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Vladimirov–Pearson operators on ζ -regular ultrametric Cantor sets

Patrick Erik Bradley 

Institute of Photogrammetry and Remote Sensing (IPF), Geodetic Institute Karlsruhe (GIK), Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany

Correspondence

Patrick Erik Bradley, Institute of Photogrammetry and Remote Sensing (IPF), Geodetic Institute Karlsruhe (GIK), Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany.
Email: bradley@kit.edu

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Abstract

A new operator for certain types of ultrametric Cantor sets is constructed using the measure coming from the spectral triple associated with the Cantor set, as well as its zeta function. Under certain mild conditions on that measure, it is shown that it is an integral operator similar to the Vladimirov–Taibleson operator on the p -adic integers. Its spectral properties are studied, and the Markov property and kernel representation of the heat kernel generated by this so-called *Vladimirov–Pearson* operator is shown, viewed as acting on a certain Sobolev space. A large class of these operators have a heat kernel and a Green function explicitly given by the ultrametric wavelets on the Cantor set, which are eigenfunctions of the operator.

1 | INTRODUCTION

For the last decades, quite some progress has been achieved in the domain of p -adic and ultrametric analysis, whereby the former has received a lot more attention than the latter. One reason is the regularity of the hierarchical structure of p -adic domains due to the additive group structure of the p -adic number fields together with the fact that their residue fields enforce a regularity on the trees associated with p -adic discs. The fractal structure of p -adic numbers allows the viewpoint of p -adic discs as examples of ultrametric Cantor sets. Pearson and Bellissard endowed every regular ultrametric Cantor set C , that is, every Cantor set together with an ultrametric which induces its topology, with a measure based on by Connes coming from a spectral triple on the Cantor set [20]. This gives them the possibility of defining an operator on the Hilbert space of L^2 -functions on C . This approach opens the door toward extending results in p -adic analysis to regular ultrametric Cantor sets. In particular, the study of new operators on these which give rise to Markov processes are in the interest of ongoing research by the author.

The first type of a p -adic operator for diffusion on the p -adic numbers is the Vladimirov operator, well studied in the book [27], and which is actually already present in the book [24]. Since then, the study of stochastic processes on the p -adic numbers emerged, cf., for example, [30, 31] or the book [32]. Viewing p -adic Brownian motion as a scaling limit has been achieved recently: on the p -adic numbers [28], on p -adic vector spaces [21], and on the p -adic integers [22]. These results rely on previous work on the study of diffusion processes on p -adic sets other than the p -adic number fields, of which the p -adic integers are a prominent example, cf. [17]. Zúñiga-Galindo provided for a method with which diffusion processes within a finite collection of p -adic discs can be studied, which jump between distinct discs [29]. This became the basis of the author's recent work for defining and studying diffusion on spaces which are locally p -adic discs,

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that is, the p -adic points of Mumford curves, cf. [3, 6, 7]. In [9], p -adic diffusion was used in order to hear the shape of a graph, in particular that of the skeleton of a Mumford curve, and in [8] it was shown that the spectrum of diffusion operators built with automorphic forms and a regular differential 1-form are sufficient to recover the genus of a Mumford curve.

As an application of a Vladimirov–Taibleson or Zúñiga approach, new forms of non-autonomous diffusion on time-dependent graphs or time-varying energy landscapes were developed [11, 18], and diffusion on finite multi-topology systems was approximated using ultrametrics [10]. In order to extend the author's recent work to more general ultrametric spaces and include interesting algebraic examples, Schottky invariant diffusion on the transcendent p -adic upper half plane was obtained in [4]. Work in progress is also the extension of boundary value problems for p -adic elliptic Parisi–Zúñiga diffusion [5] to the yet to be defined ultrametric manifolds in order to transfer methods from partial differential equations to this setting. That work relies on ideas found in [2, 23, 25, 26].

The ζ -function coming from the spectral triple associated with a regular ultrametric Cantor set (C, d) is used in this paper. It is assumed that $\zeta(s)$ has an abscissa of convergence in \mathbb{R} . Furthermore, the Connes measure μ_τ is assumed throughout to have the property that the discs defined by the children of a given vertex in the Michon tree of (C, d) are all of equal measure w.r.t. μ_τ . Under these assumptions, the following results are proven in this paper:

1. The new Vladimirov–Pearson operator D^s with $s \in \mathbb{R}$ can be represented as an integral operator on the space of test functions on a ζ -regular ultrametric Cantor set (C, d) . [Theorem 3.7]
2. The ultrametric wavelets defined in [15] are eigenfunctions of D^s , and their corresponding eigenvalues have an explicit description similar to the ones in [15]. [Corollary 3.8]
3. The operator D^s acting on $L^2(C, \mu_\tau)$ is self-adjoint and positive semi-definite. The Hilbert space $L^2(C, \mu_\tau)$ has an orthonormal basis consisting of eigenfunctions of D^s . The operator D^s is bounded if and only if $s \geq 4$. In this case, it is a compact operator, and 0 is an accumulation point of the spectrum of D^s . In the case that $s < 4$, the spectrum of D^s is a point spectrum. [Theorem 3.9]
4. The operator D^s generates a strongly continuous Markov semigroup e^{-tD^s} ($t \geq 0$) on a suitable Sobolev space $W^{1,2}(C)$. [Theorem 4.3]
5. The semigroup e^{-tD^s} ($t \geq 0$) has a kernel representation $p_t(x, \cdot)$ for $x \in C$. [Corollary 4.4]
6. In the case that $s < 4$, the Vladimirov–Pearson operator $-D^s$ has a heat kernel which is explicitly representable with the ultrametric wavelet eigenbasis of $L^2(C, \mu_\tau)$. [Theorem 4.5]
7. In the case that $s < 4$, a Green function exists for $-D^s$ of an explicit form using the ultrametric wavelets, provided the sum

$$\sum_{\lambda > 0} \lambda^{-1}$$

converges, with λ ranging over all the positive eigenvalues of D^s . [Corollary 4.6]

Section 2 gives a brief account on regular ultrametric Cantor sets. Section 3 is devoted to the construction and spectral properties of the Vladimirov–Pearson operators from the data provided in [20]. There it is also shown that the p -adic transcendent numbers are an example for Cantor sets having an irregular Michon tree for which a Connes measure can be given such that the assumptions of this paper are satisfied. Section 4 contains the proofs of the asserted properties of the semigroup associated with the Vladimirov–Pearson operator acting on the Sobolev space $W^{1,2}(C)$.

2 | REGULAR ULTRAMETRIC CANTOR SETS

Definition 2.1. A *Cantor set* is a totally disconnected compact metrisable space without isolated points.

Remark 2.2. Notice that up to homeomorphism, there is precisely one Cantor set.

Definition 2.3. A *locally compact local ultrametric Cantor set* is a second countable Hausdorff space in which each point has an open neighborhood which is a Cantor set.

Definition 2.4. An ultrametric d on a Cantor set C is *regular*, if d generates the topology of C . The pair (C, d) is then called a *regular ultrametric Cantor set*.

To a regular ultrametric Cantor set (C, d) , a weighted tree, called the *Michon tree*, can be associated whose boundary is isometrically isomorphic with C , cf. [19].

Example 2.5. A local field K is a locally compact local ultrametric Cantor set. The Michon tree associated with any disc in K is a regular rooted tree in which the number of child nodes equals the cardinality of the residue field of K . This is well-known.

Example 2.6. The transcendent p -adic upper half plane Ω_{tr} consists of the transcendent elements of the p -adic upper half plane studied in depth in [12] in the context of Shimura curves, cf. also [16]. It is a locally compact local ultrametric Cantor set, because it is covered by the orbits of transcendent p -adic numbers under the action of the p -adic absolute Galois group, and each of these is a pro-finite set [1, Theorem 3.5], meaning that it can be endowed with a regular ultrametric. In [4], Schottky invariant diffusion operators are constructed on its transcendent part Ω_{tr} . This is an example of a regular ultrametric Cantor set whose Michon tree is not regular, and even has no bound on the number of children of vertices, cf. the ongoing work [14].

The space of test functions is denoted as $D(C)$ and consists of the locally constant functions $C \rightarrow \mathbb{C}$.

3 | VLADIMIR-PEARSON OPERATORS

In [20], a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is associated with a regular ultrametric Cantor set (C, d) . Namely,

$$\mathcal{A} = C_{\text{Lip}}(C), \quad \mathcal{H} = \ell^2(V) \otimes \mathbb{C}^2, \quad D\psi(v) = \text{diam}([v])^{-1} \sigma_1 \psi(v),$$

where $C_{\text{Lip}}(C)$ is the space of Lipschitz-continuous functions $C \rightarrow \mathbb{C}$, V is the vertex set of the Michon tree associated with (C, d) , $[v]$ is the ultrametric disc in C associated with vertex $v \in V$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the first Pauli matrix. Operator D is the Dirac operator of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

According to [20, §6.1], the absolute $|D|$ of the Dirac operator D exists and has the form

$$|D|\psi(v) = \text{diam}([v])^{-1} \psi(v)$$

and is invertible, yielding the ζ -function

$$\zeta(s) = \frac{1}{2} \text{Tr}(|D|^{-s}) = \sum_{v \in V} \text{diam}([v])^s$$

as a Dirichlet series. Cantor set (C, d) is called ζ -regular, if $\zeta(s)$ converges on the half plane $\Re(s) > s_0$ for some $s_0 \in \mathbb{R}$ (called the *abscissa of convergence*), and if for any $f \in C(C)$ and $\tau \in Y(C)$, the limit

$$\lim_{s \downarrow s_0} (s - s_0) \text{Tr}(|D|^{-s} \pi_\tau(f))$$

exists. Here, $\tau : V \rightarrow C \times C$ takes any vertex v to a pair $(x, y) \in [v] \times [v]$ such that

$$\text{diam}([v]) = d(x, y),$$

and is called a *choice function*. The set of choice functions for C is denoted as $Y(C)$. The expression π_τ is a certain representation of \mathcal{A} associated with $\tau \in Y(C)$ as follows:

Given a choice function $\tau \in Y(C)$, representation

$$\pi_\tau : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$$

is constructed via

$$\pi_\tau(f)\psi(v) = \text{diag}(f(\tau_+(v)), f(\tau_-(v)))\psi(v),$$

where

$$\tau(v) = (\tau_+(v), \tau_-(v))$$

for $v \in V$. This representation is a faithful $*$ -representation [20, Prop. 7]. The measure μ_τ on (C, d) will be named *Connes measure*.

Assume now that (C, d) is ζ -regular. The measure on C defined by [20, §7.1] is then given by

$$\mu_\tau(f) = \int_C f d\mu_\tau = \lim_{s \downarrow s_0} \frac{\text{Tr}(|D|^{-s} \pi_\tau(f))}{\text{Tr}(|D|^{-s})} \quad (1)$$

for $f \in C(C)$ and a fixed $\tau \in Y(C)$. This defines a probability measure on C which is independent of the choice of a choice function $\tau \in Y(C)$, cf. [20, Thm. 3].

In order to obtain an operator for a ζ -regular ultrametric Cantor set (C, d) , it is first observed that μ_τ induces a probability measure ν on $Y(C)$, and a Dirichlet form

$$Q_s(f, g) = \frac{1}{2} \int_{Y(C)} \text{Tr} \{ |D|^{-s} [D, \pi_\tau(f)]^* [D, \pi_\tau(g)] \} d\nu(\tau)$$

on $L^2(C, \mu_\tau)$, which is closable with dense domain, cf. [20, Thm. 4].

Supposing further that the support of μ_τ is C , the theory of semigroups gives a Markovian semigroup corresponding to the Dirichlet form Q_s for any $s \in \mathbb{R}$ [13, Theorem 1.4.1]. This allows for a self-adjoint, non-positive operator Δ_s via

$$\langle -\Delta_s f, g \rangle = Q_s(f, g)$$

for f, g linear combinations of indicator functions of vertex discs $[v]$ in C , and such that $e^{t\Delta_s}$, $t \geq 0$, is the corresponding Markov semigroup. The spectrum of Δ_s is pure point, cf. [20, Prop. 9].

Generalizing the observation of [20, Prop. 10], make the following definition.

Definition 3.1. The operator, defined as

$$D^s f(z) = \frac{1}{2} \lim_{v \rightarrow z} \mu_\tau([v])^{-1} \langle 1_{[v]}, -\Delta_s f \rangle = \langle \delta_z, -\Delta_s f \rangle$$

for $s \in \mathbb{R}$ and test functions $f : C \rightarrow \mathbb{C}$, is called the *Vladimirov–Pearson operator* on the regular Cantor set (C, d) .

In what follows, it is important that μ_τ be an equity measure, that is, the value $\mu_\tau([v])$ is the same for all child vertices v of a given vertex v_0 . Define for a given vertex v of $T(C)$, the following quantity:

$$\kappa_v(s) = \sum_{w \leq v} \text{diam}([w])^s$$

for $s > s_0$, where w runs through the descendants of v , and s_0 is the abscissa of convergence of $\zeta(s)$.

Sometimes, (C, d) is a ζ -regular ultrametric Cantor set satisfying the condition

$$\kappa_v(s) = \text{diam}([v])^{s-s_0+1} \cdot \zeta(s) \quad (2)$$

for all $v \in V(T(C))$, and $s > s_0$.

Definition 3.2. An ultrametric Cantor set satisfying (2) is said to satisfy the *factorization property*.

Example 3.3 (p -adic integers). Let $C = \mathbb{Z}_p$, and $[v] = p^\ell \mathbb{Z}_p$. Then

$$\zeta(s) = \sum_{k=0}^{\infty} p^k p^{-ks}$$

with abscissa of convergence $s_0 = 1$. The factorizing property (2) is satisfied for the p -adic distance, since

$$\kappa_v(s) = p^{-\ell s} \zeta(s)$$

for $s > 1$. Hence,

$$\mu_\tau([v]) = \lim_{s \downarrow 1} \frac{\kappa_v(s)}{\zeta(s)} = p^{-\ell}$$

yields the usual Haar measure on \mathbb{Z}_p .

Example 3.4 (Level-regularity). Let T be an infinite rooted tree without dangling vertices, and assume that at for the vertices at a given level ℓ (i.e., distance from root), the number of child nodes always takes the same value $p_\ell \in \mathbb{N}$. We call this *level-regularity*. Assume that the ultrametric $d(x, y)$ on ∂T also depends only on the level of $x \wedge y$, and takes the value

$$d(x \wedge y) = \text{diam}([v_\ell]) = p_0^{-1} \cdots p_{\ell-1}^{-1}, \quad (3)$$

where the level of $x \wedge y$ equals ℓ , and if $\ell = 1$, the product (3) is the empty product equal to one. In this case, it holds true that

$$\zeta(s) = \sum_{k=0}^{\infty} p_0 \cdots p_{k-1} \text{diam}(v_k)^s = \sum_{k=0}^{\infty} p_0 \cdots p_{k-1} \cdot p_0^{-s} \cdots p_{k-1}^{-s},$$

where $v_k \in V(T)$ is any vertex at level $k \in \mathbb{N}$. Then

$$\begin{aligned} \kappa_v(s) &= \sum_{k=\ell}^{\infty} p_0 \cdots p_{k-\ell-1} \text{diam}(v_k)^s \\ &= \sum_{k=0}^{\infty} p_0 \cdots p_{k-1} \cdot p_0^{-s(1+\ell)} \cdots p_{k-1}^{-s(1+\ell)} \\ &= p_0^{-s} \cdots p_{\ell-1}^{-s} \cdot \zeta(s) = \text{diam}([v_\ell])^s \cdot \zeta(s), \end{aligned}$$

and (2) is satisfied, provided $s_0 = 1$.

Definition 3.5. An ultrametric Cantor set (C, d) set is said to be *equitizing*, if the diameters of the children of a given vertex of $T(C)$ are all the same.

Assumption 1. It is assumed that (C, d) is an equitizing ζ -regular ultrametric Cantor set.

Remark 3.6. Any ultrametric $d(x, y)$ on the boundary ∂T of a rooted tree T without dangling nodes, which depends only on the geodesic path from root to the parent vertex of $x \wedge y \in V(T)$ for $x \neq y$ in ∂T , yields an equitizing ultrametric Cantor set. Examples 3.3 and 3.4 provide examples of equitizing ultrametric Cantor sets.

Theorem 3.7. Let (C, d) be an equitizing ζ -regular Cantor set. Then, μ_τ is an equity measure, and the Vladimirov–Pearson operator on the space of test functions $D(C)$ has the form

$$D^s f(z) = \int_C \frac{d(z, y)^{s-3} (f(z) - f(y))}{\mu_\tau(S_{d(z, y)}(z))} d\mu_\tau(y)$$

for $s \in \mathbb{R}$, where $a \wedge b$ is the join of $a, b \in C$ in the Michon tree $T(C)$, $\text{ch}(v)$ the set of child nodes of vertex $v \in V$, and $S_\epsilon(x)$ the sphere of radius $\epsilon > 0$ centered in $x \in C$.

Proof. Let $\mu = \mu_\tau$. First, observe immediately that the equitizing property implies that μ is an equity measure.

From [20, §8.3] take that

$$\begin{aligned} \langle -\Delta_s 1_{[v]}, f \rangle &= \sum_{w > v} \frac{\text{diam}([w])^{s-2}}{\sum_{(u, u') \in \mathcal{G}_w} \mu([u])\mu([u'])} \\ &\quad \cdot 2 \int_{[v]} d\mu(x) \int_{[w] \cap [u_v]^c} (f(x) - f(y)) d\mu(y), \end{aligned}$$

where for $w > v$, vertex u_v is the child of w which is also an ancestor of vertex v , and where \mathcal{G}_w is the set of distinct pairs of children of vertex $w \in V$. Since $\mu = \mu_\tau$ is an equity measure, it holds true that

$$\begin{aligned} \sum_{(u, u') \in \mathcal{G}_w} \mu([u])\mu([u']) &= |\text{ch}(z \wedge y)|(|\text{ch}(z \wedge y)| - 1) \cdot \mu([u_v])^2 \\ &= \mu([w])\mu(S_{\text{diam}([w])}(z)) \\ &= \text{diam}([w])\mu(S_{\text{diam}([w])}(z)) \end{aligned}$$

for $w > v$ with $z \in [v]$, and where $S_r(z) \subset C$ is the ultrametric sphere of radius $r > 0$ (assumed to be a possible distance value) centered in $z \in C$. Hence,

$$\langle -\Delta_s 1_{[v]}, g \rangle = \sum_{w > v} \frac{\text{diam}([w])^{s-3}}{\mu(S_{\text{diam}([w])}(z))} \cdot 2 \int_{[v]} d\mu(x) \int_{[w] \cap [u_v]^c} (f(z) - f(x)) d\mu(y)$$

It now follows that

$$\frac{1}{2\mu([v])} \langle -\Delta_s 1_{[v]}, g \rangle$$

converges for $v \rightarrow z$ to

$$\begin{aligned} D^s f(z) &= \sum_{w > z} \frac{\text{diam}([w])^{s-3}}{\mu(S_{\text{diam}([w])}(z))} \cdot \int_{S_{\text{diam}([w])}(z)} (f(z) - f(y)) d\mu(y) \\ &= \int_{C \setminus \{z\}} \frac{d(z, y)^{s-3}}{\mu(S_{d(z, y)}(z))} (f(z) - f(y)) d\mu(y), \end{aligned}$$

where the last equality follows from integrating over $y \in [w]$ such that $w = y \wedge z$ for all $w > z$ which covers all of C . This proves the assertion, since $f \in D(C)$ is constant near $z \in C$. \square

Ultrametric wavelets are defined in [15, Eq. (6)], and can be written in the setting of an ultrametric Cantor set (C, d) as

$$\psi_{v, j}(x) = \mu([v])^{-\frac{1}{2}} e^{2\pi i j x_v / s_v} 1_{[v]}(x),$$

for $x \in C$, where $v \in V(T(C))$ is a vertex, $j = 1, \dots, s_v$, and s_v is the number of children of the parent vertex of v in $T(C)$. The quantity $x_v \in \{1, \dots, s_v\}$ is defined via a numbering of the children of the parent vertex of vertex v in the rooted tree $T(C)$.

Corollary 3.8. *Under Assumption 1, the ultrametric wavelets are eigenfunctions of D^s . The eigenvalue corresponding to an ultrametric wavelet $\psi_{v,j}$ with $v \succ z \in C$ and $j = 1, \dots, s_v$ equals*

$$\lambda_v = \frac{\mu_\tau([v])}{1 - s_v^{-1}} \cdot d(v, v^\sigma)^{s-4} + \int_{C \setminus [v]} \frac{d(z, y)^{s-4}}{1 - p_{y \wedge z}^{-1}} d\mu_\tau(y)$$

where v^σ is any sibling of v distinct from v in $T(C)$, and p_w equals the number of children of vertex w in $T(C)$.

Proof. Since the kernel function of the integral operator D^s depends on the distance $d(y, z)$, cf. Theorem 3.7, the result [15, Theorem 10] can be applied to the setting here. The statement about the ultrametric wavelet eigenvalue there first translates to

$$\lambda_v = \mu_\tau([v]) \frac{d(v, v^\sigma)^{s-3}}{\mu_\tau(S_{d(z, v^\sigma)}(z))} + \int_{C \setminus [v]} \frac{d(z, y)^{s-3}}{\mu_\tau(S_{d(z, y)}(z))} d\mu_\tau(y),$$

using Theorem 3.7. Now, calculate the value of $\mu_\tau(S_{d(x, z)}(z))$ for $x \neq z$ in C . By the equitizing property, this value is immediately seen to be

$$\mu_\tau(S_{d(x, z)}(z)) = (1 - p_{x \wedge z}^{-1})d(x, z), \quad (4)$$

from which the asserted eigenvalue follows. Notice that λ_v does not depend on the choices of $z \in [v]$, sibling v^σ , and also not on $j \in \{1, \dots, s_v\}$. \square

Observe from (4) that the operator D^s can also be written as

$$D^s f(x) = \int_C \frac{d(x, y)^{s-4}}{1 - p_{x \wedge y}^{-1}} (f(x) - f(y)) d\mu_\tau(y) \quad (5)$$

for $x \in C$, $s \in \mathbb{R}$, and test functions $f \in D(C)$, by inserting into the integral representation of Theorem 3.7.

Theorem 3.9. *Let (C, d) be an equitizing ζ -regular Cantor set. The operator D^s acting on $L^2(C, \mu_\tau)$ is self-adjoint and positive semi-definite. The Hilbert space $L^2(C, \mu_\tau)$ has an orthonormal basis consisting of eigenfunctions of D^s . The operator D^s is bounded if and only if $s \geq 4$. In this case, it is a compact operator, and 0 is an accumulation point of the spectrum of D^s . In the case that $s < 4$, the spectrum of D^s is a point spectrum.*

Proof. Corollary 3.8 shows that the ultrametric wavelets, together with the constant function $\mu_\tau(C)^{-\frac{1}{2}}$, form an orthonormal basis of $L^2(C, \mu_\tau)$ consisting of eigenfunctions, where the orthonormality relations between the wavelets are shown in [15, Theorem 2], and together with their mean zero property (which is immediate because of the equity property of μ_τ) yields the asserted orthonormal basis property. This implies the unitarily diagonalizability of D^s for $s \in \mathbb{R}$.

From (5), it follows that the kernel function of D^s according to Theorem 3.7 is bounded if and only if $s \geq 4$. In this case, Corollary 3.8 shows that 0 is an accumulation point in the spectrum of D^s .

Self-adjointness and positive semi-definiteness of D^s for $s \in \mathbb{R}$ follow from the eigendecomposition of $L^2(C, \mu_\tau)$, and the symmetry of the kernel function.

Since according to Lemma 3.8, the multiplicity of each eigenvalue is finite, and $L^2(C, \mu_\tau)$ is a Hilbert space, compactness follows in the case $s \geq 4$. In the other case of $s < 4$, the operator D^s is unbounded, and its spectrum contains no accumulation point, hence is a point spectrum. This proves the remaining assertions. \square

4 | VLADIMIROV–PEARSON SEMIGROUP

Here, the semigroup associated with a Vladimirov–Pearson operator D^s is studied.

Lemma 4.1. *The semigroup e^{-tD^s} acts compactly on $L^2(C, \mu_\tau)$ for $t > 0$, where $s < 4$.*

Proof. If $s < 4$, then e^{-tD^s} is clearly of trace class for $t > 0$. Hence, the operators operate compactly on the Hilbert space $L^2(C, \mu_\tau)$ in this case. \square

Define the following Sobolev space:

$$W^{1,2}(C) = \left\{ f \in L^2(C, \mu_\tau) : \|D^s f\|_{L^2(C, \mu_\tau)} < \infty \right\}$$

for $s \in \mathbb{R}$. The Sobolev norm on $W^{1,2}(C)$ is given by

$$\|f\|_{W^{1,2}(C)} = \left(\|f\|_{L^2(C, \mu_\tau)}^2 + \|D^s f\|_{L^2(C, \mu_\tau)}^2 \right)^{\frac{1}{2}}$$

for $f \in W^{1,2}(C)$.

Proposition 4.2. *The Sobolev space $W^{1,2}(C)$ is a Hilbert space.*

Proof. The proofs of [5, Proposition 4.3 and Corollary 4.4] carry over to the situation here in a simplified manner. \square

Theorem 4.3. *The operator D^s is the generator of a strongly continuous semigroup e^{-tD^s} on $W^{1,2}(C)$ for $t \geq 0$, which satisfies the Markov property.*

Proof. Since $-tD^s$ acts on the Hilbert space $L^2(C, \mu_\tau)$ for $t \geq 0$, and its eigenvalues are bounded from above (they are non-positive) by Theorem 3.9, it follows that e^{-tD^s} is a strongly continuous semigroup acting on $L^2(C, \mu_\tau)$ for $t \geq 0$.

The semigroup e^{-tD^s} with $t \geq 0$ is also a contraction semigroup, because

$$\left\| \int_0^1 e^{-\tau D^s} u \, d\tau \right\|_{L^2} \leq t \|u\|_{L^2},$$

which holds true for eigenfunctions of D^s , and then by Pythagoras for any $u \in L^2(C, \mu_\tau)$. The contraction semigroup property then follows by the same reasoning as in the proof of [5, Theorem 6.3].

Following further the proof of that theorem, one observes first the conditions

$$f \geq 0 \text{ a.e.} \Rightarrow e^{-tD^s} f \geq 0 \text{ a.e.}$$

$$f \leq 1 \text{ a.e.} \Rightarrow e^{-tD^s} f \leq 1 \text{ a.e.}$$

$$e^{-tD^s} 1_C = 1_C,$$

where the first two hold true due to the invariance of eigenfunctions under the action of \mathbb{F}_p^\times , and the third property holds true, because 1_C is an eigenfunction of D^s corresponding to eigenvalue zero.

Then, an invariant needs to be found on $W^{1,2}(C)$. Such a measure π_λ exists for each eigenspace E_λ of D^s , because E_λ is finite-dimensional and D^s is diagonal on E_λ . Define

$$\pi = \sum_{\lambda} \pi_{\lambda}$$

formally, and let

$$f = \sum_{\lambda} f_{\lambda} \phi_{\lambda}$$

be the eigendecomposition of $f \in D(C)$ with $f_{\lambda} \in \mathbb{C}$, and observe that

$$e^{-tD^s} \pi f = \sum_{\lambda} e^{-tD^s} \pi_{\lambda} f_{\lambda} \phi_{\lambda} = \sum_{\lambda} \int_C f_{\lambda} \phi_{\lambda} d\pi_{\lambda} = \int_C f d\pi,$$

which shows that π is a distribution on $D(C)$. In order to see that it is also one on $W^{1,2}(C)$, the method in the proof of [5, Theorem 6.3] simplifies here as

$$\left| \int_C f d\pi \right|^2 = \sum_{\lambda} |\langle f_{\lambda} \phi_{\lambda}, \pi_{\lambda} \rangle|^2 = \sum_{\lambda} \lambda^2 |f_{\lambda}|^2 \leq \|f\|_{W^{1,2}(C)}^2,$$

which means that

$$\sum_{\lambda} e^{-tD^s} \pi_{\lambda} \in W^{1,2}(C)'$$

is a distribution which coincides with

$$e^{-tD^s} \pi$$

together with the identity

$$\int_C f d\pi = \left(\sum_{\lambda} e^{tD^s} \pi_{\lambda} \right) f.$$

Hence, π is the distribution on $W^{1,2}(C)$ invariant under e^{-tD^s} for $t \geq 0$. This proves the assertions. \square

The analogue of [5, Corollary 6.4] holds true:

Corollary 4.4. *The semigroup e^{-tD^s} with $t \geq 0$ acting on $W^{1,2}(C)$ has a kernel representation $p_t(x, \cdot)$ for $t \geq 0$, $x \in C$, that is, the map $A \mapsto p_t(x, A)$ is a Borel measure and*

$$\int_C p_t(x, dy) f(y) = e^{-tD^s} f(x)$$

for $f \in W^{1,2}(C)$, and $x \in C$.

Proof. The proof of [5, Corollary 6.4] carries over to this case. \square

Theorem 4.5. *Assume that $s < 4$. Then, the Markov semigroup e^{-tD^s} on $W^{1,2}(C)$ has a heat kernel function $H(t, x, y) \in L^2(C) \otimes L^2(C)$ for $t > 0$, where*

$$H(t, x, y) = \sum_{\lambda} e^{-\lambda t} \phi_{\lambda}(x) \overline{\phi_{\lambda}(y)}$$

with $x, y \in C$, λ running through the eigenvalues of D^s , and ϕ_{λ} an orthonormal eigenbasis in $D(C)$ of $W^{1,2}(U)$.

Proof. First, observe that the ultrametric wavelets belong to $D(C)$ and together with the normalized constant function form an eigenbasis of $L^2(C, \mu_{\tau})$ according to Corollary 3.8 and Theorem 3.9.

Since $s < 4$, it further follows that

$$H(t, x, x) = \text{Tr} (e^{-tD^s}) < \infty$$

for $x \in C, t > 0$. Assume now that $x \neq y, t > 0$. Since

$$|\phi_\lambda(x) \overline{\phi_\lambda(y)}| \leq \mu_\tau(x \wedge y)^{-1}, \quad (6)$$

it also follow that

$$|H(t, x, y)| \leq \mu_\tau(x \wedge y)^{-1} \text{Tr} (e^{-tD^s}) < \infty$$

for $x \neq y$ in C . This shows that $H(t, x, y)$ is well-defined for $t > 0$. In fact,

$$\|H(t, \cdot, \cdot)\|_{L^2(C, \mu_\tau)^{\otimes 2}}^2 = \sum_\lambda e^{-2\lambda t} < \infty$$

shows that $H(t, \cdot, \cdot) \in L^2(C, \mu_\tau)^{\otimes 2}$ for $t > 0$ in this case, as asserted. \square

Corollary 4.6. Assume that $s < 4$. Then, the Green function associated with $-D^s$ exists and has the form

$$G(x, y) = \sum_{\lambda > 0} \lambda^{-1} \phi_\lambda(x) \overline{\phi_\lambda(y)}$$

for $x, y \in C$, provided the sum

$$\sum_{\lambda > 0} \lambda^{-1}$$

over the positive eigenvalues of D^s is convergent.

Proof. The expression for $G(x, y)$ comes from

$$G(x, y) = \int_0^\infty h(t, x, y) dt,$$

where

$$h(t, x, y) = \sum_{\lambda > 0} e^{-t\lambda} \phi_\lambda(x) \overline{\phi_\lambda(y)},$$

where λ runs through the positive eigenvalues of D^s for $s < 4$. By the convergence assumption for the sum of inverse eigenvalues, it now follows that the Green function exists for $x, y \in C$, because also of the bound (6). This proves the assertion. \square

Remark 4.7. The ultrametric wavelet eigenvalue λ_v can also be written as

$$\lambda_v = \frac{d(v, v^\sigma)^{s-4}}{(1 - p_{v \wedge v^\sigma}^{-1})^{s_v}} + \int_{C \setminus [v]} \frac{d(z, y)^{s-4}}{1 - p_{y \wedge z}^{-1}} d\mu_\tau(y)$$

with the notation from Corollary 3.8. This follows from a simple calculation. Hence,

$$\lambda_v \in O(s_v^{-1} d(v, v^\sigma)^{s-4})$$

asymptotically for $v \rightarrow z \in C$. This means that for $s < 4$, the existence of a Green function according to Corollary 4.6 can be guaranteed in many cases. Such a case occurs, for example, if, but not only if, the number s_v of children in $T(C)$ is bounded, but also in the case of the Baire distance

$$d(x, y) = 2^{-\ell(x \wedge y)}$$

with $\ell(x \wedge y)$ being the length of the largest common prefix of $x, y \in C$ in the rooted tree $T(C)$, and where $s < 4$ is such that the growth of s_v for vertex v approaching $\partial T(C)$ can be overcompensated.

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CONFLICT OF INTEREST STATEMENT

The author declares no conflicts of interest.

ORCID

Patrick Erik Bradley  <https://orcid.org/0000-0001-5595-490X>

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