Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 59-13 (2025) 99–104

From Matrices to Operators: A Tensorial View on the Legendre Tau Method for Time-Delay Systems

Tessina H. Scholl * **Lutz Gröll** *

* *Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany (e-mail: tessina.scholl@kit.edu, lutz.groell@kit.edu)*

Abstract: It is the essence of numerical methods like the Legendre tau method to approximate infinite-dimensional problems by finite-dimensional ones. Instead of functions and operators, finite-dimensional vectors and matrices occur. However, such a vector or matrix is still the coordinate representation of a function or operator that approximates the original one. The present paper considers the coordinate representations that result from the Legendre tau method for time-delay systems. A combination of two different interpretations of the coordinates turns out to be particularly helpful: a coordinate vector can be seen as representing a polynomial of degree N or as representing a polynomial of degree $N-1$ and a discontinuous end point. The fact that the associated basis functions are nonorthonormal in $L_2 \times \mathbb{R}^n$ must be dealt with. Tools from tensor algebra are used to make explicit the operators that are obtained as matrix representations from the Legendre tau method.

Keywords: Legendre tau method, spectral methods, product space $M_2 = L_2 \times \mathbb{R}^n$, polynomials, tensor algebra, metric coefficients, operator equations

1. INTRODUCTION

Once a delay $h > 0$ occurs in the system equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \tag{1}$$

$A_0, A_1 \in \mathbb{R}^{n\times n}$ (the second part of this paper assumes $n = 1$ only for notational convenience), an initial function $x_0 \in C([-h, 0], \mathbb{R}^n)$ is needed in a Cauchy problem, and the state x_t at time $t > 0$ refers to the solution segment $x_t(\theta) = x(t+\theta)$, $\theta \in [-h, 0]$. Due to the infinite-dimensional state space, numerical methods that are known from partial differential equations have proven to be useful. Spectral methods, like Chebyshev collocation or Legendre tau rely on a polynomial approximation of the state x_t, see, e.g., Hesthaven et al. (2007). The present paper focuses on the Legendre tau method. There is a wide range of applications. Ito and Teglas (1987) applied the method to time-delay systems as a numerical solution approach, as an early lumping approach to optimal control problems, and as an approach for the approximation of characteristic roots (Ito, 1985). Ito (1983) also used it for parameter estimation problems, and Hale and Sternberg (1988) for a numerical bifurcation analysis. Still, the method is clearly less widespread than the related Chebyshev collocation method, which has shown very convincing results in the meanwhile, see, e.g., Breda et al. (2016, 2005). In the last years, the Legendre tau method came into focus again as, in Scholl et al. (2024b,a), it has been proven to yield an advantageous numerical approach to Lyapunov–Krasovskii functionals for (1) that rely on operator-valued Lyapunov or algebraic Riccati equations. Recently, Provoost and Michiels (2024) used tau methods for H_2-norm approximations.

The outcome of the Legendre tau method applied to (1) is an ordinary differential equation $\dot{x}_\mathrm{c} = A_\mathrm{c} x_\mathrm{c}$. It describes the dynamics of a vector of coordinates $x_\mathrm{c}(t)$ that represents the polynomial approximation of the state x_t. Moreover, based on A_c, a matrix P_c can be computed such that $x_\mathrm{c}^\top P_\mathrm{c}\, x_\mathrm{c}$ is a numerical approximation of a quadratic Lyapunov–Krasovskii functional (Scholl et al., 2024b). As the vector x_c is a coordinate representation of a function (and thus also of the pair of the function and its end point in $L_2 \times \mathbb{R}^n \supset C \times \mathbb{R}^n$), both matrices A_c and P_c are coordinate representations of operators. Any calculation in terms of such an operator (e.g., operator norms, adjoint operators, or operator equations like operator-valued Lyapunov or algebraic Riccati equations) can be handled in terms of these matrices. However, to do that properly, special care is needed because the coordinate vectors are associated with basis functions that are not orthonormal. Moreover, the operators are not only restricted to polynomial arguments once the discretization is incorporated by which x_c is obtained from a given arbitrary function.

The objective of the present paper is to provide explicit descriptions of the operators, accompanied by a framework that enables the above motivated proper handling of their coordinate representations. To this end, we apply methods from tensor algebra. For instance, we use metric coefficients, dual bases, dyadic products, basis transformations, and index notation (Ricci calculus). Regarding the existing literature, we are neither aware of explicit applications of these tools to polynomial subspaces of $L_2 \times \mathbb{R}^n$ nor of a conscious and consequent application in the context of spectral methods. The content of the present paper is discussed in greater detail in the first author's PhD thesis (Scholl, 2024, Chapter 3 and Appendix A).

10.1016/j.ifacol.2025.10.018

2. LEGENDRE TAU METHOD

As a foundation, this section revisits the Legendre tau method. The aspired tensorial view on the outcome will be explained in the subsequent Section 3.

2.1 Discretization of a Given Function

Let a function $\phi \in C([-h,0],\mathbb{R}^n)$ be given, which might be an initial function or which might be the argument of a functional, and note that $C([-h,0],\mathbb{R}^n) \subset L_2([-h,0],\mathbb{R}^n)$. Any function $\phi \in L_2$ can be expanded in a Legendre series

$$\phi(\theta) = \sum_{k=0}^{\infty} c^k p_k(\vartheta(\theta)), \tag{2}$$

where $c^k = \frac{2}{h}\frac{2k+1}{2}\int_{-h}^{0}\phi(\theta)p_k(\vartheta(\theta))\,\mathrm{d}\theta \in \mathbb{R}^n$ is the coefficient of the k-th Legendre polynomial $p_k : [-1,1] \to \mathbb{R}$, and $\vartheta : [-h,0] \to [-1,1]$ establishes the scaling of its domain by $\vartheta(\theta) := \frac{2}{h}\theta + 1$. Still, the Legendre tau method does not simply use a truncation of that series to get a polynomial with a prescribed maximum degree N. The pointwise value $\phi(0)$ is too decisive if $\phi(\theta) = x_0(\theta)$ is an initial function of a time-delay system. That is why the last coefficient in the truncated series $\sum_{k=0}^{N} c^k p_k(\vartheta(\theta))$ is manipulated in such a way that the resulting polynomial

$$\phi^{[N]}(\theta) = \sum_{k=0}^{N} x_{\mathrm{c}}^k\, p_k(\vartheta(\theta)) \tag{3}$$

coincides in its end point at $\theta = 0$ with the original function, i.e., $\phi^{[N]}(0) = \phi(0)$. Thus, for a given function ϕ with the series coefficients $c^0, c^1, \dots$ in (2), the coordinates

$$x_{\mathrm{c}}^k := \begin{cases} c^k & \text{if } k < N, \\ \hat{x} - \displaystyle\sum_{k=0}^{N-1} c^k, \text{ with } \hat{x} = \phi(0) & \text{if } k = N, \end{cases} \tag{4}$$

are chosen in (3) since $\phi^{[N]}(0) = \sum_{k=0}^{N} x_{\mathrm{c}}^k p_k(\vartheta(0)) = \sum_{k=0}^{N} x_{\mathrm{c}}^k$ (based on the fact that all Legendre polynomials have the boundary value $p_k(\vartheta(0)) = p_k(1) = 1$). In terms of the pair of function and end point

$$\mathsf{x} = \begin{bmatrix} \phi(\cdot) \\ \phi(0) \end{bmatrix} \in L_2 \times \mathbb{R}^n =: M_2, \tag{5}$$

we denote the associated projection mapping by

$$\mathsf{x} \quad \mapsto \quad \mathrm{Proj}_{\mathrm{cont}}^{[N]}\mathsf{x} = \begin{bmatrix} \phi^{[N]}(\cdot) \\ \phi^{[N]}(0) \end{bmatrix}. \tag{6}$$

The subscript "cont" emphasizes that the second component $\phi^{[N]}(0) = \hat{x} = \phi(0)$ in (6) is a continuous end point to the polynomial (3) in the first component of (6).

Discontinuous Interpretation Of course, there is an infinite number of alternative realizations or "reconstructions" in terms of functions that provide the same discretization x_{c} in (4). In some cases, it is convenient to interpret Legendre coordinates x_{c} instead of representing $\phi^{[N]}$ from (3), rather as representing an $(N-1)$-th degree polynomial and the (possibly discontinuous) end point $\hat{x}$,

$$\phi_{\mathrm{d}}^{[N]}(\theta) = \begin{cases} \tilde{\phi}_{\mathrm{d}}^{[N]}(\theta) := \displaystyle\sum_{k=0}^{N-1} x_{\mathrm{c}}^k p_k(\vartheta(\theta)) & \text{if } \theta \in [-h,0), \\ \hat{x} := \displaystyle\sum_{k=0}^{N} x_{\mathrm{c}}^k & \text{if } \theta = 0. \end{cases} \tag{7}$$

We denote the projection of function and end point by

$$\mathsf{x} = \begin{bmatrix} \phi(\cdot) \\ \phi(0) \end{bmatrix} \quad \mapsto \quad \mathrm{Proj}_{\mathrm{d}}^{[N]}\mathsf{x} = \begin{bmatrix} \tilde{\phi}_{\mathrm{d}}^{[N]}(\cdot) \\ \phi_{\mathrm{d}}^{[N]}(0) \end{bmatrix}. \tag{8}$$

2.2 The System Dynamics to be Discretized

The retarded functional differential equation (1) only describes how the pointwise value $x(t) \in \mathbb{R}^n$ evolves with time t. How the pair of the attached function $x_t(\cdot) \in L_2$ and its end point $x(t) = x_t(0) \in \mathbb{R}^n$ evolves with time t, can be described by an abstract differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_t(\cdot) \\ x_t(0) \end{bmatrix} = \mathscr{A}\begin{bmatrix} x_t(\cdot) \\ x_t(0) \end{bmatrix} \tag{9}$$

on the space M_2. With a slight abuse of notation, consider both t and θ in $x_t(\theta)$ as independent variables in a bivariate map $(t,\theta) \mapsto x_t(\theta)$. Clearly, $x_t(\theta) = x(t+\theta)$ has equal derivatives w.r.t. both t and θ. Additionally, (1) relates $x(t) = x_t(0)$, $x(t-h) = x_t(-h)$, and $\dot{x}(t) = \frac{\partial}{\partial t}x_t(\theta)|_{\theta=0}$. Hence, $(t,\theta) \mapsto x_t(\theta) \in \mathbb{R}^n$ must obey

$$\frac{\partial}{\partial t}x_t(\theta) = \frac{\partial}{\partial \theta}x_t(\theta), \qquad \theta \in [-h,0),\, t > 0, \tag{10a}$$

$$\frac{\partial}{\partial t}x_t(0) = A_0 x_t(0) + A_1 x_t(-h), \qquad t > 0. \tag{10b}$$

Considering again only t as independent variable, $t \mapsto \begin{bmatrix} x_t(\cdot) \\ x_t(0) \end{bmatrix} \in M_2$ must correspondingly obey

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_t \\ x_t(0) \end{bmatrix} = \begin{bmatrix} x_t' \\ A_0 x_t(0) + A_1 x_t(-h) \end{bmatrix}. \tag{11}$$

To be more precise, (9) relies on the unbounded operator $\mathscr{A} : M_2 \supset D(\mathscr{A}) \to M_2$,

$$\mathscr{A}\begin{bmatrix} \phi \\ r \end{bmatrix} = \begin{bmatrix} \phi' \\ A_0 r + A_1\phi(-h) \end{bmatrix}, \tag{12}$$

$$D(\mathscr{A}) = \{\begin{bmatrix} \phi \\ r \end{bmatrix} \in M_2 : r = \phi(0), \phi' \in L_2, \phi \in AC\}$$

(denoting $\phi'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\phi(\theta)$), which is the infinitesimal generator of a C_0-semigroup (Curtain and Zwart, 2020).

2.3 The Resulting System Matrix

Spectral methods like the Legendre tau method use a polynomial ansatz for x_t in (9). Note that the upper part of $\mathscr{A}$ from (12) is only differentiation. If a polynomial is given in terms of its Legendre coordinates x_{c}, then the derivative of that polynomial is exactly described by the Legendre coordinates $(D_{\mathrm{c}} \otimes I_n)x_{\mathrm{c}}$, where, using $\vartheta' = \frac{2}{h}$,

$$D_{\mathrm{c}} = \vartheta' \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 3 & 0 & \dots & 3 \\ 0 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & \dots & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 7 & 0 & \dots & 7 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 9 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & \dots & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 15 \\ & & & & & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)} \tag{13}$$

is the Legendre differentiation matrix [1], i.e.,

$$\phi(\theta) = \sum_{k=0}^{N} x_{\mathrm{c}}^k\, p_k(\vartheta(\theta)) \tag{14}$$

$$\Longrightarrow \phi'(\theta) = \sum_{k=0}^{N} d_{\mathrm{c}}^k\, p_k(\vartheta(\theta)), \qquad \begin{bmatrix} d_{\mathrm{c}}^0 \\ \vdots \\ d_{\mathrm{c}}^N \end{bmatrix} = (D_{\mathrm{c}} \otimes I_n)\begin{bmatrix} x_{\mathrm{c}}^0 \\ \vdots \\ x_{\mathrm{c}}^N \end{bmatrix}$$

(with a matrix Kronecker product $\otimes$, which is only needed if $n > 1$).

[1] Exemplarily, D_{c} is shown for N even, otherwise a last column $[1,0,5,0,9,\dots,0,(2N-1),0]^\top$ has to be appended.

To incorporate the boundary condition, the Legendre tau method replaces in $(D_\mathrm{c} \otimes I_n)$ the last (block-) row—which is zero anyway—by entries that take account of (10b). Altogether, the resulting system matrix becomes

$$A_\mathrm{c} = D_\mathrm{c} \otimes I_n + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ A_0{+}A_1 & A_0{-}A_1 & A_0{+}A_1 & \cdots & A_0{+}(-1)^N A_1 \end{bmatrix}$$
$$+ \frac{2}{h} \begin{bmatrix} 0 & & & & & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -3 & -6 & -10 & \cdots & -\frac{N(N+1)}{2} \end{bmatrix} \otimes I_n \tag{15}$$

(see Scholl (2024) for a more detailed explanation). This system matrix defines the ordinary differential equation

$$\dot{x}_\mathrm{c}(t) = A_\mathrm{c}\, x_\mathrm{c}(t) \tag{16}$$

that describes how the coordinates $x_\mathrm{c}(t) \in \mathbb{R}^{n(N+1)}$ evolve with time t. At any time t, these Legendre coordinates $x_\mathrm{c}(t)$ shall represent an approximation of x_t.

2.4 The Approximation of $\mathscr{A}$

In fact, A_c is the coordinate representation of an operator that intends to approximate (12). How is it related to $\mathscr{A}$?

First, note that, in view of (14), the polynomial ansatz used in the above approach is unambiguous. Incorporating how the involved coordinate vector x_c is derived from a given initial function ϕ in (4), x_c in $A_\mathrm{c}x_\mathrm{c}$ represents the polynomial of degree N that is described in (3). Thus, once the ansatz is incorporated, no longer $\mathscr{A}\mathsf{x}$ but $\mathscr{A}(\mathrm{Proj}_\mathrm{cont}^{[N]}\mathsf{x})$ is considered, given $\mathsf{x} = \begin{bmatrix} \phi(\cdot) \\ \phi(0) \end{bmatrix} \in D(\mathscr{A})$.

However, how to interpret the resulting $A_\mathrm{c}x_\mathrm{c}$ is not so unambiguous. The conventional[2] point of view is that it is again a vector of Legendre coordinates that represents a polynomial of degree N. In this continuous interpretation, taking into account how A_c is built, $A_\mathrm{c}x_\mathrm{c}$ represents $\mathrm{Proj}_\mathrm{cont}^{[N]}\, \mathscr{A}\, (\mathrm{Proj}_\mathrm{cont}^{[N]}\mathsf{x})$, which we denote by $\mathscr{A}_\mathrm{cont}^{[N]}\mathsf{x}$,

$$\mathscr{A}_\mathrm{cont}^{[N]} = \mathrm{Proj}_\mathrm{cont}^{[N]}\, \mathscr{A}\, \mathrm{Proj}_\mathrm{cont}^{[N]}, \tag{17}$$

cf. (Ito and Teglas, 1987, eq. 3.13).

Alternatively, $A_\mathrm{c}x_\mathrm{c}$ can be seen as representing a polynomial and a discontinuous end point as described in (7). In this discontinuous interpretation, $A_\mathrm{c}x_\mathrm{c}$ is a coordinate representation of $\mathscr{A}_\mathrm{d}^{[N]}\mathsf{x}$, with

$$\mathscr{A}_\mathrm{d}^{[N]} = \mathrm{Proj}_\mathrm{d}^{[N]}\, \mathscr{A}\, \mathrm{Proj}_\mathrm{cont}^{[N]}, \tag{18}$$

cf. (Ito and Teglas, 1987, eq. 3.15). Notably, we even have the following result.

Theorem 2.1. The outer projection in (18) has no effect,

$$\mathscr{A}_\mathrm{d}^{[N]} = \mathscr{A}\, \mathrm{Proj}_\mathrm{cont}^{[N]}. \tag{19}$$

Proof. The inner projection $\mathrm{Proj}_\mathrm{cont}^{[N]} \begin{bmatrix} \phi \\ \phi(0) \end{bmatrix} =: \begin{bmatrix} \phi^{[N]} \\ \phi^{[N]}(0) \end{bmatrix}$ in (19) comes with a function $\phi^{[N]}$ that is a polynomial of degree at most N. Therefore, the first component $(\phi^{[N]})'$ in $\mathscr{A} \begin{bmatrix} \phi^{[N]} \\ \phi^{[N]}(0) \end{bmatrix} = \begin{bmatrix} (\phi^{[N]})' \\ A_0\phi^{[N]}(0) + A_1\phi^{[N]}(-h) \end{bmatrix}$ is already a polynomial of degree at most $N-1$, i.e., $c^k = 0$ if $k \geq N$ in (2). Thus, $\mathrm{Proj}_\mathrm{d}^{[N]}$ from (8), relying on (7) and (4), is an identity. □

[2] Lanczos' variable τ that gave the tau method its name is based on this conventional interpretation, see (Scholl, 2024, Rem. 3.6.3).

Remark 2.1. Such a result would not be possible in a collocation method. As, in contrast to (13), the last row of the differentiation matrix in interpolation coordinates[3] is nonzero, the (similarly to above) conducted replacement of that row induces some error in the derivative.

3. A TENSORIAL VIEW

Henceforth, for notational simplicity, only the scalar case $n = 1$ is considered. However, a generalization to vector-valued functions is straightforward[4]. Before being able to state the main results in Section 3.5, some preliminaries on the involved basis functions have to be discussed.

3.1 Basis Functions of Legendre Coordinates

The continuous interpretation of Legendre coordinates $x_\mathrm{c} \in \mathbb{R}^{N+1}$ from (3) gives rise to the function-value pair

$$\begin{bmatrix} \phi^{[N]}(\cdot) \\ \phi^{[N]}(0) \end{bmatrix} = \sum_{k=0}^{N} x_\mathrm{c}^k \begin{bmatrix} p_k(\vartheta(\cdot)) \\ p_k(\vartheta(0)) \end{bmatrix} =: \sum_{k=0}^{N} x_\mathrm{c}^k \mathsf{g}_{\mathrm{c},k}, \tag{20}$$

where, by $p_k(\vartheta(0)) = p_k(1) = 1$, the basis functions are

$$\mathsf{g}_{\mathrm{c},0} = \begin{bmatrix} p_0(\vartheta(\cdot)) \\ 1 \end{bmatrix}, \quad \ldots, \quad \mathsf{g}_{\mathrm{c},N} = \begin{bmatrix} p_N(\vartheta(\cdot)) \\ 1 \end{bmatrix}. \tag{21}$$

In contrast, the discontinuous interpretation from (7)

$$\begin{bmatrix} \bar{\phi}_\mathrm{d}^{[N]}(\cdot) \\ \phi_\mathrm{d}^{[N]}(0) \end{bmatrix} = \sum_{k=0}^{N-1} x_\mathrm{c}^k \begin{bmatrix} p_k(\vartheta(\cdot)) \\ 1 \end{bmatrix} + x_\mathrm{c}^N \begin{bmatrix} 0 \\ 1 \end{bmatrix} =: \sum_{k=0}^{N} x_\mathrm{c}^k \mathsf{h}_{\mathrm{c},k} \tag{22}$$

amounts to the basis functions

$$\mathsf{h}_{\mathrm{c},0} = \begin{bmatrix} p_0(\vartheta(\cdot)) \\ 1 \end{bmatrix}, \ \ldots, \ \mathsf{h}_{\mathrm{c},N-1} = \begin{bmatrix} p_{N-1}(\vartheta(\cdot)) \\ 1 \end{bmatrix}, \ \mathsf{h}_{\mathrm{c},N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{23}$$

We denote the spanned subspaces of M_2 by

$$V_\mathsf{g}^{[N]} = \mathrm{span}(\{\mathsf{g}_{\mathrm{c},0}, \ldots, \mathsf{g}_{\mathrm{c},N}\}) \subset M_2, \tag{24a}$$
$$V_\mathsf{h}^{[N]} = \mathrm{span}(\{\mathsf{h}_{\mathrm{c},0}, \ldots, \mathsf{h}_{\mathrm{c},N}\}) \subset M_2. \tag{24b}$$

Note that $M_2 = L_2 \times \mathbb{R}^n$ is naturally equipped with the inner product $\left\langle \begin{bmatrix} \phi_1 \\ r_1 \end{bmatrix}, \begin{bmatrix} \phi_2 \\ r_2 \end{bmatrix} \right\rangle_{M_2} = \langle \phi_1, \phi_2 \rangle_{L_2} + \langle r_1, r_2 \rangle_{\mathbb{R}^n}$.

Definition 3.1. (Metric coefficients). Let $\{\mathsf{g}_0, \ldots, \mathsf{g}_N\}$ be the basis of a subspace of M_2. The metric coefficients G_{jk}, with $j, k \in \{0, \ldots, N\}$, of that basis are

$$G_{jk} = \langle \mathsf{g}_j, \mathsf{g}_k \rangle_{M_2}. \tag{25}$$

For notational compactness, let

$$\mu_k := \frac{h}{2} \frac{2}{2k+1}, \quad k \in \{0, \ldots, N\}. \tag{26}$$

Lemma 3.1.
The basis $\{\mathsf{g}_{\mathrm{c},0}, \ldots, \mathsf{g}_{\mathrm{c},N}\}$ of $V_\mathsf{g}^{[N]}$ has the metric coefficients

$$G_\mathrm{c} = \mathrm{diag}([\mu_0, \ldots, \mu_{(N-1)}, \mu_N]) + \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}; \tag{27}$$

the basis $\{\mathsf{h}_{\mathrm{c},0}, \ldots, \mathsf{h}_{\mathrm{c},N}\}$ of $V_\mathsf{h}^{[N]}$ has the metric coefficients

$$H_\mathrm{c} = \mathrm{diag}([\mu_0, \ldots, \mu_{(N-1)}, 0]) + \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix}. \tag{28}$$

Proof.

$$\begin{aligned} G_{\mathrm{c},jk} &= \langle \mathsf{g}_{\mathrm{c},j}, \mathsf{g}_{\mathrm{c},k} \rangle_{M_2} \\ &= \underbrace{\langle p_j(\vartheta(\cdot)), p_k(\vartheta(\cdot)) \rangle_{L_2([-h,0],\mathbb{R})}}_{=:m_{jk}} + \underbrace{p_j(\vartheta(0))}_{1} \underbrace{p_k(\vartheta(0))}_{1} \end{aligned}$$

[3] see (Scholl, 2024, Rem. 3.6.2)

[4] Essentially, if $n > 1$, matrices of the size $(N+1) \times (N+1)$ must be expanded by a Kronecker product with I_n as in (14). Moreover, indices $j, k \in \{0, \ldots, N\}$ then refer to submatrices or subvectors like $x_c^k \in \mathbb{R}^n$ in $x_c \in \mathbb{R}^{n(N+1)}$ from (3).

with $m_{jk} = \int_{-h}^{0} p_j(\vartheta(\theta)) p_k(\vartheta(\theta))\,\mathrm{d}\theta = \mu_k$ if $j = k$ and zero otherwise. Analogously, $H_{\mathrm{c},jk} = \langle \mathsf{h}_{\mathrm{c},j}, \mathsf{h}_{\mathrm{c},k} \rangle_{M_2}$. □

3.2 Basis Functions of Mixed Coordinates

Consider the coordinate transformation

$$x_\chi = \begin{bmatrix} x_\mathrm{c}^0 \\ \vdots \\ x_\mathrm{c}^{N-1} \\ \hat{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 1 & \cdots & 1 & 1 \end{bmatrix}}_{T_{\chi\mathrm{c}}} x_\mathrm{c}, \qquad x_\mathrm{c} = \underbrace{\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ -1 & \cdots & -1 & 1 \end{bmatrix}}_{T_{\mathrm{c}\chi} = T_{\chi\mathrm{c}}^{-1}} x_\chi \tag{29}$$

($n = 1$ in this section). That is, (20) and (22) become

$$\sum_{k=0}^{N} x_\mathrm{c}^k \mathsf{g}_{\mathrm{c},k} = \sum_{k=0}^{N} x_\chi^k \mathsf{g}_{\chi,k}, \quad \sum_{k=0}^{N} x_\mathrm{c}^k \mathsf{h}_{\mathrm{c},k} = \sum_{k=0}^{N} x_\chi^k \mathsf{h}_{\chi,k}. \tag{30}$$

Lemma 3.2. If the coordinates are related by $x_\chi = T_{\chi\mathrm{c}} x_\mathrm{c}$, then the corresponding basis functions are related by $T_{\mathrm{c}\chi} = T_{\chi\mathrm{c}}^{-1}$ via

$$[\mathsf{g}_{\chi,0} \ \cdots \ \mathsf{g}_{\chi,N}] = [\mathsf{g}_{\mathrm{c},0} \ \cdots \ \mathsf{g}_{\mathrm{c},N}]\, T_{\mathrm{c}\chi}, \tag{31}$$
$$[\mathsf{h}_{\chi,0} \ \cdots \ \mathsf{h}_{\chi,N}] = [\mathsf{h}_{\mathrm{c},0} \ \cdots \ \mathsf{h}_{\mathrm{c},N}]\, T_{\mathrm{c}\chi}. \tag{32}$$

Proof. Note that (30) can be written as
$[\mathsf{g}_{\mathrm{c},0} \ \cdots \ \mathsf{g}_{\mathrm{c},N}] \underbrace{x_\mathrm{c}}_{T_{\mathrm{c}\chi} x_\chi} = [\mathsf{g}_{\chi,0} \ \cdots \ \mathsf{g}_{\chi,N}]\, x_\chi$. □

By (31), the basis that is associated with the mixed coordinates x_χ in the continuous interpretation is

$$\mathsf{g}_{\chi,0} = \mathsf{g}_{\mathrm{c},0} - \mathsf{g}_{\mathrm{c},N} = \left[\begin{smallmatrix} p_0(\vartheta(\cdot)) - p_N(\vartheta(\cdot)) \\ 0 \end{smallmatrix}\right], \quad \ldots, \tag{33}$$
$$\mathsf{g}_{\chi,N-1} = \mathsf{g}_{\mathrm{c},N-1} - \mathsf{g}_{\mathrm{c},N} = \left[\begin{smallmatrix} p_{N-1}(\vartheta(\cdot)) - p_N(\vartheta(\cdot)) \\ 0 \end{smallmatrix}\right],$$
$$\mathsf{g}_{\chi,N} = \mathsf{g}_{\mathrm{c},N} = \left[\begin{smallmatrix} p_N(\vartheta(\cdot)) \\ 1 \end{smallmatrix}\right],$$

and, by (32), in the discontinuous interpretation

$$\mathsf{h}_{\chi,0} = \mathsf{h}_{\mathrm{c},0} - \mathsf{h}_{\mathrm{c},N} = \left[\begin{smallmatrix} p_0(\vartheta(\cdot)) \\ 0 \end{smallmatrix}\right], \quad \ldots, \tag{34}$$
$$\mathsf{h}_{\chi,N-1} = \mathsf{h}_{\mathrm{c},N-1} - \mathsf{h}_{\mathrm{c},N} = \left[\begin{smallmatrix} p_{N-1}(\vartheta(\cdot)) \\ 0 \end{smallmatrix}\right],$$
$$\mathsf{h}_{\chi,N} = \mathsf{h}_{\mathrm{c},N} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right].$$

Lemma 3.3.
The basis $\{\mathsf{g}_{\chi,0}, \ldots, \mathsf{g}_{\chi,N}\}$ of $V_\mathsf{g}^{[N]}$ has the metric coefficients

$$G_\chi = \mathrm{diag}([\mu_0, \cdots, \mu_{(N-1)}, 1]) + \mu_N \begin{bmatrix} 1 & \cdots & 1 & -1 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 1 & -1 \\ -1 & \cdots & -1 & 1 \end{bmatrix}; \tag{35}$$

the basis $\{\mathsf{h}_{\chi,0}, \ldots, \mathsf{h}_{\chi,N}\}$ of $V_\mathsf{h}^{[N]}$ has the metric coefficients

$$H_\chi = \mathrm{diag}([\mu_0, \ldots, \mu_{(N-1)}, 1]). \tag{36}$$

Proof.[5] $G_{\chi,jk} = \langle \mathsf{g}_{\chi,j}, \mathsf{g}_{\chi,k} \rangle_{M_2}$ and $H_{\chi,jk} = \langle \mathsf{h}_{\chi,j}, \mathsf{h}_{\chi,k} \rangle_{M_2}$. □

As a result, whenever the discontinuous coordinate interpretation is of interest, the mixed coordinates x_χ from (29) are advantageous since the matrix of metric coefficients H_χ is diagonal, and thus $\{\mathsf{h}_{\chi,0}, \ldots, \mathsf{h}_{\chi,N}\}$ is an orthogonal (not yet orthonormal) basis of $V_\mathsf{h}^{[N]}$. Still, it should be noted that the system matrix from the Legendre tau method (Sec. 2.3), is, due to (14), naturally derived in Legendre coordinates. In particular, A_χ for mixed coordinates in

$$\dot{x}_\chi = A_\chi x_\chi \qquad \text{with } A_\chi = T_{\chi\mathrm{c}}\, A_\mathrm{c}\, T_{\mathrm{c}\chi}, \tag{37}$$

which is based on (29), is more dense than A_c from (15).

[5] Alternatively, note that $G_\chi = T_{\mathrm{c}\chi}^\top G_\mathrm{c} T_{\mathrm{c}\chi}$ and $H_\chi = T_{\mathrm{c}\chi}^\top H_\mathrm{c} T_{\mathrm{c}\chi}$ since, by (38), $\langle \mathsf{a}, \mathsf{b} \rangle_{M_2} = a_\mathrm{c}^\top G_\mathrm{c} b_\mathrm{c} = (T_{\mathrm{c}\chi} a_\chi)^\top G_\mathrm{c} (T_{\mathrm{c}\chi} b_\chi)$.

3.3 Inner Product and Dual Bases

For convenience, consider exemplarily $V_\mathsf{g}^{[N]}$. Since all of the relations below hold for any choice of basis, an indication of the basis like 'c' or 'χ' is henceforth omitted

$$\mathsf{b} = \sum_{k=0}^{N} b_\mathrm{c}^k \mathsf{g}_{\mathrm{c},k} = \sum_{k=0}^{N} b_\chi^k \mathsf{g}_{\chi,k} = \ldots \quad \xrightarrow{\text{notation}} \quad \mathsf{b} = \sum_{k=0}^{N} b^k \mathsf{g}_k.$$

The following lemmas are standard results in tensor algebra, see, e.g., Lichnerowicz (1962). We still note the single-line proofs to demonstrate how to handle elements of $V_\mathsf{g}^{[N]}$.

The metric coefficients, which are derived in the former sections, are needed when computing inner products.

Lemma 3.4. If $\mathsf{a} = \sum_{k=0}^{N} a^k \mathsf{g}_k$ and $\mathsf{b} = \sum_{k=0}^{N} b^k \mathsf{g}_k$, then

$$\langle \mathsf{a}, \mathsf{b} \rangle_{M_2} = a^\top G\, b, \qquad a = \begin{bmatrix} a^0 \\ \vdots \\ a^N \end{bmatrix}, \quad b = \begin{bmatrix} b^0 \\ \vdots \\ b^N \end{bmatrix}, \tag{38}$$

where G is the matrix of metric coefficients (25).

Proof.
$$\left\langle \sum_{j=0}^{N} a^j \mathsf{g}_j, \sum_{k=0}^{N} b^k \mathsf{g}_k \right\rangle_{M_2} = \sum_{j=0}^{N} \sum_{k=0}^{N} a^j \overbrace{\langle \mathsf{g}_j, \mathsf{g}_k \rangle_{M_2}}^{G_{jk}} b^k.$$
□

The situation is different if one of the functions is represented in terms of the dual basis functions [6] [7].

Definition 3.2. (Dual basis functions). Let $\{\mathsf{g}_0, \ldots, \mathsf{g}_N\}$ be a basis of an $(N+1)$-dimensional subspace $V_\mathsf{g}^{[N]} \subset M_2$. The dual basis functions $\underline{\mathsf{g}}^k \in V_\mathsf{g}^{[N]}$ are uniquely defined by requiring that $\forall j, k \in \{0, \ldots, N\}$:

$$\langle \mathsf{g}_j, \underline{\mathsf{g}}^k \rangle_{M_2} = \delta_j^k, \qquad \delta_j^k \overset{\text{def}}{=} \delta_{jk} = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k. \end{cases} \tag{39}$$

Any function $\mathsf{b} \in V_\mathsf{g}^{[N]}$ can equally well be represented in the original basis or in the dual basis, for which the coordinates in turn are denoted by lower indices

$$\mathsf{b} = \sum_{k=0}^{N} b^k \mathsf{g}_k = \sum_{k=0}^{N} \underline{b}_k \underline{\mathsf{g}}^k. \tag{40}$$

Lemma 3.5. Let $\mathsf{a} = \sum_{k=0}^{N} a^k \mathsf{g}_k$ and $\mathsf{b} = \sum_{k=0}^{N} \underline{b}_k \underline{\mathsf{g}}^k$. Then

$$\langle \mathsf{a}, \mathsf{b} \rangle_{M_2} = a^\top\, \underline{b}, \qquad a = \begin{bmatrix} a^0 \\ \vdots \\ a^N \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} \underline{b}_0 \\ \vdots \\ \underline{b}_N \end{bmatrix}. \tag{41}$$

Proof. Note that $\sum_k \delta_j^k\, \underline{b}_k = \underline{b}_j$ in

$$\left\langle \sum_j a^j \mathsf{g}_j, \sum_k \underline{b}_k \underline{\mathsf{g}}^k \right\rangle_{M_2} = \sum_j \sum_k a^j \overbrace{\langle \mathsf{g}_j, \underline{\mathsf{g}}^k \rangle_{M_2}}^{\delta_j^k} \underline{b}_k = \sum_j a^j \underline{b}_j.$$
□

Comparing (38) and (41) already shows that the coordinates b and $\underline{b}$ in (40) are related by

Lemma 3.6. $\underline{b} = Gb.$

Moreover, Lemma 3.2 implies

Lemma 3.7. $[\underline{\mathsf{g}}^0 \ \cdots \ \underline{\mathsf{g}}^N] = [\mathsf{g}_0 \ \cdots \ \mathsf{g}_N]\, G^{-1}.$

[6] With a slight abuse of nomenclature, the term dual basis will henceforth refer to $\{\underline{\mathsf{g}}^0, \ldots, \underline{\mathsf{g}}^N\}$, although the latter is a basis of the original space V rather than a basis of the dual space V^*, built from linear functionals—which, however, is isomorphic to V.

[7] In a tensor calculus setting, the original and the dual basis are only distinguished by the index position. We use an additional underline.

All of the above holds equally well for $V_{\mathsf{h}}^{[N]}$ from the discontinuous interpretation. That is, $\sum_{k=0}^{N} b^k \mathsf{h}_k = \sum_{k=0}^{N} \underline{b}_k \underline{\mathsf{h}}^k$, with $\langle \mathsf{h}_j, \underline{\mathsf{h}}^k \rangle_{M_2} = \delta_j^k$. These dual basis functions

$$\left[\underline{\mathsf{h}}^0 \ \cdots \ \underline{\mathsf{h}}^N\right] = \left[\mathsf{h}_0 \ \cdots \ \mathsf{h}_N\right] H^{-1} \tag{42}$$

become particularly simple in mixed coordinates, where

$$\underline{\mathsf{h}}_\chi^k = \left[\begin{smallmatrix} \frac{1}{\mu_k} p_k(\vartheta(\cdot)) \\ 0 \end{smallmatrix}\right] \text{ if } k \le N-1, \quad \text{and } \underline{\mathsf{h}}_\chi^N = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \tag{43}$$

due to (36) and (34).

3.4 Operators on Polynomial Subspaces

Consider a linear operator $\mathscr{L} : V_{\mathsf{g}}^{[N]} \to V_{\mathsf{h}}^{[N]}$ that maps a function $\mathsf{x} \in V_{\mathsf{g}}^{[N]}$ to a function $\mathscr{L}\mathsf{x} = \mathsf{y} \in V_{\mathsf{h}}^{[N]}$ by

$$\mathsf{x} = \sum_{k=0}^{N} x^k \mathsf{g}_k \ \mapsto \ \mathscr{L}\mathsf{x} := \sum_{j=0}^{N} y^j \mathsf{h}_j \text{ with } y^j = \sum_{k=0}^{N} L^j{}_k \, x^k. \tag{44}$$

The matrix that maps the coordinate representation of x to the coordinate representation of y in

$$\begin{bmatrix} y^0 \\ \vdots \\ y^N \end{bmatrix} = \begin{bmatrix} L^0{}_0 & \cdots & L^0{}_N \\ \vdots & & \vdots \\ L^N{}_0 & \cdots & L^N{}_N \end{bmatrix} \begin{bmatrix} x^0 \\ \vdots \\ x^N \end{bmatrix} \tag{45}$$

is a coordinate representation of that operator. Relying on the following definition, the operator can be made explicit.

Definition 3.3. (Dyadic product in M_2). The dyadic product (or tensor product) $\otimes$ of $\mathsf{a}, \mathsf{b} \in M_2$ is defined by

$$(\mathsf{a} \otimes \mathsf{b}) \cdot \overset{\text{def}}{=} \mathsf{a} \, \langle \mathsf{b}, \cdot \rangle_{M_2}. \tag{46}$$

Lemma 3.8. The operator in (44) can be written as

$$\mathscr{L} = \sum_{j=0}^{N} \sum_{k=0}^{N} L^j{}_k \, \mathsf{h}_j \otimes \underline{\mathsf{g}}^k. \tag{47}$$

Proof. $\mathscr{L}\mathsf{x} = \left(\sum_{j,k} L^j{}_k \, \mathsf{h}_j \otimes \underline{\mathsf{g}}^k\right)\left(\sum_l x^l \mathsf{g}_l\right)$
$\overset{(46)}{=} \sum_{j,k,l} L^j{}_k \mathsf{h}_j \underbrace{\langle \underline{\mathsf{g}}^k, \mathsf{g}_l \rangle_{M_2}}_{\delta_l^k} x^l = \sum_{j,k} L^j{}_k x^k \mathsf{h}_j = \sum_j y^j \mathsf{h}_j$. □

When restricted to arguments from the polynomial subspace $V_{\mathsf{g}}^{[N]}$, the operator $\mathscr{A}_{\mathrm{d}}$ from (18) is exactly the type of mapping considered in (44). Notably, when restricted to this polynomial subspace, even the remaining projection in (19) is without effect and $\mathscr{A}$ coincides with its approximation $\mathscr{A}_{\mathrm{d}}^{[N]}$. Altogether, if $A^j{}_k$ is the (j,k)-th entry of the system matrix (given in (15) in terms of Legendre coordinates and in (37) in terms of mixed coordinates), then the restriction of $\mathscr{A}$ to $V_{\mathsf{g}}^{[N]}$, i.e., $\mathscr{A}|_{V_{\mathsf{g}}^{[N]}} : V_{\mathsf{g}}^{[N]} \to V_{\mathsf{h}}^{[N]}$, can be written as

$$\mathscr{A}|_{V_{\mathsf{g}}^{[N]}} = \sum_{j=0}^{N} \sum_{k=0}^{N} A^j{}_k \, \mathsf{h}_j \otimes \underline{\mathsf{g}}^k. \tag{48}$$

The second type of operators we are interested in are operators in bilinear forms. Consider two functions $\mathsf{z}, \mathsf{y} \in V_{\mathsf{h}}^{[N]}$, represented by the coordinates z and y in

$$\mathsf{z} = \sum_{k=0}^{N} z^k \, \mathsf{h}_k \text{ and } \mathsf{y} = \sum_{k=0}^{N} y^k \, \mathsf{h}_k. \tag{49}$$

Let the bilinear form

$$(\mathsf{z}, \mathsf{y}) \quad \mapsto \quad \langle \mathsf{z}, \mathscr{W}\mathsf{y} \rangle_{M_2} := \sum_{j=0}^{N} \sum_{k=0}^{N} z^j W_{jk} y^k \tag{50}$$

map these two functions to the scalar value

$$\sum_{j=0}^{N} \sum_{k=0}^{N} z^j W_{jk} y^k = \begin{bmatrix} z^0 \\ \vdots \\ z^N \end{bmatrix}^\top \begin{bmatrix} W_{00} & \cdots & W_{0N} \\ \vdots & & \vdots \\ W_{N0} & \cdots & W_{NN} \end{bmatrix} \begin{bmatrix} y^0 \\ \vdots \\ y^N \end{bmatrix}. \tag{51}$$

The involved matrix is a coordinate representation of $\mathscr{W} : V_{\mathsf{h}}^{[N]} \to V_{\mathsf{h}}^{[N]}$ in the following sense.

Lemma 3.9. The operator in (50) can be written as

$$\mathscr{W} = \sum_{j=0}^{N} \sum_{k=0}^{N} W_{jk} \, \underline{\mathsf{h}}^j \otimes \underline{\mathsf{h}}^k. \tag{52}$$

Proof. $\underline{w}_j := \sum_k W_{jk} y^k$ are the coordinates of $\mathscr{W}\mathsf{y} = \left(\sum_{j,k} W_{jk} \underline{\mathsf{h}}^j \otimes \underline{\mathsf{h}}^k\right)\left(\sum_l y^l \mathsf{h}_l\right) \overset{(46)}{=} \sum_{j,k,l} W_{jk} \underline{\mathsf{h}}^j \langle \underline{\mathsf{h}}^k, \mathsf{h}_l \rangle_{M_2} y^l$ $= \sum_{j,k} W_{jk} y^k \underline{\mathsf{h}}^j = \sum_j \underline{w}_j \underline{\mathsf{h}}^j$ in the dual basis (in contrast to y^j from $\mathscr{L}\mathsf{x}$ in (47)). As a consequence, the inner product in (50) does not introduce additional metric coefficients but, cf. (41), $\langle \mathsf{z}, \mathscr{W}\mathsf{y} \rangle_{M_2} = \left\langle \sum_j z^j \mathsf{h}_j, \sum_k \underline{w}_k \underline{\mathsf{h}}^k \right\rangle_{M_2} =$ $\sum_{j,k} z^j \langle \mathsf{h}_j, \underline{\mathsf{h}}^k \rangle_{M_2} \underline{w}_k = \sum_j z^j \underline{w}_j$ as desired in (50). □

3.5 A Tensorial Description of the Operators on $L_2 \times \mathbb{R}^n$

So far only operators on the finite-dimensional subspaces $V_{\mathsf{g}}^{[N]}$ or $V_{\mathsf{h}}^{[N]}$ from (24) have been taken into account. The combination with a projection operator establishes the extension to the overall space M_2.

Lemma 3.10. The operator $\mathrm{Proj}_{\mathrm{cont}}^{[N]} : M_2 \to V_{\mathsf{g}}^{[N]}$ defined in (6) can be written as

$$\mathrm{Proj}_{\mathrm{cont}}^{[N]} = \sum_{j=0}^{N} \sum_{k=0}^{N} \delta^j{}_k \, \mathsf{g}_j \otimes \underline{\mathsf{h}}^k = \sum_{j=0}^{N} \mathsf{g}_j \otimes \underline{\mathsf{h}}^j. \tag{53}$$

Proof. Consider $\left[\begin{smallmatrix} \phi \\ \phi(0) \end{smallmatrix}\right] = \sum_{l=0}^{\infty} c^l \mathsf{g}_{\mathrm{c},l} \in M_2$, cf. (2), with $\mathsf{g}_{\mathrm{c},l} = \left[\begin{smallmatrix} p_l(\vartheta(\cdot)) \\ 1 \end{smallmatrix}\right]$. Using mixed coordinates in (53) gives

$$\mathrm{Proj}_{\mathrm{cont}}^{[N]} \left[\begin{smallmatrix} \phi \\ \phi(0) \end{smallmatrix}\right] = \left(\sum_{j=0}^{N} \mathsf{g}_{\chi,j} \otimes \underline{\mathsf{h}}_\chi^j\right) \sum_{l=0}^{\infty} c^l \mathsf{g}_{\mathrm{c},l}$$

with $\otimes$ defined in (46). By (43), it is seen that $\langle \underline{\mathsf{h}}_\chi^j, \mathsf{g}_{\mathrm{c},l} \rangle_{M_2} = \delta_l^j$ if $j \le N-1$, whereas $\langle \underline{\mathsf{h}}_\chi^N, \mathsf{g}_{\mathrm{c},l} \rangle_{M_2} = 1$ for all l. Thus,

$$\mathrm{Proj}_{\mathrm{cont}}^{[N]} \left[\begin{smallmatrix} \phi \\ \phi(0) \end{smallmatrix}\right] = \sum_{j=0}^{N-1} c^j \mathsf{g}_{\chi,j} + \underbrace{\left(\sum_{l=0}^{\infty} c^l\right)}_{=\phi(0)} \mathsf{g}_{\chi,N} = \sum_{j=0}^{N} x_\chi^j \mathsf{g}_{\chi,j}$$

with $x_\chi^j = c^j = x_{\mathrm{c}}^j$ and $x_\chi^N = \phi(0) = \hat{x}$ as in (4) and (29). □

The operator $\mathrm{Proj}_{\mathrm{cont}}^{[N]}$ in (53) is a projector on $V_{\mathsf{g}}^{[N]}$, i.e., $\mathrm{Proj}_{\mathrm{cont}}^{[N]} \mathsf{a} = \mathsf{a}$ if $\mathsf{a} = \sum_{k=0}^{N} a^k \mathsf{g}_k$. The decisive point is that $\{\underline{\mathsf{h}}^0, \ldots, \underline{\mathsf{h}}^N\}$ in (53) also satisfies the orthogonality relation

$$\forall j, k \in \{0, \ldots, N\} : \qquad \langle \underline{\mathsf{h}}^j, \mathsf{g}_k \rangle_{M_2} = \delta_k^j, \tag{54}$$

which on the subspace $V_{\mathsf{g}}^{[N]} \subset M_2$ was only satisfied by the dual basis functions $\{\underline{\mathsf{g}}^0, \ldots, \underline{\mathsf{g}}^N\}$ from (39).

Based on Lemma 3.10, we can make explicit in which sense the system matrix from the Legendre tau method is a coordinate representation of operators that approximate $\mathscr{A}$.

Theorem 3.1. The operators $\mathscr{A}_{\mathrm{d}}^{[N]} \colon M_2 \to V_{\mathsf{h}}^{[N]}$ from (19) and $\mathscr{A}_{\mathrm{cont}}^{[N]} \colon M_2 \to V_{\mathsf{g}}^{[N]}$ from (17) can be written as

$$\mathscr{A}_{\mathrm{d}}^{[N]} = \sum_{j=0}^{N}\sum_{k=0}^{N} A^j_{\ k}\ \mathsf{h}_j \otimes \underline{\mathsf{h}}^k, \quad \mathscr{A}_{\mathrm{cont}}^{[N]} = \sum_{j=0}^{N}\sum_{k=0}^{N} A^j_{\ k}\ \mathsf{g}_j \otimes \underline{\mathsf{h}}^k, \tag{55}$$

where $A^j_{\ k}$ is the (j,k)-th entry of the system matrix.

Proof. By (19), (48), and (53), $\mathscr{A}_{\mathrm{d}}^{[N]} = \mathscr{A}|_{V_{\mathsf{g}}^{[N]}}\mathrm{Proj}_{\mathrm{cont}}^{[N]} = (\sum_{j,k} A^j_{\ k}\ \mathsf{h}_j \otimes \underline{\mathsf{g}}^k)(\sum_l \mathsf{g}_l \otimes \underline{\mathsf{h}}^l) = \sum_{j,k,l} A^j_{\ k}\langle \underline{\mathsf{g}}^k, \mathsf{g}_l\rangle_{M_2}\mathsf{h}_j \otimes \underline{\mathsf{h}}^l = \sum_{j,k,l} A^j_{\ k}\delta^k_l \mathsf{h}_j \otimes \underline{\mathsf{h}}^l$. Moreover, by (19), $\mathscr{A}_{\mathrm{cont}}^{[N]} = \mathrm{Proj}_{\mathrm{cont}}^{[N]}\mathscr{A}_{\mathrm{d}}^{[N]}$, which is treated analogously. □

The projection on $V_{\mathsf{h}}^{[N]}$ from the discontinuous interpretation can be described in the same manner as Lemma 3.10.

Lemma 3.11. $\mathrm{Proj}_{\mathrm{d}}^{[N]} : M_2 \to V_{\mathsf{h}}^{[N]}$ from (8) is

$$\mathrm{Proj}_{\mathrm{d}}^{[N]} = \sum_{j=0}^{N}\sum_{k=0}^{N} \delta^j_{\ k}\mathsf{h}_j \otimes \underline{\mathsf{h}}^k = \sum_{j=0}^{N} \mathsf{h}_j \otimes \underline{\mathsf{h}}^j. \tag{56}$$

Finally, consider the approximation of a Lyapunov–Krasovskii functional $V(\phi)$ via a matrix P in

$$V(\phi) = \left\langle \begin{bmatrix} \phi(\cdot) \\ \phi(0) \end{bmatrix}, \mathscr{P} \begin{bmatrix} \phi(\cdot) \\ \phi(0) \end{bmatrix} \right\rangle_{M_2} \approx x^\top P x, \tag{57}$$

where $x = [x^0, \dots, x^N]^\top$ stems from the discretization (4) or from a coordinate transformation thereof. The involved matrix P is the coordinate representation of an operator $\mathscr{P}^{[N]}$ in the following sense.

Theorem 3.2. The quadratic form of discretization coordinates in (57) can be written as

$$x^\top P x = \left\langle \begin{bmatrix} \phi(\cdot) \\ \phi(0) \end{bmatrix}, \mathscr{P}^{[N]} \begin{bmatrix} \phi(\cdot) \\ \phi(0) \end{bmatrix} \right\rangle_{M_2} \tag{58}$$

with the operator $\mathscr{P}^{[N]} : M_2 \to M_2$,

$$\mathscr{P}^{[N]} = \sum_{j=0}^{N}\sum_{k=0}^{N} P_{jk}\ \underline{\mathsf{h}}^j \otimes \underline{\mathsf{h}}^k. \tag{59}$$

Proof. Using $W = P$ in (52), $\mathscr{P}^{[N]} = \mathrm{Proj}_{\mathrm{d}}^{[N]*}\mathscr{W}\mathrm{Proj}_{\mathrm{d}}^{[N]}$. □

Remark 3.1. Note that (58) becomes an integral formula with the typical structure of quadratic Lyapunov–Krasovskii functionals $V(\phi)$ (Scholl et al., 2024b, eq. 46) by writing out the L_2 inner products from (46), using (43) in (59). However, $x^\top P x$ is much easier to handle.

3.6 Conclusion

At the core of tensor calculus stands the idea that coordinates can be transformed arbitrarily via a change of basis, while the represented object remains the same. This is not only true in a geometrical setting but also for polynomials. The presented framework enables a concise coordinate-based treatment of functions and operators that result from projections to polynomial subspaces of M_2, independently from the choice of the polynomial basis. In Thm. 3.1, the sense by which the respective Legendre-tau-based system matrix is a coordinate representation of an operator $\mathscr{A}_{\mathrm{d}}^{[N]}$ on M_2 is made explicit. The latter has been recognized to be related to the exact infinitesimal generator $\mathscr{A}$ by no more than $\mathscr{A}_{\mathrm{d}}^{[N]} = \mathscr{A}\,\mathrm{Proj}_{\mathrm{cont}}^{[N]}$. Finally, it is shown in which sense the matrix in a quadratic form is a coordinate representation of an operator $\mathscr{P}^{[N]}$ on M_2. This understanding is decisive for Legendre-tau-based approximations of Lyapunov–Krasovskii functionals.

The presented framework of tensorial representations is helpful in various further considerations. For instance,

- to translate an operator-valued equation to a matrix-valued equation (in particular, to show that besides of (58), also $\langle \mathsf{x}, \mathscr{P}^{[N]}\mathscr{A}_{\mathrm{d}}^{[N]}\mathsf{x}\rangle_{M_2} = x^\top P A x$ holds);
- to compute adjoint operators and to conclude what self-adjointness in terms of the matrices means;
- to compute the operator norm $\|\mathscr{P}^{[N]}\|$ via P;

see (Scholl, 2024, Appendix A) for details.

REFERENCES

Breda, D., Diekmann, O., Gyllenberg, M., Scarabel, F., and Vermiglio, R. (2016). Pseudospectral discretization of nonlinear delay equations. *SIAM J. Appl. Dyn. Syst.*, 15(1), 1–23.

Breda, D., Maset, S., and Vermiglio, R. (2005). Pseudospectral differencing methods for characteristic roots of delay differential equations. *SIAM J. Sci. Comput.*, 27(2), 482–495.

Curtain, R. and Zwart, H. (2020). *Introduction to Infinite-Dimensional Systems Theory*. Springer, New York.

Hale, J.K. and Sternberg, N. (1988). Onset of chaos in differential delay equations. *J. Comput. Phys.*, 77(1), 221–239.

Hesthaven, J.S., Gottlieb, S., and Gottlieb, D. (2007). *Spectral methods for time-dependent problems*. Cambridge University Press, Cambridge.

Ito, K. (1983). The application of Legendre-tau approximation to parameter identification for delay and partial differential equations. In *The 22nd IEEE CDC*, 33–37.

Ito, K. (1985). Legendre-tau approximation for functional differential equations part III. In F. Kappel, K. Kunisch, and W. Schappacher (eds.), *Distributed Parameter Systems*, 191–212. Springer, Berlin.

Ito, K. and Teglas, R. (1987). Legendre-tau approximation for functional differential equations part II. *SIAM J. Control Optim.*, 25(6), 1379–1408.

Lichnerowicz, A. (1962). *Elements of Tensor Calculus*. Wiley, New York, 2nd edition.

Provoost, E. and Michiels, W. (2024). The Lanczos tau framework for time-delay systems: Padé approximation and collocation revisited. *SIAM J. Numer. Anal.*, 62(6), 2529–2548.

Scholl, T.H. (2024). *Stability in time-delay systems*. Ph.D. thesis, Karlsruhe Institute of Technology (KIT), Karlsruhe.

Scholl, T.H., Hagenmeyer, V., and Gröll, L. (2024a). Lyapunov-Krasovskii functionals of robust type and their Legendre-tau-based approximation. *IFAC-PapersOnLine*, 58(27), 219–224.

Scholl, T.H., Hagenmeyer, V., and Gröll, L. (2024b). What ODE-approximation schemes of time-delay systems reveal about Lyapunov–Krasovskii functionals. *IEEE Trans. Automat. Contr.*, 69(7), 4614–4629.