $j = 1, \ldots, M$, as constructed in Proposition 6.12.

Suppose λ_0 is a root of $\mathcal{E} := \mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_M$ of multiplicity $m_0 \in \mathbb{N}_0$. Then, there exists $\varrho_0 > 0$ such that for each $\varrho \in (0, \varrho_0)$ there exists $N_\varrho \in \mathbb{N}$ with $N_\varrho \geq N$ such that for all $n \in \mathbb{N}$ with $n \geq N_\varrho$ the following assertions hold true.

1. *An Evans function* $E_n \colon B_{\lambda_0}(\varrho) \to \mathbb{C}$ *associated with* (6.12) *has precisely* m_0 *roots in* $B_{\lambda_0}(\varrho)$ *(counting multiplicities).*

2. *The operator* $\mathcal{L}(u_n)$ *has point spectrum in* $B_{\lambda_0}(\varrho)$ *if and only if* $m_0 \neq 0$*. The total algebraic multiplicity of the eigenvalues of* $\mathcal{L}(u_n)$ *in* $B_{\lambda_0}(\varrho)$ *is* m_0*.*

Remark 6.15. The integer $m_0 \in \mathbb{N}_0$ in Theorem 6.14 equals the number of eigenvalues (counting algebraic multiplicities) of $\mathcal{L}(u_n)$ converging to λ_0 as $n \to \infty$. In particular, m_0 is independent of the choice of neighborhood $\mathcal{U} \subset \rho_{\mathrm{abs},\Omega}$ of λ_0 and the choice of reals $\eta_j \in \mathbb{R}$ in Theorem 6.14.

Remark 6.16. If λ_0 lies in the complement $\mathbb{C} \setminus \bigcup_{j=1}^M \sigma_{\mathrm{ess}}(\mathcal{L}(Z_j))$, then we may take $\eta_1, \ldots, \eta_{M-1} = 0$ in Theorem 6.14. The result then asserts that there exists an n-independent neighborhood $\mathcal{U}$ of λ_0 such that the point spectrum in $\mathcal{U}$ of $\mathcal{L}(u_n)$ converges, as $n \to \infty$, to the union of the point spectra of the linearizations $\mathcal{L}(Z_1), \ldots, \mathcal{L}(Z_M)$ about the primary fronts, while preserving the total algebraic multiplicity of eigenvalues. This observation proves the first assertion in Theorem 6.3.

Theorem 6.14 can be applied to show that, if one of the (weighted) linearizations about the primary fronts possesses unstable point spectrum, then so does the linearization about the multifront. However, it also serves for the purpose of counting eigenvalues, which is for instance useful for spectral (in)stability arguments in Hamiltonian systems based on Krein index theory [75, 76, 78]. We illustrate this in §6.8 by proving instability results for stationary multipulse solutions to the Gross-Pitaevskii equation with periodic potential. Notably, the control over algebraic multiplicities provided by Theorem 6.14 is essential for applying Krein index counting theory effectively. Finally, Theorem 6.14 can be used to establish strong spectral stability of the multifront in cases where Corollary 6.13 does not apply. More precisely, if both the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L}(u_n))$ and the absolute spectrum $\sigma_{\mathrm{abs},\Omega}$ in the right-most connected component of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{L}(u_n))$ are confined to the open left-half plane, then we can employ Theorem 6.14 to preclude spectrum of $\mathcal{L}(u_n)$ in n-independent compact subsets of the closed right-half plane. This leads to the following extension of Corollary 6.13.

Corollary 6.17. *Let* $M \in \mathbb{N}_{\geq 2}$*. Assume* **(H1)** *and* **(H2)***. Assume further that the following conditions hold:*

1. *The essential spectrum* $\sigma_{\mathrm{ess}}(\mathcal{L}(\chi_- v_{1,-} + \chi_+ v_{M,+}))$ *and the absolute spectrum* $\sigma_{\mathrm{abs},\Omega}$ *in the right-most connected component* Ω *of* $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{L}(\chi_- v_{1,-} + \chi_+ v_{M,+}))$ *are confined to the open-left half plane.*

2. *The Evans function* $\mathcal{E} \colon \Omega \to \mathbb{C}$ *in Theorem 6.14 has no zeros in the closed right-half plane.*

3. *There exist $N \in \mathbb{N}$ and a compact set $\mathcal{K} \subset \mathbb{C}$ such that, for each $n \in \mathbb{N}$ with $n \geq N$, there exists a stationary M-front solution u_n to* (6.1) *of the form* (6.8)*, where $\{a_n\}_n$ is a sequence in $H^k(\mathbb{R})$ converging to* 0*, such that*

$$\sigma(\mathcal{L}(u_n)) \cap \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\} \subset \mathcal{K}$$

for $n \geq N$.

Then, there exists $N_1 \in \mathbb{N}$ with $N_1 \geq N$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ the multifront u_n is strongly spectrally stable.

Remark 6.18. The conditions in Corollary 6.17 can be satisfied even if the linearization $\mathcal{L}(Z_j)$ about one of the primary fronts Z_j has unstable essential spectrum, see Figure 6.2(top). Thus, spectrally unstable primary fronts may produce strongly spectrally stable multifronts.

The observation that multifronts composed of unstable primary fronts can still be stable is not new: the phenomenon is well-studied in constant-coefficient systems [120, 130]. An advantage of the spatially periodic setting considered here is that it breaks the translational symmetry. As a result, front solutions can be *strongly* spectrally stable, eliminating the need to track eigenvalues of $\mathcal{L}(u_n)$ that converge to 0 as $n \to \infty$, cf. [126].

Remark 6.19. In systems with constant coefficients, it was shown in [130] that eigenvalues of the linearization about a multifront accumulate onto each point of the absolute spectrum as the distances between interfaces tend to infinity, see also [96, 129]. We anticipate that, using the techniques developed in [129, 130], a similar result can be established for the spatially periodic setting considered here.

Remark 6.20. Theorem 6.14 does not require that the primary fronts constituting the multifront are nondegenerate. Therefore, the theorem applies even when the linearization about a primary front is not invertible due to additional symmetries, such as translational or rotational symmetries. In §6.8, we will apply Theorem 6.14 to prove spectral instability of multipulses in a spatially periodic Gross-Pitaevskii equation, which exhibits a rotational symmetry.

The proof of Theorem 6.14 hinges on a delicate factorization procedure of the Evans function E_n. The procedure is inductive and employs the well-known identity

$$\det\left(CA^{-1}B + D\right) = \frac{(-1)^\ell}{\det(A)} \det\begin{pmatrix} -A & B \\ C & D \end{pmatrix} = \det(A^{-1}) \det\begin{pmatrix} B & -A \\ D & C \end{pmatrix} \tag{6.47}$$

in each induction step, where $A, B, C, D \in \mathbb{C}^{\ell \times \ell}$ are block matrices with A invertible and ℓ a natural number.

By Proposition 6.12, system (6.46) possesses exponential dichotomies on both half-lines for $j = 1, \ldots, M$. By applying roughness and pasting techniques, these transfer to exponential dichotomies of the first-order system

$$U' = (\mathcal{A}(x, u_n(x); \lambda) - \omega'(x))\, U, \tag{6.48}$$

associated with the eigenvalue problem $(\mathcal{L}(u_n) - \lambda)u = 0$ about the multifront u_n, on the intervals $(-\infty, nT]$, $[MnT, \infty)$, and $[jnT, (j+1)nT]$ for $j = 1, \ldots, M-1$. Here, $\omega\colon \mathbb{R} \to \mathbb{R}$ is a suitably chosen concatenation of the weight functions $\omega_{\eta_{j-1}, \eta_j}$ for $j = 1, \ldots, M$, see Figure 6.2(bottom).

Given an exponential dichotomy of (6.48) on an interval $\mathcal{I}$, the key idea is to use Lemma 6.51 to write the associated projection $P(x; \lambda)$ as

$$P(x; \lambda) = \mathcal{B}(x; \lambda) \left(\Theta(x; \lambda)^\top \mathcal{B}(x; \lambda)\right)^{-1} \Theta(x; \lambda)^\top$$

where $\mathcal{B}(x; \lambda)$ and $\Theta(x; \lambda)$ are matrices whose columns constitute a basis of $\mathrm{ran}(P(x; \lambda))$ and $\mathrm{ran}(P(x; \lambda)^\top)$, respectively. We demonstrate that

$$\Pi(x, y; \lambda) := \left(\Theta(x; \lambda)^\top \mathcal{B}(x; \lambda)\right)^{-1} \Theta(x; \lambda)^\top \mathcal{T}(x, y; \lambda) \mathcal{B}(y; \lambda) \left(\Theta(y; \lambda)^\top \mathcal{B}(y; \lambda)\right)^{-1}$$

is invertible for each $x, y \in \mathcal{I}$, where $\mathcal{T}(x, y; \lambda)$ is the evolution of system (6.48). Thus, given matrices $\mathcal{X}, \mathcal{Y}$, we can write

$$\begin{aligned} \mathcal{X}\mathcal{T}(x, y; \lambda)\mathcal{Y} &= \mathcal{X}P(x; \lambda)\mathcal{T}(x, y; \lambda)P(y; \lambda)\mathcal{Y} + \mathcal{X}\mathcal{T}(x, y; \lambda)(I - P(y; \lambda))\mathcal{Y} \\ &= \mathcal{X}\mathcal{B}(x; \lambda)\Pi(x, y; \lambda)\Theta(y; \lambda)^\top \mathcal{Y} + \mathcal{X}\mathcal{T}(x, y; \lambda)(I - P(y; \lambda))\mathcal{Y} \end{aligned} \tag{6.49}$$

for $x, y \in \mathcal{I}$. We then apply the formula (6.47) to the determinant of expressions of the form (6.49) with $A = \Pi(x, y; \lambda)^{-1}$, $B = \Theta(y; \lambda)^\top \mathcal{Y}$, $C = \mathcal{X}\mathcal{B}(x; \lambda)$ and $D = \mathcal{X}\mathcal{T}(x, y; \lambda)(I - P(y; \lambda))\mathcal{Y}$.

Specifically, we show that the Evans function E_n can be expressed, up to an invertible analytic factor, as $\det(\mathcal{X}_0(\lambda)\mathcal{T}(nT, MnT; \lambda)\mathcal{Y}_0(\lambda))$, where $\mathcal{X}_0(\lambda)$ and $\mathcal{Y}_0(\lambda)$ are appropriately chosen matrices. By applying (6.47) inductively, we obtain that E_n is, up to a nonzero analytic factor, equal to the determinant of a matrix that becomes an upper triangular block matrix as $n \to \infty$. The determinants of the diagonal blocks can be identified with the Evans functions $\mathcal{E}_1, \ldots, \mathcal{E}_M$. An application of Rouché's theorem then yields the desired approximation of the zeros of E_n by those of the product $\mathcal{E} = \mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_M$.

Proof of Theorem 6.14. This proof is structured as follows. We begin with establishing exponential dichotomies for the weighted eigenvalue problems (6.46) along the primary fronts. Then, we define a suitable weight function ω and transfer these exponential dichotomies to the unweighted and weighted eigenvalue problems (6.12) and (6.48) along the multifront. Subsequently, we define an Evans function E_n associated with (6.12). Next, we inductively establish a leading-order factorization of E_n, relating it to the product $\mathcal{E} = \mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_M$. Finally, the result follows by an application of Rouché's theorem.

Exponential dichotomies along the primary fronts. We list some consequences of Propositions 6.11 and 6.12. First of all, system (6.46) has for all $\lambda \in \mathcal{U}$ and each $j = 1, \ldots, M$ exponential dichotomies on $\mathbb{R}_\pm$ with projections $P_{j,\pm}(\pm x; \lambda), x \geq 0$ of rank l_0, where l_0 is the λ- and j-independent number of Floquet exponents in the open left-half plane (counted with algebraic multiplicity) of the systems (6.43) and (6.45). Moreover, there exist analytic functions $B_{j,\mathrm{s}} \colon \mathcal{U} \to \mathbb{C}^{km \times l_0}$ and $B_{j,\mathrm{u}} \colon \mathcal{U} \to \mathbb{C}^{km \times (km - l_0)}$ such that $B_{j,\mathrm{s}}(\lambda)$ is a basis of $\operatorname{ran}(P_{j,+}(0; \lambda))$ and $B_{j,\mathrm{u}}(\lambda)$ is a basis of $\ker(P_{j,-}(0; \lambda))$ for all $\lambda \in \mathcal{U}$ and $j = 1, \ldots, M$. The associated Evans function $\mathcal{E}_j \colon \mathcal{U} \to \mathbb{C}$ is given by

$$\mathcal{E}_j(\lambda) = \det(B_{j,\mathrm{u}}(\lambda) \mid B_{j,\mathrm{s}}(\lambda))$$

for $j = 1, \ldots, M$. Because $\mathcal{E} = \mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_M$ is analytic, there exists a closed disk $\overline{B}_{\lambda_0}(\varrho_2) \subset \mathcal{U}$ of some radius $\varrho_2 > 0$ such that λ_0 is the only root of $\mathcal{E}$ in $\overline{B}_{\lambda_0}(\varrho_2)$. Finally, there exist constants $K_0, \mu_0, \tau_0 > 0$ such that (6.46) possesses for $\lambda \in \overline{B}_{\lambda_0}(\varrho_2)$ and $j = 1, \ldots, M$ exponential dichotomies on $\mathbb{R}_\pm$ with constants $K_0, \mu_0 > 0$ and projections $\tilde{P}_{j,\pm}(\pm x; \lambda), x \geq 0$ satisfying (6.42) for each $x \geq \tau_0$. By uniqueness of the range of $P_{j,+}(0; \lambda)$ and the kernel of $P_{j,-}(0; \lambda)$, cf. [33, p. 19], $B_{j,\mathrm{s}}(\lambda)$ is a basis of $\operatorname{ran}(P_{j,+}(0; \lambda)) = \operatorname{ran}(\tilde{P}_{j,+}(0; \lambda))$ and $B_{j,\mathrm{u}}(\lambda)$ is a basis of $\ker(P_{j,-}(0; \lambda)) = \ker(\tilde{P}_{j,-}(0; \lambda))$ for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_2)$ and $j = 1, \ldots, M$.

Since $\mathcal{A}$ is T-periodic in x, system

$$U' = \Big(\mathcal{A}(x, Z_j(x - jnT); \lambda) - \omega'_{\eta_{j-1}, \eta_j}(x - jnT)\Big) U \tag{6.50}$$

is, for each $n \in \mathbb{N}$, a jnT-translation of system (6.46) for $j = 1, \ldots, M$. So, it admits for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_2)$ and $j = 1, \ldots, M$ exponential dichotomies on the half lines $(-\infty, jnT]$ and $[jnT, \infty)$ with constants K_0, μ_0 and projections $\check{P}_{j,\pm}(jnT \pm x; \lambda) = \tilde{P}_{j,\pm}(\pm x; \lambda), x \geq 0$.

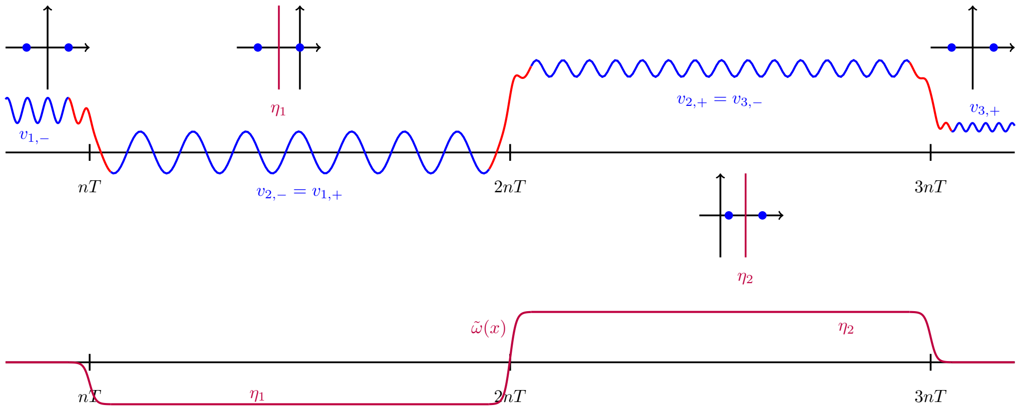

Figure 6.2: Illustration of a stationary 3-front solution (top), with insets showing the Floquet exponents (blue dots) associated with the periodic end states $v_{j,\pm}$ for $j = 1, 2, 3$, depicted in blue. The derivative of the corresponding weight ω, denoted by $\tilde{\omega}$, is shown in purple (bottom).

Weighted eigenvalue problem. Let $\tilde{\omega}\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\tilde{\omega}(x) = \begin{cases} \omega'_{\eta_0,\eta_1}(x - nT), & x \in \left(-\infty, \frac{3}{2}nT\right], \\ \omega'_{\eta_{j-1},\eta_j}(x - jnT), & x \in \left((j - \frac{1}{2})nT, (j + \frac{1}{2})nT\right], \; j = 2, \ldots, M-1, \\ \omega'_{\eta_{M-1},\eta_M}(x - MnT), & x \in \left((M - \frac{1}{2})nT, \infty\right), \end{cases}$$

see Figure 6.2(bottom). By definition of the exponential weights $\omega_{\eta_{j-1},\eta_j}, j = 1, \ldots, M$, see §6.2, the function $\tilde{\omega}$ is smooth and has support within the interval $(nT - 1, MnT + 1)$. Therefore, its primitive $\omega\colon \mathbb{R} \to \mathbb{R}$, given by

$$\omega(x) = \int_0^x \tilde{\omega}(y)\mathrm{d}y,$$

is smooth and bounded.

Denote by $T_n(x, y; \lambda)$ and $\mathcal{T}_n(x, y; \lambda)$ the evolutions of system (6.12) and (6.48), respectively. Since ω is bounded and it holds

$$T_n(x, y; \lambda) = \mathrm{e}^{\omega(x) - \omega(y)} \mathcal{T}_n(x, y; \lambda)$$

for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, system (6.12) possesses an exponential dichotomy on an interval $\mathcal{I} \subset \mathbb{R}$ with projections $P_n(x; \lambda)$ if and only if system (6.48) does.

Exponential dichotomies along the multifront. Our next step is to use roughness and pasting techniques to transfer the exponential dichotomies of system (6.50) to exponential dichotomies for system (6.48) (and thus for system (6.12)) on the intervals $(-\infty, \frac{3}{2}nT]$, $[(M - \frac{1}{2})nT, \infty)$ and $[(j-\frac{1}{2})nT, (j+\frac{1}{2})nT]$ for $j = 2, \ldots, M-1$ (only in case $M > 2$).

Let $\tilde{\chi}_{j,n} \colon \mathbb{R} \to [0,1]$ be a family of smooth cut-off functions, where $\tilde{\chi}_{1,n}$ is supported on $(-\infty, \frac{3}{2}nT + 2)$ and equal to 1 on $(-\infty, \frac{3}{2}nT+1]$, $\tilde{\chi}_{M,n}$ is supported on $((M-\frac{1}{2})nT - 2, \infty)$ and equal to 1 on $[(M-\frac{1}{2})nT - 1, \infty)$, and $\tilde{\chi}_{j,n}$ is supported on $((j-\frac{1}{2})nT-2, (j+\frac{1}{2})nT+2)$ and equal to 1 on $[(j-\frac{1}{2})nT-1, (j+\frac{1}{2})nT+1]$ for $j = 2, \ldots, M-1$ (only in case $M > 2$). We define

$$Z_{j,n} = a_n + (1 - \tilde{\chi}_{j,n}) Z_j(\cdot - jnT) + \tilde{\chi}_{j,n} w_n$$

for $j = 1, \ldots, M$, where we recall w_n is the formal concatenation of the M primary fronts, defined in (6.8). Using that $\partial_u \mathcal{A}$ is continuous, $\overline{B}_{\lambda_0}(\varrho_2)$ is compact and we have $a_n \in H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and $Z_\ell \in L^\infty(\mathbb{R})$ for $\ell = 1, \ldots, M$, the mean value theorem yields a λ- and n-independent constant $R > 0$ such that the estimate

$$\|\mathcal{A}(x, Z_{j,n}(x); \lambda) - \mathcal{A}(x, Z_j(x - jnT); \lambda)\| \leq R\delta_n \tag{6.51}$$

holds for $x \in \mathbb{R}$, $\lambda \in \overline{B}_{\lambda_0}(\varrho_2)$ and $j = 1, \ldots, M$, where we denote

$$\begin{aligned}\delta_n = \|a_n\|_{L^\infty} + \sum_{j=1}^{M} \Big(&\|\chi_-(Z_j - v_{j,-})\|_{L^\infty\left((-\infty, -\frac{1}{2}nT+2]\right)} \\ &+ \|\chi_+(Z_j - v_{j,+})\|_{L^\infty\left([\frac{1}{2}nT-2, \infty)\right)}\Big).\end{aligned}$$

Using the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, one readily observes that δ_n converges to 0 as $n \to \infty$. Thus, since for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_2)$ system (6.50) has exponential dichotomies on $(-\infty, jnT]$ and $[jnT, \infty)$ with λ- and n-independent constants K_0, μ_0 and projections $\check{P}_{j,\pm}(jnT \pm x; \lambda) = \tilde{P}_{j,\pm}(\pm x; \lambda), x \geq 0$, the estimate (6.51) and Lemma 6.58 give rise to λ- and n-independent constants $M_1, \mu_1 > 0$ and $\varrho_1 \in (0, \varrho_2)$ such that, provided $n \in \mathbb{N}$ is sufficiently large, the following two statements hold for $j = 1, \ldots, M$. First, the system

$$U' = \Big(\mathcal{A}(x, Z_{j,n}(x); \lambda) - \omega'_{\eta_{j-1},\eta_j}(x)\Big) U \tag{6.52}$$

has exponential dichotomies on $(-\infty, jnT]$ and $[jnT, \infty)$ with λ- and n-independent constants and projections $\mathcal{Q}_{j,\pm,n}(jnT \pm x; \lambda)$, $x \geq 0$ for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$. Second, the map $\mathcal{Q}_{j,\pm,n}(jnT \pm x; \cdot) \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times km}$ is analytic for each $x \geq 0$ and the estimates

$$\begin{aligned} \|\mathcal{Q}_{j,-,n}((j-\tfrac{1}{2})nT; \lambda) - \tilde{P}_{j,-}(-\tfrac{n}{2}T; \lambda)\| &\leq M_1 \left(\delta_n + \mathrm{e}^{-\mu_1 nT}\right), \\ \|\mathcal{Q}_{j,+,n}((j+\tfrac{1}{2})nT; \lambda) - \tilde{P}_{j,+}(\tfrac{n}{2}T; \lambda)\| &\leq M_1 \left(\delta_n + \mathrm{e}^{-\mu_1 nT}\right), \\ \|\mathcal{Q}_{j,\pm,n}(jnT; \lambda) - Q_{j,\pm}(\lambda)\| &\leq M_1 \delta_n \end{aligned} \tag{6.53}$$

hold for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$, where we denote by $Q_{j,+}(\lambda)$ the projection onto $\operatorname{ran}(P_{j,+}(0; \lambda))$ along $\operatorname{ran}(P_{j,+}(0; \lambda_0))^\perp$ and $Q_{j,-}(\lambda)$ is the projection onto $\ker(P_{j,-}(0; \lambda_0))^\perp$ along $\ker(P_{j,-}(0; \lambda))$. By Lemma 6.50, the maps $Q_{j,\pm} \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times km}$ are analytic for $j = 1, \ldots, M$.

Take $j \in \{1, \ldots, M\}$. Owing to [80, Section II.4.2] there exist analytic maps $\Phi_j \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times l_0}$ and $\Psi_j \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times (km - l_0)}$ such that $\Phi_j(\lambda)$ is a basis of $\operatorname{ran}(Q_{j,-}(\lambda)^\top)$ and $\Psi_j(\lambda)$ is a basis of $\operatorname{ran}(I - Q_{j,-}(\lambda)^\top) = \ker(Q_{j,-}(\lambda)^\top)$ for $\lambda \in B_{\lambda_0}(\varrho_1)$. Since $B_{j,\mathrm{u}}(\lambda)$ is a basis of $\ker(Q_{j,-}(\lambda))$, we have $\Phi_j(\lambda)^\top B_{j,\mathrm{u}}(\lambda) = 0$ for $\lambda \in B_{\lambda_0}(\varrho_1)$. Therefore, we arrive at

$$\det\left(\left(\Psi_j(\lambda) \mid \Phi_j(\lambda)\right)^\top\right) \mathcal{E}_j(\lambda) = \det\left(\Psi_j(\lambda)^\top B_{j,\mathrm{u}}(\lambda)\right) \det\left(\Phi_j(\lambda)^\top B_{j,\mathrm{s}}(\lambda)\right)$$

for $\lambda \in B_{\lambda_0}(\varrho_1)$. Clearly, the matrix $(\Psi_j(\lambda) \mid \Phi_j(\lambda))$ is invertible for all $\lambda \in B_{\lambda_0}(\varrho_1)$, and so is $\Psi_j(\lambda)^\top B_{j,\mathrm{u}}(\lambda)$ by Lemma 6.51. We conclude that $\mathcal{E}_j$ has the same zeros (including multiplicities) in $B_{\lambda_0}(\varrho_1)$ as the analytic function $\tilde{\mathcal{E}}_j \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}$ given by

$$\tilde{\mathcal{E}}_j(\lambda) = \det\left(\Phi_j(\lambda)^\top B_{j,\mathrm{s}}(\lambda)\right).$$

Take $j \in \{1, \ldots, M-1\}$. We invoke Lemma 6.48, use the T-periodicity of $Q_\pm(\cdot; \lambda)$ and employ estimates (6.42) and (6.53), to conclude that, provided $n \in \mathbb{N}$ is sufficiently large, the subspaces $\ker(\mathcal{Q}_{j,+,n}((j+\tfrac{1}{2})nT; \lambda))$ and $\operatorname{ran}(\mathcal{Q}_{j+1,-,n}((j+\tfrac{1}{2})nT; \lambda))$ are complementary and there exists a λ- and n-independent constant $M_2 > 0$ such that the projection $\check{\mathcal{P}}_{j,n}(\lambda)$ onto $\operatorname{ran}(\mathcal{Q}_{j+1,-,n}((j+\tfrac{1}{2})nT; \lambda))$ along $\ker(\mathcal{Q}_{j,+,n}((j+\tfrac{1}{2})nT; \lambda))$ is well-defined and enjoys the bound

$$\left\|\check{\mathcal{P}}_{j,n}(\lambda)\right\| \leq M_2 \tag{6.54}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$. Since $\mathcal{Q}_{j,+,n}((j+\tfrac{1}{2})nT; \cdot), \mathcal{Q}_{j+1,-,n}((j+\tfrac{1}{2})nT; \cdot) \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times km}$ are analytic, Lemma 6.50 affords analytic maps $\mathcal{B}_{j,\pm,n} \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times l_0}$ with $\operatorname{ran}(\mathcal{Q}_{j+1,-,n}((j+\tfrac{1}{2})nT; \lambda)) = \operatorname{ran}(\mathcal{B}_{j,-,n}(\lambda))$ and $\ker(\mathcal{Q}_{j,+,n}((j+\tfrac{1}{2})nT; \lambda)) = \{u \in \mathbb{C}^{km} : z^\top u =$

0 for all $z \in \mathrm{ran}(\mathcal{B}_{j,+,n}(\lambda))\}$. Therefore, Lemma 6.50 implies that $\check{\mathcal{P}}_{j,n} \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km\times km}$ is analytic.

By construction of the cut-off functions $\tilde{\chi}_{j,n}$ and the weight ω, system (6.48) coincides with (6.52) on $(-\infty, \frac{3}{2}nT]$ for $j = 1$, on $[(M - \frac{1}{2})nT, \infty)$ for $j = M$ and on $[(j - \frac{1}{2})nT, (j + \frac{1}{2})nT]$ for $j = 2, \ldots, M - 1$ (only in case $M > 2$). Hence, it follows, thanks to Lemma 6.53 and estimate (6.54), that system (6.48) admits for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$ an exponential dichotomy on $(-\infty, nT]$ with λ- and n-independent constants and projections $\mathcal{Q}_{1,-,n}(x;\lambda)$, an exponential dichotomy on $[MnT, \infty)$ with λ- and n-independent constants and projections $\mathcal{Q}_{M,+,n}(x;\lambda)$, and an exponential dichotomy on $[jnT, (j + 1)nT]$ with λ- and n-independent constants and projections

$$\mathcal{Q}_{j,\circ,n}(x;\lambda) = \mathcal{T}_n(x, (j + \tfrac{1}{2})nT; \lambda)\check{\mathcal{P}}_{j,n}(\lambda)\mathcal{T}_n((j + \tfrac{1}{2})nT, x; \lambda),$$

for $j = 1, \ldots, M - 1$. In addition, there exist λ- and n-independent constants $M_3, \mu_3 > 0$ such that, provided $n \in \mathbb{N}$ is sufficiently large, the projections obey

$$\begin{aligned} \|\mathcal{Q}_{j,\circ,n}(jnT;\lambda) - \mathcal{Q}_{j,+,n}(jnT;\lambda)\| &\leq M_3\mathrm{e}^{-\mu_3 nT}, \\ \|\mathcal{Q}_{j,\circ,n}((j+1)nT;\lambda) - \mathcal{Q}_{j+1,-,n}((j+1)nT;\lambda)\| &\leq M_3\mathrm{e}^{-\mu_3 nT} \end{aligned}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$ and $j = 1, \ldots, M - 1$. Combining the latter with (6.53), we arrive at

$$\|\mathcal{Q}_{j,\circ,n}(jnT;\lambda) - Q_{j,+}(\lambda)\|, \|\mathcal{Q}_{j,\circ,n}((j+1)nT;\lambda) - Q_{j+1,-}(\lambda)\| \leq M_3\mathrm{e}^{-\mu_3 nT} + M_1\delta_n \tag{6.55}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$ and $j = 1, \ldots, M - 1$. Finally, we observe that $\mathcal{Q}_{j,\circ,n}(x;\cdot) \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km\times km}$ is analytic, since $\check{\mathcal{P}}_{j,n}$ and $\mathcal{T}_n(x, y; \cdot)$ are analytic for $x, y \in [jnT, (j + 1)nT]$ and $j = 1, \ldots, M - 1$, cf. [78, Lemma 2.1.4].

We define the analytic maps $\Xi_{0,n}, \Xi_{j,n}, \mathcal{B}_{j,n}, \mathcal{B}_{M,n} \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km\times l_0}$ by

$$\Xi_{0,n}(\lambda) = \mathcal{Q}_{1,-,n}(nT;\lambda)^\top\Phi_1(\lambda), \qquad \Xi_{j,n}(\lambda) = \mathcal{Q}_{j,\circ,n}((j+1)nT;\lambda)^\top\Phi_{j+1}(\lambda)$$

and

$$\mathcal{B}_{j,n}(\lambda) = \mathcal{Q}_{j,\circ,n}(jnT;\lambda)B_{j,\mathrm{s}}(\lambda), \qquad \mathcal{B}_{M,n}(\lambda) = \mathcal{Q}_{M,+,n}(MnT;\lambda)B_{M,\mathrm{s}}(\lambda)$$

for $j = 1, \dots, M-1$. Let $\varrho_0 \in (0, \varrho_1)$. Employing estimates (6.53) and (6.55), using the compactness of $\overline{B}_{\lambda_0}(\varrho_0)$ and recalling that Φ_j and $B_{j,\mathrm{s}}$ are analytic on $B_{\lambda_0}(\varrho_1)$, we obtain a λ- and n-independent constant $M_4 > 0$ such that, provided $n \in \mathbb{N}$ is sufficiently large, we have

$$\begin{aligned} \|\Xi_{j-1,n}(\lambda) - \Phi_j(\lambda)\|, \|\mathcal{B}_{j,n}(\lambda) - B_{j,\mathrm{s}}(\lambda)\| &\leq M_4 \left(\mathrm{e}^{-\mu_4 nT} + \delta_n\right), \\ \|\Xi_{j-1,n}(\lambda)\|, \|\mathcal{B}_{j,n}(\lambda)\| &\leq M_4, \end{aligned} \tag{6.56}$$

for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $j = 1, \dots, M$. Because $\Phi_j(\lambda)$ and $B_{j,\mathrm{s}}(\lambda)$ are bases, the estimate (6.56) implies that, provided $n \in \mathbb{N}$ is sufficiently large, $\Xi_{0,n}(\lambda)$ forms a basis of $\mathrm{ran}(\mathcal{Q}_{1,-,n}(nT;\lambda)^\top)$, $\Xi_{j,n}(\lambda)$ is a basis of $\mathrm{ran}(\mathcal{Q}_{j,\circ,n}((j+1)nT;\lambda)^\top)$, $\mathcal{B}_{j,n}(\lambda)$ is a basis of $\mathrm{ran}(\mathcal{Q}_{j,\circ,n}(jnT;\lambda))$, and $\mathcal{B}_{M,n}(\lambda)$ is a basis of $\mathrm{ran}(\mathcal{Q}_{M,+,n}(MnT;\lambda))$ for $j = 1, \dots, M-1$ and $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$.

By [80, Section II.4.2] there exist analytic maps $\tilde{\Xi}_{j,n}, \tilde{\mathcal{B}}_{j,n} \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times l_0}$ such that $\tilde{\Xi}_{j,n}(\lambda)$ forms a basis of $\mathrm{ran}(\mathcal{Q}_{j,\circ,n}(jnT;\lambda)^\top)$ and $\tilde{\mathcal{B}}_{j,n}(\lambda)$ is a basis of $\mathrm{ran}(\mathcal{Q}_{j,\circ,n}((j+1)nT;\lambda))$ for $\lambda \in B_{\lambda_0}(\varrho_1)$ and $j = 1, \dots, M-1$. By Lemma 6.51, the matrices $\tilde{\Xi}_{j,n}(\lambda)^\top \mathcal{B}_{j,n}(\lambda)$ and $\Xi_{j,n}(\lambda)^\top \tilde{\mathcal{B}}_{j,n}(\lambda)$ are invertible for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $j = 1, \dots, M-1$. Hence, one readily verifies that it must hold

$$\begin{aligned} \mathcal{Q}_{j,\circ,n}(jnT;\lambda) &= \mathcal{B}_{j,n}(\lambda) \left(\tilde{\Xi}_{j,n}(\lambda)^\top \mathcal{B}_{j,n}(\lambda)\right)^{-1} \tilde{\Xi}_{j,n}(\lambda)^\top, \\ \mathcal{Q}_{j,\circ,n}((j+1)nT;\lambda) &= \tilde{\mathcal{B}}_{j,n}(\lambda) \left(\Xi_{j,n}(\lambda)^\top \tilde{\mathcal{B}}_{j,n}(\lambda)\right)^{-1} \Xi_{j,n}(\lambda)^\top \end{aligned} \tag{6.57}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $j = 1, \dots, M-1$. Therefore, defining $\Pi_{j,n}, \Theta_{j,n} \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{l_0 \times l_0}$ by

$$\Pi_{j,n}(\lambda) = \left(\tilde{\Xi}_{j,n}(\lambda)^\top \mathcal{B}_{j,n}(\lambda)\right)^{-1} \tilde{\Xi}_{j,n}(\lambda)^\top \mathcal{T}_n(jnT,(j+1)nT;\lambda) \tilde{\mathcal{B}}_{j,n}(\lambda) \left(\Xi_{j,n}(\lambda)^\top \tilde{\mathcal{B}}_{j,n}(\lambda)\right)^{-1}$$

and

$$\Theta_{j,n}(\lambda) = \Xi_{j,n}(\lambda)^\top \mathcal{T}_n((j+1)nT, jnT;\lambda)) \mathcal{B}_{j,n}(\lambda),$$

we deduce

$$\begin{aligned} \Theta_{j,n}(\lambda)\Pi_{j,n}(\lambda) = \Xi_{j,n}(\lambda)^\top \mathcal{Q}_{j,\circ,n}((j+1)nT;\lambda) \mathcal{T}_n((j+1)nT, jnT;\lambda) \\ \cdot \mathcal{T}_n(jnT,(j+1)nT;\lambda) \tilde{\mathcal{B}}_{j,n}(\lambda) \left(\Xi_{j,n}(\lambda)^\top \tilde{\mathcal{B}}_{j,n}(\lambda)\right)^{-1} = I \end{aligned}$$

and, similarly,

$$\Pi_{j,n}(\lambda)\Theta_{j,n}(\lambda) = I$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $j = 1, \ldots, M-1$. We conclude that $\Theta_{j,n}(\lambda)$ is invertible with inverse $\Pi_{j,n}(\lambda)$ for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $j = 1, \ldots, M-1$. Moreover, $\Phi_{j,n}$ and $\Theta_{j,n}$ are analytic for $j = 1, \ldots, M-1$, since $\Xi_{j,n}$, $\mathcal{B}_{j,n}$, $\tilde{\Xi}_{j,n}$, $\tilde{\mathcal{B}}_{j,n}$ and $\mathcal{T}_n(x, y; \cdot)$ are analytic for $x, y \in \mathbb{R}$ by [78, Lemma 2.1.4].

By [33, p. 13], we can extend the exponential dichotomy of system (6.48) on $[MnT, \infty)$ to an exponential dichotomy on $[nT, \infty)$ with projections $\mathcal{Q}_{M,+,n}(x; \lambda), x \geq nT$ by setting

$$\mathcal{Q}_{M,+,n}(x; \lambda) = \mathcal{T}_n(x, MnT; \lambda)\mathcal{Q}_{M,+,n}(MnT; \lambda)\mathcal{T}_n(MnT, x; \lambda)$$

for each $x \in [nT, MnT]$ and $\lambda \in B_{\lambda_0}(\varrho_1)$. As noted before, this implies that the unweighted system (6.12) also admits exponential dichotomies on the half lines $(-\infty, nT]$ and $[nT, \infty)$ with projections $\mathcal{Q}_{1,-,n}(x; \lambda), x \leq nT$ and $\mathcal{Q}_{M,+,n}(x; \lambda), x \geq nT$, respectively.

The Evans function for the multifront. By [80, Section II.4.2], there exist holomorphic functions $\mathcal{U}_n, \Upsilon_n \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times (km - l_0)}$ such that $\mathcal{U}_n(\lambda)$ forms a basis of $\ker(\mathcal{Q}_{1,-,n}(nT; \lambda)) = \operatorname{ran}(I - \mathcal{Q}_{1,-,n}(nT; \lambda))$ and $\Upsilon_n(\lambda)$ is a basis of $\operatorname{ran}(I - \mathcal{Q}_{1,-,n}(nT; \lambda)^\top)$ for each $\lambda \in B_{\lambda_0}(\varrho_1)$. Upon defining $\mathcal{S}_n \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}^{km \times l_0}$ by $\mathcal{S}_n(\lambda) = \mathcal{T}_n(nT, MnT; \lambda)\mathcal{B}_{M,n}(\lambda)$, we find that $\mathcal{S}_n(\lambda)$ is a basis of $\operatorname{ran}(\mathcal{Q}_{M,+,n}(nT; \lambda))$ for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, because $\mathcal{B}_{M,n}(\lambda)$ is a basis of $\operatorname{ran}(\mathcal{Q}_{M,+,n}(MnT; \lambda))$. Moreover, $\mathcal{S}_n$ is analytic, since $\mathcal{B}_{M,n}$ and $\mathcal{T}_n(nT, MnT; \cdot)$ are, cf. [78, Lemma 2.1.4]. Thus, an analytic Evans function $E_n \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}$ for system (6.12) is given by

$$E_n(\lambda) = \det(\mathcal{U}_n(\lambda) \mid \mathcal{S}_n(\lambda)).$$

Leading-order factorization of the Evans function. Our next step is to multiply $E_n(\lambda)$ with several nonzero analytic functions in order to relate it to $\mathcal{E} = \mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_M$, where $\mathcal{E}_j$ is the Evans function associated with system (6.46).

First, noting that $\Xi_{0,n}(\lambda)^\top \mathcal{U}_n(\lambda) = 0$, we compute

$$\det\left((\Upsilon_n(\lambda) \mid \Xi_{0,n}(\lambda))^\top\right) E_n(\lambda) = \det\left(\Upsilon_n(\lambda)^\top \mathcal{U}_n(\lambda)\right) \det\left(\Xi_{0,n}(\lambda)^\top \mathcal{S}_n(\lambda)\right)$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Since the matrices $(\Upsilon_n(\lambda) \mid \Xi_{0,n}(\lambda))$ and $\Upsilon_n(\lambda)^\top \mathcal{U}_n(\lambda)$ are invertible for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ by Lemma 6.51, the Evans function E_n possesses the same roots (including multiplicities) in $B_{\lambda_0}(\varrho_0)$ as the analytic function $\tilde{E}_n \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}$ given by

$$\tilde{E}_n(\lambda) = \det\left(\Xi_{0,n}(\lambda)^\top \mathcal{S}_n(\lambda)\right) = \det\left(\Xi_{0,n}(\lambda)^\top \mathcal{T}_n(nT, MnT; \lambda)\mathcal{B}_{M,n}(\lambda)\right). \tag{6.58}$$

Denote by $I_{\ell\times\ell}$ the identity matrix in $\mathbb{C}^{\ell\times\ell}$ and by $0_{\ell\times s}$ the zero matrix in $\mathbb{C}^{\ell\times s}$ for $\ell, s \in \mathbb{N}$. We claim that, provided $n \in \mathbb{N}$ is sufficiently large, there exists an analytic function $h_{j,n}\colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}$ such that

$$\tilde{E}_n(\lambda) = h_{j,n}(\lambda)\det\left(\begin{pmatrix}\tilde{B}_{j,n}(\lambda) & -\tilde{A}_j \\ \tilde{D}_{j,n}(\lambda) & \tilde{C}_{j,n}(\lambda)\end{pmatrix} + \tilde{H}_{j,n}(\lambda)\right), \tag{6.59}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $j = 1,\ldots,M-1$. Here, $\tilde{B}_{j,n}(\lambda) \in \mathbb{C}^{2^{j-1}l_0\times 2^{j-1}l_0}$ is an upper triangular $(l_0\times l_0)$-block matrix, whose blocks above the diagonal are equal to $-I_{l_0\times l_0}$ or $0_{l_0\times l_0}$ and whose diagonal contains exactly one copy of each of the blocks $\Xi_{M-1,n}(\lambda)^\top \mathcal{B}_{M,n}(\lambda),\ldots,$ $\Xi_{M-j,n}(\lambda)^\top\mathcal{B}_{M-j+1,n}(\lambda)$ and further only $(l_0\times l_0)$-identity matrices. Furthermore, $\tilde{A}_j, \tilde{C}_{j,n}(\lambda)$, $\tilde{D}_{j,n}(\lambda) \in \mathbb{C}^{2^{j-1}l_0\times 2^{j-1}l_0}$ are given by

$$\tilde{A}_j = \begin{pmatrix} I_{(2^{j-1}-1)l_0\times(2^{j-1}-1)l_0} & 0_{(2^{j-1}-1)l_0\times l_0} \\ 0_{l_0\times(2^{j-1}-1)l_0} & 0_{l_0\times l_0}\end{pmatrix},$$

$$\tilde{C}_{j,n}(\lambda) = \begin{pmatrix} I_{(2^{j-1}-1)l_0\times(2^{j-1}-1)l_0} & 0_{(2^{j-1}-1)l_0\times l_0} \\ 0_{l_0\times(2^{j-1}-1)l_0} & \Xi_{0,n}(\lambda)^\top \mathcal{T}_n(nT,(M-j)nT;\lambda)\mathcal{B}_{M-j,n}(\lambda)\end{pmatrix}$$

and

$$\tilde{D}_{j,n}(\lambda) =$$
$$\begin{pmatrix} 0_{(2^{j-1}-1)l_0\times l_0} & \cdots & 0_{(2^{j-1}-1)l_0\times l_0} \\ \Xi_{0,n}(\lambda)^\top\mathcal{T}_n(nT,(M-j)nT;\lambda)H_{j,1,n}(\lambda) & \cdots & \Xi_{0,n}(\lambda)^\top\mathcal{T}_n(nT,(M-j)nT;\lambda)H_{j,2^{j-1},n}(\lambda)\end{pmatrix},$$

for $j = 1,\ldots,M-1$ and $\lambda \in B_{\lambda_0}(\varrho_1)$. Moreover, $H_{j,\ell,n}(\lambda) \in \mathbb{C}^{l_0\times l_0}$ and $\tilde{H}_{j,n}(\lambda) \in \mathbb{C}^{2^j l_0\times 2^j l_0}$ obey the bound

$$\|H_{j,\ell,n}(\lambda)\|, \left\|\tilde{H}_{j,n}(\lambda)\right\| \leq M_5 \mathrm{e}^{-\mu_5 nT} \tag{6.60}$$

for $\ell = 1,\ldots,2^{j-1}$, $j = 1,\ldots,M-1$ and $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where $M_5, \mu_5 > 0$ are λ- and n-independent constants. Finally, $h_{j,n}$ is nonvanishing on $\overline{B}_{\lambda_0}(\varrho_0)$ for $j = 1,\ldots,M-1$.

We prove our claim inductively for $j = 1,\ldots,M-1$. In our proof we rely on the identity (6.47). We start our induction proof with considering the case $j = 1$. Here, we employ identities (6.47), (6.57) and (6.58) and use the fact that $\Theta_{M-1,n}(\lambda)$ is invertible with inverse $\Pi_{M-1,n}(\lambda)$ to derive

$$\begin{aligned}\tilde{E}_n(\lambda) &= \det\big(\Xi_{0,n}^\top \mathcal{T}_n(nT,(M-1)nT)\mathcal{B}_{M-1,n}\Pi_{M-1,n}\Xi_{M-1,n}^\top \mathcal{B}_{M,n} \\ &\qquad + \Xi_{0,n}^\top \mathcal{T}_n(nT,(M-1)nT)\mathcal{T}_n((M-1)nT,MnT)\left(I-\mathcal{Q}_{M-1,\circ,n}(MnT)\right)\mathcal{B}_{M,n}\big) \\ &= \det(\Pi_{M-1,n})\det\left(\begin{pmatrix}\Xi_{M-1,n}^\top \mathcal{B}_{M,n} & 0_{l_0\times l_0} \\ \tilde{D}_{1,n} & \Xi_{0,n}^\top \mathcal{T}_n(nT,(M-1)nT)\mathcal{B}_{M-1,n}\end{pmatrix} + \tilde{H}_{1,n}\right)\end{aligned}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where we suppressed λ-dependency on the right-hand side and denote

$$\begin{aligned}\tilde{D}_{1,n}(\lambda) &= \Xi_{0,n}(\lambda)^\top \mathcal{T}_n(nT,(M-1)nT;\lambda))H_{1,1,n}(\lambda), \\ H_{1,1,n}(\lambda) &= \mathcal{T}_n((M-1)nT,MnT;\lambda)\left(I-\mathcal{Q}_{M-1,\circ,n}(MnT;\lambda)\right)\mathcal{B}_{M,n}(\lambda),\end{aligned}$$

and

$$\tilde{H}_{1,n}(\lambda) = \begin{pmatrix}0_{l_0\times l_0} & -\Theta_{M-1,n}(\lambda) \\ 0_{l_0\times l_0} & 0_{l_0\times l_0}\end{pmatrix}.$$

Using the bound (6.56) and the fact that $\mathcal{B}_{M-1,n}(\lambda)$ is a basis of $\operatorname{ran}(\mathcal{Q}_{M-1,\circ,n}((M-1)nT;\lambda))$, we obtain, provided $n \in \mathbb{N}$ is sufficiently large, λ- and n-independent constants $M_6, \mu_6 > 0$ such that

$$\|H_{1,1,n}(\lambda)\|, \left\|\tilde{H}_{1,n}(\lambda)\right\| \leq M_6 \mathrm{e}^{-\mu_6 nT}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Finally, we note that the function $h_{1,n}\colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}$ given by $h_{1,n}(\lambda) = \det(\Pi_{M-1,n}(\lambda))$ is analytic and does not vanish on $\overline{B}_{\lambda_0}(\varrho_0)$, since $\Pi_{M-1,n}$ is analytic and $\Pi_{M-1,n}(\lambda)$ is invertible for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. We conclude that our claim is valid for $j = 1$.

Next, we perform the induction step. That is, we assume that our claim holds for some $j = j_0 \in \{1,\ldots,M-2\}$ and prove that it then also holds for $j = j_0+1$. First, we recall that $\Theta_{M-j_0-1,n}(\lambda)$ is invertible with inverse $\Pi_{M-j_0-1,n}(\lambda)$ and use

$$\begin{aligned}\mathcal{T}_n((M-j_0-1)nT,(M-j_0)nT;\lambda) &= \mathcal{B}_{M-j_0-1}(\lambda)\Pi_{M-j_0-1,n}(\lambda)\Xi_{M-j_0-1,n}(\lambda)^\top \\ &\quad + \mathcal{T}_n((M-j_0-1)nT,(M-j_0)nT;\lambda)\left(I-\mathcal{Q}_{M-j_0-1,\circ,n}((M-j_0)nT;\lambda)\right),\end{aligned}$$

cf. (6.57), to express

$$\begin{pmatrix} \tilde{B}_{j_0,n}(\lambda) & -\tilde{A}_{j_0} \\ \tilde{D}_{j_0,n}(\lambda) & \tilde{C}_{j_0,n}(\lambda) \end{pmatrix} + \tilde{H}_{j_0,n}(\lambda) = \tilde{C}_{j_0+1,n}(\lambda) A_{j_0+1,n}(\lambda)^{-1} B_{j_0+1,n}(\lambda) + D_{j_0+1,n}(\lambda) \tag{6.61}$$

with

$$A_{j_0+1,n}(\lambda) = \begin{pmatrix} I_{(2^{j_0}-1)l_0 \times (2^{j_0}-1)l_0} & 0_{(2^{j_0}-1)l_0 \times l_0} \\ 0_{l_0 \times (2^{j_0}-1)l_0} & \Theta_{M-j_0-1,n}(\lambda) \end{pmatrix},$$
$$B_{j_0+1,n}(\lambda) = \begin{pmatrix} \tilde{B}_{j_0,n}(\lambda) & -\tilde{A}_{j_0} \\ \hat{D}_{j_0+1,n}(\lambda) & \hat{C}_{j_0+1,n}(\lambda) \end{pmatrix}, \qquad D_{j_0+1,n}(\lambda) = \tilde{D}_{j_0+1,n}(\lambda) + \tilde{H}_{j_0,n}(\lambda)$$

and

$$\hat{C}_{j_0+1,n}(\lambda) = \begin{pmatrix} I_{(2^{j_0-1}-1)l_0 \times (2^{j_0-1}-1)l_0} & 0_{(2^{j_0-1}-1)l_0 \times l_0} \\ 0_{l_0 \times (2^{j_0-1}-1)l_0} & \Xi_{M-j_0-1,n}(\lambda)^\top \mathcal{B}_{M-j_0,n}(\lambda) \end{pmatrix},$$
$$\hat{D}_{j_0+1,n}(\lambda) = \begin{pmatrix} 0_{(2^{j_0-1}-1)l_0 \times l_0} & \dots & 0_{(2^{j_0-1}-1)l_0 \times l_0} \\ \Xi_{M-j_0-1,n}(\lambda)^\top H_{j_0,1,n}(\lambda) & \dots & \Xi_{M-j_0-1,n}(\lambda)^\top H_{j_0,2^{j_0-1},n}(\lambda) \end{pmatrix}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where the matrix $\tilde{D}_{j_0+1,n}(\lambda)$ is defined by setting

$$\begin{aligned}
&H_{j_0+1,\ell,n}(\lambda) = \mathcal{T}_n((M-j_0-1)nT,(M-j_0)nT)\left(I - \mathcal{Q}_{M-j_0-1,\circ,n}((M-j_0)nT)\right) H_{j_0,\ell,n},\\
&H_{j_0+1,\tilde{\ell},n}(\lambda) = 0_{l_0 \times l_0},\\
&H_{j_0+1,2^{j_0},n}(\lambda) = \mathcal{T}_n((M-j_0-1)nT,(M-j_0)nT)\\
&\qquad\qquad \times \left(I - \mathcal{Q}_{M-j_0-1,\circ,n}((M-j_0)nT)\right) \mathcal{B}_{M-j_0,n}
\end{aligned}$$

for $\ell = 1, \dots, 2^{j_0-1}$ and $\tilde{\ell} = 2^{j_0-1}+1, \dots, 2^{j_0}-1$, suppressing λ-dependency on the right-hand sides. Next, we take determinants in (6.61), use that (6.59) holds for $j = j_0$, and apply (6.47) to arrive at

$$\tilde{E}_n(\lambda) = h_{j_0,n}(\lambda) \det\left(\Pi_{M-j_0-1,n}(\lambda)\right) \det\left(\begin{pmatrix} \tilde{B}_{j_0+1,n}(\lambda) & -\tilde{A}_{j_0+1} \\ \tilde{D}_{j_0+1,n}(\lambda) & \tilde{C}_{j_0+1,n}(\lambda) \end{pmatrix} + \tilde{H}_{j_0+1,n}(\lambda)\right),$$

with

$$\tilde{B}_{j_0+1,n}(\lambda) = \begin{pmatrix} \tilde{B}_{j_0,n}(\lambda) & -\tilde{A}_{j_0} \\ 0_{2^{j_0}l_0 \times 2^{j_0}l_0} & \hat{C}_{j_0+1,n}(\lambda) \end{pmatrix}$$

and

$$\tilde{H}_{j_0+1,n}(\lambda) = \begin{pmatrix} \hat{H}_{j_0+1,n}(\lambda) & \check{H}_{j_0+1,n}(\lambda) \\ \tilde{H}_{j_0,n}(\lambda) & 0_{2^{j_0}l_0 \times 2^{j_0}l_0} \end{pmatrix},$$

where we denote

$$\hat{H}_{j_0+1,n}(\lambda) = \begin{pmatrix} 0_{2^{j_0-1}l_0 \times 2^{j_0-1}l_0} & 0_{2^{j_0-1}l_0 \times 2^{j_0-1}l_0} \\ \hat{D}_{j_0+1,n}(\lambda) & 0_{2^{j_0-1}l_0 \times 2^{j_0-1}l_0} \end{pmatrix}$$

and

$$\check{H}_{j_0+1,n}(\lambda) = \begin{pmatrix} 0_{(2^{j_0}-1)l_0 \times (2^{j_0}-1)l_0} & 0_{(2^{j_0}-1)l_0 \times l_0} \\ 0_{l_0 \times (2^{j_0}-1)l_0} & -\Theta_{M-j_0-1,n}(\lambda) \end{pmatrix}.$$

Moreover, recalling that $\mathcal{B}_{M-j_0-1,n}(\lambda)$ is a basis of $\operatorname{ran}(\mathcal{Q}_{M-j_0-1,\circ,n}((M-j_0-1)nT;\lambda))$, employing the bound (6.56), and using that (6.60) holds for $j = j_0$ and $\ell = 1,\ldots,2^{j_0-1}$, we establish λ- and n-independent constants $M_7, \mu_7 > 0$ such that, provided $n \in \mathbb{N}$ is sufficiently large, we have

$$\|H_{j_0+1,\ell,n}(\lambda)\|, \left\|\tilde{H}_{j_0+1,n}(\lambda)\right\| \leq M_7 \mathrm{e}^{-\mu_7 nT}$$

for $\ell = 1,\ldots,2^{j_0}$ and $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Finally, the function $h_{j_0+1,n} \colon B_{\lambda_0}(\varrho_1) \to \mathbb{C}$ given by $h_{j_0+1,n}(\lambda) = h_{j_0,n}(\lambda)\det(\Pi_{M-j_0-1,n}(\lambda))$ is analytic and does not vanish on $\overline{B}_{\lambda_0}(\varrho_0)$, since $h_{j_0,n}$ and $\Pi_{M-j_0-1,n}$ are analytic, $h_{j_0,n}(\lambda)$ is nonzero and $\Pi_{M-j_0-1,n}(\lambda)$ is invertible for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Therefore, our claim holds for $j = j_0 + 1$.

Inductively, we have thus established our claim for $j = 1,\ldots,M-1$ as desired. In particular, applying our claim with $j = M-1$ we find, provided $n \in \mathbb{N}$ is sufficiently large, that

$$\tilde{E}_n(\lambda) = h_{M-1,n}(\lambda)\hat{E}_n(\lambda) \tag{6.62}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ with $\hat{E}_n \colon \overline{B}_{\lambda_0}(\varrho_0) \to \mathbb{C}$ given by

$$\hat{E}_n(\lambda) = \det\left(A_{\mathrm{main},n}(\lambda) + A_{\mathrm{res},n}(\lambda)\right),$$

where we denote

$$A_{\mathrm{res},n}(\lambda) = \begin{pmatrix} 0_{2^{M-2}l_0 \times 2^{M-2}l_0} & 0_{2^{M-2}l_0 \times 2^{M-2}l_0} \\ \tilde{D}_{M-1,n}(\lambda) & 0_{2^{M-2}l_0 \times 2^{M-2}l_0} \end{pmatrix} + \tilde{H}_{M-1,n}(\lambda)$$

and where

$$A_{\mathrm{main},n}(\lambda) = \begin{pmatrix} \tilde{B}_{M-1,n}(\lambda) & -\tilde{A}_{M-1} \\ 0_{2^{M-2}l_0 \times 2^{M-2}l_0} & \tilde{C}_{M-1,n}(\lambda) \end{pmatrix} \in \mathbb{C}^{2^{M-1}l_0 \times 2^{M-1}l_0}$$

is an upper triangular $(l_0 \times l_0)$-block matrix, whose blocks above the diagonal are equal to $-I_{l_0 \times l_0}$ or to $0_{l_0 \times l_0}$, and whose diagonal contains $(l_0 \times l_0)$-identity matrices and precisely one copy of each of the blocks $\Xi_{M-1,n}(\lambda)^\top \mathcal{B}_{M,n}(\lambda), \ldots, \Xi_{0,n}(\lambda)^\top \mathcal{B}_{1,n}(\lambda)$. Hence, by estimates (6.56) and (6.60) there exist λ- and n-independent constants $M_8, \mu_8 > 0$ such that, provided $n \in \mathbb{N}$ is sufficiently large, we have

$$\|A_{\mathrm{main},n}(\lambda) - A_0(\lambda)\| \leq M_8 \left(\mathrm{e}^{-\mu_8 nT} + \delta_n\right), \; \|A_0(\lambda)\| \leq M_8, \; \|A_{\mathrm{res},n}(\lambda)\| \leq M_8 \mathrm{e}^{-\mu_8 nT}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where $A_0(\lambda)$ is the upper triangular block matrix arising by substituting the blocks $\Xi_{M-1,n}(\lambda)^\top \mathcal{B}_{M,n}(\lambda), \ldots, \Xi_{0,n}(\lambda)^\top \mathcal{B}_{1,n}(\lambda)$ in $A_{\mathrm{main},n}(\lambda)$ by the blocks $\Phi_M^\top B_{M,\mathrm{s}}(\lambda)$, $\ldots, \Phi_1^\top B_{1,\mathrm{s}}(\lambda)$, respectively. Therefore, we obtain λ- and n-independent constants $M_9, \mu_9 > 0$ such that, provided $n \in \mathbb{N}$ is sufficiently large, we have

$$\left|\hat{E}_n(\lambda) - \tilde{\mathcal{E}}(\lambda)\right| \leq M_9 \left(\mathrm{e}^{-\mu_9 nT} + \delta_n\right) \tag{6.63}$$

for $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where we denote

$$\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_1(\lambda) \ldots \tilde{\mathcal{E}}_M(\lambda).$$

Moreover, since $h_{M-1,n}$ and $\tilde{E}_n$ are analytic and $h_{M-1,n}$ does not vanish on $\overline{B}_{\lambda_0}(\varrho_0)$, we find by identity (6.62) that $\hat{E}_n$ is analytic on $B_{\lambda_0}(\varrho_0)$ and has the same zeros (including multiplicities) in $B_{\lambda_0}(\varrho_0)$ as $\tilde{E}_n$.

Application of Rouché's theorem. Let $\varrho \in (0, \varrho_0)$. Since the Evans function $\mathcal{E}_j$ has the same roots (including multiplicities) as $\tilde{\mathcal{E}}_j$ in $B_{\lambda_0}(\varrho_1)$ for $j = 1, \ldots, M$ and $\mathcal{E}$ does not vanish on $\partial B_{\lambda_0}(\varrho)$, we find that $\tilde{\mathcal{E}}$ is also nonzero on $\partial B_{\lambda_0}(\varrho)$. So, the bound (6.63) yields, provided $n \in \mathbb{N}$ is sufficiently large, that

$$\left|\hat{E}_n(\lambda) - \tilde{\mathcal{E}}(\lambda)\right| < \left|\tilde{\mathcal{E}}(\lambda)\right|$$

for all $\lambda \in \partial B_{\lambda_0}(\varrho)$. Therefore, noting that $\tilde{\mathcal{E}}$ has only one root in $B_{\lambda_0}(\varrho)$ having multiplicity m_0, Rouché's theorem implies that $\hat{E}_n$ possesses precisely m_0 zeros in $B_{\lambda_0}(\varrho)$ (counting multiplicities). Since the zeros of $\hat{E}_n$, $\tilde{E}_n$ and E_n in $B_{\lambda_0}(\varrho)$ and their multiplicities coincide, the first assertion follows. The second assertion is a direct consequence of the first by Proposition 6.12. □

6.7 Spectral analysis of periodic pulse solutions

In this section, we study the spectral stability of stationary periodic pulse solutions to (6.1). We consider an nT-periodic pulse solution u_n of the form (6.23). That is, we have $u_n = w_n + a_n$, where w_n is the formal nT-periodic extension of a primary pulse $Z = z + v \in H^k(\mathbb{R}) \oplus H^k_{\text{per}}(0,T)$, and a_n is an error term converging to 0 in $H^k_{\text{per}}(0,nT)$. Our goal is to show that spectral (in)stability properties of the primary pulse Z transfer to the periodic pulse solution u_n.

We begin by observing that Lemma 6.9 implies that, if $\mathcal{K}$ is a compact subset of the resolvent set $\rho(\mathcal{L}(Z))$, then $\mathcal{K}$ is also contained in $\rho(\mathcal{L}(u_n))$ for $n \in \mathbb{N}$ sufficiently large. Hence, if a-priori bounds preclude unstable spectrum outside of the compact region $\mathcal{K}$, then this leads to the following analogue of Corollary 6.13, which asserts that strong spectral stability of the primary pulse is inherited by the associated periodic pulse solutions.

Corollary 6.21. *Assume* **(H4)** *and* **(H5)**. *Suppose that the pulse solution $z + v$ is strongly spectrally stable. Moreover, assume that there exist a compact set $\mathcal{K} \subset \mathbb{C}$ and $N \in \mathbb{N}$ such that*

$$\sigma(\mathcal{L}_n) \cap \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\} \subset \mathcal{K}$$

for $n \geq N$, where u_n is the periodic pulse solution established in Theorem 6.8, and $\mathcal{L}_n$ is the operator $\mathcal{L}(u_n)$ or $\mathcal{L}_{per}(u_n)$. Then, there exists $N_1 \in \mathbb{N}$ with $N_1 \geq N$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ the spectrum of the operator $\mathcal{L}_n$ is confined to the open left-half plane.

In the remainder of this section, we analyze the spectra of the operators $\mathcal{L}(u_n)$ and $\mathcal{L}_{\text{per}}(u_n)$ in more detail. Our approach relies on comparing the Evans function $\mathcal{E}$ associated with $\mathcal{L}(Z)$, as constructed in Proposition 6.12, with an Evans function for the first-order formulation

$$U' = \mathcal{A}(x, u_n(x); \lambda)U \tag{6.64}$$

of the eigenvalue problem along the periodic pulse solution u_n. If $\mathcal{T}_n(x,y;\lambda)$ is the evolution of system (6.64), then this analytic Evans function $E_{n,\gamma}\colon \mathbb{C} \to \mathbb{C}$ is given by

$$E_{n,\gamma}(\lambda) = \det\left(\mathcal{T}_n(0,-\tfrac{n}{2}T;\lambda) - \gamma\mathcal{T}_n(0,\tfrac{n}{2}T;\lambda)\right)$$

for γ lying in the unit circle $S^1 \subset \mathbb{C}$, cf. [52,129]. Clearly, it holds $E_{n,\gamma}(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C}$ if and only if system (6.64) possesses a nontrivial solution $U(x)$ satisfying the boundary condition $U(-\frac{n}{2}T) = \gamma U(\frac{n}{2}T)$. Hence, λ_0 lies in the spectrum of the Bloch operator $\mathcal{L}_{\xi,\mathrm{per}}(u_n)$ if and only λ_0 is a zero of $E_{n,\mathrm{e}^{\mathrm{i}\xi nT}}$, cf. §6.2.4. In fact, it was shown in [52], see also [78, Section 8.4], that the algebraic multiplicity of λ_0 as an eigenvalue of $\mathcal{L}_{\xi,\mathrm{per}}(u_n)$ equals the multiplicity of λ_0 as a root of the Evans function $E_{n,\mathrm{e}^{\mathrm{i}\xi nT}}$. We conclude that λ_0 lies in the spectrum of $\mathcal{L}_{\mathrm{per}}(u_n)$ if and only if $\lambda_0 \in \mathbb{C}$ is a zero of $E_{n,0}$. Moreover, we have $\lambda_0 \in \sigma(\mathcal{L}(u_n))$ if and only there exists $\gamma \in S^1$ such that $E_{n,\gamma}(\lambda_0) = 0$.

The main result of this section establishes that isolated zeros of the Evans function $\mathcal{E}$ associated with the primary pulse perturb into zeros of the Evans function $E_{n,\gamma}$ of the periodic pulse solution, thereby preserving the total multiplicity. That is, isolated eigenvalues (including their algebraic multiplicities) of the linearization $\mathcal{L}(Z)$ about the primary pulse persist as eigenvalues of the Bloch operator $\mathcal{L}_{\xi,\mathrm{per}}(u_n)$ for each $\xi \in [-\frac{\pi}{nT}, \frac{\pi}{nT})$.

Theorem 6.22. *Let $z + v \in H^k(\mathbb{R}) \oplus H^k_{per}(0,T)$. Suppose there exists $N \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$ with $n \geq N$, there exists a periodic pulse solution $u_n \in H^k_{per}(0,nT)$ of the form $u_n = z_n + v + a_n$, where $\{a_n\}_n$ is a sequence with $a_n \in H^k_{per}(0,nT)$ satisfying $\|a_n\|_{H^k_{per}(0,nT)} \to 0$ as $n \to \infty$, and $z_n \in H^k_{per}(0,nT)$ is the nT-periodic extension of the function $\chi_n z$ on $\left[-\frac{n}{2}T, \frac{n}{2}T\right)$ and χ_n is the cut-off function from Theorem 6.8.*

Let Ω be a connected component of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{L}(v))$. Let $\mathcal{E}\colon \Omega \to \mathbb{C}$ be the Evans function associated with the first-order system (6.26)*, as constructed in Proposition 6.12.*

Suppose that $\lambda_0 \in \Omega$ is a root of $\mathcal{E}$ of multiplicity $m_0 \in \mathbb{N}$. Then, there exists $\varrho_0 > 0$ such that for each $\varrho \in (0,\varrho_0)$ there exists $N_\varrho \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_\varrho$ the following assertions hold true.

1. *For each $\gamma \in S^1$ the Evans function $E_{n,\gamma}$ possesses precisely m_0 roots in the disk $B_{\lambda_0}(\varrho)$ (counting multiplicities).*
2. *For each $\xi \in [-\frac{\pi}{nT}, \frac{\pi}{nT})$ the Bloch operator $\mathcal{L}_{\xi,per}(u_n)$ has precisely m_0 eigenvalues in $B_{\lambda_0}(\varrho)$ (counting algebraic multiplicities).*
3. *The operator $\mathcal{L}_{per}(u_n)$ has precisely m_0 eigenvalues in $B_{\lambda_0}(\varrho)$ (counting algebraic multiplicities).*

4. *The operator $\mathcal{L}(u_n)$ has spectrum in $B_{\lambda_0}(\varrho)$.*

Remark 6.23. Theorem 6.22 establishes the convergence of the point spectrum of the Bloch operators $\mathcal{L}_{\xi,\text{per}}(u_n)$ within n-independent compact sets $\mathcal{K} \subset \mathbb{C}\backslash\sigma_{\text{ess}}(\mathcal{L}(Z))$ to the point spectrum of $\mathcal{L}(Z)$ in $\mathcal{K}$ as $n \to \infty$. This naturally raises the question of whether spectrum of $\mathcal{L}_{\xi,\text{per}}(u_n)$ converges to the essential spectrum $\sigma_{\text{ess}}(\mathcal{L}(Z))$ as $n \to \infty$. In the case of constant coefficients, this question has been answered affirmative. Specifically, it was shown in [129] that eigenvalues of $\mathcal{L}_{\xi,\text{per}}(u_n)$ accumulate onto each point of the essential spectrum $\sigma_{\text{ess}}(\mathcal{L}(Z))$ as $n \to \infty$. Consequently, on compact subsets, the spectra of both $\mathcal{L}(u_n)$ and $\mathcal{L}_{\text{per}}(u_n)$ converge to $\sigma(\mathcal{L}(Z))$ in Hausdorff distance as $n \to \infty$. We strongly expect that, using the techniques developed in [129], a similar result can be obtained in the spatially periodic setting considered here.

Theorem 6.22 implies that spectral instability of the primary pulse is inherited by the periodic pulse solution. Furthermore, it serves as an important tool in spectral (in)stability arguments based on Krein index counting theory. In particular, we employ Theorem 6.22 in §6.8 to demonstrate the spectral and orbital stability of periodic pulse solutions to the Gross-Pitaevskii equation with a periodic potential.

As mentioned in §6.1, Theorem 6.22 was established in the constant-coefficient case in [53], using geometric dynamical systems techniques and topological arguments based on Chern numbers. The result was subsequently refined in [129] by showing that there exists an n-independent constant $\mu > 0$ such that the roots of $E_{n,\gamma}$ in $B_{\lambda_0}(\varrho)$ remain $\mathcal{O}(\mathrm{e}^{-\mu n})$-close to λ_0.

Our proof of Theorem 6.22 builds upon the approach of [129, Theorem 2]. Specifically, we employ roughness techniques to transfer exponential dichotomies on $\mathbb{R}_\pm$ for the eigenvalue problem (6.26) along the primary pulse to the system

$$U' = \mathcal{A}(x, \chi_n(x)z(x) + v(x) + a_n(x); \lambda)U. \tag{6.65}$$

System (6.65) coincides with the eigenvalue problem (6.64) along the periodic pulse u_n on a single periodicity interval $[-\frac{n}{2}T, \frac{n}{2}T]$. Denoting the projections of the exponential dichotomies of (6.65) on $\mathbb{R}_\pm$ by $\mathcal{Q}_{\pm,n}(x;\lambda)$, we construct analytic bases of $\ker(\mathcal{Q}_{-,n}(-\frac{n}{2}T;\lambda))$ and $\operatorname{ran}(\mathcal{Q}_{+,n}(\frac{n}{2}T;\lambda))$. Multiplying $E_{n,\gamma}(\lambda)$ with the nonzero determinant of the matrix formed by these basis vectors yields an approximation of the Evans function $\mathcal{E}$ associated with the primary pulse. The conclusion then follows from an application of Rouché's theorem.

Proof of Theorem 6.22. We start by collecting some facts from Proposition 6.12. First, system (6.26) possesses for each $\lambda \in \Omega$ exponential dichotomies on $\mathbb{R}_\pm$ with projections $P_\pm(\pm x;\lambda)$, $x \geq 0$ of some fixed rank l_0, which is independent of λ and x. Moreover, there exist analytic functions $B_\text{s}\colon \Omega \to \mathbb{C}^{km\times l_0}$ and $B_\text{u}\colon \Omega \to \mathbb{C}^{km\times(km-l_0)}$ such that $B_\text{s}(\lambda)$ is a basis of $\operatorname{ran}(P_+(0;\lambda))$

and $B_{\mathrm{u}}(\lambda)$ is a basis of $\ker(P_-(0;\lambda))$ for each $\lambda \in \Omega$. The associated Evans function $\mathcal{E}\colon \Omega \to \mathbb{C}$ is given by

$$\mathcal{E}(\lambda) = \det(B_{\mathrm{u}}(\lambda) \mid B_{\mathrm{s}}(\lambda)).$$

Because $\mathcal{E}$ is analytic and λ_0 is a root of $\mathcal{E}$ of finite multiplicity, there exists a closed disk $\overline{B}_{\lambda_0}(\varrho_1) \subset \Omega$ of some radius $\varrho_1 > 0$ such that λ_0 is the only root of $\mathcal{E}$ in $\overline{B}_{\lambda_0}(\varrho_1)$. Finally, there exist constants $K_0, \mu_0, \tau_0 > 0$ such that system (6.26) admits for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$ exponential dichotomies on $\mathbb{R}_\pm$ with constants $K_0, \mu_0 > 0$ and projections $\tilde{P}_\pm(\pm x;\lambda), x \geq 0$ satisfying (6.42) for each $x \geq \tau_0$, where $Q(\cdot;\lambda)$ is T-periodic. By uniqueness of exponential dichotomies, cf. [33, p. 19], $B_{\mathrm{s}}(\lambda)$ is a basis of $\operatorname{ran}(P_+(0;\lambda)) = \operatorname{ran}(\tilde{P}_+(0;\lambda))$ and $B_{\mathrm{u}}(\lambda)$ is a basis of $\ker(P_-(0;\lambda)) = \ker(\tilde{P}_-(0;\lambda))$ for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$.

Since $\partial_u \mathcal{A}$ is continuous, $\overline{B}_{\lambda_0}(\varrho_1)$ is compact, it holds $z \in H^1(\mathbb{R})$ and $v, a_n \in H^1_{\mathrm{per}}(0,nT)$, and $H^1(\mathbb{R})$ and $H^1_{\mathrm{per}}(0,nT)$ embed continuously into $L^\infty(\mathbb{R})$ with n-independent constant, we obtain by the mean value theorem a λ- and n-independent constant $R > 0$ such that

$$\|\mathcal{A}\left(x, \chi_n z(x) + v(x) + a_n(x);\lambda\right) - \mathcal{A}\left(x, z(x) + v(x);\lambda\right)\| \leq R\delta_n$$

for $x \in \mathbb{R}$ and $\lambda \in \overline{B}_{\lambda_0}(\varrho_1)$, where

$$\delta_n := \sup_{x\in\mathbb{R}} \left(\|(1-\chi_n(x))z(x)\| + \|a_n(x)\|\right)$$

converges to 0 as $n \to \infty$. So, by Lemma 6.58 there exist constants $M_1, \mu > 0$ and $\varrho_0 \in (0,\varrho_1)$ such that system (6.65) admits, provided $n \in \mathbb{N}$ is sufficiently large, exponential dichotomies on $\mathbb{R}_\pm$ with λ- and n-independent constants and projections $\mathcal{Q}_{\pm,n}(\pm x;\lambda), x \geq 0$ for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Here, the maps $\mathcal{Q}_{\pm,n}(\pm x;\cdot)\colon B_{\lambda_0}(\varrho_0) \to \mathbb{C}^{km\times km}$ are analytic for each $x \geq 0$ and the estimates

$$\begin{aligned} \|\mathcal{Q}_{\pm,n}(\pm \tfrac{n}{2}T;\lambda) - \tilde{P}_\pm(\pm \tfrac{n}{2}T;\lambda)\| &\leq M_1\left(\delta_n + \mathrm{e}^{-\mu\frac{n}{2}T}\right), \\ \|\mathcal{Q}_{\pm,n}(0;\lambda) - Q_\pm(\lambda)\| &\leq M_1\delta_n \end{aligned} \tag{6.66}$$

hold for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where $Q_+(\lambda)$ is the projection onto $\operatorname{ran}(P_+(0;\lambda))$ along $\operatorname{ran}(P_+(0;\lambda_0))^\top$ and $Q_-(\lambda)$ is the projection onto $\ker(P_-(0;\lambda_0))^\top$ along $\ker(P_-(0;\lambda))$.

Now set $\mathcal{B}_{\mathrm{s},n}(\lambda) = \mathcal{Q}_{+,n}(0;\lambda)B_{\mathrm{s}}(\lambda)$ and $\mathcal{B}_{\mathrm{u},n}(\lambda) = (I - \mathcal{Q}_{-,n}(0;\lambda))B_{\mathrm{u}}(\lambda)$. Then, $\mathcal{B}_{\mathrm{s},n}(\lambda)$ and $\mathcal{B}_{\mathrm{u},n}(\lambda)$ are analytic in λ on $B_{\lambda_0}(\varrho_0)$. Moreover, the fact that the analytic maps B_{s} and B_{u}

are bounded on the compact set $\overline{B}_{\lambda_0}(\varrho_0)$ in combination with estimate (6.66) affords a λ- and n-independent constant $M_2 > 0$ such that

$$\|B_{\mathrm{s}}(\lambda) - \mathcal{B}_{\mathrm{s},n}(\lambda)\|, \|B_{\mathrm{u}}(\lambda) - \mathcal{B}_{\mathrm{u},n}(\lambda)\| \leq M_2\delta_n, \qquad \|\mathcal{B}_{\mathrm{s},n}(\lambda)\|, \|\mathcal{B}_{\mathrm{u},n}(\lambda)\| \leq M_2, \quad (6.67)$$

for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. So, provided $n \in \mathbb{N}$ is sufficiently large, $\mathcal{B}_{\mathrm{s},n}(\lambda)$ is a basis of $\mathrm{ran}(\mathcal{Q}_{+,n}(0;\lambda))$ and $\mathcal{B}_{\mathrm{u},n}(\lambda)$ is a basis of $\ker(\mathcal{Q}_{-,n}(0;\lambda))$ for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$.

Since $Q(\cdot;\lambda)$ is T-periodic, estimates (6.42) and (6.66) and Lemma 6.48 imply that, provided $n \in \mathbb{N}$ is sufficiently large, the subspaces $\mathrm{ran}(\mathcal{Q}_{-,n}(-\frac{n}{2}T;\lambda))$ and $\ker(\mathcal{Q}_{+,n}(\frac{n}{2}T;\lambda))$ are complementary and there exists a λ- and n-independent constant $M_3 > 0$ such that the projection $\check{\mathcal{P}}_n(\lambda)$ onto $\mathrm{ran}(\mathcal{Q}_{-,n}(-\frac{n}{2}T;\lambda))$ along $\ker(\mathcal{Q}_{+,n}(\frac{n}{2}T;\lambda))$ obeys

$$\left\|\check{\mathcal{P}}_n(\lambda)\right\| \leq M_3 \qquad (6.68)$$

for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. In addition, since the functions $\mathcal{Q}_{\pm,n}(\pm\frac{n}{2}T;\cdot)\colon B_{\lambda_0}(\varrho_0) \to \mathbb{C}^{km\times km}$ are analytic, Lemma 6.50 yields analytic maps $\mathcal{B}_{1,n}, \mathcal{B}_{2,n}\colon B_{\lambda_0}(\varrho_0) \to \mathbb{C}^{km\times l_0}$ with the property that $\mathrm{ran}(\mathcal{Q}_{-,n}(-\frac{n}{2}T;\lambda)) = \mathrm{ran}(\mathcal{B}_{1,n}(\lambda))$ and $\ker(\mathcal{Q}_{+,n}(\frac{n}{2}T;\lambda)) = \{u \in \mathbb{C}^{km} : z^\top u = 0 \text{ for all } z \in \mathrm{ran}(\mathcal{B}_{2,n}(\lambda))\}$. So, it follows, again by Lemma 6.50, that $\check{\mathcal{P}}_n\colon B_{\lambda_0}(\varrho_0) \to \mathbb{C}^{km\times km}$ is analytic.

Since the evolution $\mathcal{T}_n(x,y;\lambda)$ of system (6.64) depends analytically on λ by [78, Lemma 2.1.4], the Evans function $E_{n,\gamma}$ is analytic for each $\gamma \in S^1$. Because system (6.64) coincides with system (6.65) on $[-\frac{n}{2}T, \frac{n}{2}T]$, it holds

$$E_{n,\gamma}(\lambda) = \det\left(T_n(0,-\tfrac{n}{2}T;\lambda) - \gamma T_n(0,\tfrac{n}{2}T;\lambda)\right)$$

for all $\gamma \in S^1$, where $T_n(x,y;\lambda)$ denotes the evolution of (6.65), which depends analytically on λ by [78, Lemma 2.1.4]. Define $\mathcal{H}_{n,\gamma}\colon \overline{B}_{\lambda_0}(\varrho_0) \to \mathbb{C}^{km\times km}$ by

$$\mathcal{H}_{n,\gamma}(\lambda) = \left(\left(I - \check{\mathcal{P}}_n(\lambda)\right) T_n\left(-\tfrac{n}{2}T, 0;\lambda\right)\mathcal{B}_{\mathrm{u},n}(\lambda) \mid -\gamma^{-1}\check{\mathcal{P}}_n(\lambda) T_n\left(\tfrac{n}{2}T,0;\lambda\right)\mathcal{B}_{\mathrm{s},n}(\lambda)\right)$$

for $\gamma \in S^1$. We note that $\mathcal{H}_{n,\gamma}$ is analytic on $B_{\lambda_0}(\varrho_0)$. Moreover, $T_n\left(-\frac{n}{2}T,0;\lambda\right)\mathcal{B}_{\mathrm{u},n}(\lambda)$ constitutes a basis of $\ker(\mathcal{Q}_{-,n}(-\frac{n}{2}T;\lambda))$, whereas $I - \check{\mathcal{P}}_n(\lambda)$ projects along the complementary subspace $\mathrm{ran}(\mathcal{Q}_{-,n}(-\frac{n}{2}T;\lambda))$. Hence, the first $km - l_0$ columns of $\mathcal{H}_{n,\gamma}(\lambda)$ form a basis of $\mathrm{ran}(I - \check{\mathcal{P}}_n(\lambda)) = \ker(\mathcal{Q}_{+,n}(\frac{n}{2}T;\lambda))$. Similarly, the last l_0 columns of $\mathcal{H}_{n,\gamma}(\lambda)$ constitute a basis of the complementary subspace $\mathrm{ran}(\mathcal{Q}_{-,n}(-\frac{n}{2}T;\lambda))$. Therefore, provided $n \in \mathbb{N}$ is sufficiently large, $\mathcal{H}_{n,\gamma}(\lambda)$ is invertible for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $\gamma \in S^1$.

Recall that (6.65) possesses exponential dichotomies on $\mathbb{R}_\pm$ with projections $\mathcal{Q}_{\pm,n}(\pm x;\lambda), x \geq 0$ and λ- and n-independent constants, which we denote by $C_1, \mu_1 > 0$. Combining the latter with (6.67) and (6.68), we obtain a λ- and n-independent constant $M_4 > 0$ such that

$$\begin{aligned}\left\|T_n(0,\tfrac{n}{2}T;\lambda)\left(I-\check{\mathcal{P}}_n(\lambda)\right)\right\|,\left\|T_n(0,-\tfrac{n}{2}T;\lambda)\check{\mathcal{P}}_n(\lambda)\right\| &\leq M_4\mathrm{e}^{-\mu_1\frac{n}{2}T},\\ \left\|T_n(-\tfrac{n}{2}T,0;\lambda)\mathcal{B}_{\mathrm{u},n}(\lambda)\right\|,\left\|T_n(\tfrac{n}{2}T,0;\lambda)\mathcal{B}_{\mathrm{s},n}(\lambda)\right\| &\leq M_4\mathrm{e}^{-\mu_1\frac{n}{2}T},\end{aligned} \tag{6.69}$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Therefore, using the estimates (6.67) and (6.69), we find an n-, γ- and λ-independent constant $M_5 > 0$ such that, provided $n \in \mathbb{N}$ is sufficiently large, it holds

$$\begin{aligned}\left\|\left(T_n(0,-\tfrac{n}{2}T;\lambda)-\gamma T_n(0,\tfrac{n}{2}T;\lambda)\right)\mathcal{H}_{n,\gamma}(\lambda)-\left(B_u(\lambda)\mid B_s(\lambda)\right)\right\| &\leq M_5\left(\delta_n+\mathrm{e}^{-\mu_1 nT}\right),\\ \left\|\left(T_n(0,-\tfrac{n}{2}T;\lambda)-\gamma T_n(0,\tfrac{n}{2}T;\lambda)\right)\mathcal{H}_{n,\gamma}(\lambda)\right\| &\leq M_5,\end{aligned}$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $\gamma \in S^1$. So, taking determinants, we establish an n-, γ- and λ-independent constant $M_6 > 0$ such that

$$|E_{n,\gamma}(\lambda)\det(\mathcal{H}_{n,\gamma}(\lambda))-\mathcal{E}(\lambda)| \leq M_6\left(\delta_n+\mathrm{e}^{-\mu_1 nT}\right),$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $\gamma \in S^1$.

Let $\varrho \in (0,\varrho_0)$. Since $\mathcal{E}$ does not vanish on $\partial B_{\lambda_0}(\varrho)$, the latter estimate yields, provided $n \in \mathbb{N}$ is sufficiently large, that

$$|E_{n,\gamma}(\lambda)\det(\mathcal{H}_{n,\gamma}(\lambda))-\mathcal{E}(\lambda)| < |\mathcal{E}(\lambda)|$$

for each $\lambda \in \partial B_{\lambda_0}(\varrho)$ and $\gamma \in S^1$. We recall that $\det(\mathcal{H}_{n,\gamma}(\cdot))$ is nonzero and the functions $\mathcal{E}$, $\det(\mathcal{H}_{n,\gamma}(\cdot))$ and $E_{n,\gamma}$ are analytic on the open disk $B_{\lambda_0}(\varrho_0) \subset \Omega$, which contains $\partial B_{\lambda_0}(\varrho)$. So, applying Rouché's theorem to the latter inequality and noting that λ_0 is the only root of $\mathcal{E}$ in $B_{\lambda_0}(\varrho_0)$, which has multiplicity m_0, we find that the Evans function $E_{n,\gamma}$ has precisely m_0 zeros in $B_{\lambda_0}(\varrho)$ (counting with multiplicities) for each $\gamma \in S^1$. This proves the first assertion.

The second assertion immediately follows from the first assertion by taking $\gamma = \mathrm{e}^{\mathrm{i}\xi nT}$ and applying [52, Proposition 2.5], see also [78, Lemmas 8.4.1 and 8.4.2]. Since $\mathcal{L}_{0,\mathrm{per}}(u_n) = \mathcal{L}_{\mathrm{per}}(u_n)$, the third assertion is a direct consequence of the second. Finally, the third implies the fourth assertion by evoking (6.7). □

6.8 Applications

In this section, we employ our methods to construct multifronts and periodic pulse solutions in specific prototype models and analyze their stability. To illustrate the applicability of our theory in a simple setting, we first consider a reaction-diffusion toy model. We then focus on a Klausmeier reaction-diffusion-advection system which describes the dynamics of vegetation patterns on periodic topographies [13]. Finally, we consider the Gross-Pitaevskii equation with periodic potential, which arises in the study of Bose-Einstein condensates in optical lattices [109]. Our findings are supported by numerical simulations performed with the MATLAB package `pde2path` [146].

6.8.1 A reaction-diffusion model problem

We consider the scalar reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + \varepsilon V(x)u - \sin(u), \qquad u(x,t) \in \mathbb{R},\, x \in \mathbb{R},\, t \geq 0, \tag{6.70}$$

where $V \in C^1(\mathbb{R})$ is a given real-valued potential of period $T > 0$. Here, the parameter $\varepsilon \geq 0$ measures the strength of the potential V and will serve as a bifurcation parameter.

We are interested in the existence and spectral stability of stationary multifronts and periodic pulse solutions to (6.70). Stationary solutions to (6.70) solve the ODE

$$\partial_x^2 u + \varepsilon V(x)u - \sin(u) = 0, \tag{6.71}$$

which is of the form (6.4).

Let $\underline{u} \in L^\infty(\mathbb{R})$. For the upcoming spectral stability analysis, we define the closed differential operator $L_\varepsilon(\underline{u}) \colon D(L_\varepsilon(\underline{u})) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with dense domain $D(L_\varepsilon(\underline{u})) = H^2(\mathbb{R})$ by

$$L_\varepsilon(\underline{u}) = \partial_x^2 + \varepsilon V - \cos(\underline{u}).$$

Since the operator $L_\varepsilon(\underline{u})$ is self-adjoint, its spectrum must be confined to the numerical range, leading to the following spectral a-priori bound.

Lemma 6.24. *Let $\varrho > 0$. Then, the spectrum of the operator $L(\underline{u}) \colon D(L(\underline{u})) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $D(L(\underline{u})) = H^2(\mathbb{R})$, given by*

$$L(\underline{u}) = \partial_x^2 + \underline{u},$$

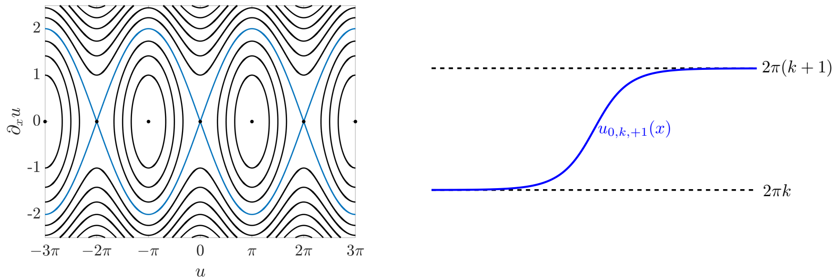

Figure 6.3: Left: phase portrait of the Hamiltonian system (6.73). Blue curves correspond to heteroclinic orbits connecting the fixed points $2\pi k$ to $2\pi(k \pm 1)$ for each $k \in \mathbb{Z}$. Black dots correspond to equilibria of the system. Right: plot of the associated front solution $u_{0,k,+1}$ to (6.72).

satisfies $\sigma(L(\underline{u})) \subset (-\infty, \varrho]$ *for all real-valued* $\underline{u} \in L^\infty(\mathbb{R})$ *with* $\|\underline{u}\|_{L^\infty} \leq \varrho$*.*

Proof. Because $L(\underline{u})$ is self-adjoint, its spectrum must be contained in the numerical range, which is confined to $(-\infty, \varrho]$. □

6.8.1.1 Existence and spectral stability of fronts for $\varepsilon = 0$

Stationary front solutions of (6.71) for $\varepsilon = 0$ correspond to heteroclinic solutions to the autonomous system

$$\partial_x^2 u - \sin(u) = 0. \tag{6.72}$$

We introduce the coordinates $(u, v)^\top = (u, \partial_x u)^\top$ and write (6.72) as the first-order system

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = J\nabla H(u, v), \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.73}$$

with Hamiltonian

$$H(u, v) = \frac{1}{2}v^2 + \cos(u).$$

Solutions to (6.73) lie on the level sets of H, see Figure 6.3. Thus, we find infinitely many heteroclinics in (6.73), connecting the fixed points $(2\pi k, 0)^\top$ to $(2\pi(k \pm 1), 0)^\top$ for all $k \in \mathbb{Z}$. The associated front solutions to (6.72) admit the explicit formula

$$u_{0,k,\pm 1}(x) = 4 \arctan\left(e^{\pm x}\right) + 2\pi \min\{k, k \pm 1\}, \qquad k \in \mathbb{Z}. \tag{6.74}$$

Fix $k \in \mathbb{Z}$. We examine the spectrum of the linearization $L_0(u_{0,k,\pm1})$ about the front solution $u_{0,k,\pm1}$ of (6.70) at $\varepsilon = 0$. A simple calculation reveals $\sigma(L_0(2\pi\ell)) = (-\infty, -1]$ for $\ell \in \mathbb{Z}$. Hence, Proposition 6.11 yields $\sigma_{\text{ess}}(L_0(u_{0,k,\pm1})) = (-\infty, -1]$. Moreover, by translational symmetry of (6.72), 0 is a simple eigenfunction of $L_0(u_{0,k,\pm1})$ with eigenfunction $u'_{0,k,\pm1}$. Since $u'_{0,k,\pm1}$ has no zeros, Sturm-Liouville theory, cf. [78, Theorem 2.3.3], yields that the front $u_{0,k,\pm1}$ is spectrally stable with simple eigenvalue $\lambda = 0$, cf. Definition 6.4.

6.8.1.2 Existence and spectral stability of fronts for $\varepsilon > 0$

In the following, we prove that the front solutions, obtained in §6.8.1.1, persist for small values of $\varepsilon > 0$ under a generic assumption on the periodic potential V, which can be checked analytically or numerically. The fronts connect T-periodic end states $v_-(\varepsilon)$ to $v_+(\varepsilon)$, see Figure 6.4. Moreover, we establish the front's spectral (in)stability and nondegeneracy.

The existence and spectral analysis of the front solutions to (6.71) consists of three steps. First, we construct their periodic end states by bifurcating from the fixed points $2\pi k$, $k \in \mathbb{Z}$ of (6.72). Second, we prove that the shifted front $u_{0,k,\pm1,\varsigma} := u_{0,k,\pm1}(\cdot - \varsigma)$ perturbs into a nondenegerate front solution of (6.71) for small $\varepsilon \neq 0$, provided that $\varsigma_0 \in \mathbb{R}$ is a simple zero of the *effective potential*

$$V_{\text{eff}}(\varsigma) = \int_{\mathbb{R}} V(x+\varsigma) u_{0,k,\pm1}(x) u'_{0,k,\pm1}(x) \mathrm{d}x. \tag{6.75}$$

In the third step, we derive a stability criterion saying that the fronts are strongly spectrally stable if $\varepsilon V'_{\text{eff}}(\varsigma_0) > 0$ and that they are spectrally unstable if $\varepsilon V'_{\text{eff}}(\varsigma_0) < 0$.

Theorem 6.25. *Let $k \in \mathbb{Z}$, $l \in \{\pm1\}$, and $T > 0$. Let $V \in C^1(\mathbb{R})$ be T-periodic. Assume that there exists $\varsigma_0 \in \mathbb{R}$ such that the effective potential $V_{\text{eff}} \colon \mathbb{R} \to \mathbb{R}$, given by* (6.75)*, has a simple zero at ς_0. Then, there exist $C, \varepsilon_0, \varrho > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ there exists a nondegenerate solution $u(\varepsilon)$ to* (6.71)*, satisfying*

$$\|u(\varepsilon) - u_{0,k,l}(\cdot - \varsigma_0)\|_{L^\infty} \leq C\varepsilon, \qquad \chi_\pm \left(u(\varepsilon) - v_\pm(\varepsilon)\right) \in H^2(\mathbb{R}), \tag{6.76}$$

where $\chi_\pm \colon \mathbb{R} \to [0,1]$ is a smooth partition of unity such that χ_+ is supported on $(-1,\infty)$ and χ_- is supported on $(-\infty, 1)$, $u_{0,k,l}$ is the front given by (6.74)*, and $v_\pm(\varepsilon) \in H^2_{per}(0,T)$ are periodic solutions to* (6.71) *with*

$$\|v_-(\varepsilon) - 2\pi k\|_{H^2_{per}(0,T)}, \|v_+(\varepsilon) - 2\pi(k+l)\|_{H^2_{per}(0,T)} \leq C\varepsilon. \tag{6.77}$$

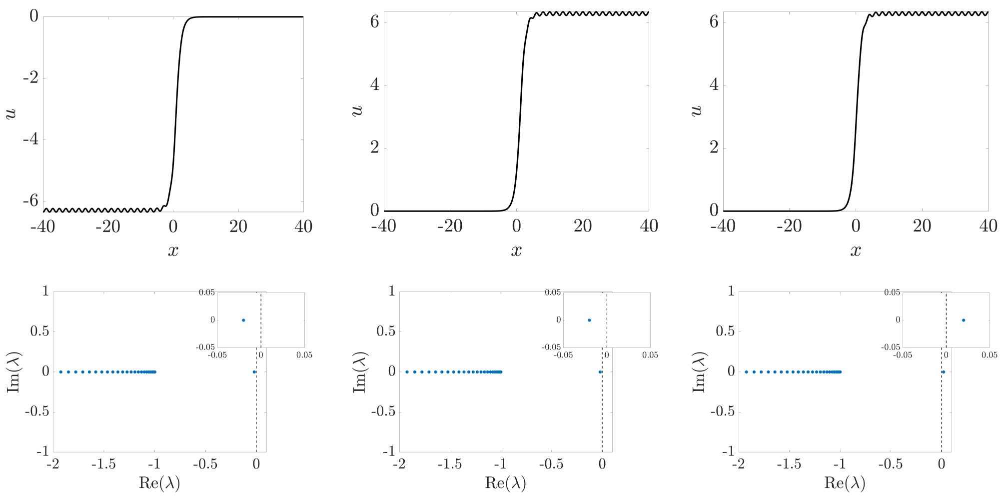

Figure 6.4: Approximations of stationary 1-front solutions to (6.70), along with their spectra, for system coefficients $\varepsilon = 0.1$ and $V(x) = \cos(\pi x)$. The insets provide a closer view of the small eigenvalues near zero. The left and middle panels depict strongly spectrally stable 1-front solutions that connect the periodic state near -2π to 0 and 0 to the periodic state near 2π, respectively. The right panel depicts a spectrally unstable front solution connecting 0 to the periodic state near 2π. The 1-front solutions are obtained through numerical continuation with the MATLAB package `pde2path` [146] by starting from the explicit 1-front solutions $u_{0,k,+1,\varsigma}$ for $k \in \{-1, 0\}$ and $\varsigma \in \mathbb{R}$.

Finally, it holds

$$\sigma(L_\varepsilon(u(\varepsilon))) \subset (-\infty, -\varrho] \cup \{\lambda_0(\varepsilon)\}$$

for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, where $\lambda_0(\varepsilon)$ is a real simple eigenvalue of $L_\varepsilon(u(\varepsilon))$ obeying the expansion

$$\lambda_0(\varepsilon) = -\varepsilon \frac{V_{\text{eff}}'(\varsigma_0)}{\|u_{0,k,l}'\|_{L^2}^2} + \mathcal{O}(\varepsilon^2).$$

Proof. We begin with the construction of the period end states $v_\pm(\varepsilon)$. To this end, we consider the nonlinear map $\mathcal{F}_{\text{per}} \colon H^2_{\text{per}}(0,T) \times \mathbb{R} \to L^2_{\text{per}}(0,T)$ given by

$$\mathcal{F}_{\text{per}}(v, \varepsilon) = v'' + \varepsilon V v - \sin(v).$$

We observe that $\mathcal{F}_{\text{per}}$ is well-defined and smooth. Fix $j \in \mathbb{Z}$. Then, we have $\mathcal{F}_{\text{per}}(2\pi j, 0) = 0$ and

$$\partial_v \mathcal{F}_{\text{per}}(2\pi j, 0) = \partial_x^2 - 1$$

is invertible. Therefore, the implicit function theorem implies that there exist $\varepsilon_1 > 0$ and a locally unique smooth map $v\colon (-\varepsilon_1, \varepsilon_1) \to H^2_{\text{per}}(0,T)$ with

$$v(0) = 2\pi j, \qquad \mathcal{F}_{\text{per}}(v(\varepsilon), \varepsilon) = 0$$

for all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. We denote by $v_+(\varepsilon)$ the locally unique periodic solution bifurcating from $2\pi(k+l)$ and by $v_-(\varepsilon)$ the periodic solution bifurcating from $2\pi k$ for $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. The bound (6.77) is a direct consequence of the smoothness of $v_\pm$.

In the next step, we construct the interface connecting the state $v_-(\varepsilon)$ to $v_+(\varepsilon)$. Accounting for the fact that the potential breaks the translational symmetry of (6.71), we impose the ansatz

$$u = v_-(\varepsilon)\chi_- + v_+(\varepsilon)\chi_+ + u_{0,k,l,\varsigma} - 2\pi k\chi_- - 2\pi(k+l)\chi_+ + w \tag{6.78}$$

for the desired front solution to (6.72), where $w \in H^2(\mathbb{R})$ is a small correction term and $u_{0,k,l,\varsigma} = u_{0,k,l}(\cdot - \varsigma)$ is the shifted front solution to (6.72). Abbreviating $A = \partial_x^2$, $\mathcal{N}(u) = -\sin(u)$, we write the existence problem (6.71) as

$$Au + \varepsilon V u + \mathcal{N}(u) = 0. \tag{6.79}$$

Inserting the ansatz (6.78) into (6.79) leads to an equation for the correction w and the shift parameter ς of the form

$$\mathcal{F}(w, \varepsilon, \varsigma) = 0 \tag{6.80}$$

with

$$\mathcal{F}(w,\varepsilon,\varsigma) = L_\varepsilon(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon))w + R(\varepsilon,\varsigma) + \check{\mathcal{N}}(w,\varepsilon,\varsigma),$$

where we denote $\tilde{v}_-(\varepsilon) = v_-(\varepsilon) - 2\pi k$, $\tilde{v}_+(\varepsilon) = v_+(\varepsilon) - 2\pi(k+l)$ and

$$\begin{aligned} R(\varepsilon,\varsigma) &= A(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon)) + \varepsilon V(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon)) \\ &\quad + \mathcal{N}(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon)), \\ \check{\mathcal{N}}(w,\varepsilon,\varsigma) &= \mathcal{N}(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon) + w) - \mathcal{N}(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon)) \\ &\quad - \mathcal{N}'(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon))w, \end{aligned}$$

for $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\varsigma \in \mathbb{R}$ and $w \in H^2(\mathbb{R})$. We have $\mathcal{F}(0,0,\varsigma_0) = 0$ and $\partial_w \mathcal{F}(0,0,\varsigma_0) = L_0(u_{0,k,l,\varsigma_0})$ for all $\varsigma_0 \in \mathbb{R}$. We recall from §6.8.1.1 that the kernel of $L_0(u_{0,k,l,\varsigma_0})$ is spanned by u'_{0,k,l,ς_0}. In particular, $L_0(u_{0,k,l,\varsigma_0})$ is not invertible. To address this, we employ Lyapunov-Schmidt reduction to solve (6.80). We note that, since $L_0(u_{0,k,l,\varsigma_0})$ is self-adjoint, its range is given by the orthogonal complement $\{u'_{0,k,l,\varsigma_0}\}^\perp$.

We proceed by obtaining bounds on the residual R and the nonlinearity $\check{\mathcal{N}}$. We employ estimate (6.77), rely on the continuous embedding $H^1_{\text{per}}(0,T) \hookrightarrow L^\infty(\mathbb{R})$, apply the mean value theorem twice, and use the identities $A(u_{0,k,l,\varsigma}) = -\mathcal{N}(u_{0,k,l,\varsigma})$ and $Av_-(\varepsilon) + \varepsilon V v_-(\varepsilon) = -\mathcal{N}(v_-(\varepsilon))$ to establish ε- and ς-independent constants $C_{1,2} > 0$ such that the pointwise estimate

$$\begin{aligned}
|R(\varepsilon,\varsigma)(x)| &= |\varepsilon V(x)\left(u_{0,k,l,\varsigma}(x) - 2\pi k\right) + \mathcal{N}\left(u_{0,k,l,\varsigma}(x) + v_-(\varepsilon)(x) - 2\pi k\right) \\
&\qquad - \mathcal{N}(u_{0,k,l,\varsigma}(x)) - (\mathcal{N}(v_-(\varepsilon)(x)) - \mathcal{N}(2\pi k))| \\
&\leq \varepsilon\|V\|_{L^\infty}|u_{0,k,l,\varsigma}(x) - 2\pi k| \\
&\quad + \sup_{|z|\leq\|u_{0,k,l,\varsigma}-2\pi k\|_{L^\infty}} |\mathcal{N}'(v_-(\varepsilon)(x)+z) - \mathcal{N}'(2\pi k + z)|\,|u_{0,k,l,\varsigma}(x) - 2\pi k| \\
&\leq \varepsilon\|V\|_{L^\infty}|u_{0,k,l,\varsigma}(x) - 2\pi k| + C_1\|v_-(\varepsilon) - 2\pi k\|_{L^\infty}|u_{0,k,l,\varsigma}(x) - 2\pi k| \\
&\leq \varepsilon C_2|u_{0,k,l,\varsigma}(x) - 2\pi k|
\end{aligned}$$

holds for $x \leq -1$, $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and $\varsigma \in \mathbb{R}$. Similarly, we find an ε- and ς-independent constant $C_3 > 0$ such that

$$|R(\varepsilon,\varsigma)(x)| \leq \varepsilon C_3|u_{0,k,l,\varsigma}(x) - 2\pi(k+l)|$$

holds for $x \geq 1$, $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and $\varsigma \in \mathbb{R}$. Moreover, using estimate (6.77), the continuous embedding $H^1_{\text{per}}(0,T) \hookrightarrow L^\infty(\mathbb{R})$, the mean value theorem, and the identities $A(u_{0,k,l,\varsigma}) = -\mathcal{N}(u_{0,k,l,\varsigma})$ and $Av_\pm(\varepsilon) = -\varepsilon V v_\pm(\varepsilon) - \mathcal{N}(v_\pm(\varepsilon))$, we obtain an ε- and ς-independent constant $C_4 > 0$ such that

$$\begin{aligned}
|R(\varepsilon,\varsigma)(x)| \leq \|A(\chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon))\|_{L^\infty} + \varepsilon\|V\|_{L^\infty}\|u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon)\|_{L^\infty} \\
+ \|\mathcal{N}(u_{0,k,l,\varsigma} + \chi_-\tilde{v}_-(\varepsilon) + \chi_+\tilde{v}_+(\varepsilon)) - \mathcal{N}(u_{0,k,l,\varsigma})\|_{L^\infty} \leq \varepsilon C_4
\end{aligned}$$

holds for $x \in [-1,1]$, $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and $\varsigma \in \mathbb{R}$. Combining the latter three estimates, we establish for each compact ε-independent subset $\mathcal{K} \subset \mathbb{R}$, an ε- and ς-independent constant $C_0 > 0$ such that

$$\|R(\varepsilon,\varsigma)\|_{L^2} \leq C_0|\varepsilon| \tag{6.81}$$

for $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and $\varsigma \in \mathcal{K}$. On the other hand, it follows from Taylor theorem, the estimate (6.77), and the continuous embedding $H^1_{\text{per}}(0,T) \hookrightarrow L^\infty(\mathbb{R})$, that there exists an ε- and ς-independent constant $C > 0$ such that

$$\|\check{\mathcal{N}}(w, \varepsilon, \varsigma)\|_{L^2} \leq C\|w\|^2_{H^2} \tag{6.82}$$

for $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\varsigma \in \mathbb{R}$, and $w \in H^2(\mathbb{R})$ with $\|w\|_{H^2} \leq 1$.

Our next step is to use Lyapunov-Schmidt reduction to solve equation (6.80). The reduction relies on a parameter-dependent decomposition of the spaces $H^2(\mathbb{R})$ and $L^2(\mathbb{R})$, which is induced by the orthogonal projections $P_\varsigma \colon H^\ell(\mathbb{R}) \to H^\ell(\mathbb{R})$ and $P_\varsigma^\perp := I - P_\varsigma$ given by

$$P_\varsigma w = \frac{\langle u'_{0,k,l,\varsigma}, w\rangle_{L^2}}{\|u'_{0,k,l,\varsigma}\|^2_{L^2}} u'_{0,k,l,\varsigma}$$

for $\ell \in \mathbb{N}_0$ and $\varsigma \in \mathbb{R}$. We decompose the problem (6.80) into a regular and singular part which we complement with a phase condition, which leads to the equivalent problem

$$P_\varsigma^\perp L_\varepsilon(u_{0,k,l,\varsigma} + \chi_- \tilde{v}_-(\varepsilon) + \chi_+ \tilde{v}_+(\varepsilon)) P_\varsigma^\perp w + P_\varsigma^\perp \left(R(\varepsilon, \varsigma) + \check{\mathcal{N}}(w, \varepsilon, \varsigma)\right) = 0, \tag{6.83}$$

$$P_\varsigma L_\varepsilon(u_{0,k,l,\varsigma} + \chi_- \tilde{v}_-(\varepsilon) + \chi_+ \tilde{v}_+(\varepsilon)) P_\varsigma^\perp w + P_\varsigma \left(R(\varepsilon, \varsigma) + \check{\mathcal{N}}(w, \varepsilon, \varsigma)\right) = 0, \tag{6.84}$$

$$P_\varsigma w = 0. \tag{6.85}$$

First, we solve the equations (6.83) and (6.85), corresponding to the regular part of the system. To this end, we define the smooth nonlinear operator $\mathcal{G} \colon H^2(\mathbb{R}) \times (-\varepsilon_1, \varepsilon_1) \times \mathbb{R} \to L^2(\mathbb{R})$ by

$$\mathcal{G}(w, \varepsilon, \varsigma) = P_\varsigma^\perp L_\varepsilon(u_{0,k,l,\varsigma} + \chi_- \tilde{v}_-(\varepsilon) + \chi_+ \tilde{v}_+(\varepsilon)) P_\varsigma^\perp w + P_\varsigma^\perp \left(R(\varepsilon, \varsigma) + \check{\mathcal{N}}(w, \varepsilon, \varsigma)\right) + P_\varsigma w.$$

We observe that $\mathcal{G}(w, \varepsilon, \varsigma) = 0$ is equivalent to the equations (6.83) and (6.85). We compute

$$\mathcal{G}(0, 0, \varsigma) = 0, \qquad \partial_w \mathcal{G}(0, 0, \varsigma) = P_\varsigma^\perp L_0(u_{0,k,l,\varsigma}) P_\varsigma^\perp + P_\varsigma$$

for $\varsigma \in \mathbb{R}$. The derivative $\partial_w \mathcal{G}(0, 0, \varsigma)$ is invertible in the space of bounded linear operators from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ for $\varsigma \in \mathbb{R}$. Therefore, the implicit function theorem yields $\varepsilon_2 \in (0, \varepsilon_1)$, an open neighborhood $W \subset H^2(\mathbb{R})$ of 0, and a smooth map $w \colon (-\varepsilon_2, \varepsilon_2) \times \mathbb{R} \to W$ such that the triple $(w, \varepsilon, \varsigma) \in U \times (-\varepsilon_2, \varepsilon_2) \times \mathbb{R}$ solves

$$\mathcal{G}(w, \varepsilon, \varsigma) = 0$$

if and only if $(w, \varepsilon, \varsigma) = (w(\varepsilon, \varsigma), \varepsilon, \varsigma)$ for $(\varepsilon, \varsigma) \in (-\varepsilon_2, \varepsilon_2) \times \mathbb{R}$. Moreover, we have $w(0, \varsigma) = 0$ for $\varsigma \in \mathbb{R}$. We plug the solution of equations (6.83) and (6.85) into (6.84) to arrive at the reduced problem

$$g(\varepsilon, \varsigma) = 0,$$

where $g\colon (-\varepsilon_2, \varepsilon_2) \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} &g(\varepsilon, \varsigma) \\ &= \left\langle u_{0,k,l,\varsigma}', L_\varepsilon\left(u_{0,k,l,\varsigma} + \chi_- \tilde{v}_-(\varepsilon) + \chi_+ \tilde{v}_+(\varepsilon)\right) P_\varsigma^\perp w(\varepsilon, \varsigma) + R(\varepsilon, \varsigma) + \check{\mathcal{N}}(w(\varepsilon, \varsigma), \varepsilon, \varsigma) \right\rangle_{L^2}. \end{aligned}$$

We solve the equation by desingularizing the smooth function g. To this end, we define the function $\tilde{g}\colon (-\varepsilon_2, \varepsilon_2) \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{g}(\varepsilon, \varsigma) = \begin{cases} \frac{g(\varepsilon, \varsigma)}{\varepsilon}, & \varepsilon \neq 0, \\ \partial_\varepsilon g(0, \varsigma), & \varepsilon = 0. \end{cases}$$

We observe that $\tilde{g}$ is smooth and obeys $\tilde{g}(0, \varsigma_0) = \partial_\varepsilon g(0, \varsigma_0) = V_{\text{eff}}(\varsigma_0) = 0$ and $\partial_\varsigma \tilde{g}(0, \varsigma_0) = \partial_\varsigma \partial_\varepsilon g(0, \varsigma_0) = V_{\text{eff}}'(\varsigma_0) \neq 0$. The implicit function theorem yields $\varepsilon_3 \in (0, \varepsilon_2)$ and a smooth function $\varsigma\colon (-\varepsilon_3, \varepsilon_3) \to \mathbb{R}$ such that $\varsigma(0) = \varsigma_0$ and

$$\tilde{g}(\varepsilon, \varsigma(\varepsilon)) = 0$$

for all $\varepsilon \in (-\varepsilon_3, \varepsilon_3)$. Consequently, the triple $(w(\varepsilon, \varsigma(\varepsilon)), \varepsilon, \varsigma(\varepsilon))$ solves the system (6.83)-(6.85) for all $\varepsilon \in (-\varepsilon_3, \varepsilon_3)$. We conclude that $u\colon (-\varepsilon_3, \varepsilon_3) \to L^\infty(\mathbb{R})$ given by

$$u(\varepsilon) = v_-(\varepsilon)\chi_- + v_+(\varepsilon)\chi_+ + u_{0,k,l,\varsigma(\varepsilon)} - 2\pi k \chi_- - 2\pi(k + l)\chi_+ + w(\varepsilon, \varsigma(\varepsilon)),$$

is the desired front solution to (6.70), which satisfies (6.76).

It remains to prove the assertions on the spectrum of the linearization operator $L_\varepsilon(u(\varepsilon))$. First, we recall from §6.8.1.1 that there exists $\varrho > 0$ such that

$$\sigma(L_0(u(0))) \subset (-\infty, -\varrho) \cup \{0\}, \tag{6.86}$$

where $0 \in \sigma(L_0(u(0)))$ is a simple eigenvalue. On the other hand, estimate (6.76) implies that $\|u(\varepsilon)\|_{L^\infty}$ is bounded by an ε-independent constant for $\varepsilon \in (-\varepsilon_3, \varepsilon_3)$. Consequently, there exists by Lemma 6.24 an ε-independent constant $\varrho_1 > 0$ such that $\sigma(L_\varepsilon(u(\varepsilon))) \subset (-\infty, \varrho_1]$. Combining the latter with the fact that $L_\varepsilon(u(\varepsilon)) - L_0(u(0))\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded operator, we infer by [80, Theorem IV.3.18] and estimate (6.86) that there exists $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\sigma(L_\varepsilon(u(\varepsilon))) \subset (-\infty, -\varrho) \cup \{\lambda_0(\varepsilon)\}$$

for all $\varepsilon \in (-\varepsilon_4, \varepsilon_4)$, where $\lambda_0(\varepsilon) \in \mathbb{R}$ is again a simple eigenvalue of $L_\varepsilon(u(\varepsilon))$. By [83, Proposition I.7.2] there exist $\varepsilon_5 \in (0, \varepsilon_4)$ and C^1-curves $\lambda_0 \colon (-\varepsilon_5, \varepsilon_5) \to \mathbb{R}$ and $z \colon (-\varepsilon_5, \varepsilon_5) \to H^2(\mathbb{R})$ with $z(0) = 0$ and $\lambda_0(0) = 0$ solving the eigenvalue problem

$$L_\varepsilon(u(\varepsilon))(u'_{0,k,l,\varsigma_0} + z(\varepsilon)) = \lambda_0(\varepsilon)(u'_{0,k,l,\varsigma_0} + z(\varepsilon))$$

for $\varepsilon \in (-\varepsilon_5, \varepsilon_5)$. Taking the derivative on both sides with respect to ε and evaluating at $\varepsilon = 0$ yields

$$L_0(u_{0,k,l,\varsigma_0})\partial_\varepsilon z(0) + V u'_{0,k,l,\varsigma_0} + \mathcal{N}''(u_{0,k,l,\varsigma_0})[\partial_\varepsilon u(0), u'_{0,k,l,\varsigma_0}] = \lambda_0'(0) u'_{0,k,l,\varsigma_0}. \tag{6.87}$$

On the other hand, differentiating the equation

$$Au(\varepsilon) + \varepsilon V u(\varepsilon) + \mathcal{N}(u(\varepsilon)) = 0$$

with respect to x and ε and subsequently setting $\varepsilon = 0$, we obtain

$$L_0(u_{0,k,l,\varsigma_0})\partial_\varepsilon \partial_x u(0) + V' u_{0,k,l,\varsigma_0} + V u'_{0,k,l,\varsigma_0} + \mathcal{N}''(u_{0,k,l,\varsigma_0})[\partial_\varepsilon u(0), u'_{0,k,l,\varsigma_0}] = 0. \tag{6.88}$$

Subtracting (6.88) from (6.87), we arrive at

$$L_0(u_{0,k,l,\varsigma_0})(\partial_\varepsilon z(0) - \partial_\varepsilon \partial_x u(0)) - V' u_{0,k,l,\varsigma_0} = \lambda_0'(0) u'_{0,k,l,\varsigma_0}.$$

Taking the L^2-scalar product of the last equation with $u'_{0,k,l,\varsigma_0} \in \ker(L_0(u_{0,k,l,\varsigma_0}))$, we establish

$$\lambda_0'(0)\|u'_{0,k,l,\varsigma_0}\|_{L^2}^2 = -\int_{\mathbb{R}} V'(x) u_{0,k,l,\varsigma_0}(x) u'_{0,k,l,\varsigma_0}(x) \mathrm{d}x = -V'_{\text{eff}}(\varsigma_0),$$

which finishes the proof. □

We observe that the front solutions, established in Theorem 6.25, obey the assumptions **(H1)** and **(H3)**. Therefore, given any collection of M such fronts with matching asymptotic end states, Theorems 6.5 and 6.14 and Corollary 6.13 yield the existence and spectral stability of multifront solutions bifurcating from the formal concatenation of these M fronts, see Figure 6.5.

Corollary 6.26. *Let $M \in \mathbb{N}$. Let $\{u_j\}_{j=1}^M$ be a sequence of front solutions to* (6.71)*, established in Theorem 6.25, with end states $v_{j,\pm}(\varepsilon)$. Assume that it holds $v_{j,+}(\varepsilon) = v_{j+1,-}(\varepsilon)$ for $j = 1, \ldots, M-1$. Then, there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq N$ there exists a nondegenerate stationary multifront solution $\tilde{u}_n$ to* (6.70) *of the form*

$$\tilde{u}_n = a_n + \sum_{j=1}^{M} \chi_{j,n} u_j(\cdot - jnT),$$

where $\chi_{j,n}, j = 1, \ldots, M$ is the smooth partition of unity defined in §6.3, and $\{a_n\}_n$ is a sequence in $H^2(\mathbb{R})$ converging to 0 *as $n \to \infty$. If the fronts u_j are strongly spectrally stable for $j = 1, \ldots, M$, then so is the multifront $\tilde{u}_n$. Moreover, if there exists $j_0 \in \{1, \ldots, M\}$ such that u_{j_0} is spectrally unstable, then so is $\tilde{u}_n$.*

Proof. Theorems 6.5 and 6.25 yield the existence of the multifronts $\tilde{u}_n$. Since the primary fronts u_j are nondegenerate for $j = 1, \ldots, M$, Theorem 6.14 yields that the multifront $\tilde{u}_n$ is also nondegenerate. Due to the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, $\|\tilde{u}_n\|_{L^\infty}$ is bounded by an n-independent constant. Therefore, the statements about the spectral (in)stability of the multifront $\tilde{u}_n$ follow from Corollary 6.13, Theorem 6.14, and Lemma 6.24. □

Applying Theorem 6.8 to the nondegenerate multifront solutions established in Corollary 6.26, we find that multifronts connecting to the same periodic end state at $\pm\infty$ are accompanied by large wavelength periodic multipulse solutions. Their spectral stability follows from Corollary 6.21, Theorem 6.22, and Lemma 6.24.

Corollary 6.27. *Let u be a multifront solution to* (6.71)*, as established in Corollary 6.26. Assume that u connects to the same periodic end state $v \in H^2_{per}(0,T)$ as $x \to \pm\infty$. Then, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ there exists a stationary nT-periodic solution u_n to* (6.70) *given by*

$$u_n(x) = \chi_n(x)u(x) + (1 - \chi_n(x))v(x) + a_n(x), \qquad x \in \left[-\tfrac{n}{2}T, \tfrac{n}{2}T\right),$$

where χ_n is the cut-off function from Theorem 6.8, and $\{a_n\}_n$ is a sequence with $a_n \in H^2_{per}(0,nT)$ satisfying $\|a_n\|_{H^2_{per}(0,nT)} \to 0$ as $n \to \infty$. Moreover, if u is strongly spectrally stable, then so is u_n. Finally, if u is spectrally unstable, then so is u_n.

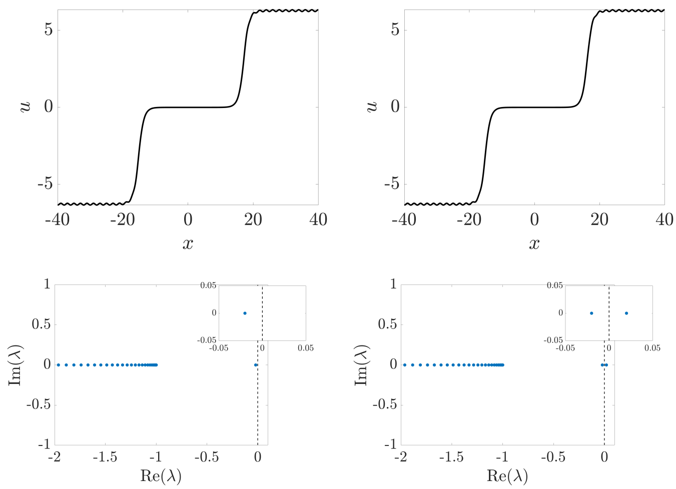

Figure 6.5: Approximations of stationary 2-front solutions to (6.70), along with their spectra, for system coefficients $\varepsilon = 0.1$ and $V(x) = \cos(\pi x)$. The insets provide a closer view of the small eigenvalues near zero. Left: a strongly spectrally stable 2-front solution obtained through numerical continuation by starting from the formal concatenation of the strongly spectrally stable front solutions depicted in the left and middle panels of Figure 6.4. Right: a spectrally unstable 2-front obtained through numerical continuation by starting from the formal concatenation of a strongly spectrally stable and a spectrally unstable 1-front solution (left and right panels of Figure 6.4).

6.8.2 A Klausmeier reaction-diffusion-advection system

We consider a Klausmeier-type model with spatially periodic coefficients as an example for a 2-component reaction-diffusion-advection system to which our theory applies. Using our methods, we rigorously establish the existence of strongly spectrally stable stationary multipulse solutions and corresponding periodic pulse solutions. These results extend the recent findings in [13], where stationary 1-pulse solutions to this Klausmeier model were constructed using singular perturbation theory and their stability was analyzed. The system of equations reads

$$\begin{aligned} \partial_t w &= \partial_x^2 w + \varepsilon\big(f(x)\partial_x w + g(x)w\big) - w - wp^2 + a, \\ \partial_t p &= d^2\partial_x^2 p - mp + wp^2, \end{aligned} \qquad \begin{pmatrix} w(x,t) \\ p(x,t) \end{pmatrix} \in \mathbb{R}^2,\ x \in \mathbb{R},\ t \geq 0$$

with parameters $d, a, m, \varepsilon > 0$ and real-valued functions $f, g \in C^1(\mathbb{R})$ with period $T > 0$. This model is employed in ecology to describe the dynamics of vegetation patterns resulting from the interaction between water w and plants p across a spatially heterogeneous terrain with periodic topography modeled by the functions f and g. Here, $d > 0$ is a diffusion coefficient, a models the amount of rain fall, $1/m$ is a quantity corresponding to the life time of plants, and $\varepsilon > 0$ measures the influence of the terrain on the vegetation dynamics. For more background on the model, including the role of the individual parameters and the functions f and g, we refer to [13] and references therein.

To fit the system in our framework, we set $\mathbf{u} = (w,p)^\top$ and write the equations as

$$\partial_t \mathbf{u} = A_\varepsilon \mathbf{u} + \mathcal{N}_\varepsilon(\mathbf{u}, x) \tag{6.89}$$

with

$$A_\varepsilon = \begin{pmatrix} \partial_x^2 + \varepsilon f \partial_x & 0 \\ 0 & d^2 \partial_x^2 \end{pmatrix}, \qquad \mathcal{N}_\varepsilon\left(\begin{pmatrix} w \\ p \end{pmatrix}, x\right) = \begin{pmatrix} \varepsilon g(x) w - w - wp^2 + a \\ -mp + wp^2 \end{pmatrix}.$$

The existence problem for stationary solutions is then given by

$$A_\varepsilon \mathbf{u} + \mathcal{N}_\varepsilon(\mathbf{u}, \cdot) = 0, \tag{6.90}$$

which is of the form (6.4). The associated linearization operator $L_\varepsilon(\underline{\mathbf{u}})\colon D(L_\varepsilon(\underline{\mathbf{u}})) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with dense domain $D(L_\varepsilon(\underline{\mathbf{u}})) = H^2(\mathbb{R})$ is given by

$$L_\varepsilon(\underline{\mathbf{u}}) = A_\varepsilon + \partial_u \mathcal{N}_\varepsilon(\underline{\mathbf{u}}, \cdot)$$

for $\underline{\mathbf{u}} \in L^\infty(\mathbb{R})$. Given that the ε-independent principal part A_0 of the operator $L_\varepsilon(\underline{\mathbf{u}})$ is sectorial, and the remainder $L_\varepsilon(\underline{\mathbf{u}}) - A_0$ is relatively A_0-bounded, cf. [44, Definition III.2.1], we obtain the following spectral a-priori bound.

Lemma 6.28. *Let $f, g \in L^\infty(\mathbb{R})$ and $C, d, a, m, \varepsilon_0 > 0$. Then, there exists a constant $\varrho > 0$, depending only on $\|f\|_{L^\infty}, \|g\|_{L^\infty}, C, d, a, m$ and ε_0, such that we have*

$$\sigma(L_\varepsilon(\underline{\mathbf{u}})) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -1\} \subset \overline{B}_0(\varrho)$$

for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and each $\underline{\mathbf{u}} \in L^\infty(\mathbb{R})$ with $\|\underline{\mathbf{u}}\|_{L^\infty} \leq C$. Here, $\overline{B}_0(\varrho)$ is the closed ball of radius ϱ centered at the origin.

Proof. On the one hand, it is well-known that the diagonal diffusion operator A_0 generates a bounded analytic semigroup, cf. [44, Example II.4.10]. On the other hand, the exposition in [44, Example III.2.2] demonstrates that the residual $L_\varepsilon(\underline{\mathbf{u}}) - A_0$ is relatively A_0-bounded with A_0-bound 0. The result then follows directly from [44, Lemma III.2.6] and its proof. □

The main goal of this section is to show that (6.89) admits stationary (periodic) multipulse solutions corresponding to (periodic sequences of) localized vegetation patches. The fundamental building blocks of these multiple pulse solutions are nondegenerate 1-pulse solutions. The existence and spectral stability of stationary 1-pulse solutions to (6.89) for small $\varepsilon > 0$ were established in [13] for a broad class of heterogeneities f and g. However, the spectral analysis in [13] assumes that f and g are localized, leaving the spectral stability and nondegeneracy of the 1-pulse solutions unaddressed for periodic f and g.

To bridge this gap, we begin by establishing the existence and spectral stability of nondegenerate stationary 1-pulse solutions to (6.89) for small $\varepsilon > 0$ and periodic f and g. This is achieved by bifurcating from even 1-pulse solutions to the unperturbed problem

$$A_0\mathbf{u} + \mathcal{N}_0(\mathbf{u}, \cdot) = 0, \tag{6.91}$$

which were constructed in [13, Theorem 2.20] using geometric singular perturbation theory in the regime $\nu := a/m \ll 1$, $d\nu^2 m^{1/2}, d/\nu^2 \leq C$ for some ν-independent constant $C > 0$. Since these solutions correspond to homoclinics to a hyperbolic equilibrium in (6.91), they are exponentially localized, see Figure 6.6. Moreover, it follows from [13, Theorem 3.2] that they are spectrally stable with simple eigenvalue $\lambda = 0$ as in Definition 6.4. With the aid of the implicit function theorem, we prove that these 1-pulse solutions persist for $\varepsilon > 0$ in case f is odd and g is even. Moreover, we show that their spectral stability is determined by the sign of a Melnikov-type integral.

Theorem 6.29. *Let $T, d, a, m > 0$. Let $f \in C^1(\mathbb{R})$ be T-periodic and odd. Let $g \in C^1(\mathbb{R})$ be T-periodic and even. Let $\mathbf{u}_0 = \mathbf{z}_0 + \mathbf{v}_0 \in H^2(\mathbb{R}) \oplus \mathbb{R}^2$ be an even solution to* (6.89) *for $\varepsilon = 0$, which is spectrally stable with simple eigenvalue $\lambda = 0$. Assume that the Melnikov integral*

$$\mathcal{M} = \int_{\mathbb{R}} \big(f'(x)\partial_x \mathbf{u}_{01}(x) + g'(x)\mathbf{u}_{01}(x)\big)\Psi_{\mathrm{ad}1}(x)\mathrm{d}x$$

does not vanish, where $\Psi_{\mathrm{ad}} \in H^2(\mathbb{R})$ spans the kernel of the adjoint operator $L_0(\mathbf{u}_0)^*$ and satisfies $\langle \partial_x \mathbf{u}_0, \Psi_{\mathrm{ad}} \rangle_{L^2} = 1$.

Then, there exist constants $C, \varepsilon_0, \eta > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ there exists a nondegenerate even solution $\mathbf{u}(\varepsilon) = \mathbf{z}(\varepsilon) + \mathbf{v}(\varepsilon) \in H^2(\mathbb{R}) \oplus H^2_{per}(0,T)$ to (6.89) *satisfying*

$$\|\mathbf{u}(\varepsilon) - \mathbf{u}_0\|_{L^\infty} \leq C\varepsilon, \qquad \|\mathbf{v}(\varepsilon) - \mathbf{v}_0\|_{H^2_{per}(0,T)} \leq C\varepsilon. \tag{6.92}$$

Furthermore, we have

$$\sigma(L_\varepsilon(\mathbf{u}(\varepsilon)) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\eta\} \cup \{\lambda_0(\varepsilon)\}$$

for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*, where* $\lambda_0(\varepsilon) \in \mathbb{R}$ *is a real simple eigenvalue of* $L_\varepsilon(\mathbf{u}(\varepsilon))$ *obeying the expansion*

$$\lambda_0(\varepsilon) = -\varepsilon\mathcal{M} + \mathcal{O}(\varepsilon^2).$$

Proof. We start with the construction of the periodic background wave $\mathbf{v}(\varepsilon)$ by perturbing from the rest state $\mathbf{v}_0$ in (6.91). The smooth function $\mathcal{F}\colon H^2_{\text{even,per}}(0,T) \times \mathbb{R} \to L^2_{\text{even,per}}(0,T)$ given by

$$\mathcal{F}(\mathbf{v}, \varepsilon) = A_\varepsilon \mathbf{v} + \mathcal{N}_\varepsilon(\mathbf{v}, \cdot)$$

obeys $\mathcal{F}(\mathbf{v}_0, 0) = 0$ and $\partial_{\mathbf{v}}\mathcal{F}(\mathbf{v}_0, 0) = L_0(\mathbf{v}_0)$. Since $\mathbf{u}_0$ is spectrally stable with simple eigenvalue $\lambda = 0$, the essential spectrum of $L_0(\mathbf{u}_0)$ is confined to the open left-half plane, which, by Proposition 6.11, is given by $\sigma_{\text{ess}}(L_0(\mathbf{u}_0)) = \sigma(L_0(\mathbf{v}_0))$. Therefore, $\partial_{\mathbf{v}}\mathcal{F}(\mathbf{v}_0, 0)$ is invertible as an operator from $H^2_{\text{even,per}}(0,T)$ into $L^2_{\text{even,per}}(0,T)$. An application of the implicit function theorem yields $\varepsilon_1 > 0$ and a smooth map $\mathbf{v}\colon (-\varepsilon_1, \varepsilon_1) \to H^2_{\text{even,per}}(0,T)$ with $\mathbf{v}(0) = \mathbf{v}_0$ such that

$$\mathcal{F}(\mathbf{v}(\varepsilon), \varepsilon) = 0$$

for all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$.

Substituting the ansatz $\mathbf{u} = \mathbf{z} + \mathbf{v}(\varepsilon)$ with $\mathbf{z} \in H^2_{\text{even}}(\mathbb{R})$ into (6.90), we arrive at

$$0 = A_\varepsilon(\mathbf{z} + \mathbf{v}(\varepsilon)) + \mathcal{N}_\varepsilon(\mathbf{z} + \mathbf{v}(\varepsilon), \cdot) = A_\varepsilon \mathbf{z} + \mathcal{N}_\varepsilon(\mathbf{z} + \mathbf{v}(\varepsilon), \cdot) - \mathcal{N}_\varepsilon(\mathbf{v}(\varepsilon), \cdot).$$

Clearly, the smooth nonlinear operator $\mathcal{G}\colon H^2_{\text{even}}(\mathbb{R}) \times (-\varepsilon_1, \varepsilon_1) \to L^2_{\text{even}}(\mathbb{R})$ given by

$$\mathcal{G}(\mathbf{z}, \varepsilon) = A_\varepsilon \mathbf{z} + \mathcal{N}_\varepsilon(\mathbf{z} + \mathbf{v}(\varepsilon), \cdot) - \mathcal{N}_\varepsilon(\mathbf{v}(\varepsilon), \cdot)$$

is well-defined and satisfies $\mathcal{G}(\mathbf{z}_0, 0) = 0$ and $\partial_{\mathbf{z}}\mathcal{G}(\mathbf{z}_0, 0) = L_0(\mathbf{u}_0)|_{H^2_{\text{even}}}$, where $L_0(\mathbf{u}_0)|_{H^2_{\text{even}}}$ is the restriction of the linear operator $L_0(\mathbf{u}_0)\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ to the subspace $H^2_{\text{even}}(\mathbb{R}) \subset H^2(\mathbb{R})$ of even functions. Since $\mathbf{u}_0$ is spectrally stable with simple eigenvalue $\lambda = 0$, the kernel of $L_0(\mathbf{u}_0)$ is spanned by the odd function $\partial_x \mathbf{u}_0$. Therefore, $\partial_{\mathbf{z}}\mathcal{G}(\mathbf{z}_0, 0) = L_0(\mathbf{u}_0)|_{H^2_{\text{even}}}$ is invertible. Hence, the implicit function theorem affords $\varepsilon_2 \in (0, \varepsilon_1)$ and a smooth map $\mathbf{z}\colon (-\varepsilon_2, \varepsilon_2) \to H^2_{\text{even}}(\mathbb{R})$ with $\mathbf{z}(0) = \mathbf{z}_0$ such that

$$\mathcal{G}(\mathbf{z}(\varepsilon), \varepsilon) = 0$$

for all $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$. In particular, we obtain a smooth map $\mathbf{u}\colon (-\varepsilon_2, \varepsilon_2) \to H^2(\mathbb{R}) \oplus H^2_{\text{per}}(0,T)$ such that $\mathbf{u}(\varepsilon) = \mathbf{z}(\varepsilon) + \mathbf{v}(\varepsilon)$ is an even solution to (6.90). The bounds (6.92) follows by the smoothness of $\mathbf{u}$ and $\mathbf{v}$ and the continuous embeddings $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and $H^1_{\text{per}}(0,T) \hookrightarrow L^\infty(\mathbb{R})$.

Next, we establish the nondegeneracy of $\mathbf{u}(\varepsilon)$ as well as its spectral (in)stability. To this end, we begin by tracing the simple eigenvalue of $L_\varepsilon(\mathbf{u}(\varepsilon))$ converging to 0 as $\varepsilon \to 0$. Since $L_0(\mathbf{u}_0)$ is spectrally stable with simple eigenvalue $\lambda = 0$, it follows from [83, Proposition I.7.2] that there exist $\varepsilon_3 \in (0, \varepsilon_2)$ and C^1-curves $\lambda_0 \colon (-\varepsilon_3, \varepsilon_3) \to \mathbb{R}$ and $\mathbf{w} \colon (-\varepsilon_3, \varepsilon_3) \to H^2(\mathbb{R})$ with $\mathbf{w}(0) = 0$ and $\lambda_0(0) = 0$ solving the eigenvalue problem

$$L_\varepsilon(\mathbf{u}(\varepsilon))(\partial_x \mathbf{u}_0 + \mathbf{w}(\varepsilon)) = \lambda_0(\varepsilon)(\partial_x \mathbf{u}_0 + \mathbf{w}(\varepsilon)).$$

Taking the derivative on both sides with respect to ε and evaluating at $\varepsilon = 0$ yields

$$L_0(\mathbf{u}_0)\partial_\varepsilon \mathbf{w}(0) + (f\partial_x + g)\begin{pmatrix}\partial_x \mathbf{u}_{01} \\ 0\end{pmatrix} + \partial_{\mathbf{uu}}\mathcal{N}_0(\mathbf{u}_0, \cdot)[\partial_\varepsilon \mathbf{u}(0), \partial_x \mathbf{u}_0] = \lambda_0'(0)\partial_x \mathbf{u}_0, \quad (6.93)$$

where we denote $\mathbf{u}_0 = (\mathbf{u}_{01}, \mathbf{u}_{02})^\top$. On the other hand, differentiating the equation

$$A_\varepsilon \mathbf{u}(\varepsilon) + \mathcal{N}_\varepsilon(\mathbf{u}(\varepsilon), \cdot) = 0$$

with respect to x and ε and subsequently setting $\varepsilon = 0$, we obtain

$$\begin{aligned} L_0(\mathbf{u}_0)\partial_\varepsilon \partial_x \mathbf{u}_0 &+ (f\partial_x + g)\begin{pmatrix}\partial_x \mathbf{u}_{01} \\ 0\end{pmatrix} \\ &+ \partial_{\mathbf{uu}}\mathcal{N}_0(\mathbf{u}_0, \cdot)[\partial_\varepsilon \mathbf{u}(0), \partial_x \mathbf{u}_0] + (f'\partial_x + g')\begin{pmatrix}\mathbf{u}_{01} \\ 0\end{pmatrix} = 0. \quad (6.94) \end{aligned}$$

Subtracting (6.94) from (6.93), we find

$$L_0(\mathbf{u}_0)\big(\partial_\varepsilon \mathbf{w}(0) - \partial_\varepsilon \partial_x \mathbf{u}(0)\big) - (f'\partial_x + g')\begin{pmatrix}\mathbf{u}_{01} \\ 0\end{pmatrix} = \lambda_0'(0)\partial_x \mathbf{u}_0.$$

Taking the L^2-scalar product of the last equation with $\Psi_{\text{ad}} \in H^2(\mathbb{R})$, we establish

$$\begin{aligned} \lambda_0'(0) = -\langle (f'\partial_x + g')\mathbf{u}_{01}, \Psi_{\text{ad}1}\rangle_{L^2} &= -\int_{\mathbb{R}} \big(f'(x)\partial_x \mathbf{u}_{01}(x) + g'(x)\mathbf{u}_{01}(x)\big)\Psi_{\text{ad}1}(x)\mathrm{d}x \\ &= -\mathcal{M} \neq 0. \end{aligned}$$

Finally, we prove that the remaining part of the spectrum of $L_\varepsilon(\mathbf{u}(\varepsilon))$ lies in the open left-half plane. Since $\mathbf{u}_0$ is spectrally stable with simple eigenvalue $\lambda = 0$, there exists a constant $\varrho > 0$ such that

$$\sigma(\mathcal{L}_0(\mathbf{u}_0)) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\varrho\} = \{0\}. \tag{6.95}$$

On the other hand, since $\|\mathbf{u}(\varepsilon)\|_{L^\infty}$ is bounded by an ε-independent constant by estimate (6.92), there exists by Lemma 6.28 an ε-independent constant $\varrho_1 > 0$ such that

$$\sigma(\mathcal{L}_\varepsilon(\mathbf{u}(\varepsilon))) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -1\} \subset \overline{B}_0(\varrho_1)$$

for $\varepsilon \in (-\varepsilon_3, \varepsilon_3)$. Combining the latter with the fact that $L_\varepsilon(\mathbf{u}(\varepsilon)) - L_0(\mathbf{u}_0)$ is a bounded operator on $L^2(\mathbb{R})$, [80, Theorem IV.3.18] and (6.95) yield $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\sigma(L_\varepsilon(\mathbf{u}(\varepsilon)) \cap \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\tfrac{1}{2}\varrho\right\}$$

contains exactly one algebraically simple eigenvalue of $L_\varepsilon(\mathbf{u}(\varepsilon))$ for $\varepsilon \in (-\varepsilon_4, \varepsilon_4)$, which is then necessarily given by $\lambda(\varepsilon) \in \mathbb{R}$. This completes the proof. □

Remark 6.30. We emphasize that the Melnikov integral $\mathcal{M}$ in Theorem 6.29 is generically nonvanishing, since the integrand is an even function of x.

Since the 1-pulse solutions, established in Theorem 6.29, satisfy the assumptions **(H1)**-**(H3)**, Theorems 6.5 and 6.14, Corollary 6.13, and Lemma 6.28 yield the existence and spectral stability of bifurcating multipulse solutions.

Corollary 6.31. *Let $M \in \mathbb{N}$. Let $\mathbf{u} \in H^2(\mathbb{R}) \oplus H^2_{per}(0,T)$ be a pulse solution to* (6.90) *as established in Theorem 6.29. Then, there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq N$ there exists a nondegenerate stationary multipulse solution $\tilde{\mathbf{u}}_n$ to* (6.89) *of the form*

$$\tilde{\mathbf{u}}_n = \mathbf{a}_n + \sum_{j=1}^{M} \chi_{j,n}\mathbf{u}(\cdot - jnT),$$

where $\chi_{j,n}, j = 1, \ldots, M$ is the smooth partition of unity defined in §6.3, and $\{\mathbf{a}_n\}_n$ is a sequence in $H^2(\mathbb{R})$ converging to 0 as $n \to \infty$. If the pulse $\mathbf{u}$ is strongly spectrally stable, then so is the multipulse $\tilde{\mathbf{u}}_n$. Moreover, if $\mathbf{u}$ is spectrally unstable, then the same holds for $\tilde{\mathbf{u}}_n$.

Proof. The existence of the multipulse $\tilde{\mathbf{u}}_n$ is a direct consequence of Theorems 6.5 and 6.29. Since the primary pulse $\mathbf{u}$ is nondegenerate, Theorem 6.14 implies that the multipulse $\tilde{\mathbf{u}}_n$ is also

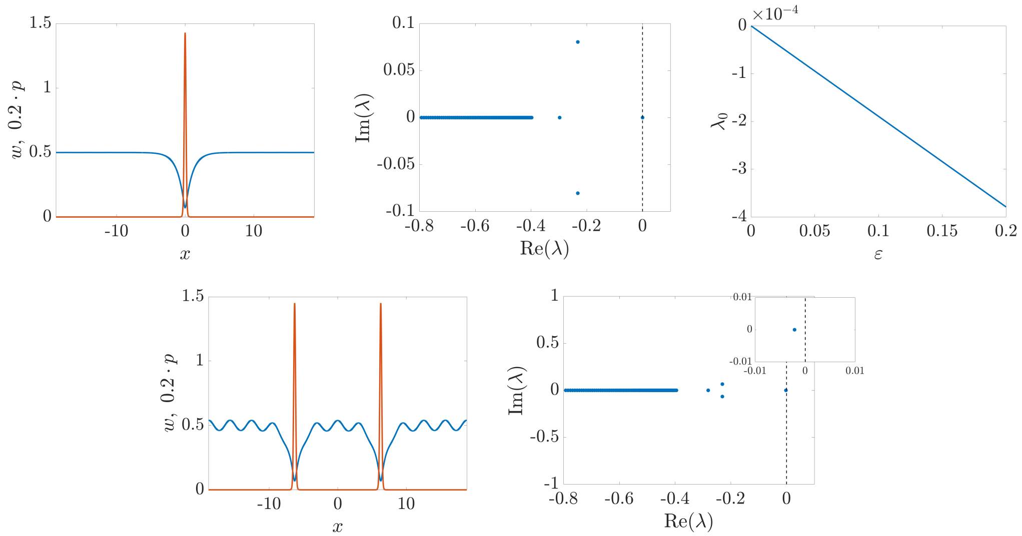

Figure 6.6: A spectrally stable stationary 1-pulse solution to (6.89) for $\varepsilon = 0$ (top left), along with its spectrum (top middle) with simple eigenvalue $\lambda = 0$. The p-component in the left panel is scaled by a factor 0.2 to improve visibility. We continued this 1-pulse solution in ε using the MATLAB package `pde2path` [146] and plotted its critical eigenvalue $\lambda_0(\varepsilon)$ as a function of ε (top right). One observes that the curve $\lambda_0(\varepsilon)$ is to leading order linear, which is in agreement with the expansion of $\lambda_0(\varepsilon)$ provided in Theorem 6.29. The bottom row depicts a strongly spectrally stable stationary 2-pulse solution to (6.89) for $\varepsilon = 1$ (bottom left), along with its spectrum (bottom right). The inset provides a closer view of the small eigenvalues near zero. The 2-pulse is obtained through numerical continuation starting from the superposition of two spectrally stable 1-pulse solutions to (6.89). The system coefficients are $d = 0.04, a = 0.5, m = 0.4, f(x) = 0.2\sin(2x)$ and $g(x) = 0.4\cos(2x)$.

nondegenerate. Thanks to the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, $\|\tilde{\mathbf{u}}_n\|_{L^\infty}$ is bounded by an n-independent constant. So, the assertions about the spectral (in)stability of $\tilde{\mathbf{u}}_n$ follow from Corollary 6.13, Theorem 6.14, and Lemma 6.28. □

The nondegenerate multipulse solutions, established in Corollary 6.31, are accompanied by large wavelength periodic multipulse solutions. Their existence and spectral (in)stability are derived from Theorems 6.8 and 6.22, Corollary 6.21, and Lemma 6.28.

Corollary 6.32. *Let* $\mathbf{u} = \mathbf{z} + \mathbf{v} \in H^2(\mathbb{R}) \oplus H^2_{per}(0,T)$ *be a multipulse solution to* (6.90)*, as established in Corollary 6.31. Then, there exists* $N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *with* $n \geq N$ *there exists a stationary* nT*-periodic solution* $\mathbf{u}_n$ *to* (6.89) *given by*

$$\mathbf{u}_n(x) = \chi_n(x)\mathbf{u}(x) + (1 - \chi_n(x))\mathbf{v}(x) + \mathbf{a}_n(x), \qquad x \in \left[-\tfrac{n}{2}T, \tfrac{n}{2}T\right),$$

where χ_n is the cut-off function from Theorem 6.8, and $\{\mathbf{a}_n\}_n$ is a sequence with $\mathbf{a}_n \in H^2_{per}(0, nT)$ satisfying $\|\mathbf{a}_n\|_{H^2_{per}(0,nT)} \to 0$ as $n \to \infty$. If $\mathbf{u}$ is strongly spectrally stable, then so is $\mathbf{u}_n$. Moreover, if $\mathbf{u}$ is spectrally unstable, then the same holds for $\mathbf{u}_n$.

6.8.3 The Gross-Pitaevskii equation

Let $T > 0$. We consider a nonlinear Schrödinger (NLS) equation with an external potential, which is commonly referred to as Gross-Pitaevskii (GP) equation and arises, for instance, as a mean-field approximation in the study of Bose-Einstein condensates in optical lattices, see [21, 82, 114] and references therein. The Gross-Pitaevskii equation is given by

$$\mathrm{i}\partial_t u = -\partial_x^2 u + \mu V(x)u + \kappa|u|^2 u, \qquad u(x,t) \in \mathbb{C},\, x \in \mathbb{R},\, t \geq 0 \tag{6.96}$$

with parameters $\kappa \in \{\pm 1\}$ and $\mu \in \mathbb{R}$, and real-valued potential $V \in C^1(\mathbb{R})$. In the context of Bose-Einstein condensation, $u(x,t)$ represents a macroscopic wave function, $|u(x,t)|^2$ is its atomic density, $V(x)$ is the external potential created by the optical lattice, μ measures the strength of the potential, and the sign of κ determines whether the nonlinear interaction of the Bose-Einstein condensates is *attractive* ($\kappa = -1$) or *repulse* ($\kappa = 1$). We say that the nonlinearity in (6.96) is *defocusing* if $\kappa = 1$ and *focusing* if $\kappa = -1$. Here, we are interested in *periodic* optical trapping lattices formed by the interference of laser beams. Thus, we consider T-periodic potentials V.

We search for time-harmonic solutions to (6.96). Thus, we insert $u(x,t) = \mathrm{e}^{\mathrm{i}\omega t}\psi(x,t)$ with $\omega \in \mathbb{R}$ into (6.96) to obtain

$$\mathrm{i}\partial_t \psi = -\partial_x^2 \psi + \mu V(x)\psi + \omega\psi + \kappa|\psi|^2\psi. \tag{6.97}$$

Real-valued stationary solutions of (6.97) then satisfy the ordinary differential equation

$$-\partial_x^2 \psi + \mu V(x)\psi + \omega\psi + \kappa\psi^3 = 0,$$

which is of the from (6.4).

In order to study the stability of stationary solutions to (6.97), we write the equation as a system

$$\partial_t \boldsymbol{\psi} = J\left(-\partial_x^2 \boldsymbol{\psi} + \mu V(x)\boldsymbol{\psi} + \omega\boldsymbol{\psi} + \kappa|\boldsymbol{\psi}|^2\boldsymbol{\psi}\right) \tag{6.98}$$

in $\boldsymbol{\psi} = (\mathrm{Re}(\psi), \mathrm{Im}(\psi))^\top$, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is skew-symmetric. System (6.98) exhibits a rotational invariance, i.e., the map $\boldsymbol{\psi} \mapsto R(\gamma)\boldsymbol{\psi}$ with

$$R(\gamma) = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) \\ -\sin(\gamma) & \cos(\gamma) \end{pmatrix}, \qquad \gamma \in \mathbb{R}$$

maps solutions of (6.98) to solutions. The advantage of the formulation (6.98) over (6.97) is that the nonlinearity is differentiable, which allows for linearization about a real-valued stationary solution $\boldsymbol{\psi} = (\psi, 0)^\top$. The associated linearization operator $L_\mu(\underline{\psi})\colon D(L_\mu(\underline{\psi})) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with dense domain $D(L_\mu(\underline{\psi})) = H^2(\mathbb{R})$ is given by

$$L_\mu(\underline{\psi}) = J \begin{pmatrix} L_{+,\mu}(\underline{\psi}) & 0 \\ 0 & L_{-,\mu}(\underline{\psi}) \end{pmatrix}$$

for $\underline{\psi} \in L^\infty(\mathbb{R})$, where $L_{\pm,\mu}(\underline{\psi})\colon D(L_{\pm,\mu}(\underline{\psi})) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $D(L_{\pm,\mu}(\underline{\psi})) = H^2(\mathbb{R})$ are defined by

$$L_{-,\mu}(\underline{\psi})\phi = -\phi'' + \mu V\phi + \omega\phi + \kappa\underline{\psi}^2\phi, \qquad L_{+,\mu}(\underline{\psi})\phi = -\phi'' + \mu V\phi + \omega\phi + 3\kappa\underline{\psi}^2\phi.$$

One observes that the second-order operators $L_{\pm,\mu}(\underline{\psi})$ are self-adjoint and bounded from below. Moreover, the spectrum of $L_\mu(\underline{\psi})$ possesses the Hamiltonian symmetry

$$\lambda \in \sigma(L_\mu(\underline{\psi})) \quad \Rightarrow \quad -\lambda, \overline{\lambda} \in \sigma(L_\mu(\underline{\psi})).$$

Therefore, a real-valued stationary solution ψ to (6.97) can only be spectrally stable if the spectrum of $L_\mu(\psi)$ is confined to the imaginary axis.

Let $n \in \mathbb{N}$. If $\psi \in H^2_{\mathrm{per}}(0, nT)$ is a real-valued nT-periodic stationary solution to (6.97), then $L_\mu(\psi)$ has nT-periodic coefficients and its action on the space $L^2_{\mathrm{per}}(0, nT)$ is well-defined.

Thus, to analyze spectral stability against co-periodic perturbations, we introduce the differential operator $L_{\mu,\mathrm{per}}(\underline{\psi})\colon D(L_{\mu,\mathrm{per}}(\underline{\psi})) \subset L^2_{\mathrm{per}}(0,nT) \to L^2_{\mathrm{per}}(0,nT)$ with dense domain $D(L_{\mu,\mathrm{per}}(\underline{\psi})) = H^2_{\mathrm{per}}(0,nT)$ by

$$L_{\mu,\mathrm{per}}(\underline{\psi}) = J \begin{pmatrix} L_{+,\mu,\mathrm{per}}(\underline{\psi}) & 0 \\ 0 & L_{-,\mu,\mathrm{per}}(\underline{\psi}) \end{pmatrix}$$

for $\underline{\psi} \in H^1(0,nT)$, where the operators $L_{\pm,\mu,\mathrm{per}}(\underline{\psi})\colon D(L_{\pm,\mu,\mathrm{per}}(\underline{\psi})) \subset L^2_{\mathrm{per}}(0,nT) \to L^2_{\mathrm{per}}(0,nT)$ with $D(L_{\pm,\mu,\mathrm{per}}(\underline{\psi})) = H^2_{\mathrm{per}}(0,nT)$ are defined by

$$L_{-,\mu,\mathrm{per}}(\underline{\psi})\phi = -\phi'' + \mu V\phi + \omega\phi + \kappa\underline{\psi}^2\phi, \qquad L_{+,\mu,\mathrm{per}}(\underline{\psi})\phi = -\phi'' + \mu V\phi + \omega\phi + 3\kappa\underline{\psi}^2\phi.$$

We recall from §6.2.4 that the spectrum of $L_{\mu,\mathrm{per}}(\underline{\psi})$ consists of isolated eigenvalues of finite algebraic multiplicity only. Moreover, due to Hamiltonian symmetry, we find that a real-valued nT-periodic stationary solution ψ to (6.97) can only be spectrally stable against co-periodic perturbations if the spectrum of $L_{\mu,\mathrm{per}}(\psi)$ is confined to the imaginary axis.

In the following, we divide our analysis in two parts. In the first part, we consider the defocusing Gross-Pitaevskii equation (6.97) with $\kappa = 1$. Here, we prove the existence of nondegenerate stationary 1-front solutions connecting periodic end states. These 1-fronts correspond to so-called *dark solitons*, established in [6,155]. We apply Theorem 6.5 to obtain stationary multifront solutions to (6.97) lying near formal concatenations of these primary 1-fronts. Theorem 6.8 then yields the existence of periodic solutions bifurcating from the formal periodic extension of these multifronts in case the multifront connects to the same periodic end state at $\pm\infty$. Since 0 lies in the essential spectrum of linearization operator, the spectral stability of these solutions is a subtle and unresolved issue, beyond the scope of this application section. We refer to [110] for a stability analysis of dark solitons in case of a localized potential.

In the second part, we consider the focusing case $\kappa = -1$. We first recall existence results from [109] and references therein, yielding nondegenerate stationary 1-pulse solutions to (6.97). Taking these so-called *gap solitons* as building blocks, we then employ Theorems 6.5 and 6.8 to construct multipulse solutions as well as periodic pulse solutions. Subsequently, we combine Theorem 6.22 with Krein index counting theory [75, 76] to establish spectral stability of the periodic pulse solutions. Our spectral analysis yields orbital stability against co-periodic perturbations through the stability theorem of Grillakis, Shatah and Strauss [64]. Finally, we combine Theorem 6.14 with Sturm-Liouville theory [158] and Krein index counting arguments to derive instability conditions for the multipulse solutions.

6.8.3.1 Fronts in the defocusing Gross-Pitaevskii equation

We consider the defocusing case $\kappa = 1$. Moreover, we assume $\omega < 0$. We search for real-valued stationary multifront solutions to (6.97) connecting periodic states at $\pm\infty$. Real-valued stationary solutions to (6.97) solve the ordinary differential equation

$$-\psi'' + \mu V(x)\psi + \omega\psi + \psi^3 = 0. \tag{6.99}$$

We aim to employ Theorem 6.5 to construct multifront solutions to (6.99) lying near formal concatenations of front solutions with a single interface. These 1-front solutions, known as dark solitons, have been rigorously constructed in [155] by leveraging a comparison principle, see also [142] for the case of a cubic-quintic nonlinearity. However, dark solitons are degenerate solutions to (6.97), obstructing an application of Theorem 6.5. The reason is that a dark soliton $\psi(\mu)$ to (6.99) necessarily connects to nonconstant periodic end states $v_\pm(\mu) \in H^2_{\text{per}}(0,T)$, which obey $L_{-,\text{per},\mu}(v_\pm(\mu))v_\pm(\mu) = 0$, since $v_\pm(\mu)$ must solve (6.99). Hence, we arrive at $0 \in \sigma(L_{-,\mu}(v_\pm(\mu))) \subset \sigma_{\text{ess}}(L_{-,\mu}(\psi(\mu))) \subset \sigma(L_\mu(\psi(\mu)))$ by (6.7) and Proposition 6.11.

Here, we circumvent this issue by regarding dark solitons as nondegenerate stationary 1-front solutions to the real-valued reaction-diffusion problem

$$\partial_t\psi = -\partial_x^2\psi + \mu V(x)\psi + \omega\psi + \psi^3, \qquad \psi(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t \geq 0, \tag{6.100}$$

which obviously admits the same stationary solutions as (6.97). In the following result, we construct nondegenerate odd 1-front solutions to (6.100) in the case of a small potential by perturbing from the black NLS soliton

$$\psi_0(x) = \sqrt{-\omega}\tanh\left(\sqrt{\frac{-\omega}{2}}x\right),$$

which solves (6.99) at $\mu = 0$.

Theorem 6.33. *Let $\omega < 0$ and $T > 0$. Let $V \in C^1(\mathbb{R})$ be even, T-periodic, and real-valued. Let $\chi_\pm\colon \mathbb{R} \to [0,1]$ be a smooth partition of unity such that χ_+ is supported on $(-1,\infty)$, χ_- is supported on $(-\infty,1)$, and we have $\chi_+(x) = \chi_-(-x)$ for all $x \in \mathbb{R}$. Assume*

$$\int_{\mathbb{R}} V'(x)\psi_0(x)\psi_0'(x)\mathrm{d}x \neq 0. \tag{6.101}$$

Then, there exist constants $C, \mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ there exists a nondegenerate stationary odd solution $\psi(\mu)$ to (6.100) *satisfying*

$$\|\psi(\mu) - \psi_0\|_{L^\infty} \leq C\mu, \qquad \chi_\pm(\psi(\mu) - v_\pm(\mu)) \in H^2(\mathbb{R}), \tag{6.102}$$

where $v_\pm(\mu) \in H^2_{per}(0,T)$ are even periodic solutions to (6.99) *obeying the bound*

$$\|v_\pm(\mu) \mp \sqrt{-\omega}\|_{H^2_{per}(0,T)} \leq C\mu. \tag{6.103}$$

Proof. The proof proceeds along the lines of the proof of Theorem 6.25 and is divided into two steps. In the first step, we construct small-amplitude periodic solutions $v_\pm(\mu)$ to (6.105) by bifurcating from the equilibria $\pm\sqrt{-\omega}$ at $\mu = 0$. In the second step, we connect these periodic solutions by an interface, which arises as a localized perturbation of the black soliton ψ_0.

Define the smooth nonlinear operator $\mathcal{F}_{\text{per}} \colon H^2_{\text{per}}(0,T) \times \mathbb{R} \to L^2_{\text{per}}(0,T)$ by

$$\mathcal{F}_{\text{per}}(v, \mu) = -v'' + (v^2 + \omega)v + \mu V v.$$

Since $\mathcal{F}_{\text{per}}(\sqrt{-\omega}, 0) = 0$ and

$$\partial_v \mathcal{F}_{\text{per}}(\sqrt{-\omega}, 0) = -\partial_x^2 - 2\omega$$

is invertible, an application of the implicit function theorem yields that there exist $\mu_1 > 0$ and a locally unique smooth function $v_+ \colon (-\mu_1, \mu_1) \to H^2_{\text{per}}(0,T)$ with $v_+(0) = \sqrt{-\omega}$ satisfying

$$\mathcal{F}_{\text{per}}(v_+(\mu), \mu) = 0$$

for all $\mu \in (-\mu_1, \mu_1)$. Symmetry of the equation (6.99) yields that $x \mapsto v_+(-x;\mu)$ is also a solution. So, by uniqueness we find $v_+(x;\mu) = v_+(-x;\mu)$ for $x \in \mathbb{R}$ and $\mu \in (-\mu_1, \mu_1)$, after shrinking $\mu_1 > 0$ if necessary. Setting $v_-(\mu) = -v_+(\mu)$, we deduce that $v_\pm(\mu) \in H^2_{\text{per}}(0,T)$ are even periodic solutions to (6.99) obeying (6.103) for $\mu \in (-\mu_1, \mu_1)$.

We proceed by constructing the interface connecting $v_-(\mu)$ and $v_+(\mu)$. For convenience, we abbreviate $A = -\partial_x^2 + \omega$ and $\mathcal{N}(\psi) = \psi^3$, and write (6.99) in the abstract form

$$A\psi + \mathcal{N}(\psi) + \mu V\psi = 0. \tag{6.104}$$

Inserting the ansatz

$$\psi = v_-(\mu)\chi_- + v_+(\mu)\chi_+ + \psi_0 + \sqrt{-\omega}\chi_- - \sqrt{-\omega}\chi_+ + \varphi$$

with error term $\varphi \in H^2_{\text{odd}}(\mathbb{R})$ into (6.104), we arrive at the equation

$$\mathcal{F}(\varphi, \mu) = 0,$$

where $\mathcal{F}\colon H^2_{\text{odd}}(\mathbb{R}) \times (-\mu_1, \mu_1) \to L^2_{\text{odd}}(\mathbb{R})$ is the smooth nonlinear operator given by

$$\mathcal{F}(\varphi, \mu) = L_{+,\mu}(\chi_- \tilde{v}_-(\mu) + \chi_+ \tilde{v}_+(\mu) + \psi_0)\varphi + R(\mu) + \check{\mathcal{N}}(\varphi, \mu)$$

with $\tilde{v}_\pm(\mu) = v_\pm(\mu) \mp \sqrt{-\omega}$ and

$$\begin{aligned} R(\mu) &= A(\psi_0 + \chi_- \tilde{v}_-(\mu) + \chi_+ \tilde{v}_+(\mu)) + \mu V(\psi_0 + \chi_- \tilde{v}_-(\mu) + \chi_+ \tilde{v}_+(\mu)) \\ &\quad + \mathcal{N}(\psi_0 + \chi_- \tilde{v}_-(\mu) + \chi_+ \tilde{v}_+(\mu)), \\ \check{\mathcal{N}}(\varphi, \mu) &= \mathcal{N}(\psi_0 + \chi_- \tilde{v}_-(\mu) + \chi_+ \tilde{v}_+(\mu) + \varphi) - \mathcal{N}(\psi_0 + \chi_- \tilde{v}_-(\mu) + \chi_+ \tilde{v}_+(\mu)) \\ &\quad - \mathcal{N}'(\psi_0 + \chi_- \tilde{v}_-(\mu) + \chi_+ \tilde{v}_+(\mu))\varphi. \end{aligned}$$

We emphasize that $\mathcal{F}$ is well-defined, because $\chi_+ \tilde{v}_+(\mu) + \chi_- \tilde{v}_-(\mu)$ and ψ_0 are odd functions.

By Sturm-Liouville theory, cf. [78, Theorem 2.3.3], the kernel of the second-order operator $L_{+,0}(\psi_0)$ is spanned by the even function $\psi_0' \in H^2(\mathbb{R})$. Hence, using that $L_{+,0}(\psi_0)$ maps odd functions to odd functions, its restriction $L_{+,0}(\psi_0)|_{H^2_{\text{odd}}}$ to the subspace of odd functions is well-defined and invertible. In addition, by employing analogous arguments as for the estimates (6.81) and (6.82) in the proof of Theorem 6.25, we find a μ-independent constant $C > 0$ such that

$$\|R(\mu)\|_{L^2} \leq C|\mu|, \qquad \|\check{\mathcal{N}}(\varphi, \mu)\|_{L^2} \leq C\|\varphi\|_{H^2}^2$$

for all $\mu \in (-\mu_1, \mu_1)$ and $\varphi \in H^2_{\text{odd}}(\mathbb{R})$ with $\|\varphi\|_{H^2} \leq 1$. We conclude that $\mathcal{F}(0,0) = 0$ and $\partial_\varphi \mathcal{F}(0,0) = L_{+,0}(\psi_0)|_{H^2_{\text{odd}}}$ is invertible. Therefore, the implicit function theorem yields $\mu_2 \in (0, \mu_1)$ and a smooth function $\varphi\colon (-\mu_2, \mu_2) \to H^2_{\text{odd}}(\mathbb{R})$ with $\varphi(0) = 0$ satisfying

$$\mathcal{F}(\varphi(\mu), \mu) = 0$$

for $\mu \in (-\mu_2, \mu_2)$. The 1-front

$$\psi(\mu) = v_-(\mu)\chi_- + v_+(\mu)\chi_+ + \psi_0 + \sqrt{-\omega}\chi_- - \sqrt{-\omega}\chi_+ + \varphi(\mu),$$

is an odd stationary solution to (6.99) for $\mu \in (-\mu_2, \mu_2)$, which obeys (6.102) by smoothness of $v_\pm$ and φ, and by exponential localization of $\chi_\pm(\psi_0 \mp \sqrt{-\omega})$ and its derivatives. The nondegeneracy of $\psi(\mu)$ as a stationary solution to (6.100) follows from (6.101) using analogous arguments as in the proof of Theorem 6.25, and is therefore omitted here. □

Remark 6.34. Let $\psi(\mu)$ be the nondegenerate stationary 1-front solution to (6.100), established in Theorem 6.33. Thanks to the reflection symmetry of (6.100), we find that $-\psi(\mu)$ is also a nondegenerate 1-front solution. It connects $v_+(\mu)$ to $v_-(\mu)$.

The nondegeneracy of the 1-front solutions $\pm\psi(\mu)$ to (6.100), established in Theorem 6.33 and Remark 6.34, permits the application of Theorems 6.5 and 6.14, yielding the existence of nondegenerate stationary multifront solutions to (6.100), see Figure 6.7.

Corollary 6.35. *Let $M \in \mathbb{N}$. Let $\{\psi_j\}_{j=1}^M$ be a sequence of front solutions to* (6.99)*, established in Theorem 6.33 and Remark 6.34, with end states $v_{j,\pm}(\mu)$. Assume that it holds $v_{j,+}(\mu) = v_{j+1,-}(\mu)$ for $j = 1, \ldots, M-1$. Then, there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq N$ there exists a nondegenerate stationary multifront solution $\tilde{\psi}_n$ to* (6.100) *of the form*

$$\tilde{\psi}_n = a_n + \sum_{j=1}^{M} \chi_{j,n}\psi_j(\cdot - jnT),$$

where $\chi_{j,n}, j = 1, \ldots, M$ is the smooth partition of unity defined in §6.3, and $\{a_n\}_n$ is a sequence in $H^2(\mathbb{R})$ converging to 0 *as $n \to \infty$.*

If the nondegenerate multifronts, obtained in Corollary 6.35, connect to the same end state at $\pm\infty$, then Theorem 6.8 yields large wavelength periodic pulse solutions approximating a formal periodic extension of the multifront.

Corollary 6.36. *Let ψ be a multifront solution to* (6.99)*, as established Corollary 6.35. Assume that ψ connects to the same periodic end state $v \in H^2_{per}(0,T)$ as $x \to \pm\infty$. Then, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ there exists a stationary nT-periodic solution ψ_n to* (6.99) *given by*

$$\psi_n(x) = \chi_n(x)\psi(x) + (1 - \chi_n(x))v(x) + a_n(x), \qquad x \in \left[-\tfrac{n}{2}T, \tfrac{n}{2}T\right),$$

where χ_n is the cut-off function from Theorem 6.8, and $\{a_n\}_n$ is a sequence with $a_n \in H^2_{per}(0,nT)$ satisfying $\|a_n\|_{H^2_{per}(0,nT)} \to 0$ as $n \to \infty$.

Remark 6.37. Under some Conley-Moser-type conditions, all bounded solutions to (6.99) can be characterized using symbolic dynamics [6]. Specifically, there exists a homeomorphism between the set of all real bounded solutions of (6.99) and the set of bi-infinite sequences of numbers $1, \ldots, N$ for some integer $N \in \mathbb{N}$. As explained in [6], this symbolic identification yields the existence of multifronts in (6.99), as well as periodic solutions featuring multiple front interfaces

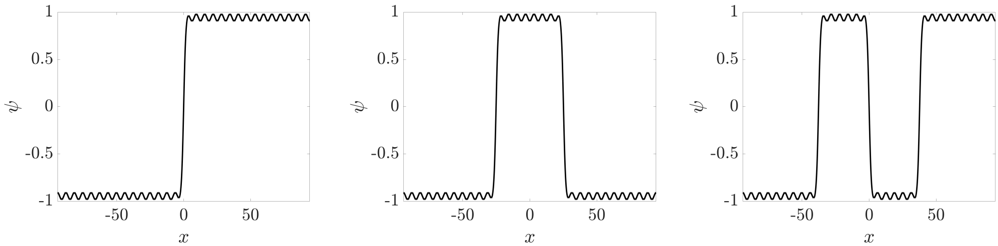

Figure 6.7: Approximations of stationary real-valued 1-, 2-, and 3-front solutions to the Gross-Pitaevskii equation (6.97) for system coefficients $\kappa = 1, \omega = -1, V(x) = 0.2\cos^2(x/2)$, and $\mu = 1$. The solutions are obtained through numerical continuation by starting from a formal concatenation of shifted black solitons $\pm\psi_0(\cdot - \varsigma)$, $\varsigma \in \mathbb{R}$, which solve (6.99) at $\mu = 0$.

on a single periodicity interval. The Conley-Moser-type conditions are verified numerically in [6] in case of the periodic potential $V(x) = \cos(2x)$ in (6.99). Notably, since we have

$$\int_{\mathbb{R}} V'(x)\psi_0(x)\psi_0'(x)\mathrm{d}x = -\frac{8\pi}{\sinh\left(\sqrt{\frac{2}{-\omega}}\,\pi\right)} \neq 0$$

for $\omega < 0$, Theorem 6.33 and Corollaries 6.35 and 6.36 rigorously establish the existence of multifronts and periodic pulse solutions to the defocusing Gross-Pitaevskii equation (6.99) with potential $V(x) = \cos(2x)$, provided $\mu > 0$ is sufficiently small.

6.8.3.2 Pulses in the focusing Gross-Pitaevskii equation

We now turn to the focusing case $\kappa = -1$. We are interested in the existence and stability of real-valued stationary multipulses and periodic pulse solutions to (6.97). Real-valued stationary solutions to (6.97) obey the ordinary differential equation

$$-\psi'' + \mu V(x)\psi + \omega\psi - \psi^3 = 0. \tag{6.105}$$

We first discuss the existence of 1-pulse solutions to (6.105). These so-called gap solitons will serve as building blocks for the construction of (periodic) multipulse solutions. We emphasize that any real-valued stationary pulse solution $\psi \in H^2(\mathbb{R}) \setminus \{0\}$ to (6.97) is degenerate, since $(0,\psi)^\top$ lies in the kernel of the operator $L_{-,\mu}(\psi)$. Therefore, we proceed as in the defocusing case and consider the associated real-valued reaction-diffusion problem

$$\partial_t\psi = -\partial_x^2\psi + \mu V(x)\psi + \omega\psi - \psi^3, \qquad \psi(x,t) \in \mathbb{R},\, x \in \mathbb{R},\, t \geq 0. \tag{6.106}$$

Existence of nondegenerate stationary 1-pulse solutions to (6.106) has been shown in different regimes for the parameters ω, μ and the potential V. For instance, Lyapunov-Schmidt reduction was employed in [109] to find bifurcating 1-pulse solutions from the family of bright NLS solitons

$$\phi_0(x; \varsigma, \omega) = \sqrt{2\omega}\,\mathrm{sech}\left(\sqrt{\omega}(x - \varsigma)\right),$$

which solve (6.105) at $\mu = 0$ and satisfy

$$\langle \partial_\omega \phi_0(\varsigma, \omega), \phi_0(\varsigma, \omega) \rangle_{L^2} = \frac{1}{\sqrt{\omega}}$$

for each $\varsigma \in \mathbb{R}$ and $\omega > 0$. Specifically, the analysis in [109, Section 3.2.3] yields the following result.

Theorem 6.38. *Let $\omega_0, T > 0$. Let $V \in C^2(\mathbb{R})$ be T-periodic and real-valued. Let $\varsigma_0 \in \mathbb{R}$ be a simple zero of the derivative of the effective potential $V_{\text{eff}} \colon \mathbb{R} \to \mathbb{R}$ given by*

$$V_{\text{eff}}(\varsigma) = \int_{\mathbb{R}} V(x + \varsigma)\phi_0(x; 0, \omega_0)^2 \mathrm{d}x.$$

Then, there exist $\mu_0 > 0$ and a smooth map $\phi \colon (-\mu_0, \mu_0) \times (\omega_0 - \mu_0, \omega_0 + \mu_0) \to H^2(\mathbb{R})$ with $\phi(0, \omega_0) = \phi_0(\varsigma_0, \omega_0)$ such that for each $\mu \in (-\mu_0, \mu_0) \setminus \{0\}$ and $\omega \in (\omega_0 - \mu_0, \omega_0 + \mu_0)$ we have that $\phi(\mu, \omega)$ is a nondegenerate stationary solution to (6.106) *satisfying*

$$\langle \partial_\omega \phi(\mu, \omega), \phi(\mu, \omega) \rangle_{L^2} > 0.$$

In addition, $L_{+,\mu}(\phi(\mu, \omega))$ has precisely one negative eigenvalue in case $\mu V_{\text{eff}}''(\varsigma_0) > 0$, whereas it has precisely two negative eigenvalues in case $\mu V_{\text{eff}}''(\varsigma_0) < 0$ (counting algebraic multiplicies). Finally, $L_{-,\mu}(\phi(\mu, \omega))$ possesses no negative eigenvalues.

On the other hand, if $\omega, \mu \in \mathbb{R}$ and $V \in C^1(\mathbb{R})$ are chosen such that ω lies in the so-called *semi-infinite gap* (s_0, ∞) where s_0 is the spectral bound of the periodic differential operator $\partial_x^2 - \mu V$ acting on $L^2(\mathbb{R})$, one can use variational methods [1, 88, 109, 138] to prove the existence of nontrivial nondegenerate stationary H^2-solutions to (6.106) arising as critical points of the Hamiltonian $\mathcal{H} \colon H^1(\mathbb{R}) \to \mathbb{R}$ given by

$$\mathcal{H}(\psi) = \frac{1}{2} \int_{\mathbb{R}} \left(|\partial_x \psi(x)|^2 + (\omega + \mu V(x)) |\psi(x)|^2 - \frac{1}{2} |\psi(x)|^4 \right) \mathrm{d}x.$$

We refer to [109] and references therein for further details on the existence of gap-soliton solutions to (6.105). In the remaining part of this section, we assume that $\omega, \mu \in \mathbb{R}$ and $V \in C^1(\mathbb{R})$ are such that a nondegenerate stationary 1-pulse solution to (6.106) exists. Theorems 6.5, 6.8 and 6.14 then readily yield the existence of associated multifronts and periodic pulse solutions to (6.105), see Figure 6.8.

Corollary 6.39. *Let $T > 0$ and $\omega, \mu \in \mathbb{R}$. Let $V \in C^1(\mathbb{R})$ be T-periodic and real-valued. Let $M \in \mathbb{N}$ and $\theta \in \{\pm 1\}^M$. Let $\psi_0 \in H^2(\mathbb{R})$ be a nondegenerate stationary solution to* (6.106). *Then, there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N$ the following assertions hold true.*

1. *There exists a nondegenerate stationary M-pulse solution to* (6.106) *of the form*

$$\psi_n = \sum_{j=1}^{M} \theta_j \psi_0(\cdot - jnT) + a_n, \tag{6.107}$$

   where $\{a_n\}_n$ is a sequence in $H^2(\mathbb{R})$ converging to 0 as $n \to \infty$.

2. *There exists an nT-periodic solution $\psi_{per,n}$ to* (6.105) *given by*

$$\psi_{per,n}(x) = \chi_n(x)\psi_0(x) + a_n(x), \qquad x \in [-\tfrac{n}{2}T, \tfrac{n}{2}T),$$

   where χ_n is the cut-off function from Theorem 6.8, and $\{a_n\}_n$ is a sequence with $a_n \in H^2_{per}(0, nT)$ satisfying $\|a_n\|_{H^2_{per}(0,nT)} \to 0$ as $n \to \infty$.

Remark 6.40. It is also possible to construct multipulse solutions lying near the formal concatenation of *different* nondegenerate stationary pulse solutions $\psi_1, \psi_2 \in H^2(\mathbb{R})$ to (6.106) using Theorem 6.5.

Remark 6.41. The existence of multipulse solutions to (6.105) is also addressed in [3, 109]. Corollary 6.39 reveals that pulse solutions are accompanied by a family of periodic solutions with large spatial period. As far as the authors are aware, existence of these so-called *soliton trains* has so far only been rigorously established in the case of the explicit periodic potential

$$V(x) = \mathrm{cn}^2\left(\frac{1}{\sqrt{2}}x; k\right) - 1, \tag{6.108}$$

where $\mathrm{cn}(x;k)$ is the Jacobi cosine function (cnoidal wave) with elliptic modulus $k \in (0,1)$, cf. [23].

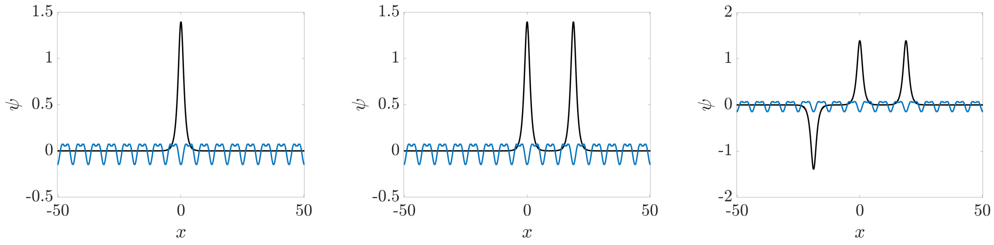

Figure 6.8: Approximations of 1-, 2-, and 3-pulse solutions to (6.105) (black) pinned to minima of the periodic potential $V(x) = -0.1\cos(x) - 0.05\cos(2x)$ (blue) for the system coefficients $\omega = 1$ and $\mu = 0.5$. The solutions are obtained through numerical continuation by starting from a formal concatenation of bright NLS solitons $\pm\phi_0(\varsigma, \omega)$ for various $\varsigma \in \mathbb{R}$, which solve (6.105) at $\mu = 0$.

For such a potential, periodic waves exist for specific values of ω and have the form

$$\psi(x) = \sqrt{\mu + k^2}\,\mathrm{cn}\left(\frac{1}{\sqrt{2}}x; k\right)$$

for $\omega = \mu + k^2 - \frac{1}{2}$ and $\mu \geq -k^2$, and

$$\psi(x) = \frac{\sqrt{\mu + k^2}}{k}\,\mathrm{dn}\left(\frac{1}{\sqrt{2}}x; k\right)$$

for $\omega = 1 + \mu k^{-2} - \frac{1}{2}k^2$ and $\mu \geq -k^2$. In the homoclinic limit $k \uparrow 1$, the period tends to infinity, the potential approaches $x \mapsto -\mu \tanh^2(x/\sqrt{2})$, and the periodic waves approximate the pulse $x \mapsto \sqrt{\mu^2 + 1}\,\mathrm{sech}(x/\sqrt{2})$ on a single periodicity interval. Thus, these periodic waves resemble periodic pulse solutions, or soliton trains, for $0 \ll k < 1$.

We proceed with analyzing the spectral stability of the stationary multipulses and periodic pulse solutions to (6.97), constructed in Corollary 6.39. To this end, we fix a nondegenerate primary pulse solution $\psi_0 \in H^2(\mathbb{R})$ to (6.106) for a frequency $\omega = \omega_0$. Since $L_{+,\mu}(\psi_0)$ is invertible, it follows directly from the implicit function theorem that ψ_0 may be continued in ω.

Lemma 6.42. *Let $T > 0$ and $\mu, \omega_0 \in \mathbb{R}$. Let $V \in C^1(\mathbb{R})$ be T-periodic and real-valued. Let $\psi_0 \in H^2(\mathbb{R})$ be a nondegenerate stationary solution to* (6.106) *at $\omega = \omega_0$. There exist $\nu > 0$ and a locally unique smooth map $\tilde{\psi}\colon (\omega_0 - \nu, \omega_0 + \nu) \to H^2(\mathbb{R})$ with $\tilde{\psi}(\omega_0) = \psi_0$ such that $\tilde{\psi}(\omega)$ is a solution to* (6.105) *for all $\omega \in (\omega_0 - \nu, \omega_0 + \nu)$.*

We impose the following assumption on the existence and spectral properties of the primary pulse.

(GP) There exist $T > 0$, $\mu, \omega_0 \in \mathbb{R}$, real-valued T-periodic $V \in C^1(\mathbb{R})$, and a nondegenerate stationary solution $\psi_0 \in H^2(\mathbb{R})$ to (6.106) at $\omega = \omega_0$ satisfying:

1. $L_{+,\mu}(\psi_0)$ has precisely one negative eigenvalue (counting algebraic multiplicities).
2. $L_{-,\mu}(\psi_0)$ has no negative eigenvalues.
3. It holds $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} \neq 0$, where $\tilde{\psi}$ is the continuation of ψ_0 with respect to ω, established in Lemma 6.42.

The spectral conditions in **(GP)** are typically imposed in the stability analysis of stationary pulse solutions to the focusing Gross-Pitaevskii equation, see, for instance, [109, Section 4] and references therein. We observe by Sturm-Liouville theory [158] that the second assertion in **(GP)** holds if and only if ψ_0 has no zeros. Moreover, Theorem 6.38 shows that, as long as the derivative of the effective potential V_{eff} has a simple zero, there exist nondegenerate pulse solutions $\psi_0 \in H^2(\mathbb{R}) \setminus \{0\}$ to (6.106), obeying the spectral conditions in **(GP)**, see also the forthcoming Remark 6.45.

The following result demonstrates that spectral stability of the periodic pulse solution $\psi_{\text{per},n}$, obtained in Corollary 6.39, is inherited from the constituting primary pulse ψ_0. Its proof employs Krein index counting theory [75, 76] to show that, if the linearization $L_{\mu,\text{per}}(\psi_{\text{per},n})$ has unstable eigenvalues, then they must be real and are separated from the imaginary axis by an n-independent spectral gap. We use Theorem 6.22 to relate the number of negative eigenvalues of the self-adjoint operators $L_{\pm,\mu,\text{per}}(\psi_{\text{per},n})$ to those of $L_{\pm,\mu}(\psi_0)$, and to show the existence of an n-independent ball centered at the origin, in which 0 is the only eigenvalue of $L_{\mu,\text{per}}(\psi_n)$. Having established that any unstable eigenvalue of $L_{\mu,\text{per}}(\psi_{\text{per},n})$ is real and bounded away from the imaginary axis, an application of Lemma 6.9, together with standard spectral a-priori bounds, rules out the presence of unstable eigenvalues of $L_{\mu,\text{per}}(\psi_{\text{per},n})$.

Theorem 6.43. *Assume* **(GP)**. *Suppose ω_0 is larger than the spectral bound of the operator $\partial_x^2 - \mu V$ acting on $L^2(\mathbb{R})$. Then, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ the following statements are equivalent:*

1. *The* 1*-pulse $\psi_0 \in H^2(\mathbb{R})$ is a spectrally stable solution to* (6.97).
2. *We have $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} > 0$.*
3. *The periodic pulse solution $\psi_{per,n} \in H^2_{per}(0, nT)$ to* (6.97)*, established in Corollary 6.39, is spectrally stable against co-periodic perturbations, i.e., the spectrum of $L_{\mu,per}(\psi_{per,n})$ is confined to the imaginary axis.*

Moreover, if one of these statements holds, then we have the following:

a. *$L_{-,\mu,per}(\psi_n)$ has no negative eigenvalues and a simple eigenvalue at* 0.

b. *$L_{+,\mu,per}(\psi_n)$ is invertible and has precisely one negative eigenvalue, which is simple.*

c. *There exist $\nu_n > 0$ and a smooth map $\tilde{\psi}_{per,n} \colon (\omega_0 - \nu_n, \omega_0 + \nu_n) \to H^2_{per}(0, nT)$ such that $\tilde{\psi}_{per,n}(\omega_0) = \psi_{per,n}$ and $\tilde{\psi}_{per,n}(\omega)$ is a solution to* (6.105) *for all $\omega \in (\omega_0 - \nu_n, \omega_0 + \nu_n)$. It holds*

$$\left\langle \partial_\omega \tilde{\psi}_{per,n}(\omega_0), \psi_{per,n} \right\rangle_{L^2_{per}(0,nT)} > 0. \tag{6.109}$$

Proof. The fact that the first two statements are equivalent follows from [109, Theorem 4.8].

We prove that the third implies the second statement by contrapositon. If $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} < 0$, then ψ_0 is spectrally unstable by [109, Theorem 4.8] with an element $\lambda \in \sigma(L_\mu(\psi))$ in the point spectrum with $\mathrm{Re}(\lambda) > 0$. Upon applying Theorem 6.22, we infer spectral instability of $\psi_{\mathrm{per},n}$, provided $n \in \mathbb{N}$ is sufficiently large.

Finally, we prove that the second implies the third statement. To this end, we assume that $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} > 0$. First, we note that the essential spectrum of $L_{\pm,\mu}(\psi_0)$ is by Proposition 6.11 given by $\sigma(L_{\pm,\mu}(0)) = \sigma(-\partial_x^2 + \mu V + \omega_0)$, which is, by assumption, confined to the positive half-line. Therefore, $L_{\pm,\mu}(0)$ is invertible and 0 does not lie in the essential spectra of $L_{\pm,\mu}(\psi_0)$ and $L_\mu(\psi_0)$. On the other hand, $L_{-,\mu}(\psi_0)\psi_0 = 0$ and Sturm-Liouville theory, cf. [78, Theorem 2.3.3], imply that 0 is a simple eigenvalue of $L_{-,\mu}(\psi_0)$. Differentiating $L_{-,\mu}(\tilde{\psi}(\mu))\tilde{\psi}(\mu) = 0$ with respect to ω and setting $\omega = \omega_0$, we obtain $L_{+,\mu}(\psi_0)\partial_\omega \tilde{\psi}(\omega_0) = -\psi_0$, cf. Lemma 6.42. Therefore, 0 is an eigenvalue of $L_\mu(\psi_0)$ whose algebraic multiplicity is at least 2. Using that $L_{-\mu}(\psi_0)$ is self-adjoint and Fredholm of index 0, observing that $0 \notin \sigma_{\mathrm{ess}}(L_\mu(\psi_0))$, and noting $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} > 0$, we find that 0 is an isolated eigenvalue of $L_\mu(\psi_0)$ of algebraic multiplicity 2. Hence, Theorem 6.22 yields $\eta_1 > 0$ and $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ the total algebraic multiplicity of the eigenvalues of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ in the ball $B_0(\eta_1)$ equals 2.

Next, we employ Krein index counting theory [75, 76] to prove the absence of eigenvalues of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ of positive real part. We start by counting negative eigenvalues of the self-adjoint operator $L_{+,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$. Since $L_{+,\mu}(\psi_0)$ has precisely one negative eigenvalue $\lambda_1 < 0$ (counting algebraic multiplicities) and ψ_0 is a nondegenerate solution to (6.106), there exists $\eta_2 > 0$ such that $\sigma(L_{+,\mu}(\psi_0)) \subset \{\lambda_1\} \cup (\eta_2, \infty)$. Moreover, since $\|\psi_{\mathrm{per},n}\|_{L^\infty}$ can be bounded by an n-independent constant by Corollary 6.39 and the continuous embedding $H^1(0, nT) \hookrightarrow L^\infty(\mathbb{R})$ with n-independent constant, there exists by Lemma 6.24 an n-independent constant $\eta_3 > 0$ such that the spectrum of the operator $L_{+,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ is confined to $[-\eta_3, \infty)$. Combining the last two sentences with Lemma 6.9 and Theorem 6.22, we find that there exists $N_2 \in \mathbb{N}$ with $N_2 \geq N_1$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$ the operator $L_{+,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ has precisely one eigenvalue in the set $[-\eta_3, \eta_2]$, which is simple and negative. In particular, this implies assertion

b. Since $L_{+,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ is invertible, the implicit function theorem yields $\nu_n > 0$ and a smooth map $\tilde{\psi}_{\mathrm{per},n} \colon (\omega_0 - \nu_n, \omega_0 + \nu_n) \to H^2_{\mathrm{per}}(0, nT)$ with $\tilde{\psi}_{\mathrm{per},n}(\omega_0) = \psi_{\mathrm{per},n}$ such that $\tilde{\psi}_{\mathrm{per},n}(\omega)$ is a solution to (6.105) for all $\omega \in (\omega_0 - \nu_n, \omega_0 + \nu_n)$.

Our next step is to count eigenvalues of the operator $L_{-,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$. Since $\psi_{\mathrm{per},n} \in H^2_{\mathrm{per}}(0, nT)$ is a nontrivial solution to (6.105), we find $L_{-,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})\psi_{\mathrm{per},n} = 0$. So, by Sturm-Liouville theory, cf. [158, Theorem 6.3.1.(8)(3)], we deduce that 0 is a simple isolated eigenvalue of $L_{-,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$. Therefore, using analogous arguments as for the operator $L_{+,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$, we infer that the facts that $L_{-,\mu}(\psi_0)$ has no negative eigenvalues and 0 is an isolated simple eigenvalue of $L_{-,\mu}(\psi_0)$ imply that there exists $N_3 \in \mathbb{N}$ with $N_3 \geq N_2$ such that $L_{-,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ has no negative eigenvalues for all $n \in \mathbb{N}$ with $n \geq N_3$. This yields assertion a.

Next, we show that the eigenvalue 0 of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ has geometric multiplicity one and algebraic multiplicity two. The fact that 0 has geometric multiplicity one follows from the analysis of the operators $L_{\pm,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$. An associated eigenfunction is given by $(0, \psi_{\mathrm{per},n})^\top$. Differentiating the equation $L_{-,\mu,\mathrm{per}}(\tilde{\psi}_{\mathrm{per},n}(\omega))\tilde{\psi}_{\mathrm{per},n}(\omega) = 0$ with respect to ω and setting $\omega = \omega_0$, we obtain the identities

$$L_{+,\mu,\mathrm{per}}(\psi_{\mathrm{per},n})\partial_\omega \tilde{\psi}_{\mathrm{per},n}(\omega_0) = -\psi_{\mathrm{per},n}, \qquad L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n}) \begin{pmatrix} \partial_\omega \tilde{\psi}_{\mathrm{per},n}(\omega_0) \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ \psi_{\mathrm{per},n} \end{pmatrix}.$$

Using that the total algebraic multiplicity of the eigenvalues of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ in $B_0(\eta_1)$ is 2, we conlcude that 0 is an eigenvalue of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ of algebraic multiplicity 2. Combining the latter with the Fredholm alternative, we arrive at $\langle \partial_\omega \tilde{\psi}_{\mathrm{per},n}(\omega_0), \psi_{\mathrm{per},n}\rangle_{L^2_{\mathrm{per}}(0,nT)} \neq 0$.

Therefore, we can apply the instability index formula from [76] to find $k_r \leq 1$ and $k_c = k_i^- = 0$, where k_r is the number of real unstable eigenvalues of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$, k_c is the number of quadruplets of eigenvalues with non-vanishing real and imaginary parts, and k_i^- is the number of purely imaginary eigenvalues of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$ with negative Krein signature [76] (all counting algebraic multiplicities). In the following, we prove $k_r = 0$, which establishes (6.109) by [76, Theorem 1]. To this end, we first show that there exists an n-independent constant $\rho > 0$ such that $\lambda \in \sigma(L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n}))$ implies $|\operatorname{Re}(\lambda)| \leq \rho$. To establish this spectral a-priori bound, we consider the principal part of $L_{\mu,\mathrm{per}}(\psi_{\mathrm{per},n})$, which is the skew-adjoint operator $A_0 \colon D(A_0) \subset L^2_{\mathrm{per}}(0, nT) \to L^2(0, nT)$ with dense domain $D(A_0) = H^2_{\mathrm{per}}(0, nT)$ given by $A_0\boldsymbol{\psi} = -J\boldsymbol{\psi}''$. By Stone's Theorem, cf. [44, Theorem 3.24], A_0 generates a unitary group on the Hilbert space $L^2_{\mathrm{per}}(0, nT)$. In particular, [44, Theorem 1.10] yields the resolvent bound

$$\left\| (A_0 - \lambda)^{-1} \boldsymbol{\psi} \right\|_{L^2_{\mathrm{per}}(0,nT)} \leq \frac{\|\boldsymbol{\psi}\|_{L^2_{\mathrm{per}}(0,nT)}}{|\operatorname{Re}(\lambda)|}, \tag{6.110}$$

for $\boldsymbol{\psi} \in L^2_{\text{per}}(0, nT)$ and $\lambda \in \mathbb{C}$ with $|\text{Re}(\lambda)| > 0$. Using that $\|\psi_{\text{per},n}\|_{L^\infty}$ is bounded by an n-independent constant, we obtain an n-independent constant $C > 0$ such that the residual $L_{\mu,\text{per}}(\psi_{\text{per},n}) - A_0$ enjoys the estimate

$$\|(L_{\mu,\text{per}}(\psi_{\text{per},n}) - A_0)\,\boldsymbol{\psi}\|_{L^2_{\text{per}}(0,nT)} \leq C\|\boldsymbol{\psi}\|_{L^2_{\text{per}}(0,nT)}$$

for $\boldsymbol{\psi} \in L^2_{\text{per}}(0, nT)$. Combining this estimate with (6.110) and [80, Theorem IV.1.16] yields an n-independent constant $\rho > 0$ such that $L_{\mu,\text{per}}(\psi_{\text{per},n}) - \lambda = A_0 - \lambda + L_{\mu,\text{per}}(\psi_{\text{per},n}) - A_0$ is bounded invertible for each $\lambda \in \mathbb{C}$ with $|\,\text{Re}(\lambda)| \geq \rho$. On the other hand, Lemma 6.9 and the spectral stability of ψ_0 imply that there exists $N_4 \in \mathbb{N}$ with $N_4 \geq N_3$ such that for all $n \in \mathbb{N}$ with $n \geq N_4$ the compact set $[-\rho, -\eta_1] \cup [\eta_1, \rho]$ lies in the resolvent set of $L_{\mu,\text{per}}(\psi_{\text{per},n})$. Combining the latter with the spectral a-priori bound and the fact that 0 is the only eigenvalue of $L_{\mu,\text{per}}(\psi_{\text{per},n})$ in $(-\eta_1, \eta_1)$, we conclude that $k_r = 0$, which establishes the third statement and (6.109). □

Using that the Gross-Pitaevskii equation (6.98) may be expressed as the Hamiltonian system

$$\partial_t \boldsymbol{\psi} = J\nabla\mathcal{H}_{\text{per}}(\boldsymbol{\psi})$$

on $H^1_{\text{per}}(0, nT)$ with Hamiltonian $\mathcal{H}_{\text{per}} \colon H^1_{\text{per}}(0, nT) \to \mathbb{R}$ given by

$$\mathcal{H}_{\text{per}}(\boldsymbol{\psi}) = \frac{1}{2}\int_0^{nT} \left(|\boldsymbol{\psi}_x(x)|^2 + (\omega + \mu V(x))\,|\boldsymbol{\psi}(x)|^2 - \frac{1}{2}|\boldsymbol{\psi}|^4\right) \mathrm{d}x,$$

it immediately follows from Theorem 6.43 and the stability theorem from Grillakis, Shatah, and Strauss [64] that the periodic pulse solutions, established in Corollary 6.39, are orbitally stable if Assumption **(GP)** holds and we have $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} > 0$.

Corollary 6.44. *Assume* **(GP)** *and* $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} > 0$. *Suppose* ω_0 *is larger than the spectral bound of the operator* $\partial_x^2 - \mu V$ *acting on* $L^2(\mathbb{R})$.

Then, there exists $N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *with* $n \geq N$ *the periodic pulse solution* $\boldsymbol{\psi}_{per,n} = (\psi_{per,n}, 0)^\top \in H^2_{per}(0, nT)$ *to* (6.98)*, where* $\psi_{per,n}$ *is established in Corollary* (6.39)*, is orbitally stable. Specifically, for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, whenever* $v_0 \in H^1_{per}(0, nT)$ *satisfies* $\|v_0\|_{H^1_{per}(0,nT)} < \delta$*, then there exists a global mild solution* $\boldsymbol{\psi} \in C\big([0,\infty), H^1_{per}(0, nT)\big)$ *to* (6.98) *with initial condition* $\boldsymbol{\psi}(0) = \boldsymbol{\psi}_{per,n} + v_0$ *obeying*

$$\inf_{\gamma\in\mathbb{R}} \|\boldsymbol{\psi}(t) - R(\gamma)\boldsymbol{\psi}_{per,n}\|_{H^1_{per}(0,nT)} < \varepsilon.$$

To the authors' best knowledge, Theorem 6.43 and Corollary 6.44 present the fist rigorous spectral and orbital stability result of any periodic wave in the Gross-Pitaevskii equation with periodic potential. In the case of the explicit periodic potential (6.108), rigorous instability results, as well as numerical simulations indicating spectral stability, can be found in [23].

Remark 6.45. For fixed $\omega_0, T > 0$, real-valued T-periodic $V \in C^2(\mathbb{R})$, and simple zero $\varsigma_0 \in \mathbb{R}$ of V_{eff}', Lemma 6.24 and Theorem 6.38 yield $\mu \in \mathbb{R} \setminus \{0\}$ such that the spectral bound of the operator $\partial_x^2 - \mu V$, acting on $L^2(\mathbb{R})$, is smaller than ω_0, and equation (6.106) possesses a nondegenerate stationary solution $\psi_0 \in H^2(\mathbb{R})$ at $\omega = \omega_0$ satisfying the spectral conditions in **(GP)**. Since it holds $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2} > 0$, the associated stationary periodic pulse solution $\psi_{\text{per},n} \in H^2_{\text{per}}(0, nT)$ to (6.97), established in Corollary 6.39, are spectrally and orbitally stable against co-periodic perturbations by Theorem 6.43 and Corollary 6.44, provided $n \in \mathbb{N}$ is sufficiently large.

Next, we turn to the spectral analysis of the multipulse solutions, obtained in Corollary 6.39. For this, we first establish the following lemma.

Lemma 6.46. *Let $T > 0$ and $\mu, \omega_0 \in \mathbb{R}$. Let $V \in C^1(\mathbb{R})$ be T-periodic and real-valued. Let $M \in \mathbb{N}$ and $\theta \in \{\pm 1\}^M$. Let $\psi_0 \in H^2(\mathbb{R})$ be a nondegenerate stationary solution to* (6.106) *at $\omega = \omega_0$. Then, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ there exist $\nu_n > 0$ and a smooth map $\tilde{\psi}_n \colon (\omega_0 - \nu_n, \omega_0 + \nu_n) \to H^2(\mathbb{R})$ such that $\tilde{\psi}_n(\omega)$ is a solution to* (6.105) *for all $\omega \in (\omega_0 - \nu_n, \omega_0 + \nu_n)$, and we have $\tilde{\psi}_n(\omega_0) = \psi_n$, where $\psi_n \in H^2(\mathbb{R})$ is the multipulse solution obtained in Corollary 6.39. Moreover, it holds*

$$\lim_{n \to \infty} \left\langle \partial_\omega \tilde{\psi}_n(\omega_0), \psi_n \right\rangle_{L^2} = M \left\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \right\rangle_{L^2}, \tag{6.111}$$

where $\tilde{\psi}$ is the continuation of ψ_0 with respect to ω, established in Lemma 6.42.

Proof. Since ψ_n is a nondegenerate stationary solution to (6.106) by Corollary 6.39, the existence of the map $\tilde{\psi}_n$ follows from the implicit function theorem. So, all that remains is to prove (6.111). Differentiating the equations $L_{-,\mu}(\tilde{\psi}(\omega))\tilde{\psi}(\omega) = 0$ and $L_{-,\mu}(\tilde{\psi}_n(\omega))\tilde{\psi}_n(\omega) = 0$ with respect to ω and setting $\omega = \omega_0$, we arrive at

$$L_{+,\mu}(\psi_0) \left[\partial_\omega \tilde{\psi}(\omega_0)\right] = -\psi_0, \qquad L_{+,\mu}(\psi_n) \left[\partial_\omega \tilde{\psi}_n(\omega_0)\right] = -\psi_n. \tag{6.112}$$

We claim that there exists a sequence $\{b_n\}_n$ in $H^2(\mathbb{R})$, converging to 0 as $n \to \infty$, with

$$\partial_\omega \tilde{\psi}_n(\omega_0) = b_n + \sum_{j=1}^{M} \theta_j \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT). \tag{6.113}$$

Inserting the ansatz for $\partial_\omega \tilde{\psi}_n(\omega_0)$ into the linearization $L_{+,\mu}(\psi_n)$ and using (6.112), we obtain

$$
\begin{aligned}
L_{+,\mu}(\psi_n)&\left(b_n + \sum_{j=1}^{M} \theta_j \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT)\right) \\
&= L_{+,\mu}(\psi_n) b_n - \sum_{j=1}^{M} \theta_j \psi_0(\cdot - jnT) \\
&\quad + \sum_{j=1}^{M} \theta_j \left(L_{+,\mu}(\psi_n) - L_{+,\mu}(\psi_0(\cdot - jnT))\right) \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT).
\end{aligned}
$$

Thus, we infer from (6.107) and (6.112) that the correction b_n has to solve the equation

$$
L_{+,\mu}(\psi_n) b_n = -a_n - \sum_{j=1}^{M} \theta_j \left(L_{+,\mu}(\psi_n) - L_{+,\mu}(\psi_0(\cdot - jnT))\right) \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT).
$$

Using that $L_{+,\mu}(\psi_0)$ is invertible, we deduce from Lemma 6.7 that, provided $n \in \mathbb{N}$ is sufficiently large, $L_{+,\mu}(\psi_n)$ is also invertible and there exists an n-independent constant $C > 0$ such that

$$
\left\| L_{+,\mu}(\psi_n)^{-1} \psi \right\|_{L^2} \leq C \|\psi\|_{L^2}
$$

for $\psi \in L^2(\mathbb{R})$. In particular,

$$
b_n = -L_{+,\mu}(\psi_n)^{-1} \left(a_n + \sum_{j=1}^{M} \theta_j \left(L_{+,\mu}(\psi_n) - L_{+,\mu}(\psi_0(\cdot - jnT))\right) \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT)\right)
$$

satisfies (6.113) and obeys

$$
\|b_n\|_{L^2} \leq C \left(\|a_n\|_{L^2} + 3 \sum_{j=1}^{M} \left\| \left(\psi_n^2 - \psi_0(\cdot - jnT)^2\right) \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT) \right\|_{L^2} \right) \to 0
$$

as $n \to \infty$ by Corollary 6.39 and the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Therefore, using (6.107) and (6.113), we arrive at

$$\begin{aligned}\left\langle \partial_\omega \tilde{\psi}_n(\omega_0), \psi_n \right\rangle_{L^2} = &\sum_{j,k=1}^{M} \left\langle \theta_j \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT), \theta_k \psi_0(\cdot - knT) \right\rangle_{L^2} + \langle b_n, a_n \rangle_{L^2} \\ &+ \sum_{j=1}^{M} \left\langle \theta_j \partial_\omega \tilde{\psi}(\omega_0)(\cdot - jnT), a_n \right\rangle_{L^2} + \sum_{k=1}^{M} \langle b_n, \theta_k \psi_0(\cdot - knT) \rangle_{L^2} \\ \to &\, M \left\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \right\rangle_{L^2}\end{aligned}$$

as $n \to \infty$. □

By combining Krein index counting theory with Theorem 6.14, we establish spectral instability conditions for the multipulse solutions ψ_n, obtained in Corollary 6.39.

Theorem 6.47. *Let $M \in \mathbb{N}$ with $M \geq 2$. Take $\theta \in \{\pm 1\}^M$. Assume* **(GP)**. *Suppose ω_0 is larger than the spectral bound of the operator $\partial_x^2 - \mu V$ acting on $L^2(\mathbb{R})$. Then, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ the following statements holds.*

1. *If $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2}$ is negative, then the M-pulse ψ_n, obtained in Corollary 6.39, is spectrally unstable.*
2. *If $\langle \partial_\omega \tilde{\psi}(\omega_0), \psi_0 \rangle_{L^2}$ is positive, then the M-pulse ψ_n, obtained in Corollary 6.39, is spectrally unstable in each of the following cases:*

   (i) ψ_n has no zeros;

   (ii) There exist $m, \ell \in \mathbb{N}$ such that $M = 2m$ and ψ_n has 2ℓ zeros;

   (iii) ψ_n is odd, V is even, and there exist $m \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$ such that $m + \ell$ is even, $M = 2m$, and ψ_n has $2\ell + 1$ zeros;

   (iv) ψ_n and V are even, and there exist $m, \ell \in \mathbb{N}$ such that $m + \ell$ is odd, $M = 2m + 1$, and ψ_n has 2ℓ zeros;

   (v) There exist $m \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$ such that $M = 2m + 1$ and ψ_n has $2\ell + 1$ zeros.

Proof. In order to prove the first assertion, we observe that Theorem 6.43 yields that ψ_0 is spectrally unstable. Therefore, provided $n \in \mathbb{N}$ is sufficiently large, ψ_n is also spectrally unstable by Proposition 6.11 and Theorem 6.14.

We proceed with proving the second assertion. For notational convenience, we denote by $n(L) \in \mathbb{N}_0$ the total algebraic multiplicity of all negative eigenvalues of an operator $L\colon D(L) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and by $z(L) \in \mathbb{N}_0$ the dimension of its kernel. Moreover, we denote by $k_r(L)$ the number of real unstable eigenvalues of L, by $k_c(L)$ the number of quadruplets of eigenvalues with non-vanishing real and imaginary parts, and by $k_i^-(L)$ the number of purely imaginary eigenvalues of L with negative Krein signature (all counting algebraic multiplicities).

To prove instability in case (i), we notice, using similar arguments as in the proof of Theorem 6.43, that $n(L_{+,\mu}(\psi_0)) = 1$ and the nondegeneracy of ψ_0 imply that $n(L_{+,\mu}(\psi_n)) = M \geq 2$ and $z(L_{+,\mu}(\psi_n)) = 0$. Thus, [109, Theorem 4.8] applies, which yields spectral instability.

Next, we consider case (ii). As in case (i), we find $n(L_{+,\mu}(\psi_n)) = 2m$ and $z(L_{+,\mu}(\psi_n)) = 0$. Moreover, combining $L_{-,\mu}(\psi_n)\psi_n = 0$ with Sturm-Liouville theory, cf. [158, Theorem 6.3.1.(8)(3)], we obtain $n(L_{-,\mu}(\psi_n)) = 2\ell$ as well as $z(L_{-,\mu}(\psi_n)) = 1$. Using Lemma 6.46, we find that the Krein index formula in [76, Theorem 1] yields

$$k_r(L_\mu(\psi_n)) + 2k_c(L_\mu(\psi_n)) + 2k_i^-(L_\mu(\psi_n)) = 2(m+\ell) - 1$$

Hence, we have $k_r(L_\mu(\psi_n)) \geq 1$, which means that ψ_n is spectrally unstable.

We proceed with case (iii). Using similar arguments as before, we obtain $n(L_{+,\mu}(\psi_n)) = 2m$, $z(L_{+,\mu}(\psi_n)) = 0$ and $n(L_{-,\mu}(\psi_n)) = 2\ell+1$, and find that the odd function ψ_n spans the kernel of $L_{-,\mu}(\psi_n)$. Since ψ_n is odd and V is even, the operators $L_{\pm,\mu}(\psi_n)$ and $L_\mu(\psi_n)$ leave the space of even functions invariant. Therefore, we can apply the index formula to these linear operators restricted to even functions, yielding

$$\begin{aligned} &k_r(L_\mu(\psi_n)|_{\text{even}}) + 2k_c(L_\mu(\psi_n)|_{\text{even}}) + 2k_i^-(L_\mu(\psi_n)|_{\text{even}}) \\ &\qquad = n(L_{+,\mu}(\psi_n)|_{\text{even}}) + n(L_{-,\mu}(\psi_n)|_{\text{even}}) = m + \ell + 1, \end{aligned}$$

where we used that by [158, Theorem 6.3.1.(8)(3)] the eigenfunctions of the Sturm-Liouville operators $L_{\pm,\mu}(\psi_n)$ are alternating between even and odd functions and that the principal eigenfunction is even. Since $m+\ell+1$ is odd, the number of real unstable eigenvalues of $L_\mu(\psi_n)$ is greater than 1, yielding spectral instability.

We turn to case (iv). Arguing as before, we have $n(L_{+,\mu}(\psi_n)|_{\text{even}}) = m+1$, $z(L_{+,\mu}(\psi_n)|_{\text{even}}) = 0$, $n(L_{-,\mu}(\psi_n)|_{\text{even}}) = \ell$, and the even function ψ_n spans the kernel of $L_{-,\mu}(\psi_n)|_{\text{even}}$. Using Lemma 6.46 and applying the index formula, this amounts to

$$k_r(L_\mu(\psi_n)|_{\text{even}}) + 2k_c(L_\mu(\psi_n)|_{\text{even}}) + 2k_i^-(L_\mu(\psi_n)|_{\text{even}}) = m + \ell + 1 - 1.$$

Using that $m+\ell$ is odd, we infer $k_r(L_\mu(\psi_n)) \geq 1$, which implies spectral instability.

Finally, the proof of spectral instability in case (v) follows as in case (ii). □

Although the first instability condition in Theorem 6.47 is well-known, cf. [109, Theorem 4.8] and [63, Theorem 6.5], the authors are not aware that the other instability conditions in Theorem 6.47 have been derived in the literature before.

6.9 A: Projections

In this appendix, we prove some auxiliary results on finite-dimensional projections, which correspond to block matrices $P \in \mathbb{C}^{n\times n}$ satisfying $P^2 = P$.

The first result asserts that, if two projections $P, Q \in \mathbb{C}^{n\times n}$ are close in norm, then their range and kernel are complementary subspaces and the associated projection onto the range of P along the kernel of Q can be bounded in terms of P and Q.

Lemma 6.48. *Let* $n \in \mathbb{N}$. *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be projections with*

$$\|P - Q\| < 1.$$

Then, $\operatorname{ran}(P)$ *and* $\ker(Q)$ *are complementary subspaces and the projection* R *onto* $\operatorname{ran}(P)$ *along* $\ker(Q)$ *obeys the bound*

$$\|R\| \leq \frac{\|P\|}{1 - \|P - Q\|}. \tag{6.114}$$

Proof. Let $v \in \operatorname{ran}(P) \cap \ker(Q)$. Then,

$$\|v\| = \|(P - Q)v\| \leq \|P - Q\|\|v\|$$

implies $v = 0$ as $\|P - Q\| < 1$. Hence, we infer $\operatorname{ran}(P) \cap \ker(Q) = \{0\}$ and, similarly, $\operatorname{ran}(Q) \cap \ker(P) = \{0\}$. So, we must have $\operatorname{rank}(Q) + \dim\ker(P) \leq n$ and, thus, we arrive at

$$\operatorname{rank}(Q) - \operatorname{rank}(P) = \operatorname{rank}(Q) + \dim\ker(P) - \dim\ker(P) - \operatorname{rank}(P) \leq n - n = 0.$$

Similarly, $\operatorname{rank}(P) + \dim\ker(Q) \leq n$ yields $\operatorname{rank}(Q) - \operatorname{rank}(P) \geq 0$. We find

$$\dim\ker(Q) + \operatorname{rank}(P) = \dim\ker(Q) + \operatorname{rank}(Q) = n$$

and conclude that $\operatorname{ran}(P)$ and $\ker(Q)$ are complementary subspaces. Now let R be the projection onto $\operatorname{ran}(P)$ along $\ker(Q)$. Since we have $RP = P$ and $RQ = R$, it holds

$$R = RQ - RP + P,$$

which readily yields the estimate (6.114). □

Next, we show that, if two projections have the same kernel and there exist bases of their ranges which are close in norm, then the projections themselves are close in norm.

Lemma 6.49. *Let* $n \in \mathbb{N}$ *and* $k \in \{1, \ldots, n\}$. *Let* $A, B, C \in \mathbb{C}^{n \times k}$. *Suppose* $C^*A \in \mathbb{C}^{k \times k}$ *is invertible and we have*

$$\|A - B\| < \frac{1}{\|(C^*A)^{-1}\| \, \|C\|}. \tag{6.115}$$

Then, the projections

$$P = A(C^*A)^{-1}C^* \tag{6.116}$$

onto $\operatorname{ran}(A)$ *along* $\operatorname{ran}(C)^\perp$ *and*

$$Q = B(C^*B)^{-1}C^* \tag{6.117}$$

onto $\operatorname{ran}(B)$ *along* $\operatorname{ran}(C)^\perp$ *are well-defined and satisfy*

$$\begin{aligned} \|P - Q\| &\leq \|A - B\| \left\|(C^*A)^{-1}\right\| \|C\| \left(1 + \frac{(\|A\| + \|A - B\|) \left\|(C^*A)^{-1}\right\| \|C\|}{1 - \|(C^*A)^{-1}\| \, \|C\| \, \|A - B\|}\right), \\ \|P\| &\leq \|A\| \| \left\|(C^*A)^{-1}\right\| \|C\| \,. \end{aligned} \tag{6.118}$$

Proof. Since C^*A is invertible, $\operatorname{ran}(A)$ and $\operatorname{ran}(C)^\perp$ are complementary subspaces and the projection P given by (6.116) is well-defined. Upon rewriting

$$C^*B = C^*A\left(I - (C^*A)^{-1}C^*(A - B)\right)$$

and recalling (6.115), expansion as a Neumann series yields that C^*B is invertible with

$$\left\|(C^*B)^{-1} - (C^*A)^{-1}\right\| \leq \frac{\left\|(C^*A)^{-1}\right\|^2 \|C\| \|A - B\|}{1 - \|(C^*A)^{-1}\| \, \|C\| \|A - B\|}. \tag{6.119}$$

Hence, the subspaces $\operatorname{ran}(B)$ and $\operatorname{ran}(C)^{\perp}$ are complementary and the projection Q given by (6.117) is well-defined. Finally, writing

$$P - Q = (A - B)(C^*A)^{-1}C^* + (A + B - A)\left((C^*A)^{-1} - (C^*B)^{-1}\right)C^*$$

and applying (6.119) yields (6.118). □

The following result shows that a projection matrix depends analytically on a parameter $\lambda \in \mathbb{C}$ if and only if its range and the real orthogonal complement of its kernel are analytic subspaces.

Lemma 6.50. *Let $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$. Let $\Omega \subset \mathbb{C}$ be open. Let $V(\lambda), U(\lambda) \subset \mathbb{C}^n$ be complementary subspaces with $\dim(V(\lambda)) = k$ for each $\lambda \in \Omega$. Denote by $P(\lambda) \in \mathbb{C}^{n\times n}$ the projection onto $V(\lambda)$ along $U(\lambda)$. Then, the map $P\colon \Omega \to \mathbb{C}^{n\times n}$ is analytic if and only if there exist analytic maps $B_1, B_2\colon \Omega \to \mathbb{C}^{n\times k}$ such that*

$$V(\lambda) = \operatorname{ran}(B_1(\lambda)), \qquad U(\lambda) = \left\{u \in \mathbb{C}^n : z^\top u = 0 \text{ for all } z \in \operatorname{ran}(B_2(\lambda))\right\} \tag{6.120}$$

for all $\lambda \in \Omega$.

The proof of Lemma 6.50 is partly based on [19, Lemma 3.3] and requires the following technical lemma.

Lemma 6.51. *Let $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$. Let $P \in \mathbb{C}^{n\times n}$ be a projection of rank k. If $B_1 \in \mathbb{C}^{n\times k}$ is a basis of $\operatorname{ran}(P)$ and $B_2 \in \mathbb{C}^{n\times k}$ is a basis of $\operatorname{ran}(P^\top)$, then $B_2^\top B_1 \in \mathbb{C}^{k\times k}$ is invertible.*

Proof. Assume that $v \in \mathbb{C}^k$ satisfies $B_2^\top B_1 v = 0$. Then, $u := B_1 v \in \operatorname{ran}(P)$ satisfies $z^\top u = 0$ for all $z \in \operatorname{ran}(B_2) = \operatorname{ran}(P^\top)$. Hence, for all $w \in \mathbb{C}^n$ it holds $0 = (P^\top w)^\top u = w^\top P u$ implying $Pu = 0$. We conclude that $u \in \ker(P) \cap \operatorname{ran}(P) = \{0\}$. Since B_1 has full rank, we have $v = 0$. □

Proof of Lemma 6.50. Assume that P is analytic. Then, by [80, Section II.4.2] there exist analytic maps $B_1, B_2\colon \Omega \to \mathbb{C}^{n\times k}$ such that $B_1(\lambda)$ is a basis of $V(\lambda)$ and $B_2(\lambda)$ is a basis of $\operatorname{ran}(P(\lambda)^\top)$ for all $\lambda \in \Omega$. Now, it follows

$$\begin{aligned} u \in \ker(P(\lambda)) = U(\lambda) \qquad &\Leftrightarrow \qquad \forall\, v \in \mathbb{C}^n : 0 = v^\top P(\lambda) u = \left(P(\lambda)^\top v\right)^\top u \\ &\Leftrightarrow \qquad \forall\, z \in \operatorname{ran}(B_2(\lambda)) : z^\top u = 0 \end{aligned}$$

for all $\lambda \in \Omega$.

Conversely, assume that there exist analytic maps $B_1, B_2 \colon \Omega \to \mathbb{C}^{n\times k}$ such that (6.120) holds for all $\lambda \in \Omega$. Let $\lambda \in \Omega$. We have $z \in \operatorname{ran}(B_2(\lambda))$ if and only if $z^\top u = 0$ for all $u \in U(\lambda) = \ker(P(\lambda))$. If $x \in \operatorname{ran}(P(\lambda)^\top)$ we find $v \in \mathbb{C}^n$ such that $x = P(\lambda)^\top v$. This yields $x^\top u = (P(\lambda)^\top v)^\top u = v^\top P(\lambda) u = 0$ for all $u \in U(\lambda)$, which proves the inclusion $\operatorname{ran}(B_2(\lambda)) \supset \operatorname{ran}(P(\lambda)^\top)$. Equality then follows from

$$\begin{aligned}\dim\operatorname{ran}(B_2(\lambda)) &= n - \dim\ker(P(\lambda)) = \dim\ker(P(\lambda))^\perp \\ &= \dim\operatorname{ran}(P(\lambda)^*) = \dim\operatorname{ran}(P(\lambda)^\top).\end{aligned}$$

Thus, by Lemma 6.51 the matrix $B_2(\lambda)^\top B_1(\lambda) \in \mathbb{C}^{k\times k}$ is invertible. Clearly,

$$\Pi(\lambda) = B_1(\lambda)\left(B_2(\lambda)^\top B_1(\lambda)\right)^{-1} B_2(\lambda)^\top \in \mathbb{C}^{n\times n} \tag{6.121}$$

is a projection with $\ker(\Pi(\lambda)) = U(\lambda)$ and $\operatorname{ran}(\Pi(\lambda)) = V(\lambda)$. So, we must have $P(\lambda) = \Pi(\lambda)$. The formula (6.121) readily yields that P is analytic. □

6.10 B: Exponential dichotomies

Exponential dichotomies are powerful tools in the spectral analysis of linear differential operators. By reformulating the associated eigenvalue problem as a first-order nonautonomous system, they can be used to characterize invertibility as well as Fredholm properties [93, 99, 101].

A linear (nonautonomous) ordinary differential equation possesses an exponential dichotomy if it admits a fundamental set of solutions that decay exponentially in either forward or backward time.

Definition 6.52. Let $n \in \mathbb{N}$, $\mathcal{J} \subset \mathbb{R}$ an interval and $A \in C(\mathcal{J}, \mathbb{C}^{n\times n})$. Denote by $T(x, y)$ the evolution operator of the linear system

$$\phi' = A(x)\phi. \tag{6.122}$$

Equation (6.122) has *an exponential dichotomy on $\mathcal{J}$ with constants $K, \mu > 0$ and projections* $P(x) \in \mathbb{C}^{n\times n}$ if for all $x, y \in \mathcal{J}$ it holds

- $P(x)T(x, y) = T(x, y)P(y)$;
- $\|T(x, y)P(y)\| \le K\mathrm{e}^{-\mu(x-y)}$ for $x \ge y$;
- $\|T(x, y)(I - P(y))\| \le K\mathrm{e}^{-\mu(y-x)}$ for $y \ge x$.

In this appendix, we establish several results on exponential dichotomies that are relevant to our analysis. For a comprehensive introduction, we refer the reader to [33].

We start with an extension of the so-called "pasting lemma", which was first established in [35, Lemma B.7]. The pasting lemma provides a tool for gluing together exponential dichotomies on two adjacent intervals.

Lemma 6.53. *Let $n \in \mathbb{N}$, $a, b, c \in \mathbb{R}$ with $a < b < c$ and $A \in C([a,c], \mathbb{C}^{n\times n})$. Suppose that equation* (6.122) *has exponential dichotomies on both $[a,b]$ and $[b,c]$ with constants $K, \mu > 0$ and projections $P_1(x), x \in [a,b]$ and $P_2(x), x \in [b,c]$, respectively. If $\ker(P_1(b))$ and $\mathrm{ran}(P_2(b))$ are complementary subspaces, then* (6.122) *has an exponential dichotomy on $[a,c]$ with constants $C, \mu > 0$ and projections $\mathcal{Q}(x) = T(x,b)QT(b,x), x \in [a,c]$, where Q is the projection onto $\mathrm{ran}(P_2(b))$ along $\ker(P_1(b))$ and $T(x,y)$ denotes the evolution of system* (6.122)*. Moreover, we have $C = K + K^2\|Q\| + K^3$ and*

$$\|P_1(a) - \mathcal{Q}(a)\| \leq CK\mathrm{e}^{-2\mu(b-a)}, \qquad \|P_2(c) - \mathcal{Q}(c)\| \leq CK\mathrm{e}^{-2\mu(c-b)}.$$

Proof. Observe that $\mathcal{Q}(x)T(x,y) = T(x,y)\mathcal{Q}(y)$ for $x, y \in [a,c]$. The exposition in [33, pp. 16-17] shows that (6.122) possesses an exponential dichotomy on the intervals $[a,b]$ and $[b,c]$ with constants $C, \mu > 0$ and projections $\mathcal{Q}(x)$. We need to show that the dichotomy estimates persist on the union $[a,c] = [a,b] \cup [b,c]$. Take $x \in [b,c]$ and $y \in [a,b]$. We estimate

$$\|T(x,y)\mathcal{Q}(y)\| \leq \|T(x,b)P_2(b)\|\|Q\|\|P_1(b)T(b,y)\| \leq K^2\|Q\|\mathrm{e}^{-\mu(x-y)} \leq C\mathrm{e}^{-\mu(x-y)},$$

where we use $P_2(b)Q = Q$ and $QP_1(b) = Q$. Similarly, one estimates $\|T(y,x)(I - \mathcal{Q}(x))\| \leq C\mathrm{e}^{-\mu(x-y)}$ for $x \in [b,c]$ and $y \in [a,b]$. Finally, using $QP_1(b) = Q$ again, we infer

$$\|P_1(a) - \mathcal{Q}(a)\| \leq \|T(a,b)(I - \mathcal{Q}(b))\| \, \|P_1(b)T(b,a)\| \leq CK\mathrm{e}^{-2\mu(b-a)}.$$

Similarly, we derive $\|P_2(c) - \mathcal{Q}(c)\| \leq CK\mathrm{e}^{-2\mu(c-b)}$. □

Next, we obtain an approximation result for the projections of two exponential dichotomies defined on the same interval.

Lemma 6.54. *Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $a < b$ and $A \in C([a,b], \mathbb{C}^{n\times n})$. Suppose equation* (6.122) *admits two exponential dichotomies on $[a,b]$ with constants $K_{1,2}, \mu_{1,2} > 0$ and projections $P_{1,2}(x)$. Then, we have*

$$\|P_1(x) - P_2(x)\| \leq K_1K_2\left(\mathrm{e}^{-(\mu_1+\mu_2)(x-a)} + \mathrm{e}^{-(\mu_1+\mu_2)(b-x)}\right),$$

for all $x \in [a, b]$.

Proof. Let $T(x, y)$ be the evolution of system (6.122). We estimate

$$\begin{aligned} &\|P_1(x) - P_2(x)\| \\ &\qquad \leq \|P_1(x)(I - P_2(x))\| + \|(I - P_1(x))P_2(x)\| \\ &\qquad \leq \|P_1(x)T(x, a)\| \, \|T(a, x)(I - P_2(x))\| + \|(I - P_1(x))T(x, b)\| \, \|T(b, x)P_2(x)\| \\ &\qquad \leq K_1 K_2 \left(\mathrm{e}^{-(\mu_1+\mu_2)(x-a)} + \mathrm{e}^{-(\mu_1+\mu_2)(b-x)}\right), \end{aligned}$$

for all $x \in [a, b]$. □

The following result establishes periodicity of the evolution operator, as well as the dichotomy projections, when the underlying system has periodic coefficients.

Lemma 6.55. *Let* $n \in \mathbb{N}$ *and* $L > 0$. *Let* $A \in C(\mathbb{R}, \mathbb{C}^{n\times n})$ *be L-periodic. Then, the evolution* $T(x, y)$ *of* (6.122) *satisfies* $T(x, y) = T(x - L, y - L)$ *for each* $x, y \in \mathbb{R}$. *Moreover, if* (6.122) *has an exponential dichotomy on* $\mathbb{R}$ *with projections* $P(x)$, *then* $P(\cdot)$ *is also L-periodic.*

Proof. By Floquet's theorem, cf. [78, Theorem 2.1.27], there exist an L-periodic function $Q\colon \mathbb{R} \to \mathbb{C}^{n\times n}$ and a matrix $B \in \mathbb{C}^{n\times n}$ such that $Q(x)$ is invertible for each $x \in \mathbb{R}$ and we have $T(x, y) = Q(x)\mathrm{e}^{B(x-y)}Q(y)^{-1}$ for each $x, y \in \mathbb{R}$. Hence, we arrive at $T(x, y) = T(x-L, y-L)$ for each $x, y \in \mathbb{R}$. Now suppose that (6.122) has an exponential dichotomy on $\mathbb{R}$. Then, the matrix B must be hyperbolic. Let $\mathcal{P}$ be the spectral projection of B onto its stable space. Then, by uniqueness of the exponential dichotomy, cf. [33, p. 19], it holds $P(x) = Q(x)\mathcal{P}Q(x)^{-1}$ for $x \in \mathbb{R}$ and, thus, P is L-periodic. □

We proceed by showing that the operator $\frac{d}{dx} - A(x)$ is invertible as a map from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$, and, in case of L-periodic coefficients of A, also as a map from $H^1_{\mathrm{per}}(0, L)$ to $L^2_{\mathrm{per}}(0, L)$, provided that (6.122) has an exponential dichotomy on $\mathbb{R}$. Moreover, we prove that the inverse operator can be bounded in terms of the dichotomy constants and the supremum norm of A.

Lemma 6.56. *Let* $n \in \mathbb{N}$ *and* $L > 0$. *Let both* $g \in C(\mathbb{R}, \mathbb{C}^n)$ *and* $A \in C(\mathbb{R}, \mathbb{C}^{n\times n})$ *be bounded. Suppose that* (6.122) *has an exponential dichotomy on* $\mathbb{R}$ *with constants* $K, \mu > 0$. *Then, the inhomogeneous problem*

$$\phi' = A(x)\phi + g(x), \tag{6.123}$$

possesses a unique bounded solution $\phi \in C^1(\mathbb{R})$. If $g \in L^2(\mathbb{R})$, then ϕ lies in $H^1(\mathbb{R})$ and obeys the estimate

$$\|\phi\|_{H^1} \leq \left((1 + \|A\|_{L^\infty}) \frac{2K}{\mu} + 1\right) \|g\|_{L^2}. \tag{6.124}$$

Moreover, if A is L-periodic and $g \in L^2_{per}(0,L)$, then ϕ lies in $H^1_{per}(0,L)$ and satisfies

$$\|\phi\|_{H^1_{per}(0,L)} \leq \left((1 + \|A\|_{L^\infty}) \frac{2K}{\mu} + 1\right) \|g\|_{L^2_{per}(0,L)}. \tag{6.125}$$

Proof. Let $T(x,y)$ be the evolution operator of (6.122). Denote by $P(x)$ the projections associated with the exponential dichotomy of (6.122) on $\mathbb{R}$. Define $\phi\colon \mathbb{R} \to \mathbb{C}^n$ by

$$\phi(x) = \int_{-\infty}^{x} T(x,y)P(y)g(y)\mathrm{d}y - \int_{x}^{\infty} T(x,y)(I - P(y))g(y)\mathrm{d}y.$$

We note that the properties of the exponential dichotomy, the fundamental theorem of calculus and the fact that g and A are bounded and continuous, readily yield that ϕ is well-defined, bounded, continuously differentiable and solves (6.123). Due to the exponential dichotomy of system (6.122) on $\mathbb{R}$ its only bounded solution is 0. Therefore, the bounded solution ϕ of (6.123) is unique. Finally, we estimate

$$\|\phi(x)\| \leq K \int_{\mathbb{R}} \mathrm{e}^{-\mu|x-y|} \|g(y)\|\mathrm{d}y = K \int_{\mathbb{R}} \mathrm{e}^{-\mu|y|} \|g(x-y)\|\mathrm{d}y, \tag{6.126}$$

for $x \in \mathbb{R}$.

Take $g \in L^2(\mathbb{R})$. Applying Young's convolution inequality to (6.126) leads to the estimate

$$\|\phi\|_{L^2} \leq \frac{2K}{\mu} \|g\|_{L^2},$$

which implies $\phi \in L^2(\mathbb{R})$. So, using the fact that ϕ solves (6.123), we establish

$$\|\phi'\|_{L^2} \leq \|A\|_{L^\infty} \|\phi\|_{L^2} + \|g\|_{L^2} \leq \left(\|A\|_{L^\infty} \frac{2K}{\mu} + 1\right) \|g\|_{L^2},$$

which proves $\phi \in H^1(\mathbb{R})$ and establishes (6.124).

Suppose A is L-periodic and $g \in L^2_{\text{per}}(0, L)$. Then, by Lemma 6.55 we deduce

$$\begin{aligned}\phi(x+L) &= \int_{-\infty}^{x+L} T(x+L,y)P(y)g(y)\mathrm{d}y - \int_{x+L}^{\infty} T(x+L,y)(I-P(y))g(y)\mathrm{d}y \\ &= \int_{-\infty}^{x+L} T(x,y-L)P(y-L)g(y-L)\mathrm{d}y \\ &\quad - \int_{x+L}^{\infty} T(x,y-L)(I-P(y-L))g(y-L)\mathrm{d}y = \phi(x)\end{aligned}$$

for $x \in \mathbb{R}$. Hence, ϕ is also L-periodic. Using (6.126) and Hölder's inequality we estimate

$$\begin{aligned}\|\phi\|^2_{L^2_{\text{per}}(0,L)} &= \int_0^L \|\phi(x)\|^2 \mathrm{d}x \le K^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-\mu(|y|+|z|)} \int_0^L \|g(x-y)\|\|g(x-z)\|\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\le K^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-\mu(|y|+|z|)} \|g\|^2_{L^2_{\text{per}}(0,L)}\mathrm{d}y\mathrm{d}z \le \frac{4K^2}{\mu^2}\|g\|^2_{L^2_{\text{per}}(0,L)},\end{aligned}$$

which proves $\phi \in L^2_{\text{per}}(0, L)$. Combining the later with the fact that ϕ solves (6.123), we obtain

$$\|\phi'\|_{L^2_{\text{per}}(0,L)} \le \|A\|_{L^\infty}\|\phi\|_{L^2_{\text{per}}(0,L)} + \|g\|_{L^2_{\text{per}}(0,L)} \le \left(\|A\|_{L^\infty}\frac{2K}{\mu} + 1\right)\|g\|_{L^2_{\text{per}}(0,L)},$$

which implies $\phi \in H^1_{\text{per}}(0, L)$ and establishes (6.125). □

Our next step is to prove that, if the linear differential operator $\mathcal{L}(\underline{u}) - \lambda$, defined in §6.2, is invertible, then the first-order formulation (6.6) of the associated eigenvalue problem has an exponential dichotomy on $\mathbb{R}$. In conjunction with lemma 6.56, this characterizes invertibility of the differential operator $\mathcal{L}(\underline{u}) - \lambda$ in terms of exponential dichotomies.

Lemma 6.57. *Let $\underline{u} \in C(\mathbb{R})$ be bounded and $\lambda \in \mathbb{C}$. If the linear operator $\mathcal{L}(\underline{u}) - \lambda$ is invertible, then system* (6.6) *has an exponential dichotomy on $\mathbb{R}$.*

Proof. By assumption there exists for each $g \in L^2(\mathbb{R})$ a unique solution $u \in H^k(\mathbb{R})$ of the resolvent problem

$$(\mathcal{L}(\underline{u}) - \lambda)\, u = g. \tag{6.127}$$

This implies that for each $\psi \in H^{k-1}(\mathbb{R})$ the inhomogeneous problem

$$U' = \mathcal{A}\,(x, \underline{u}(x); \lambda)\, U + \psi \tag{6.128}$$

possesses a solution $U \in H^1(\mathbb{R})$, which is given by $U = (u, \partial_x u - \psi_1, \ldots, \partial_x^{k-1} u - \sum_{i=1}^{k-1} \partial_x^{k-1-i}\psi_i)^\top$, where $u \in H^k(\mathbb{R})$ is the solution of the resolvent problem (6.127) with inhomogeneity

$$g = \sum_{i=1}^{k} \alpha_i \sum_{j=1}^{i} \partial_x^{i-j}\psi_j \in L^2(\mathbb{R}).$$

Let $\mathcal{J} = (-\infty, 0]$ or $\mathcal{J} = [0, \infty)$. For each $\psi \in H^{k-1}(\mathcal{J})$ we find a solution $U \in H^1(\mathcal{J})$ of (6.128). In the language of Massera and Schäffer, the pair $(H^{k-1}(\mathcal{J}), H^1(\mathcal{J}))$ is regularly admissible for equation (6.128), cf. [93, §51]. Moreover, in the ordered lattice $\mathrm{b}\mathcal{N}(\mathcal{J})$ of all Banach spaces, which are stronger than the Bochner space $L(\mathcal{J})$ of strongly measurable, locally Bochner integrable functions $h \colon \mathcal{J} \to \mathbb{C}^{km}$ endowed with the topology of convergence in mean, the space $H^{k-1}(\mathcal{J})$ is not weaker than $L^1(\mathcal{J})$, and $H^1(\mathcal{J})$ is not stronger than $L_0^\infty(\mathcal{J})$, where $L_0^\infty(\mathcal{J})$ is the L^∞-closure of all functions with compact (essential) support in $L^\infty(\mathcal{J})$ endowed with the supremum norm, see [93, §20 and §21]. That is, the pair $(H^{k-1}(\mathcal{J}), H^1(\mathcal{J}))$ is not weaker than $(L^1(\mathcal{J}), L_0^\infty(\mathcal{J}))$, cf. [93, §50]. Now, [93, Theorem 64.B] yields that system (6.6) has exponential dichotomies on both half-lines, $\mathcal{J} = (-\infty, 0]$ and $\mathcal{J} = [0, \infty)$. We denote by $P_\pm(\pm x), x \geq 0$ the associated projections.

We show that the exponential dichotomies for (6.6) on both half-lines can be pasted together to yield an exponential dichotomy for (6.6) on $\mathbb{R}$. First, we observe that it must hold $\ker(P_-(0)) \cap \mathrm{ran}(P_+(0)) = \{0\}$, since any solution $U \in H^1(\mathbb{R})$ of (6.6) with $U(0) \in \ker(P_-(0)) \cap \mathrm{ran}(P_+(0))$ is exponentially localized and, thus, generates an element $u = U_1 \in H^k(\mathbb{R})$ lying in the kernel of $\mathcal{L}(\underline{u}) - \lambda$, which is invertible. On the other hand, the adjoint problem

$$U' = -\mathcal{A}\left(x, \underline{u}(x); \lambda\right)^* U \tag{6.129}$$

has the evolution $\Phi_{\mathrm{ad}}(x, y) = \Phi(y, x)^*$, where $\Phi(x, y)$ is the evolution of (6.6). Therefore, (6.129) also possesses exponential dichotomies on both half lines with projections $I - P_\pm(\pm x)^*, x \geq 0$. Any solution $U \in H^1(\mathbb{R})$ of (6.129) with $U(0) \in \ker(I - P_-(0)^*) \cap \mathrm{ran}(I - P_+(0)^*) = \ker(P_-(0))^\perp \cap \mathrm{ran}(P_+(0))^\perp$ yields an element $u = U_k \in H^k(\mathbb{R})$ in the kernel of the adjoint operator $(\mathcal{L}(\underline{u}) - \lambda)^*$, which is invertible since $\mathcal{L}(\underline{u}) - \lambda$ is. Hence, we have $\ker(P_-(0))^\perp \cap \mathrm{ran}(P_+(0))^\perp = \{0\}$, which, in combination with $\ker(P_-(0)) \cap \mathrm{ran}(P_+(0)) = \{0\}$, implies that $\ker(P_-(0))$ and $\mathrm{ran}(P_+(0))$ are complementary subspaces. Hence, Lemma 6.53 implies that (6.6) admits an exponential dichotomy on $\mathbb{R}$. □

An important property of exponential dichotomies, often referred to as roughness or robustness, is their persistence under small perturbations, cf. [33, Section 4]. Additionally, exponential dichotomy projections can be chosen to depend analytically on parameters if the underlying system does, see [34, Appendix A] and references therein. Leveraging the results from [33, 34], we show that, if a system, depending analytically on a complex parameter λ, admits an exponential dichotomy on a half line with λ-uniform constants, then the exponential dichotomy persists under perturbations and an analytic choice of projection is always possible.

Lemma 6.58. *Let $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$. Let $\Omega \subset \mathbb{C}$ be open and let $\lambda_0 \in \Omega$. Let $A\colon [0, \infty) \times \Omega \to \mathbb{C}^{n \times n}$ be such that $A(\cdot; \lambda)$ is continuous for each $\lambda \in \Omega$ and $A(x; \cdot)$ is analytic for each $x \geq 0$. Suppose that there exist $K, \mu > 0$ such that for each $\lambda \in \Omega$ system*

$$\phi' = A(x; \lambda)\phi \tag{6.130}$$

has an exponential dichotomy on $[0, \infty)$ with constants $K, \mu > 0$ and projections $P(x; \lambda)$ of rank k. Then, there exist constants $C, \delta_0, \varrho_0, \theta > 0$ such that the closed disk $\overline{B}_{\lambda_0}(\varrho_0)$ lies in Ω and for all $\delta \in (0, \delta_0)$ and $B \in C([0, \infty), \mathbb{C}^{n \times n})$ with $\|B\|_{L^\infty} \leq \delta$ the perturbed system

$$\phi' = (A(x; \lambda) + B(x))\, \phi \tag{6.131}$$

has for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ an exponential dichotomy on $[0, \infty)$ with λ- and δ-independent constants and projections $Q(x; \lambda)$ satisfying the following properties:

1. *The map $Q(x; \cdot)\colon B_{\lambda_0}(\varrho_0) \to \mathbb{C}^{n \times n}$ is analytic for each $x \geq 0$.*
2. *We have*

$$\|Q(0; \lambda) - \tilde{P}(\lambda)\| \leq C\delta, \qquad \|\tilde{P}(\lambda)\| \leq C$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where $\tilde{P}(\lambda)$ is the projection onto $\operatorname{ran}(P(0; \lambda))$ along $\operatorname{ran}(P(0; \lambda_0))^\perp$.

3. *The estimate*

$$\|Q(x; \lambda) - P(x; \lambda)\| \leq C\left(\delta + \mathrm{e}^{-\theta x}\right) \tag{6.132}$$

holds for each $x \geq 0$ and $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$.

Proof. Since system (6.130) depends analytically on λ and the subspace $\operatorname{ran}(P(0; \lambda))$ is by [33, p. 19] uniquely determined, [34, Lemma A.2] and [80, Section II.4.2] yield that there exists an analytic map $B_\mathrm{s}\colon \Omega \to \mathbb{C}^{n \times k}$ such that $B_\mathrm{s}(\lambda)$ is a basis of $\operatorname{ran}(P(0; \lambda))$ for each $\lambda \in \Omega$. Since B_s is analytic, there exists a closed disk $\overline{B}_{\lambda_0}(\varrho_0) \subset \Omega$ of some radius $\varrho_0 > 0$ such that $\det(B_\mathrm{s}(\lambda_0)^* B_\mathrm{s}(\lambda)) \neq 0$ for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Thus, $\operatorname{ran}(P(0; \lambda_0))^\perp$ complements $\operatorname{ran}(P(0; \lambda))$ for all $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$.

By roughness of exponential dichotomies, cf. [33, Theorem 4.1], there exists a constant $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)$ and $B \in C([0, \infty), \mathbb{C}^{n \times n})$ with $\|B\|_{L^\infty} \leq \delta$, the perturbed system (6.131) admits an exponential dichotomy on $[0, \infty)$ with constants $K_1, \mu_1 > 0$, depending on K and μ only, and projections $\mathcal{Q}(x; \lambda)$ satisfying

$$\|\mathcal{Q}(x; \lambda) - P(x; \lambda)\| \leq K_1 \delta \tag{6.133}$$

for all $x \geq 0$ and $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$.

Let $\delta \in (0, \delta_0)$ and $B \in C([0, \infty), \mathbb{C}^{n \times n})$ with $\|B\|_{L^\infty} \leq \delta$. Since (6.131) depends analytically on λ and the subspace $\mathrm{ran}(\mathcal{Q}(x; \lambda))$ is by [33, p. 19] uniquely determined, [34, Lemma A.2] and [80, Section II.4.2] yield that $\mathrm{ran}(\mathcal{Q}(x; \lambda))$ possesses a basis, which is analytic in λ on $B_{\lambda_0}(\varrho_0)$. Set $\check{B}_{\mathrm{s}}(\lambda) = \mathcal{Q}(0; \lambda) B_{\mathrm{s}}(\lambda)$. Estimate (6.133), analyticity of B_{s} on Ω and compactness of $\overline{B}_{\lambda_0}(\varrho_0)$ yield a λ- and δ-independent constant $M_0 > 0$ such that it holds

$$\left\| B_{\mathrm{s}}(\lambda) - \check{B}_{\mathrm{s}}(\lambda) \right\| \leq M_0 \delta, \qquad \|B_{\mathrm{s}}(\lambda)\| \leq M_0$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. So, taking $\delta_0 > 0$ smaller if necessary, $\check{B}_{\mathrm{s}}(\lambda)$ is a basis of $\mathrm{ran}(\mathcal{Q}(0; \lambda))$ and we can arrange for

$$\left\| B_{\mathrm{s}}(\lambda) - \check{B}_{\mathrm{s}}(\lambda) \right\| \leq \frac{1}{2 \left\| (B_{\mathrm{s}}(\lambda_0)^* B_{\mathrm{s}}(\lambda))^{-1} \right\| \|B_{\mathrm{s}}(\lambda_0)\|}$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Hence, taking $\delta_0 > 0$ smaller if necessary, Lemma 6.49 demonstrates that the projections $\tilde{P}(\lambda)$ onto $\mathrm{ran}(P(0; \lambda))$ along $\mathrm{ran}(P(0; \lambda_0))^\perp$ and $Q(0; \lambda)$ onto $\mathrm{ran}(\mathcal{Q}(0; \lambda))$ along $\mathrm{ran}(P(0; \lambda_0))^\perp$ are well-defined and there exists a λ- and δ-independent constant $M_1 > 0$ such that

$$\|Q(0; \lambda) - \tilde{P}(\lambda)\| \leq M_1 \delta, \qquad \|\tilde{P}(\lambda)\|, \|Q(0; \lambda)\| \leq M_1$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$. Since $\mathrm{ran}(P(0; \lambda))$ and $\mathrm{ran}(\mathcal{Q}(0; \lambda))$ have bases which are analytic in λ on $B_{\lambda_0}(\varrho_0)$ and the subspace $\mathrm{ran}(P(0; \lambda_0))^\perp$ is independent of λ, the projections $\tilde{P}(\lambda)$ and $Q(0; \lambda)$ are analytic in λ on $B_{\lambda_0}(\varrho_0)$ by Lemma 6.50. Hence, recalling that (6.131) has an exponential dichotomy on $[0, \infty)$ with constants $K_1, \mu_1 > 0$ and projections $\mathcal{Q}(x; \lambda)$, the exposition in [33, pp. 16-17] implies that (6.131) admits an exponential dichotomy on $[0, \infty)$ with constants $C_1, \mu_1 > 0$ with $C_1 = K_1 + K_1^2 M_1 + K_1^3$ and projections $Q(x; \lambda) = T(x, 0; \lambda) Q(0; \lambda) T(0, x; \lambda)$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$, where $T(x, y; \lambda)$ is the evolution of system (6.131) which depends analytically on λ by [78, Lemma 2.1.4]. Therefore, $Q(x; \lambda)$ is for each $x \geq 0$ analytic in λ on $B_{\lambda_0}(\varrho_0)$. Finally, Lemma 6.54 yields

$$\|Q(x;\lambda) - \mathcal{Q}(x;\lambda)\| \leq 2C_1 K_1 \mathrm{e}^{-2\mu_1 x} \tag{6.134}$$

for each $\lambda \in \overline{B}_{\lambda_0}(\varrho_0)$ and $x \geq 0$. Combining (6.134) with (6.133) we arrive at (6.132), which completes the proof. □

6.11 C: Compactness of a multiplication operator

In this appendix, we prove an auxiliary compactness result for multiplication operators mapping from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$.

Lemma 6.59. *Let $g \in H^1(\mathbb{R})$. The multiplication operator $A\colon H^1(\mathbb{R}) \to L^2(\mathbb{R})$ given by $Au = gu$ is well-defined and compact.*

Proof. The fact that A is well-defined follows directly from the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Let $V \subset H^1(\mathbb{R})$ be a bounded subset and let $\varepsilon > 0$. Since the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous, there exists $K > 0$ such that for each $u \in V$ we have $\|u\|_{L^\infty} \leq K$. There exists $R > 0$ such that

$$\int_{\mathbb{R}\setminus[-R,R]} |g(x)|^2 \mathrm{d}x \leq \frac{\varepsilon^2}{2K^2},$$

implying

$$\int_{\mathbb{R}\setminus[-R,R]} |(Au)(x)|^2 \mathrm{d}x \leq \|u\|_{L^\infty}^2 \int_{\mathbb{R}\setminus[-R,R]} |g(x)|^2 \mathrm{d}x \leq \frac{\varepsilon^2}{2},$$

for all $u \in V$.

By the Rellich-Kondrachov theorem the embedding $H^1((-R,R)) \hookrightarrow L^2((-R,R))$ is compact. The set $W = \{(gu)|_{(-R,R)} : u \in V\}$ is bounded in $H^1((-R,R))$, because we have $\|(gu)|_{(-R,R)}\|_{H^1} \leq \|g\|_{H^1}\|u\|_{L^\infty} \leq K\|g\|_{H^1}$ for each $u \in V$, where $h|_{(-R,R)}$ denotes the restriction of a function $h \in H^1(\mathbb{R})$ to the interval $(-R,R)$. So, by compactness of the embedding

$H^1((-R,R)) \hookrightarrow L^2((-R,R))$, there exists a finite subset $G \subset L^2((-R,R))$ such that for each $h \in W$ there exists $f \in G$ with

$$\int_{-R}^{R} |h(x) - f(x)|^2 \mathrm{d}x \leq \frac{\varepsilon^2}{2}.$$

We conclude that for each $u \in V$ there exists $f \in G$ such that

$$\int_{\mathbb{R}} |(Au)(x) - f(x)\mathbf{1}_{(-R,R)}(x)|^2 \mathrm{d}x$$
$$= \int_{\mathbb{R}\setminus[-R,R]} |(Au)(x)|^2 \mathrm{d}x + \int_{-R}^{R} |g(x)u(x) - f(x)|^2 \mathrm{d}x \leq \varepsilon^2,$$

where $\mathbf{1}_{(-R,R)}$ is the indicator function of the interval $(-R,R)$, whence $\|Au - f\mathbf{1}_{(-R,R)}\|_{L^2} \leq \varepsilon$. We conclude that the set $A[V]$ is precompact in $L^2(\mathbb{R})$, which concludes the proof. □

7 Multi-solitons in the Lugiato-Lefever equation with periodic potential

In this chapter, we apply the theory for multiple pulse solutions and periodic waves developed in Chapter 6 to the Lugiato-Lefever equation (5.4) with a periodic potential. This equation, which was introduced in Chapter 5, reads

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + \mathrm{i}\varepsilon V(x)\partial_x u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f, \qquad (x,t) \in \mathbb{R}^2, \tag{7.1}$$

for a complex valued amplitude function $u(x,t) \in \mathbb{C}$, a T-periodic potential $V \in C^1(\mathbb{R},\mathbb{R})$ and constants $d \neq 0$, $\zeta, \varepsilon \in \mathbb{R}$, $f, \mu > 0$. We end this chapter with an outlook on modulated solitons.

7.1 Existence and stability of multi-solitons

In order to fit (7.1) into the framework of Chapter 6, we write the equation in terms of real and imaginary parts,

$$\partial_t \mathbf{u} = J\big(- d\partial_x^2 \mathbf{u} + \zeta \mathbf{u} - (\mathbf{u}_1^2 + \mathbf{u}_2^2)\mathbf{u}\big) + \varepsilon V(x)\partial_x \mathbf{u} - \mu \mathbf{u} + \mathbf{F}, \tag{7.2}$$

where the solution vector is given by $\mathbf{u} = (\mathrm{Re}(u), \mathrm{Im}(u))^\top$ and

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{F} := \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Stationary solutions of (7.2) are then found by looking at the coupled system of ordinary differential equations

$$J\big(- d\partial_x^2 \mathbf{u} + \zeta \mathbf{u} - (\mathbf{u}_1^2 + \mathbf{u}_2^2)\mathbf{u}\big) + \varepsilon V(x)\partial_x \mathbf{u} - \mu \mathbf{u} + \mathbf{F} = 0. \tag{7.3}$$

For $\underline{\mathbf{u}} \in L^\infty(\mathbb{R})$, we introduce the closed and densely defined linear operators $L(\underline{\mathbf{u}}), \mathcal{L}(\underline{\mathbf{u}})\colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$L(\underline{\mathbf{u}}) = -d\partial_x^2 + \zeta - \begin{pmatrix} 3\underline{\mathbf{u}}_1^2 + \underline{\mathbf{u}}_2^2 & 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 \\ 2\underline{\mathbf{u}}_1\underline{\mathbf{u}}_2 & \underline{\mathbf{u}}_1^2 + 3\underline{\mathbf{u}}_2^2 \end{pmatrix}, \qquad \mathcal{L}(\underline{\mathbf{u}}) = JL(\underline{\mathbf{u}}) + \varepsilon V(x)\partial_x - \mu.$$

If $\underline{\mathbf{u}} \in L^\infty(\mathbb{R})$ is a stationary solution of (7.3), then the linearization of (7.3) about $\underline{\mathbf{u}}$ equals $\mathcal{L}(\underline{\mathbf{u}})$ and the associated eigenvalue problem reads

$$\mathcal{L}(\underline{\mathbf{u}})\mathbf{u} = \lambda \mathbf{u}.$$

We emphasis that the existence problem (7.3) and associated eigenvalue problem are indeed of the form (6.4) and (6.5), respectively.

In order to construct (periodic) multi-soliton solutions to (7.1) with Theorems 6.5 and 6.8 we require the existence of a nondegenerate primary 1-soliton. In Section 5.7, we demonstrate the existence of such a 1-soliton, provided that an effective potential has a simple zero. To be more precise, the outcome of the analysis in Section 5.7 is summarized as follows.

1. In the setting of Section 5.7, assume that the effective potential $V_{\text{eff}}(\sigma)$ has a simple zero $\sigma_0 \in \mathbb{R}$. Then nondegenerate 1-soliton solutions $\underline{\mathbf{u}}_\varepsilon \in H^2(\mathbb{R}) \oplus \mathbb{R}^2$ of (7.3) for small values of ε bifurcate from the spectrally stable solitary wave solution of (7.3) for $\varepsilon = 0$ which is shifted by σ_0.
2. The 1-soliton $\underline{\mathbf{u}}_\varepsilon$ bifurcating at the shift σ_0 is strongly spectrally stable if $V_{\text{eff}}'(\sigma_0)\varepsilon > 0$ and spectrally unstable if $V_{\text{eff}}'(\sigma_0)\varepsilon < 0$.

For simplicity we set $d = 1$. Let us fix the remaining parameters $\zeta, \mu, f > 0$ and let us assume that a primary 1-soliton exists. Applying the machinery of Chapter 6 now yields families of multi-solitons and periodic pulse solutions of (7.3).

Theorem 7.1. *Let $M \in \mathbb{N}$ and let $\varepsilon > 0$ be sufficiently small. For $j = 0, \dots, M$ let $\underline{\mathbf{u}}_{\varepsilon,j} = \mathbf{z}_{\varepsilon,j} + \mathbf{v} \in H^2(\mathbb{R}) \oplus \mathbb{R}^2$ be a nondegenerate 1-soliton solution of* (7.3) *where $\mathbf{z}_{\varepsilon,j} \in H^2(\mathbb{R})$ is the localized part that depends on ε and $\mathbf{v} \in \mathbb{R}^2$ is the constant background wave that solves the system of algebraic equations $J\big(\zeta\mathbf{v} - |\mathbf{v}|^2\mathbf{v}\big) - \mu\mathbf{v} + \mathbf{F} = 0$. Then, the following statements are true.*

(i) There exist $N_0 \in \mathbb{N}, C > 0$ *such that for all* $n \geq N_0$ *there exists a multi-soliton solution* $\underline{\mathbf{u}}_{\varepsilon,n} \in H^2(\mathbb{R}) \oplus \mathbb{R}^2$ *of* (7.3) *obeying the bound*

$$\left\| \underline{\mathbf{u}}_{\varepsilon,n} - \mathbf{v} - \sum_{j=1}^{M} \mathbf{z}_{\varepsilon,j}(\cdot - jnT) \right\|_{H^2} \leq C \sum_{j=1}^{M} \|\mathbf{z}_{\varepsilon,j}\|_{H^2(\mathbb{R}\setminus[-\frac{n}{2}T+1,\frac{n}{2}T-1])}.$$

In particular, we find

$$\sum_{j=1}^{M} \|\mathbf{z}_{\varepsilon,j}\|_{H^2(\mathbb{R}\setminus[-\frac{n}{2}T+1,\frac{n}{2}T-1])} \to 0 \qquad \text{as } n \to \infty.$$

Moreover, $\underline{\mathbf{u}}_{\varepsilon,n}$ *is a nondegenerate solution of* (7.3).

(ii) There exists $L \in \mathbb{N}$ *and* $C > 0$ *such that for all* $\ell \geq L$ *there exists a* ℓT*-periodic soliton* $\underline{\mathbf{u}}_{per,\varepsilon,\ell} \in H^2_{per}(0, \ell T)$ *of* (7.3) *such that*

$$\|\underline{\mathbf{u}}_{per,\varepsilon,\ell} - \underline{\mathbf{u}}_{\varepsilon,0}\|_{H^2_{per}(0,\ell T)} \leq C \|\mathbf{z}_{\varepsilon,0}\|_{H^2(\mathbb{R}\setminus(-\frac{\ell}{6}T,\frac{\ell}{6}T))}.$$

In particular, $\|\mathbf{z}_{\varepsilon,0}\|_{H^2(\mathbb{R}\setminus(-\frac{\ell}{6}T,\frac{\ell}{6}T))} \to 0$ *as* $\ell \to \infty$*. Moreover, the solution* $\underline{\mathbf{u}}_{per,\varepsilon,\ell}$ *is nondegenerate.*

The solutions obtained in Theorem 7.1 resemble sequences of well-separated pulses pinned to the zeros of the effective potential in the spatial domain. Numerical illustrations of these multi-solitons are provided in Figure 7.1. Unlike the results presented in Chapter 5, the wavelength of the periodic solutions here is not prescribed and generally cannot be controlled. However, with additional control over the pulse distance we obtain better control over the waveform of the solutions and can even design specific types of solutions, such as 1-, 2-, or 3-soliton configurations.

Now let us proceed with the spectral stability analysis of the solutions constructed in Theorem 7.1.

Theorem 7.2. *Let* $\underline{\mathbf{u}}_{\varepsilon,n} \in H^2(\mathbb{R}) \oplus \mathbb{R}^2$ *be a* M*-soliton and* $\underline{\mathbf{u}}_{per,\varepsilon,\ell} \in H^2_{per}(0, \ell T)$ *be a periodic soliton as in Theorem 7.1 for* $\varepsilon > 0$ *and* $n \geq N_0$, $\ell \geq L$.

(i) Assume that all constituent 1-solitons $\underline{\mathbf{u}}_{\varepsilon,j}$ *with* $j = 1, \ldots, M$ *of the* M*-soliton are strongly spectrally stable. Then, the* M*-soliton* $\underline{\mathbf{u}}_{\varepsilon,n}$ *is strongly spectrally stable for* n *sufficiently large.*

(ii) Assume that one of the constituent 1-solitons $\underline{\mathbf{u}}_{\varepsilon,j_0}$ *for some* $j_0 \in \{1, \ldots, M\}$ *of the* M*-soliton is spectrally unstable. Then, the* M*-soliton* $\underline{\mathbf{u}}_{\varepsilon,n}$ *is spectrally unstable for* n *sufficiently large.*

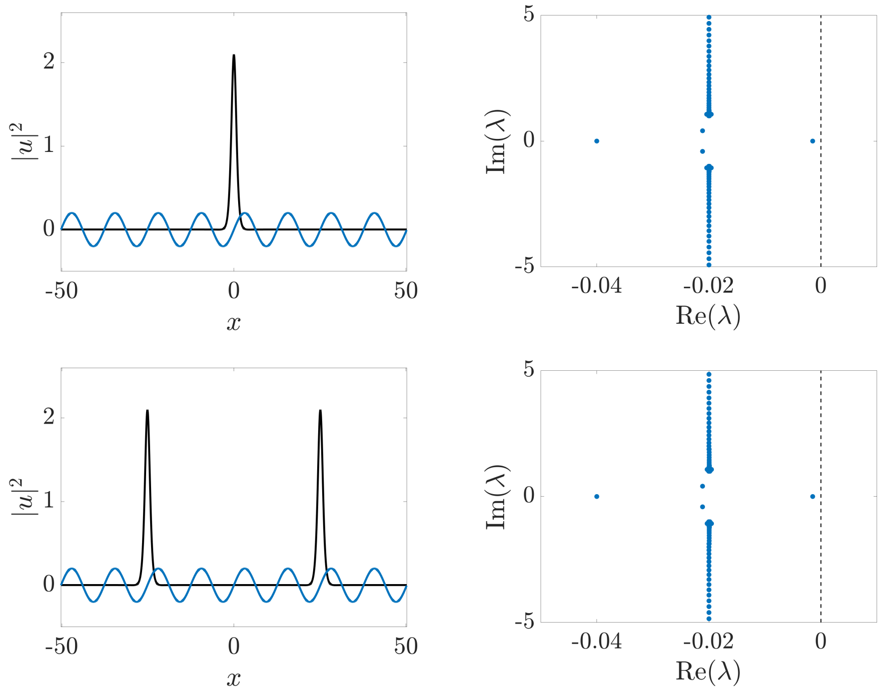

Figure 7.1: Top: stable periodic 1-soliton solution of (7.3) pinned to the zero of the periodic potential $V(x)$ (blue) with its spectrum, as established in Theorem 7.1. Bottom: corresponding stable 2-soliton and its spectrum. The system coefficients are $d = 1, \zeta = 1, f = 2\mu, \mu = 0.02, V(x) = 0.2\sin(x/2)$.

Let $\underline{\mathbf{u}}_{\varepsilon,0} \in H^2(\mathbb{R}) \oplus \mathbb{R}^2$ *be the constituent soliton of the periodic solutions* $\underline{\mathbf{u}}_{per,\varepsilon,\ell}$.

(iii) Assume that $\underline{\mathbf{u}}_{\varepsilon,0}$ *is strongly spectrally stable. Then, the periodic soliton* $\underline{\mathbf{u}}_{per,\varepsilon,\ell}$ *is strongly spectrally stable for all* $\ell \in \mathbb{N}$ *sufficiently large.*

(iv) Assume that $\underline{\mathbf{u}}_{\varepsilon,0}$ *is spectrally unstable. Then, the periodic soliton* $\underline{\mathbf{u}}_{per,\varepsilon,\ell}$ *is spectrally unstable for all* $\ell \in \mathbb{N}$ *sufficiently large.*

Proof. The proof follows from Corollaries 6.13, 6.13 and Theorems 6.14 and 6.22 as soon as we have established high-frequency resolvent estimates on the linearization to reduce the stability problem to a n- and ℓ-independent compact subset of $\mathbb{C}$. This is done in Lemma 7.3 below and thus the claim readily follows. □

Lemma 7.3. *Let $C > 0$. There exist $\varepsilon_0, \rho_1, \rho_2 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the spectral inclusion*

$$\sigma(\mathcal{L}(\mathbf{u})) \subset \left\{-\frac{3\mu}{2} \leq \operatorname{Re}(\lambda) \leq -\frac{\mu}{2}\right\} \cup \{|\operatorname{Re}(\lambda)| \leq \rho_1, |\operatorname{Im}(\lambda)| \leq \rho_2\}$$

holds for all $\mathbf{u} \in L^\infty(\mathbb{R})$ with $\|\mathbf{u}\|_{L^\infty} \leq C$.

Proof. We proof the equivalent statement that there exists $\tilde{\rho}_1 > 0$ such that

$$\sigma(JL(\mathbf{u}) + \varepsilon V(x)\partial_x) \subset \left\{|\operatorname{Re}(\lambda)| \leq \frac{\mu}{2}\right\} \cup \{|\operatorname{Re}(\lambda)| \leq \tilde{\rho}_1, |\operatorname{Im}(\lambda)| \leq \rho_2\}$$

for all $\mathbf{u}$ with $\|\mathbf{u}\|_{L^\infty} \leq C$. Let $\|\mathbf{u}\|_{L^\infty} \leq C$ and $|\varepsilon| \leq \varepsilon_0$ for some $\varepsilon_0 > 0$. We start with some preliminary computations to simplify the problem. For $\mathbf{u} \in L^\infty(\mathbb{R})$ we define the matrix

$$\mathbf{M}(\mathbf{u}) := \begin{pmatrix} 3\mathbf{u}_1^2 + \mathbf{u}_2^2 & 2\mathbf{u}_1\mathbf{u}_2 \\ 2\mathbf{u}_1\mathbf{u}_2 & \mathbf{u}_1^2 + 3\mathbf{u}_2^2 \end{pmatrix}.$$

Then we can write $L(\mathbf{u}) = -\partial_x^2 + \zeta - \mathbf{M}(\mathbf{u})$. We introduce the multiplication operator

$$R\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$$

where $\theta(x) = -\frac{\varepsilon}{2}\int_0^x V(y)\mathrm{d}y$. Conjugation with R then yields

$$R\left(JL(\mathbf{u}) + \varepsilon V(x)\partial_x\right)R^\top = J\left(-\partial_x^2 + \zeta - R\mathbf{M}(\mathbf{u})R^\top\right) + \varepsilon V(x)R\partial_x(R^\top) + RJ\partial_x^2(R^\top).$$

We observe that $T := \varepsilon V(x)R\partial_x(R^\top) + RJ\partial_x^2(R^\top) \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded operator with $\|T\|_{L^2 \to L^2} \lesssim |\varepsilon|$ and that the first order operator $\varepsilon V(x)\partial_x$ has been transformed to a bounded operator on L^2. For the moment we drop the operator T as it is a small bounded perturbation for small ε and only consider the operator $J\left(-\partial_x^2 + \zeta - R\mathbf{M}(\mathbf{u})R^\top\right)$. Notice that $(R\mathbf{M}(\mathbf{u})R^\top)^* = R\mathbf{M}(\mathbf{u})R^\top$ which implies that $J\left(-\partial_x^2 + \zeta - R\mathbf{M}(\mathbf{u})R^\top\right)$ is a linear Hamiltonian operator. Next, diagonalization of the matrix J yields

$$J = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} =: S^*\widetilde{J}S$$

and after another conjugation with the unitary operator $S \in \mathbb{C}^{2\times 2}$ we find

$$SJ\left(-\partial_x^2 + \zeta - R\mathbf{M}(\mathbf{u})R^\top\right)S^* = \tilde{J}\left(-\partial_x^2 + \zeta - SR\mathbf{M}(\mathbf{u})R^\top S^*\right).$$

Again, the multiplication operator $SR\mathbf{M}(\mathbf{u})R^\top S^*$ is self-adjoint and both conjugations leave the spectrum of the operator invariant. We define $\widetilde{\mathbf{M}}(\mathbf{u}) := SR\mathbf{M}(\mathbf{u})R^\top S^*$ and start bounding the inverse of $\widetilde{J}(-\partial_x^2 + \zeta - \widetilde{\mathbf{M}}(\mathbf{u}))$.

Clearly, $-\widetilde{J}\partial_x^2 \colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a densely defined skew-adjoint operator, thus generating a unitary group on the Hilbert space $L^2(\mathbb{R})$ by Stone's theorem, see [44, Theorem 3.24]. In particular, from [44, Theorem 1.10] we obtain the bound

$$\left\| \left(-\widetilde{J}\partial_x^2 - \lambda\right)^{-1} \right\|_{L^2 \to L^2} \le \frac{1}{|\operatorname{Re}(\lambda)|},$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) \neq 0$. The residual $\tilde{J}\left(\zeta - SR\mathbf{M}(\mathbf{u})R^\top S^*\right)$ enjoys the estimate

$$\left\| \tilde{J}\left(\zeta - SR\mathbf{M}(\mathbf{u})R^\top S^*\right)\mathbf{v} \right\|_{L^2} \le C_1\left(\zeta + \|\mathbf{u}\|_{L^\infty}^2\right)\|\mathbf{v}\|_{L^2},$$

for $\mathbf{v} \in H^2(\mathbb{R})$ and some positive ε- and $\mathbf{u}$-independent constant $C_1 > 0$. Here we used the embedding $H^2(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Now, we define the $\mathbf{u}$-independent constant

$$\tilde{\rho}_1 := C_1\left(\zeta + C^2\right) + 1 < \infty.$$

Then, $\tilde{J}\left(-\partial_x^2 + \zeta - SR\mathbf{M}(\underline{\mathbf{u}}_n)R^\top S^*\right) - \lambda = -J\partial_x^2 - \lambda + \tilde{J}\left(\zeta - SR\mathbf{M}(\mathbf{u})R^\top S^*\right)$ is by [80, Theorem IV.1.16] bounded invertible for each $\lambda \in \mathbb{C}$ with $|\operatorname{Re}(\lambda)| \le \tilde{\rho}_1$ with ε- and $\mathbf{u}$-independent bound.

Next, we note that for any real-valued $\mathbf{u} \in L^\infty(\mathbb{R})$ the operator $\tilde{J}\left(-\partial_x^2 + \zeta - SR\mathbf{M}(\mathbf{u})R^\top S^*\right)$ is of the form

$$\widetilde{J}\begin{pmatrix} A_+(\mathbf{u}) & B(\mathbf{u}) \\ B(\mathbf{u})^* & A_-(\mathbf{u}) \end{pmatrix},$$

where $A_\pm(\mathbf{u}) \colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are the self-adjoint operators given by

$$A_+(\mathbf{u})u = -u'' + \zeta u - (SR\mathbf{M}(\mathbf{u})R^\top S^*)_{11}u,$$
$$A_-(\mathbf{u})u = -u'' + \zeta u - (SR\mathbf{M}(\mathbf{u})R^\top S^*)_{22}u$$

and

$$B(\mathbf{u}) \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad B(\mathbf{u})u = -(SR\mathbf{M}(\mathbf{u})R^\top S^*)_{12}u$$

is a bounded multiplication operator. Using integration by parts we infer the lower bound

$$\langle A_\pm(\mathbf{u})u, u\rangle_{L^2} \geq \|u'\|_{L^2}^2 - \left(\zeta + C_2\|\mathbf{u}\|_{L^\infty}^2\right)\|u\|_{L^2}^2,$$

for $u \in H^2(\mathbb{R})$ and some positive ε- and $\mathbf{u}$-independent constant $C_2 > 0$. Now we define the ε- and $\mathbf{u}$-independent constant

$$\gamma := \zeta + C_2 C^2 + 1 > 0,$$

and observe that the operators $A_\pm(\mathbf{u})$ have lower bound $-\gamma$. Similarly, it follows that the operator norm of $B(\mathbf{u})$ on $L^2(\mathbb{R})$ is bounded by an ε- and $\mathbf{u}$-independent constant. Therefore, [15, Theorem 4] yields an ε- and $\mathbf{u}$-independent constant $\rho_2 > 0$ such that all high-frequency spectrum

$$\left\{\lambda \in \sigma\left(\tilde{J}\left(-\partial_x^2 + \zeta - SR\mathbf{M}(\mathbf{u})R^\top S^*\right)\right) : |\mathrm{Im}(\lambda)| \geq \rho_2\right\}$$

must be purely imaginary. Hence,

$$\left\{\lambda \in \mathbb{C} : |\mathrm{Re}(\lambda)| \geq \tfrac{\mu}{2}, |\mathrm{Im}(\lambda)| \geq \rho_2\right\}$$

lies in the resolvent set of the linear operator $\tilde{J}\left(-\partial_x^2 + \zeta - SR\mathbf{M}(\mathbf{u})R^\top S^*\right)$ and the proof of [15, Theorem 4] yields that the inverse is uniformly bounded in ε and $\mathbf{u}$. Finally, by a Neumann series argument we infer that for sufficiently small values of ε_0 the operator $J\left(-\partial_x^2 + \zeta - R\mathbf{M}(\mathbf{u})R^\top\right) + \varepsilon V(x)R\partial_x(R^\top) + RJ\partial_x^2(R^\top) - \lambda$ has a bound inverse for

$$\lambda \in \left\{\lambda \in \mathbb{C} : |\mathrm{Re}(\lambda)| \geq \tfrac{\mu}{2}, |\mathrm{Im}(\lambda)| \geq \rho_2\right\} \cup \{|\,\mathrm{Re}(\lambda)| \geq \tilde{\rho}_1\}.$$

This proves the claim. □

The spectral result is confirmed by the numerical simulations in the right panels of Figure 7.1. We see that the strongly spectrally stable 1-soliton yields a strongly spectrally stable 2-soliton solution through concatenation.

Remark 7.4. Let us remark that Lemma 7.3 extends the previous Lemma 5.16 from co-periodic to localized perturbations.

7.2 Outlook: modulated solitons

We close this chapter with an outlook on modulated solitons. This covers the cases where the potential $V(x)$ does not admit zeros. We note, that the first order derivative multiplied by the potential acts as a transport term with an x-dependent velocity. If the potential is constant, $V(x) = c$ for all $x \in \mathbb{R}$, then the traveling wave ansatz $u(x,t) = u_0(x - \varepsilon ct)$ transforms (7.1) to the stationary LLE for the profile u_0,

$$-du_0'' + (\zeta - \mathrm{i}\mu)u_0 - |u_0|^2 u_0 + \mathrm{i}f = 0. \tag{7.4}$$

This means that traveling wave solutions of (7.1) and stationary solutions of the (7.4) are in a 1-to-1 correspondence. The situation becomes more complicated if $V(x)$ is not constant. Then, traveling waves generically cease to exist and we expect to find stationary or modulated waves, depending on the form of the potential. We explain this in more detail using a formal perturbation analysis. For $0 < \varepsilon \ll 1$ we search for solutions of the form

$$u(x,t) = u_0(s(x,t)) + \varepsilon v(x,t), \qquad s(x,t) = x - \sigma(t), \qquad v(x,0) = 0,$$

where u_0 is a stable soliton of (7.4), which is strongly localized around $x = 0$, v is a correction, and the position satisfies the ODE

$$\sigma'(t) = -\varepsilon V(\sigma(t)), \qquad \sigma(0) = 0.$$

Plugging the solution ansatz into (7.1) yields

$$\mathrm{i}\varepsilon v_t(x,t) = \varepsilon L_{u_0(s)} v(x,t) + \underbrace{\mathcal{N}(\varepsilon v(x,t)) + \mathrm{i}\big(\sigma'(t) + \varepsilon V(x)\big)u_0'(s(x,t))}_{=:\widetilde{\mathcal{N}}(x,t)}. \tag{7.5}$$

Here $L_{u_0} v = -dv'' + (\zeta - \mathrm{i}\mu)v - 2|u_0|^2 v - u_0^2 \overline{v}$ is the linearization about u_0 and $\mathcal{N}(\varepsilon v) = \mathcal{O}(\varepsilon^2)$. Since we assume that u_0 is strongly spatially localized around $x = 0$, we find $u_0'(s) \approx 0$ for $|s| \geq \varepsilon$. If $|s| < \varepsilon$, we have $x \approx \sigma(t)$. This implies that $\big(\sigma'(t) + \varepsilon V(x)\big)u_0'(s)$ approximately vanishes by the choice $\sigma'(t) = -\varepsilon V(\sigma(t))$. This means that equation (7.5) reduces to an equation for the perturbation with a small x- and t-dependent forcing,

$$\mathrm{i}v_t = L_{u_0(s)} v + \underbrace{\varepsilon^{-1}\widetilde{\mathcal{N}}(x,t)}_{\text{small}}, \qquad v(x,0) = 0.$$

Since u_0 is a stable solution, we expect solutions of the linear equation $\mathrm{i}v_t = L_{u_0(s)} v$ to decay (or remain bounded) and this should persist under the small perturbation $\varepsilon^{-1}\widetilde{\mathcal{N}}(x,t)$. Consequently, we should find localized solutions of (7.1), that satisfy the approximation

$$u(x,t) \approx u_0(x - \sigma(t)), \qquad \text{with} \qquad \sigma'(t) = -\varepsilon V(\sigma(t)).$$

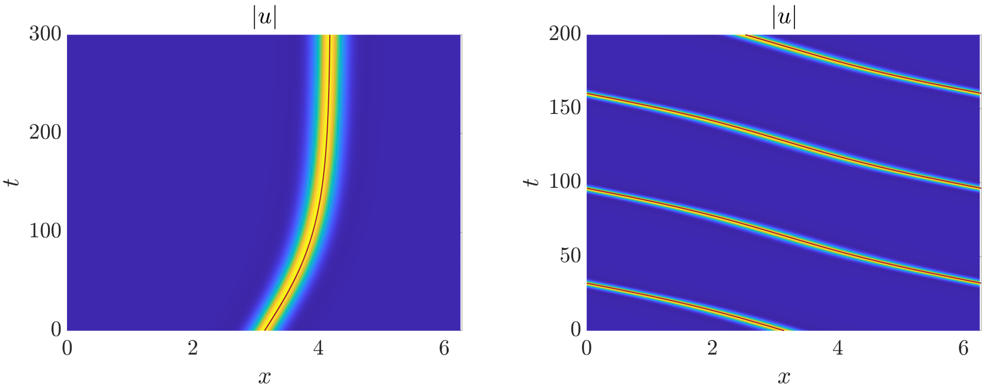

Figure 7.2: Space-time plots of spatially localized solutions of (7.1). The red curves show the approximate positions $(\sigma(t), t)$ of the intensity maxima of the solutions predicted by the ODE $\sigma' = -\varepsilon V(\sigma)$. In the left panel, the potential is given by $\varepsilon V(x) = 0.01 + 0.02\cos(x)$, which has a simple zero at $\sigma_0 = 4/3\pi$. Since $V'(\sigma_0) > 0$, we observe that the localized state gets attracted and pinned to this zero. In the right panel, the potential is given by $\varepsilon V(x) = 0.1 + 0.02\cos(x)$ and has no zero. Consequently, oscillatory behavior of the solution is observed. The remaining parameters are $d = 0.1, \zeta = 4.81, \mu = 1, f = 2$.

If the approximation formula holds, the solutions converge to a stable steady state if V has a simple zero σ_0 with $V'(\sigma_0) > 0$, as in this case σ_0 is an attractor in the ODE for σ. This observation is in agreement with the previous results of Chapter 5 where pinning was observed for zeros of $V_{\text{eff}}(x) \approx V(x)$. In contrast, if V is not sign-changing then the solutions oscillate in time with x-dependent velocity. Thus we observe modulated solitons. In Figure 7.2 we show numerical simulations of (7.1) with a localized soliton of (7.4) as initial condition. The red curves plot the positions $\sigma(t)$ predicted by the ODE $\sigma' = -\varepsilon V(\sigma)$. We find, that the numerics strongly support our approximation formula for $u(x,t)$, as $\sigma(t)$ closely follows the position of the intensity maxima of the solution. The left panel shows pinning to a simple zero of V and the right panel shows oscillatory behavior for a sign definite V. Motivated by these observations, we formulate the following conjecture.

Conjecture 7.5 (Locking Range Conjecture)**.** *If the LLE* (7.1) *admits a stationary soliton solution, then the potential* V *has a sign change.*

In this chapter and in Chapter 5 we make a first step towards the proof of the locking range conjecture 7.5 by establishing a connection between the existence of soliton solutions and sign changes of V. However, proving or disproving the conjecture remains an open problem.

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Eidesstattliche Erklärung:

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Karlsruhe, den 19. August 2025