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Sums of Distances from a Point to the Faces of Polyhedra. Generalizations of Viviani's Theorem

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Abstract. Theorems of Viviani and Walser, Vargyas on sums of distances from a point to the sides of polygons are generalized for polyhedra in $\mathbb{R}^n$ ($n \geq 3$).

Key Words: Viviani's theorem, signed distance from a point to lines and planes, polygon, polyhedron, equifacial tetrahedron, insphere radius, Platonic solid, polyhedron in $\mathbb{R}^n$

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1 Introduction

Viviani proved

Theorem 1. *In a triangle, the sum of the distances from an interior point P to the sides of the triangle is constant, i.e., independent of P if and only if the triangle is equilateral.*

The distance from a point P to a straight line (to a plane) is $|\vec{n} \cdot \overrightarrow{PQ}|$ where $\vec{n}$ and Q are a unit normal vector and an arbitrary point of the line (the plane), respectively. In the present paper the restriction to interior points is removed by defining the *signed distance*

$$d = \vec{n} \cdot \overrightarrow{PQ} \tag{1}$$

where $\vec{n}$ is understood to be an *outward* normal unit vector of a polygon or polyhedron. By this definition, d is positive for interior points P of convex polygons and polyhedra. In contrast, for interior points P of nonconvex polygons and polyhedra as well as for exterior points P of (convex or nonconvex) polygons and polyhedra, d is positive or zero or negative depending on the location of P.

Walser and Vargyas [3] proved the more general

Theorem 2. *In a polygon with $m \geq 3$ sides, the sum of the signed distances from a point P to the sides of the polygon is constant, i.e., independent of P, if and only if the sum of the outward normal unit vectors $\vec{n}_k$ $(k = 1, \ldots, m)$ of the sides of the polygon is zero:*

$$\sum_{k=1}^{m} \vec{n}_k = \vec{0}. \tag{2}$$

Kawasaki et al. [2] and Walser [4] proved that in a regular tetrahedron the sum of the signed distances from a point P to the faces of the tetrahedron is constant. Katsuura [1] proved

Theorem 3. *In a tetrahedron, the sum of the signed distances from a point P to the faces of the tetrahedron is constant if and only if the tetrahedron is equifacial.*

The proofs given in [1, 2] and [4] are complicated. In the present paper, Theorems 2 and 3 are shown to be special cases of a general theorem on sums of signed distances from a point to sides of polygons and to faces of polyhedra.

2 Generalization of Theorem 2

Theorem 4. *Let Π be a convex or nonconvex polygon with $m \geq 3$ sides or a convex or nonconvex polyhedron with $m \geq 4$ faces and with outward normal unit vectors $\vec{n}_k$ $(k = 1, \ldots, m)$ of sides (faces). The sum Σ_d of the signed distances from a point P to the m sides (faces) of Π is, independent of P, constant if and only if condition* (2) *is satisfied.*

Proof. Repeating an argument used in [3] it suffices to show that the difference $\Sigma_{d1} - \Sigma_{d2}$ of the sums of signed distances from two points P_1 and P_2, respectively, to the sides (faces) is zero. According to Eq. (1)

$$\Sigma_{d1} - \Sigma_{d2} = \sum_{k=1}^{m} \vec{n}_k \cdot (\overrightarrow{P_1Q_k} - \overrightarrow{P_2Q_k}) = \overrightarrow{P_1P_2} \cdot \sum_{k=1}^{m} \vec{n}_k. \tag{3}$$

The difference is, independent of $\overrightarrow{P_1P_2}$, zero if and only if condition (2) is satisfied. □

Regular m-gons and polygons with $n \geq 2$ pairs of parallel sides are special polygons satisfying condition (2). The five Platonic solids are special polyhedra satisfying condition (2).

The proof technique of Theorem 4 suggests an immediate extension to $\mathbb{R}^n$. Consider a hyperpolyhedron[1] $\Pi \subset \mathbb{R}^n$ formed by $m \geq n+1$ affine hyperplanes. The signed distance d from a point $P \in \mathbb{R}^n$ to a hyperplane is the product $d = \vec{n} \cdot \overrightarrow{PQ}$, where $\vec{n} \in \mathbb{R}^n$ is an outward normal unit vector and Q an arbitrary point of the hyperplane.

Theorem 4.1. *Given a hyperpolyhedron $\Pi \subset \mathbb{R}^n$ formed by $m \geq n+1$ affine hyperplanes and with corresponding outward normal unit vectors $\vec{n}_k \in \mathbb{R}^n$ $(k = 1, \ldots, m)$. The sum Σ_d of the signed distances from an arbitrary point $P \in \mathbb{R}^n$ to the m hyperplanes of Π is, independent of P, constant if and only if condition* (2) *is satisfied.*

[1] A hyperpolyhedron is a closed, bounded, connected subset of $\mathbb{R}^n$ with boundary formed by faces of $m \geq n+1$ intersecting affine hyperplanes.

3 Elementary proofs of Theorems 1 and 3

Proof 1: Let O and ϱ be the center and the radius, respectively, of the incircle of a triangle (of the insphere of a tetrahedron). As before, $m = 3$ ($m = 4$) is the number of sides of the triangle (of faces of the tetrahedron). According to Eq. (1)

$$\Sigma_d = \sum_{k=1}^{m} \vec{n}_k \cdot (\overrightarrow{PO} + \varrho \vec{n}_k) = \overrightarrow{PO} \cdot \sum_{k=1}^{m} \vec{n}_k + m\varrho. \tag{4}$$

The sum is, independent of P, the constant $m\varrho$ if and only if condition (2) is satisfied. The triangle must be equilateral. In what follows it is shown that the tetrahedron must be equifacial, i.e., its faces must be congruent triangles. For better understanding, what follows, an equifacial tetrahedron should be drawn embedded in a rectangular box (a cuboid) such that the six edges of the tetrahedron are diagonals of the six faces of the box (cf. Fig. 1 in [1]). With A, B, C, D being the vertices of the tetrahedron the edge lengths satisfy the conditions $|AB| = |CD|$, $|AC| = |BD|$ and $|AD| = |BC|$. Furthermore, with F being the face area of the tetrahedron, the outward normal unit vectors are given by the equations

$$\overrightarrow{CA} \times \overrightarrow{CB} = 2F\vec{n}_1, \quad \overrightarrow{AC} \times \overrightarrow{AD} = 2F\vec{n}_2, \quad \overrightarrow{DB} \times \overrightarrow{DA} = 2F\vec{n}_3, \quad \overrightarrow{BD} \times \overrightarrow{BC} = 2F\vec{n}_4.$$

With this, (2) is the condition

$$\overrightarrow{CA} \times (\overrightarrow{CB} + \overrightarrow{DA}) + \overrightarrow{DB} \times (\overrightarrow{DA} + \overrightarrow{CB}) = \vec{0} \quad \text{or} \quad (\overrightarrow{CA} + \overrightarrow{DB}) \times (\overrightarrow{DA} + \overrightarrow{CB}) = \vec{0}.$$

This condition is satisfied, since $(\overrightarrow{CA} + \overrightarrow{DB}) = (\overrightarrow{DA} + \overrightarrow{CB})$. □

Proof 2. Let $\vec{r}_k$ ($k = 1, \ldots, m$) and $\vec{r}_P$ be the position vectors of the vertices $V_1, \ldots, V_m$ of the triangle (the tetrahedron) and of P, respectively. Let, furthermore, the side (the face) opposite V_k and its outward normal unit vector be denoted S_k and $\vec{n}_k$, respectively. This has the effect that the vertex V_m is a point common to $S_1, \ldots, S_{m-1}$. According to Eq. (1) the signed distance from P to S_k is

$$d_k = \begin{cases} \vec{n}_k \cdot (\vec{r}_m - \vec{r}_P) & (k = 1, \ldots, m-1) \\ \vec{n}_k \cdot (\vec{r}_1 - \vec{r}_P) = \vec{n}_k \cdot (\vec{r}_m - \vec{r}_P + \vec{r}_1 - \vec{r}_m) & (k = m). \end{cases} \tag{5}$$

Hence the sum of the signed distances is

$$\Sigma_d = (\vec{r}_m - \vec{r}_P) \cdot \sum_{k=1}^{m} \vec{n}_k + \vec{n}_m \cdot (\vec{r}_1 - \vec{r}_m). \tag{6}$$

Σ_d is independent of P if and only if condition (2) is satisfied. □

Equations (4) and (6) yield for the insphere radius of the equifacial tetrahedron the well-known formula (V = volume of the rectangular box)

$$\varrho = -\frac{1}{8F} \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \frac{V}{4F}. \tag{7}$$

Equation (4) is a proof of

Theorem 5. *Let* Π *be a polygon with* $m \geq 3$ *sides tangent to a circle of radius* ϱ *or a polyhedron with* $m \geq 4$ *faces tangent to a sphere of radius* ϱ*. The sum* Σ_d *of the signed distances from an arbitrary point* P *to the* m *sides (faces) of* Π *is the constant* $m\varrho$ *if and only if condition* (2) *is satisfied.*

The five Platonic solids are special polyhedra satisfying the conditions stated in this theorem. With a being the edge length, the sums Σ_d are $(a/3)\sqrt{6} \approx .8165a$ (tetrahedron), $3a$ (cube), $(4a/3)\sqrt{6} \approx 3.266a$ (octahedron), $3a\sqrt{10 + 22/\sqrt{5}} \approx 13.36a$ (dodecahedron) and $5a(\sqrt{3} + \sqrt{5/3}) \approx 15.12a$ (icosahedron).

Theorem 5 can be extended to

Theorem 5.1. *Given a hyperpolyhedron* $\Pi \subset \mathbb{R}^n$ *formed by* $m \geq n+1$ *affine hyperplanes tangent to a hypersphere of radius* ϱ*. The sum* Σ_d *of the signed distances from an arbitrary point* $P \in \mathbb{R}^n$ *to the* m *hyperplanes of* Π *is, independent of* P*, the constant* $m\varrho$ *if and only if condition* (2) *is satisfied.*

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