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Research article

Maxwell equations with localized internal damping: strong and polynomial stability

Serge Nicaise[1,*] **and Roland Schnaubelt**[2]

[1] CÉRAMATHS/DMATHS and FR CNRS 2037, Université Polytechnique Hauts-de-France, 59313 Valenciennes Cedex 9, France

[2] Department of Mathematics, Karlsruhe Institute of Technology, P.O. Box 6980, 76049 Karlsruhe, Germany

* **Correspondence:** Email: Serge.Nicaise@uphf.fr; Tel: +32327511927.

Abstract: We study the Maxwell system with localized conductivity σ and the boundary conditions of a perfect conductor on a simply connected domain Ω, assuming that there are no electric charges off the support of σ. For matrix-valued permittivity ε and permeability μ, we show strong stability of the underlying semigroup by checking the spectral criteria of the Arendt–Batty–Lyubich–Vũ Theorem. If $\varepsilon = \mu = 1$, Ω is the cube $(0, \pi)^3$ and $\operatorname{supp} \sigma$ contains a strip, the semigroup is polynomially stable of rate $\frac{1}{2}$. To derive this result, we establish the resolvent estimate of the Borichev–Tomilov Theorem using an orthonormal basis of eigenfunctions of the Maxwell operator for $\sigma = 0$.

Keywords: stability; Maxwell system; localized conductivity; ABLV theorem; Borichev–Tomilov theorem
Mathematics Subject Classification: 35L60, 35Q

1. Introduction and main results

The Maxwell system expresses the fundamental laws of electromagnetism. Here the conductivity σ plays an important role in many materials such as metals since it induces the current $-\sigma E$ in the system that acts as a damping. Like for the wave equation, it is an intriguing question how the strength of the damping is related to the support of σ. However, this problem is far less studied for the Maxwell equations. If the conductivity is strictly positive on 'large parts' of the domain one typically obtains exponential decay and related observability estimates. See e.g., [1–3], or [4] in the linear case and [5] for nonlinear material laws. Among the results on boundary conductivity, we also mention [6–11]. So far, only for constant coefficients $\varepsilon = \mu = 1$ exponential decay was established under the 'geometric control condition' using techniques from microlocal analysis, see [4]. There are almost no results on strong or polynomial stability in cases where the geometric control condition fails. In this work we show strong stability for a fairly general class of linear autonomous systems, and polynomial stability with rate $\frac{1}{2}$ if $\operatorname{supp}\sigma$ contains a strip of the cube $(0,\pi)^3$ and $\varepsilon = \mu = 1$. Our approach is based on spectral and resolvent criteria.

Let Ω be a bounded and open subset of $\mathbb{R}^3$ with a Lipschitz boundary Γ and outer unit normal ν. On Ω, we consider the Maxwell system

$$\varepsilon\partial_t E(t) = \operatorname{curl} H(t) - \sigma E(t), \quad \mu\partial_t H(t) = -\operatorname{curl} E(t), \quad \text{in } (0,\infty)\times\Omega. \tag{1.1}$$

Here the permittivity ε, the permeability μ, and the conductivity σ belong to $L^\infty(\Omega,\mathbb{R}^{3\times3}_{\text{sym}})$ and satisfy

$$\varepsilon,\mu \geq \eta\mathbb{I}, \quad \sigma \geq 0 \tag{1.2}$$

for some $\eta > 0$. We complement (1.1) with the electric boundary condition

$$E(t)\times\nu = 0 \quad \text{in } (0,\infty)\times\Gamma \tag{1.3}$$

and with the initial conditions $E(0) = E_0$ and $H(0) = H_0$.

Further, Ω_0 denotes the open subset of Ω consisting of points $x \in \Omega$ such that $\sigma = 0$ a.e. in a neighborhood of x, and we set

$$\Omega_+ = \Omega \setminus \overline{\Omega_0}.$$

Recall that the essential support of σ is defined by $\operatorname{supp}_{\text{ess}}(\sigma) = \Omega\setminus\Omega_0 = \overline{\Omega_+}$, with relative closure in Ω.

The kernel of curl contains the gradient fields, which are a severe obstacle for regularity and compactness properties, and they lead to fixed vectors preventing decay of solutions to (1.1). To get rid of them, one needs divergence conditions for the data and restrictions on the topology of Ω; cf. Theorem 2.10' in [12]. These difficulties constitute a major difference between the scalar wave equation and the Maxwell system, besides the complexity of the system and its boundary conditions. Throughout we require that the electric charge density $\operatorname{div}(\varepsilon E_0)$ vanish on Ω_0, there is no 'magnetic charge density' $\operatorname{div}(\mu H_0)$ on Ω, and the normal trace $\nu\cdot\mu H_0$ is zero on Γ. As noted in the next section, these conditions are preserved by the solutions of (1.1). However, the internal conductivity will produce a non-zero charge density $\operatorname{div}(\varepsilon E_0)$ on Ω_+, which is a serious difficulty in our analysis. (This does not happen in the case of boundary damping, which is easier in this respect.)

In this setting the Maxwell system is solved by a contraction semigroup $e^{t\mathcal{A}}$ with generator $\mathcal{A}$, see Lemma 2.1. After Section 2, when dealing with the long-term behavior, we have to impose stronger assumptions on the domain and the coefficients. We need that Ω is simply connected, that Ω_+ and Ω_0 are non-empty and have a Lipschitz boundary, and that $\partial\Omega_0\cap\Gamma$ is a Lipschitz submanifold of Γ. Moreover, ε and μ have to be Lipschitz, and σ is scalar. Under these conditions, in Section 3 we show that $i\mathbb{R}\setminus\{0\}$ belongs to the resolvent set of $\mathcal{A}$. Here injectivity follows from a unique continuation result for the time-harmonic Maxwell system. Surjectivity is proven by means of Lax–Milgram and a Fredholm argument, which exploits a compact embedding for the H-fields in our state space. We stress that we cannot expect a compact resolvent of $\mathcal{A}$ since there could be electric charges on Ω_+. We further describe the kernel of $\mathcal{A}$ and show that it is the orthogonal complement of its range. Because of the Arendt–Batty–Lyubich–Vũ theorem, in Theorem 3.6 we then obtain the strong stability of the restriction of $e^{t\mathcal{A}}$ to $(\ker\mathcal{A})^\perp$. A variant of this fact under partly different hypotheses was shown by Eller in [1] without invoking spectral theory, see also [13] for the case of boundary damping.

With some effort we can then show the closedness of the range of $\mathcal{A}$ in Proposition 4.4, assuming also that Γ is connected, σ is strictly positive on Ω_+, and certain geometric constraints on Ω_0, see

(A0)–(A3). This fact is based on a Poincaré-type inequality for the curl in Lemma 4.2, which exploits the charge-freeness on Ω_0 to counteract the electric charges on Ω_+. In our main result, Theorem 5.2 we further specialize to the cube $\Omega = (0,\pi)^3$ (see Remark 5.6 for a variant), coefficients $\varepsilon = \mu = 1$, and damping regions Ω_+ that contain a strip $(0,\pi)^2 \times (a,b)$. We can then show that the kernel of $\mathcal{A}$ is polynomially stable with rate $\frac{1}{2}$; i.e.,

$$\|e^{t\mathcal{A}}U_0 - \mathcal{P}U_0\|_{\mathcal{H}} \le C\, t^{-\frac{1}{2}} \|U_0\|_{D(\mathcal{A})}, \quad \forall\, U_0 \in D(\mathcal{A}), \ \ t \ge 1, \tag{1.4}$$

for the orthogonal projection $\mathcal{P}$ onto $\ker(\mathcal{A}) = R(\mathcal{A})^\perp$. We are aware of only one related result. In [14] Phung treated general cylinders $D \times (-\rho,\rho)$ with a damping region around the lateral boundary $\partial D \times (-\rho,\rho)$. However, in this result there is no information about the rate; whereas the proofs seem to give much smaller values than our $\frac{1}{2}$. The method in [14] is completely different than ours, using Fourier integral operators and observation-type estimates.

To show (1.4), we check the resolvent condition of the Borichev–Tomilov theorem via a contradiction argument. By this approach, for the 2D wave equation, an analogous result was already shown in [15]. There one could use trigonometric polynomials as an orthonormal base, which here have to be replaced by the TE- and TM-modes for the Maxwell system with $\sigma = 0$, see [16]. Our argument involves correction terms to make our solutions completely divergence-free, and the calculations take advantage of the (complicated) structure of the eigenfunctions.

We finish this introduction with some notation used in the remainder of the paper. Let us first recall important Hilbert spaces based on the div and curl operators, where $\theta \in \{\varepsilon,\mu\}$. The traces below are defined in $H^{-1/2}(\Gamma)$, see Theorems 2.2.18 and 2.2.24 in [17].

$$\begin{aligned}
H(\operatorname{curl},\Omega) &= \{E \in L^2(\Omega)^3 : \operatorname{curl} E \in L^2(\Omega)^3\},\\
H_0(\operatorname{curl},\Omega) &= \{E \in H(\operatorname{curl},\Omega) : E \times \nu = 0 \ \text{ on } \Gamma\},\\
H(\operatorname{div}_\theta,\Omega) &= \{F \in L^2(\Omega)^3 : \operatorname{div}(\theta F) \in L^2(\Omega)\},\\
H_0(\operatorname{div}_\theta,\Omega) &= \{F \in H(\operatorname{div}_\theta,\Omega) : (\theta F)\cdot \nu = 0 \ \text{ on } \Gamma\},\\
H(\operatorname{div}_\theta = 0,\Omega) &= \{F \in L^2(\Omega)^3 : \operatorname{div}(\theta F) = 0\},\\
H_0(\operatorname{div}_\theta = 0,\Omega) &= H_0(\operatorname{div}_\theta,\Omega) \cap H(\operatorname{div}_\theta = 0,\Omega),\\
X_{N,\theta}(\Omega) &= H_0(\operatorname{curl},\Omega) \cap H(\operatorname{div}_\theta,\Omega),\\
X_{T,\theta}(\Omega) &= H(\operatorname{curl},\Omega) \cap H_0(\operatorname{div}_\theta,\Omega),\\
K_{N,\theta}(\Omega) &= \{u \in X_{N,\theta}(\Omega) : \operatorname{curl} u = 0 \text{ and } \operatorname{div}(\theta u) = 0\},\\
K_{T,\theta}(\Omega) &= \{u \in X_{T,\theta}(\Omega) : \operatorname{curl} u = 0 \text{ and } \operatorname{div}(\theta u) = 0\}.
\end{aligned}$$

All are endowed with the natural Hilbertian norm. We omit the subscript θ if we take $\theta = \mathbb{I}$ in the above formulas.

Let $s \in \mathbb{R}$, $D \subset \Omega'$ or $D \subset \partial\Omega'$, where $\Omega' \subset \mathbb{R}^3$ has a Lipschitz boundary. The usual norm and semi-norm of $H^s(D)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. For $s = 0$ we drop the index s. By $A \lesssim B$, we mean that there exists a constant $C > 0$ independent of A, B, and the time variable t such that $A \le CB$.

2. Well-posedness of the problem

We first discuss the Gauß' laws for E and H. Let $(E,H)^\top$ in $C([0,\infty),L^2(\Omega))\cap C^1([0,\infty),H^{-1}(\Omega))$ solve (1.1). Applying the (distributional) divergence to the first equation of (1.1) in Ω_0, we obtain

$$\partial_t \operatorname{div}(\varepsilon E(t)) = 0 \ \text{ in } \Omega_0, \quad \forall\ t>0.$$

Consequently, if the initial electric field satisfies

$$\operatorname{div}(\varepsilon E_0) = 0 \ \text{ in } \Omega_0, \tag{2.1}$$

then it follows

$$\operatorname{div}(\varepsilon E(t)) = 0 \text{ in } \Omega_0, \quad \forall\ t>0.$$

Similarly, the second equation yields

$$\partial_t \operatorname{div}(\mu H(t)) = 0 \ \text{ in } \Omega, \quad \forall\ t>0.$$

Hence, the assumption

$$\operatorname{div}(\mu H_0) = 0 \ \text{ in } \Omega, \tag{2.2}$$

leads to

$$\operatorname{div}(\mu H(t)) = 0 \ \text{ in } \Omega, \quad \forall\ t>0.$$

Assume also $E\in C([0,\infty),H_0(\operatorname{curl},\Omega))$. Then Corollary 3.1.6 of [17] yields $\nu\cdot\operatorname{curl}E=0$ on Γ. So we deduce as above that

$$\nu\cdot H = 0 \ \text{ on } \Gamma, \quad \forall\ t>0, \qquad \text{if } \ \nu\cdot H_0 = 0 \ \text{ on } \Gamma. \tag{2.3}$$

We often use these properties, e.g., in the context of Helmholtz decompositions.

These arguments justify introducing the state space

$$\mathcal{H} = H_\sigma(\operatorname{div}_\varepsilon = 0,\Omega)\times H_0(\operatorname{div}_\mu = 0,\Omega),$$

which is a Hilbert space with the inner product

$$((E,H)^\top,(E',H')^\top)_{\mathcal{H}} = \int_\Omega (\varepsilon E\cdot\bar{E}' + \mu H\cdot\bar{H}'), \quad \forall\ (E,H)^\top,(E',H')^\top\in\mathcal{H},$$

of $L^2_\varepsilon(\Omega)^3\times L^2_\mu(\Omega)^3$, where for shortness we have set

$$H_\sigma(\operatorname{div}_\varepsilon = 0,\Omega) = \{E\in L^2(\Omega)^3 \ : \ \operatorname{div}(\varepsilon E) = 0 \ \text{ in } \Omega_0\}, \tag{2.4}$$

Note that this space is not a subspace of $H(\operatorname{div}_\varepsilon,\Omega)$.

To treat (1.1) on the space $\mathcal{H}$, we define the closed operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \mathcal{H}\cap(H_0(\operatorname{curl},\Omega)\times H(\operatorname{curl},\Omega)),$$
$$\mathcal{A}(E,H)^\top = (\varepsilon^{-1}(\operatorname{curl}H-\sigma E), -\mu^{-1}\operatorname{curl}E)^\top.$$

This operator generates a C_0-semigroup $e^{t\mathcal{A}}$ of contractions in $\mathcal{H}$ because it is maximally dissipative, as shown in the next lemma. One could probably weaken the assumptions on Ω in this generation result, but we do not want to pursue this issue to focus on stability properties.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain and $\varepsilon, \mu, \sigma \in L^\infty(\Omega, \mathbb{R}^{3\times 3}_{sym})$ satisfy* (1.2). *Then $\mathcal{A}$ is maximally dissipative in $\mathcal{H}$.*

Proof. The proof is quite standard except for the maximality, where we have to take into account the divergence constraint in Ω_0. First, Green's formula yields the core dissipation identity

$$(\mathcal{A}(E,H)^\top, (E,H)^\top)_{\mathcal{H}} = -\int_\Omega \sigma E \cdot \bar{E}, \quad \forall\,(E,H)^\top \in D(\mathcal{A}), \tag{2.5}$$

so that $\mathcal{A}$ is dissipative since σ is positive semi-definite.

To show maximality of $\mathcal{A}$, fix $\lambda > 0$ and $(F,G)^\top \in \mathcal{H}$. We are looking for $(E,H)^\top \in D(\mathcal{A})$ satisfying

$$\lambda(E,H)^\top - \mathcal{A}(E,H)^\top = (F,G)^\top. \tag{2.6}$$

or equivalently

$$\lambda\varepsilon E - \operatorname{curl} H + \sigma E = \varepsilon F, \qquad \lambda\mu H + \operatorname{curl} E = \mu G.$$

We derive a second-order version. The second equation leads to

$$H = \lambda^{-1} G - \lambda^{-1}\mu^{-1} \operatorname{curl} E. \tag{2.7}$$

Inserting this expression into the first equation above, we get

$$\lambda\varepsilon E + \lambda^{-1} \operatorname{curl}(\mu^{-1} \operatorname{curl} E) + \sigma E = \varepsilon F + \lambda^{-1} \operatorname{curl} G. \tag{2.8}$$

To solve this problem, we thus look for a function $E \in H_0(\mathrm{curl}, \Omega)$ satisfying

$$a_\lambda(E, E') = \int_\Omega (\varepsilon F \cdot \bar{E}' + \lambda^{-1} G \cdot \operatorname{curl} \bar{E}'), \quad \forall\, E' \in H_0(\mathrm{curl}, \Omega), \tag{2.9}$$

$$a_\lambda(E, E') := \int_\Omega ((\lambda\varepsilon + \sigma)E \cdot \bar{E}' + \lambda^{-1}\mu^{-1} \operatorname{curl} E \cdot \operatorname{curl} \bar{E}').$$

The sesquilinear form a_λ is continuous, symmetric, and coercive in $H_0(\mathrm{curl}, \Omega)$ as

$$a_\lambda(E, E) \geq \int_\Omega (\lambda\eta |E|^2 + (\lambda \|\mu\|_\infty)^{-1} |\operatorname{curl} E|^2).$$

Therefore, (2.9) has a unique solution $E \in H_0(\mathrm{curl}, \Omega)$. Testing by $E' \in \mathcal{D}(\Omega)^3$, we find that E satisfies (2.8) in the distributional sense.

To check the divergence constraint for E, into (2.9) we insert the gradient of the 0-extension $\tilde{\psi}$ of a test function $\psi \in \mathcal{D}(\Omega_0)$, obtaining

$$\int_\Omega (\lambda\varepsilon E + \sigma E - \varepsilon F) \cdot \nabla\tilde{\psi} = 0.$$

Since σ and ψ have disjoint supports, it follows

$$\int_{\Omega_0} (\lambda\varepsilon E - \varepsilon F) \cdot \nabla\psi = 0, \quad \forall\, \psi \in \mathcal{D}(\Omega_0),$$

which means

$$\lambda\,\mathrm{div}(\varepsilon E) = \mathrm{div}(\varepsilon F) = 0 \quad \text{in } \Omega_0$$

distributionally, where $F \in H_\sigma(\mathrm{div}_\varepsilon = 0, \Omega)$ is used.

Now we can define H by (2.7), so that the second component of (2.6) is true. This map then belongs to $H_0(\mathrm{div}_\mu = 0, \Omega)$ because of $G \in H_0(\mathrm{div}_\mu = 0, \Omega)$ and $E \in H_0(\mathrm{curl}, \Omega)$, see Corollary 3.1.16 in [17] for the normal trace. Finally, (2.8) becomes

$$\lambda\varepsilon E - \mathrm{curl}\, H + \sigma E = \varepsilon F,$$

which shows that $(E, H)^\top$ is contained in $D(\mathcal{A})$ and solves (2.6). □

3. Strong stability

One simple way to prove the strong stability of (1.1) is provided by the following theorem due to Arendt–Batty and Lyubich–Vũ (see [18, 19]).

Theorem 3.1. *Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by A on X. Assume that $(T(t))_{t\geq 0}$ is bounded and that no eigenvalues of A lie on the imaginary axis. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is strongly stable in the sense that*

$$\lim_{t\to\infty} T(t)x = 0, \quad \forall\, x \in X.$$

Since the resolvent $\mathcal{A}$ is not necessarily compact, we have to analyze the full spectrum on the imaginary axis. This is done in the next lemmas, using the following assumption.

(H) Let $\Omega \subseteq \mathbb{R}^3$ be open, bounded, and simply connected; $\Omega_+, \Omega_0 \neq \emptyset$, $\Gamma = \partial\Omega$, $\partial\Omega_+$ and $\partial\Omega_0$ be Lipschitz, and $\partial\Omega_0 \cap \Gamma$ be a Lipschitz submanifold of Γ. Moreover, let $\varepsilon, \mu \in W^{1,\infty}(\Omega, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ and $\sigma \in L^\infty(\Omega, \mathbb{R})$ satisfy (1.2).

We note that some of the following auxiliary results could be proven in greater generality, but for the sake of conciseness we restrict ourselves to the setting of Theorem 3.6, which is the aim of this section. By ν_0 we denote the outer unit normal of $\partial\Omega_0$.

Lemma 3.2. *Let **(H)** hold. Then $i\omega\mathbb{I} - \mathcal{A}$ is injective for each $\omega \in \mathbb{R} \setminus \{0\}$.*

Proof. Fix $\omega \in \mathbb{R} \setminus \{0\}$, and let $(E, H)^\top \in \ker(i\omega\mathbb{I} - \mathcal{A})$. This pair then satisfies

$$i\omega\varepsilon E - \mathrm{curl}\, H + \sigma E = 0, \tag{3.1}$$

$$i\omega\mu H + \mathrm{curl}\, E = 0. \tag{3.2}$$

First, the dissipation property (2.5) yields

$$0 = \Re((i\omega\mathbb{I} - \mathcal{A})(E, H)^\top, (E, H)^\top)_{\mathcal{H}} = \int_\Omega \sigma|E|^2$$

and therefore

$$\sigma E = 0 \ \text{ in } \Omega, \tag{3.3}$$

implying also that

$$E = 0 \ \text{ in } \Omega_+. \tag{3.4}$$

Coming back to (3.1), we find

$$i\omega\varepsilon E - \operatorname{curl} H = 0,$$

and therefore εE is divergence-free. By (3.4) and (3.2), the fields $(E, H)^\top$ vanish on a ball in Ω. (Recall that Ω_+ is open and non-empty.) Hence, they are zero on Ω by the unique continuation result Corollary 1.2 in [20] and a standard argument using that Ω is connected. (See the text following this result, and also Corollary 1.2 in [21] for an earlier version.) Here we need that ε and μ are Lipschitz. □

We next determine the kernel of $\mathcal{A}$, which could be non-zero because of initial charges located in Ω_0.

Lemma 3.3. *Let **(H)** hold. We then have*

$$\ker \mathcal{A} = \tilde{K}_{N,\varepsilon}(\Omega_0) \times \{0\}$$

with the space

$$\tilde{K}_{N,\varepsilon}(\Omega_0) := \{\tilde{E} : E \in K_{N,\varepsilon}(\Omega_0)\},$$

where $\tilde{E}$ is the extension of E by zero outside Ω_0.

Proof. Let $(\tilde{E}, H)^\top \in \ker \mathcal{A}$. As in the previous proof, $\tilde{E}$ then satisfies (3.3) and (3.4), which yields

$$\operatorname{curl} H = 0 = \operatorname{curl} \tilde{E}. \tag{3.5}$$

Hence, H belongs to $K_{T,\mu}(\Omega)$ and is thus zero by Dominguez' theorem as Ω is simply connected, see Theorem 6.2.5 in [17] or Proposition 3.14 in [22] if $\mu = 1$. Furthermore, since $\tilde{E} = 0$ on Ω_+ and $\tilde{E} \in H_0(\operatorname{curl}, \Omega)$, Propositions 2.2.32 and 2.1.60 of [17] show that $\tilde{E} \times \nu_0 = 0$ on $\partial\Omega_0$. (The extra regularity of $\partial\Omega_0$ is used here.) So $(\tilde{E}, H)^\top \in \mathcal{H}$ and (3.5) imply that E is contained in $K_{N,\varepsilon}(\Omega_0)$.

For the converse inclusion, take $(\tilde{E}, H)^\top \in \tilde{K}_{N,\varepsilon}(\Omega_0) \times \{0\}$. Then $\tilde{E}$ belongs to $H_0(\operatorname{curl}, \Omega)$, and $\operatorname{curl} \tilde{E} = 0$ by Proposition 2.2.32 of [17]. As a result, $(\tilde{E}, H)^\top$ is contained in the kernel of $\mathcal{A}$. □

For the proof of the surjectivity of $i\omega\mathbb{I} - \mathcal{A}$, we cannot use the same arguments as in Lemma 2.1 because $a_{i\omega}$ is either not coercive in $H_0(\operatorname{curl}, \Omega)$ anymore if $\omega \neq 0$, or not defined if $\omega = 0$. Instead, the case $\omega \neq 0$ is treated via a compact perturbation combined with the injectivity property of Lemma 3.2. The case $\omega = 0$ is more delicate and will be considered in the next section under additional assumptions.

Lemma 3.4. *Let **(H)** hold. Then $i\omega\mathbb{I} - \mathcal{A}$ is surjective for each $\omega \in \mathbb{R} \setminus \{0\}$.*

Proof. Fix $\omega \in \mathbb{R} \setminus \{0\}$ and $(F, G)^\top \in \mathcal{H}$. We are looking for $(E, H)^\top \in D(\mathcal{A})$ with

$$i\omega(E, H)^\top - \mathcal{A}(E, H)^\top = (F, G)^\top. \tag{3.6}$$

or equivalently

$$i\omega\varepsilon E - \operatorname{curl} H + \sigma E = \varepsilon F, \tag{3.7}$$

$$i\omega\mu H + \operatorname{curl} E = \mu G. \tag{3.8}$$

The first equation can be rewritten as

$$E = (\sigma\mathbb{I} + i\omega\varepsilon)^{-1}(\operatorname{curl} H + \varepsilon F). \tag{3.9}$$

Inserting this expression in the second equation, we infer

$$i\omega\mu H + \operatorname{curl}\left((\sigma\mathbb{I} + i\omega\varepsilon)^{-1}(\operatorname{curl} H + \varepsilon F)\right) = \mu G.$$

We thus look for a function $H \in X_{T,\mu}(\Omega)$ solving

$$b_{i\omega}(H, H') = \int_\Omega (\mu G \cdot \bar{H}' - ((\sigma\mathbb{I} + i\omega\varepsilon)^{-1}\varepsilon F) \cdot \operatorname{curl} \bar{H}'), \quad \forall\, H' \in X_{T,\mu}(\Omega), \tag{3.10}$$

$$b_{i\omega}(H, H') := \int_\Omega (i\omega\mu H \cdot \bar{H}' + (\sigma\mathbb{I} + i\omega\varepsilon)^{-1} \operatorname{curl} H \cdot \operatorname{curl} \bar{H}' + \operatorname{div}(\mu H)\operatorname{div}(\mu\bar{H}')).$$

This sesquilinear form is not coercive on $X_{T,\mu}(\Omega)$, but we use a perturbation argument. Namely, we introduce the form

$$c_{i\omega}(H, H') = \int_\Omega (H \cdot \bar{H}' + (\sigma\mathbb{I} + i\omega\varepsilon)^{-1} \operatorname{curl} H \cdot \operatorname{curl} \bar{H}' + \operatorname{div}(\mu H)\operatorname{div}(\mu\bar{H}')),$$

on $X_{T,\mu}(\Omega)$ and show that $e^{i\theta}c_{i\omega}$ is coercive for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Indeed, we have

$$\begin{aligned}\Re(e^{i\theta}c_{i\omega}(H, H)) = \int_\Omega \Big[&\omega \sin\theta(\varepsilon(\sigma^2\mathbb{I} + \omega^2\varepsilon^2)^{-1} \operatorname{curl} H) \cdot \operatorname{curl} \bar{H} \\ &+ \cos\theta(|H|^2 + |\operatorname{div}(\mu H)|^2 + (\sigma(\sigma^2\mathbb{I} + \omega^2\varepsilon^2)^{-1}\operatorname{curl} H) \cdot \operatorname{curl} \bar{H})\Big]\end{aligned}$$

since $\sigma\mathbb{I}$ and ε commute. Therefore if $\omega > 0$, we choose $\theta \in (0, \frac{\pi}{2})$ so that $\cos\theta > 0$ and $\sin\theta > 0$ so that

$$\Re(e^{i\theta}c_{i\omega}(H, H)) \geq C \int_\Omega (|H|^2 + |\operatorname{div}(\mu H)|^2 + |\operatorname{curl} H|^2),$$

with a positive constant C that depends on ω, θ and σ. On the contrary, if $\omega < 0$, we chose $\theta \in (-\frac{\pi}{2}, 0)$ so that $\cos\theta > 0$ and $\sin\theta < 0$ and find the same estimate with another positive constant C. We next define the continuous operators

$$\mathcal{B}_{i\omega} \colon X_{T,\mu}(\Omega) \to X_{T,\mu}(\Omega)',\ H \mapsto \mathcal{B}_{i\omega}H; \quad \mathcal{C}_{i\omega} \colon X_{T,\mu}(\Omega) \to X_{T,\mu}(\Omega)',\ H \mapsto \mathcal{C}_{i\omega}H,$$

by setting

$$\langle \mathcal{B}_{i\omega}H, H' \rangle = b_{i\omega}(H, H'), \quad \langle \mathcal{C}_{i\omega}H, H' \rangle = c_{i\omega}(H, H'), \quad \forall\, H, H' \in X_{T,\mu}(\Omega).$$

Thanks to the Lax–Milgram lemma, the coercivity of $e^{i\theta}c_{i\omega}$ implies that the operator $\mathcal{C}_{i\omega}$ is an isomorphism. Observe that

$$\mathcal{B}_{i\omega} - \mathcal{C}_{i\omega} = i\omega\mu - \mathbb{I},$$

and that $X_{T,\mu}(\Omega)$ is compactly embedded into $L^2(\Omega)^3$ due to Theorem 7.5.3 in [17]. Hence, $\mathcal{B}_{i\omega}$ is a Fredholm operator of index zero.

To show $\ker \mathcal{B}_{i\omega} = \{0\}$, we take $H \in \ker \mathcal{B}_{i\omega}$. Then it fulfills

$$\int_\Omega (i\omega\mu H \cdot \bar{H}' + (\sigma\mathbb{I} + i\omega\varepsilon)^{-1} \operatorname{curl} H \cdot \operatorname{curl} \bar{H}' + \operatorname{div}(\mu H)\operatorname{div}(\mu\bar{H}')) = 0 \tag{3.11}$$

for all $H' \in X_{T,\mu}(\Omega)$. Let us first prove that μH is divergence free (cf. [23, Theorem 1.2] or [24, Theorem 1.1]). We introduce the divergence-form operator $\Delta^{\mu}_{\mathrm{Dir}}$ by

$$\Delta^{\mu}\psi = \operatorname{div}(\mu\nabla\psi), \quad D(\Delta^{\mu}_{\mathrm{Dir}}) = \{\psi \in H^1_0(\Omega) | \Delta^{\mu}\psi \in L^2(\Omega)\}$$

with the Dirichlet boundary condition. For $\varphi \in D(\Delta^{\mu}_{Dir})$, in (3.11) we take $H' = \nabla\varphi$ and obtain

$$\int_{\Omega} (i\omega\mu H \cdot \nabla\bar{\varphi} + \operatorname{div}(\mu H) \operatorname{div}(\mu\nabla\bar{\varphi})) = 0. \tag{3.12}$$

Green's formula then yields

$$\int_{\Omega} \operatorname{div}(\mu H)(i\omega\bar{\varphi} - \operatorname{div}(\mu\nabla\bar{\varphi})) = 0.$$

Since the range of the operator $i\omega\mathbb{I} - \Delta^{\mu}_{\mathrm{Dir}}$ is the whole $L^2(\Omega)$, we deduce that H belongs to $H_0(\operatorname{div}_{\mu} = 0, \Omega)$.

Coming back to (3.11), we set

$$E = (\sigma\mathbb{I} + i\omega\varepsilon)^{-1} \operatorname{curl} H \in L^2(\Omega)^3.$$

Then (3.1) is true, and (3.11) leads to

$$\int_{\Omega} \left(i\omega\mu H \cdot \bar{H}' + E \cdot \operatorname{curl} \bar{H}'\right) = 0, \quad \forall\, H' \in X_{T,\mu}(\Omega). \tag{3.13}$$

Inserting $H' \in \mathcal{D}(\Omega)^3 \subseteq X_{T,\mu}(\Omega)$, we find that also (3.2) holds and thus $E \in H(\operatorname{curl}, \Omega)$.

To show $E \times \nu = 0$ on Γ, we extend (3.13) to test functions in $H^1(\Omega)^3$, namely

$$\int_{\Omega} \left(i\omega\mu H \cdot \bar{\Phi} + E \cdot \operatorname{curl} \bar{\Phi}\right) = 0, \quad \forall\, \Phi \in H^1(\Omega)^3. \tag{3.14}$$

Indeed, for a given $\Phi \in H^1(\Omega)^3$ we can consider its Helmholtz decomposition

$$\Phi = \nabla\psi + H'$$

with $\psi \in H^1(\Omega)$, $H' \in X_{T,\mu}(\Omega)$, and $\operatorname{div}(\mu H') = 0$, see (6.37) in [17]. Consequently,

$$\int_{\Omega} (i\omega\mu H \cdot \bar{\Phi} + E \cdot \operatorname{curl} \bar{\Phi}) = i\omega \int_{\Omega} \mu H \cdot \nabla\bar{\psi} + \int_{\Omega} (i\omega\mu H \cdot \bar{H}' + E \cdot \operatorname{curl} \bar{H}').$$

The second term on the right-hand side is zero due to (3.13), while its first term disappears using Green's formula and $H \in H_0(\operatorname{div}_{\mu} = 0, \Omega)$. This proves (3.14).

In (3.14) we can apply Green's formula and (3.2) to find

$$\langle E \times \nu, \Phi \rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0, \quad \forall\, \Phi \in H^1(\Omega)^3,$$

in the $H^{-\frac{1}{2}}$-$H^{\frac{1}{2}}$-duality. The surjectity of the trace map then yields

$$E \times \nu = 0 \ \text{ on } \Gamma. \tag{3.15}$$

Since the definition of E directly implies that

$$\operatorname{div}(\varepsilon E) = 0 \text{ in } \Omega_0,$$

we conclude that the fields $(E, H)^\top$ belong to $\ker(i\omega\mathbb{I} - \mathcal{A})$. Lemma 3.2 then shows $E = H = 0$, i.e., $\ker \mathcal{B}_{i\omega} = \{0\}$.

Now we can come back to (3.10) and conclude that this problem has a unique solution $H \in X_{T,\mu}(\Omega)$. We then define $E \in L^2(\Omega)^3$ by (3.9) and have to show that $(E, H)^\top$ belongs to $D(\mathcal{A})$ and solves (3.6).

First, (3.7) is trivially satisfied by the definition of E. The divergence-free property of E directly follows from this identity and $\operatorname{div}(\varepsilon F) = 0$ on Ω_0. Next in (3.10) we take test functions $H' = \nabla\varphi$ with $\varphi \in D(\Delta^\mu_{Dir})$. Then H satisfies (3.12) since $G \in H_0(\operatorname{div}_\mu = 0, \Omega)$. As before, we infer $\operatorname{div}(\mu H) = 0$. Finally, equations (3.9) and (3.10) lead to

$$\int_\Omega (i\omega\mu H \cdot \bar{H}' + E \cdot \operatorname{curl} \bar{H}') = \int_\Omega \mu G \cdot \bar{H}', \quad \forall\, H' \in X_{T,\mu}(\Omega).$$

As for $G = 0$ above, we arrive at (3.8) and (3.15). This shows that $(E, H)^\top$ belongs to $D(\mathcal{A})$ and solves (3.6). □

The surjectivity of $\mathcal{A}$ is a delicate question, so at this stage we first prove that $\ker \mathcal{A}$ is orthogonal to the range of $\mathcal{A}$.

Lemma 3.5. *Let* **(H)** *hold. For the scalar product in $\mathcal{H}$, we then have*

$$R(\mathcal{A})^\perp = \ker \mathcal{A}.$$

Proof. Green's formula and Lemma 3.3 imply

$$\ker \mathcal{A} \subseteq R(\mathcal{A})^\perp.$$

To show the converse inclusion, let $(E', H')^\top \in R(\mathcal{A})^\perp$. So $(E', H')^\top \in \mathcal{H}$ satisfies

$$\int_\Omega ((\operatorname{curl} H - \sigma E) \cdot \bar{E}' - \operatorname{curl} E \cdot \bar{H}') = 0, \quad \forall\, (E, H)^\top \in D(\mathcal{A}). \tag{3.16}$$

In a first step, we take an arbitrary $\Psi \in \mathcal{D}(\Omega)^3$ and consider the unique solution $\varphi \in H^1_0(\Omega_0)$ of

$$\int_{\Omega_0} \varepsilon \nabla\varphi \cdot \nabla\bar{\chi} = \int_{\Omega_0} \varepsilon \Psi \cdot \nabla\bar{\chi}, \quad \forall\, \chi \in H^1_0(\Omega_0).$$

Let $\tilde{\varphi}$ be the extension of φ by zero outside Ω_0. Then the pair $(\Psi - \nabla\tilde{\varphi}, 0)^\top$ belongs to $D(\mathcal{A})$. Note that $\sigma\nabla\tilde{\varphi} = 0$ by the supports. Using $(\Psi - \nabla\tilde{\varphi}, 0)^\top$ in (3.16), we thus obtain

$$\int_\Omega (\sigma\Psi \cdot \bar{E}' + \operatorname{curl} \Psi \cdot \bar{H}') = 0.$$

Since Ψ is arbitrary in $\mathcal{D}(\Omega)^3$, we find

$$\operatorname{curl} H' + \sigma E' = 0. \tag{3.17}$$

In a similar manner, pick $\Phi \in \mathcal{D}(\Omega)^3$ and consider the unique solution $\varphi \in H^1(\Omega)/\mathbb{C}$ of

$$\int_\Omega \mu \nabla\varphi \cdot \nabla\bar{\chi} = \int_\Omega \mu\Phi \cdot \nabla\bar{\chi}, \quad \forall \chi \in H^1(\Omega). \tag{3.18}$$

Then the pair $(0, \Phi - \nabla\varphi)^\top$ is contained in $D(\mathcal{A})$. Inserting it into (3.16), we deduce

$$\int_\Omega \operatorname{curl}\Phi \cdot \bar{E}' = 0 \tag{3.19}$$

which means

$$\operatorname{curl} E' = 0. \tag{3.20}$$

Now we repeat this argument with $\Phi \in H^1(\Omega)^3$ and $\varphi \in H^1(\Omega)/\mathbb{C}$ solving (3.18). This again yields (3.19). Green's formula and (3.20) then lead to

$$\langle E' \times \nu, \Phi \rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0,$$

and therefore

$$E' \times \nu = 0 \ \text{ on } \Gamma.$$

This property combined with (3.17) and (3.20) implies that $(E', H') \in D(\mathcal{A})$. Using Green's formula, we also infer

$$0 = \int_\Omega ((\operatorname{curl} H' + \sigma E') \cdot \bar{E}' - \operatorname{curl} E' \cdot \bar{H}') = \int_\Omega \sigma |E'|^2.$$

Since then $\sigma E' = 0$, we conclude $\operatorname{curl} H' = 0$ and thus $\mathcal{A}(E', H')^\top = 0$. □

This proof indeed shows that $D(\mathcal{A}^*) = D(\mathcal{A})$ and

$$\mathcal{A}^*(E', H')^\top = (-(\operatorname{curl} H' + \sigma E'), \operatorname{curl} E')^\top, \quad \forall (E', H')^\top \in D(\mathcal{A}^*). \tag{3.21}$$

In order to formulate a strong stability result, using Lemma 3.5 we introduce $\mathcal{H}_\perp = (\ker \mathcal{A})^\perp = \overline{R(\mathcal{A})}$ in $\mathcal{H}$ and the restriction $\mathcal{A}_\perp$ of $\mathcal{A}$ to $\mathcal{H}_\perp$ defined by

$$\mathcal{A}_\perp(E, H)^\top = \mathcal{A}(E, H)^\top, \quad \forall (E, H)^\top \in D(\mathcal{A}_\perp) = D(\mathcal{A}) \cap \mathcal{H}_\perp.$$

Clearly, $\mathcal{A}_\perp$ maps $D(\mathcal{A}_\perp)$ into $\mathcal{H}_\perp$. Since $e^{t\mathcal{A}}$ leaves invariant $\overline{R(\mathcal{A})}$, its restriction to this space is generated by $\mathcal{A}_\perp$. The above lemmas and Theorem 3.1 yield the following strong stability result.

Theorem 3.6. *Let **(H)** hold. Then $\mathcal{A}_\perp$ has no eigenvalue on $i\mathbb{R}$ and*

$$i\mathbb{R} \setminus \{0\} \subseteq \rho(\mathcal{A}_\perp).$$

Therefore the semigroup generated by $\mathcal{A}_\perp$ is strongly stable in $\mathcal{H}_\perp$, i. e.,

$$\lim_{t\to\infty} e^{t\mathcal{A}} x = 0, \forall x \in \mathcal{H}_\perp.$$

Strong stability was already shown in Theorem 4.1 of [1] also for anisotropic Lipschitz coefficients ε and μ on a bounded Lipschitz domain, but for charge-free data. In contrast to our result, σ has to be strictly positive on its support and Lipschitz, though it can be matrix-valued. On the other hand, we have to require that σ is scalar, that Ω is simply connected, and that $\partial\Omega_0$ has some extra regularity. The conditions on the initial fields seem to differ.

4. The closed range property

Lemma 3.5 shows that $0 \in \rho(\mathcal{A}_\perp)$ if and only if $R(\mathcal{A})$ is closed. In this section we give some (sufficient) additional conditions that guarantee this property, namely (see Figures 1-4 for some illustrations):

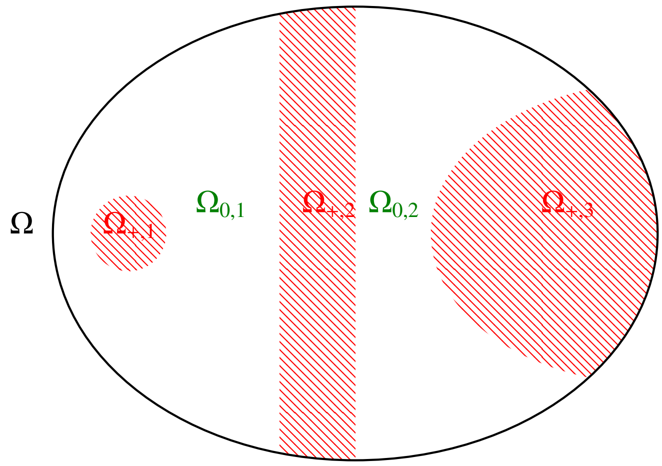

Figure 1. An illustration of our assumptions **(A0)–(A2)** with Ω_+ in red and Ω_0 in white.

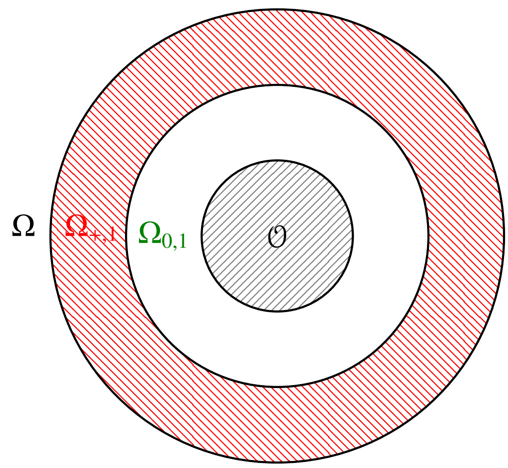

Figure 2. The assumption **(A0)** is not satisfied: the boundary of Ω is not connected since $\mathcal{O} \subseteq \mathbb{R}^3 \setminus \Omega$.

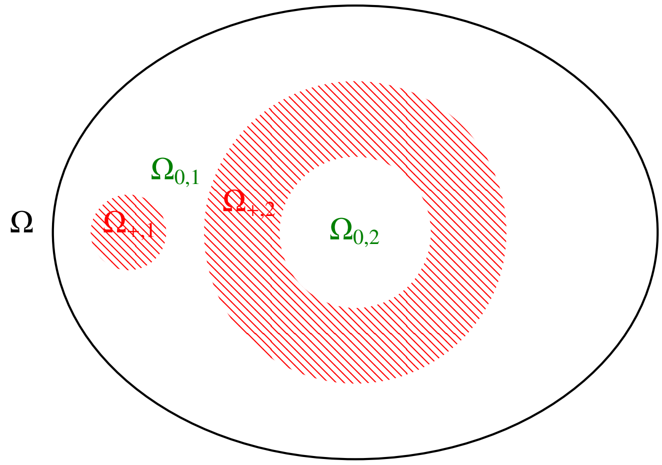

Figure 3. The assumption **(A1)** is not satisfied: the boundary of $\Omega_{+,2}$ is not connected.

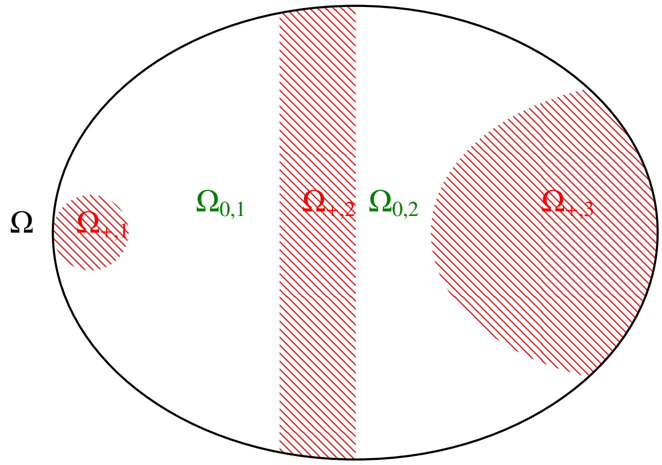

Figure 4. The assumption **(A2)** is not satisfied: meas $\overline{\Omega_{+,1}} \cap \Gamma = 0$.

(A0) Let **(H)** hold and Γ be connected. Assume there exist a finite set $\mathcal{K}$ and disjoint,* connected, open, and non-empty sets $\Omega_{+,k}$ with Lipschitz boundaries Γ_k such that

$$\Omega_+ = \bigcup_{k\in\mathcal{K}} \Omega_{+,k}.$$

(A1) If $\overline{\Omega_{+,k}} \cap \Gamma = \emptyset$, then its boundary Γ_k is assumed to be connected. Denote by $\mathcal{K}_{\text{int}}$ the index set of such subdomains.
(A2) If $\overline{\Omega_{+,k}} \cap \Gamma \neq \emptyset$, then we assume that this set is of positive (relative) measure. We will set $\mathcal{K}_{\text{bdy}} = \mathcal{K} \setminus \mathcal{K}_{\text{int}}$.
(A3) There exists a positive constant σ_- such that

$$\sigma \geq \sigma_- \text{ on } \Omega_{+,k}, \quad \forall\, k \in \mathcal{K}. \tag{4.1}$$

*meaning that $\overline{\Omega_{+,k}} \cap \overline{\Omega_{+,k'}} = \emptyset$ if $k \neq k'$

The connected, disjoint, and open components of Ω_0 are denoted by $\Omega_{0,j}$ for $j \in \mathcal{J}$, the index set $\mathcal{J}$ being finite due to our assumptions. For each $j \in \mathcal{J}$, the boundary of $\Omega_{0,j}$ is not necessarily connected. Following the notations from [22], we then decompose $\partial\Omega_{0,j}$ as

$$\partial\Omega_{0,j} = \bigcup_{i=0}^{I_j} \Gamma_{j,i},$$

where $\Gamma_{j,i}$ are the different connected components of $\partial\Omega_{0,j}$ and $\Gamma_{j,0}$ is the boundary of the sole unbounded connected component of $\mathbb{R}^3 \setminus \Omega_{0,j}$. Then, according to Lemma 3 in [25] or Proposition 6.1.1 in [17], see also Proposition 3.18 in [22] for $\varepsilon = 1$ and Theorem 4.12 in [26] for related investigations, the space $K_{N,\varepsilon}(\Omega_{0,j})$ has a finite dimension $I_j := \dim K_{N,\varepsilon}(\Omega_{0,j})$ equal to the number of connected components of $\partial\Omega_{0,j}$ minus 1. For $i \in \mathfrak{I}_j := \{1, \cdots, I_j\}$, we introduce the unique solution $\varphi_{0,j,i} \in H^1(\Omega_{0,j})$ of

$$\begin{aligned} \operatorname{div}(\varepsilon\nabla\varphi_{0,j,i}) &= 0 \quad \text{in } \Omega_{0,j}, \\ \varphi_{0,j,i} &= 1 \quad \text{on } \Gamma_{j,i}, \\ \varphi_{0,j,i} &= 0 \quad \text{on } \Gamma_{j,i'}, \quad \forall\, i' \neq i. \end{aligned} \tag{4.2}$$

Then $K_{N,\varepsilon}(\Omega_{0,j})$ is spanned by the set of functions $\nabla\varphi_{0,j,i}$ with $i = 1, \cdots, I_j$ since they belong to $K_{N,\varepsilon}(\Omega_{0,j})$ and are linearly independent. These functions are now extended into $\tilde{\varphi}_{0,j,i}$ to the whole Ω by

$$\tilde{\varphi}_{0,j,i} = \begin{cases} 1 & \text{in } U_{j,i}, \\ 0 & \text{on } U_{j,i'}, i' \neq i, \end{cases}$$

for the connected components $U_{j,i}$ of $\Omega \cap (\mathbb{R}^3 \setminus \Omega_{0,j})$ with $i = 0, \cdots, I_j$, where $U_{j,0}$ is included into the sole unbounded connected component of $\mathbb{R}^3 \setminus \Omega_{0,j}$. With these notations, Lemma 3.3 directly leads to the next result.

Corollary 4.1. *Let* **(A0)–(A2)** *hold. Then we have*

$$\ker\mathcal{A} = \tilde{K}_{N,\varepsilon}(\Omega_0) \times \{0\} = \operatorname{span}\{(\nabla\tilde{\varphi}_{0,j,i}, 0)^\top : i = 1, \cdots, I_j, j \in \mathcal{J}\}.$$

Inspired by [27, Lemma 2.2] and [26, Lemma 4.20] for $\varepsilon = 1$, we first prove a Poincaré-type inequality.

Lemma 4.2. *Under assumptions* **(A0)–(A2)***, it holds*

$$\|E\|_{\Omega_0} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega} \tag{4.3}$$

for all $E \in H_\sigma(\operatorname{div}_\varepsilon = 0, \Omega) \cap H_0(\operatorname{curl}, \Omega) \cap (\tilde{K}_{N,\varepsilon}(\Omega_0))^\perp$*, where* $(\tilde{K}_{N,\varepsilon}(\Omega_0))^\perp$ *means here the orthogonal complement of* $\tilde{K}_{N,\varepsilon}(\Omega_0)$ *in* $L^2_\varepsilon(\Omega)^3$.

Proof. Fix $E \in H_\sigma(\operatorname{div}_\varepsilon = 0, \Omega) \cap H_0(\operatorname{curl}, \Omega) \cap (\tilde{K}_{N,\varepsilon}(\Omega_0))^\perp$. Since Ω is simply connected, by Theorem 6.1.4 in [17] (or Theorem 3.17 in [22] if $\varepsilon = 1$) there exists a unique $w \in X_{N,\varepsilon}(\Omega) \cap H(\operatorname{div}_\varepsilon = 0, \Omega)$ such that

$$\operatorname{curl} w = \operatorname{curl} E \ \text{ in } \Omega,$$

and

$$\|w\|_{H(\operatorname{curl},\Omega)} \lesssim \|\operatorname{curl} E\|_{\Omega}. \tag{4.4}$$

Further, as Γ is connected, Proposition 3.3.9 of [17] provides a map $\varphi \in H_0^1(\Omega)$ with

$$E = w - \nabla\varphi \text{ in } \Omega. \tag{4.5}$$

This identity and (4.4) directly imply that

$$\|\nabla\varphi\|_{\Omega_+} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}. \tag{4.6}$$

A priori this estimate gives no control on the L^2-norm of φ in Ω_+, needed below to apply the trace theorem. But Poincaré's inequality and assumption **(A2)** yield

$$\|\varphi\|_{1,\Omega_{+,k}} \lesssim \|\nabla\varphi\|_{\Omega_{+,k}} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}, \quad \forall\, k \in \mathcal{K}_{\text{bdy}}. \tag{4.7}$$

Our goal is to prove a similar estimate for $k \in \mathcal{K}_{\text{int}}$. For that purpose, by (4.5) and the fact that εw and εE are divergence free in Ω_0, we have

$$\operatorname{div}(\varepsilon\nabla\varphi) = 0 \text{ in } \Omega_0. \tag{4.8}$$

Moreover, since $E \in (\tilde{K}_{N,\varepsilon}(\Omega_0))^\perp$, Corollary 4.1 leads to

$$\int_{\Omega_{0,j}} \varepsilon E \cdot \nabla\varphi_{0,j,i} = 0, \quad \forall\, i = 1, \cdots, I_j,\, j \in \mathcal{J}. \tag{4.9}$$

Now for $j \in \mathcal{J}$ and $i \in \mathcal{I}_j$, we note that there exists a unique index

$$k_{j,i} \in \mathcal{K}_{\text{int}} \text{ such that } \Gamma_{j,i} = \partial\Omega_{+,k_{j,i}}.$$

We fix $j \in \mathcal{J}$, $i \in \mathcal{I}_j$, and $k_{j,i} \in \mathcal{K}_{\text{int}}$ as above. The divergence-free property of w in $\Omega_{+,k_{j,i}}$ implies

$$\int_{\Gamma_{j,i}} \varepsilon w \cdot \nu_0 = 0.$$

Using also $\varphi_{0,j,i} = 1$ on $\Gamma_{j,i}$, Green's formula then yields

$$\int_{\Omega_{0,j}} \varepsilon w \cdot \nabla\varphi_{0,j,i} = -\int_{\Omega_{0,j}} \operatorname{div}(\varepsilon w)\varphi_{0,j,i} + \int_{\partial\Omega_{0,j}} (\varepsilon w \cdot \nu_0)\varphi_{0,j,i} = 0. \tag{4.10}$$

Combining the formulas (4.5), (4.9), and (4.10), we deduce that

$$\int_{\Omega_{0,j}} \nabla\varphi \cdot \varepsilon\nabla\varphi_{0,j,i} = 0, \quad \forall\, i \in \mathcal{I}_j,\, j \in \mathcal{J}.$$

Green's formula then directly leads to

$$\int_{\partial\Omega_{0,j}} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} = 0,$$

$$\sum_{i' \in \mathcal{I}_j} \int_{\Gamma_{j,i'}} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} = -\int_{\Gamma_{j,0}} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i}, \quad \forall\, i \in \mathcal{I}_j.$$

Since $\varphi = 0$ on Γ, the integral on the right reduces to the intersection between $\Gamma_{j,0}$ and some Γ_k with $k \in \mathcal{K}_{\text{bdy}}$, and therefore

$$\sum_{i'\in\mathcal{I}_j} \int_{\Gamma_{j,i'}} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} = -\sum_{k\in\mathcal{K}_{\text{bdy}}} \int_{\Gamma_{j,0}\cap\Gamma_k} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i}, \quad \forall\, i \in \mathcal{I}_j.$$

Observe that

$$\left| \int_{\Gamma_{j,0}\cap\Gamma_k} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} \right| \lesssim \|\varphi\|_{0,\Gamma_k} \lesssim \|\varphi\|_{1,\Omega_{+,k}}.$$

by Cauchy–Schwarz and the trace theorem. In view of (4.7), we arrive at

$$\left| \sum_{i'\in\mathcal{I}_+} \int_{\Gamma_{j,i'}} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} \right| \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}, \quad \forall\, i \in \mathcal{I}_j. \tag{4.11}$$

Setting $\mathcal{O}_{+,j} = \bigcup_{i\in\mathcal{I}_j} \Omega_{+,k_{j,i}}$, we assert that the Poincaré inequality

$$\|\varphi\|_{1,\mathcal{O}_{+,j}} \lesssim \|\nabla\varphi\|_{\mathcal{O}_j} + \sum_{i\in\mathcal{I}_j} \left| \sum_{i'\in\mathcal{I}_j} \int_{\Gamma_{j,i'}} \varphi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} \right| \tag{4.12}$$

holds. Indeed, by a contradiction argument and the compact embedding of $H^1(\mathcal{O}_{+,j})$ into $L^2(\mathcal{O}_{+,j})$, it suffices to show that if this right-hand side is zero, then the left-hand side has to vanish. So let the right-hand side be zero. Then the nullity of the first term yields

$$\varphi = c_{i'} \ \text{ on } \Omega_{+,k_{j,i'}}$$

for some $c_{i'} \in \mathbb{C}$ and all $i' \in \mathcal{I}_j$. ¿From the vanishing second term we thus obtain

$$\sum_{i'\in\mathcal{I}_j} \int_{\Gamma_{j,i'}} c_{i'}\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} = 0, \quad \forall\, i \in \mathcal{I}_j.$$

On $\Omega_{0,j}$ we define the function

$$\psi = \sum_{i'\in\mathcal{I}_j} c_{i'}\varphi_{0,j,i'}$$

and note that it is equal to c_i on $\Gamma_{j,i}$ for $i \in \mathcal{I}_j$ and equal to zero on the remaining part of $\partial\Omega_{0,j}$ by the properties of $\varphi_{0,j,i'}$, see (4.2). Hence, the previous identity means that

$$\int_{\partial\Omega_{0,j}} \psi\, \nu_0 \cdot \varepsilon\nabla\varphi_{0,j,i} = 0, \quad \forall\, i \in \mathcal{I}_j.$$

Since $\operatorname{div}(\varepsilon\nabla\varphi_{0,j,i}) = 0$, Green's formula yields

$$\int_{\Omega_{0,j}} \nabla\psi \cdot \varepsilon\nabla\varphi_{0,j,i} = 0, \quad \forall\, i \in \mathcal{I}_j.$$

Multiplying this equality by $\overline{c_i}$ and summing on $i \in \mathcal{I}_j$, we get

$$\int_{\Omega_{0,j}} \nabla\psi \cdot \varepsilon\nabla\bar{\psi} = 0.$$

This fact implies that $\psi = 0$ (because $\psi = 0$ on $\Gamma_{j,0}$), and then $c_i = 0$ for all $i \in \mathcal{I}_j$ by the linear independence of $\{\varphi_{0,j,i'} : i' \in \mathcal{I}_j\}$. We have shown (4.12).

Inserting (4.11) into (4.12) and using (4.6), we derive

$$\|\varphi\|_{1,\mathcal{O}_{+,j}} \lesssim \|\nabla\varphi\|_{\mathcal{O}_{+,j}} + \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}, \quad \forall\, j \in \mathcal{J}.$$

Since any $\Omega_{+,k}$ with $k \in \mathcal{K}_{\text{int}}$ is included into one $\mathcal{O}_{+,j}$, this estimate implies that

$$\|\varphi\|_{1,\Omega_{+,k}} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}, \quad \forall\, k \in \mathcal{K}_{\text{int}}. \tag{4.13}$$

Let us again fix $j \in \mathcal{J}$. Taking into account (4.8), we deduce that

$$\|\nabla\varphi\|_{\Omega_{0,j}} \lesssim \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega_{0,j})} \lesssim \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{j,0})} + \sum_{i\in\mathcal{I}_j} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{j,i})}, \tag{4.14}$$

recalling that $\partial\Omega_{0,j} = \bigcup_{i=0}^{I_j} \Gamma_{j,i}$ and that the sets $\Gamma_{j,i}$ are disjoint. On one hand, for $i \in \mathcal{I}_j$, the component $\Gamma_{j,i}$ is a part of some Γ_k with $k \in \mathcal{K}_{\text{int}}$. Then the trace theorem and (4.13) lead to

$$\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{j,i})} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}, \quad \forall\, i \in \mathcal{I}_j. \tag{4.15}$$

On the other hand, as $\varphi = 0$ on Γ, for $\Gamma_{j,0}$ we have

$$\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{j,0})} = \sum_{k\in\mathcal{K}_{\text{bdy}}} \|\varphi\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_{j,0}\cap\Gamma_k)},$$

cf. Definition 2.1.53 in [17]. Therefore, again by the trace theorem, we deduce

$$\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{j,0})} \lesssim \sum_{k\in\mathcal{K}_{\text{bdy}}} \|\varphi\|_{1,\Omega_{+,k}},$$

so that (4.7) yields

$$\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{j,0})} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}. \tag{4.16}$$

Estimates (4.14), (4.15), and (4.16) show

$$\|\nabla\varphi\|_{\Omega_{0,j}} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}, \quad \forall\, j \in \mathcal{J},$$

and therefore

$$\|\nabla\varphi\|_{\Omega_0} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}.$$

Combined with (4.6), we arrive at

$$\|\nabla\varphi\|_{\Omega} \lesssim \|E\|_{\Omega_+} + \|\operatorname{curl} E\|_{\Omega}.$$

This inequality, (4.4), and the decomposition (4.5) lead to the assertion (4.3). □

From this result we deduce the core property to show the closedness of $R(\mathcal{A})$.

Lemma 4.3. *Under assumptions* **(A0)–(A3)***, we have*

$$\|(E,H)^\top\|_{\mathcal{H}} \lesssim \|\mathcal{A}(E,H)^\top\|_{\mathcal{H}}, \quad \forall\,(E,H)^\top \in D(\mathcal{A}_\perp). \tag{4.17}$$

Proof. Suppose that (4.17) does not hold. Then there exist fields $(E_n,H_n)^\top \in D(\mathcal{A}_\perp)$ satisfying

$$\|(E_n,H_n)^\top\|_{\mathcal{H}} = 1, \quad \forall\, n \in \mathbb{N}, \tag{4.18}$$

and

$$\|\mathcal{A}(E_n,H_n)^\top\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty. \tag{4.19}$$

The dissipation identity (2.5) and (4.18) imply

$$\int_\Omega \sigma|E_n|^2 \leq \|\mathcal{A}(E_n,H_n)^\top\|_{\mathcal{H}}.$$

Using properties (4.1) and (4.19), we infer

$$\|E_n\|_{\Omega_+} \to 0 \quad \text{as } n \to \infty.$$

The limit (4.19) then leads to

$$\|\operatorname{curl} E_n\|_\Omega + \|\operatorname{curl} H_n\|_\Omega \to 0 \quad \text{as } n \to \infty. \tag{4.20}$$

¿From these two statements and Lemma 4.2, we conclude

$$\|E_n\|_\Omega \to 0 \quad \text{as } n \to \infty.$$

As Ω is simply connected and $H_n \in H_0(\operatorname{div}_\mu = 0, \Omega)$, Theorem 6.2.5 of [17] gives

$$\|H_n\|_\Omega \lesssim \|\operatorname{curl} H_n\|_\Omega,$$

so that (4.20) yields

$$\|H_n\|_\Omega \to 0 \quad \text{as } n \to \infty.$$

We now arrive at a contradiction with (4.18). □

Proposition 4.4. *Under assumptions* **(A0)–(A3)***, the range* $R(\mathcal{A})$ *is closed.*

Proof. As $R(\mathcal{A}) = R(\mathcal{A}_\perp)$, it suffices to prove the closedness of $R(\mathcal{A}_\perp)$. This property follows immediately from the estimate (4.17) using Theorem IV.1.6 in [28]. □

Combined with Lemma 3.5 and Corollary 3.6, the above result shows that $\mathcal{A}_\perp$ has no spectrum on $i\mathbb{R}$.

Corollary 4.5. *Under assumptions* **(A0)–(A3)***,* $\mathcal{A}_\perp$ *is invertible and therefore*

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_\perp).$$

5. A polynomial stability result in a cube

Our theorem on polynomial decay is based on the next result shown in Theorem 2.4 of [29]. See also [15, 30, 31] for weaker variants, partly in more general settings.

Lemma 5.1. *A bounded C_0 semigroup $(e^{t\mathcal{L}})_{t\geq 0}$ on a Hilbert space X satisfies*

$$\|e^{t\mathcal{L}}U_0\|_X \leq C\,t^{-\frac{1}{l}}\|U_0\|_{D(\mathcal{L})}, \quad \forall\, U_0 \in D(\mathcal{L}),\ \ t \geq 1,$$

for some constants $C, l > 0$ if we have

$$\rho(\mathcal{L}) \supseteq \{i\beta \,\big|\, \beta \in \mathbb{R}\} = i\mathbb{R} \tag{5.1}$$

and

$$\limsup_{|\beta|\to\infty} \frac{1}{\beta^l}\,\|(i\beta - \mathcal{L})^{-1}\| < \infty. \tag{5.2}$$

We will combine this criterion with the Fourier series expansion technique developed in Example 3 of [15] for the scalar wave equation, which has to be adapted to our Maxwell system. For that reason, in this section we restrict ourselves to the case of a domain Ω that is the cube $(0,\pi)^3$ and to $\varepsilon = \mu = 1$. In such a case, the Maxwell eigenmodes are explicitly known due to [16]. To recall these results, we introduce the Maxwell operator $\mathcal{A}_0$ on $\mathcal{H}_0 := H(\mathrm{div} = 0, \Omega) \times H_0(\mathrm{div} = 0, \Omega)$ by

$$\mathcal{A}_0(E, H)^\top = (\mathrm{curl}\, H, -\,\mathrm{curl}\, E)^\top, \quad \forall\, (E, H)^\top \in D(\mathcal{A}_0),$$

with

$$D(\mathcal{A}_0) = (X_N(\Omega) \cap H(\mathrm{div} = 0, \Omega)) \times (X_T(\Omega) \cap H(\mathrm{div} = 0, \Omega)).$$

As a direct consequence of Lemma 2.1, Theorem 3.5, and Section 4 of [16], the eigenfunctions of $\mathcal{A}_0$ are made of two families, namely the TE and TM modes:

1. **TE modes**. For all $k = (k_1, k_2, k_3) \in K^{TE} = \{(k_1, k_2) \in \mathbb{N}^2 : k_1 + k_2 > 0\} \times \mathbb{N}^*$, we set[†]

$$\begin{aligned} \kappa_k^{TE} &= \sqrt{k_1^2 + k_2^2 + k_3^2}, \\ \Psi_k^{TE,\pm} &= (E_k^{TE}, \mp(i\kappa_k^{TE})^{-1}\,\mathrm{curl}\, E_k^{TE})^\top, \\ E_k^{TE}(x_1, x_2, x_3) &= w_{k_3}^{\mathrm{Dir}}(x_3)\begin{pmatrix} \mathrm{curl}_\perp\, v_{k_1,k_2}^{\mathrm{Neu}}(x_1, x_2) \\ 0 \end{pmatrix}. \end{aligned} \tag{5.3}$$

Here $w_{k_3}^{\mathrm{Dir}}(x_3) = \sqrt{2/\pi}\,\sin(k_3 x_3)$ are the orthonormal eigenvectors of the Laplace operator with Dirichlet boundary condition on the interval $(0,\pi)$,

$$v_{k_1,k_2}^{\mathrm{Neu}}(x_1, x_2) = \tfrac{2}{\pi}\cos(k_1 x_1)\cos(k_2 x_2)$$

are the orthonormal eigenvectors of the Laplace operator with the Neumann boundary condition on the square $(0,\pi)^2$, and $\mathrm{curl}_\perp$ is the two-dimensional curl of a scalar field, i.e.,

$$\mathrm{curl}_\perp\, v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}.$$

[†] $\mathbb{N}^*$ means $\mathbb{N} \setminus \{0\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

2. **TM modes**. For all $k = (k_1, k_2, k_3) \in K^{TM} = (\mathbb{N}^*)^2 \times \mathbb{N}$, we set

$$\begin{aligned} \kappa_k^{TM} &= \sqrt{k_1^2 + k_2^2 + k_3^2}, \\ \Psi_k^{TM,\pm} &= (E_k^{TM}, \mp(i\kappa_k^{TM})^{-1} \operatorname{curl} E_k^{TM})^\top, \\ E_k^{TM}(x_1, x_2, x_3) &= \partial_3 w_{k_3}^{\mathrm{Neu}}(x_3) \begin{pmatrix} \nabla_\perp v_{k_1,k_2}^{\mathrm{Dir}}(x_1, x_2) \\ 0 \end{pmatrix} \\ &\quad + (k_1^2 + k_2^2) w_{k_3}^{\mathrm{Neu}}(x_3) \begin{pmatrix} 0 \\ v_{k_1,k_2}^{\mathrm{Dir}}(x_1, x_2) \end{pmatrix}. \end{aligned} \tag{5.4}$$

Here $w_{k_3}^{\mathrm{Neu}}(x_3) = \sqrt{2/\pi}\cos(k_3 x_3)$ are the orthonormal eigenvectors of the Laplace operator with the Neumann boundary condition on the interval $(0, \pi)$,

$$v_{k_1,k_2}^{\mathrm{Dir}}(x_1, x_2) = \tfrac{2}{\pi} \sin(k_1 x_1) \sin(k_2 x_2)$$

are the orthonormal eigenvectors of the Laplace operator with Dirichlet boundary condition on the square $(0, \pi)^2$, and $\nabla_\perp$ is the two-dimensional gradient of a scalar field, i.e.,

$$\nabla_\perp v = \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix}.$$

The family $\{E_k^{TE}, k \in K^{TE}\} \cup \{E_k^{TM}, k \in K^{TE}\}$ forms an orthogonal basis of $L^2(\Omega)^3$. Completeness is shown in Theorem 3.5 of [16], and the orthogonality is easily checked. With these notations, the spectrum of $\mathcal{A}_0$ is given by

$$\sigma(\mathcal{A}_0) = \{\pm i\kappa_k^{TE}\}_{k \in K^{TE}} \cup \{\pm i\kappa_k^{TM}\}_{k \in K^{TM}}.$$

For all $k \in K^{TE}$, $\Psi_k^{TE,\pm}$ is the eigenvector of $\mathcal{A}_0$ for the eigenvalue $\pm i\kappa_k^{TE}$, while for all $k \in K^{TM}$, $\Psi_k^{TM,\pm}$ is the eigenvector of $\mathcal{A}_0$ for the eigenvalue $\pm i\kappa_k^{TM}$. Finally, the set

$$\{\Psi_k^{TE,+}\}_{k \in K^{TE}} \cup \{\Psi_k^{TE,-}\}_{k \in K^{TE}} \cup \{\Psi_k^{TM,+}\}_{k \in K^{TM}} \cup \{\Psi_k^{TM,-}\}_{k \in K^{TM}}$$

forms an orthogonal basis of $\mathcal{H}_0$.

We still have to normalize these eigenfunctions. To this aim, we first note that

$$\begin{aligned} \operatorname{curl}_\perp v_{k_1,k_2}^{\mathrm{Neu}}(x_1, x_2) &= \frac{2}{\pi} \begin{pmatrix} -k_2 \cos(k_1 x_1) \sin(k_2 x_2) \\ k_1 \sin(k_1 x_1) \cos(k_2 x_2) \end{pmatrix}, \\ \operatorname{curl} E_k^{TE}(x_1, x_2, x_3) &= \begin{pmatrix} \partial_3 w_{k_3}^{\mathrm{Dir}}(x_3) \partial_1 v_{k_1,k_2}^{\mathrm{Neu}}(x_1, x_2) \\ \partial_3 w_{k_3}^{\mathrm{Dir}}(x_3) \partial_2 v_{k_1,k_2}^{\mathrm{Neu}}(x_1, x_2) \\ -w_{k_3}^{\mathrm{Dir}}(x_3) \Delta_\perp v_{k_1,k_2}^{\mathrm{Neu}}(x_1, x_2) \end{pmatrix} \\ &= \frac{2^{3/2}}{\pi^{3/2}} \begin{pmatrix} -k_1 k_3 \cos(k_3 x_3) \sin(k_1 x_1) \cos(k_2 x_2) \\ -k_2 k_3 \cos(k_3 x_3) \cos(k_1 x_1) \sin(k_2 x_2) \\ (k_1^2 + k_2^2) \sin(k_3 x_3) \cos(k_1 x_1) \cos(k_2 x_2) \end{pmatrix}, \end{aligned} \tag{5.5}$$

and hence $\|E_k^{TE}\|_\Omega^2 = k_1^2 + k_2^2$ as well as $\|\operatorname{curl} E_k^{TE}\|_\Omega^2 = (k_1^2 + k_2^2)(\kappa_k^{TE})^2$. Setting $s_j(x) = \sin(k_j x_j)$ and $c_j(x) = \cos(k_j x_j)$, we further compute

$$\nabla_\perp v_{k_1,k_2}^{\mathrm{Dir}}(x_1, x_2) = \frac{2}{\pi} \begin{pmatrix} k_1 \cos(k_1 x_1) \sin(k_2 x_2) \\ k_2 \sin(k_1 x_1) \cos(k_2 x_2) \end{pmatrix}, \tag{5.6}$$

$$\operatorname{curl} E_k^{TE}(x) = \frac{2^{3/2}}{\pi^{3/2}} \begin{pmatrix} k_2(k_1^2+k_2^2)c_3(x)s_1(x)c_2(x) + k_2k_3^2c_3(x)s_1(x)c_2(x) \\ -k_1k_3^2c_3(x)c_1(x)s_2(x) - k_1(k_1^2+k_2^2)c_3(x)c_1(x)s_2(x) \\ -k_1k_2k_3s_3(x)c_1(x)c_2(x) + k_1k_2k_3s_3(x)c_1(x)c_2(x) \end{pmatrix}$$

and obtain $\|E_k^{TM}\|_\Omega^2 = (k_1^2+k_2^2)(\kappa_k^{TE})^2$ as well as $\|\operatorname{curl} E_k^{TM}\|_\Omega^2 = (k_1^2+k_2^2)(\kappa_k^{TE})^4$. This results in the norms

$$\begin{aligned} &\|\Psi_k^{TE,\pm}\|_{\mathcal{H}_0}^2 = 2(k_1^2+k_2^2), \quad \forall\, k=(k_1,k_2,k_3)\in K^{TE}, \\ &\|\Psi_k^{TM,\pm}\|_{\mathcal{H}_0}^2 = 2(k_1^2+k_2^2)(k_1^2+k_2^2+k_3^2), \quad \forall\, k=(k_1,k_2,k_3)\in K^{TE}. \end{aligned}$$

With the normalized functions

$$\hat{\Psi}_k^{TE,\pm} := \frac{1}{\sqrt{2(k_1^2+k_2^2)}}\Psi_k^{TE,\pm}, \qquad \hat{\Psi}_k^{TM,\pm} := \frac{1}{\kappa_k^{TM}\sqrt{2(k_1^2+k_2^2)}}\Psi_k^{TE,\pm}, \tag{5.7}$$

the set

$$\{\hat{\Psi}_k^{TE,+}\}_{k\in K^{TE}} \cup \{\hat{\Psi}_k^{TE,-}\}_{k\in K^{TE}} \cup \{\hat{\Psi}_k^{TM,+}\}_{k\in K^{TM}} \cup \{\hat{\Psi}_k^{TM,-}\}_{k\in K^{TM}}$$

then forms an orthonormal basis of $\mathcal{H}_0$. Clearly some eigenvalues may coincide, but all have finite multiplicity. We thus rearrange them in increasing order (on the imaginary axis), writing

$$\begin{aligned} &\sigma(\mathcal{A}_0) = \{i\lambda_\ell\}_{\ell\in\mathbb{Z}^*} \\ &\text{with} \quad \lambda_{-1} < 0 < \lambda_1, \quad \lambda_\ell < \lambda_{\ell+1}, \quad \lambda_{-(\ell+1)} < \lambda_{-\ell}, \quad \forall\, \ell\in\mathbb{N}^*. \end{aligned}$$

Denoting the multiplicity of $i\lambda_\ell$ by N_ℓ, we let $\Phi_{\ell,j}$, $j=1,\dots,N_\ell$, be the N_ℓ orthornormal eigenvectors associated with $i\lambda_\ell$. If $\lambda_\ell > 0$ (resp. $\lambda_\ell < 0$), these eigenvectors are equal to $\hat{\Psi}_k^{TE,+}$ (resp. $\hat{\Psi}_k^{TE,-}$) with $\kappa_k^{TE} = \lambda_\ell$ and $k\in K^{TE}$, or equal to $\hat{\Psi}_k^{TM,+}$ (resp. $\hat{\Psi}_k^{TM,-}$) with $\kappa_k^{TM} = \lambda_\ell$ and $k\in K^{TM}$.

We now state the main result of this paper, which provides polynomial stability of $e^{t\mathcal{A}}$ on the cube if the damping region contains a strip that is parallel to x_1-x_2-plane.

Theorem 5.2. *Assume that* Ω *is the cube* $(0,\pi)^3$ *and that*

$$\Omega_{a,b} := (0,\pi)^2\times(a,b) \subseteq \Omega_+$$

for some $0\le a<b\le\pi$ *with* $b-a<\pi$. *Assume further that the assumptions* **(A0)** *to* **(A3)** *of the previous section are satisfied. Then* $\mathcal{A}_\perp$ *satisfies* (5.1) *and* (5.2) *with* $l=2$*, and consequently we have*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \lesssim t^{-\frac{1}{2}}\|U_0\|_{D(\mathcal{A})}, \quad \forall\, U_0\in D(\mathcal{A}_\perp), \ \ t\ge 1.$$

Proof. Property (5.1) has been checked in Corollary 4.5. So it remains to show (5.2) with $l=2$. For that purpose, suppose that (5.2) with $l=2$ is wrong. Hence, there exists a sequence $\{(\lambda_n, U_n=(E_n,H_n))\}_{n\ge1}$ in $\mathbb{R}\times D(\mathcal{A}_\perp)$ with $\beta_n>0$ satisfying

$$\beta_n\to+\infty \quad \text{as } n\to\infty, \qquad \|U_n\|_{\mathcal{H}} = 1, \quad \forall\, n\ge 1, \tag{5.8}$$

and

$$\beta_n^2(\, i\beta_nU_n - \mathcal{A}U_n) = (F_n,G_n)\to 0 \ \text{ in } \mathcal{H}. \tag{5.9}$$

In particular, there is an index $N \in \mathbb{N}^*$ such that

$$\beta_n \geq 1, \quad \forall\, n \geq N. \tag{5.10}$$

First, the dissipativity property (2.5) yields

$$\beta_n^2 \int_\Omega \sigma |E_n|^2 \to 0 \ \text{ as } n \to \infty. \tag{5.11}$$

Detailing (5.9), we obtain

$$\begin{aligned} \beta_n^2(i\beta_n E_n - \operatorname{curl} H_n + \sigma E_n) = F_n \to 0 \quad \text{in } L^2(\Omega)^3, \\ \beta_n^2(i\beta_n H_n + \operatorname{curl} E_n) = G_n \to 0 \quad \text{in } L^2(\Omega)^3. \end{aligned} \tag{5.12}$$

Next, since (E_n, H_n) may not belong to $D(\mathcal{A}_0)$, for a correction we will use the unique solution $\varphi_n \in H_0^1(\Omega)$ of

$$\int_\Omega \nabla\varphi_n \cdot \nabla\bar{\chi} = \int_\Omega E_n \cdot \nabla\bar{\chi}, \quad \forall\, \chi \in H_0^1(\Omega). \tag{5.13}$$

For an arbitrary $\chi \in H_0^1(\Omega)$, the first identity in (5.12) leads to

$$\int_\Omega (i\beta_n E_n - \operatorname{curl} H_n + \sigma E_n) \cdot \nabla\bar{\chi} = \beta_n^{-2} \int_\Omega F_n \cdot \nabla\bar{\chi}.$$

Since Green's formula yields

$$\int_\Omega \operatorname{curl} H_n \cdot \nabla\bar{\chi} = 0,$$

we can express the right-hand side of (5.13) by

$$\int_\Omega E_n \cdot \nabla\bar{\chi} = -i \int_\Omega (\beta_n^{-3} F_n - \beta_n^{-1} \sigma E_n) \cdot \nabla\bar{\chi}$$

so that (5.13) is rewritten as

$$\int_\Omega \nabla\varphi_n \cdot \nabla\bar{\chi} = -i \int_\Omega (\beta_n^{-3} F_n - \beta_n^{-1} \sigma E_n) \cdot \nabla\bar{\chi}, \quad \forall\, \chi \in H_0^1(\Omega).$$

Taking $\chi = \varphi_n$ and using the Cauchy–Schwarz inequality, we deduce

$$\|\nabla\varphi_n\|_\Omega \leq \beta_n^{-3} \|F_n\|_\Omega + \beta_n^{-1} \|\sigma E_n\|_\Omega.$$

Poincaré's inequality, (5.11), and (5.12) then imply

$$\|\varphi_n\|_{1,\Omega} \lesssim \|\nabla\varphi_n\|_\Omega \lesssim \beta_n^{-2} o(1). \tag{5.14}$$

We now set $\hat{E}_n = E_n - \nabla\varphi_n$. Observe that the pair $\hat{U}_n := (\hat{E}_n, H_n)^\top$ belongs to $D(\mathcal{A}_0)$. Since

$$\beta_n(i\beta_n \hat{U}_n - \mathcal{A}_0 \hat{U}_n) = \beta_n(i\beta_n U_n - \mathcal{A} U_n) + (\sigma\beta_n E_n - i\beta_n^2 \nabla\varphi_n, 0)^\top,$$

the limits (5.9), (5.11), and (5.14) show

$$\beta_n(i\beta_n\hat{U}_n - \mathcal{A}_0\hat{U}_n) \to 0 \text{ in } \mathcal{H}. \tag{5.15}$$

Moreover, from (5.11), (5.14), and **(A3)**, we infer

$$\|\beta_n\hat{E}_n\|_{\Omega_+} \to 0 \quad \text{as } n \to \infty, \tag{5.16}$$

while (5.8) and (5.14) lead to

$$\lim_{n\to\infty} \|\hat{U}_n\|_{\mathcal{H}} = 1.$$

Therefore, there exists $N_0 \in \mathbb{N}^*$ such that

$$\|\hat{U}_n\|_{\mathcal{H}}^2 \geq 3/4, \quad \forall\, n \geq N_0. \tag{5.17}$$

Next we write $\hat{U}_n$ in the basis $\{\Phi_{\ell,j}\}$, i.e.,

$$\hat{U}_n = \sum_{\ell\in\mathbb{Z}^*}\sum_{j=1}^{N_\ell} \alpha_{\ell,j}^{(n)}\Phi_{\ell,j}$$

with the coefficients $\alpha_{\ell,j}^{(n)} = (\hat{U}_n, \Phi_{\ell,j})_{\mathcal{H}}$, so that its norm is given by

$$\|\hat{U}_n\|_{\mathcal{H}}^2 = \sum_{\ell\in\mathbb{Z}^*}\sum_{j=1}^{N_\ell} |\alpha_{\ell,j}^{(n)}|^2.$$

Then (5.15) means that

$$\lim_{n\to\infty} \beta_n \sum_{\ell\in\mathbb{Z}^*}\sum_{j=1}^{N_\ell} (\beta_n - \lambda_\ell)\alpha_{\ell,j}^{(n)}\Phi_{\ell,j} \to 0 \quad \text{in } \mathcal{H}_0,$$

or equivalently

$$\beta_n^2 \sum_{\ell\in\mathbb{Z}^*}\sum_{j=1}^{N_\ell} |(\beta_n - \lambda_\ell)\alpha_{\ell,j}^{(n)}|^2 \to 0 \quad \text{as } n \to \infty.$$

Hence, for any $\epsilon > 0$ there exists $M_\epsilon \in \mathbb{N}^*$ such that

$$\beta_n^2 \sum_{\ell\in\mathbb{Z}^*}\sum_{j=1}^{N_\ell} |(\beta_n - \lambda_\ell)\alpha_{\ell,j}^{(n)}|^2 \leq \epsilon, \quad \forall\, n \geq M_\epsilon. \tag{5.18}$$

Before going on with the proof, we notice that Lemma 3.1 of [15] can be transferred to our setting. Recall that the indices N, N_0, and M_ϵ have been fixed in (5.10), (5.17), and (5.18).

Lemma 5.3. *For every $\epsilon \in (0, \frac{1}{288})$ and $n \in \mathbb{N}^*$ with $n \geq \max\{N_0, N, M_\epsilon\}$, there exists an index $k(\epsilon, n) \in \mathbb{N}^*$ such that*

$$\beta_n^2|\beta_n - \lambda_{k(\epsilon,n)}|^2 \leq 2\epsilon, \tag{5.19}$$

$$\beta_n^2|\beta_n - \lambda_k|^2 \geq \frac{1}{144}, \quad \forall\, k \neq k(\epsilon, n). \tag{5.20}$$

Proof of Lemma 5.3. The proof follows the arguments of Lemma 3.1 of [15], but differs a bit from it; therefore, we give the details.

We fix $\epsilon \in (0, \frac{1}{288})$, and take $n \in \mathbb{N}^*$ with $n \geq \max\{N_0, N, M_\epsilon\}$. If (5.19) did not hold, we would obtain $\beta_n^2 |\beta_n - \lambda_k|^2 > 2\epsilon$ for all $k \in \mathbb{N}^*$. This inequality contradicts (5.18) because of (5.10) and (5.17). Observe that $\lambda_{k(\epsilon,n)}$ has to be positive, because negative $\lambda_{k(\epsilon,n)}$ fulfill

$$|\beta_n - \lambda_{k(\epsilon,n)}| \geq \beta_n,$$

and then (5.10) leads to the contradiction

$$\beta_n^2 |\beta_n - \lambda_{k(\epsilon,n)}|^2 \geq \beta_n^3 \geq 1.$$

To show (5.20), we first note that (5.19) and the triangle inequality imply

$$\beta_n |\beta_n - \lambda_k| \geq \beta_n |\lambda_k - \lambda_{k(\epsilon,n)}| - \sqrt{2\epsilon} \tag{5.21}$$

for $k \in \mathbb{Z}^*$. The inequality $|\lambda_k - \lambda_{k(\epsilon,n)}| \geq 1$ leads to

$$\beta_n |\lambda_k - \lambda_{k(\epsilon,n)}| \geq 1$$

and thus (5.20). This case covers all indices $k < 0$ as $\lambda_{k(\epsilon,n)} \geq 1$.

So it suffices to treat $k \in \mathbb{N}^*$ with $k \neq k(\epsilon, n)$ and $|\lambda_k - \lambda_{k(\epsilon,n)}| < 1$. Notice that

$$|\lambda_k^2 - \lambda_{k(\epsilon,n)}^2| \geq 1 \tag{5.22}$$

since λ_ℓ^2 is always a positive integer. Using this fact, $\lambda_{k(\epsilon,n)} \geq 1$ and (5.19), we compute

$$\beta_n |\lambda_k - \lambda_{k(\epsilon,n)}| = \frac{\beta_n |\lambda_k^2 - \lambda_{k(\epsilon,n)}^2|}{\lambda_k + \lambda_{k(\epsilon,n)}} \geq \frac{\beta_n}{3\lambda_{k(\epsilon,n)}} \geq \frac{\beta_n^2}{3(\beta_n^2 + \sqrt{2\epsilon})}.$$

The property (5.10) and the assumption on ϵ give

$$\frac{\beta_n^2}{3(\beta_n^2 + \sqrt{2\epsilon})} \geq \frac{1}{6}.$$

Combined with (5.21), we conclude (5.20) via

$$\beta_n |\beta_n - \lambda_k| \geq \frac{1}{6} - \sqrt{2\epsilon} \geq \frac{1}{12}. \qquad \square$$

We continue with the proof of Theorem 5.2, where we use ϵ, n and $\lambda_{k(\epsilon,n)}$ from Lemma 5.3. Inserting (5.20) into (5.18), we obtain

$$\frac{1}{144} \sum_{\ell \in \mathbb{Z}^*, \ell \neq k(\epsilon,n)} \sum_{j=1}^{N_\ell} |\alpha_{\ell,j}^{(n)}|^2 + \beta_n^2 |\beta_n - \lambda_{k(\epsilon,n)}|^2 \sum_{j=1}^{N_{k(\epsilon,n)}} |\alpha_{k(\epsilon,n),j}^{(n)}|^2 \leq \epsilon. \tag{5.23}$$

For the function

$$\Psi_{n,\epsilon} := \sum_{j=1}^{N_{k(\epsilon,n)}} \alpha_{k(\epsilon,n),j}^{(n)} \Phi_{k(\epsilon,n),j}$$

estimate (5.23) yields

$$\|\hat{U}_n - \Psi_{n,\epsilon}\|^2_{\mathcal{H}} \leq 144\epsilon. \tag{5.24}$$

This fact and (5.17) lead to

$$\|\Psi_{n,\epsilon}\|^2_{\mathcal{H}} = \sum_{j=1}^{N_{k(\epsilon,n)}} |\alpha^{(n)}_{k(\epsilon,n),j}|^2 \geq \frac{3}{4} - 144\epsilon \geq \frac{1}{4}. \tag{5.25}$$

Recall that $\Omega_{a,b} = (0,\pi)^2 \times (a,b) \subseteq \Omega_+$. From (5.24) we infer the bound

$$\|\hat{E}_n - (\Psi_{n,\epsilon})_1\|^2_{\Omega} \leq 144\epsilon, \tag{5.26}$$

where $(\Psi_{n,\epsilon})_1$ means the first component of $\Psi_{n,\epsilon}$. In view of the form of $\Phi_{k(\epsilon,n),j}$, this component is given by

$$\begin{aligned}(\Psi_{n,\epsilon})_1 = &\sum_{k\in K^{TE}:|k|^2=\lambda^2_{k(\epsilon,n)}} \frac{\alpha_k^{n,TE,+}}{\sqrt{2(k_1^2+k_2^2)}} E_k^{TE} \\ &+ \sum_{k\in K^{TM}:|k|^2=\lambda^2_{k(\epsilon,n)}} \frac{\alpha_k^{n,TM,+}}{\sqrt{2(k_1^2+k_2^2+k_3^2)(k_1^2+k_2^2)}} E_k^{TM},\end{aligned}$$

where $\alpha_k^{n,TE,+} = (\hat{U}_n, \hat{\Psi}^{TE,+})_{\mathcal{H}}$ and $\alpha_k^{n,TM,+} = (\hat{U}_n, \hat{\Psi}^{TM,+})_{\mathcal{H}}$. We minorate the L^2-norm of $(\Psi_{n,\epsilon})_1$. First, using the orthogonality of E_k^τ and $E_{k'}^\tau$ for $\tau \in \{TE, TM\}$ it can be expressed as

$$\begin{aligned}\|(\Psi_{n,\epsilon})_1\|^2_{\Omega} \geq &\sum_{k\in K^{TE}:|k|^2=\lambda^2_{k(\epsilon,n)}} \frac{|\alpha_k^{n,TE,+}|^2}{2(k_1^2+k_2^2)} \|E_k^{TE}\|^2_{\Omega_{a,b}} \\ &+ \sum_{k\in K^{TM}:|k|^2=\lambda^2_{k(\epsilon,n)}} \frac{|\alpha_k^{n,TM,+}|^2}{2(k_1^2+k_2^2+k_3^2)(k_1^2+k_2^2)} \|E_k^{TM}\|^2_{\Omega_{a,b}}.\end{aligned} \tag{5.27}$$

By formulas (5.3) and (5.5), the first norm on the right is given by

$$\|E_k^{TE}\|^2_{\Omega_{a,b}} = \int_a^b |w_{k_3}^{\mathrm{Dir}}(x_3)|^2 \int_{(0,\pi)^2} |\operatorname{curl}_\perp v_{k_1,k_2}^{\mathrm{Neu}}(x_1,x_2)|^2 = (k_1^2+k_2^2)\int_a^b |w_{k_3}^{\mathrm{Dir}}(x_3)|^2.$$

Since $b-a>0$, we can find a constant $\delta>0$ depending on a and b such that

$$\int_a^b |w_{k_3}^{\mathrm{Dir}}(x_3)|^2 = \frac{2}{\pi}\int_a^b \sin^2(k_3x_3) \geq \delta, \quad \forall\, k_3 \in \mathbb{N}^*. \tag{5.28}$$

These properties imply that

$$\|E_k^{TE}\|^2_{\Omega_{a,b}} \geq \delta(k_1^2+k_2^2). \tag{5.29}$$

In a similar manner, from (5.4) and (5.6) we infer

$$\|E_k^{TM}\|^2_{\Omega_{a,b}} = \int_a^b |\partial_3 w_{k_3}^{\mathrm{Neu}}(x_3)|^2 \int_{(0,\pi)^2} |\nabla_\perp v_{k_1,k_2}^{\mathrm{Dir}}(x_1,x_2)|^2 \;+ (k_1^2+k_2^2)^2$$

$$= k_3^2(k_1^2 + k_2^2)\frac{2}{\pi}\int_a^b \sin^2(k_3 x_3) \ + (k_1^2 + k_2^2)^2$$

Combined with (5.28), it follows

$$\|E_k^{TM}\|^2_{\Omega_{a,b}} \geq k_3^2\delta(k_1^2 + k_2^2) + (k_1^2 + k_2^2)^2 \geq \min\{1, \delta\}(k_1^2 + k_2^2)(k_1^2 + k_2^2 + k_3^2).$$

Inserting this estimate and (5.29) in (5.27) and applying (5.25), we conclude

$$\begin{aligned}\|(\Psi_{n,\epsilon})_1\|^2_{\Omega_{a,b}} &\geq \frac{\min\{1,\delta\}}{2}\Big[\sum_{k\in K^{TE}:|k|^2=\lambda^2_{k(\epsilon,n)}} |\alpha_k^{n,TE,+}|^2 + \sum_{k\in K^{TM}:|k|^2=\lambda^2_{k(\epsilon,n)}} |\alpha_k^{n,TM,+}|^2\Big] \\ &= \frac{\min\{1,\delta\}}{2}\|\Psi_{n,\epsilon}\|^2_{\mathcal{H}} \geq \frac{\min\{1,\delta\}}{8}.\end{aligned}$$

This inequality contradicts (5.16) and (5.26) by fixing small $\epsilon > 0$ and large n. □

We add immediate consequences of the above theorem.

Corollary 5.4. *Under the assumptions of Theorem 5.2, let $\mathcal{P}$ be the orthogonal projection in $\mathcal{H}$ onto* $\ker(\mathcal{A}) = R(\mathcal{A})^\perp = \mathcal{H}_\perp^\perp$. *Then we have*

$$\|e^{t\mathcal{A}}U_0 - \mathcal{P}U_0\|_{\mathcal{H}} \leq C\, t^{-\frac{1}{2}}\|U_0\|_{D(\mathcal{A})}, \quad \forall\, U_0 \in D(\mathcal{A}), \ \ t \geq 1.$$

Proof. Let $U_0 \in D(\mathcal{A})$. We then obtain $(\mathbb{I} - \mathcal{P})\mathcal{A}U_0 = \mathcal{A}U_0$ and $\mathcal{A}\mathcal{P}U_0 = 0$, so that $\mathcal{A}$ and $\mathcal{P}$ commute. Since $e^{t\mathcal{A}}\mathcal{P}U_0 = \mathcal{P}U_0$ for $t \geq 0$ and $U_0 - \mathcal{P}U_0 \in D(\mathcal{A}_\perp)$, we can apply Theorem 5.2 to $e^{t\mathcal{A}}U_0 - \mathcal{P}U_0 = e^{t\mathcal{A}}(U_0 - \mathcal{P}U_0)$. □

Example 5.5. Among the manifold examples of Ω_+ satisfying the assumptions of Theorem 5.2, we only mention two.

1) Let $\Omega_+ = (0,\pi)^2 \times \bigcup_{i=1}^I (a_i, a_{i+1})$ with $I \in \mathbb{N}^*$ and $a_1 \geq 0$, $a_{I+1} \leq \pi$, $a_i < a_{i+1}$, for all $i = 1, \cdots, I$ with $a_2 - a_1 < \pi$ if $I = 1$. In that case we have $\ker\mathcal{A} = \{0\}$ since the components $\Omega_{0,j}$ have connected boundaries, cf. Corollary 4.1.

2) Let $\Omega_+ = B \cup ((0,\pi)^2 \times (a,b))$ with $0 \leq a < b \leq \pi$, $b - a < \pi$, and a ball $B \subseteq (0,\pi)^3$ being disjoint with $(0,\pi)^2 \times (a,b)$. In that case, Ω_0 is made of two connected components, $\Omega_{0,1}$ and $\Omega_{0,2}$, the first one with a connected boudary, while the boundary of the second one has two connected components. Therefore, $\ker\mathcal{A}$ is one-dimensional.

Remark 5.6. In view of the discussion at the end of Section 4 of [16], Theorem 5.2 remains valid for a parallelipiped $\Omega = (0,\ell_1\pi) \times (0,\ell_2\pi) \times (0,\ell_3\pi)$ with

$$(0,\ell_1\pi) \times (0,\ell_2\pi) \times (a,b) \subset \Omega_+,$$

with $0 \leq a < b \leq \ell_3\pi$ with $b - a < \ell_3\pi$ as soon as the ratio $\frac{\ell_j^2}{\ell_k^2}$ is a rational number, for any $j,k \in \{1,2,3\}$. Indeed, in that case the spectrum of $\mathcal{A}_0$ is equal to $\{i\lambda_\ell\}_{\ell\in\mathbb{Z}^*}$, where

$$\lambda_\ell = \pm\sqrt{\frac{k_1^2}{\ell_1^2} + \frac{k_2^2}{\ell_2^2} + \frac{k_3^2}{\ell_3^2}},$$

for some $k_1, k_2, k_3 \in \mathbb{N}$ with $k_1 + k_2 + k_3 > 0$. Therefore the gap condition

$$|\lambda_\ell^2 - \lambda_{\ell'}^2| \geq \frac{p}{\alpha m_1 m_2 m_3}$$

holds for all $\lambda_\ell \neq \lambda_{\ell'}$, writing $\ell_j^2 = \frac{m_j}{p}\alpha$ for some $\alpha > 0$ and $p, m_j \in \mathbb{N}^*$, $j = 1, 2, 3$. (Compare inequality (5.22) above.)

Author contributions

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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