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Preference diversity

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Abstract
How can we assess the diversity of a group of decision makers? Identifying decision makers with their preferences, we address this question by applying the multi-attribute approach developed by Nehring and Puppe (2002) to sets of preferences. Specifically, we provide a repertoire of alternative models to measure the diversity of sets of preferences. The proposed models are purely ordinal and are characterized in terms of the different properties that a preference order need to satisfy in order to contribute to the diversity of a given set of preference orderings.

Keywords Diversity · Committees · Sets of Preferences

JEL Classification D71

1 Introduction

An obviously relevant and important characteristic of groups of agents is their diversity. Frequently, diversity is considered to be desirable by itself, but even setting the potential intrinsic value of diversity aside, it is widely acknowledged that a group's diversity has important implications on its quality and performance. Presumably, this holds for groups of experts under incomplete information, but it certainly is true if a group needs to serve, among others, purposes of fair representation.

But how to assess, or measure, the diversity of a group of agents? The present paper aims to provide solutions to this problem in the case of *groups of decision makers*, or *committees*, characterized by their preferences over a fixed set of alternatives. We thus follow standard approaches to the modeling of economic agents as being characterized by their preferences.[1] Groups of decision makers are thus characterized by the set of preferences held by their members, and the problem of measuring their diversity

[1] In general equilibrium theory, individual endowments would also be part of the description of agents' characteristics. Differential individual endowments can be integrated in the analysis, but for simplicity we do not include them in our present approach.

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transforms into the problem of measuring the diversity of a set of preferences. To address this problem, we employ the general multi-attribute approach to measuring the diversity of a set of objects developed by Nehring and Puppe (2002).

According to the multi-attribute model, the diversity of a set of objects is given by the number and the weight of the different attributes realized by the set. For instance, potentially relevant attributes of preferences are: 'having alternative x as top element', 'having the set $\{x_1, ..., x_k\}$ as the set of k top alternatives', 'ranking alternative x above y', 'ranking alternative x at least h positions above alternative y' etc. Accordingly, the diversity of a set of preferences, i.e. a group of agents, will depend on whether or not there is an agent who has x as her top alternative, ranks x above y, etc. The main purpose of the present paper is to offer a repertoire of different sub-models of the multi-attribute approach to diversity using these and related attributes, and to provide a 'roadmap' clarifying the logical interrelations between them, thus helping the modeler to choose among them.

Importantly, we do not use the full power of the multi-attribute approach and confine ourselves here to a simple class of *ordinal* comparisons. Specifically, we only ask under which circumstances the addition of a given preference ordering (i.e. the inclusion of an additional agent) enhances the diversity of a given set of preferences (i.e. the diversity of a group of agents), and we characterize families of relevant attributes in terms of these ordinal comparisons. Naturally, we can thus only qualitatively distinguish different sub-models by their ordinal properties and not by their cardinal implications.

The two basic sub-models are the *plurality model* on the one hand, and the *Condorcet model* on the other. The plurality model only looks at the top alternatives and asks how many of them occur in a given group of agents. By contrast, the Condorcet model looks at binary comparisons of alternatives and asks how many different binary comparisons are realized in a given group of agents. We then study various extensions and combinations of these two models. The plurality model is naturally extended by asking which *sets* of alternatives occur at the top of the preference orderings in a group, and the Condorcet model is extended in a natural way by distinguishing the *rank difference* of two alternatives in the preference orderings occurring in a group. In total, we consider and characterize 8 different models and investigate their logical interrelations.

An important limitation of our approach is that we do not address the potential multiplicity of preference orderings occurring in a group of agents. In this respect, we follow the analysis of Nehring and Puppe (2002), and focus on the *existence value* of a certain attribute and neglect the frequency of its occurrence. This is justified either if the group of agents is small in comparison to the underlying population that it is supposed to represent, or by assuming that preferences are approximately equally distributed in the population. But clearly, the neglect of attribute and/or preference frequency, respectively, represents a major limitation of our approach. First steps towards the inclusion of frequency information in the multi-attribute approach have been taken by Nehring and Puppe (2009), and it would certainly be worthwhile to incorporate the relevant considerations in the context of preference diversity in future work.

Related literature

As noted, our general approach is based on the multi-attribute model of Nehring and Puppe (2002), see also Nehring and Puppe (2003, 2009) for extensions and applications. Since we focus here on ordinal properties of diversity functions, there is also a connection to the ordinal concept of 'diversity relations' introduced by Galindo and Ülkü (2020).

The problem of preference diversity has been addressed in the literature mainly in the context of *preference profiles*, see Gehrlein et al. (2013), Hashemi and Endriss (2014), Karpov (2017) and the references therein. In this context, the potential multiplicity of preference orderings is an essential characteristic of the problem by design, and cannot be abstracted from as in the present approach. Our hope is that even though our analysis has no immediate implications on the problem of assessing the diversity of preference profiles, our results will help to understand, and illuminate, the various conditions and results derived in the existing literature.

More distantly related is the literature on 'preference cohesiveness', initiated by Bosch (2006) and further built upon by Alcalde-Unzu and Vorsatz (2013, 2016); Alcantud et al. (2015). This literature addresses the polar opposite problem to the measurement of preference diversity, namely measuring 'agreement,' or 'similarity' of preferences of the members of a group, see also Xue et al. (2020) for a recent contribution.

Overview of paper

The remaining paper is organized as follows. The next section reviews the basic multi-attribute framework of Nehring and Puppe (2002) and sets the stage for the later analysis. Section 3 develops of our main models and results. Section 4 clarifies the logical interrelations between the different models. Section 5 addresses the problem of finding maximally diverse committees. Section 6 demonstrates that our modeling strategy, while quite general in principle, does impose non-trivial limitations on the admissible diversity measures. Concretely, we show that a *prima facie* plausible criterion, the number of Pareto optimal alternatives, does *not* represent an admissible diversity measure. Section 7 concludes.

2 Background: the multi-attribute approach

Let Z be a finite set of objects. We want to measure the diversity of any subset of objects $S \subseteq Z$. The idea behind the multi-attribute approach of Nehring and Puppe (2002) is to base the diversity value $v(S)$ on the number and the weight of the different attributes realized by the objects in S. Specifically, let A be any attribute that can be possessed by the objects in Z. It is convenient to identify an attribute simply with its extension, i.e. with the set of objects that possess it. Thus, any subset $A \subseteq Z$ can be viewed as an 'attribute,' namely the (extensionally unique) attribute possessed by exactly the objects in A. Every attribute $A \subseteq Z$ is assigned a weight $\lambda_A \geq 0$, and the diversity of a set $S \subseteq Z$ is computed according to

$$v(S) \; = \sum_{A: A \cap S \neq \emptyset} \lambda_A. \tag{1}$$

A function that can be represented as in (1) with a non-negative weighting function λ will be referred to as a **diversity function**. An immediate implication of the non-negativity of the attribute weights λ_A in formula (1) is that diversity is *monotone* in the sense that

$$S \subseteq T \implies v(S) \leq v(T). \tag{2}$$

The following lemma provides a fundamental connection between set functions and their attribute weighting functions (see Nehring and Puppe 2002, Facts 2.1 and 2.2).

Lemma 1 *For any function $v : 2^Z \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$ there exists a unique function $\lambda : 2^Z \longrightarrow \mathbb{R}$, the* conjugate Moebius inverse, *such that $\lambda_\emptyset = 0$ and, for all $S \subseteq Z$, $v(S) = \sum_{A: A \cap S \neq \emptyset} \lambda_A$. The conjugate Moebius inverse is given as follows; for all non-empty $A \subseteq Z$,*

$$\lambda_A \; = \sum_{S: S \subseteq A} (-1)^{\#(A \setminus S)+1} v(Z \setminus S). \tag{3}$$

Furthermore, we have $\lambda_A \geq 0$ for all $A \subseteq Z$ if and only if the diversity function v is monotone and totally submodular *(cf. Nehring and Puppe 2002, p.1165).*

The most basic instance of the 'total submodularity' condition in this result is the following (simple) submodularity condition. For all $S, T \subseteq Z$, and all $z \in Z$,

$$S \subseteq T \implies v(S \cup \{z\}) - v(S) \geq v(T \cup \{z\}) - v(T). \tag{4}$$

Submodularity requires that the marginal contribution of an additional object decreases as the set to which it is added increases. This appears to make much sense in the context of diversity, since it reflects the intuition that it is the harder for an object to contribute to the diversity of a set the more diverse that set already is. In any case, the submodularity condition (4) is a direct consequence of the non-negativity of the conjugate Moebius inverse of every diversity function. This follows from noting that by (1)

$$v(S \cup \{z\}) - v(S) \; = \sum \{\lambda_A \mid z \in A \text{ and } A \cap S = \emptyset\},$$

which is clearly decreasing in S.

An important distinction is between the attributes that, for a given diversity function v, receive zero weight as opposed to those that receive strictly positive weight. We refer to the latter as the set of *relevant attributes* and we denote them by

$$\Lambda_v \; := \{A \subseteq Z \mid \lambda_A > 0\},$$

where the function λ is derived from v as in (3) above.

Definition 1 For any set of relevant attributes $\Lambda \subseteq 2^Z$, the family of all diversity functions $v : 2^Z \longrightarrow \mathbb{R}$ with $\Lambda_v = \Lambda$ will be referred to as the **model** associated with Λ. For simplicity, and since no confusion can arise, we will often identify a model with Λ itself, i.e. with the set of its relevant attributes.

If $\Lambda' \subseteq \Lambda$ we say that the model Λ' is *coarser* than the model Λ, and that Λ is *finer* than Λ'.

The qualitative properties of suitable models crucially depend on the application and the context. For instance, in the case of biodiversity the 'hierarchical model' and its generalizations have been shown to be useful (Nehring and Puppe 2002, 2003); on the other hand, in the case of sociological diversity multi-dimensional models seem to be natural (see Nehring and Puppe 2002). The application to preference diversity requires still different models to which we turn now.

3 Diversity of preference sets

We now apply the multi-attribute approach to the case in which Z is given by the set of all linear orderings over a finite set $X = \{x_1, \ldots, x_m\}$ of m alternatives; throughout, we assume $m \geq 3$. A generic linear ordering over X is denoted by $\succ$, and the set of all linear orderings over X by $\mathcal{L}(X)$. In the following, subsets of preferences are denoted by $S \subseteq \mathcal{L}(X)$. Depending on the context, a preference set may be identified with the set of admissible preferences of a society, or with the actual preferences of a representative committee of decision makers, or the preferences of a group of consultants, etc.

3.1 The plurality model

For all $x \in X$, denote by $A_x \subset \mathcal{L}(X)$ the set of all orderings that have x as their top alternative. The model

$$\text{PM} := \{A_x \mid x \in X\}$$

is referred to as the **plurality model**. Thus, the only relevant attributes in the plurality model are of the form 'having x as top alternative' for each alternative $x \in X$, and consequently the only relevant issue in the assessment of the diversity of a preference set is whether or not it contains an ordering with a given top alternative. Every diversity function in the plurality model evidently satisfies the following monotonicity condition. For all $S \subseteq \mathcal{L}(X)$ and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \begin{cases} \text{the top alternative of } \succ \text{ is different} \\ \text{from the top alternatives of all orders in } S. \end{cases} \quad (5)$$

The following condition strengthens the implication in (5) by requiring that the diversity of a preference set should increase strictly *only if* the added preference ordering adds a new top alternative to it. Every diversity function in the plurality model evidently satisfies this monotonicity equivalence condition as well. For all $S \subseteq \mathcal{L}(X)$ and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \begin{cases} \text{the top alternative of } \succ \text{ is different} \\ \text{from the top alternatives of all orders in } S. \end{cases} \quad (6)$$

As is easily verified, every diversity function of the model that extends PM by the attribute $A = A_x \cup A_y$ also satisfies the monotonicity equivalence condition (6). Moreover, this condition is satisfied by the diversity functions of many further extensions of PM. Our first result shows that the plurality model is in fact the coarsest model that satisfies the monotonicity equivalence (6).

To establish this result, we first prove the following general characterization result. First, observe that any model Λ^* satisfies the following generalized *monotonicity equivalence* condition. For all Λ^*, all v with $\Lambda_v = \Lambda^*$ and all $S \subseteq \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \exists\, A \in \Lambda^* \text{ with } \succ \in A \text{ and } A \cap S = \emptyset. \quad (7)$$

That is, the diversity of a preference set increases if and only if the added preference ordering adds a new, previously unrealized attribute from the model Λ^*.

Theorem 1 *A model Λ^* is the coarsest model Λ satisfying (7) for all v with $\Lambda_v = \Lambda$ if and only if no attribute $A \in \Lambda^*$ can be represented as a union of other attributes in Λ^* distinct from A.*

Proof We first show that if no attribute $A \in \Lambda^*$ can be represented as a union of other attributes in Λ^* distinct from A, then Λ^* is indeed the coarsest model satisfying (7). Suppose to the contrary that there exists a coarser model $\Lambda' \subset \Lambda^*$ which still satisfies (7). Consider any $A^* \in \Lambda^* \setminus \Lambda'$, and let $S = \mathcal{L}(X) \setminus A^*$. By (7), for every $\succ \in A^*$, we have $v(S \cup \{\succ\}) > v(S)$. Since we assume that Λ' satisfies (7) and $A^* \notin \Lambda'$ there has to be an attribute $A^{\succ} \in \Lambda'$ with $\succ \in A^{\succ} \subset A^*$ for every $\succ \in A^*$. But then evidently $A^* = \bigcup\{A^{\succ} | \succ \in A^*\}$, contrary to our assumption. Hence, no such coarser model exists, and Λ^* is the coarsest model satisfying (7).

Conversely, suppose that Λ^* is the coarsest model satisfying (7), and assume, by way of contradiction, that there exists an attribute $A^* \in \Lambda^*$ which can be represented as a union of other attributes in $\Lambda^* \setminus \{A^*\}$. Define $\Lambda' = \Lambda^* \setminus \{A^*\}$. We demonstrate that under this assumption Λ' still satisfies (7), contradicting the minimality of Λ^*.

Indeed, consider any $S \subset \mathcal{L}(X)$ and $\succ \in \mathcal{L}(X)$ such that $v(S \cup \{\succ\}) > v(S)$. This implies that there exists $A \in \Lambda'$ such that $\succ \in A$ and $A \cap S = \emptyset$. Since $\Lambda' \subseteq \Lambda^*$, the left hand side of (7) implies the right hand side.

For the converse direction, consider any $A \in \Lambda^*$ with $\succ \in A$ and $A \cap S = \emptyset$. If $A \neq A^*$, then $A \in \Lambda'$ and $v(S \cup \{\succ\}) > v(S)$. If $A = A^*$, then by assumption $A = A^* = \bigcup_{A' \subset A} A'$. Since $S \cap A = \emptyset$, it follows that $S \cap A' = \emptyset$ for all A', and for each $\succ \in A^*$ there exists an A' such that $\succ \in A'$, implying that $v(S \cup \{\succ\}) > v(S)$. Hence, Λ' satisfies (7), contradicting the minimality of Λ^*. Consequently, no attribute $A \in \Lambda^*$ can be represented as a union of other attributes in Λ^* distinct from A. □

We next provide three corollaries that facilitate the verification of whether or not a given model satisfies the union condition in Theorem 1.

Corollary 1 *A model Λ^* is the coarsest model satisfying* (7) *if Λ^* constitutes a partition of $\mathcal{L}(X)$.*

Corollary 2 *A model Λ^* is the coarsest model satisfying* (7) *if all of its attributes have equal cardinality.*

Corollary 3 *A model Λ^* is the coarsest model satisfying* (7) *if no attribute $A \in \Lambda^*$ is a strict subset of any other attribute in Λ^*.*

For PM, Corollary 1 applies directly and hence PM constitutes the coarsest model satisfying the monotonicity equivalence condition (6).

Moreover, the argument in the proof of Theorem 1 shows that any diversity function satisfying (6) must give strictly positive weight to all attributes of the form A_x. At the same time, it can be shown that every diversity function additionally assigning strictly positive weight to *supersets* of these attributes continues to satisfy (6). Thus, the restriction of being the coarsest model is indeed necessary to characterize the plurality model.

Counting the number of top alternatives appears to be a good starting point for measuring diversity, but it is arguably not completely satisfactory in examples like the following:

$$S \cup \{\succ\} = \left\{\begin{array}{ccc|c} a & a & c & c \\ b & c & a & b \\ c & b & b & a \end{array}\right\}. \tag{8}$$

Here, $v(S) = v(S \cup \{\succ\})$ seems counter intuitive. While the order $\succ$ does not add a new top to the set S, it does add the binary comparison $b \succ a$ to the preference set (all orders in S place a above b). In order to capture the intuition that this should increase the diversity, let us introduce the following 'Condorcet model.'

3.2 The Condorcet model

For all $x, y \in X$, denote by $A_{xy} \subset \mathcal{L}(X)$ the set of all orderings that rank alternative x above y. The model

$$\text{CM} := \{A_{xy} \mid x, y \in X, x \neq y\}.$$

is referred to as the **Condorcet model**. Thus, the only relevant attributes in the Condorcet model are of the form 'ranking alternative x above y' for each pair of alternatives $x, y \in X$, and consequently the only relevant issue in the assessment of the diversity of a preference set is whether or not it contains an ordering with a given binary comparison. Every diversity function in the Condorcet model evidently satisfies the following monotonicity equivalence condition. For all S and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \exists x, y \in X \text{ with } x \succ y \text{ and } y \succ' x \text{ for all } \succ' \in S. \tag{9}$$

Moreover, CM is the coarsest model satisfying the monotonicity equivalence condition (9) since Corollary 2 applies.

As appealing as the intuitive notion of counting binary relations may seem, the Condorcet model appears to be too coarse as a complete model for measuring diversity. For instance, the Condorcet model implies that a set of any two completely reversed orderings is already maximally diverse independently of the number of alternatives m. Therefore, in the following sections, we extend and combine the two basic notions of the plurality and the Condorcet model in order to define more refined models. First, we extend the plurality model to the 'rising rank' and the 'rank' models; then we extend the Condorcet model to the 'increasing rank difference' and the 'rank difference' models. Finally, we consider the 'top alternatives' model, and conclude this section with the 'cardinality' model.

3.3 The rising rank model

For all $x \in X$, denote by $A_x^{-k} \subset \mathcal{L}(X)$ the set of all orderings that have x as one of their top k alternatives. The model

$$\text{RRM} := \{A_x^{-k} \mid k = 1, ..., m-1, x \in X\}$$

is referred to as the **rising rank model** (note that we exclude the trivial attribute $A_x^{-m} = \mathcal{L}(X)$). Thus, the only relevant attributes in the rising rank model are of the form 'having x as one of the k top alternatives' for each alternative $x \in X$, and consequently the only relevant issue in the assessment of the diversity of a preference set is whether or not it contains, for any $x \in X$, an order with alternative x as one of its k top alternatives. Every diversity function in the rising rank model evidently satisfies the following monotonicity equivalence condition. For all S and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \exists\, x \in X \text{ with } \text{rank}(\succ, x) < \text{rank}(\succ', x) \text{ for all } \succ' \in S. \tag{10}$$

Moreover, RRM is the coarsest model satisfying the monotonicity equivalence condition (10). In contrast to PM and CM, there is no immediately verifiable corollary of Theorem 1 for RRM. Therefore, we establish directly that RRM satisfies the union condition of Theorem 1.

Proposition 2 *In RRM, there exists no attribute A_x^{-k} that can be represented as a union of other attributes in RRM distinct from A_x^{-k}.*

The proof is provided in the appendix.

3.4 The rank model

For all $x \in X$, denote by $A_x^k \subset \mathcal{L}(X)$ the set of all orderings that have x (exactly) as their k-th ranked alternative. The model

$$\text{RM} := \{A_x^k \mid k = 1, ..., m, x \in X\}$$

is referred to as the **rank model**. Thus, the only relevant attributes in the rank model are of the form 'having x as k-th ranked alternative' for each alternative $x \in X$, and consequently the only relevant issue in the assessment of the diversity of a preference set is whether or not it contains an ordering with a given alternative on the k-th rank. Every diversity function in the rank model evidently satisfies the following monotonicity equivalence condition. For all S and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \exists x \in X \text{ with } \text{rank}(\succ, x) \neq \text{rank}(\succ', x) \text{ for all } \succ' \in S. \tag{11}$$

Moreover, RM is the coarsest model satisfying the monotonicity equivalence condition (11) since Corollary 2 applies.

3.5 The increasing rank difference model

For all $x, y \in X$, denote by $A_{xy}^{+k} \subset \mathcal{L}(X)$ the set of all orderings that have alternative x at least k ranks above y. The model

$$\text{IRDM} := \{A_{xy}^{+k} \mid k = 1, ..., m - 1, x, y \in X\}$$

is referred to as the **increasing rank difference model**. Thus, the only relevant attributes in the increasing rank difference model are of the form 'ranking x at least k positions above alternative y' for each pair of alternatives $x, y \in X$, and consequently the only relevant issue in the assessment of the diversity of a preference set is whether or not it contains an ordering with a sufficiently large rank difference between two alternatives. Every diversity function in the increasing rank difference model evidently satisfies the following monotonicity equivalence condition. For all S and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \begin{cases} \text{for some } x, y \in X, \ x \succ y \text{ and} \\ \text{rank}(\succ, y) - \text{rank}(\succ, x) > \text{rank}(\succ', y) - \text{rank}(\succ', x) \\ \text{for all } \ \succ' \in S. \end{cases} \tag{12}$$

Moreover, for $m > 3$, IRDM is the coarsest model satisfying the monotonicity equivalence condition (12).[2] As for RRM, there is no immediately verifiable corollary of Theorem 1 for IRDM. Therefore, we establish directly that IRDM satisfies the union condition of Theorem 1.

Proposition 3 *For $m > 3$, there exists no attribute A_{xy}^{+k} in IRDM that can be represented as a union of other attributes in IRDM distinct from A_{xy}^{+k}.*

The proof is provided in the appendix.

3.6 The rank difference model

For all $x, y \in X$, denote by $A_{xy}^{k} \subset \mathcal{L}(X)$ the set of all orderings that have alternative x exactly k ranks above y. The model

$$\text{RDM} := \{A_{xy}^{k} \mid k = 1, ..., m - 1, x, y \in X, x \neq y\}$$

[2] For $m = 3$, the cardinality model introduced in Subsection 3.8 below is the coarsest model.

is referred to as the **rank difference model**. Thus, the only relevant attributes in the rank difference model are of the form ‘ranking alternative x exactly k positions above y’ for each pair of alternatives $x, y \in X$, and consequently the only relevant issue in the assessment of the diversity of a preference set is whether or not it contains an ordering with a given rank difference between two alternatives. Every diversity function in the rank difference model evidently satisfies the following monotonicity equivalence condition. For all S and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \begin{cases} \text{for some } x, y \in X,\ x \succ y \text{ and} \\ \text{rank}(\succ, y) - \text{rank}(\succ, x) \neq \text{rank}(\succ', y) - \text{rank}(\succ', x) \\ \text{for all } \succ' \in S. \end{cases} \tag{13}$$

Moreover, for $m > 3$, RDM is the coarsest model satisfying the monotonicity equivalence condition (13).[3] This follows from Corollary 3, which applies as a consequence of the following proposition.

Proposition 4 *For $m > 3$, the model RDM contains no attribute that is a strict subset of any other attribute.*

The proof is provided in the appendix.

3.7 The top alternatives model

For all $T \subseteq X$, denote by $A_T^k \subseteq \mathcal{L}(X)$ the set of all orderings that have the set T (of cardinality k) as the set of their top k alternatives. The model

$$\text{TAM} := \{A_T^k \mid k = 1, \ldots, m-1, T \subset X, |T| = k\}$$

is referred to as the **top alternatives model** (note that we again exclude the trivial attribute A_X^m). Thus, the only relevant attributes in the top alternatives model are of the form ‘having the set T as the set of k top alternatives’ for each subset $T \subset X$, and consequently the only relevant issue in the assessment of the diversity of a preference set is whether or not it contains an ordering with a given set of k top alternatives. Every diversity function in the top alternatives model evidently satisfies the following monotonicity equivalence condition. For all S and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \begin{cases} \text{for some } k = 1, \ldots, m-1, \\ \text{the top } k \text{ elements of } \succ \text{ are different} \\ \text{from the top } k \text{ elements of all orders in } S. \end{cases} \tag{14}$$

Moreover, TAM is the coarsest model satisfying the monotonicity equivalence condition (14). This follows from Corollary 3, which applies as a consequence of the following proposition.

Proposition 5 *The model TAM contains no attribute that is a strict subset of any other attribute.*

[3] For $m = 3$, again the cardinality model from Subsection 3.8 is the coarsest model.

The proof is provided in the appendix.

3.8 The cardinality model

For all $\succ \in \mathcal{L}(X)$, denote by $A_{\succ} \subset \mathcal{L}(X)$ the singleton set $A_{\succ} = \{\succ\}$. The model

$$\text{CdM} := \{A_{\succ} \mid \succ \in \mathcal{L}(X)\}$$

is referred to as the **cardinality model**. Thus, the only relevant attributes in the cardinality model are of the form 'being identical to the order $\succ$' for each order $\succ \in \mathcal{L}(X)$, and consequently the only relevant issue in the assessment of the diversity of a preference set is, for every given order, whether or not it contains that order. If all attributes of the form $A_{\succ}$ receive the same weight, the diversity of a set of preferences is thus fully determined by its cardinality.

Every diversity function in the cardinality model evidently satisfies the following monotonicity equivalence condition. For all S and all $\succ \in \mathcal{L}(X)$,

$$v(S \cup \{\succ\}) > v(S) \iff \succ \notin S. \tag{15}$$

Moreover, CdM is the coarsest model satisfying the monotonicity equivalence condition (15) since Corollary 1 applies.

Clearly, a model satisfying (15) can have more relevant attributes than the singleton sets. For instance, any diversity function in the complete model $\Lambda = 2^{\mathcal{L}(X)} \setminus \emptyset$ satisfies (15).

4 Model interrelations

The eight proposed models are interrelated by their monotonicity properties. Together they span a model space, depicted in Figure 1, which is bounded by the cardinality model on one side and by the weak monotonicity condition (2), labelled WM in Fig. 1, on the other side. The cardinality model satisfies all potential monotonicity conditions (but, of course not the corresponding monotonicity equivalences).[4] On the other hand, all models satisfy the monotonicity condition WM.

In Fig. 1, a model W 'entails' another model V, indicated by an arrow from W to V if and only if model W satisfies all monotonicity properties that model V satisfies, i.e. if and only if, for all S and $\succ$, $v(S \cup \{\succ\}) > v(S)$ for all v in model V implies that $v(S \cup \{\succ\}) > v(S)$ holds for all v in model W.

Theorem 6 *A model W satisfies all monotonicity properties of a model V if and only if for all $A_V \in \Lambda_V$ and all $\succ \in A_V$, there exists $A_W \in \Lambda_W$ such that $\succ \in A_W \subseteq A_V$.*

Proof Consider any case in which $v(S \cup \{\succ\}) > v(S)$ holds for model V. Then, $\succ \in A_V$ for some $A_V \in \Lambda_V$ not yet realized by S; by assumption, $\succ \in A_W \subseteq A_V$

[4] In the case $m = 3$, the model CdM satisfies the same monotonicity equivalences as the models IRDM and RDM; indeed, in that case CdM $\subsetneq$ IRDM, RDM.

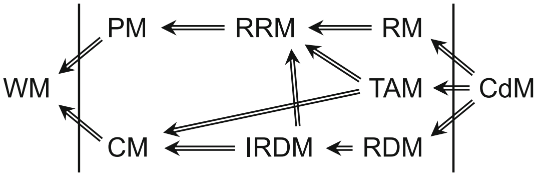

Fig. 1 Model space by monotonicity. The model space is constrained by weak monotonicity (2) and the cardinality model. In the model space, a model W entails another model V if and only if model W satisfies all monotonicity properties of model V

for some $A_W \in \Lambda_W$. This implies $v(S \cup \{\succ\}) > v(S)$ if v is a member of model W. Consequently, model W satisfies all monotonicity properties that model V satisfies.

Now suppose, conversely, that model W satisfies all monotonicity properties that model V satisfies. Then, $v(S \cup \{\succ\}) > v(S)$ for v in model V implies $v(S \cup \{\succ\}) > v(S)$ for v in model W. In particular, this implication holds for all $S = \mathcal{L}(X) \setminus A_V$ and $\succ \in A_V$ with $A_V \in \Lambda_V$. Since $v(S \cup \{\succ\}) > v(S)$ for v in model W, there must exist an attribute $A_W \in \Lambda_W$ such that $\succ \in A_W$ and not yet realized by $S = \mathcal{L}(X) \setminus A_V$; hence, $A_W \subseteq A_V$. □

5 Designing maximally diverse committees

In applications, a natural problem is to design committees of maximal diversity. In the following, we derive some basic facts about this problem for our models. Concretely, we focus on finding maximally diverse preference sets S for any fixed number of elements n, given the number of alternatives $m = \#X$, and a particular model Λ.

Clearly, in the absence of any further restrictions on the attribute weighting function (other than non-negativity) not much can be said about this problem. A natural restriction that allows one to derive a number of basic facts about maximally diverse sets is the following neutrality condition. Let $\pi : X \longrightarrow X$ be a bijection; for each order $\succ$ on X, denote by $\succ^\pi$ the order defined by $x \succ^\pi y :\Leftrightarrow \pi(x) \succ \pi(y)$, and for each set S of orders, denote by $\pi(S) := \{\succ^\pi \mid \succ \in S\}$. A diversity function is said to be *neutral* if, for all S and all bijections π,

$$v(\pi(S)) = v(S).$$

Note that a diversity function v is neutral if and only if its attribute weighting function (i.e. the conjugate Moebius inverse) is neutral in the sense that, for all attributes $A \subseteq \mathcal{L}(X)$ and all bijections π, $\lambda_A = \lambda_{\pi(A)}$. (This follows from the explicit formula (3) for the attribute weighting function.) Hence, every neutral diversity function induces a number K of equivalence classes of relevant attributes $\mathcal{A}_k$, $k = 1, ..., K$, such that, for all k, $A, A' \in \mathcal{A}_k$ if and only if $A' = \pi(A)$ for some bijection π. By neutrality, all attributes in $\mathcal{A}_k$ must receive the same weight, denoted by λ_k.

In the following, we will consider neutral diversity functions belonging to one of the models considered in the previous section. By neutrality, all single preference orders $\succ \in \mathcal{L}(X)$ receive the same diversity value $v(\{\succ\})$. Consequently, the diversity $v(S)$

of a set $S = \{\succ_1, ..., \succ_n\}$ of n different preferences is bounded above by

$$v(S) \;\le\; \sum_{i=1}^{n} v(\{\succ_i\}) \;=\; n \cdot v(\{\succ\}),$$

where the first inequality follows from subadditivity of any diversity function, and the second equality from the neutrality assumption. But due to the fact that different preferences can share common attributes, this bound can be improved upon. Specifically, we obtain the following inequality. For all $k = 1, ..., K$, denote by $a_k := \#\mathcal{A}_k$, and by ℓ_k the maximal number of attributes that any single preference order can possess in attribute class $\mathcal{A}_k$; then, for all S,

$$v(S) \le \sum_{k=1}^{K} b_k \lambda_k \qquad \text{with} \qquad b_k = \begin{cases} n\ell_k & \text{if } n\ell_k \le a_k \\ a_k & \text{if } n\ell_k > a_k \end{cases} \tag{16}$$

(note that $b_k = \min\{n\ell_k, a_k\}$). Our strategy to find maximally diverse sets of preferences is now simply to check if we can satisfy (16) with equality for some S; if yes, S must be maximally diverse.

Let us illustrate inequality (16) in the RRM with $m = 3$ and $X = \{a, b, c\}$. In the RRM there are then two classes of attributes: $\mathcal{A}_1 = \{A_x^{-1} \mid x \in X\}$ and $\mathcal{A}_2 = \{A_x^{-2} \mid x \in X\}$. For instance, the attributes A_a^{-1} ('having a at the top') and A_a^{-2} ('having a at the first or second rank') are given by

$$A_a^{-1} = \begin{Bmatrix} a & a \\ b & c \\ c & b \end{Bmatrix} \qquad \text{and} \qquad A_a^{-2} = \begin{Bmatrix} a & a & b & c \\ b & c & a & a \\ c & b & c & b \end{Bmatrix},$$

respectively. In this example, we have $a_1 = a_2 = 3$ since each class has three elements (one for each $x \in \{a, b, c\}$). Moreover, we have $\ell_1 = 1$ since each given preference order can only satisfy one of the three attributes in $\mathcal{A}_1$, and $\ell_2 = 2$ since each given preference order satisfies exactly two of the three attributes in $\mathcal{A}_2$. Inequality (16) now says that the diversity $v(S)$ of a set S with n elements is never larger than $\min\{n, 3\} \cdot \lambda_1 + \min\{2n, 3\} \cdot \lambda_2$, where λ_k is the (common) weight of all attributes in class $\mathcal{A}_k$ for $k = 1, 2$. In particular, if a set S with three elements achieves diversity $3(\lambda_1 + \lambda_2)$ it must have maximal diversity. This is the case, for instance, for the set

$$S = \begin{Bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{Bmatrix}.$$

For each of the five models PM, CM, RRM, RM, and CdM, there exists for every combination of n and number of alternatives m a set S of n preferences that satisfies the equality of (16). Table 1 displays for $m = 3$ and each of the five models the smallest set that satisfies all attributes of the respective model. Each set and all of their subsets satisfy the equality of (16) for their respective model. Furthermore, all supersets of

Table 1 Smallest set satisfying all attributes of one of the five models PM, CM, RRM, RM, and CdM for $m = 3$ alternatives. The smallest sets are not necessarily unique. The dots are placeholders for every possible combination of missing alternatives

$\text{PM}: \begin{Bmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Bmatrix}$	$\text{CM}: \begin{Bmatrix} a & c \\ b & b \\ c & a \end{Bmatrix}$	$\text{RRM/RM}: \begin{Bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{Bmatrix}$	$\text{CdM}: \begin{Bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Bmatrix}$

these sets satisfy (16) with equality as well. In fact, the design of such maximally diverse sets is quite straightforward and it is easily seen that, if a set does not satisfy (16) with equality for one of these five models it cannot be maximally diverse. In summary, the satisfaction of (16) with equality is a necessary and sufficient condition of maximality of a set of preferences for all neutral diversity functions belonging to one of the five models PM, CM, RRM, RM, and CdM.

For PM, the smallest set satisfying all attributes places each alternative exactly once on top. Therefore, the set has size $n = m$ and all ranks beside the top rank do not influence the diversity of the set. For CM, the smallest set possessing all attributes consists of any two polarized preference orders. Every preference order possesses exactly half of all attributes. Thus, inverting all relations of a given preference order results in the realization of remaining half of all attributes. For RRM and RM, the smallest set satisfying all attributes places each alternative exactly once at each rank. For RM, this design is necessary, but for RRM there also exist other possibilities to design such a set. For CdM, the smallest set satisfying all attributes obviously is $\mathcal{L}(X)$. CdM has the special property that no pairs of preference orders share any attribute and therefore, any preference set S is maximally diverse given its size n.

Remark: For *combinations* of the above five models the task of finding maximally diverse sets is more complex. Consider, for instance, a neutral diversity function in the model $\Lambda = \text{CM} \cup \text{RM}$; then it might not be possible to satisfy (16) with equality. For instance, let $n = 2$, and $m = 3$ and consider the sets

$$S = \begin{Bmatrix} a & c \\ b & b \\ c & a \end{Bmatrix} \quad \text{and} \quad S' = \begin{Bmatrix} a & b \\ b & c \\ c & a \end{Bmatrix}. \tag{17}$$

The set S realizes all six attributes of CM but only five different attributes of RM (because b is on second rank in both orders in S). On the other hand, S' realizes six attributes of RM but only five attributes of CM (because b is above c in both orders in S'). It is easily verified that no set containing two orders can realize more attributes from the model $\text{CM} \cup \text{RM}$; in particular (16) cannot be satisfied with equality, and the answer to the question if S or S' is more diverse depends on the cardinal values of the weights of the different attributes.

It is an open problem if (16) can be satisfied with equality for all n and m and all neutral diversity functions in the models TAM and IRDM. For IRDM, the following example illustrates the difficulty. Consider for $m = 5$ the two sets

$$S = \begin{Bmatrix} a & e \\ b & d \\ c & c \\ d & b \\ e & a \end{Bmatrix} \quad \text{and} \quad S' = \begin{Bmatrix} a & d & c \\ b & e & e \\ c & b & a \\ d & a & d \\ e & c & b \end{Bmatrix}; \tag{18}$$

both sets are maximally diverse given their respective size, but their general design principle is not at all evident. In particular, it can be shown that the set S is not contained in *any* maximally diverse set of cardinality $n = 3$.

For RDM, the bound (16) cannot always be satisfied with equality. For instance, as is easily verified, for $n = m = 3$, there is no set that realizes all six attributes of the first attribute class $\{A^1_{xy} \mid x \neq y\}$, although every single preference order does realize exactly two different attributes of this class. Consequently, the bound (16) cannot be used in case of the model RDM to test for maximal diversity. Finding algorithms that yield maximally diverse sets for various models is a worthwhile task, but this is left for future research.

6 On the number of Pareto optimal alternatives as a measure of diversity

A *prima facie* attractive proposal is to measure the diversity of a set of preferences in terms of the number of Pareto optimal alternatives. In the present section, we argue that this proposal does in fact not satisfy a fundamental tenet of diversity measurement; indeed, the number of Pareto optimal alternatives may exhibit *increasing* marginal contributions to the value of a set. The set function defined by the number of Pareto optimal alternatives thus nicely illustrates by negative example the submodularity assumption satisfied by all our models.

Specifically, for each set $S \subseteq \mathcal{L}(X)$, denote by

$$\begin{aligned} v_{\mathrm{PO}}(S) &:= \#\{x \in X \mid \text{for no } y \in X,\ y \succ x \text{ for all } \succ \in S\} \\ &= \#\{x \in X \mid \text{for all } y \in X \setminus \{x\},\ \text{there exists } \succ \in S \text{ such that } x \succ y\}, \end{aligned}$$

in other words, $v_{\mathrm{PO}}(S)$ simply counts the number of Pareto optimal alternatives given the preferences in S. It is easily verified that the function $v_{\mathrm{PO}}(\cdot)$ is monotone in the sense of (2), but it fails the submodularity condition (4). This can be seen as follows. Let S consist of the single order $a \succ' b \succ' c$ and let $T \supset S$ contain in addition the order $b \succ'' c \succ'' a$. Clearly, we have $v_{\mathrm{PO}}(S) = 1$ and $v_{\mathrm{PO}}(T) = 2$. What happens if we add the order $a \succ c \succ b$ to S and to T, respectively? We have

$$S \cup \{\succ\} = \left\{\begin{array}{c|c} a & a \\ b & c \\ c & b \end{array}\right\} \quad \text{and} \quad T \cup \{\succ\} = \left\{\begin{array}{cc|c} a & b & a \\ b & c & c \\ c & a & b \end{array}\right\},$$

hence $v_{\text{PO}}(S \cup \{\succ\}) = 1$ and $v_{\text{PO}}(T \cup \{\succ\}) = 3$. Therefore,

$$v_{\text{PO}}(S \cup \{\succ\}) - v_{\text{PO}}(S) = 0$$
$$v_{\text{PO}}(T \cup \{\succ\}) - v_{\text{PO}}(T) = 1,$$

in violation of the submodularity condition (4).

The violation of submodularity of the function $v_{\text{PO}}(\cdot)$ is reflected in its Moebius inverse which displays negative values. Recall that, for all distinct $x, y \in X$, $A_{xy} \subset \mathcal{L}(X)$ is the attribute '$x \succ y$', i.e. the set of all orders that place x above y; moreover, for all pairwise distinct triples $x, y, z \in X$, denote by $A_{xy \vee xz} \subset \mathcal{L}(X)$ the attribute '($x \succ y$ or $x \succ z$)', i.e. the set of all orders that place x above y or x above z (or both). Direct computation using formula (3) yields the following values for the conjugate Moebius inverse $\lambda_{\text{PO}}(\cdot)$ of $v_{\text{PO}}(\cdot)$ if $\#X = 3$:

$$\lambda_{\text{PO}}(A) = \begin{cases} +1 & \text{if } A = A_{xy} \text{ for some } x, y \in X \\ -1 & \text{if } A = A_{xy \vee xz} \text{ for some } x, y, z \in X \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

In particular, the value of attributes of the form '($x \succ y$ or $x \succ z$)' (which in the case of three alternatives coincides with the attribute 'x is not the last alternative') is negative.[5] Since in the multi-attribute model no attribute can diminish the diversity value, the number of Pareto optimal alternatives does thus not qualify as a candidate for the measurement of diversity.

7 Conclusion

In this paper, we have explored the possibility of measuring the diversity of sets of preferences within the general framework of the multi-attribute approach of Nehring and Puppe (2002). We have proposed a number of *prima facie* reasonable models of preference diversity. Evidently, the present analysis provides only a first step towards the general problem of measuring the diversity of committees, or groups of individuals, characterized by their preferences. Indeed, much is left for future work. We briefly comment on directions that we deem worthwhile.

First, the eight models proposed here should be assessed with concrete application in mind. It may well turn out that different applications call for different models. It may also turn out that *combinations* of our models are sometimes more useful than the pure version presented here. From the general perspective of the multi-attribute model such combinations (i.e. unions of the underlying sets of relevant attributes) pose no mathematical difficulty.

Secondly, while we view our purely ordinal approach conceptually appealing due to its simplicity, complementing the analysis with cardinal information, i.e. the magnitude of attribute weights, is certainly a worthwhile task.[6] For instance, a model that may be

[5] A formula similar to, but naturally more complex than (19) can be derived for general X.

[6] Ordinal properties of diversity functions such as the monotonicity equivalences used here to characterize various models can also be described in terms of 'diversity relations' as introduced by Galindo and Ülkü

interesting to explore in future work is a cardinal version of the Top Alternative Model (TAM) in which the (positive) weights of the attributes A_T^k are *decreasing* in k. Such a specification would reflect the intuition that the number of different top alternatives has a larger weight in the diversity assessment than the number of different second-best alternatives, etc.

Finally, our analysis is in fact silent on the question of the diversity of *profiles* of preferences. This is perhaps the most serious limitation of our approach. But we also believe that the simple and special case in which every order occurs exactly once in every set to be evaluated in terms of diversity must be well understood first, before one can attack the more difficult setting in which orders may have copies of different multiplicity. In any case, we hope that our approach will inspire solution to this more difficult problem. Ultimately, it would be desirable to understand the complex relation between the diversity of *elections* and their outcomes, a problem that has recently received some attention (Faliszewski et al. 2019; Szufa et al. 2020; Boehmer et al. 2021).

Appendix A Proofs

Proof of Proposition 2. Let $A_x^{-k} \in$ RRM. Consider any attribute $A_y^{-k'}$ with $y \neq x$. By construction, $A_y^{-k'}$ contains at least one order in which x appears on the last rank, which contradicts inclusion in A_x^{-k} for $k \leq m-1$. Hence, $A_y^{-k'} \not\subseteq A_x^{-k}$.

Next, consider $A_x^{-k'}$ with $k' > k$. By definition $|A_x^{-k'}| > |A_x^{-k}|$ and thus $A_x^{-k'} \not\subseteq A_x^{-k}$.

Now consider the collection $\{A_x^{-k'} : k' < k\}$. Each $A_x^{-k'} \subset A_x^{-k}$ by definition, but their union omits all orders placing x precisely on the k-th rank. Consequently, $\bigcup_{k'<k} A_x^{-k'} \subset A_x^{-k}$, and the attribute A_x^{-k} cannot be represented as a union of other attributes in RRM. □

Proof of Proposition 3. Fix $m > 3$ and an attribute $A_{xy}^{+k} \in$ IRDM. We show that for each such A_{xy}^{+k}, there exists at least one order $\succ \in A_{xy}^{+k}$ such that $\succ \notin A_{wz}^{+k'}$ for any proper subset of A_{xy}^{+k}.[7] To establish this, we argue that for each binary relation $w \succ z$ realized by $\succ$ other than the defining relation $x \succ y$, there exists an alternative order $\succ' \notin A_{xy}^{+k}$ that realizes the same binary relation with at least the same rank difference. In consequence, the attributes defined by this binary relation and all rank differences smaller than or equal to the rank difference in $\succ'$ cannot lie entirely within A_{xy}^{+k}. We distinguish two cases:

For $k > 1$, consider any $\succ \in A_{xy}^{+k}$ in which x is ranked exactly k positions above y. Define the two orders $\succ^*$ and $\succ^{**}$. $\succ^*$ results from $\succ$ by moving y one rank up and $\succ^{**}$ by moving x one rank down. By construction, both $\succ^*$ and $\succ^{**}$ reduce the rank difference between x and y, implying $\succ^*, \succ^{**} \notin A_{xy}^{+k}$. We now verify that the binary

(2020); for the precise relation between ordinal diversity relations and totally submodular diversity functions see also Wong et al. (1991); Ryan (2014).

[7] This is equivalent to showing that A_{xy}^{+k} cannot be represented as a union of other attributes in IRDM distinct from A_{xy}^{+k}.

comparisons and corresponding rank differences in $\succ$, other than that between x and y, are preserved in $\succ^*$ or $\succ^{**}$ with at least the same rank difference.

In $\succ^*$, all rank differences are unchanged except those involving y or the alternative it swapped positions with. For y's swap partner and the alternatives above y, the rank difference to y decreases. However, in $\succ^{**}$, those same rank differences are restored or increased, except the one to x. An analogous argument applies to y's swap partner, whose rank differences may decrease in $\succ^*$, but are recovered in $\succ^{**}$.

Hence, every binary relation in $\succ$ (apart from the pair (x, y)) appears in at least one of $\succ^*$ or $\succ^{**}$, with a rank difference greater than or equal to that in $\succ$. Therefore, none of $\succ$'s attributes is a strict subset of A_{xy}^{+k}.

For $k = 1$, let $\succ \in A_{xy}^{+1}$ be such that the rank difference between x and y is exactly one, and assume x is not placed first and y is not placed last. Let $\succ'$ denote the order obtained by swapping x and y. Then $\succ' \notin A_{xy}^{+1}$.

$\succ'$ preserves all rank differences apart the ones containing x or y. For alternatives ranked below x, the swap reduces their distance to x by one. To restore the original rank distances, construct an order where both x and y are moved one position up. A symmetric argument holds for alternatives above y: moving both x and y one rank down restores those rank differences. Consequently, all attributes of $\succ$ are not a strict subset of A_{xy}^{+1}.

We conclude that no attribute $A_{xy}^{+k} \in$ IRDM can be represented as a union of other attributes in IRDM distinct from A_{xy}^{+k}. □

Remark: The restriction $m > 3$ is essential for Proposition 3. When $m = 3$, the conditions x not ranked first and y not ranked last, in the case $k = 1$, cannot be simultaneously satisfied, rendering the construction above infeasible. In this case, CdM is the coarsest model.

Proof of Proposition 4. We proceed by contradiction. Consider any attribute $A_{xy}^{k} \in$ RDM. Suppose there exists another attribute $A_{wz}^{k'} \neq A_{xy}^{k}$ such that $A_{wz}^{k'} \subset A_{xy}^{k}$. If $k' \leq k$, then $|A_{wz}^{k'}| \geq |A_{xy}^{k}|$, contradicting the strict inclusion.

Now consider $k' > k$. We distinguish five cases based on the choices for w and z in $A_{wz}^{k'}$.

- If $w = x$, $z = y$, then all orders in $A_{wz}^{k'}$ feature a rank difference of k' between x and y, whereas all orders in A_{xy}^{k} have a rank difference of k between x and y; hence $A_{wz}^{k'} \not\subset A_{xy}^{k}$.
- If $w = y$, $z = x$, there exists an order $\succ \in A_{wz}^{k'}$ placing y exactly k' ranks above x, which cannot belong to A_{xy}^{k}.
- Similarly, if $w = y$, $z \neq x$, or $w \neq y$, $z = x$, then in each case $A_{wz}^{k'}$ contains an order placing y exactly k ranks above x, which is not included in A_{xy}^{k}.
- If $w, z \notin \{x, y\}$, all attributes $A_{wz}^{k'}$ that include an order placing x exactly k ranks above y must also include an order with the reverse placement; hence $A_{wz}^{k'} \not\subset A_{xy}^{k}$.

- If $w \neq x, z = y$, or $w = x, z \neq y$, then $A^{k'}_{wz}$ contains orders where x is ranked $k^* \neq k', k$ positions above y, which violates the inclusion in A^{k}_{xy}.[8]

Therefore, no attribute $A^{k'}_{wz}$ is a strict subset of A^{k}_{xy}, completing the proof. □

Proof of Proposition 5. We proceed by contradiction. Let A^{k}_{T} be an arbitrary attribute in TAM. Suppose there exists a distinct attribute $A^{k'}_{T'} \subset A^{k}_{T}$.

If $k' < k$, then $A^{k'}_{T'}$ contains orders with some alternative $x \in X \setminus T'$ and $x \notin T$ ranked exactly in position k, which cannot be the case for A^{k}_{T}.

If $k' = k$, the attributes have equal cardinality, and $A^{k'}_{T'} \neq A^{k}_{T}$, violating the inclusion.

If $k' > k$, then $A^{k'}_{T'}$ includes orders with some $x \in T'$ ranked among the top k positions, but $x \notin T$, contradicting $A^{k'}_{T'} \subset A^{k}_{T}$.

Thus, no attribute in TAM is a strict subset of another, completing the proof. □

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References

Alcalde-Unzu J, Vorsatz M (2013) Measuring the cohesiveness of preferences: an axiomatic analysis. Soc Choice Welfare 41:965–988. https://doi.org/10.1007/s00355-012-0716-9

Alcalde-Unzu J, Vorsatz M (2016) Do we agree? Measuring the cohesiveness of preferences. Theor Decis 80:313–339. https://doi.org/10.1007/s11238-015-9494-z

Alcantud JCR, de Andrés Calle R, Cascón JM (2015) Pairwise dichotomous cohesiveness measures. Group Decis Negot 24:833–854. https://doi.org/10.1007/s10726-014-9415-5

Boehmer N, Bredereck R, Faliszewski P et al (2021) Putting a compass on the map of elections. In: Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, https://doi.org/10.24963/ijcai.2021/9

Bosch R (2006) Characterizations on voting rules and consensus measures. PhD thesis, Tilburg University

[8] This last case explains why RDM is the coarsest model only for $m > 3$; for $m = 3$ no rank $k^* \neq k', k$ exists, and the strict subset relation holds.

Faliszewski P, Skowron P, Slinko A, et al (2019) How similar are two elections? In: Proceedings of the AAAI Conference on Artificial Intelligence, https://doi.org/10.1609/aaai.v33i01.33011909
Galindo J, Ülkü L (2020) Diversity relations over menus. Soc Choice Welfare 55:229–242
Gehrlein WV, Moyouwou I, Lepelley D (2013) The impact of voters' preference diversity on the probability of some electoral outcomes. Math Soc Sci 66(3):352–365
Hashemi V, Endriss U (2014) Measuring diversity of preferences in a group. In: Proceedings of the 21st European Conference on Artificial Intelligence (ECAI)
Karpov A (2017) Preference diversity orderings. Group Decis Negot 26(4):753–774. https://doi.org/10.1007/s10726-017-9532-z
Nehring K, Puppe C (2002) A theory of diversity. Econometrica 70(3):1155–1198. https://doi.org/10.1111/1468-0262.00321
Nehring K, Puppe C (2003) Diversity and dissimilarity in lines and hierarchies. Math Soc Sci 45(2):167–183. https://doi.org/10.1016/S0165-4896(03)00024-6
Nehring K, Puppe C (2009) Diversity. In: The Handbook of Rational and Social Choice. Oxford University Press, p 298–322
Ryan M (2014) Belief functions and preference for flexibility. Econ Bull 34(1):581–588
Szufa S, Faliszewski P, Skowron P, et al (2020) Drawing a map of elections in the space of statistical cultures. In: Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems
Wong SKM, Yao YY, Bollmann P et al (1991) Axiomatization of qualitative belief structure. IEEE Trans Syst Man Cybern 21(4):726–734
Xue Z, Lin Z, Wang H et al (2020) Quantifying consensus of rankings based on q-support patterns. Inf Sci 518:396–412. https://doi.org/10.1016/j.ins.2019.12.070