Original Article

Improved Error Bounds for Approximations of High-Frequency Wave Propagation in Nonlinear Dispersive Media

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Abstract

We analyze nonlinear Friedrichs systems where the differential equation and the initial data contain the inverse of a small parameter ε, which causes oscillations with wavelengths proportional to ε in time and space. In diffractive optics, such problems have to be solved on time intervals with length of $\mathcal{O}(1/\varepsilon)$. Approximating the solution numerically with a standard method is hopeless, because traditional methods require an extremely fine resolution in time and space. A possible alternative is to replace the original problem by a new system of partial differential equation, which is more suitable for numerical computations but still yields a sufficiently accurate approximation. Such models are often based on the *slowly varying envelope approximation* or generalizations thereof. For applications in nonlinear optics, a rigorous analysis of the accuracy of such approximations is of utmost importance. We show that under a number of natural assumptions the error of the slowly varying envelope approximation is proportional to ε^2. For a higher-order generalization, we improve the error bound from $\mathcal{O}(\varepsilon^2)$ to $\mathcal{O}(\varepsilon^3)$. Both proofs are based on a careful analysis of the nonlinear interaction between oscillatory and nonoscillatory terms, and on a priori bounds for certain "parts" of the approximations, which are identified by suitable projections.

Keywords

high-frequency wave propagation, nonlinear Friedrichs system, diffractive geometric optics, slowly varying envelope approximation, error bounds, Maxwell–Lorentz system

1 Introduction

1.1 High-Frequency Wave Propagation

High-frequency wave propagation in nonlinear, dispersive media can be modeled by Friedrichs systems of the form:

$$\partial_t u + \mathbf{A}(\partial)u + \frac{1}{\varepsilon}\mathbf{E}u = \varepsilon T(u,u,u), \qquad t \in (0, t_{\text{end}}/\varepsilon],\ x \in \mathbb{R}^d, \tag{1.1a}$$

$$u(0,x) = p(x)\mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot x)/\varepsilon} + c.c., \tag{1.1b}$$

with a trilinear nonlinearity $T : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and a differential operator

$$\mathbf{A}(\partial) = \sum_{\ell=1}^{d} \mathbf{A}_\ell \partial_\ell, \tag{1.2}$$

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($d, n \in \mathbb{N}$). We assume that the matrices $\mathbf{A}_1, \dots, \mathbf{A}_d \in \mathbb{R}^{n\times n}$ in (1.2) are symmetric, and that $\mathbf{E} \in \mathbb{R}^{n\times n}$ in (1.1a) is skew-symmetric. In the initial data, a smooth and localized envelope function $p : \mathbb{R}^d \to \mathbb{C}^n$ is multiplied by a phase with a given wave vector $\boldsymbol{\kappa} \in \mathbb{R}^d \setminus \{0\}$. Here and below, "$X$ + c.c." means $X + \overline{X}$, where $\overline{X}$ is the complex conjugate of X. An important example in this class of problems is the Maxwell–Lorentz system, which is a classical model for the propagation of light in a Kerr medium; cf. Colin and Lannes (2009), Colin et al. (2005), Donnat and Rauch (1997a), Donnat and Rauch (1997b), Joly et al. (1996), and Lannes (1998, 2011).

The partial differential equation (PDE) (1.1a), the initial data in (1.1b), and the time interval involve a small positive parameter $\varepsilon \ll 1$. Although the nonlinearity in (1.1a) is multiplied by ε, the problem (1.1) is strongly nonlinear, because the length of the time interval is proportional to ε^{-1}. In fact, by rescaling $\tau = \varepsilon t$ and $w(\tau, x) = u(t, x)$, we could convert (1.1a) into the equivalent form:

$$\partial_\tau w + \frac{1}{\varepsilon}\mathbf{A}(\partial)w + \frac{1}{\varepsilon^2}\mathbf{E}w = T(w, w, w), \qquad \tau \in (0, t_{\text{end}}],\ x \in \mathbb{R}^d, \tag{1.3}$$

where the nonlinear term and the time interval do not depend on ε anymore. However, we will consider the original version (1.1), which is the representation considered, for example, in Baumstark (2022), Baumstark and Jahnke (2023), Baumstark et al. (2024), Colin and Lannes (2009), and Lannes (2011).

The small parameter ε accounts for different scales in time and space. The terms $\mathrm{e}^{\pm \mathrm{i}(\boldsymbol{\kappa}\cdot x)/\varepsilon}$ in the initial data cause spatial oscillations with wavelength of $\mathcal{O}(\varepsilon)$, whereas p changes on a scale of $\mathcal{O}(1)$, roughly speaking. As a consequence, the solution $u(t, x)$ is a wave packet with a high-frequency carrier wave modulated by a smooth envelope. Concerning the evolution in time, the initial value problem (1.1) is scaled in such a way that nonlinear and diffractive effects appear on long time intervals of length $t_{\text{end}}/\varepsilon$ for some $t_{\text{end}} > 0$, whereas the envelope of the wave packet propagates with speed $\mathcal{O}(1)$. The solution itself, however, evolves on a third scale, because the linear part $\mathbf{A}(\partial)u + (1/\varepsilon)\mathbf{E}u$ of the PDE causes rapid oscillations in time with wavelength of $\mathcal{O}(\varepsilon)$. Because of the highly oscillatory nature and the long time interval, an attempt to approximate the vector-valued solution $u : [0, t_{\text{end}}/\varepsilon] \times \mathbb{R}^d \to \mathbb{R}^n$ of (1.1) numerically with a traditional method is bound to fail, because the time and space discretizations would require an extremely fine resolution and hence an impracticable runtime. The equivalent form (1.3) does not have any advantage for numerical approximation, because the ε-independent time interval is counterbalanced by an extra factor $1/\varepsilon$ in each term apart from the time derivative.

Remark 1.1. In this paper, we consider wave packets where the wavelength of the oscillations is much shorter than the scale on which the envelope varies. This assumption excludes short or chirped pulses, which have been analyzed, for example, in Alterman and Rauch (2000, 2003), Barrailh and Lannes (2002), Chung et al. (2005), Colin and Lannes (2009), Colin et al. (2005), and Lannes (2011).

1.2 Slowly Varying Envelope Approximation (SVEA) and Generalizations

A feasible approach is to replace (1.1) by a different model, which can be solved numerically with significantly less efforts and at the same time provide a decent approximation to u. Such models are often based on the SVEA or generalizations thereof, which are derived as follows. For every $\beta \in \mathbb{R}^d$, the matrix

$$\mathbf{A}(\beta) = \sum_{\ell=1}^{d} \beta_\ell \mathbf{A}_\ell \in \mathbb{R}^{n\times n}$$

is symmetric, and

$$\mathcal{L}(\alpha, \beta) = -\alpha I + \mathbf{A}(\beta) - \mathrm{i}\mathbf{E} \in \mathbb{C}^{n\times n} \tag{1.4}$$

is Hermitian for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$. Let $\boldsymbol{\kappa} \in \mathbb{R}^d \setminus \{0\}$ be the (given) wave vector, which appears in (1.1b), and

$$\text{let } \omega = \omega(\boldsymbol{\kappa}) \text{ be an eigenvalue of } \mathrm{A}(\boldsymbol{\kappa}) - \mathrm{i}\mathbf{E} \tag{1.5}$$

Then, $\mathcal{L}(\omega, \boldsymbol{\kappa})$ has a nontrivial kernel, and the pair $(\omega, \boldsymbol{\kappa})$ is said to fulfill the dispersion relation. We assume the following.

Assumption 1.2. (*i*) *The kernel of* $\mathcal{L}(\omega, \boldsymbol{\kappa})$ *is one-dimensional.*
(*ii*) *The function p in (1.1b) has the structure*

$$p = p_0 + \varepsilon p_1 \qquad \text{with } p_0(x) \in \ker\left(\mathcal{L}(\omega, \boldsymbol{\kappa})\right) \text{ a.e.} \tag{1.6}$$

and $p_0, p_1 \in L^\infty(\mathbb{R}^d, \mathbb{C}^n)$.

Assumption (i) is only made in order to keep the notation simple; cf. Remark 4.6. Assumption (ii) is a polarization condition, which was also imposed in a similar way in Baumstark and Jahnke (2023), Colin and Lannes (2009, Theorem 1), Lannes (2011, Theorem 2.15), and other works.

As in Baumstark and Jahnke (2023), we seek an approximation of the form:

$$u(t,x) \approx \widetilde{u}^{(m)}(t,x) = \sum_{j \in \mathcal{J}^{(m)}} \mathrm{e}^{\mathrm{i}j(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} u_j(t,x), \qquad u_{-j} = \overline{u_j}, \tag{1.7}$$

for $\mathcal{J}^{(m)} = \{\pm 1, \pm 3, \dots, \pm m\}$, where $m \in \mathbb{N}$ is an odd integer. The condition $u_{-j} = \overline{u_j}$ ensures that $\widetilde{u}^{(m)}$ is real-valued. If we substitute (1.7) into (1.1), then the trilinear nonlinearity generates *higher harmonics*, that is, terms with prefactor $\mathrm{e}^{\mathrm{i}j(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon}$ for $|j| > m$. These terms appear only on the right-hand side of (1.1a), because all terms on the left-hand side are linear. Ignoring higher harmonics and then comparing terms on both sides yields the PDE system

$$\begin{aligned} \partial_t u_j + \frac{\mathbf{i}}{\varepsilon}\mathcal{L}(j\omega, j\boldsymbol{\kappa})u_j + \mathbf{A}(\partial)u_j &= \varepsilon \sum_{j_1+j_2+j_3=j} T(u_{j_1}, u_{j_2}, u_{j_3}) \\ \text{for } j \in \mathcal{J}_+^{(m)} = \mathcal{J}^{(m)} \cap \mathbb{N}, \quad t \in (0, t_{\mathrm{end}}/\varepsilon], \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.8a}$$

with initial conditions

$$u_1(0,\cdot) = p, \qquad u_j(0,\cdot) = 0 \text{ for } j \in \mathcal{J}_+^{(m)} \setminus \{1\}. \tag{1.8b}$$

The sum on the right-hand side of (1.8a) is to be taken over the set

$$\left\{J = (j_1, j_2, j_3) \in (\mathcal{J}^{(m)})^3 : \#J := j_1 + j_2 + j_3 = j\right\},$$

and T is now the trilinear extension of the real nonlinearity from (1.1a) to $T : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$. It is sufficient to consider positive $j \in \mathcal{J}_+^{(m)} = \mathcal{J}^{(m)} \cap \mathbb{N}$ instead of $j \in \mathcal{J}^{(m)}$ in (1.8a), because the u_j with negative index j are obtained from the condition $u_{-j} = \overline{u_j}$. For $m = 1$ and $\mathcal{J}^{(1)} = \{-1, 1\}$, we obtain the SVEA

$$u(t,x) \approx \widetilde{u}^{(1)}(t,x) = \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} u_1(t,x) + c.c., \tag{1.9}$$

with u_1 being the solution of

$$\begin{aligned} \partial_t u_1 + \frac{\mathbf{i}}{\varepsilon}\mathcal{L}(\omega, \boldsymbol{\kappa})u_1 + \mathbf{A}(\partial)u_1 &= \varepsilon \sum_{j_1+j_2+j_3=1} T(u_{j_1}, u_{j_2}, u_{j_3}) \qquad (1.10a) \\ &= \varepsilon \left(T(u_1, u_1, u_{-1}) + T(u_1, u_{-1}, u_1) + T(u_{-1}, u_1, u_1)\right), \\ u_1(0,\cdot) &= p, \qquad (1.10b) \end{aligned}$$

as a special case of (1.7) and (1.8). Note that the initial data in (1.8b) are smooth, nonoscillatory functions, in contrast to (1.1b). Hence, solutions to (1.8) can be discretized in space on a ε-independent grid, which is a significant advantage over (1.1). However, typical solutions of (1.8) do still oscillate in time due to the term $(\mathrm{i}/\varepsilon)\mathcal{L}(j\omega, j\boldsymbol{\kappa})u_j$ in (1.8a).

For the error of the SVEA (1.9), (1.10a), and (1.10b), the bound

$$\sup_{t \in [0, t_{\mathrm{end}}/\varepsilon]} \|u(t,\cdot) - \widetilde{u}^{(1)}(t,\cdot)\|_{L^\infty(\mathbb{R}^d, \mathbb{R}^n)} \leq C\varepsilon \tag{1.11}$$

was shown in Colin and Lannes (2009, Section 2.2). Under additional assumptions, one can replace the PDE (1.10a) by a nonlinear Schrödinger equation without spoiling the error bound (1.11); cf. Colin and Lannes (2009, Corollary 2) and also

Colin (2002), Donnat et al. (1996), Joly et al. (1998), Lannes (2011) and Schneider and Uecker (2017). This nonlinear Schrödinger equation has the advantage that it does not involve ε at all when considered in a comoving coordinate system, and that it only has to be solved on the ε-independent time interval $[0, t_{\mathrm{end}}]$. Hence, standard numerical methods can be used to solve the nonlinear Schrödinger equation numerically, which then yields an $\mathcal{O}(\varepsilon)$-approximation to $\widetilde{u}^{(1)}$ and, via (1.11), to the solution of (1.1).

In this paper, we consider the situation where an error of $\mathcal{O}(\varepsilon)$ is not small enough to be acceptable. For the approximation

$$u(t,x) \approx \widetilde{u}^{(3)}(t,x) = \left(\mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot x-\omega t)/\varepsilon} u_1(t,x) + \mathrm{e}^{3\mathrm{i}(\boldsymbol{\kappa}\cdot x-\omega t)/\varepsilon} u_3(t,x)\right) + c.c.$$

we have already shown the error bound

$$\sup_{t\in[0,t_\star/\varepsilon]} \|u(t,\cdot) - \widetilde{u}^{(3)}(t,\cdot)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \le C\varepsilon^2 \tag{1.12}$$

for some $t_\star \in (0, t_{\mathrm{end}}]$ in Baumstark and Jahnke (2023). The proof is rather long and technical because of the complicated self-interaction of the oscillatory solution via the nonlinearity. Moreover, the approximation $\widetilde{u}^{(3)}$ is more complicated than (1.9) because of the additional coefficient function u_3.

1.3 Main Results

Numerical experiments indicate that the estimates (1.11) and (1.12) *are both not optimal*; see Sections 4.2 and 5.3. In this work, we will prove the following stronger error bounds.

Main Results (Informal Version). *Under a number of assumptions the inequalities*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|u(t,\cdot) - \widetilde{u}^{(1)}(t,\cdot)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \le C\varepsilon^2, \tag{1.13}$$

$$\sup_{t\in[0,t_\star/\varepsilon]} \|u(t,\cdot) - \widetilde{u}^{(3)}(t,\cdot)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \le C\varepsilon^3, \tag{1.14}$$

hold for some $t_\star \in (0, t_{\mathrm{end}}]$ and with a constant $C \ge 0$, which does not depend on ε.

Mathematically rigorous formulations of these two results with a detailed specification of the assumptions are given in Theorems 4.3 and 5.7. The bound (1.13) explains the error behavior which appears in numerical examples where a reference solution can be computed. Moreover, this inequality shows that the SVEA yields a *significantly higher accuracy than the classical nonlinear Schrödinger approximation*, which has an error of $\mathcal{O}(\varepsilon)$. This fact was apparently not known until now. The numerical experiment and the discussion in Section 4.1 indicate that the result is sharp in the sense that a higher convergence order can in general not be expected; cf. Remark 4.5. The second error bound (1.14) states that in applications where an error of $\mathcal{O}(\varepsilon^2)$ is still too large, the refined approximation $\widetilde{u}^{(3)}$ offers the possibility of reducing the error down to $\mathcal{O}(\varepsilon^3)$ at the cost of higher computational work. Our error bounds are not only crucial for estimating the accuracy of the reduced models in nonlinear optics, but also for deciding whether the SVEA $\widetilde{u}^{(1)}$ or the refined approximation $\widetilde{u}^{(3)}$ should be used in a specific application.

Natural questions are whether and under which additional assumptions the approximations $\widetilde{u}^{(m)}$ with $m > 3$ have a higher order of convergence than $\widetilde{u}^{(3)}$, and how much the order of convergence improves. Since these questions have been discussed by the first author in Section 4.4 of his PhD thesis (Baumstark, 2022), we only give a short summary of the most important findings. Numerical experiments for $d = 1$ suggest that increasing m does indeed increase the order of convergence. Unfortunately, it is almost impossible to perform significant numerical tests for $d > 1$, because computing a highly accurate reference solution would require immense computational ressources. On the analytical side, we were not able to prove error bounds for $\widetilde{u}^{(m)}$ with $m > 3$ and $d > 1$ without making certain nonresonance assumptions. These nonresonance assumptions, however, are not realistic as they do not hold for the Maxwell–Lorentz system, which is the most important application in the problem class (1.1). We will return to this issue in Subsection 5.3.

1.3.1 Delineation From Other Results. In Donnat and Rauch (1997a), Joly et al. (1993, 2000), Rauch (2012), and other contributions, asymptotic expansions of solutions to problems similar to (1.1) have been analyzed in the regime of geometric optics, that is, for time intervals of length $\mathcal{O}(1)$. This differs from the regime of diffractive geometric optics, where the PDE system has to be solved on time intervals of length $\mathcal{O}(1/\varepsilon)$, which is the situation we consider here. Approximations in diffractive geometric optics have been constructed in Donnat et al. (1996) and Joly et al. (1998) for semilinear and quasilinear systems with a more general nonlinearity, but with $\varepsilon\mathbf{E}$ instead of $\mathbf{E}/\varepsilon$ in Donnat et al. (1996) and with

$\mathbf{E} = 0$ in Joly et al. (1998). Quasilinear systems with dispersion and dispersive problems with bilinear nonlinearity are approximated in Lannes (1998) and Colin (2002), respectively, but without an explicit rate of convergence. In Baumstark et al. (2024), we have constructed modulated Fourier expansions for (1.1) with *nonlinear* polarization of the initial data. This approach is likewise based on the ansatz (1.7) and (1.8), but the nonlinear polarization considered in Baumstark et al. (2024) means that p_1 depends on p_0 in (1.6), which excludes, for example, the case $p_1 = 0$. In the present work, p_1 and p_0 are completely independent. Recently, another related but different way to construct approximations to (1.1) was analyzed in Logioti et al. (2025). As in Baumstark et al. (2024), the approach does not require the solution of highly oscillatory PDEs, and in contrast to Baumstark et al. (2024), it can be extended to nonpolarized initial data. However, even for an accuracy of $\mathcal{O}(\varepsilon^{1/2})$ a large system of coupled equations has to be solved. The book (Schneider & Uecker, 2017) provides an extensive analysis of the approximation of PDEs by nonlinear Schrödinger equations and other modulation equations.

1.4 Structure of the Paper and Notation

In Section 2, we specify the analytical framework, we review results on local wellposedness of (1.1) and (1.8), and we introduce a transformation of the coefficient functions u_j, which was already employed in Baumstark and Jahnke (2023). The proofs of the error bounds (1.13) and (1.14) rely on the fact that for a certain projection $\mathcal{P}_\varepsilon$ the Fourier transform $\widehat{u}_1$ of the coefficient function u_1 can be decomposed into an essentially nonoscillatory part $\mathcal{P}_\varepsilon \widehat{u}_1$ and an oscillatory but "small" part $(I - \mathcal{P}_\varepsilon)\widehat{u}_1$. For the SVEA (i.e. for $m = 1$), we compile the corresponding results in Section 3. Then, in Section 4, we prove the error bound (1.13) for the SVEA, and we corroborate this result by a numerical experiment. In Section 5, we turn to the case $m = 3$. We show the error bound (1.14) and we give reasons why we observe an even better rate of convergence in a numerical example with a one-dimensional Klein–Gordon system.

1.4.1 Notation. Throughout the text, $v \cdot w = v^* w$ is the Euclidean scalar product of $v, w \in \mathbb{C}^n$, and $|v|_q$ is the q norm of v. The identity in $\mathbb{R}^{n\times n}$ is denoted by I. For space- and time-dependent functions $f = f(t,x)$, we will often denote the mapping $x \mapsto f(t,x)$ by $f(t)$ instead of $f(t,\cdot)$. Likewise, we will omit the second argument of the Fourier transform $\widehat{f}(t,k)$ of such a function. From now on, we will use the short-hand notation L^1 and L^∞ for $L^1(\mathbb{R}^d, \mathbb{C}^n)$ and $L^\infty(\mathbb{R}^d, \mathbb{C}^n)$, respectively. The symbol $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, whereas i appears as an index in a few formulas.

2 Analytical Setting

2.1 Wiener Algebra and Evolution Equations in Fourier Space

As in Baumstark (2022), Baumstark and Jahnke (2023), Baumstark et al. (2024), Colin and Lannes (2009), and Lannes (2011), we will analyze the accuracy in the Wiener algebra

$$W = \left\{ f \in \left(\mathcal{S}'(\mathbb{R}^d)\right)^n : \widehat{f} \in L^1 \right\}, \qquad \|f\|_W = \|\widehat{f}\|_{L^1} = \int_{\mathbb{R}^d} |\widehat{f}(k)|_2 \,\mathrm{d}k \tag{2.1}$$

of vector-valued functions. Here and below, $\widehat{f} = \mathcal{F}f$ denotes the Fourier transform

$$(\mathcal{F}f)(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\mathrm{i}k\cdot x} \mathrm{d}x$$

of f. For $s \in \mathbb{N}_0$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, we define $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ and

$$W^s = \{ f \in W : \partial^\alpha f \in W \text{ for all } \alpha \in \mathbb{N}_0^d, |\alpha|_1 \le s \}, \qquad \|f\|_{W^s} = \sum_{|\alpha|_1 \le s} \|\partial^\alpha f\|_W.$$

It is well-known that W^s is a Banach algebra with continuous embedding $W \hookrightarrow L^\infty$, cf. Colin and Lannes (2009, Proposition 1) and Lannes (2011, Proposition 3.2).

In order to work in the Wiener algebra, we apply the Fourier transform to the PDE system (1.8a). This yields

$$\partial_t \widehat{u}_j(t,k) + \frac{\mathrm{i}}{\varepsilon} \mathcal{L}_j(\varepsilon k) \widehat{u}_j(t,k) = \varepsilon \sum_{\#J=j} \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(t,k), \tag{2.2a}$$

$$j \in \mathcal{J}_+^{(m)}, \quad t \in (0, t_{\mathrm{end}}/\varepsilon], \quad k \in \mathbb{R}^d$$

with initial conditions

$$\widehat{u}_1(0,\cdot) = \widehat{p}, \qquad \widehat{u}_j(0,\cdot) = 0 \quad \text{for } j \in \mathcal{J}_+^{(m)} \setminus \{1\} \tag{2.2b}$$

and the notation

$$\begin{aligned} \mathcal{L}_j(\theta) &= \mathcal{L}(j\omega, j\boldsymbol{\kappa} + \theta) = \mathcal{L}_j(0) + \mathbf{A}(\theta), \qquad j \in \mathcal{J}_+^{(m)}, \\ \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(k) &= \mathcal{F}\Big(T(u_{j_1}, u_{j_2}, u_{j_3})\Big)(k) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T(\widehat{u}_{j_1}(k^{(1)}), \widehat{u}_{j_2}(k^{(2)}), \widehat{u}_{j_3}(k - k^{(1)} - k^{(2)}))\, \mathrm{d}k^{(2)} \mathrm{d}k^{(1)}, \end{aligned} \tag{2.3}$$

cf. Baumstark and Jahnke (2023, Section 2.2). In (2.3), we have used that by definition the mapping $\beta \mapsto \mathbf{A}(\beta)$ is linear. With the shorthand notation

$$K = \left(k^{(1)}, k^{(2)}, k^{(3)}\right) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d, \qquad \#K := k^{(1)} + k^{(2)} + k^{(3)} \in \mathbb{R}^d, \tag{2.4a}$$

and

$$\begin{aligned} &\int_{\#K=k} T(\widehat{u}_{j_1}(k^{(1)}), \widehat{u}_{j_2}(k^{(2)}), \widehat{u}_{j_3}(k^{(3)}))\, \mathrm{d}K \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T(\widehat{u}_{j_1}(k^{(1)}), \widehat{u}_{j_2}(k^{(2)}), \widehat{u}_{j_3}(k - k^{(1)} - k^{(2)}))\, \mathrm{d}k^{(2)}\, \mathrm{d}k^{(1)}, \end{aligned} \tag{2.4b}$$

the Fourier transform of the nonlinearity can be expressed as

$$\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(k) = (2\pi)^{-d} \int_{\#K=k} T(\widehat{u}_{j_1}(k^{(1)}), \widehat{u}_{j_2}(k^{(2)}), \widehat{u}_{j_3}(k^{(3)}))\, \mathrm{d}K. \tag{2.5}$$

Later we will often use that

$$\left\| \mathcal{T}(\widehat{f}_1, \widehat{f}_2, \widehat{f}_3) \right\|_{L^1} \le C_{\mathcal{T}} \|\widehat{f}_1\|_{L^1} \|\widehat{f}_2\|_{L^1} \|\widehat{f}_3\|_{L^1} \tag{2.6}$$

with a constant $C_{\mathcal{T}}$, which depends on $\mathcal{T}$ and on n. Via trilinearity, we obtain that

$$\begin{aligned} \left\| \mathcal{T}(\widehat{f}_1, \widehat{f}_2, \widehat{f}_3) - \mathcal{T}(\widehat{g}_1, \widehat{g}_2, \widehat{g}_3) \right\|_{L^1} \le\ & C_{\mathcal{T}} \|\widehat{f}_1 - \widehat{g}_1\|_{L^1} \|\widehat{f}_2\|_{L^1} \|\widehat{f}_3\|_{L^1} \\ & + C_{\mathcal{T}} \|\widehat{g}_1\|_{L^1} \|\widehat{f}_2 - \widehat{g}_2\|_{L^1} \|\widehat{f}_3\|_{L^1} \\ & + C_{\mathcal{T}} \|\widehat{g}_1\|_{L^1} \|\widehat{g}_2\|_{L^1} \|\widehat{f}_3 - \widehat{g}_3\|_{L^1}. \end{aligned} \tag{2.7}$$

We set $u_{-j} = \overline{u_j}$ throughout, which implies that $\widehat{u}_{-j}(t,k) = \overline{\widehat{u}_j(t,-k)}$. The system (2.2a) can be extended to $j \in \mathcal{J}^{(m)}$ (including negative indices) if we define

$$\mathcal{L}_{-j}(\theta) = -\overline{\mathcal{L}_j(-\theta)} \qquad \text{for } j \in \mathcal{J}_+^{(m)}. \tag{2.8}$$

2.2 Local Wellposedness

The polarization condition (Assumption 1.2(ii)) is not needed to prove existence and uniqueness of solutions to the original problem (1.1) and the PDE system (1.8). For the sake of consistency, however, we always allow for ε-dependent initial data of the form:

$$p = p_0 + \varepsilon p_1 \qquad \text{with } p_0, p_1 \in W^\sigma \tag{2.9}$$

for some $\sigma \in \mathbb{N}$. The value of σ will be specified whenever we refer to (2.9).

Lemma 2.1 (Local wellposedness of (1.1))**.** *If $p_0, p_1 \in W$, then there is a $t_{\mathrm{end}} > 0$ such that for every $\varepsilon \in (0,1]$ the original problem (1.1) with $p = p_0 + \varepsilon p_1$ has a unique mild solution $u \in C([0, t_{\mathrm{end}}/\varepsilon), W)$, which is uniformly bounded, that is, there is a constant $c > 0$ such that*

$$\sup_{t \in [0, t_{\mathrm{end}}/\varepsilon]} \|u(t)\|_W \le c \qquad \textit{for all } \varepsilon \in (0,1].$$

We omit the proof, because Lemma 2.1 can be shown with the usual fixed-point argument. Other proofs for well-posedness of (1.1) via approximation by the SVEA are given in Colin and Lannes (2009, Theorem 1) and Lannes (2011, Theorem 3.8).

Lemma 2.2 (Local wellposedness of (1.8))**.** *Let $m \in \mathbb{N}$ be an odd integer.*

(*i*) *If (2.9) holds with $\sigma = 0$ and $C_{u,0} > \|p_0\|_W + \|p_1\|_W$, then there is a $t_{\mathrm{end}} > 0$ such that for every $\varepsilon \in (0,1]$ the system (1.8) has a unique mild solution*

$$\{u_j\}_{j\in\mathcal{J}_+^{(m)}}, \qquad u_j \in C([0, t_{\mathrm{end}}/\varepsilon), W),$$

which is uniformly bounded, that is,

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|u_j(t)\|_W \le C_{u,0} \qquad \text{for all } j \in \mathcal{J}_+^{(m)} \text{ and all } \varepsilon \in (0,1].$$

(*ii*) *If (2.9) holds with $\sigma = 1$, then the mild solution on $[0, t_{\mathrm{end}}/\varepsilon]$ is a classical solution with*

$$u_j \in C^1([0, t_{\mathrm{end}}/\varepsilon], W) \cap C([0, t_{\mathrm{end}}/\varepsilon], W^1), \qquad j \in \mathcal{J}_+^{(m)},$$
$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|u_j(t)\|_{W^1} \le C_{u,1}.$$

(*iii*) *If (2.9) holds with $\sigma \in \{2,3\}$, then*

$$u_j \in C^{\sigma-\ell}([0, t_{\mathrm{end}}/\varepsilon], W^\ell) \quad \text{for every } \ell = 0,\dots,\sigma, \qquad j \in \mathcal{J}_+^{(m)},$$
$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|u_j(t)\|_{W^\sigma} \le C_{u,\sigma}. \tag{2.10}$$

The constants $C_{u,\sigma}$, $\sigma \in \{0,1,2,3\}$, depend on the nonlinearity T and on t_{end}, $\|p_0\|_{W^\sigma}$, $\|p_1\|_{W^\sigma}$, but not on $\varepsilon \in (0,1]$.

For $m = 3$, a slightly different version of this result was shown in Baumstark and Jahnke (2023, Lemma 2.3). The extension to arbitrary odd m is straightforward; cf. Baumstark (2022, Lemma 3.6.1). Wellposedness of the SVEA ($m = 1$) was already proven in Colin and Lannes (2009, Theorem 1) and Lannes (2011, Theorem 3.8).

Although t_{end} does in general not have the same value in Lemmas 2.1 and 2.2, we will henceforth assume that solutions to (1.1) and (1.8) exist on the *same* interval $[0, t_{\mathrm{end}}/\varepsilon]$, as suggested by our notation. This is not a restriction as one can always consider the smaller one of the two possibly different intervals.

2.3 Eigendecompositions

The highly oscillatory behavior of the coefficient functions $\widehat{u}_j$ originates from the linear part $(\mathrm{i}/\varepsilon)\mathcal{L}_j(\varepsilon k)\widehat{u}_j(t,k)$ in (2.2a). It is thus not surprising that the eigendecomposition of $\mathcal{L}_j(\theta) = \mathcal{L}(j\omega, j\boldsymbol{\kappa} + \theta)$ plays a crucial role in our analysis. As in Baumstark and Jahnke (2023, Assumption 2.2), we assume the following.

Assumption 2.3. (*i*) *The matrix $\mathcal{L}(0,\beta) = \mathbf{A}(\beta) - \mathrm{i}\mathbf{E}$ has a smooth eigendecomposition: if $\omega_\ell(\beta)$ is an eigenvalue of $\mathcal{L}(0,\beta)$ for some $\ell \in \{1,\dots,n\}$, then $\omega_\ell \in C^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{R})$, and there is a corresponding eigenvector $\boldsymbol{\phi}_\ell(\beta)$ with $\boldsymbol{\phi}_\ell \in C^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{C}^n)$. With no loss of generality, we assume that $|\boldsymbol{\phi}_\ell(\beta)|_2 = 1$ for all β and all $\ell = 1,\dots,n$. The enumeration is chosen in such a way that $\omega = \omega_1(\boldsymbol{\kappa})$ in (1.5).*

(*ii*) *Every eigenvalue $\omega_\ell(\beta)$ of $\mathcal{L}(0,\beta)$ is globally Lipschitz continuous, that is, there is a constant C such that*

$$|\omega_\ell(\widetilde{\beta}) - \omega_\ell(\beta)| \le C|\widetilde{\beta} - \beta|_1 \qquad \text{for all } \widetilde{\beta}, \beta \in \mathbb{R}^d \text{ and } \ell = 1,\dots,n.$$

(*iii*) *The eigenvalue $\omega = \omega_1(\boldsymbol{\kappa})$ is bounded away from the other eigenvalues: There is a constant $C > 0$ such that*

$$|\omega - \omega_\ell(\beta)| \ge C \qquad \text{for all } \beta \in \mathbb{R}^d \text{ and } \ell = 2,\dots,n.$$

Assumption (i) corresponds to Assumption 2 in Colin and Lannes (2009), whereas Assumption (iii) is a part of Assumption 3 in Colin and Lannes (2009).

Remark 2.4. Explicit formulas for the eigenvalues in case of the Maxwell–Lorentz system and the Klein–Gordon system are given in Colin and Lannes (2009, Examples 3 and 4), and one can check that the Assumptions (i) and (ii) on the eigenvalues are true. Assumption (iii) is true if we choose ω to be the largest or smallest eigenvalue in (1.5).

For $j \in \mathcal{J}_+^{(m)}$ and every $\theta \in \mathbb{R}^d$ let

$$\mathcal{L}_j(\theta) = \Psi_j(\theta)\Lambda_j(\theta)\Psi_j^*(\theta) \tag{2.11a}$$

be the eigendecomposition of (2.3): the real diagonal matrix

$$\Lambda_j(\theta) = \operatorname{diag}(\lambda_{j1}(\theta), \dots, \lambda_{jn}(\theta)) \in \mathbb{R}^{n\times n} \tag{2.11b}$$

contains the eigenvalues $\lambda_{j\ell}(\theta) \in \mathbb{R}$ of $\mathcal{L}_j(\theta)$, and

$$\Psi_j(\theta) = \big(\psi_{j1}(\theta) \mid \cdots \mid \psi_{jn}(\theta)\big) \in \mathbb{C}^{n\times n} \tag{2.11c}$$

is unitary with the corresponding normalized eigenvectors $\psi_{j\ell}(\theta) \in \mathbb{C}^n$ in its columns. By Assumption 1.2(i) $\mathcal{L}_1(0) = \mathcal{L}(\omega, \boldsymbol{\kappa})$ has a one-dimensional kernel, and we choose the enumeration of the eigenvalues and eigenvectors in such a way that $\lambda_{11}(0) = 0$ and $\ker \mathcal{L}_1(0) = \operatorname{span}\{\psi_{11}(0)\}$. Equation (2.8) implies that $\Psi_{-j}(\theta) = \overline{\Psi_j(-\theta)}$ and $\Lambda_{-j}(\theta) = -\overline{\Lambda_j(-\theta)} = -\Lambda_j(-\theta)$.

The matrices $\mathcal{L}(0, j\boldsymbol{\kappa} + \theta)$ and $\mathcal{L}(j\omega, j\boldsymbol{\kappa} + \theta) = -j\omega I + \mathcal{L}(0, j\boldsymbol{\kappa} + \theta)$ have the same eigenvectors, and their eigenvalues $\omega_\ell(j\boldsymbol{\kappa} + \theta)$ and $\lambda_{j\ell}(\theta) = -j\omega + \omega_\ell(j\boldsymbol{\kappa} + \theta)$ differ only by a shift. Hence, it follows from Assumption 2.3 that $\lambda_{j\ell} \in C^\infty(\mathbb{R}^d \setminus \{-j\boldsymbol{\kappa}\}, \mathbb{R})$ and $\psi_{j\ell} \in C^\infty(\mathbb{R}^d \setminus \{-j\boldsymbol{\kappa}\}, \mathbb{C}^n)$ with

$$|\lambda_{j\ell}(\widetilde{\theta}) - \lambda_{j\ell}(\theta)| \le C|\widetilde{\theta} - \theta|_1 \qquad \text{for all } \widetilde{\theta}, \theta \in \mathbb{R}^d, \tag{2.12}$$

$$|\lambda_{1\ell}(\theta)| \ge C \qquad \text{for all } \theta \in \mathbb{R}^d \text{ and } \ell = 2, \dots, n. \tag{2.13}$$

2.4 Transformation of the Coefficient Functions

The strategy in the proofs of (1.13) and (1.14) is, roughly speaking, to distinguish the oscillatory "parts" of the solution from the nonoscillatory ones, and to carefully analyze how these parts interact in the nonlinearity. For this purpose, the following transformation was introduced in Baumstark and Jahnke (2023).

Let $\widehat{U}^{(m)} = \{\widehat{u}_j\}_{j\in\mathcal{J}_+^{(m)}}$ be the solution of (2.2) for $\varepsilon \in (0, 1]$. For every $t \ge 0$ and $k \in \mathbb{R}^d$, we define

$$z_j(t,k) = S_{j,\varepsilon}(t,k)\widehat{u}_j(t,k), \qquad z_{-j}(t,k) = \overline{z_j(t,-k)}, \qquad j \in \mathcal{J}_+^{(m)} \tag{2.14}$$

with transformation matrix

$$S_{j,\varepsilon}(t,k) = \exp\left(\frac{\mathrm{i}t}{\varepsilon}\Lambda_j(\varepsilon k)\right)\Psi_j^*(\varepsilon k) = \Psi_j^*(\varepsilon k)\exp\left(\frac{\mathrm{i}t}{\varepsilon}\mathcal{L}_j(\varepsilon k)\right), \qquad j \in \mathcal{J}_+^{(m)}, \tag{2.15a}$$

$$S_{-j,\varepsilon}(t,k) := \overline{S_{j,\varepsilon}(t,-k)}. \tag{2.15b}$$

It follows from (2.14) and (2.2a) that

$$\partial_t z_j(t) = \varepsilon \sum_{\#J=j} F\left(t, \widehat{U}^{(m)}, J\right), \qquad \widehat{U}^{(m)} = \{\widehat{u}_j\}_{j\in\mathcal{J}_+^{(m)}}, \tag{2.16}$$

where F is given by

$$F(t, \widehat{U}^{(m)}, J) = S_{j,\varepsilon}(t)\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(t), \qquad J = (j_1, j_2, j_3) \in (\mathcal{J}^{(m)})^3, \quad j = \#J. \tag{2.17}$$

By means of the inverse transform $\widehat{u}_j(t,k) = S^*_{j,\varepsilon}(t,k)z_j(t,k)$ we could turn (2.16) into a closed system of evolution equations for $\{z_j\}_{j\in\mathcal{J}^{(m)}}$, but with a rather complicated right-hand side. The initial conditions are

$$z_j(0,k) = S_{j,\varepsilon}(0,k)\widehat{u}_j(0,k) = \begin{cases} \Psi_1^*(\varepsilon k)\widehat{p}(k) & \text{if } j = 1, \\ 0 & \text{if } j \in \mathcal{J}_+^{(m)} \setminus \{1\}, \end{cases} \tag{2.18}$$

according to (2.2b), (2.14), and (2.15).

The transformations (2.14) and (2.15) are motivated by the fact that in the *linear* case, the exact solution of (2.2) is

$$\widehat{u}_j(t,k) = S^*_{j,\varepsilon}(t,k)z_j(0,k) \qquad \text{for } \mathcal{T}(\cdot,\cdot,\cdot) = 0,$$

because $z_j(t) = z_j(0)$ is constant in time for $\mathcal{T}(\cdot,\cdot,\cdot) = 0$ according to (2.16) and (2.17). But even in the nonlinear case $\mathcal{T}(\cdot,\cdot,\cdot) \neq 0$ the right-hand side of (2.16) is formally only $\mathcal{O}(\varepsilon)$ instead of $\mathcal{O}(1/\varepsilon)$ in (2.2a), because the linear part $(\mathrm{i}/\varepsilon)\mathcal{L}_j(\varepsilon k)\widehat{u}_j(t,k)$ is cancelled by the transformation. The transformed functions z_j do still oscillate in time, but the oscillations appear on a much smaller scale, and in this sense, z_j is smoother than $\widehat{u}_j$.

2.5 Projectors

Recall that by Assumption 1.2(i), the matrix $\mathcal{L}(\omega,\boldsymbol{\kappa}) = \mathcal{L}_1(0)$ has a one-dimensional kernel spanned by $\psi_{11}(0)$. This is the reason why the first eigenspace of the matrix $\mathcal{L}_1(\varepsilon k) = \mathcal{L}_1(0) + \varepsilon\mathbf{A}(k)$, which appears in (2.2a) will play a special role in our analysis. We denote the orthogonal projection onto this eigenspace by

$$\widehat{w} \mapsto \mathcal{P}_\varepsilon\widehat{w}, \qquad \mathcal{P}_\varepsilon(k) = \psi_{11}(\varepsilon k)\psi^*_{11}(\varepsilon k) \in \mathbb{C}^{n\times n} \tag{2.19}$$

and the projector onto the orthogonal complement by $\mathcal{P}^\perp_\varepsilon = I - \mathcal{P}_\varepsilon$. Assumption 1.2(ii) is equivalent to $\mathcal{P}^\perp_0\widehat{p} = \varepsilon\mathcal{P}^\perp_0\widehat{p}_1$, and for $p_0, p_1 \in W^1$ it was shown in the proof of Lemma 3 in Colin and Lannes (2009) that

$$\|\mathcal{P}^\perp_\varepsilon\widehat{p}\|_{L^1} \leq C\varepsilon(\|p_1\|_W + \|\nabla p\|_W) \leq C\varepsilon(\|p_0\|_{W^1} + \|p_1\|_{W^1}). \tag{2.20}$$

For the transformed function (2.14), we obtain from (2.15) that

$$\mathcal{P}_\varepsilon(k)\widehat{u}_1(t,k) = \psi_{11}(\varepsilon k)\exp\left(-\frac{\mathrm{i}t}{\varepsilon}\lambda_{11}(\varepsilon k)\right)z_{11}(t,k) = S^*_{1,\varepsilon}(t,k)Pz_1(t,k), \tag{2.21}$$

where $z_{11}(t,k)$ is the first entry of $z_1(t,k) \in \mathbb{C}^n$ and

$$P : \mathbb{C}^n \to \mathbb{C}^n, \quad (w_1,\dots,w_n)^\top \mapsto (w_1,0,\dots,0)^\top \tag{2.22}$$

is the orthogonal projection of a vector $\mathbf{w}$ onto $\mathrm{span}\{(1,0,\dots,0)^\top\}$. For $P^\perp = (I-P)$ the estimate (2.20) yields

$$\|P^\perp z_1(0,\cdot)\|_{L^1} \leq C\varepsilon(\|p_0\|_{W^1} + \|p_1\|_{W^1}), \tag{2.23}$$

because with (2.21), we obtain

$$P^\perp z_1(0,\cdot) = z_1(0,\cdot) - Pz_1(0,\cdot) = S_{1,\varepsilon}(0,k)\mathcal{P}^\perp_\varepsilon(k)\widehat{u}_1(0,k).$$

2.6 Useful Identities and Inequalities

Throughout, we will frequently use the following facts. Since we have chosen the Euclidean vector norm $|\cdot|_2$ to define $\|\cdot\|_{L^1}$ in (2.1), the norm $\|\widehat{f}\|_{L^1}$ of $\widehat{f} \in L^1$ is invariant under multiplication of $\widehat{f}(k) \in \mathbb{C}^n$ with a unitary matrix $S(k) \in \mathbb{C}^{n\times n}$. This means, in particular, that for the transformed functions $z_j(t,k) = S_{j,\varepsilon}(t,k)\widehat{u}_j(t,k)$ from (2.14) the identities

$$|z_j(t,k)|_2 = |\widehat{u}_j(t,k)|_2, \qquad \|z_j(t)\|_{L^1} = \|\widehat{u}_j(t)\|_{L^1} = \|u_j(t)\|_W \tag{2.24}$$

and, via (2.21), the equations

$$|Pz_1(t,k)|_2 = |\mathcal{P}_\varepsilon(k)\widehat{u}_1(t,k)|_2, \qquad \|Pz_1(t)\|_{L^1} = \|\mathcal{P}_\varepsilon\widehat{u}_1(t)\|_{L^1},$$
$$|P^\perp z_1(t,k)|_2 = |\mathcal{P}_\varepsilon^\perp(k)\widehat{u}_1(t,k)|_2, \qquad \|P^\perp z_1(t)\|_{L^1} = \|\mathcal{P}_\varepsilon^\perp\widehat{u}_1(t)\|_{L^1} \tag{2.25}$$

hold for all $t \geq 0$, $k \in \mathbb{R}^d$, and $\varepsilon \in (0,1]$. Moreover, we will use that for all $w \in \mathbb{C}^n$ and $f \in L^1$ the inequalities

$$|Pw|_2 \leq |w|_2, \qquad \|Pf\|_{L^1} \leq \|f\|_{L^1}, \tag{2.26}$$
$$|\mathcal{P}_\varepsilon(k)w|_2 \leq |w|_2, \qquad \|\mathcal{P}_\varepsilon f\|_{L^1} \leq \|f\|_{L^1} \tag{2.27}$$

hold, as well as the same inequalities with P and $\mathcal{P}_\varepsilon$ replaced by $P^\perp$ and $\mathcal{P}_\varepsilon^\perp$, respectively.

3 Why $\mathcal{P}_\varepsilon\widehat{u}_1$ is Smooth and $\mathcal{P}_\varepsilon^\perp\widehat{u}_1$ is Small in the SVEA

In this and the next section, we analyze the SVEA (1.9)–(1.10), which corresponds to setting

$$m = 1, \qquad \mathcal{J}^{(m)} = \mathcal{J}^{(1)} = \{-1, 1\}, \qquad \mathcal{J}_+^{(1)} = \{1\}$$

in (1.7), (1.8), and (2.2), respectively. Our main goal is to prove the error bound (1.13), which will be achieved in Section 4; cf. Theorem 4.3. This proof is based on a number of auxiliary results, which we compile now. We start by quoting two important inequalities from Colin and Lannes (2009).

Lemma 3.1. *Let $m = 1$, let $\sigma = 1$ in (2.9), and let u_1 be the classical solution of (1.10), which was established in Lemma 2.2(ii). Under Assumptions 2.3 and 1.2(i), there is a constant C such that*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(t)\|_{L^1} \leq C. \tag{3.1}$$

The constant C depends on $C_{u,1}$ from (2.10) and thus also on t_{end}, but not on $\varepsilon \in (0,1]$.

Proof. See Colin and Lannes (2009, Lemma 2). □

Proposition 3.2. *Let $m = 1$ and let u_1 be the classical solution of (1.10) with initial data of the form (2.9) with $\sigma = 1$. Under Assumptions 1.2 and 2.3, there is a constant C such that*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|\mathcal{P}_\varepsilon^\perp\widehat{u}_1(t)\|_{L^1} \leq C\varepsilon \tag{3.2}$$

for all $\varepsilon \in (0,1]$.

Proof. See Colin and Lannes (2009, Lemma 3). □

In Baumstark (2022), a similar result was shown without Assumption 2.3(iii), but on a possibly smaller interval $[0, t_\star/\varepsilon]$ for some $t_\star \leq t_{\mathrm{end}}$.

These results can be interpreted as follows. The term $(\mathrm{i}/\varepsilon)\mathcal{L}_1(\varepsilon k)\widehat{u}_1(t,k)$ in (2.2a) suggests that formally $\partial_t\widehat{u}_1 = \mathcal{O}(1/\varepsilon)$. Lemma 3.1 shows, however, that the time derivative of the *projected* part $\mathcal{P}_\varepsilon\widehat{u}_1$ is bounded uniformly in ε. Hence, we can consider $\mathcal{P}_\varepsilon\widehat{u}_1$ as "the nonoscillatory part of $\widehat{u}_1$," although strictly speaking this interpretation is not correct, because oscillations in $\mathcal{P}_\varepsilon\widehat{u}_1$ can still be detected on a very small scale; cf. Remark 3.6 at the end of this subsection.

For the time derivatives of the other part $\mathcal{P}_\varepsilon^\perp\widehat{u}_1 = \widehat{u}_1 - \mathcal{P}_\varepsilon\widehat{u}_1$ a corresponding result does not hold, which means that $\partial_t\mathcal{P}_\varepsilon^\perp\widehat{u}_1 = \mathcal{O}(1/\varepsilon)$ in general. Proposition 3.2 shows, however, that $\|\mathcal{P}_\varepsilon^\perp\widehat{u}_1(t)\|_{L^1} = \mathcal{O}(\varepsilon)$ even on the long time interval $[0, t_{\mathrm{end}}/\varepsilon]$. Hence, we can think of $\mathcal{P}_\varepsilon^\perp\widehat{u}_1$ as "small but oscillatory" in the sense that its time derivative is much larger than $\mathcal{P}_\varepsilon^\perp\widehat{u}_1$ itself. Exploiting the different properties of $\mathcal{P}_\varepsilon\widehat{u}_1$ and $\mathcal{P}_\varepsilon^\perp\widehat{u}_1$ will be crucial in the proof of Theorem 4.3 in Section 4. Before that, we have to extend Lemma 3.1 and Proposition 3.2 to a stronger norm.

Let D_μ denote the Fourier multiplier $(D_\mu \widehat{w})(k) = \mathrm{i}k_\mu \widehat{w}(k)$ for $\mu \in \{1, \dots, d\}$. We want to show that under stronger regularity assumptions Proposition 3.2 remains true when $\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)$ is replaced by $D_\mu \mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)$; cf. Proposition 3.4. This corresponds to an extension of the inequality (3.2) from

$$\|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)\|_{L^1} = \|\mathcal{F}^{-1}(\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t))\|_W$$

to the stronger norm

$$\|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)\|_{L^1} + \sum_{\mu=1}^{d} \|D_\mu \mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)\|_{L^1} = \|\mathcal{F}^{-1}(\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t))\|_{W^1}.$$

As a first step, we prove the following counterpart of Lemma 3.1.

Lemma 3.3. *Let $m = 1$, let $\sigma = 2$ in (2.9), and let u_1 be the classical solution of (1.10), which was established in Lemma 2.2(iii). Under Assumptions 2.3 and 1.2(i), there is a constant C such that*

$$\sup_{t \in [0, t_{\mathrm{end}}/\varepsilon]} \|\partial_t D_\mu \mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} \leq C.$$

The constant C depends on $C_{u,2}$ from (2.10) and thus also on t_{end}, but not on $\varepsilon \in (0, 1]$.

Proof. The proof is similar to the proof of Lemma 3.1. We choose $\mu \in \{1, \dots, d\}$ and apply $D_\mu \mathcal{P}_\varepsilon(k)$ to both sides of (2.2a) with $j = m = 1$. This yields

$$\partial_t D_\mu \mathcal{P}_\varepsilon(k) \widehat{u}_1(t,k) = -\frac{\mathrm{i}}{\varepsilon} D_\mu \mathcal{P}_\varepsilon(k) \mathcal{L}_1(\varepsilon k) \widehat{u}_1(t,k) + \varepsilon D_\mu \mathcal{P}_\varepsilon(k) \sum_{\#J=1} \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(t,k) \tag{3.3}$$

for all $t \in (0, t_{\mathrm{end}}/\varepsilon]$ and $k \in \mathbb{R}^d$. The first term on the right-hand side is

$$-\frac{\mathrm{i}}{\varepsilon} D_\mu \mathcal{P}_\varepsilon(k) \mathcal{L}_1(\varepsilon k) \widehat{u}_1(t,k) = -\frac{\mathrm{i}}{\varepsilon} \lambda_{11}(\varepsilon k) \mathcal{P}_\varepsilon(k) D_\mu \widehat{u}_1(t,k) \tag{3.4}$$

because of (2.19) and (2.11). The Lipschitz continuity (2.12) of the eigenvalues and the fact that $\lambda_{11}(0) = 0$ yield

$$|\lambda_{11}(\varepsilon k)| = |\lambda_{11}(\varepsilon k) - \lambda_{11}(0)| \leq C\varepsilon |k|_1,$$

and together with (3.4) and (2.27), this gives

$$\begin{aligned} \left\| \frac{\mathrm{i}}{\varepsilon} D_\mu \mathcal{P}_\varepsilon \mathcal{L}_1(\varepsilon \cdot) \widehat{u}_1(t) \right\|_{L^1} &\leq C \int_{\mathbb{R}^d} |k|_1 |\mathcal{P}_\varepsilon(k) D_\mu \widehat{u}_1(t,k)|_2 \, \mathrm{d}k \\ &\leq C \int_{\mathbb{R}^d} |k|_1^2 |\widehat{u}_1(t,k)|_2 \, \mathrm{d}k = C \cdot C_{u,2}, \end{aligned} \tag{3.5}$$

with $C_{u,2}$ from Lemma 2.2(iii). For the nonlinear term on the right-hand side of (3.3), we have

$$D_\mu \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right) = \mathcal{T}\left(D_\mu \widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right) + \mathcal{T}\left(\widehat{u}_{j_1}, D_\mu \widehat{u}_{j_2}, \widehat{u}_{j_3}\right) + \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, D_\mu \widehat{u}_{j_3}\right), \tag{3.6}$$

which corresponds to the product rule. Since there are three multiindices $J \in (\mathcal{J}^{(1)})^3$ with $\#J = 1$, namely $(1, 1, -1), (1, -1, 1), (-1, 1, 1)$, we obtain with (2.6) and (3.6)

$$\varepsilon \left\| D_\mu \mathcal{P}_\varepsilon \sum_{\#J=1} \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(t) \right\|_{L^1} \leq 9\varepsilon C_{\mathcal{T}} C_{u,1}^3. \tag{3.7}$$

The assertion follows by combining (3.3), (3.5), (3.7), and using that $\varepsilon \leq 1$ by assumption. □

With Lemma 3.3, we can now show the following extension of Proposition 3.2.

Proposition 3.4. *Let $m = 1$ and let u_1 be the classical solution of (1.10) with initial data of the form (2.9) with $\sigma = 2$. Under the assumptions of Proposition 3.2 there is a constant C such that*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|D_\mu \mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)\|_{L^1} \le C\varepsilon \tag{3.8}$$

for all $\varepsilon \in (0,1]$ and all $\mu \in \{1,\dots,d\}$.

Proof. Choose a fixed $\mu \in \{1,\dots,d\}$ and set

$$\widehat{v}(t,k) = (D_\mu \mathcal{P}_\varepsilon^\perp \widehat{u}_1)(t,k) = \mathrm{i}k_\mu \mathcal{P}_\varepsilon^\perp(k)\widehat{u}_1(t,k). \tag{3.9}$$

We apply $D_\mu \mathcal{P}_\varepsilon^\perp$ to (2.2) with $m = j = 1$ and use that $\mathcal{P}_\varepsilon^\perp$ commutes with $\mathcal{L}_1(\varepsilon k)$. This yields

$$\begin{aligned}\partial_t \widehat{v}(t) + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}_1(\varepsilon\,\cdot)\widehat{v}(t) &= \varepsilon \sum_{\#J=1} D_\mu \mathcal{P}_\varepsilon^\perp \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(t),\\ \widehat{v}(0) &= D_\mu \mathcal{P}_\varepsilon^\perp \widehat{p}\end{aligned}$$

with $\mathcal{L}_1(\varepsilon\,\cdot)$ denoting $k \mapsto \mathcal{L}_1(\varepsilon k)$. Now we adapt the proof of Proposition 3.2. With Duhamel's formula and the short-hand notation

$$\mathcal{T}(\widehat{u}_J) = \mathcal{T}(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}) \qquad \text{for } J = (j_1, j_2, j_3),$$

we obtain

$$\widehat{v}(t) = \widehat{v}^{[1]}(t) + \widehat{v}^{[2]}(t) + \widehat{v}^{[3]}(t)$$

with the three terms

$$\begin{aligned}\widehat{v}^{[1]}(t) &= \exp\left(-\frac{\mathrm{i}t}{\varepsilon}\mathcal{L}_1(\varepsilon\,\cdot)\right) D_\mu \mathcal{P}_\varepsilon^\perp \widehat{p},\\ \widehat{v}^{[2]}(t) &= \varepsilon \sum_{\#J=1} \int_0^t \exp\left(\frac{\mathrm{i}(s-t)}{\varepsilon}\mathcal{L}_1(\varepsilon\,\cdot)\right) D_\mu \mathcal{P}_\varepsilon^\perp \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_J(s)\right)\,\mathrm{d}s,\\ \widehat{v}^{[3]}(t) &= \varepsilon \sum_{\#J=1} \int_0^t \exp\left(\frac{\mathrm{i}(s-t)}{\varepsilon}\mathcal{L}_1(\varepsilon\,\cdot)\right) D_\mu \mathcal{P}_\varepsilon^\perp \left[\mathcal{T}\left(\widehat{u}_J(s)\right) - \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_J(s)\right)\right]\mathrm{d}s.\end{aligned}$$

We will show that

$$\|\widehat{v}^{[\eta]}(t)\|_{L^1} \le c_1\varepsilon + c_2\varepsilon \int_0^t \left\|\widehat{v}(s)\right\|_{L^1}\mathrm{d}s \qquad \text{for } t \in [0, t_{\mathrm{end}}/\varepsilon] \text{ and } \eta = 1,2,3 \tag{3.10}$$

with constants $c_1 \ge 0$ and $c_2 \ge 0$, which do not depend on $\varepsilon \in (0,1]$. If (3.10) is true, then applying Gronwall's lemma and using that $\varepsilon t \le t_{\mathrm{end}}$ proves that $\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|\widehat{v}(t)\|_{L^1} \le C\varepsilon$ which, via (3.9), is equivalent to (3.8).

For the first term $\widehat{v}^{[1]}(t)$, the inequality (2.20) implies

$$\left\|\widehat{v}^{[1]}(t)\right\|_{L^1} = \|\mathcal{P}_\varepsilon^\perp D_\mu \widehat{p}\|_{L^1} \le C\varepsilon(\|p_0\|_{W^2} + \|p_1\|_{W^2}),$$

which verifies (3.10) for $\eta = 1$ (with $c_2 = 0$).

For the third term $\widehat{v}^{[3]}(t)$, we infer with (2.27), (3.6), (2.7), and Proposition 3.2 that

$$\begin{aligned}\|\widehat{v}^{[3]}(t)\|_{L^1} &\le 3\varepsilon \int_0^t \left\|D_\mu \mathcal{T}\left(\widehat{u}_J(s)\right) - D_\mu \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_J(s)\right)\right\|_{L^1}\mathrm{d}s\\ &\le C\varepsilon \int_0^t \left(\left\|\widehat{u}_1(s) - \mathcal{P}_\varepsilon \widehat{u}_1(s)\right\|_{L^1} + \left\|D_\mu \widehat{u}_1(s) - D_\mu \mathcal{P}_\varepsilon \widehat{u}_1(s)\right\|_{L^1}\right)\mathrm{d}s\\ &= C\varepsilon \int_0^t \left(\left\|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(s)\right\|_{L^1} + \left\|\widehat{v}(s)\right\|_{L^1}\right)\mathrm{d}s\\ &\le c_1\varepsilon + c_2\varepsilon \int_0^t \left\|\widehat{v}(s)\right\|_{L^1}\mathrm{d}s\end{aligned}$$

with constants c_1, c_2, which depend on $C_{\mathcal{T}}$, $C_{u,1}$, and in case of c_1 also on the constant from (3.2).

Now we consider the second term $\widehat{v}^{[2]}(t)$. Since there are three multiindices $J \in (\mathcal{J}^{(1)})^3$ with $\#J = 1$, we obtain

$$\begin{aligned}\|\widehat{v}^{[2]}(t)\|_{L^1} &\leq 3\varepsilon \int_{\mathbb{R}^d} \Big| \int_0^t \exp\left(\frac{\mathrm{i}(s-t)}{\varepsilon}\mathcal{L}_1(\varepsilon k)\right) \left(D_\mu \mathcal{P}_\varepsilon^\perp \mathcal{T}(\mathcal{P}_\varepsilon \widehat{u}_J)\right)(s,k)\, \mathrm{d}s\Big|_2 \,\mathrm{d}k \\ &= 3\varepsilon \int_{\mathbb{R}^d} \Big| \int_0^t \exp\left(\frac{\mathrm{i}(s-t)}{\varepsilon}\mathcal{L}_1(\varepsilon k)\right) \mathcal{P}_\varepsilon^\perp(k)\widehat{q}(s,k)\, \mathrm{d}s\Big|_2 \,\mathrm{d}k \end{aligned} \tag{3.11}$$

with the abbreviation

$$\widehat{q}(s,k) = \left(D_\mu \mathcal{T}(\mathcal{P}_\varepsilon \widehat{u}_J)\right)(s,k). \tag{3.12}$$

The goal is now to integrate by parts to gain one additional factor ε, which is then used to compensate the long time interval. However, this requires some care, because the matrix $\mathcal{L}_1(0) = \mathcal{L}(\omega, \boldsymbol{\kappa})$ is singular; see (1.5) or Assumption 1.2(i). What saves us here is the projector $\mathcal{P}_\varepsilon^\perp(k)$ in (3.11). For every $k \in \mathbb{R}^d$, the restriction of $\mathcal{L}_1(\varepsilon k)$ to the subspace $\mathcal{P}_\varepsilon^\perp(k)\mathbb{C}^n$ is given by

$$\begin{aligned}&\mathcal{L}_1^\perp(\varepsilon k) : \mathcal{P}_\varepsilon^\perp(k)\mathbb{C}^n \to \mathcal{P}_\varepsilon^\perp(k)\mathbb{C}^n, \\ &\mathcal{L}_1^\perp(\varepsilon k) = \mathcal{L}_1(\varepsilon k)\mathcal{P}_\varepsilon^\perp(k) = \sum_{\ell=2}^n \lambda_{1\ell}(\varepsilon k)\psi_{1\ell}(\varepsilon k)\psi_{1\ell}^*(\varepsilon k).\end{aligned}$$

By (2.13), this mapping is regular with uniformly bounded inverse

$$\left(\mathcal{L}_1^\perp(\varepsilon k)\right)^{-1} : \mathcal{P}_\varepsilon^\perp(k)\mathbb{C}^n \to \mathcal{P}_\varepsilon^\perp(k)\mathbb{C}^n, \qquad \left(\mathcal{L}_1^\perp(\varepsilon k)\right)^{-1} = \sum_{\ell=2}^n \frac{1}{\lambda_{1\ell}(\varepsilon k)}\psi_{1\ell}(\varepsilon k)\psi_{1\ell}^*(\varepsilon k).$$

The presence of $\mathcal{P}_\varepsilon^\perp(k)$ in (3.11) allows us to replace $\mathcal{L}_1(\varepsilon k)$ by $\mathcal{L}_1^\perp(\varepsilon k)$ and to integrate by parts in the inner integral of (3.11). This yields

$$\begin{aligned}&\Big| \int_0^t \exp\left(\frac{\mathrm{i}(s-t)}{\varepsilon}\mathcal{L}_1(\varepsilon k)\right) \mathcal{P}_\varepsilon^\perp(k)\widehat{q}(s,k)\, \mathrm{d}s\Big|_2 \\ &\quad\leq \Big|\frac{\varepsilon}{\mathrm{i}}\left(\mathcal{L}_1^\perp(\varepsilon k)\right)^{-1}\widehat{q}(t,k) - \frac{\varepsilon}{\mathrm{i}}\exp\left(-\frac{\mathrm{i}t}{\varepsilon}\mathcal{L}_1^\perp(\varepsilon k)\right)\left(\mathcal{L}_1^\perp(\varepsilon k)\right)^{-1}\widehat{q}(0,k)\Big|_2 \\ &\qquad + \Big|\frac{\varepsilon}{\mathrm{i}}\int_0^t \exp\left(\frac{\mathrm{i}(s-t)}{\varepsilon}\mathcal{L}_1^\perp(\varepsilon k)\right)\left(\mathcal{L}_1^\perp(\varepsilon k)\right)^{-1}\partial_t\widehat{q}(s,k)\, \mathrm{d}s\Big|_2 \\ &\quad\leq C\varepsilon\Big(|\widehat{q}(t,k)|_2 + |\widehat{q}(0,k)|_2\Big) + C\varepsilon\int_0^t |\partial_t\widehat{q}(s,k)|_2\, \mathrm{d}s,\end{aligned}$$

and substituting this into (3.11) leads to

$$\|\widehat{v}^{[2]}(t)\|_{L^1} \leq C\varepsilon^2\Big(\|\widehat{q}(t)\|_{L^1} + \|\widehat{q}(0)\|_{L^1}\Big) + C\varepsilon t_{\mathrm{end}} \sup_{s\in[0,t_{\mathrm{end}}/\varepsilon]} \|\partial_t\widehat{q}(s)\|_{L^1}. \tag{3.13}$$

With (3.12), (3.6), and (2.6), we obtain that

$$\|\widehat{q}(t)\|_{L^1} = \|\left(D_\mu \mathcal{T}(\mathcal{P}_\varepsilon \widehat{u}_J)\right)(t)\|_{L^1} \leq 3C_{\mathcal{T}}C_{u,1}^3, \qquad \|\widehat{q}(0)\|_{L^1} \leq 3C_{\mathcal{T}}C_{u,1}^3, \tag{3.14}$$

and that

$$\begin{aligned}\|\partial_t\widehat{q}(s)\|_{L^1} &= \|\partial_t D_\mu \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_J(s)\right)\|_{L^1} \\ &\leq 6C_{\mathcal{T}}\|\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(s)\|_{L^1}\|D_\mu\mathcal{P}_\varepsilon\widehat{u}_1(s)\|_{L^1}\|\mathcal{P}_\varepsilon\widehat{u}_1(s)\|_{L^1} \\ &\quad + 3C_{\mathcal{T}}\|\partial_t D_\mu\mathcal{P}_\varepsilon\widehat{u}_1(s)\|_{L^1}\|\mathcal{P}_\varepsilon\widehat{u}_1(s)\|_{L^1}^2.\end{aligned}$$

Since $\|\partial_t \mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1}$ and $\|\partial_t D_\mu \mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1}$ are uniformly bounded by Lemmas 3.1 and 3.3, respectively, this shows that $\|\partial_t \widehat{q}(s)\|_{L^1}$ is uniformly bounded in $s \in [0, t_{\mathrm{end}}/\varepsilon]$ and $\varepsilon \in (0,1]$. Combining this with (3.14) and (3.13) yields (3.10) for $\eta = 2$. This completes the proof. □

Before closing this section, we prove that even the second time derivative of $\mathcal{P}_\varepsilon \widehat{u}_1(t)$ is uniformly bounded. This somewhat simple observation will be crucial for showing the error bound for the SVEA; cf. (4.28) in step 4 of the proof of Theorem 4.3.

Lemma 3.5. *Let $m = 1$, let $\sigma = 2$ in (2.9), and let u_1 be the classical solution of (1.10). Under Assumptions 2.3 and 1.2(i), the second time derivative of $\mathcal{P}_\varepsilon \widehat{u}_1(t)$ is uniformly bounded, that is, there is a constant C such that*

$$\sup_{t \in [0, t_{\mathrm{end}}/\varepsilon]} \|\partial_t^2 \mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} \le C.$$

The constant C depends on $C_{u,2}$ from (2.10) and thus also on t_{end}, but not on ε.

Proof. Applying $\mathcal{P}_\varepsilon(k)\partial_t$ on both sides of (2.2a) with $j = m = 1$ gives

$$\mathcal{P}_\varepsilon(k)\partial_t^2 \widehat{u}_1(t,k) = -\frac{\mathrm{i}}{\varepsilon}\mathcal{P}_\varepsilon(k)\mathcal{L}_1(\varepsilon k)\partial_t \widehat{u}_1(t,k) + \varepsilon \mathcal{P}_\varepsilon(k) \sum_{\#J=1} \partial_t \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(t,k)$$

for $t \in (0, t_{\mathrm{end}}/\varepsilon]$ and $k \in \mathbb{R}^d$. By adapting the arguments from the proof of Lemma 3.3, we arrive at the bound

$$\left|\frac{\mathrm{i}}{\varepsilon}\mathcal{P}_\varepsilon \mathcal{L}_1(\varepsilon k)\partial_t \widehat{u}_1(t,k)\right|_2 \le C|k|_1 \left|\mathcal{P}_\varepsilon \partial_t \widehat{u}_1(t,k)\right|_2 = C \sum_{\mu=1}^{d} \left|\partial_t D_\mu \mathcal{P}_\varepsilon \widehat{u}_1(t,k)\right|_2$$

for the first term. For the nonlinear term, the product rule yields

$$\partial_t \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right) = \mathcal{T}\left(\partial_t \widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right) + \mathcal{T}\left(\widehat{u}_{j_1}, \partial_t \widehat{u}_{j_2}, \widehat{u}_{j_3}\right) + \mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \partial_t \widehat{u}_{j_3}\right),$$

and with (2.6) we obtain

$$\|\partial_t^2 \mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} \le C \sum_{\mu=1}^{d} \left\|\partial_t D_\mu \mathcal{P}_\varepsilon \widehat{u}_1(t)\right\|_{L^1} + 9\varepsilon C_{\mathcal{T}} \|\partial_t \widehat{u}_1(t)\|_{L^1} \|\widehat{u}_1(t)\|_{L^1}^2.$$

Now the assertion follows from Lemma 3.3 and the fact that $\|\partial_t \widehat{u}_1(t)\|_{L^1} \le C\varepsilon^{-1}$. □

Remark 3.6. By taking more derivatives of (2.2a) and proceeding as in the proof of Lemma 3.5, it can be shown that $\|\partial_t^\ell \mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} = \mathcal{O}(\varepsilon^{2-\ell})$ for $\ell \ge 3$. Hence, higher-order time derivatives are *not* uniformly bounded, which means that our interpretation of $\mathcal{P}_\varepsilon \widehat{u}_1$ as the nonoscillatory part of $\widehat{u}_1$ is only true to a certain extent.

4 Convergence Analysis for the SVEA

With the results from the previous section we are now in a position to prove the error bound (1.13), where $\widetilde{u}^{(1)}$ is the SVEA (1.9)–(1.10). We assume that p has the form (2.9) with $\sigma = 2$. Then, by Lemmas 2.1 and 2.2, there is a constant C_u such that

$$\sup_{\varepsilon \in (0,1]} \sup_{t \in [0, t_{\mathrm{end}}/\varepsilon]} \|u(t)\|_W \le C_u \qquad \text{and} \qquad \sup_{\varepsilon \in (0,1]} \sup_{t \in [0, t_{\mathrm{end}}/\varepsilon]} \|\widetilde{u}^{(1)}(t)\|_W \le C_u. \tag{4.1}$$

The error bound requires the following assumption on the eigenvalues of $\mathcal{L}_j(0) = \mathcal{L}(j\omega, j\boldsymbol{\kappa})$.

Assumption 4.1 (Nonresonance condition)**.** *The matrix $\mathcal{L}(3\omega, 3\boldsymbol{\kappa})$ is regular and has no common eigenvalues with $\mathcal{L}_1(0) = \mathcal{L}(\omega, \boldsymbol{\kappa})$, that is, $\lambda_{3i}(0) \ne \lambda_{1\ell}(0)$ for all $i, \ell \in \{1, \dots, n\}$.*

Remark 4.2. As mentioned earlier, explicit formulas for the eigenvalues in case of the Klein–Gordon system and the Maxwell–Lorentz system can be found in Colin and Lannes (2009, Examples 3 and 4). For these applications, one can check that Assumption 4.1 holds if the chosen eigenvalue $\omega = \omega(\boldsymbol{\kappa})$ is not constant with respect to $\boldsymbol{\kappa}$.

4.1 Improved Error Bound for the SVEA

The following theorem is our first main result. It states that the SVEA converges with *second* order. We recall that the SVEA (1.9)–(1.10) is identical to (1.7)–(1.8) with $m = 1$ and $\mathcal{J}^{(1)} = \{-1, 1\}$.

Theorem 4.3 (Error bound for the SVEA)**.** *Suppose that (2.9) holds with $\sigma = 2$, and let u be the solution of (1.1). Let u_1 be the classical solution of (1.10) established in part (iii) of Lemma 2.2, and let $\widetilde{u}^{(1)}$ be the approximation defined in (1.9). Under Assumptions 1.2, 2.3, and 4.1, there is a constant C such that*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|u(t) - \widetilde{u}^{(1)}(t)\|_W \le C\varepsilon^2, \tag{4.2}$$

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|u(t) - \widetilde{u}^{(1)}(t)\|_{L^\infty} \le C\varepsilon^2. \tag{4.3}$$

The following lemma is a preparation for the proof of Theorem 4.3.

Lemma 4.4. *Let R be the residual of the approximation $\widetilde{u}^{(1)}$, that is,*

$$R = \varepsilon T(\widetilde{u}^{(1)}, \widetilde{u}^{(1)}, \widetilde{u}^{(1)}) - \Big(\partial_t \widetilde{u}^{(1)}(t,x) + \mathbf{A}(\partial)\widetilde{u}^{(1)} + \frac{1}{\varepsilon}\mathbf{E}\,\widetilde{u}^{(1)}\Big),$$

and let $\widehat{R}$ be its Fourier transform. If the inequality

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \Big\| \int_0^t \exp\Big((s-t)\Big(\mathrm{i}\mathbf{A}(\cdot) + \frac{1}{\varepsilon}\mathbf{E}\Big)\Big)\widehat{R}(s)\,\mathrm{d}s\Big\|_{L^1} \le C\varepsilon^2 \tag{4.4}$$

holds with a constant C, which does not depend on ε, then the error bounds (4.2) and (4.3) are true.

Proof of Lemma 4.4. We will prove that (4.4) implies (4.2). Then, the second error bound (4.3) follows directly from (4.2) via the embedding $W \hookrightarrow L^\infty$. By definition of R, the difference $\delta = u - \widetilde{u}^{(1)}$ is the solution of

$$\partial_t \delta = -\mathbf{A}(\partial)\delta - \frac{1}{\varepsilon}\mathbf{E}\delta + \varepsilon\left[T(u,u,u) - T(\widetilde{u}^{(1)}, \widetilde{u}^{(1)}, \widetilde{u}^{(1)})\right] + R, \tag{4.5a}$$

$$\delta(0) = 0. \tag{4.5b}$$

Now we investigate the structure of the residual. By (1.9), the approximation $\widetilde{u}^{(1)}$ can be expressed as

$$\widetilde{u}^{(1)}(t,x) = \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} u_1(t,x) + \mathrm{e}^{-\mathrm{i}(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} u_{-1}(t,x) = \sum_{j\in\mathcal{J}^{(1)}} \mathrm{e}^{\mathrm{i}j(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} u_j(t,x).$$

Substituting this into the left-hand side of (1.1) and using (1.10a) yields

$$\begin{aligned}
&\partial_t \widetilde{u}^{(1)}(t,x) + \mathbf{A}(\partial)\widetilde{u}^{(1)}(t,x) + \frac{1}{\varepsilon}\mathbf{E}\,\widetilde{u}^{(1)}(t,x) \\
&\quad = \sum_{j\in\mathcal{J}^{(1)}} \mathbf{e}^{\mathbf{i}j(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon}\Big(\partial_t u_j(t,x) + \frac{\mathbf{i}}{\varepsilon}\mathcal{L}(j\omega, j\boldsymbol{\kappa})u_j(t,x) + \mathbf{A}(\partial)u_j(t,x)\Big) \\
&\quad = \varepsilon \sum_{j\in\mathcal{J}^{(1)}} \sum_{\#J=j} \mathrm{e}^{\mathrm{i}j(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} T(u_{j_1}, u_{j_2}, u_{j_3})(t,x),
\end{aligned} \tag{4.6}$$

whereas on the right-hand side of (1.1), we obtain

$$\begin{aligned}
\varepsilon T(\widetilde{u}^{(1)}, \widetilde{u}^{(1)}, \widetilde{u}^{(1)})(t,x) &= \varepsilon \sum_{J\in(\mathcal{J}^{(1)})^3} \mathrm{e}^{\mathrm{i}\#J(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} T(u_{j_1}, u_{j_2}, u_{j_3})(t,x) \\
&= \varepsilon \sum_{\substack{j \text{ odd} \\ |j|\le 3}} \sum_{\#J=j} \mathrm{e}^{\mathrm{i}j(\boldsymbol{\kappa}\cdot x - \omega t)/\varepsilon} T(u_{j_1}, u_{j_2}, u_{j_3})(t,x).
\end{aligned} \tag{4.7}$$

The only difference between (4.6) and (4.7) is that the terms with $j = \pm 3$ are missing in (4.6). These terms are exactly the higher harmonics, which were omitted in the derivation of (1.8) and hence of the SVEA. Equations (4.6) and (4.7) yield the representation

$$R(t,x) = \varepsilon \sum_{j\in\{\pm 3\}} \sum_{\#J=j} \mathrm{e}^{\mathrm{i}j(\boldsymbol{\kappa}\cdot x-\omega t)/\varepsilon} T(u_{j_1}, u_{j_2}, u_{j_3})(t,x)$$

of the residual.

Since (4.2) is equivalent to

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|\widehat{\delta}(t)\|_{L^1} \le C\varepsilon^2, \tag{4.8}$$

we need an evolution equation for $\widehat{\delta} = \mathcal{F}\delta$. In Fourier space, (4.5a) reads

$$\partial_t \widehat{\delta}(t,k) = -\left(\mathrm{i}\mathbf{A}(k) + \frac{1}{\varepsilon}\mathbf{E}\right)\widehat{\delta}(t,k) + \varepsilon \mathcal{G}\left(\mathcal{F}u, \mathcal{F}\widetilde{u}^{(1)}\right)(t,k) + \widehat{R}(t,k) \tag{4.9}$$

with

$$\begin{aligned}
\mathcal{G}\left(\mathcal{F}u, \mathcal{F}\widetilde{u}^{(1)}\right) &= \mathcal{T}(\mathcal{F}u, \mathcal{F}u, \mathcal{F}u) - \mathcal{T}\left(\mathcal{F}\widetilde{u}^{(1)}, \mathcal{F}\widetilde{u}^{(1)}, \mathcal{F}\widetilde{u}^{(1)}\right), \\
\widehat{R}(t,k) &= \varepsilon \sum_{j\in\{\pm 3\}} \sum_{\#J=j} \mathcal{F}\left(T(u_{j_1}, u_{j_2}, u_{j_3})\mathrm{e}^{\mathrm{i}j\boldsymbol{\kappa}\cdot x/\varepsilon}\right)(t,k)\mathrm{e}^{-\mathrm{i}j\omega t/\varepsilon} \\
&= \varepsilon \sum_{j\in\{\pm 3\}} \sum_{\#J=j} \mathcal{T}(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3})\left(t, k - \frac{j\boldsymbol{\kappa}}{\varepsilon}\right)\mathrm{e}^{-\mathrm{i}j\omega t/\varepsilon},
\end{aligned} \tag{4.10}$$

and with $\mathcal{T}$ defined in (2.5). We apply Duhamel's formula to (4.9) and use that $\widehat{\delta}(0,k) = 0$ to obtain

$$\begin{aligned}
\widehat{\delta}(t,k) = \varepsilon &\int_0^t \exp\left((s-t)\left(\mathrm{i}\mathbf{A}(k) + \frac{1}{\varepsilon}\mathbf{E}\right)\right)\mathcal{G}\left(\mathcal{F}u(s), \mathcal{F}\widetilde{u}^{(1)}(s)\right)(k)\,\mathrm{d}s \\
&+ \int_0^t \exp\left((s-t)\left(\mathrm{i}\mathbf{A}(k) + \frac{1}{\varepsilon}\mathbf{E}\right)\right)\widehat{R}(s,k)\,\mathrm{d}s.
\end{aligned} \tag{4.11}$$

For every $k \in \mathbb{R}^d$, the matrix $\mathrm{i}\mathbf{A}(k) + \mathbf{E}/\varepsilon$ is skew-Hermitian, and hence $\exp(t(\mathrm{i}\mathbf{A}(k) + \mathbf{E}/\varepsilon))$ is unitary for every $t \in \mathbb{R}$. The first term on the right-hand side of (4.11) can thus be bounded in L^1 by

$$\begin{aligned}
&\varepsilon \int_0^t \int_{\mathbb{R}^d} \left|\exp\left((s-t)\left(\mathrm{i}\mathbf{A}(k) + \frac{1}{\varepsilon}\mathbf{E}\right)\right)\mathcal{G}\left(\mathcal{F}u(s), \mathcal{F}\widetilde{u}^{(1)}(s)\right)(k)\right|_2 \mathrm{d}k\,\mathrm{d}s \\
&= \varepsilon \int_0^t \int_{\mathbb{R}^d} \left|\mathcal{G}\left(\mathcal{F}u(s), \mathcal{F}\widetilde{u}^{(1)}(s)\right)(k)\right|_2 \mathrm{d}k\,\mathrm{d}s \\
&\le 3C_{\mathcal{T}}C_u^2\,\varepsilon \int_0^t \|\widehat{\delta}(s)\|_{L^1}\,\mathrm{d}s.
\end{aligned} \tag{4.12}$$

In the last step, we have used (2.7) and (4.1). Now it follows from (4.11), (4.12), and (4.4) that

$$\|\widehat{\delta}(t)\|_{L^1} \le CC_u^2\,\varepsilon \int_0^t \|\widehat{\delta}(s)\|_{L^1}\,\mathrm{d}s + C\varepsilon^2,$$

and applying Gronwall's lemma yields the desired inequality (4.8) with a constant, which depends on C_u and t_{end}. □

The central task in the proof of Theorem 4.3 is thus to prove (4.4). Equation (4.10) shows that $\|\widehat{R}(s)\|_{L^1} = \mathcal{O}(\varepsilon)$, but straightforward estimates yield only

$$\begin{aligned}
\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \left\|\int_0^t \exp\left((s-t)\left(\mathrm{i}\mathbf{A}(\cdot) + \frac{1}{\varepsilon}\mathbf{E}\right)\right)\widehat{R}(s)\,\mathrm{d}s\right\|_{L^1} &\le \sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \int_0^t \left\|\widehat{R}(s)\right\|_{L^1}\mathrm{d}s \\
&\le \frac{t_{\mathrm{end}}}{\varepsilon} \sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \left\|\widehat{R}(s)\right\|_{L^1} \le C.
\end{aligned}$$

Compared to this simple bound, we have to gain a factor of ε^2. This is where the real work starts.

Proof of Theorem 4.3. In view of Lemma 4.4, we have to prove (4.4). The strategy, notation, and presentation are very similar to the proof of Theorem 4.2 in Baumstark and Jahnke (2023), but there are some crucial differences which we point out below. □

Step 1. In this step, we express the integral term from (4.4) in an appropriate way. We use (1.4), (2.3), and (4.10) to obtain

$$\begin{aligned}
&\int_0^t \exp\left((s-t)\left(\mathrm{i}\mathbf{A}(k)+\frac{1}{\varepsilon}\mathbf{E}\right)\right)\widehat{R}(s,k)\,\mathrm{d}s \\
&\quad=\varepsilon\int_0^t \exp\left((s-t)\left(\mathrm{i}\mathbf{A}(k)+\frac{1}{\varepsilon}\mathbf{E}\right)\right)\sum_{j\in\{\pm 3\}}\sum_{\#J=j}\mathcal{T}(\widehat{u}_{j_1},\widehat{u}_{j_2},\widehat{u}_{j_3})\left(s,k-\frac{j\boldsymbol{\kappa}}{\varepsilon}\right)\mathrm{e}^{-\mathrm{i}j\omega s/\varepsilon}\,\mathrm{d}s \\
&\quad=\varepsilon\mathrm{e}^{-\mathrm{i}j\omega t/\varepsilon}\sum_{j\in\{\pm 3\}}\sum_{\#J=j}\int_0^t \exp\left(\frac{\mathrm{i}}{\varepsilon}(s-t)\mathcal{L}(j\omega,\varepsilon k)\right)\mathcal{T}(\widehat{u}_{j_1},\widehat{u}_{j_2},\widehat{u}_{j_3})\left(s,k-\frac{j\boldsymbol{\kappa}}{\varepsilon}\right)\mathrm{d}s \\
&\quad=\varepsilon\mathrm{e}^{-\mathrm{i}j\omega t/\varepsilon}\sum_{j\in\{\pm 3\}}\sum_{\#J=j}\int_0^t \exp\left(\frac{\mathrm{i}}{\varepsilon}(s-t)\mathcal{L}_j(\varepsilon k')\right)\mathcal{T}(\widehat{u}_{j_1},\widehat{u}_{j_2},\widehat{u}_{j_3})(s,k')\,\mathrm{d}s
\end{aligned}$$

with the shifted variable $k'=k-j\boldsymbol{\kappa}/\varepsilon$. In order to keep the notation simple, we write again k instead of k' in the following. Since later we integrate over k, the difference between k and k' does not really matter.

With (2.17) and (2.15), we can represent the integrand as

$$\begin{aligned}
\exp\left(\frac{\mathrm{i}}{\varepsilon}(s-t)\mathcal{L}_j(\varepsilon k)\right)\mathcal{T}(\widehat{u}_{j_1},\widehat{u}_{j_2},\widehat{u}_{j_3})(s,k)&=\exp\left(-\frac{\mathrm{i}t}{\varepsilon}\mathcal{L}_j(\varepsilon k)\right)\Psi_j(\varepsilon k)F(s,\{\widehat{u}_1\},J)(k)\\
&=S^*_{j,\varepsilon}(t,k)F(s,\{\widehat{u}_1\},J)(k).
\end{aligned}$$

Since $S^*_{j,\varepsilon}(t)$ is unitary and does not depend on s, it follows that the term which appears on the left-hand side of (4.4) can be bounded by

$$\left\|\int_0^t \exp\left((s-t)\left(\mathrm{i}\mathbf{A}(\cdot)+\frac{1}{\varepsilon}\mathbf{E}\right)\right)\widehat{R}(s)\,\mathrm{d}s\right\|_{L^1}\leq\varepsilon\sum_{j\in\{\pm 3\}}\sum_{\#J=j}\left\|\int_0^t F(s,\{\widehat{u}_1\},J)\,\mathrm{d}s\right\|_{L^1}. \tag{4.13}$$

Step 2. The goal in this and the following steps is to prove that

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]}\sum_{j\in\{\pm 3\}}\sum_{\#J=j}\left\|\int_0^t F(s,\{\widehat{u}_1\},J)\,\mathrm{d}s\right\|_{L^1}\leq C\varepsilon. \tag{4.14}$$

If (4.14) holds, then the crucial inequality (4.4) follows via[1] (4.13), which then completes the proof of (4.2). The sum in (4.14) is taken over multiindices $J\in(\mathcal{J}^{(1)})^3=\{1,-1\}^3$ with $\#J=j\in\{3,-3\}$. There are only two possibilities, namely $J=(1,1,1)$, $j=3$ and $J=-(1,1,1)$, $j=-3$. Since both cases can be treated *mutatis mutandis*, we will only consider the first one, that is, $J=(1,1,1)$, $j=3$, and thus

$$F(s,\{\widehat{u}_1\},J)=S_{3,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_1,\widehat{u}_1,\widehat{u}_1\right)(s).$$

We have to show that

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]}\left\|\int_0^t S_{3,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_1,\widehat{u}_1,\widehat{u}_1\right)(s)\,\mathrm{d}s\right\|_{L^1}\leq C\varepsilon. \tag{4.15}$$

In order to use Proposition 3.2, we decompose the nonlinearity into eight parts

$$\begin{aligned}\mathcal{T}\left(\widehat{u}_1,\widehat{u}_1,\widehat{u}_1\right) &= \mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)+\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1\right)\\ &\quad+\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)+\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)\\ &\quad+\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1\right)+\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1\right)\\ &\quad+\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)+\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1\right).\end{aligned}$$

The last four terms are those where $\mathcal{P}_\varepsilon^\perp\widehat{u}_1$ appears in at least two of the three arguments of $\mathcal{T}(\cdot,\cdot,\cdot)$. These terms are $\mathcal{O}(\varepsilon^2)$ because of Proposition 3.2, and their contribution to the left-hand side of (4.14) can be estimated in a straightforward way. For example, (2.6) and the fact that $S_{3,\varepsilon}(s)$ is unitary yields

$$\begin{aligned}&\left\|\int_0^t S_{3,\varepsilon}(s)\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(s)\,\mathrm{d}s\right\|_{L^1}\\ &\quad\le C_{\mathcal{T}}\int_0^t\left(\|\mathcal{P}_\varepsilon^\perp\widehat{u}_1(s)\|_{L^1}\,\|\mathcal{P}_\varepsilon^\perp\widehat{u}_1(s)\|_{L^1}\,\|\mathcal{P}_\varepsilon\widehat{u}_1(s)\|_{L^1}\right)\mathrm{d}s\le Ct\varepsilon^2\le Ct_{\mathrm{end}}\varepsilon.\end{aligned}$$

For the first four parts of $\mathcal{T}(\widehat{u}_1,\widehat{u}_1,\widehat{u}_1)$ the analysis is much more involved. We have to prove that

$$\left\|\int_0^t S_{3,\varepsilon}(s)\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(s)\,\mathrm{d}s\right\|_{L^1}\le C\varepsilon, \tag{4.16}$$

$$\left\|\int_0^t S_{3,\varepsilon}(s)\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(s)\,\mathrm{d}s\right\|_{L^1}\le C\varepsilon, \tag{4.17}$$

because bounds for the terms involving $\mathcal{T}(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1)$ and $\mathcal{T}(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon^\perp\widehat{u}_1)$ can be shown in the same way as (4.16).

Step 3. In this step, we prove (4.16). To accomplish this, we have to identify the oscillatory "parts" of the integrand. We use that (2.21), (2.14), and (2.15) yield the representation

$$\mathcal{P}_\varepsilon^\perp\widehat{u}_1(t,k)=S_{1,\varepsilon}^*(t,k)P^\perp z_1(t,k)=\sum_{\ell=2}^n\exp\left(-\frac{\mathrm{i}t}{\varepsilon}\lambda_{1\ell}(\varepsilon k)\right)z_{1\ell}(t,k)\psi_{1\ell}(\varepsilon k), \tag{4.18}$$

where again $\lambda_{1\ell}(\varepsilon k)$ is the ℓ-th eigenvalue of $\mathcal{L}_1(\varepsilon k)$ and $\psi_{1\ell}(\varepsilon k)$ is the corresponding eigenvector, as defined in (2.11). Combining (4.18) with (2.15) and (2.5) results in

$$\begin{aligned}&\int_0^t S_{3,\varepsilon}(s,k)\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(s,k)\,\mathrm{d}s\\ &\quad=\frac{1}{(2\pi)^d}\int_0^t\exp\left(\frac{\mathrm{i}s}{\varepsilon}\Lambda_3(\varepsilon k)\right)\Psi_3^*(\varepsilon k)\int_{\#K=k}T\left(\mathcal{P}_\varepsilon^\perp\widehat{u}_1(s,k^{(1)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)\,\mathrm{d}K\,\mathrm{d}s\\ &\quad=\sum_{\ell=2}^n\int_{\#K=k}\int_0^t\exp\left(\frac{\mathrm{i}s}{\varepsilon}\left[\Lambda_3(\varepsilon k)-\lambda_{1\ell}(\varepsilon k^{(1)})I\right]\right)f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\,\mathrm{d}K\end{aligned}$$

with the shorthand notation from (2.4), and with

$$f_{\varepsilon,\ell}(s,K)=\frac{1}{(2\pi)^d}\Psi_3^*(\varepsilon k)T\left(z_{1\ell}(s,k^{(1)})\psi_{1\ell}(\varepsilon k^{(1)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right),$$

where $k = \#K$. Taking the norm yields

$$\begin{aligned}
&\Big\| \int_0^t S_{3,\varepsilon}(s)\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s)\,\mathrm{d}s\Big\|_{L^1} \\
&\quad = \int_{\mathbb{R}^d} \Big| \int_0^t S_{3,\varepsilon}(s,k)\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s,k)\,\mathrm{d}s\Big|_2 \,\mathrm{d}k \\
&\quad \le \sum_{\ell=2}^{n} \int_{\mathbb{R}^d} \int_{\#K=k} \Big| \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\big[\Lambda_3(\varepsilon k) - \lambda_{1\ell}(\varepsilon k^{(1)})I\big]\Big) f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2 \,\mathrm{d}K\,\mathrm{d}k. \qquad (4.19)
\end{aligned}$$

Now we focus on the inner integral. The exponential function

$$s \mapsto \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\big[\Lambda_3(\varepsilon k) - \lambda_{1\ell}(\varepsilon k^{(1)})I\big]\Big)$$

in (4.19) oscillates if all diagonal entries of the diagonal matrix $\Lambda_3(\varepsilon k) - \lambda_{1\ell}(\varepsilon k^{(1)})I$ are bounded away from zero, but we cannot expect this to be true for all $k, k^{(1)} \in \mathbb{R}^d$. For this reason, we define

$$\begin{aligned}
\Delta_\ell(\theta, \theta^{(1)}) &= \Lambda_3(\theta) - \lambda_{1\ell}(\theta^{(1)})I, \qquad \text{for } \theta, \theta^{(1)} \in \mathbb{R}^d, \\
g_{\varepsilon,\ell}(s,K) &= \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\big[\Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0)\big]\Big) f_{\varepsilon,\ell}(s,K)
\end{aligned}$$

and reformulate the inner integral in (4.19) as

$$\Big| \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\big[\Lambda_3(\varepsilon k) - \lambda_{1\ell}(\varepsilon k^{(1)})I\big]\Big) f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2 = \Big| \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Delta_\ell(0,0)\Big) g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2. \qquad (4.20)$$

By Assumption 4.1, the diagonal matrix

$$\Delta_\ell(0,0) = \Lambda_3(0) - \lambda_{1\ell}(0)I = \mathrm{diag}\Big(\lambda_{31}(0) - \lambda_{1\ell}(0), \dots, \lambda_{3n}(0) - \lambda_{1\ell}(0)\Big)$$

is regular for all ℓ. Hence, we can now integrate by parts to obtain

$$\begin{aligned}
&\Big| \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Delta_\ell(0,0)\Big) g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2 \\
&\quad = \Big|\frac{\varepsilon}{\mathrm{i}}\Delta_\ell(0,0)^{-1}\Big(\exp\Big(\frac{\mathrm{i}t}{\varepsilon}\Delta_\ell(0,0)\Big) g_{\varepsilon,\ell}(t,K) - g_{\varepsilon,\ell}(0,K)\Big)\Big|_2 \\
&\qquad + \Big|\frac{\varepsilon}{\mathrm{i}}\Delta_\ell(0,0)^{-1} \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Delta_\ell(0,0)\Big)\partial_t g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2 \\
&\quad \le C\varepsilon\Big(|g_{\varepsilon,\ell}(t,K)|_2 + |g_{\varepsilon,\ell}(0,K)|_2\Big) + C\varepsilon \int_0^t \Big|\partial_t g_{\varepsilon,\ell}(s,K)\Big|_2\,\mathrm{d}s. \qquad (4.21)
\end{aligned}$$

By definition of $g_{\varepsilon,\ell}$, we have

$$\begin{aligned}
\partial_t g_{\varepsilon,\ell}(s,K) = \frac{\mathrm{i}}{\varepsilon}\big[\Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0)\big] g_{\varepsilon,\ell}(s,K) \\
+ \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\big[\Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0)\big]\Big)\partial_t f_{\varepsilon,\ell}(s,K),
\end{aligned}$$

and since $|g_{\varepsilon,\ell}(s,K)|_2 = |f_{\varepsilon,\ell}(s,K)|_2$ this yields

$$\Big|\partial_t g_{\varepsilon,\ell}(s,K)\Big|_2 \le \frac{C}{\varepsilon}\Big|\Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0)\Big|_2 |f_{\varepsilon,\ell}(s,K)|_2 + |\partial_t f_{\varepsilon,\ell}(s,K)|_2.$$

With (4.21), (4.20), and (4.19), we infer that

$$\left\| \int_0^t S_{3,\varepsilon}(s)\mathcal{T}\left(\mathcal{P}_\varepsilon^\perp \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s)\, \mathrm{d}s \right\|_{L^1} \leq C\varepsilon \Big(X_1(t,\varepsilon) + X_2(t,\varepsilon) + X_3(t,\varepsilon) \Big)$$

with

$$\begin{aligned}
X_1(t,\varepsilon) &= \sum_{\ell=2}^n \int_{\mathbb{R}^d} \int_{\#K=k} \Big(|g_{\varepsilon,\ell}(t,K)|_2 + |g_{\varepsilon,\ell}(0,K)|_2 \Big)\, \mathrm{d}K\, \mathrm{d}k, \\
X_2(t,\varepsilon) &= \sum_{\ell=2}^n \int_{\mathbb{R}^d} \int_{\#K=k} \int_0^t \frac{1}{\varepsilon} \Big| \Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0) \Big|_2 |f_{\varepsilon,\ell}(s,K)|_2 \, \mathrm{d}s\, \mathrm{d}K\, \mathrm{d}k, \\
X_3(t,\varepsilon) &= \sum_{\ell=2}^n \int_{\mathbb{R}^d} \int_{\#K=k} \int_0^t |\partial_t f_{\varepsilon,\ell}(s,K)|_2 \, \mathrm{d}s\, \mathrm{d}K\, \mathrm{d}k.
\end{aligned}$$

In order to complete the proof of (4.16), we have to show that $X_1(t,\varepsilon)$, $X_2(t,\varepsilon)$, and $X_3(t,\varepsilon)$ are uniformly bounded in $\varepsilon \in (0,1]$ and $t \in [0, t_{\mathrm{end}}/\varepsilon]$. For $X_2(t,\varepsilon)$ and $X_3(t,\varepsilon)$, this is not obvious because of the integration over the possibly long time interval $[0,t]$ with $t \leq t_{\mathrm{end}}/\varepsilon$. We use that

$$\begin{aligned}
\sum_{\ell=2}^n |f_{\varepsilon,\ell}(s,K)|_2 &\leq C \sum_{\ell=2}^n \left| T\Big(z_{1\ell}(s,k^{(1)}) \psi_{1\ell}(\varepsilon k^{(1)}), \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(2)}), \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(3)}) \Big) \right|_2 \\
&\leq C \Big| \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s,k^{(1)}) \Big|_2 \Big| \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(2)}) \Big|_2 \Big| \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(3)}) \Big|_2
\end{aligned}$$

holds, because of the normalization $|\psi_{1\ell}(\varepsilon k^{(1)})|_2 = 1$ and the fact that

$$\sum_{\ell=2}^n |z_{1\ell}(s,k^{(1)})| = |P^\perp z_1(s,k^{(1)})|_1 \leq C |P^\perp z_1(s,k^{(1)})|_2 = C \Big| \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s,k^{(1)}) \Big|_2$$

by (2.25). With $|g_{\varepsilon,\ell}(s,K)|_2 = |f_{\varepsilon,\ell}(s,K)|_2$ this implies that

$$\begin{aligned}
&\sum_{\ell=2}^n \int_{\mathbb{R}^d} \int_{\#K=k} |g_{\varepsilon,\ell}(s,K)|_2 \, \mathrm{d}K\, \mathrm{d}k \\
&\quad \leq C \int_{\mathbb{R}^d} \int_{\#K=k} \left(\Big| \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s,k^{(1)}) \Big|_2 \Big| \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(2)}) \Big|_2 \Big| \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(3)}) \Big|_2 \right) \mathrm{d}K\, \mathrm{d}k \\
&\quad = C \int_{\mathbb{R}^d} \Big| \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s,k^{(1)}) \Big|_2 \, \mathrm{d}k^{(1)} \int_{\mathbb{R}^d} \Big| \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(2)}) \Big|_2 \, \mathrm{d}k^{(2)} \int_{\mathbb{R}^d} \Big| \mathcal{P}_\varepsilon \widehat{u}_1(s,k^{(3)}) \Big|_2 \, \mathrm{d}k^{(3)} \\
&\quad = C \| \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s) \|_{L^1} \| \mathcal{P}_\varepsilon \widehat{u}_1(s) \|_{L^1}^2 \\
&\quad \leq C\varepsilon
\end{aligned}$$

for all $s \in [0, t_{\mathrm{end}}/\varepsilon]$ due to Proposition 3.2, Lemma 2.2(i), and (2.27). This shows, in particular, that $X_1(t,\varepsilon)$ is uniformly bounded[2] in $\varepsilon \in (0,1]$ and $t \in [0, t_{\mathrm{end}}/\varepsilon]$.

For $X_2(t,\varepsilon)$, we use that the Lipschitz continuity (2.12) of the eigenvalues yields

$$\begin{aligned}
\Big| \Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0) \Big|_2 &\leq \Big| \Lambda_3(\varepsilon k) - \Lambda_3(0) \Big|_2 + \Big| \lambda_{1\ell}(\varepsilon k^{(1)}) - \lambda_{1\ell}(0) \Big| \\
&\leq C\varepsilon (|k|_1 + |k^{(1)}|_1),
\end{aligned}$$

and the ε in the second line compensates the factor $1/\varepsilon$ in $X_2(t,\varepsilon)$. For $K = (k^{(1)}, k^{(2)}, k^{(3)})$ with $k = \#K = k^{(1)} + k^{(2)} + k^{(3)}$, we have that $|k|_1 \leq |k^{(1)}|_1 + |k^{(2)}|_1 + |k^{(3)}|_1$. Hence, it follows that

$$\begin{aligned} X_2(t,\varepsilon) &= \sum_{\ell=2}^{n} \int_{\mathbb{R}^d} \int_{\#K=k} \int_0^t \frac{1}{\varepsilon} \left| \Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0) \right|_2 |\, f_{\varepsilon,\ell}(s,K)|_2 \, \mathrm{d}s \, \mathrm{d}K \, \mathrm{d}k \\ &\leq C \sum_{\ell=2}^{n} \int_{\mathbb{R}^d} \int_{\#K=k} \int_0^t (|k^{(1)}|_1 + |k^{(2)}|_1 + |k^{(3)}|_1) |\, f_{\varepsilon,\ell}(s,K)|_2 \, \mathrm{d}s \, \mathrm{d}K \, \mathrm{d}k \end{aligned}$$

and proceeding as before yields with Lemma 2.2(ii) and (2.27) that

$$\begin{aligned} X_2(t,\varepsilon) &\leq C \sum_{\mu=1}^{d} \int_0^t \left(\|D_\mu \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s)\|_{L^1} \|\mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1}^2 + 2\|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(s)\|_{L^1} \|D_\mu \mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1} \|\mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1} \right) \mathrm{d}s \\ &\leq C \sum_{\mu=1}^{d} \int_0^t \|D_\mu \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s)\|_{L^1} \, \mathrm{d}s + C \int_0^t \|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(s)\|_{L^1} \, \mathrm{d}s. \end{aligned}$$

Since both integrands are $\mathcal{O}(\varepsilon)$ according to Propositions 3.2 and 3.4, respectively, the right-hand side is uniformly bounded for $t \in [0, t_{\mathrm{end}}/\varepsilon]$.

In a similar way, one can show that

$$\begin{aligned} X_3(t,\varepsilon) &= \sum_{\ell=2}^{n} \int_{\mathbb{R}^d} \int_{\#K=k} \int_0^t |\partial_t \, f_{\varepsilon,\ell}(s,K)|_2 \, \mathrm{d}s \, \mathrm{d}K \, \mathrm{d}k \\ &\leq C \sum_{\ell=2}^{n} \int_0^t \left(\|\partial_t P^\perp z_1(s)\|_{L^1} \|\mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1}^2 + 2\|P^\perp z_1(s)\|_{L^1} \|\partial_t \mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1} \|\mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1} \right) \mathrm{d}s. \end{aligned}$$

Since $\|P^\perp z_1(s)\|_{L^1} = \|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(s)\|_{L^1} \leq C\varepsilon$ by Proposition 3.2, since $\|\partial_t \mathcal{P}_\varepsilon \widehat{u}_1(s)\|_{L^1}$ is uniformly bounded by Lemma 3.1, and since

$$\|\partial_t P^\perp z_1(s)\|_{L^1} \leq \|\partial_t z_1(s)\|_{L^1} \leq \varepsilon \sum_{\#J=1} \|\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(t)\|_{L^1} \leq C\varepsilon$$

by (2.16) and (2.17), we conclude that $X_3(t,\varepsilon)$ is uniformly bounded, too. We have thus shown the inequality (4.16).

Step 4. In this step, we prove (4.17). For the proof of (4.16) in the previous step, it was crucial that $\mathcal{P}_\varepsilon^\perp \widehat{u}_1$ appears in one of the arguments of $\mathcal{T}$, because this allowed us to use Propositions 3.2 and 3.4. In (4.17), however, this is not possible, because all three arguments of $\mathcal{T}$ are $\mathcal{P}_\varepsilon \widehat{u}_1$ instead of $\mathcal{P}_\varepsilon^\perp \widehat{u}_1$. Hence, we have to proceed in a different way. The crucial observation is that $\mathcal{P}_\varepsilon \widehat{u}_1$ and thus also $\mathcal{T}(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1)$ are nonoscillatory in the sense that the first two time derivatives of $\mathcal{P}_\varepsilon \widehat{u}_1$ are uniformly bounded according to Lemmas 3.1, 3.3, and 3.5. The only oscillatory function on the left-hand side of (4.17) is $S_{3,\varepsilon}(s)$. The strategy is now to integrate by parts *twice*, which generates a factor ε each time. One of these factors is then used to compensate the long time interval.

We set $\Delta_3(\varepsilon k) = \Lambda_3(\varepsilon k) - \Lambda_3(0)$ and

$$f_\varepsilon(t,k) = \exp\left(\frac{\mathrm{i}t}{\varepsilon}\Delta_3(\varepsilon k)\right) \Psi_3^*(\varepsilon k) \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(t,k).$$

With (2.15), we obtain the representation

$$\left\| \int_0^t S_{3,\varepsilon}(s) \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s) \, \mathrm{d}s \right\|_{L^1} = \left\| \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\Lambda_3(0)\right) f_\varepsilon(s) \, \mathrm{d}s \right\|_{L^1}$$

of the left-hand side of (4.17). By Assumption 4.1, the matrix $\mathcal{L}_3(0) = \mathcal{L}(3\omega, 3\boldsymbol{\kappa})$ and thus also $\Lambda_3(0)$ is invertible. Hence, we can integrate by parts twice and obtain

$$\begin{aligned}\Big\| \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Lambda_3(0)\Big) f_\varepsilon(s)\,\mathrm{d}s\Big\|_{L^1} &\le C\varepsilon\Big[\|f_\varepsilon(0)\|_{L^1} + \|f_\varepsilon(t)\|_{L^1} + \Big\| \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Lambda_3(0)\Big)\,\partial_t f_\varepsilon(s)\,\mathrm{d}s\Big\|_{L^1}\Big] \\ &\le C\varepsilon\Big[\|f_\varepsilon(0)\|_{L^1} + \|f_\varepsilon(t)\|_{L^1} + \varepsilon\|\partial_t f_\varepsilon(0)\|_{L^1} + \varepsilon\|\partial_t f_\varepsilon(t)\|_{L^1} \\ &\quad + \varepsilon\Big\| \int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Lambda_3(0)\Big)\,\partial_t^2 f_\varepsilon(s)\,\mathrm{d}s\Big\|_{L^1}\Big]. \end{aligned} \tag{4.22}$$

Now, we have to show that all terms inside the big bracket […] are uniformly bounded in $t \in [0, t_{\mathrm{end}}/\varepsilon]$ and $\varepsilon \in (0, 1]$. As a preparatory step, we note that applying (2.6), (2.27), and the product rule yields

$$\Big\|\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t)\Big\|_{L^1} \le C_{\mathcal{T}}\|\widehat{u}_1(t)\|_{L^1}^3, \tag{4.23}$$

$$\Big\|D_\mu\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t)\Big\|_{L^1} \le 3C_{\mathcal{T}}\Big\|D_\mu\widehat{u}_1(t)\Big\|_{L^1}\|\widehat{u}_1(t)\|_{L^1}^2, \tag{4.24}$$

$$\begin{aligned}\Big\|D_\mu D_\nu\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t)\Big\|_{L^1} &\le C_{\mathcal{T}}\Big(6\Big\|D_\mu\widehat{u}_1(t)\Big\|_{L^1}\Big\|D_\nu\widehat{u}_1(t)\Big\|_{L^1} \\ &\quad + 3\Big\|D_\mu D_\nu\widehat{u}_1(t)\Big\|_{L^1}\Big\|\widehat{u}_1(t)\Big\|_{L^1}\Big)\Big\|\widehat{u}_1(t)\Big\|_{L^1}\end{aligned} \tag{4.25}$$

for all $\mu, \nu \in \{1, \dots, d\}$. The right-hand side of (4.23), (4.24), and (4.25) is uniformly bounded by Lemma 2.2(iii). In a similar way, we obtain the inequalities

$$\Big\|\partial_t\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t)\Big\|_{L^1} \le 3C_{\mathcal{T}}\Big\|\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(t)\Big\|_{L^1}\|\widehat{u}_1(t)\|_{L^1}^2, \tag{4.26}$$

$$\begin{aligned}\Big\|D_\mu\partial_t\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t)\Big\|_{L^1} &\le C_{\mathcal{T}}\Big(6\Big\|D_\mu\widehat{u}_1(t)\Big\|_{L^1}\Big\|\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(t)\Big\|_{L^1} \\ &\quad + 3\Big\|D_\mu\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(t)\Big\|_{L^1}\Big\|\widehat{u}_1(t)\Big\|_{L^1}\Big)\Big\|\widehat{u}_1(t)\Big\|_{L^1},\end{aligned} \tag{4.27}$$

$$\begin{aligned}\Big\|\partial_t^2\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t)\Big\|_{L^1} &\le 3C_{\mathcal{T}}\Big(\Big\|\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(t)\Big\|_{L^1}^2 \\ &\quad + \Big\|\partial_t^2\mathcal{P}_\varepsilon\widehat{u}_1(t)\Big\|_{L^1}\|\widehat{u}_1(t)\|_{L^1}\Big)\|\widehat{u}_1(t)\|_{L^1},\end{aligned} \tag{4.28}$$

and applying Lemmas 3.1, 3.3, and 3.5 yields uniform boundedness of the right-hand sides of (4.26), (4.27), and (4.28).

Since the matrix $\exp((\mathrm{i}t/\varepsilon)\Delta_3(\varepsilon k))\Psi_3^*(\varepsilon k)$ is unitary, (4.23) implies that $\|f_\varepsilon(t)\|_{L^1}$ is uniformly bounded. Taking the time derivative of $f_\varepsilon(t)$ gives

$$\begin{aligned}\partial_t f_\varepsilon(t,k) &= f_\varepsilon^{[1,1]}(t,k) + f_\varepsilon^{[1,2]}(t,k), \\ f_\varepsilon^{[1,1]}(t,k) &= \frac{\mathrm{i}}{\varepsilon}\Delta_3(\varepsilon k) f_\varepsilon(t,k), \\ f_\varepsilon^{[1,2]}(t,k) &= \exp\Big(\frac{\mathrm{i}t}{\varepsilon}\Delta_3(\varepsilon k)\Big)\Psi_3^*(\varepsilon k)\partial_t\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t,k).\end{aligned} \tag{4.29}$$

The fact that Λ_3 is globally Lipschitz continuous by (2.12) yields

$$\Big|\frac{\mathrm{i}}{\varepsilon}\Delta_3(\varepsilon k)\Big|_2 = \frac{1}{\varepsilon}|\Lambda_3(\varepsilon k) - \Lambda_3(0)|_2 \le C|k|_1$$

with a constant which does not depend on ε and k. Using again that $\exp((\mathrm{i}t/\varepsilon)\Delta_3(\varepsilon k))\Psi_3^*(\varepsilon k)$ is a unitary matrix gives

$$\begin{aligned}
|f_\varepsilon^{[1,1]}(t)|_2 &\le C|k|_1 |f_\varepsilon(t,k)|_2 \le C|k|_1 \left| \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(t,k) \right|_2 \\
&= C \sum_{\mu=1}^{d} \left| D_\mu \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(t,k) \right|_2, \\
|f_\varepsilon^{[1,2]}(t)|_2 &\le |\partial_t \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(t,k)|_2,
\end{aligned} \tag{4.30}$$

and by combining this with (4.29), (4.24), and (4.26), we infer that $\varepsilon \|\partial_t f_\varepsilon(t)\|_{L^1} \le C\varepsilon$. This $\mathcal{O}(\varepsilon)$ estimate is even better than the uniform boundedness which we require at this point. Finally, we show uniform boundedness of the integral term in (4.22). Since $t \in [0, t_{\mathrm{end}}/\varepsilon]$, we can use that

$$\varepsilon \left\| \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\Lambda_3(0)\right) \partial_t^2 f_\varepsilon(s)\, \mathrm{d}s \right\|_{L^1} \le t_{\mathrm{end}} \sup_{s\in[0,t_{\mathrm{end}}/\varepsilon]} \|\partial_t^2 f_\varepsilon(s)\|_{L^1}$$

with

$$\begin{aligned}
\partial_t^2 f_\varepsilon(s) &= \partial_t^2 \Big(\exp\left(\frac{\mathrm{i}t}{\varepsilon}\Delta_3(\varepsilon k)\right) \Psi_3^*(\varepsilon k) \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s,k) \Big) \\
&= f_\varepsilon^{[2,1]}(s,k) + 2 f_\varepsilon^{[2,2]}(s,k) + f_\varepsilon^{[2,3]}(s,k), \\
f_\varepsilon^{[2,1]}(s,k) &= \left(\frac{\mathrm{i}}{\varepsilon}\Delta_3(\varepsilon k)\right)^2 f_\varepsilon(s,k), \\
f_\varepsilon^{[2,2]}(s,k) &= \frac{\mathrm{i}}{\varepsilon}\Delta_3(\varepsilon k) f_\varepsilon^{[1,2]}(s,k), \\
f_\varepsilon^{[2,3]}(s,k) &= \exp\left(\frac{\mathrm{i}t}{\varepsilon}\Delta_3(\varepsilon k)\right) \Psi_3^*(\varepsilon k) \partial_t^2 \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s,k).
\end{aligned}$$

Proceeding as before yields

$$\begin{aligned}
|f_\varepsilon^{[2,1]}(s,k)|_2 &\le C|k|_1^2 |f_\varepsilon(s,k)|_2 \\
&= C|k|_1^2 \left| \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s,k) \right|_2 \\
&= C \sum_{\mu=1}^{d} \sum_{\nu=1}^{d} \left| D_\mu D_\nu \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s,k) \right|_2,
\end{aligned}$$

and hence uniform boundedness of $\|f_\varepsilon^{[2,1]}(s)\|_{L^1}$ follows from (4.25). In a similar way, we obtain with (4.30)

$$\begin{aligned}
|f_\varepsilon^{[2,2]}(s,k)|_2 &\le C|k|_1 |f_\varepsilon^{[1,2]}(s,k)|_2 \\
&\le C|k|_1 |\partial_t \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s,k)|_2 \\
&= C \sum_{\mu=1}^{d} |D_\mu \partial_t \mathcal{T}\left(\mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s,k)|_2,
\end{aligned}$$

such that (4.27) yields uniform boundedness of $\|f_\varepsilon^{[2,2]}(s)\|_{L^1}$. Uniform boundedness of $\|f_\varepsilon^{[2,3]}(s)\|_{L^1}$ follows from (4.28). We have thus shown that all terms in the big bracket […] in (4.22) are uniformly bounded, which completes the proof of (4.17).

According to step 2, the inequalities (4.16) and (4.17) imply the bound (4.15) and hence (4.4). Now the assertion of Theorem 4.3 follows from Lemma 4.4.

Remark 4.5. The proof shows that, in general, the error of the SVEA cannot be expected to be smaller than $\mathcal{O}(\varepsilon^2)$. We have seen in Lemma 4.4 that the accuracy is determined by the right-hand side of (4.4), and in order to improve this

inequality, we have to replace (4.16) and (4.17) by something better[3]. But this is impossible, which can be seen in the proof of (4.17) in step 4. Since $\sup_{t\in[0,t_{\text{end}}/\varepsilon]} \|\mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} = \mathcal{O}(1)$ it follows that

$$\|f_\varepsilon(t)\|_{L^1} = \|\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(t)\|_{L^1} = \mathcal{O}(1), \tag{4.31}$$

and as a consequence, the right-hand side of (4.22) cannot be smaller than $\mathcal{O}(\varepsilon)$. We would like to point out that (4.31) is not the only bottleneck in the proof, and that there are many terms for which a better bound is not feasible. The only way to achieve a higher accuracy is thus to change the approximation, that is, to use (1.7) and (1.8) with $m > 1$. This is the topic of Section 5.

Remark 4.6. We have assumed throughout that the kernel of $\mathcal{L}(\omega, \boldsymbol{\kappa})$ is one-dimensional; see Assumption 1.2(i). In case of the Maxwell–Lorentz system, however, most of the eigenvalues of $\mathcal{L}(0, \boldsymbol{\kappa}) = \mathrm{A}(\boldsymbol{\kappa}) - \mathrm{i}\mathbf{E}$ occur with multiplicity 2, as pointed out in Baumstark (2022, Example 3.2.4), and if the eigenvalue ω chosen in (1.5) has multiplicity 2, then $\mathcal{L}(\omega, \boldsymbol{\kappa}) = -\omega I + \mathcal{L}(0, \boldsymbol{\kappa})$ has a two-dimensional kernel. For this reason, we would like to emphasize that the only purpose of Assumption 1.2(i) is to keep the notation simple, and that all results and proofs in this work could be adapted to cases where the dimension of the kernel is two or larger, as has been done in Baumstark (2022). If the kernel of $\mathcal{L}(\omega, \boldsymbol{\kappa})$ has dimension 2, then $\ell = 2, \dots, n$ in Assumption 2.3(iii) has to be replaced by $\ell = 3, \dots, n$, and the definitions of the projectors (2.19) and (2.22) have to be modified in an obvious way. Likewise, $\sum_{\ell=2}^{n} \dots$ has to be replaced by $\sum_{\ell=3}^{n} \dots$ in the proofs of Proposition 3.4 and Theorem 4.3.

4.2 Numerical Experiment

We illustrate Theorem 4.3 by a numerical example. As a model problem, we use a Klein–Gordon system in one space dimension; cf. Example 2 in Colin and Lannes (2009) and Example 1.5 in Lannes (2011). This system is a special case of (1.1a) with

$$d = 1, \qquad n = 2, \qquad \mathbf{A}(\partial_x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x, \qquad \mathbf{A}(\boldsymbol{\kappa}) = \begin{pmatrix} 0 & \boldsymbol{\kappa} \\ \boldsymbol{\kappa} & 0 \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}.$$

We set

$$t_{\mathbf{end}} = 1, \qquad \boldsymbol{\kappa} = 1.2, \qquad \gamma = 0.7, \qquad T(f_1, f_2, f_3) = (f_1 \cdot f_2)\mathbf{E}f_3.$$

The eigenvalues of $\mathrm{A}(\boldsymbol{\kappa}) - \mathrm{i}\mathbf{E} \in \mathbb{C}^{2\times 2}$ are $\pm\sqrt{\boldsymbol{\kappa}^2 + \gamma^2} \approx \pm 1.3892$, and we select $\omega = \omega(\boldsymbol{\kappa})$ to be the one with the positive sign. For the initial data in (1.1b), we choose $p(x) = \mathrm{e}^{-(x-0.5)^2}\nu$ with $\nu \in \ker(\mathcal{L}(\omega, \boldsymbol{\kappa}))$, such that the polarization condition (Assumption 1.2(ii)) holds with $p = p_0$ and $p_1 = 0$. The initial data and the values for $\boldsymbol{\kappa}$ and γ were chosen more or less arbitrarily. The numerical results reported below remained qualitatively the same for other parametrizations we have tested.

Since numerical approximations of (1.1) and (1.10) can only be computed on a bounded domain, we switch to comoving coordinates

$$\xi = x - c_g t, \qquad v(t, \xi) = u(t, x), \qquad v_1(t, \xi) = u_1(t, x)$$

with group velocity $c_g = \nabla\omega(\boldsymbol{\kappa}) = \boldsymbol{\kappa}/\omega(\boldsymbol{\kappa})$. For $d = 1$, this turns (1.1) into

$$\partial_t v + \mathbf{A}(\partial_\xi)v - c_g\partial_\xi v + \frac{1}{\varepsilon}\mathbf{E}v = \varepsilon T(v, v, v), \qquad t \in (0, t_{\text{end}}/\varepsilon],\ \xi \in \mathbb{R}, \tag{4.32a}$$

$$v(0, \xi) = p(\xi)\mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\xi)/\varepsilon} + c.c., \tag{4.32b}$$

and (1.9)–(1.10) into

$$\begin{aligned} v(t, \xi) \approx \widetilde{v}^{(1)}(t, \xi) &= \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\xi + (\boldsymbol{\kappa}c_g - \omega)t)/\varepsilon} v_1(t, \xi) + c.c., \\ \partial_t v_1 + \frac{\mathbf{i}}{\varepsilon}\mathcal{L}(\omega, \boldsymbol{\kappa})v_1 + \mathbf{A}(\partial_\xi)v_1 - c_g\partial_\xi v_1 &= \varepsilon \sum_{j_1+j_2+j_3=1} T(v_{j_1}, v_{j_2}, v_{j_3}), \\ v_1(0, \cdot) &= p. \end{aligned} \tag{4.33}$$

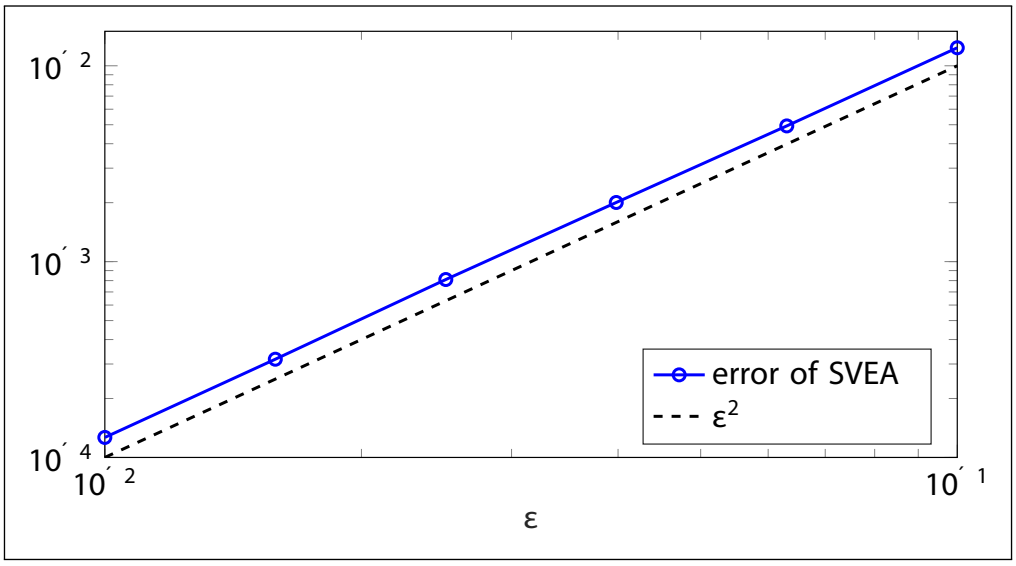

Figure 1. Accuracy of the Slowly Varying Envelope Approximation (SVEA) for Different Values of ε. See Text for Details.

Then, we replace $\xi \in \mathbb{R}$ by $\xi \in [-64, 64]$ with periodic boundary conditions and approximate v_1 with a Strang splitting method with very small step-size ($t_{\text{end}}/10^5$) and mesh-width ($128/2^{14} = 2^{-7}$). Inserting this numerical approximation of v_1 into (4.33) yields a numerical approximation to $\widetilde{v}^{(1)}$, which is then compared with a numerical approximation to the solution of (4.32). As we have explained in the introduction, such an approximation can unfortunately not be obtained by applying a standard method to (4.32) in a straightforward way, because the highly oscillatory solution behavior imposes a very fine discretization in time and space, which causes huge computational costs even in one space dimension. As a remedy, we have used (1.7) and (1.8) with $m = 5$ to compute a reference solution in comoving coordinates.

Figure 1 shows the numerical counterpart of

$$\sup_{t\in[0,t_{\text{end}}/\varepsilon]} \|v(t,\cdot) - \widetilde{v}^{(1)}(t,\cdot)\|_{L^\infty}$$

for different values of ε (blue line) in logarithmic axes. Comparing with $\varepsilon \mapsto \varepsilon^2$ (black dashed line) shows that the error is proportional to ε^2, as predicted by Theorem 4.3.

5 Convergence Analysis for $m = 3$

In this section, we analyze the approximation (1.7) with

$$m = 3 \qquad \text{and} \qquad \mathcal{J}^{(3)} = \{\pm 1, \pm 3\}.$$

As mentioned in the introduction, we have shown in Baumstark and Jahnke (2023) that $\widetilde{u}^{(3)}$ approximates the exact solution u of (1.1) up to an error of $\mathcal{O}(\varepsilon^2)$; cf. (1.12). In this section, we prove that actually the error is only $\mathcal{O}(\varepsilon^3)$ if (2.9) holds with $\sigma = 3$.

By definition the approximation

$$\begin{aligned}\widetilde{u}^{(3)}(t,x) &= \sum_{j\in\mathcal{J}^{(3)}} \mathrm{e}^{\mathrm{i}j(\boldsymbol{\kappa}\cdot x-\omega t)/\varepsilon} u_j(t,x) \\ &= \left(\mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot x-\omega t)/\varepsilon} u_1(t,x) + \mathrm{e}^{3\mathrm{i}(\boldsymbol{\kappa}\cdot x-\omega t)/\varepsilon} u_3(t,x)\right) + c.c.\end{aligned}$$

is based on two functions u_1, u_3 and their complex conjugates $u_{-j} = \overline{u_j}$. These functions u_1, u_3 are determined by the coupled system (1.8), and thus u_1 (which now depends on u_3) is *not* the same as u_1 in the case $m = 1$. As a consequence, we cannot readily use the auxiliary results concerning u_1, which we have shown in Lemmas 3.1, 3.3, and 3.5. However, these results can be extended to the case $m = 3$ with little effort. We summarize this in the following lemma.

Lemma 5.1. *Let $m = 3$, let $\sigma = 1$ in (2.9), and let $\{u_1, u_3\}$ be the classical solution of (1.8).*

(*i*) *Under Assumptions 2.3 and 1.2(i), there is a constant C independent of $\varepsilon \in (0, 1]$ such that*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|\partial_t \mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} \le C, \tag{5.1}$$

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|\partial_t D_\mu \mathcal{P}_\varepsilon \widehat{u}_1(t)\|L^1 \le C. \tag{5.2}$$

(*ii*) *If in addition $\sigma = 2$ in (2.9), then there is a constant C such that*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|\partial_t^2 \mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} \le C. \tag{5.3}$$

(*iii*) *If in addition $\sigma = 3$ in (2.9), then there is a constant C such that*

$$\sup_{t\in[0,t_{\mathrm{end}}/\varepsilon]} \|D_\mu \partial_t^2 \mathcal{P}_\varepsilon \widehat{u}_1(t)\|_{L^1} \le C. \tag{5.4}$$

Proof. The bound (5.1) was shown in Baumstark and Jahnke (2023, Lemma 3.5). To show (5.2) and (5.3), the proofs of Lemmas 3.3 and 3.5 carry over almost verbatim. The only difference is that for $m = 3$ the sum $\sum_{\#J=1} \mathcal{T}(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3})$ contains more terms than for $m = 1$, for example, $\mathcal{T}(\widehat{u}_3, \widehat{u}_{-1}, \widehat{u}_{-1})$, and thus the constants in the very last inequality of each proof change a bit. The proof of (5.4) is more complicated than the proof of (5.3), because new terms arise due to the presence of D_μ, but these terms do not cause any essential new difficulty. □

5.1 Bounds on the Coefficient Functions

As a first step, we prove that for $m = 3$ it is still true that $\|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)\|_{L^1} = \mathcal{O}(\varepsilon)$, and that in addition $\|\widehat{u}_3(t)\|_{L^1} = \mathcal{O}(\varepsilon^2)$; cf. Corollary 5.4. For this purpose, we define the scaled norm $|||\cdot|||_\varepsilon$ of a pair $Y^{(3)} = \{y_1, y_3\}$ of functions $y_j \in L^1$ by

$$|||Y^{(3)}|||_\varepsilon = 2\|Py_1\|_{L^1} + \frac{2}{\varepsilon}\|P^\perp y_1\|_{L^1} + \frac{2}{\varepsilon^2}\|y_3\|_{L^1}. \tag{5.5}$$

In Baumstark and Jahnke (2023, Equation (3.3)), we have used a similar definition, but with factor $2/\varepsilon$ instead of $2/\varepsilon^2$ in the last term. This difference is important. The motivation for multiplying every term on the right-hand side with 2 is that then equation (5.9) below holds true.

Proposition 5.2. *Suppose that the initial data in (1.8b) have the form (2.9) with $\sigma = 2$. Let $U^{(3)} = \{u_1, u_3\}$ be the classical solution of (1.8) with $m = 3$ and let $\widehat{U}^{(3)} = \{\widehat{u}_1, \widehat{u}_3\}$. Let z_1 and z_3 be the transformed functions defined in (2.14), and set $Z^{(3)} = \{z_1, z_3\}$. For every sufficiently large $r > 0$ there is a $t_\star \in (0, t_{\mathrm{end}}]$ such that under the Assumptions 1.2 and 2.3*

$$\sup_{t\in[0,t_\star/\varepsilon]} |||Z^{(3)}(t)|||_\varepsilon \le r \qquad \text{for all } \varepsilon \in (0, 1]. \tag{5.6}$$

The constant $t_\star$ depends on t_{end}, r, $C_{u,2}$, and $C_{\mathcal{T}}$, on the inverse of the nonzero eigenvalues of $\Lambda_1(0)$, and on the Lipschitz constant in (2.12), but not on ε.

Remark 5.3. The proof yields an explicit formula for $t_\star$; cf. (5.14). Numerical computations indicate that this formula is way too pessimistic in most cases, but for our goals it is sufficient that for every r there is a $t_\star$ such that (5.6) holds, and that $t_\star$ does not depend on ε. The number $t_\star$ obtained from (5.14) is positive only if $r > C_\bullet$, where $C_\bullet$ is a constant which appears in the proof. This is what we mean by "sufficiently large r."

Before we prove Proposition 5.2, we note that the following corollary is an immediate consequence of (2.25), (2.24), (5.5), and (5.6).

Corollary 5.4. *Under the assumptions of Proposition 5.2 the bounds*

$$\sup_{t\in[0,t_\star/\varepsilon]} \|\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)\|_{L^1} = \sup_{t\in[0,t_\star/\varepsilon]} \|P^\perp z_1(t)\|_{L^1} \le C\varepsilon,$$
$$\sup_{t\in[0,t_\star/\varepsilon]} \|\widehat{u}_3(t)\|_{L^1} = \sup_{t\in[0,t_\star/\varepsilon]} \|z_3(t)\|_{L^1} \quad \le C\varepsilon^2,$$

hold with a constant independent of $\varepsilon \in (0,1]$.

Corollary 5.4 reveals that Proposition 5.2 can be understood as an extension of Proposition 3.2 from $m = 1$ to the case $m = 3$. However, a substantial difference between the two cases is the fact that the proof of Proposition 3.2 (see Colin and Lannes 2009, Lemma 3) is based on Gronwall's lemma, whereas the proof of Proposition 5.2 requires other techniques. The reason is, roughly speaking, that for $m = 3$ there are two functions, $\mathcal{P}_\varepsilon^\perp \widehat{u}_1(t)$ and $\widehat{u}_3(t)$, which we have to estimate simultaneously. This is also the reason why Proposition 5.2 refers to a possibly smaller interval $[0, t_\star/\varepsilon]$ instead of $[0, t_{\text{end}}/\varepsilon]$.

Proof of Proposition 5.2. We integrate (2.16) for $m = 3$ from 0 to $t \in [0, t_{\text{end}}/\varepsilon]$. This yields

$$|||Z^{(3)}(t)|||_\varepsilon \le |||Z^{(3)}(0)|||_\varepsilon + 2\sum_{\#J=1}\left(\varepsilon\Big\|\int_0^t PF(s,\widehat{U}^{(3)},J)\,\mathrm{d}s\Big\|_{L^1} + \Big\|\int_0^t P^\perp F(s,\widehat{U}^{(3)},J)\,\mathrm{d}s\Big\|_{L^1}\right)$$
$$+ \frac{2}{\varepsilon}\sum_{\#J=3}\Big\|\int_0^t F(s,\widehat{U}^{(3)},J)\,\mathrm{d}s\Big\|_{L^1} \tag{5.7}$$

with F defined in (2.17). Since $z_3(0) = 0$ by (2.18), it follows from (2.23) that

$$|||Z^{(3)}(0)|||_\varepsilon = 2\|Pz_1(0)\|_{L^1} + \frac{2}{\varepsilon}\|P^\perp z_1(0)\|_{L^1} \le C(\|p_0\|_{W^1} + \|p_1\|_{W^1}).$$

Now, we define

$$a_{\pm1}(t) = \|Pz_1(t)\|_{L^1} + \frac{1}{\varepsilon}\|P^\perp z_1(t)\|_{L^1} \qquad \text{and} \qquad a_{\pm3}(t) = \frac{1}{\varepsilon^2}\|z_3(t)\|_{L^1} \tag{5.8}$$

and note that

$$\sum_{j\in\mathcal{J}^{(3)}} a_j(s) = 2a_1(s) + 2a_3(s) = |||Z^{(3)}(s)|||_\varepsilon \tag{5.9}$$

by (5.5). Our goal is to prove that there are constants $C_\star$ and $\widehat{C}$ such that for all $t \in [0, t_{\text{end}}/\varepsilon]$ the inequality

$$\varepsilon\Big\|\int_0^t PF(s,\widehat{U}^{(3)},J)\,\mathrm{d}s\Big\|_{L^1} + \Big\|\int_0^t P^\perp F(s,\widehat{U}^{(3)},J)\,\mathrm{d}s\Big\|_{L^1} \le C_\star + \widehat{C}\varepsilon\int_0^t \prod_{i=1}^3 a_{j_i}(s)\,\mathrm{d}s \tag{5.10}$$

holds for every $J = (j_1, j_2, j_3) \in (\mathcal{J}^{(3)})^3$ with $\#J = 1$, and that

$$\frac{1}{\varepsilon}\Big\|\int_0^t F(s,\widehat{U}^{(3)},J)\,ds\Big\|_{L^1} \le C_\star + \widehat{C}\varepsilon\int_0^t \prod_{i=1}^3 a_{j_i}(s)\,\mathrm{d}s \tag{5.11}$$

holds for every $J = (j_1, j_2, j_3) \in (\mathcal{J}^{(3)})^3$ with $\#J = 3$. Substituting (5.10) and (5.11) into (5.7) yields

$$\begin{aligned}
|||Z^{(3)}(t)|||_\varepsilon &\le C_\bullet + 2\widehat{C}\varepsilon \sum_{j\in\{1,3\}} \sum_{\#J=j} \int_0^t \prod_{i=1}^3 a_{j_i}(s)\,\mathrm{d}s \\
&\le C_\bullet + \widehat{C}\varepsilon \sum_{j\in\mathcal{J}^{(3)}} \sum_{\#J=j} \int_0^t \prod_{i=1}^3 a_{j_i}(s)\,\mathrm{d}s \\
&= C_\bullet + \widehat{C}\varepsilon \int_0^t \Bigg(\sum_{j\in\mathcal{J}^{(3)}} a_j(s) \Bigg)^3 \mathrm{d}s \\
&= C_\bullet + \widehat{C}\varepsilon \int_0^t |||Z^{(3)}(s)|||_\varepsilon^3\,\mathrm{d}s
\end{aligned} \tag{5.12}$$

by (5.9). The constant $C_\bullet$ depends on $\|p_0\|_{W^1}$, $\|p_1\|_{W^1}$, $C_\star$, and the (finite) number of multiindices J with $\#J = 1$ and $\#J = 3$, respectively. Now let $t_\star \in (0, t_{\mathrm{end}}]$ be a number to be determined below. Then, (5.12) implies that

$$|||Z^{(3)}(t)|||_\varepsilon \le C_\bullet + \widehat{C}t_\star \sup_{s\in[0,t_\star/\varepsilon]} |||Z^{(3)}(s)|||_\varepsilon^3 \qquad \text{for all } t \in [0, t_\star/\varepsilon]. \tag{5.13}$$

If we choose $t_\star$ in such a way that the right-hand side of this inequality is not larger than r for some $r > C_\bullet$, then we can infer from (5.13) that $|||Z^{(3)}(t)|||_\varepsilon \le r$ for all $t \in [0, t_\star/\varepsilon]$. Hence, the desired inequality (5.6) holds with

$$t_\star = \frac{r - C_\bullet}{\widehat{C}r^3}. \tag{5.14}$$

To prove the first inequality (5.10), we can adapt the arguments from Baumstark and Jahnke (2023, Section 3.2.2), because the fact that $a_3(t)$ was defined with a different prefactor in Baumstark and Jahnke (2023, equation (3.7)) does not matter for this part. To complete the proof of Proposition 5.2, we have to show (5.11). In Baumstark and Jahnke (2023, Section 3.2.1), we have proven such a bound, but without the factor $1/\varepsilon$ on the left-hand side. Let $\#J = 3$ and recall that

$$F(s, \widehat{U}^{(3)}, J) = S_{j,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(s), \qquad j = \#J = 3$$

according to (2.17). We first consider the (easy) case where $|J|_1 > \#J = 3$ and thus $|J|_1 \ge 5$ because $|J|_1$ is an odd integer. In this case, (2.6) implies

$$\begin{aligned}
\frac{1}{\varepsilon}\Big\| \int_0^t F(s, \widehat{U}^{(3)}, J)\,\mathrm{d}s\Big\|_{L^1} &= \frac{1}{\varepsilon}\int_0^t \Big\|\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(s)\Big\|_{L^1}\,\mathrm{d}s \\
&\le \frac{C_\mathcal{T}}{\varepsilon}\int_0^t \prod_{i=1}^3 \|\widehat{u}_{j_i}(s)\|_{L^1}\,\mathrm{d}s \\
&= \frac{C_\mathcal{T}}{\varepsilon}\int_0^t \prod_{i=1}^3 \|z_{j_i}(s)\|_{L^1}\,\mathrm{d}s \\
&= C_\mathcal{T}\varepsilon^{|J|_1-4}\int_0^t \prod_{i=1}^3 \left(\varepsilon^{1-|j_i|}\|z_{j_i}(s)\|_{L^1}\right)\mathrm{d}s \\
&\le C_\mathcal{T}\varepsilon\int_0^t \prod_{i=1}^3 a_{j_i}(s)\,\mathrm{d}s,
\end{aligned}$$

which is an estimate of the form (5.11) with $C_\star = 0$. In the last step, we have used that $|J|_1 - 4 \ge 1$ and $\varepsilon^{1-|j_i|}\|z_{j_i}(s)\|_{L^1} \le a_{j_i}(s)$ by definition (5.8).

Now let $|J|_1 = \#J = 3$, which is only true for $J = (1, 1, 1)$. Since

$$F\left(s, \widehat{U}^{(3)}, (1,1,1)\right) = S_{3,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_1, \widehat{u}_1, \widehat{u}_1\right)(s),$$

we have to show that

$$\frac{1}{\varepsilon}\Big\| \int_0^t S_{3,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_1,\widehat{u}_1,\widehat{u}_1\right)(s)\,\mathrm{d}s\Big\|_{L^1} \le C_\star + \widehat{C}\varepsilon\int_0^t a_1^3(s)\,\mathrm{d}s. \tag{5.15}$$

At this point, it seems that the inequality (4.15), which we have shown in steps 2–4 of the proof of Theorem 4.3, readily implies (5.15) with $\widehat{C}=0$. This is not quite true, because (4.15) refers to the case $m=1$, not $m=3$, and we have pointed out at the beginning of this section that $\widehat{u}_1$ is not the same function in these two cases. But the parts (i) and (ii) of Lemma 5.1 ensure that for $m=3$ the function $\widehat{u}_1$ has still all the properties which were used to prove (4.15), and this allows us to use that proof verbatim. □

Before we proceed, we have to extend Corollary 5.4 to a stronger norm as in Section 3. The following result is the counterpart of Proposition 3.4 in the case $m=3$.

Proposition 5.5. *Suppose that the assumptions of Proposition 5.2 hold, and that in addition (2.9) is true with $\sigma=3$. Then, the bounds*

$$\begin{aligned}\sup_{t\in[0,t_\star/\varepsilon]}\|D_\mu\mathcal{P}_\varepsilon^\perp\widehat{u}_1(t)\|_{L^1} &= \sup_{t\in[0,t_\star/\varepsilon]}\|D_\mu P^\perp z_1(t)\|_{L^1} &&\le C\varepsilon,\\ \sup_{t\in[0,t_\star/\varepsilon]}\|D_\mu\widehat{u}_3(t)\|_{L^1} &= \sup_{t\in[0,t_\star/\varepsilon]}\|D_\mu z_3(t)\|_{L^1} &&\le C\varepsilon^2,\end{aligned}$$

hold with a constant independent of $\varepsilon\in(0,1]$.

Proof. Using the higher regularity and, in particular, (5.4), the bound

$$\sup_{t\in[0,t_\star/\varepsilon]}|||D_\mu Z^{(3)}(t)|||_\varepsilon \le C \qquad \text{for all } \varepsilon\in(0,1],\,\mu\in\{1,\dots,d\}.$$

can be shown with standard techniques. Then, the assertion follows from the definition (5.5). □

5.2 Improved Error Bound for $m=3$

For the error analysis of $\widetilde{u}^{(3)}$, we need a second nonresonance condition similar to Assumption 4.1.

Assumption 5.6 (Nonresonance condition)**.** *The matrix $\mathcal{L}_5(0)=\mathcal{L}(5\omega,5\boldsymbol{\kappa})$ is regular and has no common eigenvalues with $\mathcal{L}_3(0)=\mathcal{L}(3\omega,3\boldsymbol{\kappa})$, that is, $\lambda_{5i}(0)\neq\lambda_{3\ell}(0)$ for all $i,\ell=1,\dots,n$.*

We are now in a position to formulate and prove our second main result.

Theorem 5.7 (Error bound for $m=3$)**.** *Let p have the form (2.9) with $\sigma=3$ and let u be the solution of (1.1). Let $\widetilde{u}^{(3)}$ be the approximation defined in (1.7) with $m=3$. Under Assumptions 1.2, 2.3, 4.1, and 5.6 there is a constant C such that*

$$\sup_{t\in[0,t_\star/\varepsilon]}\|u(t)-\widetilde{u}^{(3)}(t)\|_W \le C\varepsilon^3, \tag{5.16}$$

$$\sup_{t\in[0,t_\star/\varepsilon]}\|u(t)-\widetilde{u}^{(3)}(t)\|_{L^\infty} \le C\varepsilon^3. \tag{5.17}$$

Proof. We use the proofs of Theorem 4.2 in Baumstark and Jahnke (2023) and of Theorem 4.3 in the present paper as a blueprint and focus on what has to be changed. In Baumstark and Jahnke (2023, proof of Theorem 4.2), we have shown that the Fourier transform $\widehat{\delta}$ of $\delta=u-\widetilde{u}^{(3)}$ is the solution of

$$\partial_t\widehat{\delta}(t,k) = -\left(\mathrm{i}\mathbf{A}(k)+\frac{1}{\varepsilon}\mathbf{E}\right)\widehat{\delta}(t,k)+\varepsilon\mathcal{G}\left(\mathcal{F}u,\mathcal{F}\widetilde{u}^{(3)}\right)(t,k)+\widehat{R}(t,k)$$

with

$$\mathcal{G}\left(\mathcal{F}u, \mathcal{F}\widetilde{u}^{(3)}\right) = \mathcal{T}(\mathcal{F}u, \mathcal{F}u, \mathcal{F}u) - \mathcal{T}\left(\mathcal{F}\widetilde{u}^{(3)}, \mathcal{F}\widetilde{u}^{(3)}, \mathcal{F}\widetilde{u}^{(3)}\right),$$
$$\widehat{R}(t,k) = \varepsilon \sum_{|j|\in\{5,7,9\}} \sum_{\#J=j} \mathcal{T}(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}) \left(t, k - \frac{j\boldsymbol{\kappa}}{\varepsilon}\right) \mathrm{e}^{-\mathrm{i}j\omega t/\varepsilon},$$

and with $\mathcal{T}$ defined by (2.5). Our main task is to prove that

$$\sup_{t\in[0,t_\star/\varepsilon]} \left\| \int_0^t \exp\left((s-t)\left(\mathrm{i}\mathbf{A}(\cdot) + \frac{1}{\varepsilon}\mathbf{E}\right)\right) \widehat{R}(s)\,\mathrm{d}s \right\|_{L^1} \leq C\varepsilon^3 \tag{5.18}$$

uniformly in $\varepsilon \in (0,1]$. If (5.18) holds, then the estimate (5.16) can be shown by applying Duhamel's formula as in the proof of Theorem 4.3, and (5.17) follows from the embedding $W \hookrightarrow L^\infty$.

In Baumstark and Jahnke (2023, proof of Theorem 4.2), we have already derived the inequality

$$\left\| \int_0^t \exp\left((s-t)\left(\mathrm{i}A(\cdot) + \frac{1}{\varepsilon}E\right)\right) \widehat{R}(s)\,\mathrm{d}s \right\|_{L^1} \leq \varepsilon \sum_{|j|\in\{5,7,9\}} \sum_{\#J=j} \left\| \int_0^t S_{j,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(s)\,\mathrm{d}s \right\|_{L^1}.$$

In order to prove (5.18), we thus have to show that

$$\sum_{|j|\in\{5,7,9\}} \sum_{\#J=j} \left\| \int_0^t S_{j,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(s)\,\mathrm{d}s \right\|_{L^1} \leq C\varepsilon^2 \tag{5.19}$$

with a constant C which does not depend on ε nor on $t \in [0, t_\star/\varepsilon]$.

As before, we consider several cases. First, suppose that $|j| \in \{7,9\}$. If $J = (j_1, j_2, j_3) \in (\mathcal{J}^{(3)})^3$ with $\#J = j$, then at least two of the three entries must have a modulus of 3, such that with Corollary 5.4 we even obtain the bound

$$\begin{aligned}\left\| \int_0^t S_{j,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(s)\,\mathrm{d}s \right\|_{L^1} &\leq \frac{t_\star}{\varepsilon} \sup_{s\in[0,t_\star/\varepsilon]} \left\|\mathcal{T}\left(\widehat{u}_{j_1}, \widehat{u}_{j_2}, \widehat{u}_{j_3}\right)(s)\right\|_{L^1} \\ &\leq \frac{Ct_\star}{\varepsilon} \sup_{s\in[0,t_\star/\varepsilon]} \prod_{i=1}^{3} \|\widehat{u}_{j_i}(s)\|_{L^1} \leq C\varepsilon^3. \end{aligned} \tag{5.20}$$

If $|j| = 5$ and $\#J = j$ but $|J|_1 > j$ (e.g. if $j = 5$ and $J = (3,-1,3)$), we can proceed in the same way. The difficult case is the one where $|j| = 5 = \#J = |J|_1$. We consider only $j = 5$ and $J = (3,1,1)$, because all other such combinations can be treated analogously. Now we cannot use (5.20), because Corollary 5.4 yields only $\prod_{i=1}^3 \|\widehat{u}_{j_i}(s)\|_{L^1} = \|\widehat{u}_3(s)\|_{L^1}\|\widehat{u}_1(s)\|_{L^1}^2 \leq C\varepsilon^2$, which is not enough due to the factor $t_\star/\varepsilon$ in (5.20).

Since $\widehat{u}_1(t) = \mathcal{P}_\varepsilon\widehat{u}_1(t) + \mathcal{P}_\varepsilon^\perp\widehat{u}_1(t)$ and since $\sup_{t\in[0,t_\star/\varepsilon]} \|\mathcal{P}_\varepsilon^\perp\widehat{u}_1(t)\|_{L^1} \leq C\varepsilon$ by Corollary 5.4, the problem boils down to showing the bound

$$\left\| \int_0^t S_{5,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_3, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(s)\,\mathrm{d}s \right\|_{L^1} \leq C\varepsilon^2. \tag{5.21}$$

To prove this, we use similar techniques as in step 3 of the proof of Theorem 4.3. The strategy is again to identify the oscillatory "parts" of the integrand.

We use the representation

$$\widehat{u}_3(t,k) = S_{3,\varepsilon}^*(t,k)z_3(t,k) = \sum_{\ell=1}^{n} \exp\left(-\frac{\mathrm{i}t}{\varepsilon}\lambda_{3\ell}(\varepsilon k)\right) z_{3\ell}(t,k)\psi_{3\ell}(\varepsilon k), \tag{5.22}$$

which follows from (2.11), (2.14), and (2.15). With (2.15), (2.4), and (2.5), this allows us to reformulate the integral in (5.21) as

$$\begin{aligned}
&\int_0^t S_{5,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_3, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(s)\,\mathrm{d}s \\
&\quad= \frac{1}{(2\pi)^d}\int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\Lambda_5(\varepsilon k)\right)\Psi_5^*(\varepsilon k)\int_{\#K=k} T\left(\widehat{u}_3(s,k^{(1)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)\,\mathrm{d}K\,\mathrm{d}s \\
&\quad= \sum_{\ell=1}^n \int_{\#K=k}\int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\left[\Lambda_5(\varepsilon k) - \lambda_{3\ell}(\varepsilon k^{(1)})I\right]\right) f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\,\mathrm{d}K
\end{aligned}$$

with

$$f_{\varepsilon,\ell}(s,K) = \frac{1}{(2\pi)^d}\Psi_5^*(\varepsilon k)T\left(z_{3\ell}(s,k^{(1)})\psi_{3\ell}(\varepsilon k^{(1)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)$$

and $k = \#K$. The left-hand side of (5.21) can thus be bounded by

$$\begin{aligned}
&\left\| \int_0^t S_{5,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_3, \mathcal{P}_\varepsilon\widehat{u}_1, \mathcal{P}_\varepsilon\widehat{u}_1\right)(s)\,\mathrm{d}s\right\|_{L^1} \\
&\quad\le \sum_{\ell=1}^n \int_{\mathbb{R}^d}\int_{\#K=k} \left| \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\left[\Lambda_5(\varepsilon k) - \lambda_{3\ell}(\varepsilon k^{(1)})I\right]\right) f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\right|_2 \mathrm{d}K\,\mathrm{d}k. \qquad (5.23)
\end{aligned}$$

After setting

$$\begin{aligned}
\Delta_\ell(\theta,\theta^{(1)}) &= \Lambda_5(\theta) - \lambda_{3\ell}(\theta^{(1)})I \qquad \text{for } \theta,\theta^{(1)} \in \mathbb{R}^d, \\
g_{\varepsilon,\ell}(s,K) &= \exp\left(\frac{\mathrm{i}s}{\varepsilon}\left[\Delta_\ell(\varepsilon k, \varepsilon k^{(1)}) - \Delta_\ell(0,0)\right]\right) f_{\varepsilon,\ell}(s,K), \qquad (5.24)
\end{aligned}$$

the inner integral reads

$$\left| \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\left[\Lambda_5(\varepsilon k) - \lambda_{3\ell}(\varepsilon k^{(1)})I\right]\right) f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\right|_2 = \left| \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\Delta_\ell(0,0)\right) g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\right|_2.$$

By Assumption 5.6, the diagonal matrix $\mathbf{\Delta}_\ell(0,0) = \Lambda_5(0) - \lambda_{3\ell}(0)I$ is regular such that we can integrate by parts and obtain

$$\begin{aligned}
\left| \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(0,0)\right) g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\right|_2 &\le C\varepsilon\left(|g_{\varepsilon,\ell}(t,K)|_2 + |g_{\varepsilon,\ell}(0,K)|_2\right) \\
&\quad + C\varepsilon\left| \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(0,0)\right)\partial_t g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\right|_2. \qquad (5.25)
\end{aligned}$$

The term $C\varepsilon(|g_{\varepsilon,\ell}(t,K)|_2 + |g_{\varepsilon,\ell}(0,K)|_2)$ on the right-hand side leads to a contribution of $\mathcal{O}(\varepsilon^3)$ in (5.23), because

$$\begin{aligned}
\sum_{\ell=1}^n |g_{\varepsilon,\ell}(t,K)|_2 &= \sum_{\ell=1}^n |f_{\varepsilon,\ell}(s,K)|_2 \\
&\le C\sum_{\ell=1}^n \left| T\left(z_{3\ell}(s,k^{(1)})\psi_{3\ell}(\varepsilon k^{(1)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)\right|_2 \\
&\le C|z_3(s,k^{(1)})|_2|\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)})|_2|\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})|_2 \\
&\le C\varepsilon^2
\end{aligned}$$

by Corollary 5.4. For the other term on the right-hand side of (5.25), the product rule gives

$$\begin{aligned}
&C\varepsilon\Big|\int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(0,0)\Big)\,\partial_t g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2\\
&\quad\le C\varepsilon\Big|\int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(0,0)\Big)\,\frac{\mathrm{i}}{\varepsilon}\big[\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})-\mathbf{\Delta}_\ell(0,0)\big]g_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2\\
&\qquad+C\varepsilon\Big|\int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(0,0)\Big)\exp\Big(\frac{\mathrm{i}s}{\varepsilon}\big[\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})-\mathbf{\Delta}_\ell(0,0)\big]\Big)\,\partial_t f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2\\
&\quad\le C\varepsilon\Big|\int_0^t \frac{\mathrm{i}}{\varepsilon}\big[\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})-\mathbf{\Delta}_\ell(0,0)\big]\exp\Big(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})\Big)f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2\\
&\qquad+C\varepsilon\Big|\int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})\Big)\,\partial_t f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2. \qquad (5.26)
\end{aligned}$$

The Lipschitz continuity (2.12) of the eigenvalues yields

$$\begin{aligned}
\Big|\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})-\mathbf{\Delta}_\ell(0,0)\Big|_2 &\le \Big|\Lambda_5(\varepsilon k)-\Lambda_5(0)\Big|_2+\Big|\lambda_{3\ell}(\varepsilon k^{(1)})-\lambda_{3\ell}(0)\Big|\\
&\le C\varepsilon(|k|_1+|k^{(1)}|_1),
\end{aligned}$$

and together with Corollary 5.4 it can be shown that the first term in (5.26) causes a contribution of $\mathcal{O}(\varepsilon^2)$ in (5.23).

Unfortunately, the second term in (5.26) requires a bit more efforts. By definition of $f_{\varepsilon,\ell}$, we formally have

$$\begin{aligned}
&\partial_t f_{\varepsilon,\ell}(s,K)\\
&\quad=\frac{1}{(2\pi)^d}\Psi_5^*(\varepsilon k)\Big[T\left(\partial_t z_{3\ell}(s,k^{(1)})\psi_{3\ell}(\varepsilon k^{(1)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)\\
&\qquad+T\left(z_{3\ell}(s,k^{(1)})\psi_{3\ell}(\varepsilon k^{(1)}),\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)\\
&\qquad+T\left(z_{3\ell}(s,k^{(1)})\psi_{3\ell}(\varepsilon k^{(1)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}),\partial_t\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)\Big]\\
&\quad=\frac{1}{(2\pi)^d}\Psi_5^*(\varepsilon k)T\left(\partial_t z_{3\ell}(s,k^{(1)})\psi_{3\ell}(\varepsilon k^{(1)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}),\mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right)+\mathcal{O}(\varepsilon^2)
\end{aligned}$$

because $\partial_t\mathcal{P}_\varepsilon\widehat{u}_1$ is uniformly bounded by (5.1) and $z_{3\ell}(s,k^{(1)})=\mathcal{O}(\varepsilon^2)$ due to Corollary 5.4. Proving the desired bound for the $\mathcal{O}(\varepsilon^2)$-part of $\partial_t f_{\varepsilon,\ell}(s,K)$ in the second term in (5.26) is straightforward because the factor ε compensates the integral. The difficulty is that $\partial_t z_{3\ell}(s,k^{(1)})$ is not $\mathcal{O}(\varepsilon^2)$ in general. We can only infer from (2.16), (2.17), and Corollary 5.4 that

$$\begin{aligned}
\partial_t z_3(t)&=\varepsilon\sum_{\#J=3}S_{3,\varepsilon}(t)\mathcal{T}\left(\widehat{u}_{j_1},\widehat{u}_{j_2},\widehat{u}_{j_3}\right)(t)\\
&=\varepsilon S_{3,\varepsilon}(t)\mathcal{T}\left(\widehat{u}_1,\widehat{u}_1,\widehat{u}_1\right)(t)+\mathcal{O}(\varepsilon^3)\\
&=\varepsilon S_{3,\varepsilon}(t)\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(t)+\mathcal{O}(\varepsilon^2).
\end{aligned}$$

The ℓ-th entry of the dominating part of $\partial_t z_3(t,k^{(1)})$ is thus

$$\begin{aligned}
&\varepsilon\Big[S_{3,\varepsilon}(t,k^{(1)})\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(t,k^{(1)})\Big]_\ell\\
&\quad=\varepsilon\Big[\exp\Big(\frac{\mathrm{i}t}{\varepsilon}\Lambda_3(\varepsilon k^{(1)})\Big)\Psi_3^*(\varepsilon k^{(1)})\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(t,k^{(1)})\Big]_\ell\\
&\quad=\varepsilon\exp\Big(\frac{\mathrm{i}t}{\varepsilon}\lambda_{3\ell}(\varepsilon k^{(1)})\Big)\psi_{3\ell}^*(\varepsilon k^{(1)})\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(t,k^{(1)})\\
&\quad=\varepsilon\exp\Big(\frac{\mathrm{i}t}{\varepsilon}\lambda_{3\ell}(\varepsilon k^{(1)})\Big)\varphi_\varepsilon(t,k^{(1)})
\end{aligned}$$

with the abbreviation

$$\varphi_\varepsilon(t,k^{(1)})=\psi_{3\ell}^*(\varepsilon k^{(1)})\mathcal{T}\left(\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1,\mathcal{P}_\varepsilon\widehat{u}_1\right)(t,k^{(1)}).$$

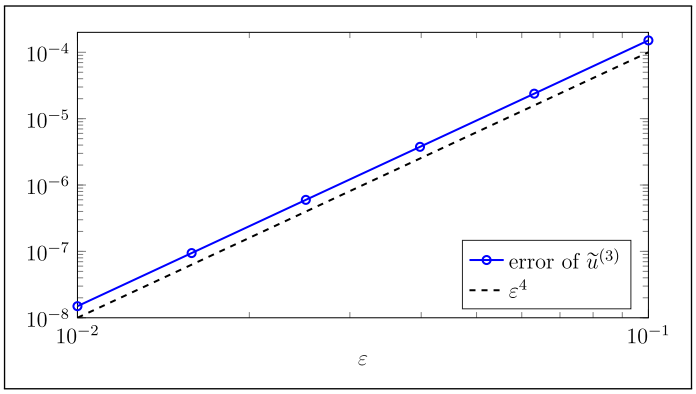

Figure 2. Accuracy of $\widetilde{u}^{(3)}$ for Different Values of ε. Parameters, Data, and Discretizations are the Same as in the Numerical Experiment Described in Section 4.2.

All in all, it follows that

$$\partial_t f_{\varepsilon,\ell}(s,K) = \varepsilon \exp\left(\frac{\mathrm{i}s}{\varepsilon}\lambda_{3\ell}(\varepsilon k^{(1)})\right)\Phi_\varepsilon(s,K) + \mathcal{O}(\varepsilon^2) \tag{5.27}$$

with

$$\Phi_\varepsilon(s,K) = \frac{1}{(2\pi)^d}\Psi_5^*(\varepsilon k)T\left(\varphi_\varepsilon(s,k^{(1)})\psi_{3\ell}(\varepsilon k^{(1)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(2)}), \mathcal{P}_\varepsilon\widehat{u}_1(s,k^{(3)})\right).$$

Substituting the right-hand side of (5.27) into the second term in (5.26) yields

$$\begin{aligned} &C\varepsilon\Big|\int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})\right)\partial_t f_{\varepsilon,\ell}(s,K)\,\mathrm{d}s\Big|_2 \\ &\quad = C\varepsilon^2\Big|\int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)})\right)\exp\left(\frac{\mathrm{i}s}{\varepsilon}\lambda_{3\ell}(\varepsilon k^{(1)})\right)\Phi_\varepsilon(s,K)\,\mathrm{d}s\Big|_2 + \mathcal{O}(\varepsilon^2) \\ &\quad = C\varepsilon^2\Big|\int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\Lambda_5(\varepsilon k)\right)\Phi_\varepsilon(s,K)\,\mathrm{d}s\Big|_2 + \mathcal{O}(\varepsilon^2) \end{aligned}$$

because by definition $\mathbf{\Delta}_\ell(\varepsilon k,\varepsilon k^{(1)}) = \Lambda_5(\varepsilon k) - \lambda_{3\ell}(\varepsilon k^{(1)})I$; see (5.24). In order to show uniform boundedness of

$$\Big|\int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Lambda_5(\varepsilon k)\Big)\Phi_\varepsilon(s,K)\,\mathrm{d}s\Big|_2 = \Big|\int_0^t \exp\Big(\frac{\mathrm{i}s}{\varepsilon}\Lambda_5(0)\Big)\exp\Big(\frac{\mathrm{i}s}{\varepsilon}\big[\Lambda_5(\varepsilon k) - \Lambda_5(0)\big]\Big)\Phi_\varepsilon(s,K)\,\mathrm{d}s\Big|_2,$$

we can use integration by parts again, because $\Lambda_5(0)$ is regular by Assumption 5.6 and the time derivative of $\exp(\mathrm{i}s/\varepsilon[\Lambda_5(\varepsilon k) - \Lambda_5(0)])\Phi_\varepsilon(s,K)$ is uniformly bounded. This completes the proof of Theorem 5.7. □

5.3 Numerical Experiment and Discussion

We have repeated the numerical experiment described in Section 4.2 with $m = 3$ instead of $m = 1$, and with $t_\star = t_{\mathrm{end}} = 1$. Figure 2 shows that in this example, the numerical counterpart of the error $\sup_{t\in[0,t_\star/\varepsilon]}\|u(t) - \widetilde{u}^{(3)}(t)\|_{L^\infty}$ scales like ε^4, which is better than what the error bound (5.17) in Theorem 5.7 predicts. We believe, however, that this advantageous error behavior cannot be expected, in general, and we briefly sketch the reasons.

If we want to improve (5.17) in such a way that ε^3 is replaced by ε^4, then instead of (5.19) we have to prove that

$$\sum_{|j|\in\{5,7,9\}}\sum_{\#J=j}\Big\|\int_0^t S_{j,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_{j_1},\widehat{u}_{j_2},\widehat{u}_{j_3}\right)(s)\,\mathrm{d}s\Big\|_{L^1} \le C\varepsilon^3.$$

As before, the critical indices are those where $|j| = 5 = \#J = |J|_1$, and we consider again $j = 5$ and $J = (3, 1, 1)$ as an example. Now instead of (5.21), we have to show that

$$\left\| \int_0^t S_{5,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_3, \mathcal{P}_\varepsilon^\perp \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s)\,\mathrm{d}s \right\|_{L^1} \le C\varepsilon^3 \tag{5.28a}$$

$$\text{and} \quad \left\| \int_0^t S_{5,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_3, \mathcal{P}_\varepsilon \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s)\,\mathrm{d}s \right\|_{L^1} \le C\varepsilon^3. \tag{5.28b}$$

We will now explain why the first inequality (5.28a) cannot be true in general. Since $\widehat{u}_3 = \mathcal{O}(\varepsilon^2)$ and $\mathcal{P}_\varepsilon^\perp \widehat{u}_1 = \mathcal{O}(\varepsilon)$ by Corollary 5.4, the integrand is formally $\mathcal{O}(\varepsilon^3)$, but since $t \in [0, t_{\mathrm{end}}/\varepsilon]$ we need one more factor of ε to compensate the long integration interval. By (2.15) and (2.5), the integral in (5.28a) reads

$$\begin{aligned}
&\int_0^t S_{5,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_3, \mathcal{P}_\varepsilon^\perp \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s)\,\mathrm{d}s \\
&\quad = \frac{1}{(2\pi)^d} \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\Lambda_5(\varepsilon k)\right) \Psi_5^*(\varepsilon k) \int_{\#K=k} T\left(\widehat{u}_3(s, k^{(1)}), \mathcal{P}_\varepsilon^\perp \widehat{u}_1(s, k^{(2)}), \mathcal{P}_\varepsilon \widehat{u}_1(s, k^{(3)})\right)\,\mathrm{d}K\,\mathrm{d}s.
\end{aligned}$$

After substituting (4.18), (5.22), and $\mathcal{P}_\varepsilon \widehat{u}_1(s, k^{(3)}) = \psi_{11}(\varepsilon k^{(3)}) \mathrm{e}^{-\mathrm{i}s\lambda_{11}(\varepsilon k^{(3)})/\varepsilon} z_{11}(s, \varepsilon k^{(3)})$ we obtain

$$\begin{aligned}
&\int_0^t S_{5,\varepsilon}(s)\mathcal{T}\left(\widehat{u}_3, \mathcal{P}_\varepsilon^\perp \widehat{u}_1, \mathcal{P}_\varepsilon \widehat{u}_1\right)(s)\,\mathrm{d}s \\
&\quad = \sum_{\ell_1=1}^n \sum_{\ell_2=2}^n \int_{\#K=k} \int_0^t \exp\left(\frac{\mathrm{i}s}{\varepsilon}\left[\Lambda_5(\varepsilon k) - \left(\lambda_{3\ell_1}(\varepsilon k^{(1)}) + \lambda_{1\ell_2}(\varepsilon k^{(2)}) + \lambda_{11}(\varepsilon k^{(3)})\right) I\right]\right) \\
&\qquad\qquad f_{\varepsilon,\ell_1,\ell_2}(s, K)\,\mathrm{d}s\,\mathrm{d}K,
\end{aligned}$$

with a smooth function $f_{\varepsilon,\ell_1,\ell_2}$. (Details do not matter at this point.) In order to generate an ε via integration by parts, we need that the diagonal matrix

$$\Lambda_5(0) - \left(\lambda_{3\ell_1}(0) + \lambda_{1\ell_2}(0) + \lambda_{11}(0)\right) I$$

is regular. Since $\lambda_{11}(0) = 0$, this is equivalent to the condition

$$\lambda_{5\ell}(0) - \lambda_{3\ell_1}(0) - \lambda_{1\ell_2}(0) \neq 0 \qquad \text{for all } \ell, \ell_1, \ell_2 \in \{1, \dots, n\}, \quad \ell_2 \neq 1. \tag{5.29}$$

This is a nonresonance condition similar to what we have assumed in Assumptions 4.1 and 5.6, but now with three terms. In contrast to those assumptions, however, (5.29) is *not* true in case of the Klein–Gordon system with $d > 1$, nor for the Maxwell–Lorentz system, as we will show now. In these applications, the eigenvalues $\omega_\ell(\beta)$ of $\mathcal{L}(0, \beta) = \mathbf{A}(\beta) - \mathrm{i}\mathbf{E}$ have the following properties:

(P1) The largest eigenvalue $\omega_1(\beta)$ is related to the smallest eigenvalue $\omega_n(\beta)$ by $\omega_n(\beta) = -\omega_1(\beta)$.
(P2) $\mathcal{L}(0, \beta)$ has at least one vanishing eigenvalue, that is, there is an index $\ell_\bullet$ with $1 \neq \ell_\bullet \neq n$ and $\omega_{\ell_\bullet}(\beta) = 0$ for all β.

Recall that $\omega = \omega(\boldsymbol{\kappa})$ is an eigenvalue of $\mathcal{L}(0, \boldsymbol{\kappa}) = \mathrm{A}(\boldsymbol{\kappa}) - \mathrm{i}\mathbf{E}$ (cf. (1.5)), and suppose that we have chosen $\omega = \omega_1(\boldsymbol{\kappa})$. By definition, the eigenvalues of

$$\mathcal{L}_j(0) = \mathcal{L}(j\omega, j\boldsymbol{\kappa}) = -j\omega I + \mathcal{L}(0, j\boldsymbol{\kappa})$$

are $\lambda_{j\ell}(0) = -j\omega + \omega_\ell(j\boldsymbol{\kappa})$. If we choose $\ell = \ell_1 = \ell_\bullet$ and $\ell_2 = n$ in (5.29), then we obtain

$$\begin{aligned}
\lambda_{5\ell_\bullet}(0) - \lambda_{3\ell_\bullet}(0) - \lambda_{1n}(0) &= (-5\omega + \omega_{\ell_\bullet}(5\boldsymbol{\kappa})) - (-3\omega + \omega_{\ell_\bullet}(3\boldsymbol{\kappa})) - (-\omega + \omega_n(\boldsymbol{\kappa})) \\
&= -5\omega + 0 + 3\omega - 0 + \omega - (-\omega_1(\boldsymbol{\kappa})) \\
&= -\omega + \omega_1(\boldsymbol{\kappa}) = 0,
\end{aligned}$$

which shows that the nonresonance condition (5.29) is not true. This is only one counterexample among many others. The corresponding nonoscillatory terms in the integrand cause contributions of $\mathcal{O}(t\varepsilon^3)$, which eventually leads to a contribution

of $\mathcal{O}(\varepsilon^2)$ instead of $\mathcal{O}(\varepsilon^3)$ on the left-hand side of (5.28a). Similar resonance problems appear also in the integral in (5.28b), such that this inequality cannot be true for the applications mentioned above.

A noteworthy exception is the Klein–Gordon system in *one* space dimension ($d = 1$, $n = 2$), which we have used in our numerical experiments. Here, the two eigenvalues of the matrix $\mathcal{L}(0, \boldsymbol{\kappa}) = \mathrm{A}(\boldsymbol{\kappa}) - \mathrm{i}\mathbf{E} \in \mathbb{C}^{2\times 2}$ are $w = w_1(\boldsymbol{\kappa}) = \sqrt{\boldsymbol{\kappa}^2 + \gamma^2}$ and $w_2(\boldsymbol{\kappa}) = -w_1(\boldsymbol{\kappa})$, as we have mentioned in Section 4.2. These eigenvalues have property (P1), but not property (P2), such that the counterexample does not apply. We conjecture that in this special case, one could indeed prove that (5.17) even holds with ε^4 instead of ε^3 on the right-hand side, which is the behavior observed in Figure 2.

This discussion raises the question if the convergence behavior predicted by Theorem 5.7 could be observed in a numerical example with a *two-dimensional* Klein–Gordon equation, because then the eigenvalues have also the property (P2). The problem is that in order to test the accuracy of the approximation $u \approx \widetilde{u}^{(3)}$, the PDEs (1.1) and (1.8) have to be solved numerically with such a high precision that the numerical error is negligible compared to the analytical error. But approximating u with sufficiently high precision by applying a standard method to (1.1) was already hopeless in one space dimension (cf. Section 4.2), and computing a reference solution via (1.7) and (1.8) with $m = 5$ was already extremely expensive in the one-dimensional case, because the functions u_j still oscillate in time. For these reasons, we were not able to produce a reliable numerical example in two space dimensions.

The approach to approximate the solution u of (1.1) via (1.7) and (1.8) has the advantage that the coefficient functions u_j do not oscillate in *space*. This gives us the possibility to use a space discretization where the number of grid points depends only on the regularity of $u_j(t, \cdot)$, but not on $1/\varepsilon$. To realize the full potential of this approach, however, it is important to develop tailor-made time integrators for (1.8), which use nonstandard techniques to handle the oscillations in *time*, and which are far more efficient than traditional schemes such as the splitting method used in our numerical examples. In a joint work with Johanna Mödl (KIT), the second author has recently constructed and analyzed such a tailor-made time integrator; cf. Jahnke and Mödl (2024).

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Notes

1. Note that the right-hand side of (4.13) contains a factor ε, which was omitted on the left-hand side of (4.14).
2. In fact, we have even shown that $X_1(t, \varepsilon) \leq C\varepsilon$ for all $\varepsilon \in (0, 1]$ and $t \in [0, t_{\mathrm{end}}/\varepsilon]$.
3. In addition, a number of terms which were estimated in a straightforward way in our proof would require a more sophisticated analysis.

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