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RECONSTRUCTING WAKE FUNCTIONS USING HAÏSSINSKI DISTRIBUTIONS FROM MULTIPLE BUNCH CHARGES

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Abstract

Accurate knowledge of wake functions is crucial in accelerator physics, serving as the cornerstone for understanding intra-bunch interactions and for controlling or mitigating instabilities that limit accelerator performance. Haïssinski distributions, which describe the steady-state longitudinal bunch density, are intrinsically determined by the wake function experienced by the bunch. While these distributions are typically computed from a given wake function, we investigate the inverse problem: extracting the wake function directly from measured Haïssinski distributions.

In this theoretical work, we introduce a novel method to reconstruct wake functions by utilizing Haïssinski distributions obtained at multiple bunch charges. By combining these profiles into an overdetermined system, we address challenges posed by the inverse problem, which is sensitive to noise and discretization errors. Here, our preliminary results suggest that the use of regularization techniques may help achieve more stable reconstructions of the wake function.

INTRODUCTION

Wake functions represent a fundamental quantity in accelerator physics, governing the self-interactions within electron bunches and thereby influencing their dynamical behavior in storage rings. They play a crucial role, particularly concerning the longitudinal microbunching instabilities that emerge at high bunch charges [1, 2]. These instabilities define operational thresholds, separating accelerator performance into distinct regimes [3]. On one side of these thresholds lies classical stable operation, whereas on the other side, accelerators enter a non-equilibrium regime characterized by intense bursts of coherent synchrotron radiation, mainly in the THz region [4]. In this nonlinear regime, accelerator dynamics become complex but not chaotic, revealing distinct and reproducible emission patterns which can be visualized with spectrograms [5]. Recently, we have demonstrated how corrugated structures impact these fingerprint patterns, highlighting the crucial role of the wake function [6]. The longitudinal dynamics in these scenarios is fundamentally described by the Vlasov-Fokker-Planck equation, in which the wake function acts as a key parameter governing intra-bunch interactions and consequently shaping the bunch's equilibrium distribution and spectral fingerprints.

Precise knowledge of the wake function is therefore highly desirable. However, its direct experimental determination represents a challenging inverse problem, which can be approached via Fourier-based deconvolution methods applied to individually measured Haïssinski distributions. Such an approach is inherently ill-posed and noise-sensitive.

Building upon recent work by Zhou et al. [7], where impedance reconstruction from Haïssinski solutions was explored, this contribution extends the reconstruction methodology. We propose and analyze an approach that leverages multiple Haïssinski distributions obtained at different bunch charges to improve the inverse reconstruction of wake functions. By combining these distributions into an overdetermined system, we utilize advanced mathematical techniques — namely the Moore-Penrose pseudoinverse coupled with regularization methods — to achieve a stable and consistent reconstruction of wake functions.

In this paper, we first present our theoretical framework and numerical methods. We then apply these methods to synthetic data derived from an impedance model of the SuperKEKB low energy ring, as introduced in [7], to reconstruct the wake function from computed Haissinski distributions. The corresponding impedance spectrum is obtained via Fourier transform and compared with the reference impedance from the same study. This comparison illustrates the feasibility of the reconstruction and suggests the potential for future experimental validation.

METHODS

Haïssinki distributions describe delicate equilibrium states in storage rings where the self-interaction of the electron bunch balances the phase space rotation of the phase focusing effect. From the Vlasov–Fokker–Planck equation, one can derive that the Haïssinski distribution $\lambda(q)$ of the normalized bunch profile satisfies the following relation [8]:

$$\frac{d\lambda}{dq} + [q - F(q, \lambda(\cdot))]\,\lambda = 0 \tag{1a}$$

with

$$F(q, \lambda(\cdot)) = I \int_{-\infty}^{\infty} W(q - q')\lambda(q')dq' \tag{1b}$$

and

$$\int_{-\infty}^{\infty} \lambda(q)dq = 1. \tag{1c}$$

Here, $q = z/\sigma_z = (s - s_0)/\sigma_{z0}$ denotes a normalized longitudinal coordinate, where z is the longitudinal position

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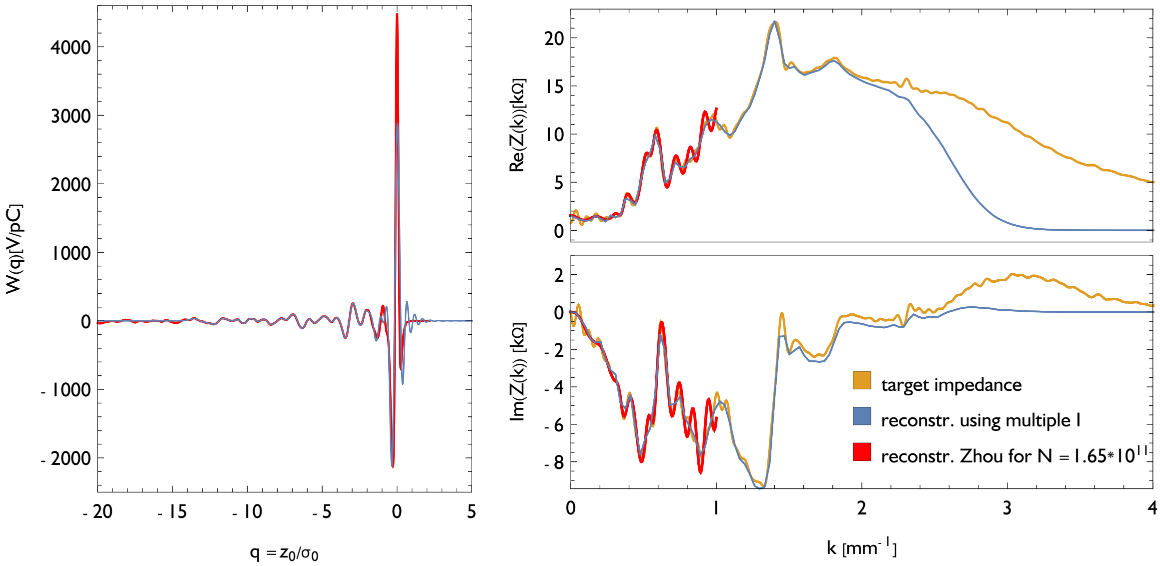

Figure 1: Comparison of reconstructed wake functions and impedance spectra for the SuperKEKB low energy ring, with reference data from Zhou et al. [7]. Left: Target wake function (red), extracted from the theoretical impedance provided by Zhou et al., compared with our reconstructed wake function (blue), obtained from multiple simulated Haïssinski distributions at various bunch charges. Right: Comparison of theoretical impedance from Zhou et al. (yellow), *single-current reference* impedance (red), re-derived by Zhou from one Haïssinski profile in the range up to $k = 1\,\mathrm{mm}^{-1} \mathrel{\hat{=}} 48$ GHz with the scaling parameter $N = 1.65 \times 10^{11} \sim I$ via their Eq. (26); and our impedance (blue) reconstructed with the proposed multi-current inverse method.

(or arc length s along the reference orbit) of a particle relative to the synchronous particle, and σ_{z0} is the zero-current bunch length. The parameter I denotes the normalized current [7, 8].

To reconstruct the wake function $W(q)$ from discrete Haïssinski profiles, we first compute $F(q)$ numerically using the identity (see e.g. Eq. (24) in reference [7]):

$$F(q) = \frac{d \ln \lambda(q)}{dq} + q, \tag{2}$$

which can be derived from Eq. (1).

Given a Haïssinski distribution sampled at discrete positions $\lambda_i = \lambda(q_i)$ for $i = -h, ..., j$, we can evaluate $F_s = F(q_s)$ at a subset of indices $s = l, ..., m$. To compute the vector $F = (F_l, ...F_m)^T$, we discretize the convolution integral in expression (1b) using a fixed step size Δq, yielding $F_s \approx I \sum_{k'=-h}^{j} W(q_s - q_{k'})\lambda(q_{k'})\Delta q$. Collecting terms across multiple F_s, we can express the relationship as a matrix-vector multiplication:

$$\begin{pmatrix} F_l \\ F_{l+1} \\ \vdots \\ F_{m-1} \\ F_m \end{pmatrix} = I \cdot \Delta q \begin{pmatrix} \lambda_j & \dots & \lambda_{-h} & 0 & 0 & \dots & 0 \\ 0 & \lambda_j & \dots & \lambda_{-h} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_j & \dots & \lambda_{-h} & 0 \\ 0 & \dots & 0 & 0 & \lambda_j & \dots & \lambda_{-h} \end{pmatrix} \cdot (W_{l-j}, W_{l-j+1}, W_{l-j+2}, \dots, W_{m+h})^T \tag{3}$$

In this formulation, we assume negligible contributions from $\lambda(q)$ outside the sampled interval $q \in [q_{-h}, ..., q_j]$, using zero-padding where necessary. Equation (3) contains $m - l + 1$ known values of F, but more unknown values $m + h - l + j + 1 > m - l + 1$ on the right-hand side.

To obtain an overdetermined system, we combine Haïssinski distributions obtained at different currents $I^{(k)}$ into a stacked system of equations:

$$\begin{pmatrix} \mathbf{F}^{(1)} \\ \mathbf{F}^{(2)} \\ \vdots \\ \mathbf{F}^{(n)} \end{pmatrix} = \begin{pmatrix} \mathbf{\Lambda}^{(1)} \\ \mathbf{\Lambda}^{(2)} \\ \vdots \\ \mathbf{\Lambda}^{(n)} \end{pmatrix} \begin{pmatrix} W_{l-j} \\ W_{l-j+1} \\ \vdots \\ W_{m+h} \end{pmatrix}, \tag{4}$$

Here, each $\mathbf{\Lambda}^{(k)}$ is a matrix similar to the one in Eq. (3), scaled by $I^{(k)}\Delta q$, and $\mathbf{F}^{(k)}$ represents the corresponding F-vector. The unknown vector $\mathbf{W} = (W_{l-j}, ..., W_{m+h})$ remains constant for all currents.

To construct the system of equations in (4), one may choose a broad Haïssinski distribution $(\lambda_{-h} \dots \lambda_j)$ as a starting point to define the first set of equations. We then include only those rows in the stacked system for which all non-negligible values of $\lambda^{(k)}(q)$ required on the right-hand side are available. Rows corresponding to truncated distributions due to finite matrix size are excluded to avoid artifacts.

Since Eq. (4) stems from an ill-posed inverse problem, we apply Tikhonov regularization to stabilize the solution [9]. Instead of solving the system $\mathbf{F} = \mathbf{\Lambda}\mathbf{W}$ directly, we compute the regularized estimate

$$\mathbf{W}_{\mathrm{reg}} = (\mathbf{\Lambda}^T\mathbf{\Lambda} + \alpha I)^{-1}\mathbf{\Lambda}^T\mathbf{F}, \tag{5}$$

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where $\alpha > 0$ is the regularization parameter, I is the identity matrix, and $\mathbf{W}_{\text{reg}}$ is the regularized solution, which is our final estimate for the wake function. In Eq. (5), the operator $(\mathbf{\Lambda}^T\mathbf{\Lambda} + \alpha I)^{-1}\mathbf{\Lambda}^T$ serves as a Tikhonov-regularized approximation to the Moore–Penrose pseudoinverse.

RESULTS AND DISCUSSION

To test our method, we implemented the procedure described above using high-resolution Haïssinski distributions generated for four different bunch charges. These distributions were computed from theoretical wake function data of the SuperKEKB low energy ring as published in [7], employing the algorithm developed by Warnock and Bane [8]. For the final calculation, we used 6001 sampling points for each of the four currents considered in [7]. The Haïssinski distributions were evaluated over the interval $q \in [-10, 10]$.

The reconstructed wake function employing our procedure was then transformed into its frequency-domain representation via the Fourier transform, allowing direct comparison with the impedance spectrum provided by Zhou et al. [7] as a quality measure for our reconstruction.

Figure 1 displays two primary comparisons: Left panel – wake function: The red curve is the target wake function obtained from Zhou *et al.*, while the blue curve is our reconstruction based on multiple simulated Haïssinski profiles at different bunch charges. The use of Tikhonov regularization suppresses high-frequency content and therefore slightly broadens sharp features of the reconstructed wake. This broadening can be mitigated by choosing a smaller regularization parameter α; however, doing so reduces numerical damping and makes the inversion more susceptible to noise amplification and instability.

Right panel – impedance: A detailed comparison of impedance spectra separated into real and imaginary components is displayed. Here we include three curves: the theoretical impedance spectrum computed by Zhou et al. (yellow), a single-current Haïssinski-inversion impedance provided by Zhou (red) at $N = 1.65\times10^{11}$, and the impedance spectrum reconstructed from our method using multiple Haïssinski profiles (blue). Our reconstruction shows good to excellent agreement with the original impedance over a broad frequency range.

At higher frequencies ($k \gtrsim 2\ \text{mm}^{-1}$), the reconstructed impedance begins to deviate significantly from the reference spectrum. This behavior can be attributed to the rapid decay of the bunch spectrum $\tilde{\lambda}(k)$, which follows a Gaussian tail and limits the observation of high-frequency components. As discussed by Zhou et al. [7], the inversion process in this regime becomes increasingly sensitive to noise. While the Tikhonov regularization mitigates this issue, it starts to suppress spectral content at higher k, contributing to the observed discrepancy. A related consequence is the appearance of oscillatory ringing features in the reconstructed wake function.

For this reconstruction, we used an empirical value of 10^{-12} for the regularization parameter. Additional tests using different sampling rates for the Haïssinski distribution suggest that a finer resolution can significantly enhance the reconstruction of high-frequency components of the impedance. A precise estimate of the impact of sampling on frequency resolution and a method for optimal α selection such as L-curve analysis [10] could be explored in future work.

Our method assumes that Haïssinski distributions are precisely measured. In practice, a key challenge will be to determine the correct origin of the normalized coordinate q/σ_{z0}. One possible approach is to analyze the mean position of the Haïssinski distributions, which should converge to $q = 0$ as $I \to 0$ [7]. In future studies, it would be valuable to investigate the influence of both systematic and statistical errors in the q coordinate that may arise in real-world measurements.

SUMMARY

Using theoretical calculations, we demonstrated that precise measurements of the Haïssinski distribution in storage rings may enable the reconstruction of longitudinal wake functions. By combining data from multiple bunch currents and applying a suitable regularization scheme, we addressed the inherent ill-posedness of the inverse problem.

Future work may explore whether experimental techniques — such as streak camera diagnostics or electro-optical sampling [11] — can provide Haïssinski profiles with sufficient resolution and signal-to-noise ratio to support this reconstruction method. In this context, accurate determination of the coordinate origin $q = 0$ and fine sampling in the normalized q coordinate are likely to be crucial.

The reconstruction of wake functions using Haïssinski distributions is particularly promising for scenarios where direct measurements of wakefields are impractical. Moreover, it offers a complementary approach that can yield insights even when full-scale simulations of complex accelerator structures are computationally prohibitive.

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