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Joannis Alexopoulos

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Referent: Dr. Björn de Rijk
Korreferenten: Prof. Dr. Guido Schneider
Prof. Dr. Michael Plum

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Kurzfassung

Das Ziel dieser Arbeit ist es, nichtlineare Stabilitätsergebnisse periodischer Wellenzüge zu beweisen, indem jegliche Lokalisierungs- und Periodizitätsannahmen zu Störungen in bestimmten dissipativen Modellen aufgehoben werden. Dies weicht von früheren Ergebnissen in der Literatur ab (die sich auf die Lokalisierung von Störungen stützen) und zeigt, dass periodische Wellenzüge robuster sind als bisher angenommen. Mathematisch lässt sich die allgemeine Aufgabe wie folgt zusammenfassen: Wie lässt sich aus generischen Spektraleigenschaften die nichtlineare Stabilität einer bestimmten Modellklasse beweisen? Aufgrund der fehlenden Lokalisierung und der Herausforderungen, die sich aus den Modellen selbst ergeben, wie z. B. unvollständige Parabolizität oder dispersive Eigenschaften, sind etablierte Rahmenwerke nicht mehr anwendbar, Techniken müssen verfeinert werden und neue Ideen sind erforderlich. Wir beweisen neue Ergebnisse zur nichtlinearen Stabilität periodischer Wellenzüge anhand von drei Paradigmenmodellen. Wir verwenden einen Modulierungsansatz, um die nichtlineare Stabilität gegenüber vollständig nichtlokalisierten Störungen im FitzHugh-Nagumo-System zu zeigen. Für parabolische Reaktions-Diffusions-Systeme zeigen wir ein Modulationsstabilitätsergebnis gegenüber vollständig nichtlokalisierten Störungen. Insbesondere lassen wir große Phasenmodulationen zu, ohne jegliche Lokalisierung der anfänglichen Phasenmodulation anzunehmen. Inspiriert durch das sogenannte Zahnproblem in der nichtlinearen Schrödinger-Gleichung betrachten wir die Lugiato-Lefever-Gleichung und Störungen, die aus Summen von koperiodischen und lokalisierten Störungen bestehen. Der Anhang widmet sich einem separaten Ergebnis, das durch Herausforderungen motiviert ist, die sich bei der Beweisführung für das Ergebnis im FitzHugh-Nagumo-System ergeben haben. Wir befassen uns mit der Fourier- und Laplace-Inversionsformel und finden Bedingungen, die eine lokal gleichmäßige Konvergenz dieser Inversionsformeln garantieren.

Abstract

The purpose of this thesis is to prove nonlinear stability results of periodic wave trains by lifting any localization and periodicity assumptions on perturbations in certain dissipative models. This diverges from previous results established in the literature (which rely on localization of perturbations) and shows that periodic wave trains are more robust than always assumed. Mathematically, the general task boils down to: given a specific class of models, how to conclude nonlinear stability from generic spectral properties? Due to the lack of localization and challenges coming from the models themselves, such as incomplete parabolicity or dispersive characteristics, well-established frameworks do not apply anymore, techniques have to be refined and new ideas are required. We prove new nonlinear stability results of periodic wave trains along three paradigm models. The first model is devoted to the FitzHugh-Nagumo system for which we adopt a modulational approach to show nonlinear stability against fully nonlocalized perturbations. For parabolic reaction diffusion systems, we show a modulational stability result against fully nonlocalized perturbations. In particular, we allow for large phase modulations without assuming any localization on the initial phase modulation. Inspired by the so-called tooth problem in the nonlinear Schrödinger equation, we consider the Lugiato-Lefever equation and perturbations consisting of sums of co-periodic plus localized perturbations. The appendix is devoted to a detached result which is motivated by challenges arising in the proof of the result in the FitzHugh-Nagumo system. We deal with the Fourier- and Laplace inversion formula and find conditions to guarantee locally uniform convergence of these inversion formulae.

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CHAPTER 1

Introduction

We consider an n-dimensional system of partial differential equations (PDE) with spatial variable $x \in \mathbb{R}$ and time variable $t \in [0, \infty)$. As a concrete example, the reader may think of a reaction-diffusion system $\partial_t u = D\partial_x^2 u + f(u)$ with $D \in \mathbb{R}^{n\times n}$ positive definite and $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth. Let $\phi_0 : \mathbb{R} \to \mathbb{R}^n$ be a smooth T-periodic function. A *periodic wave train* is a solution of the PDE which can be written in the form

$$\phi_0(k_0 x - \omega_0 t), \quad (x,t) \in \mathbb{R} \times [0, \infty), \tag{1.1}$$

where ϕ_0 is the *periodic profile* (see Figure 1.2), $k_0 \neq 0$ is called the *wavenumber* and $\omega_0 \in \mathbb{R}$ is the *time frequency*. By the quotient $c_0 = \frac{\omega_0}{k_0}$, we denote the *speed of the wave train*. If $c_0 = 0$, we say that (1.1) is a *standing* wave. We are interested in the *nonlinear L^∞-stability of periodic wave trains* which we informally formulate as:

Every solution of the (nonlinear) PDE starting close to the periodic wave train with respect to a class of perturbations from a Banach space $Y \hookrightarrow L^\infty(\mathbb{R})$, exists globally and stays L^∞-close to the wave train for all times.

Furthermore, we are curious about the asymptotic behavior:

Do the perturbed solutions converge to the wave train and at which rate as time tends to infinity?

The interest in studying nonlinear stability of periodic or other patterns is not solely of mathematical nature. Many phenomena from biology, chemistry, physics or ecology are described through nonlinear PDEs and therefore the question whether certain patterns such as periodic wave trains exist in these PDEs are of crucial interest. To show nonlinear stability now justifies that these patterns can actually be observed and accredits them a physical relevance.

We formalize the notion of nonlinear L^∞-stability in Definition 1.0.1 and give an illustration in Figure 1.1.

Definition 1.0.1 (Nonlinear L^∞-stability). To prevent confusion at this stage of the thesis, we suppose that $k_0 = 1$ and $c_0 = 0$. We fix a Banach space $X \hookrightarrow L^\infty(\mathbb{R})$ such that $\phi_0 \in X$. To formalize the notion of nonlinear L^∞-stability, we note that every PDE which we consider in this thesis can be written as *semilinear evolution equation*

$$\partial_t u = \mathcal{L}u + \mathcal{N}(u) \tag{1.2}$$

where $\mathcal{L}$ generates a C_0-semigroup on X, $\mathcal{N} : X \to X$ is locally Lipschitz continuous and ϕ_0 is a stationary solution of (1.2). Let $Y \hookrightarrow X$ be a Banach space. We define: the periodic profile ϕ_0 is called *nonlinearly L^∞-stable against perturbations from Y* if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $v_0 \in Y$ with $||v_0||_Y < \delta$, the mild solution $u(t)$ of (1.2) with initial condition $u(0) = \phi_0 + v_0$ exists globally and $||u(t) - \phi_0||_{L^\infty} < \varepsilon$ for all $t \geq 0$.

We remark on some aspects of Definition 1.0.1. First, it can be seen as a variant for semilinear evolution equations of the classical notion of Lyapunov stability for ordinary differential equations. For all considerations in this thesis, we can assume sufficient regularity on the initial perturbations to obtain classical solutions. Whenever we mention 'nonlinear stability', the reader may refer to the notion given in Definition 1.0.1. For standard references on semilinear evolution equations, we refer to [22, 79].

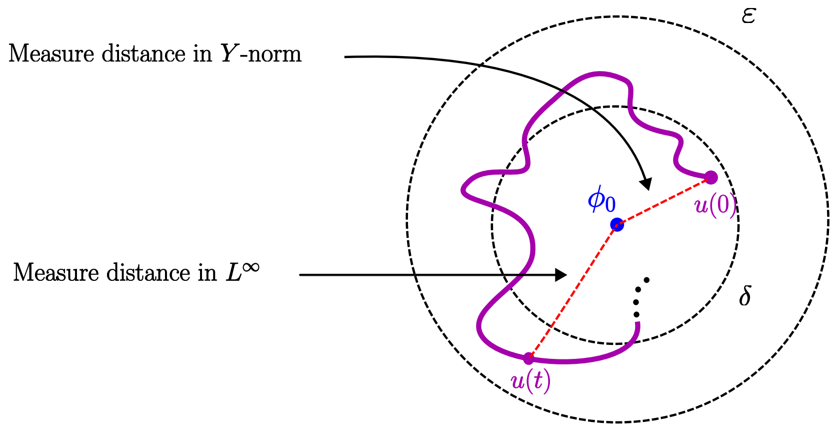


Figure 1.1.: A simplified illustration of Definition 1.0.1 in two dimensions.

Nonlinear stability and asymptotic behavior crucially depend on the class of perturbations against which one perturbs the wave train. We refer to Figure 1.2 for an illustration of important types of perturbations. A *co-periodic* perturbation is T-periodic, i.e., it has the same periodic as the profile. Typical examples for *localized* perturbation spaces are $H^k(\mathbb{R})$ or $L^1(\mathbb{R}) \cap H^k(\mathbb{R})$, $k \in \mathbb{N}$. A space of *nonlocalized* perturbations should include functions which are neither periodic nor localized. A common choice is the space $C^m_{\mathrm{ub}}(\mathbb{R})$, $m \in \mathbb{N}_0$, which contains all bounded, m-times differentiable functions whose derivatives are bounded and uniformly continuous.[1] In particular, $C^m_{\mathrm{ub}}(\mathbb{R})$ contains every C^∞-function with bounded derivatives and we therefore say that $C^m_{\mathrm{ub}}(\mathbb{R})$ is a class of *fully nonlocalized* perturbations.

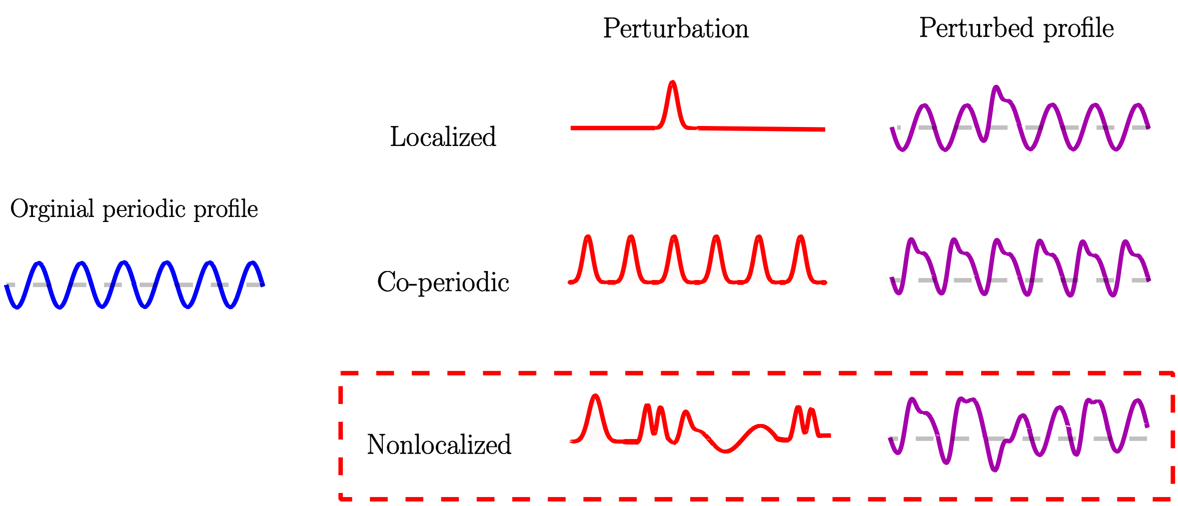


Figure 1.2.: Illustration of different types of perturbations (restricted to one component). The focus of this thesis is on nonlocalized perturbations.

[1]This space forms a Banach space when equipped with the standard $W^{m,\infty}(\mathbb{R})$-norm.

The purpose of this introduction is twofold. First, we give an overview of the current state of the literature. We introduce the models that we consider in this thesis and *informally* state our main results from Chapters 3 to 5 and Appendix A. The second purpose is to set up the modulational approach, which we follow throughout this thesis. For this intention, we introduce the abstract framework into which the presented models are embedded. This allows us to discuss certain challenges more explicitly. In particular, we summarize different approaches to linear estimates in Section 1.5. In Chapter 2, we discuss toy examples to illustrate crucial ideas and techniques. All in all, the introduction together with Chapter 2 are meant to support reading of the remaining chapters which contain the main mathematical contributions of this thesis and their proofs.

1.1. Literature overview

The research on nonlinear stability in general and specifically of periodic wave trains is a highly active area of research with long tradition. Periodic patterns are one of the most fundamental patterns which arise through the instability of a homogeneous state. Such a pattern-forming mechanism was first discovered by Turing in the pioneering work [99] from 1952 for a linear reaction-diffusion system describing phenomena in morphogenesis. In 1977, Swift and Hohenberg [98] proposed their model

$$\partial_t u = -(1+\partial_x^2)^2 u + \mu u - u^3, \quad u \in \mathbb{R}, \tag{1.3}$$

as reduced equation to describe effects in thermal fluctuations in fluid flows.[2] This equation is one of the simplest nonlinear model admitting pattern formation. In fact, if μ exceeds the critical value $\mu_* = 0$, the homogeneous state $u \equiv 0$ becomes unstable and periodic patterns arise. The equation (1.3) is a paradigm example for which the Ginzburg-Landau equation

$$\partial_t u = \partial_x^2 u + u - |u|^2 u, \quad u \in \mathbb{C}, \tag{1.4}$$

can be rigorously derived as amplitude equation. The validation of (1.4) as amplitude equation was also rigorously justified for other dissipative systems such as the Taylor-Couette problem or reaction-diffusion systems. We refer to [94] for a systematic and detailed presentation of this topic and further references. The Ginzburg-Landau formalism suggests an important heuristic:

If we can show the existence of periodic waves and their stability for (1.4), then we can also show them for a system for which (1.4) serves as amplitude equation.

The first rigorous nonlinear stability result of periodic waves against localized perturbations was obtained in the Ginzburg-Landau equation for the periodic roll waves $x \mapsto \sqrt{1-q^2}e^{iqx}$, $q \in (-1,1)$, against localized perturbations. This is due to Collet, Eckmann and Epstein [26] in 1992. Switching to polar coordinates, the associated perturbation equations admit constant coefficients and the spectrum can be computed explicitly with stable parameter regime $q^2 < \frac{1}{3}$. Furthermore, in contrast to general reaction-diffusion systems, the analysis of (1.4) simplifies due to translational, reflection and gauge symmetry. In the same year, Bricmont and Kupiainen publish their paper [16] where they introduce the renormalization group approach and extend the results from [26]. They could allow for small wavenumber-offsets giving rise to linear

[2] An interesting side note due to [27] is that the Swift-Hohenberg equation is already formulated in unpublished manuscripts of Turing.

spatial growth of the phase modulation. With the help of the renormalization group approach, fundamental progress has been made for the nonlinear stability analysis of periodic wave trains in a rich variety of models beyond the Ginzburg-Landau equation, i.e., in absence of gauge invariance. We particularly mention Schneider's pioneering work on nonlinear stability in the Swift-Hohenberg equation [91] from 1996 and the Taylor-Couette problem [93] from 1998. Finally, the concept of *diffusive spectral stability* became standard as starting point to obtain nonlinear stability; an illustration is given in Figure 1.3 and we refer to Section 1.3 for an introduction of this concept.

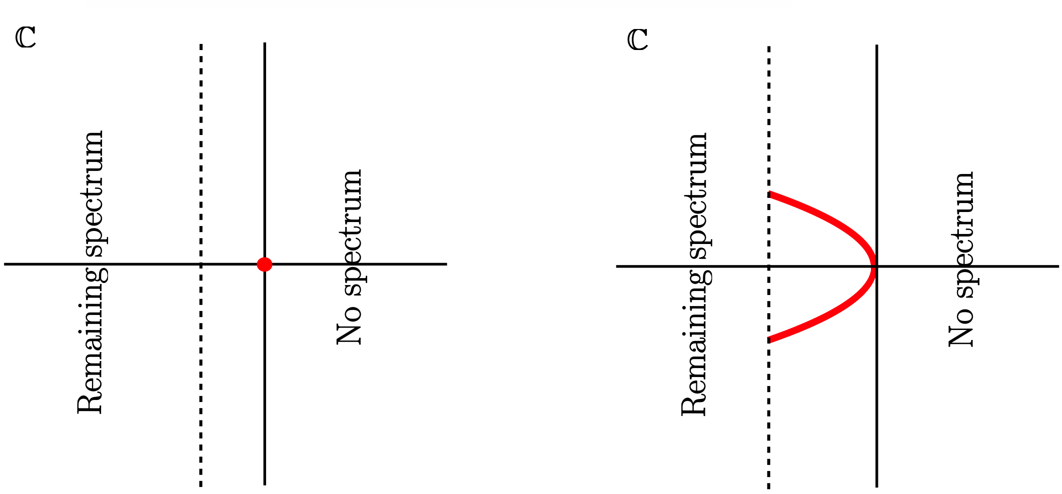


Figure 1.3.: We schematically illustrate two scenarios of spectra of the linearization of a PDE about the wave trains. Left: this is the case when considering the spectrum on co-periodic functions, i.e., functions with the same period as the periodic profile, where only discrete point spectrum occurs. The eigenvalue in zero originates from the translational invariance of the system, that is, ϕ_0' is an eigenfunction to the eigenvalue 0. A standard reference on nonlinear stability against co-periodic perturbations is [64]. The central aspect to show nonlinear stability is to isolate this eigenvalue which reduces the critical dynamics to an ordinary differential equation as the remaining spectrum is uniformly confined to left half plane. Right: a schematic illustration of what we understand as *diffusive spectral stability* which lies at the core of the nonlinear stability analysis against localized and (fully) nonlocalized perturbations, see Section 1.4. Instead of an ordinary differential equation, the critical quadratic curve of the spectrum (red) gives rise to a (convective) heat equation capturing the critical dynamics induced by translational invariance.

Between 2011 and 2014, Johnson, Noble, Rodrigues and Zumbrun published a series of papers demonstrating nonlinear stability of periodic wave trains in more general settings based on a modulational approach. For parabolic reaction-diffusion systems nonlinear stability of periodic wave trains against $L^1 \cap H^k$-perturbations is established in [55] under diffusive spectral stability assumptions. They extend their theory by allowing for nonlocalized phase modulations which connect constant states at spatial $\pm\infty$, c.f. [56, 57]. We refer to such a result as *modulational stability*. These considerations ultimately apply to viscous conservation laws by taking into account that the spectrum close to the origin may consist of multiple critical curves [59].

We further mention the work of Jung [62] using a Green's function approach and the work of Scheel and Wu [90] using an approach via normal forms to show nonlinear stability of periodic wave trains in parabolic reaction-diffusion systems.

The first result on modulational stability for parabolic reaction-diffusion systems can be traced back to [89] relying on the renormalization group approach. The phase modulation there may be chosen as explicit shock wave solution of a viscous Hamilton-Jacobi equation

$$\partial_t u = d\partial_x^2 u + a\partial_x u + \nu(\partial_x u)^2 \tag{1.5}$$

with constants $d > 0$ and $a, \nu \in \mathbb{R}$. This result may be referred to as *diffusive mixing* of the periodic wave train, i.e., two distinct values of the phase modulation at $\pm\infty$ are mixed through the (nonlinear) diffusion equation (1.5). In 2019, Iyer and Sandstede extend this result to allow for *large* phase offsets. We also refer to Figure 1.4.

The first nonlinear stability result against fully nonlocalized perturbations is due to Hilder, de Rijk and Schneider in [51] from 2022. There, the authors establish nonlinear stability of the roll waves in the Ginzburg-Landau equation against perturbations from $C^m_{\text{ub}}(\mathbb{R})$. Since the linearization about the origin has constant coefficients, they are in the position to filtrate modes with the help of the Fourier transform. A variant to the Fourier transform is to apply the Bloch transform to find mode filtration in $L^2(\mathbb{R})$ whenever the linearization admits periodic coefficients. This procedure gets more complicated for nonlocalized perturbations. This is circumvented in [85] by following a Green's function approach to show nonlinear stability periodic waves in two spatial dimensions whose perturbations are nonlocalized in one spatial direction. Solely relying on parabolic smoothing, a nonlinear stability theory is established in [84] for parabolic reaction-diffusion systems exploiting a modulational approach. A central achievement was to control the perturbed viscous Hamilton-Jacobi equation, which captures the most critical dynamics, with the aid of the Cole-Hopf transform. We refer to Sections 1.4 and 2.3 for insights on this decisive aspect of the analysis.

In this thesis, we present new results on the nonlinear stability of periodic wave trains against nonlocalized perturbations along three paradigm models arising in various disciplines. One particular concern is to extend the theory [84] beyond the parabolic setting to enclose other phenomenological models such as the FitzHugh-Nagumo system and Lugiato-Lefever equation. Another crucial contribution of this thesis is an extended theory of the state-of-the-art modulational stability results which all rely on spatial localization. We summarize our new results and relate them to the existing literature in the next section.

1.2. New results and embedding into the literature

We start by briefly discussing which results we can expect. Reconsidering Figure 1.3, we detect a lack of a spectral gap due to translation invariance of the system and therefore the evolution generated by the linearization about the wave train is at best bounded from a perturbation space into itself in virtue of the Hille-Yosida theorem. This suggests that in general we can only expect nonlinear stability of the periodic wave trains but no convergence rates for the perturbations as $t \to \infty$ without assuming localization assumptions on the initial perturbation.

We *informally* state and discuss the main results of the thesis. We refer to the corresponding chapters for *rigorous* formulations of the theorems. Furthermore, we refrain from discussing regularity assumptions and whether we can control higher derivatives in the informal statements. For details on these aspects, we again refer to the corresponding chapters.

A paradigm model from biology is the **FitzHugh-Nagumo system**

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = \begin{pmatrix} \partial_x^2 u \\ 0 \end{pmatrix} + \begin{pmatrix} u(1-u)(u-\mu) - v \\ \varepsilon(u - \gamma v - \mu) \end{pmatrix}, \quad (x,t) \in \mathbb{R} \times [0,\infty), \quad u(x,t), v(x,t) \in \mathbb{R} \quad (1.6)$$

with parameters $\mu \in \mathbb{R}$, $\gamma > 0$ and $\varepsilon > 0$. The system (1.6) serves as phenomenological simplification of the Hodgkin-Huxley model describing signal propagation in nerve fibers [77]. Assuming $k_0 = 1$ and that the periodic wave profile ϕ_0 moves with nonzero speed $c_0 = \omega_0$, we perform a coordinate change $\zeta = x - \omega_0 t$ which yields the system

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = \begin{pmatrix} \partial_\zeta^2 u \\ 0 \end{pmatrix} + c_0 \begin{pmatrix} \partial_\zeta u \\ \partial_\zeta v \end{pmatrix} + \begin{pmatrix} u(1-u)(u-\mu) - v \\ \varepsilon(u - \gamma v - \mu) \end{pmatrix}, \quad (\zeta,t) \in \mathbb{R} \times [0,\infty). \quad (1.7)$$

We observe that (1.7) is now a damped transport equation in the principal part of the second component. **Chapter 3** is the content of the preprint [5] which is a joint work with Björn de Rijk. We informally state its main result.

Informal statement of main result in Chapter 3. *Let $\phi_0 : \mathbb{R} \to \mathbb{R}^2$ be a smooth and diffusively spectrally stable (see Figure 1.3), stationary solution to (1.7). Then, there exist $C, \delta > 0$ such that for every $\tilde{w}_0 \in C^3_{ub}(\mathbb{R}) \times C^2_{ub}(\mathbb{R})$ with $E_0 := ||\tilde{w}_0||_{C^3_{ub}(\mathbb{R}) \times C^2_{ub}(\mathbb{R})} < \delta$ there exists a unique classical global solution $w(t) = (u(t), v(t))^T$ of (1.7) with $w(0) = \tilde{w}_0 + \phi_0$ fulfilling the estimate*

$$||w(t) - \phi_0||_{L^\infty} \leq C E_0$$

for all $t \geq 0$.

In particular, ϕ_0 is nonlinearly L^∞-stable against perturbations from $C^3_{ub}(\mathbb{R}) \times C^2_{ub}(\mathbb{R})$.[3]

As coupling of a scalar reaction-diffusion equation and a linear ordinary differential equation, the system (1.6) admits remarkably rich dynamics and gives rise to many types of patterns. An interesting class of patterns are pattern-forming fronts which connect a constant state in leading edge to a periodic pattern in the wake. A guiding paper from 2024 on their nonlinear stability against localized perturbations is [8] which also includes a theorem on the nonlinear stability of purely periodic wave trains. To the author's best knowledge, this is the first nonlinear stability result of periodic wave trains against localized perturbations in the FitzHugh-Nagumo system. Building on their considerations, our result is the first to lift any assumption on localization of the initial perturbation in a dissipative, non-parabolic system. We emphasize that our analysis presents a robust theory that does not depend on the specific features of the FitzHugh-Nagumo system, meaning that our developed scheme might be applied to other dissipative systems which satisfy the following assumptions

(A1) The linearization about the wave train generates a C_0-semigroup on $C_{\mathrm{ub}}(\mathbb{R})$;

[3] In Chapter 3, we show a stronger result by only assuming that the initial perturbations have to be small in L^∞ while keeping the regularity assumptions.

(A2) The periodic profile is diffusively spectrally stable.

It is an interesting task for future research to establish such a general theory for systems supporting (A1)-(A2). A crucial challenge in this approach is to control regularity in the arising quasilinear iteration scheme on which we remark in the introductory Sections 1.4 and 2.2. On the other hand, the lack of parabolicity and the question on how to filtrate modes in $C_{\mathrm{ub}}(\mathbb{R})$ are challenges in developing a suitable framework and proving linear decay rates. We address them in Section 1.5.

In **Chapter 4**, we study parabolic reaction-diffusion systems which are of the form

$$\partial_t u = Du_{xx} + f(u), \quad (x,t) \in \mathbb{R} \times [0,\infty), \quad u(x,t) \in \mathbb{R}^n \tag{1.8}$$

with positive definite $D \in \mathbb{R}^{n\times n}$ and smooth $f : \mathbb{R}^n \to \mathbb{R}^n$. The main result of the preprint [4] (jointly with Björn de Rijk) is devoted to modulational stability. We heuristically motivate the meaning of this concept. If ϕ_0 is a stationary periodic solution of (1.8), then translational invariance of (1.8) implies that also the constantly shifted wave train $\phi_0(\cdot + s)$ is a stationary periodic solution of (1.8) as well for every $s \in \mathbb{R}$. In particular, $\phi_0(\cdot+s)$ is diffusively spectrally stable whenever ϕ_0 is. Therefore, at the onset of pattern formation, one expects that not only a pure periodic wave could be observed but also modulated versions of them. Such modulated variants are in general given as $\phi(\cdot + \gamma_0)$ with bounded initial phase modulation $\gamma_0 : \mathbb{R} \to \mathbb{R}$ and we ask whether they are stable in an appropriate sense. We give an informal statement of our result, see also Figure 1.4.

Informal statement of main result in Chapter 4. *Let $\phi_0 : \mathbb{R} \to \mathbb{R}^n$ be a smooth and diffusively spectrally stable periodic wave train to (1.8). For simplicity, we set $k_0 = 1$ and $\omega_0 = 0$ in (1.1). Fix any $M \geq 1$. Then, for every small $\alpha \geq 0$ there exist some $C, \varepsilon > 0$ such that for every initial phase modulation $\gamma_0 \in C^1_{ub}(\mathbb{R})$ and $u_0 \in C_{ub}(\mathbb{R})$ with $||\gamma_0||_{L^\infty} \leq M$ and $E_0 := ||\partial_x\gamma_0||_{L^\infty} + ||u_0 - \phi_0(\cdot + \gamma_0)||_{L^\infty} < \varepsilon$ there exist a unique classical global solution $u(t)$ of (1.8) with $u(0) = u_0$ and some globally defined modulation function $\gamma(t)$ with $\gamma(0) = \gamma_0$ such that*

$$||u(t) - \phi_0(\cdot + \gamma(t))||_{L^\infty} \leq CE_0^\alpha (1+t)^{-\frac{1}{2}+\alpha}$$

for all $t \geq 0$. Moreover, there are $a, \nu \in \mathbb{R}$ and $d > 0$ (see (P4)) such that we can approximate

$$\left|\left|\partial_x^j(\gamma(t) - \check{\gamma}(t))\right|\right|_{L^\infty} \leq CE_0^{\frac{2}{3}\alpha}(1+t)^{-\frac{j}{2}}, \quad j = 0,1, \tag{1.9}$$

for all $t \geq 0$, where $\check{\gamma}(t)$ is the solution of the viscous Hamilton-Jacobi equation

$$\partial_t\check{\gamma} = d\partial_x^2\check{\gamma} + a\partial_x\check{\gamma} + \nu(\partial_x\check{\gamma})^2, \quad (x,t) \in \mathbb{R} \times [0,\infty),$$

with $\check{\gamma}(0) = \gamma_0$.

An interesting observation in this theorem is that if we formally set $\gamma(t) \equiv 0$, then we obtain nonlinear stability of the original wave train. Therefore, modulational stability can be considered as a generalized concept of nonlinear stability. We note that we can also use the explicit solution of the viscous Hamilton-Jacobi equation as modulation function: by (1.9) and the mean value theorem, we obtain

$$||u(t) - \phi_0(\cdot + \check{\gamma}(t))||_{L^\infty} \leq CE_0^{\frac{2}{3}\alpha}, \quad t \geq 0.$$

The reason for the α-exponent is an interpolation argument which allows to balance the smallness of $\partial_x\gamma_0$ and decay of $\partial_x e^{\partial_x^2 t}$ from $C_{\rm ub}(\mathbb{R})$ to $C_{\rm ub}(\mathbb{R})$. This idea is paradigmatically executed in Section 2.3. There, we also indicate where the estimate (1.9) originates from.

Importantly, assuming higher regularity on the initial perturbation, we expect that our scheme also applies for the FitzHugh-Nagumo by direct adaptions from Chapter 3.

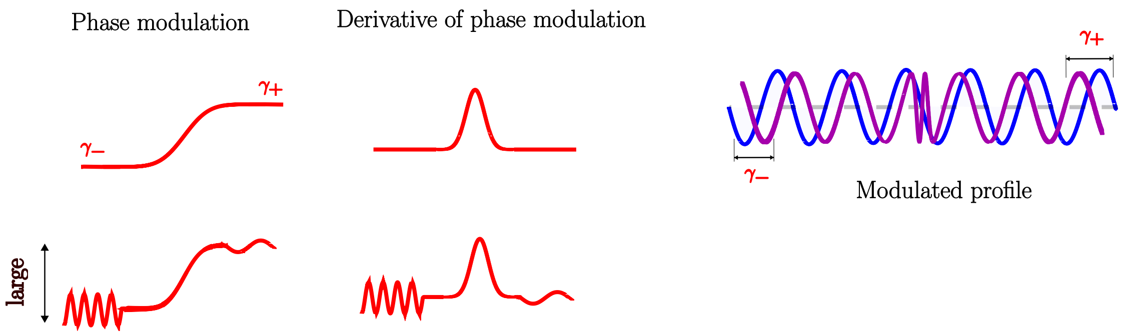


Figure 1.4.: Illustration of different initial scenarios where the scenario in the first row is studied in [54, 57, 89]. The purple profile is the modulated profile with indicated phase shifts at $x = \pm\infty$. In Chapter 4 (scenario: second row in figure), we drop the smallness assumptions from [89] and [57] on $||\gamma_0||_{L^\infty}$ but also any localization assumption on the initial conditions from [57] and [54]. The figure indicates that we can allow for slow oscillations of the phase modulation at spatial $\pm\infty$ in Chapter 4.

Example. Take $L > 0$ and set $\gamma_0(x) = L(\sin(\delta x) + \arctan(\delta x))$, $x \in \mathbb{R}$. Now, we set $\delta > 0$ sufficiently small while L may be large. This a concrete example of an initial phase modulation which is covered by our result in Chapter 4 but not by any earlier result since $\partial_x\gamma_0$ does not lie in $L^p(\mathbb{R})$ for any $1 \le p < \infty$. In general, one can consider each $g_0 \in C^1_{\rm ub}(\mathbb{R})$ and choose $\gamma_0(x) = g_0(\delta x)$ for sufficiently small $\delta > 0$. □

The third model under investigation, in **Chapter 5**, is the **Lugiato-Lefever equation** which reads as

$$\partial_t u = -\beta i \partial_x^2 u - (1 + i\alpha)u + i|u|^2 u + F, \quad (x,t) \in \mathbb{R} \times [0,\infty), \quad u(x,t) \in \mathbb{C} \tag{1.10}$$

with parameters $\beta \in \{-1, 1\}, \alpha \in \mathbb{R}$ and $F > 0$. Mathematically, (1.10) is a variant of the cubic nonlinear Schrödinger equation with an additional damping term $-u$ and a constant forcing term $F > 0$. In 1987, Lugiato and Lefever have derived this equation as a model for nonlinear optical systems [72] and identified it as paradigm model for pattern formation.

The equation (1.10) has recently been studied in [46] where it was shown that periodic standing waves are nonlinearly stable against localized perturbations. In the earlier work [96], nonlinear stability is shown against co-periodic perturbations. The Chapter 5 consists of the preprint [2] where we unify both theories by perturbing against *co-periodic plus localized* perturbations. These sums constitute a non-trivial class of nonlocalized perturbations. Considerations from fiber optics [67], where combinations of localized and co-periodic effects naturally occur, motivate the study of these types of perturbations. We informally state the main result of this chapter.

Informal statement of main result in Chapter 5. *Let $\phi_0 : \mathbb{R} \to \mathbb{R}^2$ be a smooth and T-periodic, stationary solution of (1.10). In addition, assume that ϕ_0 is diffusively spectrally stable. Then, there exist some $C, \varepsilon > 0$ such that for every $\tilde{u}_0 \in H^6_{per}(0,T) \oplus H^3(\mathbb{R})$ with $E_0 := ||\tilde{u}_0||_{H^6_{per}(0,T)\oplus H^3(\mathbb{R})} < \varepsilon$ there exists a unique classical global solution $u(t)$ of (1.10) with $u(0) = \tilde{u}_0 + \phi_0$ fulfilling the estimate*

$$||u(t) - \phi_0||_{L^\infty} \leq CE_0$$

for all $t \geq 0$.

In particular, ϕ_0 is nonlinearly L^∞-stable against perturbations from $H^6_{per}(0,T) \oplus H^3(\mathbb{R})$.[4]

To construct a solution of (1.10) with initial datum $w_0 + v_0 \in L^2_{\text{per}}(0,T) \oplus L^2(\mathbb{R})$, one first solves (1.10) for $w(t)$ with $w(0) = w_0 \in L^2_{\text{per}}(0,T)$ and then derives the perturbation equation for $v(t)$ which can be solved in $L^2(\mathbb{R})$ such that $w(t) + v(t)$ is again a solution of (1.10) with $w(0) + v(0) = w_0 + v_0$. Therefore, the main idea of the proof of the nonlinear stability result is to perform an iterative L^2-scheme on the perturbation equations of $v(t)$, similarly as in [46] and [105], where the novel difficulty is to control v-w-mix terms. We resolve this difficulty by establishing a suitable modulation ansatz which allows to rely on the orbital stability result from [96] for the co-periodic perturbation $w(t)$.

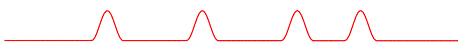
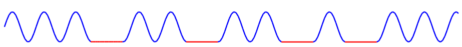
	Small initial perturbation	Reference
$L^2_{\text{per}}(0,T)$		[96] (2018)
$L^2(\mathbb{R})$		[46] (2022)
$L^2_{\text{per}}(0,T) \oplus L^2(\mathbb{R})$		Chapter 5

Table 1.1.: A related question to the one in Chapter 5 is whether solutions of the cubic nonlinear Schrödinger equation exist globally for initial data from $L^2_{\text{per}}(0,T) \oplus L^2(\mathbb{R})$. This problem is also referred to as *tooth problem* in the literature and the red lines in the last row might be seen as 'knocked out teeth'. For a discussion on the tooth problem, we refer to [24, 69].

The reader might wonder whether we can apply our scheme for the FitzHugh-Nagumo system from Chapter 3 to establish a nonlinear stability result against fully nonlocalized perturbations in the case of the Lugiato-Lefever equation. A first complication to this suggestion is that (A1) is violated as shown in [14]. An alternative choice of spaces to guarantee that $e^{i\partial_x^2 t}$ generates a semigroup on a space of fully nonlocalized perturbations are the so-called modulation spaces. We briefly discuss the possibility of such an extension in Chapter 5. In general, global existence for Schrödinger-type equations with fully nonlocalized initial data is a widely open problem.

[4] We systematically introduce the spaces $H^k_{\text{per}}(0,T) \oplus H^l(\mathbb{R})$ at the end of Chapter 5.

The **Appendix** A is devoted to a detached result on the Laplace transform: during the work on [5], we were confronted with the validity of the complex inversion formula of the Laplace transform applied to convolutions of C_0-semigroups to generate high-frequency estimates, see Chapter 3 and Section 1.5. This was the motivation for the preprint [3] (to appear in *Archiv der Mathematik*). We show that for a vector-valued function the condition of Lipschitz continuity in [6] can be relaxed to local Hölder continuity to obtain complex inversion of the Laplace transform with locally uniform convergence. In particular, this result applies for C_0-semigroups on Favard spaces.

Notation.

- Let S be a set, and let $A, B\colon S \to \mathbb{R}$. Throughout the thesis, the expression '$A(x) \lesssim B(x)$ for $x \in S$', means that there exists a constant $C > 0$, independent of x, such that $A(x) \leq CB(x)$ holds for all $x \in S$. We also write '$A(x) = O(B(x))$' instead of '$A(x) \lesssim B(x)$'.
- Let X be a normed space and let S be a set. Let $A(t) : X \to X$ for $t \in S$ and $B : S \to \mathbb{R}$. The expression '$A(t) = O_{X\to X}(B(t))$' means that that there exists a constant $C > 0$, independent of x and t, such that $||A(t)||_X \leq CB(t)||x||_X$ holds for all $x \in X$ and $t \in S$.

Comment on notational consistency. The notation within each Chapter 3 to 5 is consistent and can differ between the chapters. This means for example that the phase modulation throughout Chapter 3 is called Ψ while we call it γ in the others. The reason is that the Chapters 3 to 5 mainly consist of the corresponding preprints.

1.3. Abstract framework: existence and spectrum

The systems (1.6), (1.8) and the equation (1.10) (by splitting into real and imaginary parts) can be written as a semilinear system which is of the form

$$\partial_t u = \mathcal{D}\partial_x^2 u + f(u), \quad (x,t) \in \mathbb{R} \times [0,\infty), \quad u(x,t) \in \mathbb{R}^n \tag{1.11}$$

with $n \in \mathbb{N}$, smooth $f : \mathbb{R}^n \to \mathbb{R}^n$ and a matrix $\mathcal{D} \in \mathbb{R}^{n\times n}$.

If $\mathcal{D}$ is semi-positive definite, then we call (1.11) a reaction-diffusion system. In this case, (1.11) becomes *parabolic* if $\mathcal{D}$ is positive definite and we call it *incomplete parabolic* otherwise.

Existence. Let $k_0 \neq 0$, $\omega_0 \in \mathbb{R}$ and $T > 0$. We assume the existence of a T-periodic profile $\phi_0 : \mathbb{R} \to \mathbb{R}^n$ such that the wave train

$$\phi_0(k_0 x - \omega_0 t), \quad (x,t) \in \mathbb{R} \times [0,\infty),$$

is a solution of (1.11).

The comoving frame. The coordinate change $\zeta = k_0 x - \omega_0 t$ transforms (1.11) into

$$\partial_t u = k_0^2 \mathcal{D}\partial_\zeta^2 u + \omega_0 \partial_\zeta u + f(u), \quad (\zeta,t) \in \mathbb{R} \times [0,\infty), \quad u(\zeta,t) \in \mathbb{R}^n. \tag{1.12}$$

The periodic profile ϕ_0 is now a stationary solution of (1.12). Measuring the deviation $v(\zeta,t) = u(\zeta,t) - \phi_0(\zeta)$ of a solution u of (1.12) from ϕ_0, we arrive at the perturbation equation

$$\partial_t v = \mathcal{L}_0 v + \mathcal{R}(v)$$

where the linearization $\mathcal{L}_0$ of (1.12) about ϕ_0 is given by

$$\mathcal{L}_0 v = k_0^2 \mathcal{D}\partial_\zeta^2 v + \omega_0 \partial_\zeta v + f'(\phi_0)v$$

and the nonlinear residual reads as

$$\mathcal{R}(v) = f(v+\phi_0) - f(\phi_0) - f'(\phi_0)v.$$

The linearization $\mathcal{L}_0$ is posed on $C_{\mathrm{ub}}(\mathbb{R})$ with $\mathrm{dom}(\mathcal{L}_0) \subset C_{\mathrm{ub}}(\mathbb{R})$ resp. on $L^2(\mathbb{R})$ with $\mathrm{dom}(\mathcal{L}_0) \subset L^2(\mathbb{R})$.

Spectral preliminaries. We introduce the Bloch operators

$$\mathcal{L}(\xi) = e^{-i\xi\cdot}\mathcal{L}_0 e^{i\xi\cdot}, \quad \xi \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right),$$

posed on $L^2_{\mathrm{per}}(0,T)$ with $\mathrm{dom}(\mathcal{L}(\xi)) \subset L^2_{\mathrm{per}}(0,T)$. It holds that $\mathcal{L}(\xi)$ has compact resolvent and its spectrum consists of isolated eigenvalues of finite algebraic multiplicity. Furthermore, we have the identities

$$\sigma(\mathcal{L}_0) := \sigma_{C_{\mathrm{ub}}}(\mathcal{L}_0) = \sigma_{L^2}(\mathcal{L}_0) = \bigcup_{\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})} \sigma_{L^2_{\mathrm{per}}(0,T)}(\mathcal{L}(\xi)). \tag{1.13}$$

We refer to [39, 83] and [29] for details. The spectrum of $\sigma_{C_{\mathrm{ub}}}(\mathcal{L}_0)$ and $\sigma_{L^2}(\mathcal{L}_0)$ qualitatively differ: while $\sigma_{C_{\mathrm{ub}}}(\mathcal{L}_0)$ consists of a continuum of eigenvalues, $\sigma_{L^2}(\mathcal{L}_0)$ is purely essential.

We make the following assumptions on the spectrum which we refer to as *diffusive spectral stability*:

(D1) It holds $\sigma(\mathcal{L}_0) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\}$;

(D2) There exists $\theta > 0$ such that for any $\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})$ we have $\mathrm{Re}\ \sigma_{L^2_{\mathrm{per}}(0,T)}(\mathcal{L}(\xi)) \leq -\theta\xi^2$;

(D3) 0 is a simple eigenvalue of $\mathcal{L}(0)$.

Due to translational invariance of (1.12) and (D3), we have

$$\ker_{L^2_{\mathrm{per}}(0,T)}(\mathcal{L}(0)) = \mathrm{span}\{\phi_0'\}, \tag{1.14}$$

whenever ϕ_0 is nonconstant.This gives rise to one critical spectral curve $\lambda_c(\xi)$, $\xi \in (-\xi_0, \xi_0)$, for some $\xi_0 \in (0, \frac{\pi}{T})$ touching the origin and which is confined to the left half plane, see Figure 1.5. Due to the necessary eigenvalue in the origin of $\mathcal{L}(0)$, we note that the assumptions (D1)-(D3) are the generically most stable configuration one can hope for in dissipative systems. We informally summarize some consequences of (D1)-(D3) and refer to [29] for their justifications:

(P1) There exist a neighborhood $U \subset \mathbb{R}$ of k_0 and functions $\phi\colon \mathbb{R} \times U \to \mathbb{R}^n$ and $\omega\colon U \to \mathbb{R}$ with $\phi(\cdot; k_0) = \phi_0$ and $\omega(k_0) = \omega_0$ such that

$$u_k(x,t) = \phi(kx - \omega(k)t; k),$$

is a solution to (1.11) of period T for each wavenumber $k \in U$. We can arrange for $\langle \widetilde{\Phi}_0, \phi_0' \rangle_{L^2_{\mathrm{per}}(0,T)} = 1$, where $\widetilde{\Phi}_0$ is an eigenfunction to the eigenvalue 0 of the adjoint Bloch operator $\mathcal{L}(0)^*$.

(P2) There exist eigenfunctions Φ_ξ with $\Phi_0 = \phi_0'$ associated to the simple eigenvalue $\lambda_c(\xi)$ of $\mathcal{L}(\xi)$, i.e., $\mathcal{L}(\xi)\Phi_\xi = \lambda_c(\xi)\Phi_\xi$ for $\xi \in (-\xi_0, \xi_0)$.

(P3) There exist eigenfunctions $\tilde{\Phi}_\xi$ associated to the simple eigenvalue $\overline{\lambda_c(\xi)}$ of the adjoint operator $\mathcal{L}(\xi)^*$, i.e., $\mathcal{L}(\xi)^*\tilde{\Phi}_\xi = \overline{\lambda_c(\xi)}\tilde{\Phi}_\xi$ for $\xi \in (-\xi_0, \xi_0)$.

(P4) The expansions

$$\left|\lambda_c(\xi) - \mathrm{i}a\xi + d\xi^2\right| \lesssim |\xi|^3, \qquad \left\|\Phi_\xi - \phi_0' - \mathrm{i}k_0\xi\partial_k\phi(\cdot;k_0)\right\|_{H^m(0,1)} \lesssim |\xi|^2,$$

hold for $\xi \in (-\xi_0, \xi_0)$ with coefficients $a \in \mathbb{R}$ and $d > 0$.

One can explicitly determine

$$a = \omega_0 - k_0\omega'(k_0), \quad d = k_0^2\langle\tilde{\Phi}_0, D\phi_0' + 2k_0 D\partial_{\zeta k}\phi(\cdot;k_0)\rangle_{L^2_{\mathrm{per}(0,T)}}.$$

We also introduce $\nu = -\frac{1}{2}k_0\omega''(k_0)$ whose relevance becomes apparent later.

The coefficient a is referred to as group velocity of the wave train in the literature.

Remark 1.3.1. *We have the symmetry property: if $\lambda \in \sigma(\mathcal{L}_0)$, then $\bar{\lambda} \in \sigma(\mathcal{L}_0)$. This is due to the observation that $\mathcal{L}_0$ has real coefficients. In particular, $\lambda_c(-\xi) = \overline{\lambda_c(\xi)}$, $\xi \in (-\xi_0, \xi_0)$. In case that the system admits reflection symmetry and the periodic profile is even, i.e., $\phi(x) = \phi(-x)$, we additionally have $\lambda_c(-\xi) = \lambda_c(\xi)$ and therefore $\lambda_c(\xi) \in \mathbb{R}$ for $x \in (-\xi_0, \xi_0)$.*

We now embed the presented models into this abstract framework:

- The **FitzHugh-Nagumo system** is a reaction-diffusion system of incomplete parabolic type where $\mathcal{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The existence of periodic traveling waves is established in [95] by exploiting the slow-fast structure of the system. In [33], a spectral analysis of the resulting linearization is performed in the case $\gamma = 0$ for which diffusive spectral stability cannot hold due to the lack of damping in the second component. Another recent type of periodic wave trains is shown to exist with justified spectral assumptions (D1)-(D3) in [71]. This is to the author's best knowledge the first result proving existence of periodic wave trains which are diffusively spectrally stable for $\gamma > 0$.

- As example for **parabolic reaction-diffusion systems**, one considers the Brusselator model

$$\begin{aligned} \partial_t u &= d_1\partial_x^2 u + a - (\beta+1)u + u^2 v, \\ \partial_t v &= d_2\partial_x^2 v + \beta u - u^2 v, \end{aligned} \qquad (x,t) \in \mathbb{R}\times[0,\infty), \quad (u(x,t), v(x,t)) \in \mathbb{R}^2 \quad (1.15)$$

  with parameters $d_1, d_2 > 0$ and $a, \beta \in \mathbb{R}$. For this system, it is known through [97] that diffusively spectrally stable periodic wave trains exist in a certain paramater regime.

- The **Lugiato-Lefever equation** can be written in form (1.11) by switching to the variables $(\mathrm{Re}(u), \mathrm{Im}(u))$. This gives rise to $\mathcal{D} = -\beta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and a smooth reaction term $F : \mathbb{R}^2 \to \mathbb{R}^2$. The model (1.10) admits periodic standing waves and the associated linearization obeys (D1)-(D3) as shown in [28] and [13] exploiting Lyapunov Schmidt-reduction and bifurcation arguments. They are typically even due to the reflection symmetry of the Lugiato-Lefever equation.

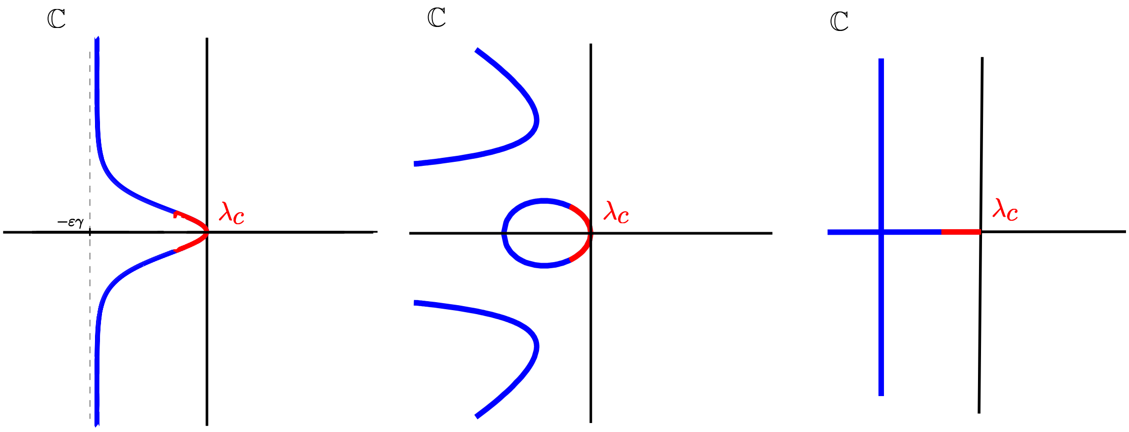


Figure 1.5.: Qualitative illustration of spectra supported by (D1)-(D3) with the critical touching at the origin. Left: the spectrum of FitzHugh-Nagumo system behaves asymptotically parallel to the imaginary axis with an asymptote $-\varepsilon\gamma$. Middle: image of the spectrum in parabolic reaction-diffusion system which in particular lies in a sector. Right: due to the reflection symmetry of the Lugiato-Lefever equation and the evenness of the standing wave, the critical curve is confined to the real axis.

Remark 1.3.2 (The scalar case). *It is crucial to work with **systems** of equations, i.e., $n \geq 2$, as the condition ((D1)) never holds in scalar equations. This follows from Sturm-Liouville theory [64] which tells us that the eigenfunction to the largest eigenvalue of $\mathcal{L}(0)$ is necessarily zero-free. If ϕ_0 is nonconstant, then the eigenfunction ϕ_0' to the zero eigenvalue has at least one zero and therefore there has to be an unstable eigenvalue. Through identity (1.13), this leads to nonlinear instability of non-trivial periodic wave trains against perturbations from $L^2(\mathbb{R})$ and $C_{ub}(\mathbb{R})$.*

1.4. The modulational approach

We consider a solution $u(t)$ of (1.12). By the touching of the spectrum of $\mathcal{L}_0$ at the origin, we cannot expect decay of the semigroup $e^{\mathcal{L}_0 t}$ from some Banach space X into itself by the Hille-Yosida theorem. This obstructs an immediate argument through iterative estimates on the Duhamel formula of the perturbation $u(t) - \phi_0$. The principal idea of the modulational approach is to *capture the critical touching of the spectrum at the origin on the linear and nonlinear level*. For this purpose, we introduce a phase modulation $\gamma(t)$[5] and the *inverse-modulated perturbation* of $u(t)$ given by

$$w(\zeta, t) = u(\zeta - \gamma(\zeta, t), t) - \phi_0(\zeta), \quad (\zeta, t) \in \mathbb{R} \times [0, \infty).$$

The inverse-modulated perturbation w satisfies a quasilinear system of the form

$$(\partial_t - \mathcal{L}_0)(w - \phi_0'\gamma) = \tilde{\mathcal{R}}(w, \gamma_\zeta, \gamma_t), \tag{1.16}$$

[5]We use γ to denote the modulation function which should not be confused with the constant in the FitzHugh-Nagumo system. In Chapter 3, we call the phase modulation ψ.

where we emphasize that the nonlinear residual $\tilde{\mathcal{R}}$ contains only derivatives of γ. We refer to [8, 55] for the derivation of (1.16). To exploit the structure of (1.16), one splits the linear evolution as

$$e^{\mathcal{L}_0 t} = S_r(t) + (\phi_0' + k_0\partial_k\phi(\cdot;k_0)\partial_\zeta)s_p(t) \tag{1.17}$$

by relying on (D1)-(D3). In Section 1.5, we discuss different approaches yielding the decomposition (1.17) and decay rates on $s_p(t)$ and $S_r(t)$. The requirement on such a decomposition is that $s_p(t)$ captures the linear critical dynamics caused by the parabolic touching of the critical spectral curve λ_c at the origin of the spectrum. Therefore, $s_p(t)$ qualitatively behaves as the (convective) heat semigroup $e^{(d\partial_\zeta^2 + a\partial_\zeta)t}$ with respect to decay and smoothing properties while we expect higher-order decay of $S_r(t)$.

We write $u_0 = u(0)$ and set $w_0 = u_0 - \phi_0$. Let $0 \leq \chi \leq 1$ be a smooth function such that $\chi \equiv 0$ on $[0,1]$ and $\chi \equiv 1$ on $[2,\infty)$. To capture the most critical behavior of $w(t)$, we define the phase modulation function $\gamma(t)$ by the Duhamel formula[6]

$$\gamma(t) = \chi(t)s_p(t)w_0 + \int_0^t \chi(t-s)s_p(t-s)\tilde{\mathcal{R}}(s)\,ds, \tag{1.18}$$

where we abbreviate $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(w, \gamma_\zeta, \gamma_t)$. Furthermore, using (1.17), we introduce the variable

$$z(t) = w(t) - k_0\partial_k\phi(\cdot;k_0)\gamma_\zeta(t) \tag{1.19}$$

which satisfies the Duhamel formula

$$z(t) = S_r(t)w_0 + \int_0^t S_r(t-s)\tilde{\mathcal{R}}(s)\,ds + \text{remaining terms}. \tag{1.20}$$

This gives rise to the expectation that $z(t)$ admits higher-order decay since $S_r(t)$ decays at higher-order, c.f. Section 1.5. We may also assume that the remaining terms decay at higher-order. The genuinely most critical term that might occur in the nonlinearity $\tilde{\mathcal{R}}$ is γ_ζ^2 as $\tilde{\mathcal{R}}$ contains only derivatives of γ and we may assume that $z(t)$, $\gamma_t(t)$ and higher derivatives of $\gamma(t)$ decay at higher-order. In summary, the most critical part of our stability analysis is governed by a perturbed viscous Hamilton-Jacobi

$$\partial_t\gamma = d\partial_\zeta^2\gamma + a\partial_\zeta\gamma + \nu(\partial_\zeta\gamma)^2 + \text{h.o.t} \tag{1.21}$$

with $a \in \mathbb{R}$ and $d > 0$ from the second order Taylor expansion of the critical spectral curve λ_c in (P4). Furthermore, it is possible to show that $\nu \in \mathbb{R}$ is precisely the value as defined in (P4), c.f. [84]. The particularly interesting scenario is when $\nu \neq 0$ since the remaining nonlinear terms in (1.21) are of higher-order. In this case, the nonlinearity $\nu(\partial_\zeta\gamma)^2$ can be removed via the Cole-Hopf transform and we give a comprehensive discussion of this procedure in Section 2.3.

Regularity control. Due to the quasilinearity of (1.16), spatial derivatives of w appear in the residual nonlinearity $\tilde{\mathcal{R}}$ and a crucial challenge is to control these derivatives. This is usually and also here achieved by *nonlinear damping estimates* on the forward-modulated perturbation $\mathring{u}(\zeta,t) = u(\zeta,t) - \phi_0(\zeta + \gamma(\zeta,t), t)$ which satisfies again a semilinear system. Finally, one

[6]To guarantee that $\gamma(0) = \gamma_0$ for a prescribed γ_0, one needs to introduce an additional correction term. We suppress this correction term here, as it does not affect our explanations and we refer to Chapter 4 for details.

relates $||\partial_\zeta^k w(t)||_{L^\infty}$ to $||\partial_\zeta^k \mathring{u}(t)||_{L^\infty}$ to control derivatives of the inverse-modulate perturbation $w(t)$, see also [105]. Analogously, one argues for $z(t)$ and the modified forward-modulated perturbation $\mathring{z}(\zeta,t) = u(\zeta,t) - \phi_0(\zeta + \gamma(\zeta,t)(1+\partial_\zeta\gamma(\zeta,t)); k_0(1+\partial_\zeta\gamma(\zeta,t)))$ which are used in Chapters 3 and 4. These damping estimates contain pure L^∞- norms. We derive them by relying on uniformly local Sobolev norms which allow to define weighted energies for which we can show suitable energy estimates.[7] The regularity control via damping estimates is a robust approach and allows us to extend the analysis from [84] on parabolic systems, which solely relies on parabolic smoothing, to non-parabolic systems as done in Chapter 3 for the FitzHugh-Nagumo system. We provide an illustration of the used technique along toy examples in Section 2.2. Another possibility to control derivatives are the use of so-called *tame estimates* which are exploited in [85] or [46]. We do not go into detail on this approach here.

Similar as for the unmodulated perturbation, we note that the forward-modulated perturbation $\mathring{u}$ is not suitable to close an iteration argument on its Duhamel formula. The reason is that the linear evolution of the semilinear system satisfied by $\mathring{u}$ is again given through $e^{\mathcal{L}_0 t}$ and the structure on the left hand side of (1.17), which allows to isolate the critical contribution $s_p(t)$, is not present anymore.

Closing a nonlinear iteration argument. The intuition why an iteration argument closes by following the modulational approach is as follows. Considering the propagator $s_p(t)$, which behaves similar to the heat semigroup, one has $\partial_\zeta s_p(t) = O_{C_{\mathrm{ub}}\to C_{\mathrm{ub}}}(t^{-\frac{1}{2}})$ for $t > 0$. Since only derivatives of γ appear in the nonlinearity $\tilde{\mathcal{R}}$, we execute nonlinear iterative estimates on $\partial_\zeta\gamma(t)$ by taking the derivative of (1.18) with respect to the spatial variable ζ. Using the linear predictions for the decay rates $||\partial_\zeta\gamma(t)||_{L^\infty} = O((1+t)^{-\frac{1}{2}})$ and $||z(t)||_{L^\infty} = O((1+t)^{-1})$ and assuming that the term $(\partial_\zeta\gamma(t))^2$ is removed, the most critical terms that might appear in $\tilde{\mathcal{R}}(s)$ are for example $(\partial_\zeta\gamma(s))^3$ and $z(s)\partial_\zeta\gamma(s)$. Now, one estimates

$$\begin{aligned}&\int_0^t (1+t-s)^{-\frac{1}{2}}\Big(||\partial_\zeta\gamma(s)||_{L^\infty}^3 + ||z(s)||_{L^\infty}||\partial_\zeta\gamma(s)||_{L^\infty}\Big)\,ds\\ &\lesssim \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{3}{2}}\,ds \lesssim (1+t)^{-\frac{1}{2}}, \quad t\geq 0.\end{aligned}$$

We conclude that we get back the predicted decay $(1+t)^{-\frac{1}{2}}$ on $\partial_\zeta\gamma(t)$ for the nonlinear contributions $(\partial_\zeta\gamma(s))^3$ and $z(s)\partial_\zeta\gamma(s)$.

We refer to Sections 2.1 and 2.3 for explicit examples on how such an argument works.

Controlling the umodulated perturbation. After closing a nonlinear iteration argument, we arrive at $||\partial_\zeta^j\gamma(t)||_{L^\infty} \lesssim (1+t)^{-\frac{j}{2}}||w_0||_{L^\infty}$, $j=0,1$, for sufficiently small $||w_0||_{L^\infty}$. Now, one observes that $||w(t)||_{L^\infty}$ is controlled by the behavior of $||\gamma_\zeta(t)||_{L^\infty}$ through (1.19) and since $||z(t)||_{L^\infty}$ admits higher-order decay. By the relation of $\mathring{u}(t)$ and $w(t)$, we also obtain the control $||\mathring{u}(t)||_{L^\infty} \lesssim (1+t)^{-\frac{1}{2}}||w_0||_{L^\infty}$. With the help of the mean value theorem, we arrive at

$$||u(t)-\phi_0||_{L^\infty} \leq ||\mathring{u}(t)|| + ||\phi_0 - \phi_0(\cdot+\gamma(t))||_{L^\infty} \lesssim ||w_0||_{L^\infty}$$

providing nonlinear L^∞-stability.

[7] The uniformly local L^2-space $L^2_{\mathrm{ul}}(\mathbb{R})$ contains all $L^2_{\mathrm{loc}}(\mathbb{R})$-functions such that the uniformly local L^2-norm $||u||_{L^2_{\mathrm{ul}}} := \sup_{y\in\mathbb{R}}\int_{\mathbb{R}}\frac{2}{2+(x-y)^2}u(x)^2\,dx$ is finite and $||u(\cdot+y)-u(\cdot)||_{L^2_{\mathrm{ul}}} \to 0$ as $y\to\infty$. Analogously, one defines the spaces $H^m_{\mathrm{ul}}(\mathbb{R})$, $m\in\mathbb{N}$. Furthermore, the embeddings $C^m_{\mathrm{ub}} \hookrightarrow H^m_{\mathrm{ul}} \hookrightarrow C^{m-1}_{\mathrm{ub}}$ hold, see [94].

1.5. Approaches to linear decomposition and estimates

To the end of this chapter, we address the question of how we come from the spectral assumptions (D1)-(D3) to linear estimates and the decomposition (1.17) of the semigroup $e^{\mathcal{L}_0 t}$. We distinguish between three approaches, which can be converted partially into each other: the Bloch transform, pointwise estimates on the Green's function and the complex inversion formula. Each approach has advantages and limits which we briefly discuss, in particular, in view of nonlocalized data.

Bloch transform. Since $\mathcal{L}_0$ has T-periodic coefficients, one can rewrite the semigroup, see e.g. [29], as

$$e^{\mathcal{L}_0 t}v(\zeta) = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{i\xi\zeta}e^{\mathcal{L}(\xi)t}\check{v}(\xi,\zeta)\,d\xi, \quad \check{v}(\xi,\zeta) = \sum_{l\in\mathbb{Z}} e^{\frac{2\pi i l\zeta}{T}}\hat{v}\Big(\xi + \frac{2\pi l}{T}\Big), \quad v\in\mathcal{S}(\mathbb{R}), \tag{1.22}$$

where $\hat{v}$ denotes the Fourier transform of v and $\mathcal{L}(\xi)$ are the Bloch operators defined in Section 1.3. The function $\check{v}$ is called the Bloch transform of v.

The representation (1.22) allows us to partition the Bloch modes ξ into modes which are close to zero (low frequencies) and uniformly away from zero (high frequencies); this type of decomposition is referred to as what we call *mode filtration*. Now, we explain this in some more detail and formally isolate the principal propagator $s_p(t)$ in (1.17).

Denote by $\Pi(\xi)$ the spectral projection onto the eigenspace $\operatorname{span}\{\Phi_\xi\}$ and let $0\le\rho\le 1$ be a smooth function with support in $(-\xi_0,\xi_0)$ and $\rho\equiv 1$ on $(-\frac{\xi_0}{2},\frac{\xi_0}{2})$. One splits

$$\begin{aligned} S_e(t)v(\zeta) =& \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}(1-\rho(\xi))e^{i\xi\zeta}e^{\mathcal{L}(\xi)t}\check{v}(\xi,\zeta)\,d\xi \\ &+ \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\rho(\xi)e^{i\xi\zeta}e^{\mathcal{L}(\xi)t}(1-\Pi(\xi))\check{v}(\xi,\zeta)\,d\xi \end{aligned} \tag{1.23}$$

and

$$S_c(t)v(\zeta) = e^{\mathcal{L}_0 t}v(\zeta) - S_e(t)v(\zeta) = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\rho(\xi)e^{i\xi\zeta}e^{\mathcal{L}(\xi)t}\Pi(\xi)\check{v}(\xi,\zeta)\,d\xi.$$

First, we have the identity

$$e^{\mathcal{L}(\xi)t}\Pi(\xi)\check{v}(\xi,\zeta) = e^{\lambda_c(\xi)t}\Phi_\xi(\zeta)\langle\tilde{\Phi}_\xi,\check{v}(\xi,\cdot)\rangle_{L^2_{\mathrm{per}(0,T)}}.$$

Furthermore, inserting the Bloch transform of v and rearranging terms, we obtain

$$\begin{aligned} \langle\tilde{\Phi}_\xi,\check{v}(\xi,\cdot)\rangle_{L^2_{\mathrm{per}(0,T)}} &= \int_{\mathbb{R}} e^{-i\xi\bar{\zeta}}v(\bar{\zeta})\left(\sum_{l\in\mathbb{Z}} e^{-\frac{2\pi i l\bar{\zeta}}{T}}\frac{1}{T}\int_0^T \tilde{\Phi}^*_\xi(\zeta)e^{\frac{2\pi i l\zeta}{T}}\,d\zeta\right)d\bar{\zeta} \\ &= \int_{\mathbb{R}} e^{-i\xi\bar{\zeta}}\tilde{\Phi}^*_\xi(\bar{\zeta})v(\bar{\zeta})\,d\bar{\zeta}, \end{aligned}$$

where we used the Fourier series of $\tilde{\Phi}^*_\xi$. We summarize

$$S_c(t)v(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\rho(\xi)e^{i\xi(\zeta-\bar{\zeta})+\lambda_c(\xi)t}\Phi_\xi(\zeta)\tilde{\Phi}^*_\xi(\bar{\zeta})\,d\xi v(\bar{\zeta})\,d\bar{\zeta}. \tag{1.24}$$

The expressions (1.23) and (1.24) may be extended to L^2-functions due to the density of the Schwartz functions. The Bloch representation (1.22) has the advantage to trace back the derivation of linear L^2-estimates to L^2_{per}-theory by the Hausdorff-Young inequality. This is particularly reflected by using the Gearhart-Prüss theorem to derive exponential decay of $||S_e(t)||_{L^2\to L^2}$ in [45, 55–57].

In view of our C_{ub}-theory, an important observation is that the low-frequency representation (1.24) is defined for $v \in C_{\mathrm{ub}}(\mathbb{R})$ and one is in the position to trade powers of Bloch modes against time decay, see [84]. As rule of thumb, one trades powers of ξ^k to decay $t^{-\frac{k}{2}}$ as $t \to \infty$. Therefore, invoking the expansion (P4), we arrive at

$$S_c(t) = (\phi_0' + k_0\phi_k(k_0;\cdot)\partial_\zeta)s_p(t) + O_{C_{\mathrm{ub}}\to C_{\mathrm{ub}}}((1+t)^{-1}) \tag{1.25}$$

with

$$s_p(t)v(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \rho(\xi)e^{i\xi(\zeta-\bar{\zeta})+\lambda_c(\xi)t}\tilde{\Phi}^*_\xi(\bar{\zeta})\,d\xi v(\bar{\zeta})\,d\bar{\zeta}. \tag{1.26}$$

Finally, with the help of (1.25), we formally arrive at $S_r(t) = S_e(t) + O_{C_{\mathrm{ub}}\to C_{\mathrm{ub}}}((1+t)^{-1})$ in (1.17). In Remark 1.5.2, we briefly comment on the possibility to treat $S_e(t)v$, given through (1.23), as distribution for $v \in C_{\mathrm{ub}}(\mathbb{R})$. As we explain below, we can derive estimates in Chapter 3 on $e^{\mathcal{L}_0 t} - S_c(t)$ without the need for any distributional consideration.

Pointwise estimates on Green's function. The idea is to establish pointwise estimates on the Green's function G of $\partial_t v = \mathcal{L}_0 v$ where we represent

$$e^{\mathcal{L}_0 t}v(\zeta) = \int_{\mathbb{R}} G(\zeta,\bar{\zeta},t)v(\bar{\zeta})\,d\bar{\zeta}.$$

The main advantage of this approach is that it is possible to derive estimates on the Green's functions for equations with coefficients which are not necessarily constant or periodic. Furthermore, the Green's function approach allows to derive explicit decay rates on the linear evolution between L^p-spaces. We refer to [104] as guiding literature in the context of shock waves stability. In the context of periodic coefficient operators, Green's functions approaches are derived and exploited in [62] and [85]. We note that as soon as the linear operator $\mathcal{L}_0$ is not sectorial, one cannot guarantee for an integrable Green's function. This can already be observed for the evolution equation $\partial_t v = \partial_\zeta v$ where G is represented by a δ-distribution.

Complex inversion formula. Whenever a linear operator $\mathcal{L}_0 : \mathrm{dom}(\mathcal{L}_0) \to X$ generates a C_0-semigroup with growth bound $\eta_0 \in \mathbb{R}$, we represent

$$e^{\mathcal{L}_0 t}v = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\eta-iR}^{\eta+iR} e^{\lambda t}(\lambda-\mathcal{L}_0)^{-1}v\,d\lambda, \quad v \in \mathrm{dom}(\mathcal{L}_0), \quad \eta > \eta_0. \tag{1.27}$$

This allows a representation of the semigroup $e^{\mathcal{L}_0 t}$ which is not restricted to specific patterns and to specific properties of the linear operators. On a formal level, we can solve the resolvent problem $v = (\lambda - \mathcal{L})f_\lambda$ for $\lambda \in \rho(\mathcal{L}_0)$, which gives rise to a spatial Green's function representation

$$f_\lambda = \int_{\mathbb{R}} G_\lambda(\cdot,\bar{\zeta})v(\bar{\zeta})\,d\bar{\zeta}.$$

Inserting this into (1.27) and swapping the order of integration, one formally arrives at the temporal Green's function

$$G(\zeta, \bar{\zeta}, t) = \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} G_\lambda(\zeta, \bar{\zeta}) \, d\lambda.$$

A main idea in Chapter 3 is to combine the approach through the complex inversion formula with the low-frequency analysis of the Bloch representation. Instead of leveraging [104], see also [8], which rely on well-selected spatio-temporal contours to deduce linear decay rates, we show that the contour close to the origin (orange in Figure 1.6) is equal to the low-frequency part (1.24) modulo exponentially decaying terms. This follows by investigating $e^{\lambda t} G_\lambda(\zeta, \bar{\zeta}, t)$ for $|\lambda| \ll 1$, with the help of Floquet theory and the use of exponential dichotomies where the non-degeneracy assumption $a \neq 0$, see (H2), is crucial. The latter guarantees the pointwise analyticity of G_λ for $|\lambda| \ll 1$. The advantage is to apply previously derived estimates on (1.24) from [84, 85], and in particular, we can isolate $s_p(t)$ in form of (1.26) and explicitly relate $s_p(t)$ to the convective heat equation $\partial_t u = d\partial_\zeta^2 u + a\partial_\zeta u$ by expanding λ_c in the spirit of (P4). As described in Section 1.4, this allows us to reduce the question on how to control $\gamma(t)$, as defined in (1.18), to the analysis of a perturbed viscous Hamilton-Jacobi equation which can be performed as illustrated in Section 2.3. The contours uniformly bounded away from the origin (green in Figure 1.6) give rise to exponentially decaying contributions. The proof of these estimates can be traced back to the validity of the complex inversion formula for convolutions of semigroups. This observation originates from a Neumann series expansion of the resolvent $(\lambda - \mathcal{L}_0)^{-1}$ for $\mathrm{Im}(\lambda) \gg 1$ where the critical terms appear as compositions of simpler resolvents in the components, respectively. We refer to Chapter 3 for details. In summary, we circumvent discussions on distributional interpretations in the context of a potential Green's function ansatz or in the treatment of $S_e(t)$ within the Bloch representation while still relying on the representations (1.25) and (1.26).

Remark 1.5.1 (Possible extensions). *For future research, it is interesting to adapt the described procedure to other situations where the group velocity a can be zero or multiple critical spectral curves appear. The latter then rather leads to a Whitham system, see [59], than a viscous Hamilton-Jacobi equation for which one cannot immediately apply the Cole-Hopf transform which makes the nonlinear stability analysis against nonlocalized perturbations more difficult.*

Since the analysis in Chapter 3 of the contributions from green contours in Figure 1.6 only relies on the boundedness of ϕ_0, our approach suggests that we can allow for extended classes of perturbations for other patterns, such as pattern-forming fronts, whenever we can establish identification procedures for the corresponding spectral situations.

Remark 1.5.2 (Distributional interpretation of $S_e(t)$ on $C_{\mathbf{ub}}$). *For $v \in C_{ub}(\mathbb{R})$, one can interpret (1.23) as tempered distribution to make sense out of it. A possibility is to define the distribution*

$$\mathcal{S}(\mathbb{R}) \ni \phi \mapsto F(\phi) := \int_{\mathbb{R}} v(\zeta) S_e(t) \phi(\zeta) \, d\zeta$$

for fixed $v \in C_{ub}(\mathbb{R})$ and $t \geq 0$. As $S_c(t)$ and $e^{\mathcal{L}_0 t}$ are defined on $C_{ub}(\mathbb{R})$, one would aim to evaluate

$$||e^{\mathcal{L}_0 t} v - S_c(t) v||_{L^\infty} = \sup_{\phi \in \mathcal{S}(\mathbb{R}), ||\phi||_{L^1} = 1} |F(\phi)|$$

due to the fact that $L^\infty(\mathbb{R})$ is the dual of of $L^1(\mathbb{R})$. Assuming such an identity, we expect similar challenges to those already faced in the FitzHugh-Nagumo system, see Chapter 3 and [8], when one tries to establish

$$\sup_{\phi\in\mathcal{S}(\mathbb{R}),||\phi||_{L^1}=1} |F(\phi)| \leq e^{-\Theta t}||v||_{L^\infty}$$

for some $\Theta > 0$. This is justified by the observation that the L^2_{per}-theory on the Bloch operators, as used in [56], is not applicable anymore.[8]

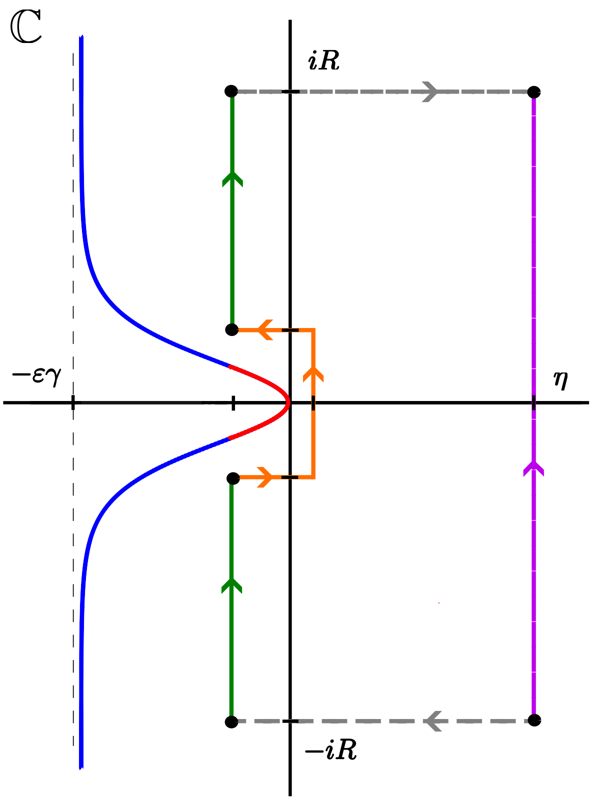


Figure 1.6.: Invoking Cauchy's integral theorem for fixed $R \geq 1$, one decomposes $\int_{\eta-iR}^{\eta+iR} e^{\lambda t}(\lambda - \mathcal{L})^{-1}v\, d\lambda$ by deforming the purple contour. The contributions of the gray contours vanish as $R \to \infty$. The orange contour close to the origin corresponds to the low frequency part and the green ones to exponentially damped contributions.

[8] In virtue of the Hausdorff-Young inequality, we expect to require $L^q((-\frac{\pi}{T}, \frac{\pi}{T}), L^p_{\text{per}}(0,T))$-estimates on the Bloch transform of ϕ and $L^p_{\text{per}}(0,T) \to L^p_{\text{per}}(0,T)$-estimates on the Bloch operators to estimate $S_e(t)\phi$ for suitable $q, p \neq 2$. We leave the details on this distributional approach for future work.

CHAPTER 2

Toy examples

We discuss various nonlinear variants of the heat equation as toy examples. In the first section, we discuss different results on localized and nonlocalized perturbations for the equation of the form $\partial_t u = \partial_x^2 u + u^p$. In the second section, we present the use of weighted energies to show damping estimates only containing L^∞-norms. This is again done along a scalar nonlinear heat equation. Then, we study a perturbed viscous Hamilton-Jacobi equation and show how the application of the Cole-Hopf transform allows us to control solutions. This represents a key aspect of the analysis in Chapters 3 and 4.

2.1. Localized versus nonlocalized perturbations

Let $p \in \mathbb{N}$, $p > 1$. As an introductory example to illustrate differences of nonlocalized compared to localized perturbations, we consider the scalar nonlinear heat equation

$$\partial_t u = \partial_x^2 u + u^p, \quad (x,t) \in \mathbb{R} \times [0,\infty), \quad u(x,t) \in \mathbb{R} \tag{2.1}$$

with respect to the zero solution. The first observation is that the linear evolution equation $\partial_t u = \partial_x^2 u$ gives rise to the estimates

$$||e^{\partial_x^2 t}||_{L^1 \to L^\infty} \lesssim t^{-\frac{1}{2}}, \quad ||e^{\partial_x^2 t}||_{L^\infty \to L^\infty}, ||e^{\partial_x^2 t}||_{L^1 \to L^1} \lesssim 1, \quad t > 0.$$

For a solution $u(t)$ of the nonlinear problem (2.1), this indicates that we expect for $||u(t)||_{L^\infty}$ at best decay $(1+t)^{-\frac{1}{2}}$ for initial data from $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and no decay for initial data from $u_0 \in L^\infty(\mathbb{R})$.

For the standard local well-posedness of (2.1) for initial data $u_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $u_0 \in C_{\mathrm{ub}}(\mathbb{R}) \cap L^1(\mathbb{R})$, we refer to [73]. We establish the following results.

Proposition 2.1.1. *The following statements are true.*

(i) *Let $p \in \mathbb{N}$, $p > 3$. For all $\varepsilon > 0$ there exists $u_0 \in C_{ub}(\mathbb{R})$ with $||u_0||_{L^\infty} < \varepsilon$ and $0 < T < \infty$ such that the unique classical solution*

$$u \in C((0,T); C_{ub}^2(\mathbb{R})) \cap C^1((0,T); C_{ub}(\mathbb{R})) \cap C([0,T); C_{ub}(\mathbb{R}))$$

of (2.1) with $u(0) = u_0$ blows up as $t \uparrow T$, i.e., $\limsup_{t \uparrow T} ||u(t)||_{L^\infty} = \infty$.

(ii) *Let $p \in \mathbb{N}$, $p > 3$. There exist constants $C, \varepsilon > 0$ such that for all $u_0 \in C_{ub}(\mathbb{R})$ with $||u_0||_{L^1 \cap L^\infty} < \varepsilon$ there exists a unique classical global solution*

$$u \in C((0,T); C_{ub}^2(\mathbb{R})) \cap C^1((0,T); C_{ub}(\mathbb{R})) \cap C([0,T); C_{ub}(\mathbb{R})))$$

of (2.1) with $u(0) = u_0$ satisfying

$$||u(t)||_{L^\infty} \le C(1+t)^{-\frac{1}{2}} ||u_0||_{L^1 \cap L^\infty}, \quad t \ge 0.$$

(iii) *Let $p \in \{2,3\}$. For all non-negative, non-zero $u_0 \in C_{ub}(\mathbb{R}) \cap L^1(\mathbb{R})$ there exists $0 < T < \infty$ such that the unique classical solution*

$$u \in C((0,T); C^2_{ub}(\mathbb{R})) \cap C^1((0,T); C_{ub}(\mathbb{R})) \cap C([0,T); C_{ub}(\mathbb{R}))$$

of (2.1) with $u(0) = u_0$ blows up as $t \uparrow T$.

Proof. For the proof of (i), we fix any $\alpha > 0$. Then, we directly observe $u(t) = ((p-1)(\alpha - t))^{-\frac{1}{p-1}}$ is a solution of (2.1) with initial datum $u_0 = (\alpha(p-1))^{-\frac{1}{p-1}}$ which blows up as $t \uparrow \alpha$. To show (ii), let $u_0 \in C_{\mathrm{ub}}(\mathbb{R}) \cap L^1(\mathbb{R})$. There exist $\tau_{\max} > 0$ and a unique maximally defined solution

$$u \in C([0, \tau_{\max}); C_{\mathrm{ub}}(\mathbb{R}) \cap L^1(\mathbb{R}))$$

satisfying the Duhamel formula

$$u(t) = e^{\partial_x^2 t} u_0 + \int_0^t e^{\partial_x^2 (t-s)} u(s)^p \, ds, \quad t \in [0, \tau_{\max}). \tag{2.2}$$

We set

$$\eta(t) = \sup_{0 \le s \le t} \left(||u(s)||_{L^1} + (1+s)^{\frac{1}{2}} ||u(s)||_{L^\infty} \right), \quad 0 \le t < \tau_{\max},$$

where the weights in front of the norms are the decay rates predicted by the linear evolution. We note that η is continuous and monotonically increasing. Fix $t \in [0, \tau_{\max})$ such that $\eta(t) \le \frac{1}{2}$ and let $0 \le s \le t$. Using (2.2), we perform the iterative estimates

$$\begin{aligned} ||u(s)||_{L^\infty} &\lesssim (1+s)^{-\frac{1}{2}} ||u_0||_{L^1 \cap L^\infty} + \int_0^s (s-\tau)^{-\frac{1}{2}} ||u(\tau)||_{L^1} ||u(\tau)||_{L^\infty}^{p-1} \, d\tau \\ &\lesssim (1+s)^{-\frac{1}{2}} ||u_0||_{L^1 \cap L^\infty} + \eta(t)^p \int_0^s (s-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{p-1}{2}} \, d\tau \\ &\lesssim (1+s)^{-\frac{1}{2}} (||u_0||_{L^1 \cap L^\infty} + \eta(t)^p) \end{aligned}$$

and

$$||u(s)||_{L^1} \lesssim ||u_0||_{L^1} + \int_0^s ||u(\tau)||_{L^1} ||u(\tau)||_{L^\infty}^{p-1} \, d\tau \lesssim ||u_0||_{L^1} + \eta(t)^p$$

where we used that $\frac{p-1}{2} > 1$ and the monotonicity of η. We summarize

$$\eta(t) \le C(\eta(0) + \eta(t)^2)$$

for some $C > 0$ and all $t \in [0, \tau_{\max})$ with $\eta(t) \le \frac{1}{2}$. We invoke the following

Lemma 2.1.2. *Let $I = [0, a)$ with $a > 0$ and $\alpha \in (0, 1]$. Let $\eta : I \to \mathbb{R}$ be continuous and monotonically increasing. If there exists a constant $C \ge 1$ such that*

$$\eta(t) \le C(\eta(0)^\alpha + \eta(t)^2)$$

for all $t \in I$ with $\eta(t) \le \frac{1}{2}$, then there exists an $\varepsilon > 0$ such that if $\eta(0) < \varepsilon$, then it follows

$$\eta(t) < 2C\eta(0)^\alpha$$

for all $t \in I$.

Proof. Suppose that $\eta(0) < \min\Big\{(\frac{1}{4C})^{\frac{1}{2-\alpha}}, (\frac{1}{4C})^{\frac{1}{\alpha}}\Big\}$. Assume in contradiction to the claim that there exists some $t \in I$ such that $\eta(t) > 2C\eta(0)^\alpha$. Then, by continuity and monotonicity of η, there exists some $t_0 \in I$ with $\eta(t_0) = 2C\eta(0)^\alpha \leq \frac{1}{2}$. By assumption, we find

$$\eta(t_0) \leq C\eta(0)^\alpha + (2C)^2\eta(0)^{2-\alpha}\eta(0)^\alpha < 2C\eta(0)^\alpha.$$

This contradicts the choice of t_0 and we deduce $\eta(t) < 2C\eta(0)^\alpha$ for all $t \in I$. ☐

We deduce that there exists $\varepsilon > 0$ such that for $||u_0||_{L^1\cap L^\infty} < \varepsilon$, we have

$$\eta(t) \leq 2C||u_0||_{L^1\cap L^\infty}$$

for all $t \in [0, \tau_{\max})$. This implies $\tau_{max} = \infty$ and in particular

$$||u(t)||_{L^\infty} \leq 4C(1+t)^{-\frac{1}{2}}||u_0||_{L^1\cap L^\infty}, \quad t \geq 0,$$

concluding (ii). For (iii), we refer to [36] ($p = 2$) and to [50] ($p = 3$). We also refer to [102] for further considerations and higher dimensional variants of this proposition and to [26] for fractional exponents. ☐

Proposition 2.1.1 (i) and (iii) show the impossibility of nonlinear stability of the origin in scalar heat equations of the form (2.1) against all C_{ub}-perturbations. In the case of localized perturbations, the proposition teaches that we need the nonlinearity to be strong enough to obtain the nonlinear stability of the origin. This begs the question of whether one can establish a general nonlinear stability theory of periodic wave trains at all, even against localized perturbations, as the nonlinearity of the perturbation equations are genuinely quadratic. Thinking of an uncoupled system with nonlinearities of the type u^p, Proposition 2.1.1 applies and the zero solution blows up against certain C_{ub}-perturbations. In this situation, (1.14) does not meet which illustrates that our analysis decisively hinges on the assumption that the periodic profile is nonconstant. This rather leads to the viscous Hamilton-Jacobi equation, see Section 2.3, as proper toy model to describe challenges from Chapters 3 and 4.

Remark 2.1.3 (Classification of nonlinearities). *Following [17], we consider the general nonlinear heat equation $\partial_t u = \partial_x^2 u + u^{n_1}(\partial_x u)^{n_2}(\partial_x^2 u)^{n_3}$ and introduce the index $d_F = n_1 + 2n_2 + 3n_3 - 3$. Then, the nonlinearity is classified as irrelevant if $d_F > 0$, marginal for $d_F = 0$ and relevant otherwise. Along this classification, several results are shown in [17] for localized data. We also refer to [86, 87] for results along this classification.*

Remark 2.1.4 (Discussion of other cases). *For marginal nonlinearities, the sign in front of the nonlinearity can be decisive in question of nonlinear stability.*

(i) *Let $u_0 \in H^1(\mathbb{R})$. Considering $\partial_t u = \partial_x^2 u - u^3$, one finds*

$$\frac{1}{2}\partial_t||u||_{L^2}^2 \leq -||\partial_x u||_{L^2}^2 - ||u||_{L^4}^4 \leq 0$$

and

$$\frac{1}{2}\partial_t||\partial_x u||_{L^2}^2 \leq -||\partial_{xx} u||_{L^2}^2 - 3||u\partial_x u||_{L^2}^2 \leq 0$$

by multiplying the equation with u and integrating and analogously for its derivative by multiplying with $\partial_x u$. This shows

$$\partial_t ||u||_{H^1}^2 \leq 0$$

and Grönwall's inequality and the embedding $H^1 \hookrightarrow L^\infty$ yield

$$||u(t)||_{L^\infty} \leq C||u_0||_{H^1}$$

for all $t \geq 0$.

(ii) *We discuss other situations which are not covered by Proposition 2.1.1. Let u be a solution of $\partial_t u = \partial_x^2 u + u^2$ with $u_0 \leq 0$. Then, one immediately has $||u(t)||_{L^\infty} \leq ||u_0||_{L^\infty}$ by the comparison principle. Investigating $\partial_t u = \partial_x^2 u - u^2$, one observes that $v = -u$ satisfies $\partial_t v = \partial_x^2 v + v^2$ and Fujita's result [36] applies whenever $u_0 \leq 0$ while the previous consideration applies for $u_0 \geq 0$.*

2.2. Nonlocalized energy estimates

We demonstrate the possibility to adapt the approach via energy estimates of Remark 2.1.4 (i) to a pure L^∞-situation. This prevents the use of the comparison principle which is in general not applicable for systems. As illustrative example, we consider again the weakly damped heat equation

$$\partial_t u = \partial_x^2 u - u^3, \quad (x,t) \in \mathbb{R} \times [0,\infty), \quad u(x,t) \in \mathbb{R}. \tag{2.3}$$

For the sake of readability, we also write $u_t = \partial_t u$, $u_x = \partial_x u$ and so forth.

Let $u_0 \in C_{\text{ub}}(\mathbb{R})$ and set $E_0 = ||u_0||_{L^\infty} \leq \frac{1}{2}$.

Short time argument. There exist $T_{\max} > 0$ and a unique maximally defined solution $u \in C([0, T_{\max}); C_{\text{ub}}(\mathbb{R}))$ satisfying

$$u(t) = e^{\partial_x^2 t} u_0 - \int_0^t e^{\partial_x^2 (t-s)} u(s)^3 \, ds. \tag{2.4}$$

Due to continuity, there is some small $0 < t_0 < T_{\max}$ such that $\sup_{0 \leq t \leq t_0} ||u(t)||_{L^\infty} \leq 2E_0$. Taking the spatial derivative of (2.4), we estimate

$$||\partial_x u(t_0)||_{L^\infty} \lesssim t_0^{-\frac{1}{2}} E_0.$$

We conclude that there exists a constant $C \geq 1$ such that

$$||u(t_0)||_{W^{1,\infty}} \leq CE_0 \text{ and } \sup_{0 \leq t \leq t_0} ||u(t)||_{L^\infty} \leq 2E_0.$$

We note that u is smooth on $(0, T_{\max}) \times \mathbb{R}$ by standard parabolic theory, c.f. [73].

Nonlinear damping. Setting $\rho(x) = 2(2+x^2)^{-1}$, we observe that

$$|\rho'|, |\rho''| \leq \rho. \tag{2.5}$$

Fix $0 < \delta \leq 1$. We set $E_{t_0} = ||u(t_0)||_{W^{1,\infty}}$ and introduce the weighted energies

$$E(y,t) = \frac{1}{2}\int_{\mathbb{R}} (u(x,t)^2 + u_x(x,t)^2)\rho(\delta(x-y))\,dx, \quad y \in \mathbb{R},$$

and set

$$E(t) = \sup_{y\in\mathbb{R}} E(y,t), \quad t_0 \leq t < T_{\max}.$$

We have the following properties: there exist δ- and t-independent constants $C_1 \geq 1$ and $C_2 > 0$ such that

$$\int_{\mathbb{R}} \rho(\delta x)\,dx \leq \frac{C_1}{\delta}, \tag{2.6}$$

$$E(t) \leq \frac{C_1}{\delta}||u(x,t)||^2_{W^{1,\infty}}, \quad t_0 \leq t < T_{\max}, \tag{2.7}$$

$$E(t) \geq \frac{C_2}{\delta}||u(x,t)||^2_{L^\infty}, \quad t_0 \leq t < T_{\max}, \tag{2.8}$$

see [94]. We determine

$$\partial_t u_x = \partial_x^2 u_x - 3u^2 u_x,$$

write $v = u_x$ and fix $y \in \mathbb{R}$, $t_0 \leq t < T_{\max}$. Now, we show an energy estimate by splitting

$$\begin{aligned}\partial_t E(y,t) &= \int_{\mathbb{R}} u(x,t)u_t(x,t)\rho(\delta(x-y))\,dx + \int_{\mathbb{R}} v(x,t)v_t(x,t)\rho(\delta(x-y))\,dx\\ &=: E_1(y,t) + E_2(y,t).\end{aligned}$$

We start to estimate $E_1(y,t)$, that is, we use integration by parts to obtain

$$\begin{aligned}E_1(y,t) &\leq -\int_{\mathbb{R}} u(x,t)^4\rho(\delta(x-y))\,dx - \delta\int_{\mathbb{R}} \frac{1}{2}\partial_x(u(x,t)^2)\rho'(\delta(x-y))\,dx\\ &\quad -\int_{\mathbb{R}} v(x,t)^2\rho(\delta(x-y))\,dx\\ &\leq -\int_{\mathbb{R}} u(x,t)^4\rho(\delta(x-y))\,dx + \frac{\delta^2}{2}\int_{\mathbb{R}} u(x,t)^2\rho''(\delta(x-y))\,dx\\ &\quad -\int_{\mathbb{R}} v(x,t)^2\rho(\delta(x-y))\,dx.\end{aligned}$$

Using (2.5) and (2.6), we estimate

$$\begin{aligned}E_1(y,t) &\leq \int_{\mathbb{R}} u(x,t)^2\left(\frac{\delta^2}{2} - u(x,t)^2\right)\rho(\delta(x-y))\,dx\\ &\quad -\int_{\mathbb{R}} v(x,t)^2\rho(\delta(x-y))\,dx\\ &\leq \frac{\delta^4}{2}\int_{\{u^2\leq\delta^2\}} \rho(\delta(x-y))\,dx - \frac{\delta^2}{2}\int_{\{u^2\geq\delta^2\}} u(x,t)^2\rho(\delta(x-y))\,dx\\ &\quad -\int_{\mathbb{R}} v(x,t)^2\rho(\delta(x-y))\,dx\\ &\leq \frac{C_1\delta^3}{2} - \frac{\delta^2}{2}\int_{\mathbb{R}} u(x,t)^2\rho(\delta(x-y))\,dx - \int_{\mathbb{R}} v(x,t)^2\rho(\delta(x-y))\,dx.\end{aligned} \tag{2.9}$$

Similarly, we estimate $E_2(y,t)$. Using integration by parts, we find

$$\begin{aligned} E_2(y,t) &\leq -3\int_{\mathbb{R}} u(x,t)^2 v(x,t)^2 \rho(\delta(x-y))\,dx + \frac{\delta^2}{2}\int_{\mathbb{R}} v(x,t)^2 \rho''(\delta(x-y))\,dx \\ &\leq \frac{\delta^2}{2}\int_{\mathbb{R}} v(x,t)^2 \rho(\delta(x-y))\,dx. \end{aligned} \tag{2.10}$$

Using that $0 < \delta \leq 1$, we summarize

$$\partial_t E(y,t) \leq -\frac{\delta^2}{2}E(y,t) + \frac{C_1\delta^3}{2}, \quad t_0 \leq t < T_{\max},$$

and invoking Grönwall's inequality and taking the supremum in $y \in \mathbb{R}$ yield

$$\begin{aligned} E(t) &\leq e^{-\frac{\delta^2}{2}t}E_{t_0} + C_1\delta^3\int_{t_0}^t e^{-\frac{\delta^2}{2}(t-s)}\,ds \\ &\leq e^{-\frac{\delta^2}{2}t}E_{t_0} + 4C_1\delta, \quad t_0 \leq t < T_{\max}. \end{aligned}$$

Invoking (2.8), we arrive at

$$||u(t)||_{L^\infty}^2 \leq \frac{1}{C_2}e^{-\frac{\delta^2}{2}t}\delta E_{t_0} + \frac{4C1}{C_2}\delta^2, \quad t_0 \leq t < T_{\max}. \tag{2.11}$$

Setting $\delta = E_0$ and using (2.7), we conclude

$$||u(t)||_{L^\infty} \leq \left(8\frac{CC_1}{C_2}\right)^{\frac{1}{2}} E_0, \quad t_0 \leq t < T_{\max}.$$

In combination with the short-time estimate, we have shown

$$||u(t)||_{L^\infty} \lesssim E_0, \quad t \geq 0.$$

An important observation due to parabolicity is that the term

$$-\int_{\mathbb{R}} v(x,t)^2 \rho(\delta(x-y))\,dx$$

in (2.9) absorbs the term

$$\frac{\delta^2}{2}\int_{\mathbb{R}} v(x,t)^2 \rho(\delta(x-y))\,dx$$

in (2.10) for small δ.

We note that, we can now take the $\limsup_{t\to\infty}$ in (2.11) first and then let $\delta \downarrow 0$ which tells us that

$$||u(t)||_{L^\infty} \to 0 \text{ as } t \to \infty.$$

Regularity control in non-parabolic situations. An important aspect for the presented weighted energy argument is that one does not necessarily require parabolicity when the damping in the principle part is sufficiently strong. We consider the example

$$\partial_t u = \partial_x u - u + u^2 + (\partial_x u)f + f^2. \tag{2.12}$$

We aim to describe a procedure without relying on the special structure of (2.12) which means that we avoid to solve $\partial_t u = \partial_x u - u + (\partial_x u)f$ by the method of characteristics. This equation is more representative to the situation of the FitzHugh-Nagumo system in the context of Section 1.4 by interpreting $f(t)$ as the spatial derivative of the phase modulation and $u(t)$ playing the role of the inverse-modulated perturbation. Indeed, when ignoring the nonlinear term u^2, (2.12) occurs in this form in the perturbation equations of the inverse modulated perturbation in Chapter 3. The term $\partial_x u(t)$ also appears in the Duhamel formula of $z(t)$ in (1.20). For simplicity, we set $f(t) = e^{\partial_x^2 t} f_0$. We suppose that $||f_0||_{L^\infty} \leq M$ for some $M > 0$ and $||\partial_x f_0||_{W^{1,\infty}}$ is sufficiently small. When one tries to estimate $\partial_x u(t)$ through the Duhamel formula associated to (2.12) and the linear evolution $\partial_t u = \partial_x^2 u - u$, one necessarily obtains the term $\partial_{xx} u(t)$ which we also do not control a priori. How to control derivatives in this situation? One principle idea here is to control $||\partial_x u(t)||_{L^\infty}$ through $f(t)$ and $||u(t)||_{L^\infty}$. In this situation, one can establish an estimate of the form

$$||\partial_x u(t)||_{L^\infty}^2 \leq C\Big(e^{-\Theta t}||u_0||_{W^{2,\infty}}^2 + \int_0^t e^{-\Theta(t-s)}||\partial_\zeta f(s)||_{W^{1,\infty}}^2\, ds\Big), \tag{2.13}$$

for suitable $C = C(M) > 0$ and $\Theta > 0$ by a piori assuming that $\sup_{0\leq s\leq t}||\partial_x u(s)||_{L^\infty}$ is bounded and $\sup_{0\leq s\leq t}||u(s)||_{L^\infty} \leq \frac{1}{2}$. This can again be shown using weighted energies as in the previous example and by remarking that $||u(t)||_{L^\infty}$ is considered to be small such that the term $-u$ dominates u^2. We refrain from executing the details. Such an estimate is referred to as what we call *regularity control* since we are in the position to control derivatives by variables which we control by iterative estimates on their Duhamel formulae.

2.3. L^∞-STABILITY IN THE VISCOUS HAMILTON-JACOBI EQUATION

Let $\nu \in \mathbb{R}\setminus\{0\}$. We consider the perturbed viscous Hamilton-Jacobi equation

$$\partial_t u = \partial_x^2 u + \nu(\partial_x u)^2 + (\partial_x u)^3, \quad (x,t) \in \mathbb{R}\times[0,\infty), \quad u(x,t) \in \mathbb{R} \tag{2.14}$$

and establish a result for (2.14) controlling $||\partial_\zeta u(t)||_{L^\infty}$ with possibly large initial data. Subsequently, we give two interesting consequences of our analysis. The first one shows that we can relate the solution of (2.14) to an explicit solution of the unperturbed viscous Hamilton-Jacobi equation. As second consequence, we deduce nonlinear stability whenever the initial datum is sufficiently small. We refrain from discussing the case $\nu = 0$ since it does not require the application of the Cole-Hopf transform and can be resolved with direct iterative estimates on the Duhamel formula, see also Chapter 4 for details.

Assume $||u_0||_{W^{1,\infty}} \leq M$ for some $M \geq 1$. Considering the linear evolution equation $\partial_t u = \partial_x^2 u$, a key idea here is to estimate

$$\begin{aligned} ||\partial_x e^{\partial_x^2 t}u_0||_{L^\infty} &\lesssim \Big(||\partial_x e^{\partial_x^2 t}||_{W^{1,\infty}\to L^\infty}||u_0||_{W^{1,\infty}}\Big)^{1-2\alpha}\Big(||e^{\partial_x^2 t}||_{L^\infty\to L^\infty}||\partial_x u_0||_{L^\infty}\Big)^{2\alpha} \\ &\lesssim (1+t)^{-\frac{1}{2}+\alpha}||\partial_x u_0||_{L^\infty}^{2\alpha}, \quad t \geq 0, \end{aligned}$$

for $\alpha \in (0, \frac{1}{2})$. For small α, we can exploit the decay $(1+t)^{-\frac{1}{2}+\alpha}$ while still relying on the smallness of $||\partial_x u_0||_{L^\infty}^{2\alpha}$.

Proposition 2.3.1. *Fix any $M > 0$ and $\alpha \in [0, \frac{1}{6})$. Then, there exist $C, \varepsilon > 0$ such that for all $u_0 \in C_{ub}^1(\mathbb{R})$ with $||\partial_x u_0||_{L^\infty} < \varepsilon$ and $||u_0||_{L^\infty} \leq M$ there exists a unique classical solution*

$$u \in C^1((0,\infty), C_{ub}(\mathbb{R})) \cap C((0,\infty), C_{ub}^2(\mathbb{R})) \cap C([0,\infty), C_{ub}(\mathbb{R}))$$

to (2.14) with $u(0) = u_0$ *satisfying*

$$||\partial_x u(t)||_{L^\infty} \leq C||\partial_x u_0||_{L^\infty}^{2\alpha}(1+t)^{-\frac{1}{2}+\alpha}, \quad ||u(t)||_{L^\infty} \leq C \tag{2.15}$$

for all $t \geq 0$.

Proof. Let $u_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$. Local well-posedness is again given through [73] and therefore there exist $T_{\max} > 0$ and a unique maximally defined solution

$$u \in C([0, T_{\max}); C^1_{\mathrm{ub}}(\mathbb{R}))$$

which is classical and satisfies (2.14) with $u(0) = u_0$. We apply the Cole-Hopf transform to $u(t)$, i.e., we look at $v(t) = e^{\nu u(t)}$ which satisfies

$$\partial_t v = \partial_x^2 v + (\partial_x u)^3(\nu v). \tag{2.16}$$

Let $\alpha \in (0, \frac{1}{6})$. We set the template function to

$$\eta(t) = \sup_{0\leq s\leq t}(1+s)^{\frac{1}{2}-\alpha}(||\partial_x v(s)||_{L^\infty} + ||\partial_x u(s)||_{L^\infty}), \quad 0 \leq t < T_{\max},$$

which is continuous and montonically increasing. Integrating (2.16), we obtain the Duhamel formula

$$v(t) = e^{\partial_x^2 t}v(0) + \nu\int_0^t e^{\partial_x^2(t-s)}(\partial_x u(s))^3 v(s)\,ds, \quad 0 \leq t < T_{\max}. \tag{2.17}$$

The maximum principle entails

$$\inf_{t\geq 0} e^{\partial_x^2 t}v(0) \geq e^{\partial_x^2 t}\Big(v(0) - e^{-|\nu|M}\Big) + e^{-|\nu|M} \geq e^{-|\nu|M} =: d,$$

and we establish two a piori bounds assuming $\eta(t) \leq \frac{c}{2}$ with

$$c = \min\left\{\left(|\nu|\int_0^\infty \frac{1}{(1+s)^{\frac{3}{2}-\frac{3}{\alpha}}}\,ds + 1\right)^{-1}, \frac{d}{2e^{|\nu|M}}\right\}.$$

We will use that $\frac{3}{2} - 3\alpha > 1$.

(I) We derive an a piori upper bound on v: for $0 \leq s \leq t$, we find

$$\begin{aligned} ||v(s)||_{L^\infty} &\leq ||v(0)||_{L^\infty} + \sup_{0\leq\tau\leq t}(||v(\tau)||_{L^\infty})\eta(t)^3|\nu|\int_0^t \frac{1}{(1+\tau)^{\frac{3}{2}-3\alpha}}\,d\tau \\ &\leq e^{|\nu|M} + \frac{1}{2}\sup_{0\leq\tau\leq t}||v(\tau)||_{L^\infty}. \end{aligned}$$

We conclude

$$\sup_{0\leq s\leq t}||v(s)||_{L^\infty} \leq 2e^{|\nu|M}$$

for all $0 \leq t < T_{\max}$ with $\eta(t) \leq \frac{c}{2}$.

(II) Using the upper bound, we show an a priori lower bound. We infer

$$\inf_{0\leq s\leq t} v(s) \geq d - \sup_{0\leq s\leq t}||v(s)||_{L^\infty}\eta(t)^3|\nu|\int_0^t \frac{1}{(1+s)^{\frac{3}{2}-3\alpha}}\,ds \geq \frac{d}{2}$$

for all $0 \leq t < T_{\max}$with $\eta(t) \leq \frac{c}{2}$.

Using (I), we now find

$$\begin{aligned}||\partial_x v(s)||_{L^\infty} &\lesssim (1+s)^{-\frac{1}{2}+\alpha}||\partial_x v(0)||_{L^\infty}^{2\alpha} + \eta(t)^3 \int_0^s (s-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}+3\alpha}\, d\tau \\ &\lesssim (1+s)^{-\frac{1}{2}+\alpha}(||\partial_x v(0)||_{L^\infty}^{2\alpha} + \eta(t)^3),\end{aligned} \tag{2.18}$$

for $0 \leq s \leq t < T_{\max}$ with $\eta(t) \leq \frac{c}{2}$. Next, we estimate[1]

$$\begin{aligned}\sup_{0\leq s\leq t}(1+s)^{\frac{1}{2}-\alpha}||\partial_x u(s)||_{L^\infty} &\lesssim \sup_{0\leq s\leq t}(1+s)^{\frac{1}{2}-\alpha}||\partial_x v(s)||_{L^\infty}\frac{1}{\inf_{0\leq s\leq t}|v(s)|} \\ &\leq \frac{2}{d}\sup_{0\leq s\leq t}(1+s)^{\frac{1}{2}-\alpha}||\partial_x v(s)||_{L^\infty} \\ &\lesssim ||\partial_x v(0)||_{L^\infty}^{2\alpha} + \eta(t)^3,\end{aligned} \tag{2.19}$$

thanks to (II) and (2.18). Finally, we have

$$||\partial_x v(0)||_{L^\infty} \lesssim ||\partial_x u_0||_{L^\infty}.$$

In summary, we have shown

$$\eta(t) \lesssim ||\partial_x u_0||_{L^\infty}^{2\alpha} + \eta(t)^2,$$

for $0 \leq t < T_{\max}$ with $\eta(t) \leq \frac{c}{2}$. Lemma 2.1.2 implies

$$\eta(t) \lesssim ||\partial_x u_0||_{L^\infty}^{2\alpha}, \quad 0 \leq t < T_{\max},$$

for $\alpha \in (0, \frac{1}{6})$. Furthermore, using (I) and (II), we have

$$||u(t)||_{L^\infty} \leq 2|\log(2)| + 2|\nu|M, \quad 0 \leq t < T_{\max},$$

and therefore no blow-up in finite time is possible. This implies $T_{\max} = \infty$. We have shown the result for $\alpha \in (0, \frac{1}{6})$. Using the just shown result with $\alpha = \frac{1}{12}$ for the nonlinearities in (2.17), we obtain

$$\begin{aligned}||\partial_x v(t)||_{L^\infty} &\lesssim \frac{||v(0)||_{W^{1,\infty}}}{(1+t)^{\frac{1}{2}}} + ||\partial_x u_0||_{L^\infty}^{\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{3}{2}+\frac{1}{4}}\, ds \\ &\lesssim \frac{1}{(1+t)^{\frac{1}{2}}}, t \geq 0.\end{aligned}$$

Proceeding as in (2.19), one also establishes the estimate on $||\partial_x u(t)||_{L^\infty}$. This completes the proof of the proposition. □

We can explicitly relate the solution from Proposition 2.3.1 to a solution of the unperturbed viscous Hamilton-Jacobi equation.

Corollary 2.3.2. *Fix any $M > 0$ and $\beta \in (0,1)$. Then, there exist $C, \varepsilon > 0$ such that for all $u_0 \in C^1_{ub}(\mathbb{R})$ with $||\partial_x u_0||_{L^\infty} < \varepsilon$ and $||u_0||_{L^\infty} \leq M$ there exists a unique classical solution*

$$u \in C^1((0,\infty), C_{ub}(\mathbb{R})) \cap C((0,\infty), C^2_{ub}(\mathbb{R})) \cap C([0,\infty), C_{ub}(\mathbb{R}))$$

to (2.14) with $u(0) = u_0$ satisfying

$$\left|\left|\partial_x^j\left(u(t) - \frac{1}{\nu}\log\left(e^{\partial_x^2 t}e^{\nu u_0}\right)\right)\right|\right|_{L^\infty} \leq C||\partial_x u_0||_{L^\infty}^{\beta}(1+t)^{-\frac{j}{2}}, \quad j = 0,1, \tag{2.20}$$

for all $t \geq 0$.

[1]Note that we suppress the dependence by M and ν of the constants in all estimates.

Proof. Let $\alpha \in (0, \frac{1}{6})$. Let v be as in the proof of Proposition 2.3.1. Set $w(t) = e^{\partial_x^2 t} e^{\nu u_0}$ and note that $w(t)$ is a solution of the heat equation with initial datum $w(0) = e^{\nu u_0}$. Therefore, using (2.17) and applying Proposition 2.3.1, we find

$$||\partial_x^j (v(t) - w(t))||_{L^\infty} \lesssim ||\partial_x u_0||_{L^\infty}^{6\alpha} \int_0^t (t-s)^{-\frac{j}{2}} (1+s)^{-\frac{3}{2}+3\alpha}\, ds \lesssim ||\partial_x u_0||_{L^\infty}^{6\alpha} (1+t)^{-\frac{j}{2}},$$

for $j = 0, 1$ and all $t \geq 0$. Noting that $w(t)$ and $v(t)$ are uniformly bounded away from zero, the mean value inequality gives

$$\begin{aligned} \left|\left|\partial_x^j \left(u(t) - \frac{1}{\nu} \log\left(e^{\partial_x^2 t} e^{\nu u_0}\right)\right)\right|\right|_{L^\infty} &= \left|\left|\partial_x^j \left(\frac{1}{\nu} \log(v(t)) - \frac{1}{\nu} \log(w(t))\right)\right|\right|_{L^\infty} \\ &\lesssim ||\partial_x^j (v(t) - w(t))||_{L^\infty} \lesssim ||\partial_x u_0||_{L^\infty}^{6\alpha} (1+t)^{-\frac{j}{2}} \end{aligned}$$

for $j = 0, 1$ and all $t \geq 0$. □

Whenever we assume that $||u_0||_{L^\infty}$ also is sufficiently small, we can apply Proposition 2.3.1 to infer nonlinear L^∞-stability.

Corollary 2.3.3. *Let $\beta \in (0,1)$. Then, there exist $C, \varepsilon > 0$ such that for all $u_0 \in C_{ub}^1(\mathbb{R})$ with $||u_0||_{W^{1,\infty}} < \varepsilon$, there exists a unique classical solution*

$$u \in C^1((0,\infty), C_{ub}(\mathbb{R})) \cap C((0,\infty), C_{ub}^2(\mathbb{R})) \cap C([0,\infty), C_{ub}(\mathbb{R}))$$

to (2.14) with $u(0) = u_0$ satisfying

$$||u(t)||_{L^\infty} \leq C ||u_0||_{W^{1,\infty}}^\beta \text{ and } ||\partial_x u(t)||_{L^\infty} \leq C ||u_0||_{W^{1,\infty}}^\beta (1+t)^{-\frac{1}{2}} \tag{2.21}$$

for all $t \geq 0$.

Proof. Let $w = v - 1$ with v from the proof of Proposition 2.3.1. Then w solves

$$\partial_t w = \partial_x^2 w + \nu (\partial_x u)^3 (w+1)$$

with initial datum $w(0) = v(0) - 1$. We first note that

$$||w(0)||_{L^\infty} \lesssim ||u_0||_{L^\infty}.$$

Let $\alpha \in (0, \frac{1}{6})$ and assume $||u_0||_{W^{1,\infty}} \leq 1$. Applying Proposition 2.3.1 to the Duhamel formula of $w(t)$, we find

$$\begin{aligned} ||\partial_x^j w(s)||_{L^\infty} \lesssim &(1+s)^{-\frac{1}{2}} ||u_0||_{W^{1,\infty}} \\ &+ ||\partial_x u_0||_{L^\infty}^{6\alpha} \left(\sup_{0 \leq s \leq t} w(s) + 1 \right) \int_0^t \frac{1}{(t-s)^{\frac{j}{2}}} \frac{1}{(1+s)^{\frac{3}{2}-3\alpha}}\, ds, \end{aligned} \tag{2.22}$$

for every $0 \leq s \leq t$ and whenever $||\partial_x u_0||_{L^\infty}$ is sufficiently small. Taking $||\partial_x u_0||_{L^\infty}$ smaller if necessary, (2.22) provides first

$$\sup_{0 \leq s \leq t} ||w(s)||_{L^\infty} \lesssim ||u_0||_{W^{1,\infty}}^{6\alpha}$$

and then

$$||\partial_x w(t)||_{L^\infty} \lesssim ||u_0||_{W^{1,\infty}}^{6\alpha} (1+t)^{-\frac{1}{2}}$$

for all $t \geq 0$. Choosing $||u_0||_{W^{1,\infty}}^{6\alpha}$ so small that $\sup_{t\geq 0} ||w(t)||_{L^\infty} \leq \frac{1}{2}$, the mean value theorem implies

$$\begin{aligned} ||\partial_x^j u(t)||_{L^\infty} &= \left|\left|\frac{1}{\nu}\partial_x^j(\log(w(t)+1))\right|\right|_{L^\infty} \\ &\lesssim ||\partial_x^j w(t)||_{L^\infty}, \quad j = 0,1 \quad t \geq 0, \end{aligned}$$

which shows the claim. □

One can reach the exponent $\beta = 1$ in Corollary 2.3.3 by again closing a nonlinear iteration argument. We refrain from doing so here.

Remark 2.3.4 (Localized data). *In the spirit of Remark 2.1.3 and ignoring the higher order term $(\partial_x u)^3$, the equation (2.14) is of irrelevant type. Considering an initial datum u_0 with $\partial_x u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and let $u(t)$ be the corresponding solution of (2.14) with $u(0) = u_0$, we choose the template function*

$$\eta(t) = \sup_{0\leq s\leq t} (1+s)^{-\frac{1}{4}} ||\partial_x u(s)||_{H^1}.$$

We obtain the estimates

$$\begin{aligned} ||\partial_x u(s)||_{H^1} &\lesssim (1+s)^{-\frac{1}{4}} ||\partial_x u_0||_{H^1\cap L^1} + \eta(t)^2 \int_0^s \frac{1}{(s-\tau)^{\frac{3}{4}}} \frac{1}{(1+\tau)^{\frac{1}{2}}}\, d\tau \\ &\lesssim (1+s)^{-\frac{1}{4}} \Big(||\partial_x u_0||_{H^1\cap L^1} + \eta(t)^2 \Big), \quad 0 \leq s \leq t, \end{aligned}$$

using Hölder's inequality to bound $||(\partial_x u)^2||_{L^1} \lesssim ||\partial_x u||_{L^2}^2$ and the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ to bound $||(\partial_x u)^3||_{L^1} \lesssim ||\partial_x u||_{H^1}^3$. If $||\partial_x u_0||_{H^1\cap L^1}$ is sufficiently small, then an iteration argument can be closed without exploiting the Cole-Hopf transform. This is the case in earlier works such as [57] or [105] where the phase modulations are necessarily L^∞-small.

CHAPTER 3

NONLINEAR STABILITY OF PERIODIC WAVE TRAINS IN THE FITZHUGH-NAGUMO SYSTEM AGAINST FULLY NONLOCALIZED PERTURBATIONS

This chapter is the content of the preprint [5] and is a joint work with Björn de Rijk.

——————————— Start of Paper ———————————

Abstract. Recently, a nonlinear stability theory has been developed for wave trains in reaction-diffusion systems relying on pure L^∞-estimates. In the absence of localization of perturbations, it exploits diffusive decay caused by smoothing together with spatio-temporal phase modulation. In this paper, we advance this theory beyond the parabolic setting and propose a scheme designed for general dissipative semilinear problems. We present our method in the context of the FitzHugh-Nagumo system. The lack of parabolicity and localization complicates mode filtration in L^∞-spaces using the Floquet-Bloch transform. Instead, we employ the inverse Laplace representation of the semigroup generated by the linearization to uncover high-frequency damping, while leveraging a novel link to the Floquet-Bloch representation for the smoothing low-frequency part. Another challenge arises in controlling regularity in the quasilinear iteration scheme for the modulated perturbation. We address this by extending the method of nonlinear damping estimates to nonlocalized perturbations using uniformly local Sobolev norms.

Keywords. Periodic waves; nonlinear stability; fully nonlocalized perturbations; FitzHugh-Nagumo system; inverse Laplace transform; Floquet-Bloch analysis; uniformly local Sobolev spaces; Cole-Hopf transform
Mathematics Subject Classification (2020). 35B10; 35B35; 35K57; 44A10

3.1. INTRODUCTION

We study the nonlinear stability of traveling periodic waves against bounded, fully nonlocalized perturbations in the FitzHugh-Nagumo (FHN) system

$$\begin{aligned} \partial_t u &= u_{xx} + u(1-u)(u-\mu) - v, \\ \partial_t v &= \varepsilon(u - \gamma v - \mu), \end{aligned} \tag{3.1}$$

with $x \in \mathbb{R}, t \geq 0$ and parameters $\mu \in \mathbb{R}$ and $\gamma, \varepsilon > 0$. The FHN system was originally proposed as a simplification of the Hodgkin-Huxley model describing signal propagation in nerve fibers [34, 77, 78]. Mathematically, system (3.1) is a coupling between a scalar bistable reaction-diffusion equation and a linear ordinary differential equation and is thereby one of the simplest[1] models, which can, and does, exhibit stable spatio-temporal patterns. In fact, exploiting the slow-fast structure of system (3.1) arising for $0 < \varepsilon \ll 1$, a large variety of (spectrally) stable patterns and nonlinear waves have been rigorously constructed using tools from geometric singular perturbation theory, such as fast traveling pulses [49, 60, 61, 103], pulses with oscillatory tails [19, 20], periodic wave trains [21, 33, 95] and pattern-forming fronts [21] connecting such pulse trains to the homogeneous rest state $(\mu, 0)$.

Due to its remarkably rich dynamics, yet simple structure, the FHN system is widely recognized as a paradigmatic model for far-from-equilibrium patterns in excitable and oscillatory media. It has, in small variations, been employed across various scientific disciplines to explain phenomena such as the onset of turbulence in fluids [11], oxidation processes on platinum surfaces [9, 76], and heart arrhythmias [74].

The simplest and most fundamental spatio-temporal patterns exhibited by (3.1) are periodic traveling waves, or *wave trains*. Writing (3.1) as a degenerate reaction-diffusion system

$$\partial_t \mathbf{u} = D\mathbf{u}_{xx} + F(\mathbf{u}), \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F(\mathbf{u}) = \begin{pmatrix} u(1-u)(u-\mu) - v \\ \varepsilon(u - \gamma v - \mu) \end{pmatrix}, \tag{3.2}$$

in $\mathbf{u} = (u, v)^\top$, wave trains are solutions to (3.2) of the form $\mathbf{u}_0(x,t) = \phi_0(x - c_0 t)$ with smooth periodic profile function $\phi_0 \colon \mathbb{R} \to \mathbb{R}^2$ and propagation speed $c_0 \in \mathbb{R}$. Upon switching to the co-moving frame $\zeta = x - c_0 t$, in which system (3.2) reads

$$\partial_t \mathbf{u} = D\mathbf{u}_{\zeta\zeta} + c_0 \mathbf{u}_\zeta + F(\mathbf{u}), \tag{3.3}$$

we find that ϕ_0 is a stationary solution to (3.3).

Wave-train solutions to (3.2) have been constructed in the oscillatory regime with $0 < \mu < \frac{1}{2}$ and $0 < \varepsilon \ll \gamma \ll 1$, as well as in the excitable regime with $\mu < 0$ and $0 < \varepsilon \ll \gamma \ll 1$, using geometric singular perturbation theory and blow-up techniques, see [21, 95] and Remark 3.1.1. The associated profile functions consist of steep jumps interspersed with long transient states, where the profile varies slowly. Accordingly, these wave trains correspond to highly nonlinear far-from-equilibrium patterns. It has recently been argued theoretically and demonstrated numerically [21] that some of these wave trains are selected by compactly supported perturbations of the unstable rest state $(\mu, 0)$ in the oscillatory regime and, thus, play a pivotal role in pattern formation away from onset.

In this paper, we focus on the dynamical, or nonlinear, stability of wave trains as solutions to (3.2). The nonlinear stability theory for wave trains in spatially extended dissipative problems such as (3.2) has been rapidly developing over the past decades. The general approach is to first linearize the system about the wave train, obtain bounds on the C_0-semigroup generated by the linearization and then close a nonlinear argument by iterative estimates on the associated Duhamel formulation. A standard issue is that the linearization is a periodic differential operator acting on an unbounded domain, which possesses continuous spectrum touching the imaginary axis at the origin due to translational invariance. The lack of a spectral gap prevents, in contrast to the case of a finite domain with periodic boundary conditions, exponential convergence of the perturbed solution towards a translate of the original profile.

To overcome this issue a common strategy is to decompose the semigroup generated by the linearization in a diffusively decaying low-frequency part and an exponentially damped

[1] We note that Sturm-Liouville theory implies that all periodic traveling waves in real scalar reaction-diffusion equations are unstable.

high-frequency part, cf. [55]. The critical diffusive behavior caused by translational invariance can then be captured by introducing a spatio-temporal phase modulation, whose leading-order behavior is given by a viscous Hamilton-Jacobi equation [29]. The modulated perturbation obeys a quasilinear equation depending only on *derivatives* of the phase modulation, which thus satisfy a perturbed Burgers' equation. Observing that small, sufficiently localized initial data in a (perturbed) viscous Burgers' equation decay diffusively, cf. [100, Theorem 1] or [17, Theorem 4], suggests that the critical dynamics in a nonlinear iteration scheme, tracking the modulated perturbation variable and derivatives of the phase, can be controlled. This observation has led to a series of nonlinear stability results of wave trains against localized perturbations in general (nondegenerate) reaction-diffusion systems [55, 56, 62, 63, 89] relying on renormalization group theory [89], pointwise estimates [62, 63] or L^1-H^k-estimates [55, 56] to close the nonlinear iteration. We note that, since only derivatives of the phase enter in the nonlinear iteration and thus need to be localized, one could allow for a nonlocalized phase modulation, cf. [56, 63, 89]. With the aid of periodic-coefficient damping estimates to obtain high-frequency resolvent bounds and control regularity in the quasilinear iteration scheme, the method employing L^1-H^k-estimates could be extended beyond the parabolic setting to general dissipative semilinear problems (and some quasilinear problems) such as the St. Venant equations [58, 88], the Lugiato-Lefever equation [46, 105] and the FHN system [8].

Recently, a novel approach was developed [51, 84] to establish nonlinear stability of wave trains in (nondegenerate) reaction-diffusion systems, which employs pure L^∞-estimates to close the nonlinear iteration, thereby lifting all localization assumptions on perturbations. In contrast to previous methods, diffusive decay cannot be realized by giving up localization, but emanates from smoothing action of the analytic semigroup generated by the linearization about the wave train. The Cole-Hopf transform is then applied to the equation for the phase to eliminate the critical Burgers'-type nonlinearity, which cannot be readily controlled by diffusive smoothing.

In this paper, we extend the approach developed in [51, 84] beyond the parabolic framework by proving nonlinear stability of wave trains in the FHN system (3.2) against C_{ub}-perturbations. The incomplete parabolicity of (3.2) in combination with lack of localization of perturbations presents novel challenges in our analysis. These challenges involve the decomposition of the C_0-semigroup and the control of regularity. We explain the main ideas on how to address these challenges in §3.1.3 after we have stated our main result in §3.1.2.

Remark 3.1.1. Let $\mu < 0$, $\gamma \geq 0$ and $\varepsilon > 0$, so that we are in the excitable regime. Upon rescaling time, space, the variables u and v, and the system parameters ε, μ and γ by setting

$$\tilde{x} = (1-\mu)x, \quad \tilde{t} = (1-\mu)^2 t, \quad \tilde{u} = \frac{u-\mu}{1-\mu}, \quad \tilde{v} = \frac{v}{(1-\mu)^3},$$
$$\tilde{\varepsilon} = \frac{\varepsilon}{(1-\mu)^4}, \quad \tilde{\gamma} = (1-\mu)^2\gamma, \quad \tilde{\mu} = -\frac{\mu}{1-\mu},$$

we arrive at the equivalent formulation

$$\begin{aligned} \partial_{\tilde{t}}\tilde{u} &= \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}(1-\tilde{u})(\tilde{u}-\tilde{\mu}) - \tilde{v}, \\ \partial_{\tilde{t}}\tilde{v} &= \tilde{\varepsilon}(\tilde{u} - \tilde{\gamma}\tilde{v}), \end{aligned} \tag{3.4}$$

of the FHN system (3.1). Here, we have $\tilde{\mu} \in (0,1)$, $\tilde{\gamma} \geq 0$ and $\tilde{\varepsilon} > 0$. We note that the formulation (3.4) of the FHN system has been used in the existence and spectral stability analysis of wave trains and traveling pulses in the excitable regime, cf. [19, 33, 60, 61, 95, 103].

3.1.1. Assumptions on the wave train and its spectrum

Here, we formulate the hypotheses for our main result. The first hypothesis concerns the existence of the wave train.

(H1) There exist a speed $c_0 \in \mathbb{R}$ and a period $T > 0$ such that (3.2) admits a wave-train solution $\mathbf{u}_0(x,t) = \phi_0(x - c_0 t)$, where the profile function $\phi_0 \colon \mathbb{R} \to \mathbb{R}^2$ is nonconstant, smooth and T-periodic.

We note that wave-train solutions have been shown to exist, i.e., (H1) has been verified, in the excitable regime with $\mu < 0 \leq \gamma \ll 1$ and $0 < \varepsilon \ll 1$, cf. [95], and in the oscillatory regime with $0 < \mu < \frac{1}{2}$ and $0 < \varepsilon \ll \gamma \ll 1$, cf. [21].

Next, we specify our spectral assumptions on the wave train $\mathbf{u}_0$. Linearizing (3.3) about its stationary solution ϕ_0 yields the T-periodic differential operator $\mathcal{L}_0 \colon D(\mathcal{L}_0) \subset C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ given by

$$\mathcal{L}_0 \mathbf{w} = D\mathbf{w}_{\zeta\zeta} + c_0 \mathbf{w}_\zeta + F'(\phi_0)\mathbf{w} \tag{3.5}$$

with domain $D(\mathcal{L}_0) = C^2_{\mathrm{ub}}(\mathbb{R}) \times C^1_{\mathrm{ub}}(\mathbb{R})$, where $C^m_{\mathrm{ub}}(\mathbb{R})$ denotes for $m \in \mathbb{N}_0$ the space of bounded and uniformly continuous functions, which are m times differentiable and whose m derivatives are also bounded and uniformly continuous. We endow $C^m_{\mathrm{ub}}(\mathbb{R})$ with the standard $W^{m,\infty}$-norm, so that it is a Banach space.

The spectrum of $\mathcal{L}_0$ is determined by the family of Bloch operators

$$\mathcal{L}(\xi)\mathbf{w} = D(\partial_\zeta + \mathrm{i}\xi)^2\mathbf{w} + c_0(\partial_\zeta + \mathrm{i}\xi)\mathbf{w} + F'(\phi_0)\mathbf{w}, \qquad \xi \in \mathbb{C}$$

posed on $L^2_{\mathrm{per}}(0,T)$ with domain $D(\mathcal{L}(\xi)) = H^2_{\mathrm{per}}(0,T) \times H^1_{\mathrm{per}}(0,T)$. Since $\mathcal{L}(\xi)$ has compact resolvent, its spectrum consists of isolated eigenvalues of finite multiplicity. The spectrum of $\mathcal{L}_0$ can then be characterized as

$$\sigma(\mathcal{L}_0) = \bigcup_{\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})} \sigma(\mathcal{L}(\xi)), \tag{3.6}$$

cf. [39]. We require that the following standard *diffusive spectral stability* assumptions, cf. [55, 84, 89, 92], are satisfied.

(D1) We have $\sigma(\mathcal{L}_0) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\}$;

(D2) There exists a constant $\theta > 0$ such that for any $\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})$ we have $\mathrm{Re}\, \sigma(\mathcal{L}(\xi)) \leq -\theta\xi^2$;

(D3) 0 is a simple eigenvalue of $\mathcal{L}(0)$.

The main result of [7] establishes diffusive spectral stability of wave trains in (3.1) in the oscillatory regime $(3 - \sqrt{5})/6 < \mu < \frac{1}{2}$ and $0 < \varepsilon \ll \gamma \ll 1$. On the other hand, a spectral analysis of wave trains in the excitable regime with $\mu < 0$, $\gamma = 0$ and $0 < \varepsilon \ll 1$ can be found in [33].[2]

It is a consequence of translational invariance that 0 is an eigenvalue of the Bloch operator $\mathcal{L}(0)$ with associated eigenfunction ϕ_0'. Assumption (D3) then states that the kernel of $\mathcal{L}(0)$

[2]Although the spectral assumptions (D1) and (D3) are verified in [33], we emphasize that the fact that $\gamma = 0$ yields a lack of damping in the second component of (3.1), causing the spectrum of the linearization to asymptote to $\mathrm{i}\mathbb{R}$ at infinity. In particular, the spectrum is not bounded away from the imaginary axis away from 0 and the assumption (D2) does not hold, prohibiting diffusive spectral stability.

is spanned by ϕ_0'. In this case 0 is also a simple eigenvalue of the adjoint operator $\mathcal{L}(0)^*$. We denote by $\widetilde{\Phi}_0 \in H^2_{\mathrm{per}}(0,T) \times H^1_{\mathrm{per}}(0,T)$ the corresponding eigenfunction satisfying

$$\langle \widetilde{\Phi}_0, \phi_0' \rangle_{L^2(0,T)} = 1.$$

An application of the implicit function theorem in combination with Assumption (D3) readily yields that the wave train can be continued with respect to the wavenumber, cf. [29, Section 4.2].

Proposition 3.1.2. *Assume* (H1) *and* (D3)*. Then, there exists a constant* $r_0 \in (0,1)$ *and smooth functions* $\phi\colon \mathbb{R} \times [1-r_0, 1+r_0] \to \mathbb{R}^2$ *and* $\omega\colon [1-r_0,1+r_0] \to \mathbb{R}$ *with* $\phi(\cdot;1) = \phi_0$ *and* $\omega(1) = c_0$ *such that* $\phi(\cdot;k)$ *is* T*-periodic and*

$$\mathbf{u}_k(x,t) = \phi(kx - \omega(k)t; k)$$

is a solution to (3.2) *for each wavenumber* $k \in [1-r_0, 1+r_0]$*. By shifting the wave train if necessary, we can arrange for*

$$\langle \widetilde{\Phi}_0, \partial_k \phi(\cdot;1) \rangle_{L^2(0,T)} = 0.$$

The curve $\omega\colon [1-r_0,1+r_0] \to \mathbb{R}$ from Proposition 3.1.2 describes the relationship between the temporal frequency $\omega(k)$ and the wavenumber k of the T/k-periodic wave train $\mathbf{u}_k$ and is called the *nonlinear dispersion relation*.

Because the Bloch operators $\mathcal{L}(\xi)$ depend analytically on the Floquet exponent ξ and 0 is a simple eigenvalue of $\mathcal{L}(0)$ by Hypothesis (D3), it follows by standard analytic perturbation theory, see e.g. [66], that the 0-eigenvalue can be continued to a simple eigenvalue $\lambda_c(\xi)$ of $\mathcal{L}(\xi)$ for ξ close to 0. The curve $\lambda_c(\xi)$ is analytic and necessarily touches the imaginary axis in a quadratic tangency by Hypothesis (D2). Using Lyapunov-Schmidt reduction, the eigenvalue $\lambda_c(\xi)$, as well as the associated eigenfunction, can be expanded in ξ about $\xi = 0$, cf. [29, Section 4.2] or [57, Section 2].[3] We record these facts in the following result.

Proposition 3.1.3. *Assume* (H1) *and* (D1)-(D3)*. There exist a constant* $C > 0$*, open balls* $V_1, V_2 \subset \mathbb{C}$ *centered at* 0 *and an analytic function* $\lambda_c\colon V_1 \to \mathbb{C}$ *such that the following assertions hold.*

(i) $\lambda_c(\xi)$ *is a simple eigenvalue of* $\mathcal{L}(\xi)$ *for each* $\xi \in V_1$*. An associated eigenfunction* Φ_ξ *of* $\mathcal{L}(\xi)$ *lies in* $H^m_{\mathrm{per}}(0,T)$ *for any* $m \in \mathbb{N}_0$*, is analytic in* ξ*, satisfies* $\Phi_0 = \phi_0'$ *and fulfills*

$$\langle \widetilde{\Phi}_0, \Phi_\xi \rangle_{L^2(0,T)} = 1$$

for $\xi \in V_1$*.*

(ii) *It holds* $\sigma(\mathcal{L}_0) \cap V_2 = \{\lambda_c(\xi) : \xi \in V_1 \cap \mathbb{R}\} \cap V_2$*.*

(iii) *The complex conjugate* $\overline{\lambda_c(\xi)}$ *is a simple eigenvalue of the adjoint* $\mathcal{L}(\xi)^*$ *for any* $\xi \in V_1$*. An associated eigenfunction* $\widetilde{\Phi}_\xi$ *lies in* $H^m_{\mathrm{per}}(0,T)$ *for any* $m \in \mathbb{N}_0$*, is smooth in* ξ *and satisfies*

$$\langle \widetilde{\Phi}_\xi, \Phi_\xi \rangle_{L^2(0,T)} = 1$$

for $\xi \in V_1$*.*

[3] For the purpose of our current analysis, it suffices to expand the eigenvalue $\lambda_c(\xi)$ up to second order and the associated eigenvector up to first order. We refer to Remark 3.1.5 for further details.

(iv) *We have*

$$\lambda_c'(\xi) = 2\mathrm{i}\langle \widetilde{\Phi}_\xi, D(\partial_\zeta + \mathrm{i}\xi)\Phi_\xi\rangle_{L^2(0,T)} + \mathrm{i}c_0$$

and the expansions

$$\left|\lambda_c(\xi) + \mathrm{i}c_g\xi + d\xi^2\right| \leq C|\xi|^3, \qquad \|\Phi_\xi - \phi_0' - \mathrm{i}\xi\partial_k\phi(\cdot;1)\|_{H^m(0,T)} \leq C|\xi|^2, \tag{3.7}$$

hold for $\xi \in V_1$ *with coefficients*

$$\begin{aligned} c_g &= -2\langle \widetilde{\Phi}_0, D\phi_0''\rangle_{L^2(0,T)} - c_0 = \omega'(1) - c_0 \in \mathbb{R}, \\ d &= \langle \widetilde{\Phi}_0, D\phi_0' + 2D\partial_{\zeta k}\phi(\cdot;1)\rangle_{L^2(0,T)} > 0. \end{aligned} \tag{3.8}$$

The function λ_c in Proposition 3.1.3 is called the *linear dispersion relation*. The coefficient c_g in (3.7) is the *group velocity* of the wave train and provides the speed at which perturbations are transported along the wave train (in the frame moving with the speed c_0), cf. [29]. We make the generic assumption that the wave train has nonzero group velocity. By reversing space $x \to -x$ in (3.2) we may then without loss of generality assume that the group velocity is negative.

(H2) Assuming, in accordance with Hypothesis (D3), that 0 is a simple eigenvalue of $\mathcal{L}(0)$, the group velocity c_g, defined in (3.8), is negative.

On the linear level, the interpretation of Assumptions (D1)-(D3) and (H2) is that perturbations decay diffusively and are transported to the left along the wave train, i.e., there is an *outgoing diffusive mode* at the origin, cf. [8, Section 2.1]. In [21], it was shown that the group velocity of the wave trains is negative in the oscillatory regime with $0 < \mu < \frac{1}{2}$ and $0 < \varepsilon \ll \gamma \ll 1$.

Another important consequence of Assumption (H2) is that the linear dispersion relation λ_c is invertible in the point $\xi = 0$. Hence, for $|\lambda|$ sufficiently small, the periodic eigenvalue problem $(\mathcal{L}_0 - \lambda)\mathbf{w} = 0$ has a single Floquet exponent converging to 0 as $\lambda \to 0$. In our stability analysis we exploit this fact to relate the inverse Laplace representation of the low-frequency part of the semigroup generated by $\mathcal{L}_0$ with the Floquet-Bloch representation, see §3.3.4.

3.1.2. Main result

We are now ready to present our main result, which establishes Lyapunov stability of diffusively spectrally stable wave trains in the FHN system against C_{ub}-perturbations. Furthermore, it yields convergence of the perturbed solution towards a modulated wave train, where the phase modulation can be approximated by a solution of a viscous Hamilton-Jacobi equation.

Theorem 3.1.4. *Assume* (H1), (H2) *and* (D1)-(D3). *Fix a constant* $K > 0$. *Then, there exist constants* $\alpha, \epsilon_0, M > 0$ *such that, whenever* $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$ *satisfies*

$$E_0 := \|\mathbf{w}_0\|_{L^\infty} < \epsilon_0, \qquad \|\mathbf{w}_0\|_{C^3_{\mathrm{ub}} \times C^2_{\mathrm{ub}}} < K,$$

there exist a smooth function $\psi \in C^\infty([0,\infty) \times \mathbb{R}, \mathbb{R})$ *with* $\psi(0) = 0$ *and* $\psi(t) \in C^m_{\mathrm{ub}}(\mathbb{R})$ *for each* $m \in \mathbb{N}_0$ *and* $t \geq 0$ *and a unique classical global solution*

$$\mathbf{u} \in C\big([0,\infty), C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,\infty), C^1_{\mathrm{ub}}(\mathbb{R})\big) \tag{3.9}$$

to (3.3) *with initial condition* $\mathbf{u}(0) = \phi_0 + \mathbf{w}_0$*, which obey the estimates*

$$\|\mathbf{u}(t) - \phi_0\|_{L^\infty} \leq ME_0, \tag{3.10}$$

$$\|\mathbf{u}(t) - \phi_0(\cdot + \psi(\cdot,t))\|_{L^\infty} \leq \frac{ME_0}{\sqrt{1+t}}, \tag{3.11}$$

$$\|\mathbf{u}(t) - \phi_0(\cdot + \psi(\cdot,t)(1+\psi_\zeta(\cdot,t)); 1+\psi_\zeta(\cdot,t))\|_{L^\infty} \leq ME_0 \frac{\log(2+t)}{1+t} \tag{3.12}$$

and

$$\begin{aligned} \|\psi(t)\|_{L^\infty} \leq ME_0, \qquad \|\psi_\zeta(t)\|_{L^\infty}, \|\partial_t\psi(t)\|_{L^\infty} \leq \frac{ME_0}{\sqrt{1+t}}, \\ \|\psi_{\zeta\zeta}(t)\|_{C^4_{\mathrm{ub}}}, \|\partial_t\psi_\zeta(t)\|_{C^3_{\mathrm{ub}}} \leq ME_0 \frac{\log(2+t)}{1+t} \end{aligned} \tag{3.13}$$

for all $t \geq 0$*. Moreover, there exists a unique classical global solution* $\breve{\psi} \in C\big([0,\infty), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,\infty), C_{\mathrm{ub}}(\mathbb{R})\big)$ *with initial condition* $\breve{\psi}(0) = \widetilde{\Phi}_0^* \mathbf{w}_0$ *of the viscous Hamilton-Jacobi equation*

$$\partial_t \breve{\psi} = d\breve{\psi}_{\zeta\zeta} - c_g \breve{\psi}_\zeta + \nu \breve{\psi}_\zeta^2 \tag{3.14}$$

with coefficients (3.8) *and*

$$\begin{aligned} \nu = -\tfrac{1}{2}\omega''(1) = \big\langle \widetilde{\Phi}_0, D(\phi_0'' + 2\partial_{\zeta kk}\phi(\cdot;1)) + \tfrac{1}{2}F''(\phi_0)\big(\partial_k\phi(\cdot;1), \partial_k\phi(\cdot;1)\big)\big\rangle_{L^2(0,T)} \\ - 2\big\langle \widetilde{\Phi}_0, D\phi_0''\big\rangle_{L^2(0,T)} \big\langle \widetilde{\Phi}_0, \partial_{\zeta k}\phi(\cdot;1)\big\rangle_{L^2(0,T)}, \end{aligned} \tag{3.15}$$

satisfying

$$t^{\frac{j}{2}} \left\| \partial_\zeta^j \Big(\psi(t) - \breve{\psi}(t) \Big) \right\|_{L^\infty} \leq ME_0 \left(E_0^\alpha + \frac{\log(2+t)}{\sqrt{1+t}} \right) \tag{3.16}$$

for $j = 0,1$ *and* $t \geq 0$.

We compare Theorem 3.1.4 with earlier nonlinear stability results [51, 84] of wave trains in nondegenerate reaction-diffusion systems against C_{ub}-perturbations. First of all, we retrieve the same diffusive decay rates as in the reaction-diffusion case. It is argued in [84, Section 6.1] that these decay rates are sharp (up to possibly a logarithm). Second, we do require more regular initial data than in [84], where initial conditions $\mathbf{w}_0$ in $C_{\mathrm{ub}}(\mathbb{R})$ are considered. The reason is as follows. The lack of parabolic smoothing naturally leads one to consider initial data $\mathbf{w}_0$ from the domain $C^2_{\mathrm{ub}}(\mathbb{R}) \times C^1_{\mathrm{ub}}(\mathbb{R})$ of the diffusion-advection operator $\mathcal{L}_0$, so that the perturbed solution $\mathbf{u}(t)$ of the semilinear evolution problem (3.3) with initial condition $\mathbf{u}(0) = \phi_0 + \mathbf{w}_0$ is classical. Moreover, we lose one additional degree of regularity due to the embedding of uniformly local Sobolev spaces in C_{ub}-spaces, cf. [94, Section 8.3.1], which are used to obtain a nonlinear damping estimate to control regularity in the scheme, see §3.1.3 below for more details. We emphasize that we only require our initial data to be *bounded* in $(C^3_{\mathrm{ub}} \times C^2_{\mathrm{ub}})$-norm and, similar as in [84], to be small in L^∞-norm. This contrasts with earlier nonlinear stability results [46, 58, 88] of wave trains in semilinear (nonparabolic) problems and is due to the use of Gagliardo-Nirenberg interpolation in the nonlinear damping estimate, see Remark 3.4.10 for more details.

The approximation of the phase modulation $\psi(t)$ by a solution to the viscous Hamilton-Jacobi equation (3.14) was also found in the reaction-diffusion case in [84]. Thus, independent of the precise structure and smoothing properties of the underlying system, the viscous Hamilton-Jacobi equation arises as governing equation for the phase modulation, whose coefficients are fully determined by the first and second-order terms in the expansion of the linear and nonlinear dispersion relations. We refer to [29] for further details. Important to note is that once the diffusive spectral stability assumptions are violated, e.g. due to the presence of additional conservation laws, the governing equation of the phase modulation can change, cf. [59].

3.1.3. Strategy of proof and main challenges

We prove Theorem 3.1.4 by extending the L^∞-theory, which was recently developed in [51, 84] and applied to establish nonlinear stability of wave trains in reaction-diffusion systems against C_{ub}-perturbations, beyond the parabolic setting. Here, we outline the strategy of proof and explain how we address the novel challenges arising due to incomplete parabolicity.

To prove Theorem 3.1.4, we wish to control the perturbation $\widetilde{\mathbf{w}}(t) = \mathbf{u}(t) - \phi_0$ over time, which obeys the semilinear equation

$$(\partial_t - \mathcal{L}_0)\widetilde{\mathbf{w}} = \widetilde{\mathcal{N}}(\widetilde{\mathbf{w}}), \tag{3.17}$$

where $\mathcal{L}_0$ is the linearization of (3.3) about ϕ_0 given by (3.5) and $\widetilde{\mathcal{N}}(\widetilde{\mathbf{w}})$ is the nonlinear residual given by

$$\widetilde{\mathcal{N}}(\widetilde{\mathbf{w}}) = F(\phi_0 + \widetilde{\mathbf{w}}) - F(\phi_0) - F'(\phi_0)\widetilde{\mathbf{w}}.$$

We will establish that $\mathcal{L}_0$ generates a C_0-semigroup $\mathrm{e}^{\mathcal{L}_0 t}$, which, due to the fact that $\mathcal{L}_0$ has spectrum up to the imaginary axis $\mathrm{i}\mathbb{R}$, does not exhibit decay as an operator on $C_{\mathrm{ub}}(\mathbb{R})$, thus obstructing a standard nonlinear stability argument.

In earlier works [55, 62, 89], considering the nonlinear stability of wave trains in reaction-diffusion systems against localized perturbations, this issue was addressed by employing its Floquet-Bloch representation to decompose the semigroup generated by the linearization and introducing a spatio-temporal phase modulation to capture the critical diffusive behavior. More precisely, one considers the *inverse-modulated perturbation*

$$\mathbf{w}(\zeta, t) = \mathbf{u}(\zeta - \psi(\zeta, t), t) - \phi_0(\zeta), \tag{3.18}$$

where the spatio-temporal phase modulation $\psi(\zeta, t)$ is determined a posteriori. The inverse-modulated perturbation satisfies a *quasilinear* equation of the form

$$(\partial_t - \mathcal{L}_0)(\mathbf{w} + \phi_0'\psi - \psi_\zeta \mathbf{w}) = N(\mathbf{w}, \mathbf{w}_\zeta, \mathbf{w}_{\zeta\zeta}, \psi_\zeta, \partial_t\psi, \psi_{\zeta\zeta}, \psi_{\zeta\zeta\zeta}), \tag{3.19}$$

where N is nonlinear in its variables. One decomposes the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ into a principal part of the form $\phi_0' S_p(t)$, where $S_p(t)$ decays diffusively, and a residual part exhibiting higher order temporal decay. Finally, one chooses the phase modulation $\psi(t)$ in (3.18) in such a way that it captures the most critical contributions in the Duhamel formulation of (3.19), allowing one to close a nonlinear iteration argument in ψ_ζ, ψ_t and $\mathbf{w}$. The leading-order dynamics of the phase modulation ψ is then given by a viscous Hamilton-Jacobi equation, cf. [29] and Remark 3.1.5.

The above approach has successfully been extended to the nonlinear stability analysis of periodic traveling waves against L^2-localized perturbations in nonparabolic dissipative problems such as the St. Venant equations [58, 88] and the Lugiato-Lefever equation [46] using resolvent estimates and the Gearhart-Prüss theorem to render exponential decay of the high-frequency part of the C_0-semigroup.

In the nonlinear stability analysis of wave trains in reaction-diffusion systems against C_{ub}-perturbations in [84], the decomposition was carried out on the level of the temporal Green's function, which is C^2 and exponentially localized, thus circumventing an application of the Floquet-Bloch transform to nonlocalized functions, which is only defined in the sense of tempered distributions. This leads to an explicit representation of the low-frequency part of the semigroup as in [55] and control on the high-frequency part by pointwise Green's function estimates established in [62].

For nonelliptic operators, such as $\mathcal{L}_0$, the temporal Green's function is typically a distribution, complicating a potential decomposition via the Floquet-Bloch transform. We address this challenge by taking inspiration from [8] and employing its inverse Laplace representation, given by the complex inversion formula

$$\mathrm{e}^{\mathcal{L}_0 t}\mathbf{w} = \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\eta - \mathrm{i}R}^{\eta+\mathrm{i}R} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{w}\,\mathrm{d}\lambda \tag{3.20}$$

with $\eta, t > 0$ and $\mathbf{w} \in D(\mathcal{L}_0)$, to decompose the semigroup. By partitioning and deforming the integration contour in (3.20), we write the semigroup as the sum of a high- and low-frequency part. Here, we associate the high-frequency part of the semigroup with pieces of the deformed contour integral where $|\operatorname{Im}(\lambda)| \gg 1$, i.e., where $\mathrm{e}^{\lambda t}$ rapidly oscillates, and the low-frequency part of the semigroup with pieces of the deformed contour integral where $|\lambda| \ll 1$.

As the space of perturbations $C_{\text{ub}}(\mathbb{R})$ does not admit any Hilbert-space structure, we cannot rely on the Gearhart-Prüss theorem (or leverage the sectoriality of the linearization) to establish a spectral mapping property. Therefore, we instead use the expansion of the resolvent as a Neumann series for $\lambda \in \mathbb{C}$ with $|\operatorname{Im}(\lambda)| \gg 1$, which was established in [8], to control the high-frequency part of the semigroup. The leading-order terms in the Neumann series expansion of resolvent are not absolutely integrable over the high-frequency parts of the contour in (3.20) and, thus, the question of how to control these terms is not straightforward. Here, we cannot resort to the arguments in [8] which rely on test functions, since these are not dense in $C_{\text{ub}}(\mathbb{R})$. Instead, we identify the critical terms in the Neumann series expansion of $(\lambda - \mathcal{L}_0)^{-1}$ as products of resolvents of simple diffusion and advection operators. The corresponding terms in the inverse Laplace formula then correspond to *convolutions* of the C_0-semigroups generated by these diffusion and advection operators. As far as the authors are aware, the observation that the complex inversion formula holds for convolutions of C_0-semigroups is novel and is therefore of its own interest, cf. [43]. All in all, we obtain that the high-frequency part of the semigroup is exponentially decaying on $C_{\text{ub}}(\mathbb{R})$.

To render decay of the low-frequency part of the semigroup one must rely on diffusive smoothing in the case of nonlocalized perturbations. The diffusive decay rates of the low-frequency part are not strong enough to control the critical nonlinear term $\nu(\psi_\zeta)^2$ in the perturbed viscous Hamilton-Jacobi equation satisfied by ψ. In [84], this difficulty has been addressed by further decomposing the low-frequency part of the semigroup via its Floquet-Bloch representation and relating its principal part to the convective heat semigroup $\mathrm{e}^{(d\partial_\zeta^2 - c_g\partial_\zeta)t}$, which allows to apply the Cole-Hopf transform to eliminate the critical $(\psi_\zeta)^2$-term.

Here, we link the inverse Laplace representation of the low-frequency part with the Floquet-Bloch representation from [84] modulo exponentially decaying terms, while exploiting the nonzero group velocity of the wave train, cf. Assumption (H2). This allows us to harness the decomposition of and estimates on the low-frequency part of the semigroup from [84]. We emphasize that, to the authors' knowledge, such a link has not been established before and is interesting in its own right.

After applying the Cole-Hopf transform to the equation of the phase modulation ψ to eliminate the critical nonlinear term, the decay of all remaining linear and nonlinear terms is strong enough to close a nonlinear iteration argument in ψ_ζ, ψ_t and $\mathbf{w}$. Yet, the equation for the inverse-modulated perturbation is quasilinear and an apparent loss of derivatives must be addressed to control regularity in the nonlinear argument. This is a standard issue in the nonlinear stability of wave trains and it has been recognized that, as long as the underlying equation is semilinear, such a loss of derivatives can be addressed by considering the unmodulated perturbation or to the so-called *forward-modulated perturbation*

$$\mathring{\mathbf{w}}(\zeta,t) = \mathbf{u}(\zeta,t) - \phi_0(\zeta + \psi(\zeta,t)), \tag{3.21}$$

which measures the deviation from the modulated wave train, cf. [105]. Both the unmodulated perturbation $\widetilde{\mathbf{w}}(t)$ and the forward-modulated perturbation $\mathring{\mathbf{w}}(t)$ obey a semilinear equation in which no derivatives are lost, yet where decay is too slow to close an independent iteration scheme. However, by relating $\widetilde{\mathbf{w}}(t)$ (or $\mathring{\mathbf{w}}(t)$) to the inverse-modulated perturbation $\mathbf{w}(t)$ regularity can be controlled in the nonlinear iteration scheme. Regularity control can then be obtained by showing that $\widetilde{\mathbf{w}}(t)$ (or $\mathring{\mathbf{w}}(t)$) obeys a so-called *nonlinear damping estimate* [59, 105], which is an energy estimate bounding the H^m-norm of the solution for some $m \in \mathbb{N}$ in terms of the H^m-norm of its initial condition and the L^2-norm of the solution. A nonlinear damping estimate for the forward-modulated perturbation has been derived in the setting of the FHN system in [8, Proposition 8.6].

A second option is to control regularity by deriving tame estimates on derivatives of $\widetilde{\mathbf{w}}(t)$ (or $\mathring{\mathbf{w}}(t)$) via its Duhamel formulation [46, 84, 85]. In the absence of parabolic smoothing, the advantage of using nonlinear damping estimates is that they yield sharp bounds on derivatives and typically require less regular initial data, as can for instance be seen by comparing [105, Theorem 6.2] with [46, Theorem 1.3]. In the case of nonlocalized perturbations, one has so far been compelled to the second approach using tame estimates, cf. [84, 85], since the lack of localization prohibits the use of L^2-energy estimates. Motivated by the possibility to accommodate less regular initial data, we control regularity in this work by extending the method of nonlinear damping estimates to uniformly local Sobolev norms, see [94, Section 8.3.1], which allow for nonlocalized perturbations. On top of that, we work with a slightly modified version of the forward-modulated perturbation given by

$$\begin{aligned}\mathring{\mathbf{z}}(\zeta,t) &:= \mathbf{u}(\zeta,t) - \phi(\zeta + \psi(\zeta,t)(1+\psi_\zeta(\zeta,t)); 1+\psi_\zeta(\zeta,t))\\ &= \mathring{\mathbf{w}}(\zeta,t) + \phi_0(\zeta+\psi(\zeta,t)) - \phi(\zeta + \psi(\zeta,t)(1+\psi_\zeta(\zeta,t)); 1+\psi_\zeta(\zeta,t))\\ &= \widetilde{\mathbf{w}}(\zeta,t) + \phi_0(\zeta) - \phi(\zeta + \psi(\zeta,t)(1+\psi_\zeta(\zeta,t)); 1+\psi_\zeta(\zeta,t)),\end{aligned} \tag{3.22}$$

which again satisfies a semilinear equation in which no derivatives are lost and is well-defined as long as $\|\psi_\zeta(t)\|_{L^\infty}$ is sufficiently small, cf. Proposition 3.1.2. The reason is that $\mathring{\mathbf{z}}(t)$ and its derivatives exhibit stronger decay than $\mathring{\mathbf{w}}(t)$, cf. [84, Corollary 1.4]. Having sharper bounds on derivatives, it is no longer necessary to move derivatives in the Duhamel formulation from the nonlinearity to the slowly decaying principal low-frequency part $S_p(t)$ of the semigroup as in [84]. This provides a significant simplification with respect to [84] as the computation and estimation of commutators between the operators $S_p(t)$ and $\partial_\zeta^m, m \in \mathbb{N}$, is no longer necessary.

Thus, using uniformly local Sobolev norms[4], we obtain a nonlinear damping estimate for the modified forward-modulated perturbation $\mathring{\mathbf{z}}(t)$ and our nonlinear iteration scheme can also be closed from the perspective of regularity. This then leads to the proof of Theorem 3.1.4.

Remark 3.1.5. *It was already observed in [29] that the coefficients of the viscous Hamilton-Jacobi equation* (3.14)*, governing the leading-order phase dynamics, can be expressed in terms of the coefficients of the second-order expansion of the linear and nonlinear dispersion relations $\lambda_c(\xi)$ and $\omega(k)$, cf. Propositions 3.1.2 and 3.1.3 and identity* (3.15)*. In the current setting of fully nonlocalized perturbations [84], it is important to identify the leading-order Hamilton-Jacobi dynamics of the phase modulation as this allows for an application of the Cole-Hopf transform to eliminate the most critical nonlinear term. In contrast, in the nonlinear stability analyses [46, 55, 58, 62, 88] of wave trains against localized perturbations, it is not necessary to determine the leading-order phase dynamics explicitly. The derivation of the viscous Hamilton-Jacobi equation in the current setting can be found in §3.4.3 and exploits the characterization of the first-order term in the expansion of the eigenfunction Φ_ξ as the derivative of the family of wave trains $\phi(\cdot;k)$, established in Proposition 3.1.2, with respect to the wavenumber k, cf. Proposition 3.1.3.*

Remark 3.1.6. *The nonlinear damping estimate, used in the proof of Theorem 3.1.4, leads to estimates on* derivatives *of the (modulated) perturbation. Specifically, we can replace the L^∞-norms in estimates* (3.10)-(3.12) *by $(C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}})$-norms upon substituting E_0 by its fractional power $E_0^{\frac{1}{5}}$.*[5]

Here, the occurrence of the fractional power is a consequence of the use of Gagliardo-Nirenberg interpolation in the nonlinear damping estimate, see Remark 3.4.10. In addition, we note that, although our initial perturbation $\mathbf{w}_0$ lies in $C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$, we do not control the associated norm in our nonlinear stability analysis, since we lose one degree of regularity by embedding of uniformly local Sobolev spaces in C_{ub}-spaces. Nevertheless, by considering more regular initial data in Theorem 3.1.4, it is possible to track higher-order derivatives in the nonlinear argument. More precisely, taking $m \in \mathbb{N}$ and $\mathbf{w}_0 \in C^{m+3}_{\mathrm{ub}}(\mathbb{R}) \times C^{m+2}_{\mathrm{ub}}(\mathbb{R})$ with $\|\mathbf{w}_0\|_{C^{m+3}_{\mathrm{ub}} \times C^{m+2}_{\mathrm{ub}}} < K$ in Theorem 3.1.4, we find

$$\mathbf{u} \in C\big([0,\infty), C^{m+3}_{\mathrm{ub}}(\mathbb{R}) \times C^{m+2}_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,\infty), C^{m+1}_{\mathrm{ub}}(\mathbb{R})\big).$$

and the estimates (3.10)-(3.13) *can be upgraded to*

$$\begin{aligned}
\|\mathbf{u}(t) - \phi_0\|_{C^{m+2}_{\mathrm{ub}} \times C^{m+1}_{\mathrm{ub}}} &\leq M E_0^{\alpha_m}, \\
\|\mathbf{u}(t) - \phi_0(\cdot + \psi(\cdot,t))\|_{C^{m+2}_{\mathrm{ub}} \times C^{m+1}_{\mathrm{ub}}} &\leq \frac{M E_0^{\alpha_m}}{\sqrt{1+t}}, \\
\|\mathbf{u}(t) - \phi_0(\cdot + \psi(\cdot,t)(1+\psi_\zeta(\cdot,t)); 1+\psi_\zeta(\cdot,t))\|_{C^{m+2}_{\mathrm{ub}} \times C^{m+1}_{\mathrm{ub}}} &\leq M E_0^{\alpha_m} \frac{\log(2+t)}{1+t},
\end{aligned}$$

where $\alpha_m > 0$ depends on m only, and

$$\|\partial_t \psi_\zeta(t)\|_{C^{3+m}_{\mathrm{ub}}} \leq \frac{M E_0}{\sqrt{1+t}}, \qquad \|\psi_{\zeta\zeta}(t)\|_{C^{4+m}_{\mathrm{ub}}} \leq M E_0 \frac{\log(2+t)}{1+t}$$

for all $t \geq 0$. For the sake of clarity of exposition and in order to reduce the amount of technicalities, we have chosen to only consider $(C^3_{\mathrm{ub}} \times C^2_{\mathrm{ub}})$-regular initial data only in our nonlinear stability analysis.

[4]We note that uniformly local Sobolev norms have also been used in other works, e.g. [38], to make energy estimate methods available in L^∞-spaces.

[5]In fact, we can also take $\alpha = \frac{1}{5}$ in (3.16).

3.1.4. Outline

This paper is organized as follows. In §3.2, we analyze the resolvent associated with the linearization $\mathcal{L}_0$ of (3.3) about the wave train. In §3.3 we decompose the C_0-semigroup $e^{\mathcal{L}_0 t}$ and derive associated estimates with the aid of the inverse Laplace representation and establish a Floquet-Bloch representation for its critical low-frequency part. In §3.4, we set up our nonlinear iteration scheme and derive a nonlinear damping estimate. We close the nonlinear argument and prove our main result, Theorem 3.1.4, in §3.5. We conclude in §3.6 by discussing the wider applicability of our method to achieve nonlinear stability of wave trains against fully nonlocalized perturbations in semilinear dissipative problems. Appendix 3.A is devoted to background material on the vector-valued Laplace transform. In particular, we prove that its complex inversion formula holds for convolutions of C_0-semigroups. Finally, we relegate the derivation of the equation for the modified forward-modulated perturbation to Appendix 3.B.

Notation. Let S be a set, and let $A, B\colon S \to \mathbb{R}$. Throughout the paper, the expression "$A(x) \lesssim B(x)$ for $x \in S$", means that there exists a constant $C > 0$, independent of x, such that $A(x) \leq CB(x)$ holds for all $x \in S$.

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3.2. Resolvent analysis

This section is devoted to the study of the resolvent and serves as preparation to derive pure L^∞-estimates on the high- and low-frequency components of the semigroup given by (3.20). That is, we collect and prove properties of $(\lambda - \mathcal{L}_0)^{-1}$ in the regimes $|\operatorname{Im}(\lambda)| \gg 1$ and $|\lambda| \ll 1$. Our refined low-frequency analysis of the resolvent is the starting point to link the inverse Laplace representation to the Floquet-Bloch representation of the low-frequency part of the semigroup.

3.2.1. Low-frequency resolvent analysis and decomposition

We consider the resolvent problem

$$(\mathcal{L}_0 - \lambda)\mathbf{w} = \mathbf{g} \tag{3.23}$$

with $\mathbf{w} = (u, v)^\top$ and $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ for λ in a small ball $B(0, \delta) \subset \mathbb{C}$ of radius $\delta > 0$ centered at the origin. We proceed as in [8] and write (3.23) as a first-order system

$$\psi' = A(\zeta; \lambda)\psi + G \tag{3.24}$$

in $\psi = (u, u_\zeta, v)^\top$ with inhomogeneity $G = (0, \mathbf{g})^\top$ and coefficient matrix

$$A(\zeta; \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - f'(u_0) & -c_0 & 1 \\ -\frac{\varepsilon}{c_0} & 0 & \frac{\varepsilon\gamma + \lambda}{c_0} \end{pmatrix},$$

where u_0 is the first-component of the wave train $\phi_0 = (u_0, v_0)^\top$ and $f(u) = u(1-u)(u-\mu)$ is the cubic nonlinearity in the FHN system (3.1).

The coefficient matrix $A(\cdot;\lambda)$ is T-periodic for each $\lambda \in \mathbb{C}$. Thus, we can apply Floquet theory, cf. [64, Section 2.1.3], to establish a T-periodic change of coordinates, which is locally analytic in λ, converting the homogeneous problem

$$\psi' = A(\zeta;\lambda)\psi \tag{3.25}$$

into a constant-coefficient system.

Proposition 3.2.1. *Assume* (H1). *For $\delta > 0$ sufficiently small, there exist maps $Q\colon \mathbb{R} \times B(0,\delta) \to \mathbb{C}^{3\times 3}$ and $M\colon B(0,\delta) \to \mathbb{C}^{3\times 3}$ such that the evolution $T(\zeta,\bar{\zeta};\lambda)$ of* (3.25) *can be expressed as*

$$T(\zeta,\bar{\zeta};\lambda) = Q(\zeta;\lambda)^{-1} \mathrm{e}^{M(\lambda)(\zeta-\bar{\zeta})} Q(\bar{\zeta};\lambda).$$

Here, $Q(\cdot;\lambda)$ is smooth and T-periodic for each $\lambda \in B(0,\delta)$. Moreover, M and $Q(\zeta;\cdot)$ are analytic for each $\zeta \in \mathbb{R}$.

An eigenvalue $\nu(\lambda)$ of the monodromy matrix $M(\lambda)$ is called a *spatial Floquet exponent.* It gives rise to a solution $\psi(\zeta;\lambda) = \mathrm{e}^{\nu(\lambda)\zeta} p(\zeta;\lambda)$ of (3.25), where $p(\cdot;\lambda)$ is T-periodic. Thus, translating back to the eigenvalue problem $(\mathcal{L}_0 - \lambda)\mathbf{w} = 0$, one readily observes that for each $\xi \in \mathbb{C}$ a point $\lambda \in B(0,\delta)$ is a (temporal) eigenvalue of the Bloch operator $\mathcal{L}(\xi)$ if and only if $\mathrm{i}\xi$ is an eigenvalue of $M(\lambda)$. The spectral decomposition (3.6) then implies that a point $\lambda \in B(0,\delta)$ lies in $\sigma(\mathcal{L}_0)$ if and only if $M(\lambda)$ has a purely imaginary eigenvalue.

Proposition 3.1.3 yields balls $V_1, V_2 \subset \mathbb{C}$ centered at 0 and a holomorphic map $\lambda_c\colon V_1 \to \mathbb{C}$ such that $\mathcal{L}(\xi)$ has a simple eigenvalue $\lambda_c(\xi)$ for each $\xi \in V_1$ and it holds $\sigma(\mathcal{L}_0) \cap V_2 = \{\lambda_c(\xi) : \xi \in \mathbb{R} \cap V_1\} \cap V_2$. Since we have $\lambda_c'(0) = -\mathrm{i}c_g \neq 0$ by Assumption (H2), the implicit function theorem implies, provided $\delta > 0$ is sufficiently small, that for each $\lambda \in B(0,\delta)$ the matrix $M(\lambda)$ possesses precisely one simple eigenvalue $\nu_c(\lambda)$ in V_1. These observations readily lead to the following proposition.

Proposition 3.2.2. *Assume* (H1), (H2) *and* (D1)-(D3). *There exist constants $C, \delta > 0$ and a holomorphic map $\nu_c\colon B(0,\delta) \to \mathbb{C}$ satisfying the following assertions.*

(i) *$\nu_c(\lambda)$ is a simple spatial Floquet exponent associated with the T-periodic first-order problem* (3.25) *for each $\lambda \in B(0,\delta)$.*

(ii) *A point $\lambda \in B(0,\delta)$ lies in $\sigma(\mathcal{L}_0)$ if and only if $\nu_c(\lambda)$ is purely imaginary.*

(iii) *We have $\nu_c(\lambda_c(\xi)) = \mathrm{i}\xi$ for each $\xi \in V_1$ such that $\lambda_c(\xi) \in B(0,\delta)$.*

(iv) *The expansion*

$$\left|\nu_c(\lambda) + \frac{1}{c_g}\lambda\right| \leq C|\lambda|^2$$

holds for all $\lambda \in B(0,\delta)$.

(v) *For $\lambda \in B(0,\delta)$ to the right of $\sigma(\mathcal{L}_0)$ we have* $\mathrm{Re}(\nu_c(\lambda)) > 0$.

Propositions 3.2.1 and 3.2.2 imply that for $\lambda \in B(0,\delta)$ system (3.25) has an exponential dichotomy on $\mathbb{R}$ if and only if there are no purely imaginary Floquet exponents, which is the case precisely if λ lies in the resolvent set $\rho(\mathcal{L}_0)$. Hence, taking $\lambda \in B(0,\delta) \cap \rho(\mathcal{L}_0)$ and letting $P^s(\lambda)$ and $P^u(\lambda)$ be the spectral projections onto the stable and unstable subspaces of $M(\lambda)$, we can express the spatial Green's function associated with (3.25) as

$$\mathcal{G}(\zeta,\bar{\zeta};\lambda) = Q(\zeta;\lambda)^{-1}\mathrm{e}^{M(\lambda)(\zeta-\bar{\zeta})}\Big(P^s(\lambda)\mathbf{1}_{(-\infty,\zeta]}(\bar{\zeta}) - P^u(\lambda)\mathbf{1}_{[\zeta,\infty)}(\bar{\zeta})\Big)Q(\bar{\zeta};\lambda)$$

where $\mathbf{1}_{(-\infty,\zeta]}$ and $\mathbf{1}_{[\zeta,\infty)}$ are indicator functions. Introducing the matrices

$$\Pi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Pi_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & c_0^{-1} \end{pmatrix}.$$

to translate between the original formulation (3.23) and the first-order formulation (3.24) of the resolvent problem, we find that the unique solution of (3.23) is now given by

$$\Big((\mathcal{L}_0 - \lambda)^{-1}\mathbf{g}\Big)(\zeta) = \mathbf{w}(\zeta;\lambda) = \int_{\mathbb{R}} \Pi_2 \mathcal{G}(\zeta,\bar{\zeta};\lambda)\Pi_3 \mathbf{g}(\bar{\zeta})\,\mathrm{d}\bar{\zeta}.$$

By Proposition 3.2.2 the spatial Floquet exponent $\nu_c(\lambda)$ is a simple eigenvalue of $M(\lambda)$ and all other spatial Floquet exponents are bounded away from $\mathrm{i}\mathbb{R}$ for $\lambda \in B(0,\delta)$. Therefore, the spectral projection $P^{cu}(\lambda)$ of $M(\lambda)$ onto the eigenspace associated with $\nu_c(\lambda)$ is defined for all $\lambda \in B(0,\delta)$. For $\lambda \in B(0,\delta)$ to the right of $\sigma(\mathcal{L}_0)$ it holds $\mathrm{Re}(\nu_c(\lambda)) > 0$ and we can decompose $P^u(\lambda) = P^{uu}(\lambda) + P^{cu}(\lambda)$. This then leads to the desired resolvent decomposition for small λ.

Proposition 3.2.3. *Assume* (H1), (H2) *and* (D1)-(D3). *There exist constants $C,\delta > 0$ and a holomorphic map $S_e^0\colon B(0,\delta) \to \mathcal{B}(C_{\mathrm{ub}}(\mathbb{R}))$ such that for $\lambda \in B(0,\delta)$, $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ and $\zeta \in \mathbb{R}$ we have*

$$\begin{aligned}\Big((\mathcal{L}_0 - \lambda)^{-1}\mathbf{g}\Big)(\zeta) = &- \int_{\mathbb{R}} \Pi_2 Q(\zeta;\lambda)^{-1}\mathrm{e}^{\nu_c(\lambda)(\zeta-\bar{\zeta})}\mathbf{1}_{[\zeta,\infty)}(\bar{\zeta})P^{cu}(\lambda)Q(\bar{\zeta};\lambda)\Pi_3\mathbf{g}(\bar{\zeta})\,\mathrm{d}\bar{\zeta} \\ &+ \Big(S_e^0(\lambda)\mathbf{g}\Big)(\zeta)\end{aligned} \tag{3.26}$$

and it holds

$$\Big\|S_e^0(\lambda)\mathbf{g}\Big\|_{L^\infty} \leq C\|\mathbf{g}\|_{L^\infty}.$$

In order to later relate the inverse Laplace representation of the low-frequency part of the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ to its Floquet-Bloch representation, we prove the following technical lemma showing that the expression $\Pi_2 Q(\zeta;\lambda)^{-1}P^{cu}(\lambda)Q(\bar{\zeta};\lambda)\Pi_3$ in (3.26) can be written as a product of solutions of the eigenvalue problem $(\mathcal{L}_0 - \lambda)\mathbf{w} = 0$ and its adjoint $(\mathcal{L}_0 - \lambda)^*\mathbf{w} = 0$.

Lemma 3.2.4. *Assume* (H1), (H2) *and* (D1)-(D3). *There exist a constant $\delta > 0$ and functions $\Psi, \tilde{\Psi}\colon \mathbb{R} \times B(0,\delta) \to \mathbb{C}^2$ satisfying*

$$\Pi_2 Q(\zeta;\lambda)^{-1}P^{cu}(\lambda)Q(\bar{\zeta};\lambda)\Pi_3 = \Psi(\zeta;\lambda)\tilde{\Psi}(\bar{\zeta};\lambda)^* \tag{3.27}$$

for $\zeta,\bar{\zeta} \in \mathbb{R}$ and $\lambda \in B(0,\delta)$. Moreover, $\Psi(\cdot;\lambda)$ and $\tilde{\Psi}(\cdot;\lambda)$ are smooth and T-periodic for each $\lambda \in B(0,\delta)$ and $\Psi(\zeta;\cdot)\tilde{\Psi}(\bar{\zeta};\cdot)^*$ is analytic for each $\zeta,\bar{\zeta} \in \mathbb{R}$. Finally, we have

$$\Psi(\cdot;\lambda_c(\xi)) = \Phi_\xi, \qquad \lambda_c'(\xi)\tilde{\Psi}(\cdot;\lambda_c(\xi)) = \mathrm{i}\widetilde{\Phi}_\xi \tag{3.28}$$

for $\xi \in V_1$ such that $\lambda_c(\xi) \in B(0,\delta)$, where Φ_ξ and $\widetilde{\Phi}_\xi$ are defined in Proposition 3.1.3.

Proof. Let $\lambda \in B(0,\delta)$. By Propositions 3.2.1 and 3.2.2 the monodromy matrix $M(\lambda)$ has a simple eigenvalue $\nu_c(\lambda)$, provided $\delta > 0$ is sufficiently small. Let $w_1(\lambda)$ be an associated eigenvector. Moreover, let $\widetilde{w}_1(\lambda)$ be an eigenvector associated with the simple eigenvalue $\overline{\nu_c(\lambda)}$ of the adjoint matrix $M(\lambda)^*$. The spectral projection $P^{cu}(\lambda)$ onto the eigenspace of $M(\lambda)$ associated with $\nu_c(\lambda)$ is now given by

$$P^{cu}(\lambda) = \frac{w_1(\lambda)\widetilde{w}_1(\lambda)^*}{\langle \widetilde{w}_1(\lambda), w_1(\lambda)\rangle}.$$

Since $\nu_c(\lambda)$ is simple for each $\lambda \in B(0,\delta)$, the map $P^{cu}\colon B(0,\delta) \to \mathbb{C}^{3\times 3}$ is holomorphic by standard analytic perturbation theory [66, Section II.1.4].

We define $\Psi, \tilde{\Psi}\colon \mathbb{R} \times B(0,\delta) \to \mathbb{C}^2$ by

$$\Psi(\zeta;\lambda) = \Pi_2 v_1(\zeta;\lambda), \qquad v_1(\zeta;\lambda) := \frac{Q(\zeta;\lambda)^{-1} w_1(\lambda)}{\langle \widetilde{\Phi}_{-\mathrm{i}\nu_c(\lambda)}, \Pi_2 Q(\cdot;\lambda)^{-1} w_1(\lambda)\rangle_{L^2(0,T)}}$$

and

$$\tilde{\Psi}(\zeta;\lambda) = \Pi_3^* v_2(\zeta;\lambda), \qquad v_2(\zeta;\lambda) := \frac{Q(\zeta;\lambda)^* \widetilde{w}_1(\lambda)}{\langle w_1(\lambda), \widetilde{w}_1(\lambda)\rangle} \langle \Pi_2 Q(\cdot;\lambda)^{-1} w_1(\lambda), \widetilde{\Phi}_{-\mathrm{i}\nu_c(\lambda)}\rangle_{L^2(0,T)}.$$

Then, $\Psi(\cdot;\lambda)$ and $\tilde{\Psi}(\cdot;\lambda)$ are smooth and T-periodic for each $\lambda \in B(0,\delta)$ by Proposition 3.2.1. One readily observes that (3.27) holds for $\zeta, \bar{\zeta} \in \mathbb{R}$ and $\lambda \in B(0,\delta)$. Moreover, since $Q(\zeta;\cdot)$, $Q(\bar{\zeta};\cdot)$ and P^{cu} are analytic by Proposition 3.2.1, so is $\Psi(\zeta;\cdot)\tilde{\Psi}(\bar{\zeta};\cdot)^*$ for each $\zeta, \bar{\zeta} \in \mathbb{R}$.

Next, we observe that the evolution $T_{\mathrm{ad}}(\zeta,\bar{\zeta};\lambda)$ of the adjoint problem

$$\vartheta' = -A(\zeta;\lambda)^* \vartheta, \tag{3.29}$$

of (3.25) is given by $T_{\mathrm{ad}}(\zeta,\bar{\zeta};\lambda) = T(\bar{\zeta},\zeta;\lambda)^*$, where $T(\zeta,\bar{\zeta};\lambda)$ is the evolution of (3.25). So, since $\nu_c(\lambda)$ is an eigenvalue of $M(\lambda)$ with associated eigenvector $w_1(\lambda)$ and $-\overline{\nu_c(\lambda)}$ is an eigenvalue of $-M(\lambda)^*$ with associated eigenvector $\widetilde{w}_1(\lambda)$, we obtain, by Proposition 3.2.1, that $\psi(\zeta;\lambda) = \mathrm{e}^{\nu_c(\lambda)\zeta} v_1(\zeta;\lambda)$ and $\vartheta(\zeta;\lambda) = \mathrm{e}^{-\overline{\nu_c(\lambda)}\zeta} v_2(\zeta;\lambda)$ are solutions of (3.25) and (3.29), respectively. Consequently, $\mathbf{w}(\zeta;\lambda) = \mathrm{e}^{\nu_c(\lambda)\zeta}\Psi(\zeta;\lambda)$ and $\widetilde{\mathbf{w}}(\zeta;\lambda) = \mathrm{e}^{-\overline{\nu_c(\lambda)}\zeta}\tilde{\Psi}(\zeta;\lambda)$ solve the eigenvalue problems $(\mathcal{L}_0 - \lambda)\mathbf{w} = 0$ and $(\mathcal{L}_0 - \lambda)^*\widetilde{\mathbf{w}} = 0$, respectively. Therefore, $\Psi(\cdot;\lambda), \tilde{\Psi}(\cdot;\lambda) \in H^2_{\mathrm{per}}(0,T)$ are nontrivial solutions of the eigenvalue problems $(\mathcal{L}(-\mathrm{i}\nu_c(\lambda)) - \lambda)\mathbf{w} = 0$ and $(\mathcal{L}(-\mathrm{i}\nu_c(\lambda)) - \lambda)^*\widetilde{\mathbf{w}} = 0$, respectively. Now, let $\xi \in V_1$ be such that $\lambda_c(\xi) \in B(0,\delta)$. Then, we find with the aid of Proposition 3.2.2 that $\Psi(\zeta;\lambda_c(\xi))$ and $\tilde{\Psi}(\zeta;\lambda_c(\xi))$ lie in $\ker(\mathcal{L}(\xi) - \lambda_c(\xi))$ and $\ker((\mathcal{L}(\xi) - \lambda_c(\xi))^*)$, respectively, which are spanned by Φ_ξ and $\widetilde{\Phi}_\xi$, respectively, by Proposition 3.1.3. Hence, on the one hand, the gauge condition $\langle \widetilde{\Phi}_\xi, \Psi(\zeta;\lambda_c(\xi))\rangle_{L^2(0,T)} = 1 = \langle \widetilde{\Phi}_\xi, \Phi_\xi\rangle_{L^2(0,T)}$, cf. Proposition 3.1.3, implies $\Phi_\xi = \Psi(\cdot;\lambda_c(\xi))$. On the other hand, there exists $\kappa_\xi \in \mathbb{C}\setminus\{0\}$ such that $\tilde{\Psi}(\cdot;\lambda_c(\xi)) = \kappa_\xi \widetilde{\Phi}_\xi$. So, all that remains to show is that $\kappa_\xi = \mathrm{i}/\lambda_c'(\xi)$.

First, using that $\psi(\zeta;\lambda) = \mathrm{e}^{\nu_c(\lambda)\zeta} v_1(\zeta;\lambda)$ and $\vartheta(\zeta;\lambda) = \mathrm{e}^{-\overline{\nu_c(\lambda)}\zeta} v_2(\zeta;\lambda)$ are solutions of (3.25) and (3.29), respectively, and we have $\nu_c(\lambda_c(\xi)) = \mathrm{i}\xi$ by Proposition 3.2.2, we obtain

$$v_1(\zeta;\lambda_c(\xi)) = \begin{pmatrix} \Phi_{1,\xi} \\ \mathrm{i}\xi\Phi_{1,\xi} + \Phi_{1,\xi}' \\ \Phi_{2,\xi} \end{pmatrix}, \qquad v_2(\zeta;\lambda_c(\xi)) = \kappa_\xi \begin{pmatrix} (c_0 - \mathrm{i}\xi)\widetilde{\Phi}_{1,\xi} - \widetilde{\Phi}_{1,\xi}' \\ \widetilde{\Phi}_{1,\xi} \\ c_0\widetilde{\Phi}_{2,\xi} \end{pmatrix}.$$

Finally, evoking Proposition 3.1.3, integrating by parts and using $1 = \langle \widetilde{\Phi}_\xi, \Phi_\xi \rangle_{L^2(0,T)}$, we arrive at

$$\begin{aligned}\kappa_\xi^{-1} &= \kappa_\xi^{-1} \langle v_2(\cdot;\lambda), v_1(\cdot;\lambda) \rangle_{L^2(0,T)} \\ &= \langle (c_0 - \mathrm{i}\xi)\widetilde{\Phi}_{1,\xi} - \widetilde{\Phi}'_{1,\xi}, \Phi_{1,\xi} \rangle_{L^2(0,T)} + \langle \widetilde{\Phi}_{1,\xi}, \mathrm{i}\xi\Phi_{1,\xi} + \Phi'_{1,\xi} \rangle_{L^2(0,T)} + \langle c_0 \widetilde{\Phi}_{2,\xi}, \Phi_{2,\xi} \rangle_{L^2(0,T)} \\ &= c_0 + 2\langle \widetilde{\Phi}_\xi, D(\partial_\zeta + \mathrm{i}\xi)\Phi_\xi \rangle_{L^2(0,T)} = -\mathrm{i}\lambda_c'(\xi),\end{aligned}$$

which concludes the proof. □

3.2.2. High-frequency resolvent analysis

We consider the resolvent $(\lambda - \mathcal{L}_0)^{-1}$ in the high-frequency regime. The spectrum of $\mathcal{L}_0$ away from the origin is by Proposition 3.1.3 confined to the left-half plane with uniform distance from the imaginary axis, which allows us to deform the high-frequency parts of the integration contour in (3.20) into the left-half plane away from the imaginary axis and the spectrum. Specifically, this leads us to consider the contours connecting $b \pm \mathrm{i}\varpi_0$ with $b \pm \mathrm{i}R$ for some $b < 0$ and $R > \varpi_0 > 0$. Since these contours are unbounded as $R \to \infty$, we require a more refined understanding of the resolvent to secure exponential decay on the high-frequency contributions of the corresponding contour integrals.[6] The idea from [8] is to expand the resolvent $(\lambda - \mathcal{L}_0)^{-1}$ as a Neumann series in $|\operatorname{Im}(\lambda)|^{-\frac{1}{2}}$ for $|\operatorname{Im}(\lambda)| \gg 1$. It turns out that it suffices to explicitly identify the first three terms in this expansion, since a remainder of order $\mathcal{O}(|\operatorname{Im}(\lambda)|^{-\frac{3}{2}})$ is integrable. These three leading-order terms can be expressed as products of the resolvents of the simpler operators $\mathcal{L}_1 \colon C_{\mathrm{ub}}(\mathbb{R}) \subset C^2_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ and $\mathcal{L}_2 \colon C_{\mathrm{ub}}(\mathbb{R}) \subset C^1_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ given by

$$\mathcal{L}_1 = \partial_{\zeta\zeta}, \qquad \mathcal{L}_2 = c_0\partial_\zeta - \varepsilon\gamma.$$

Before stating the outcome of the expansion procedure in [8], we provide the following standard result showing that $\mathcal{L}_1$ and $\mathcal{L}_2$ generate C_0-semigroups and providing bounds on their resolvents.

Lemma 3.2.5. *The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed, densely defined and generate C_0-semigroups on $C_{\mathrm{ub}}(\mathbb{R})$. Morover, there exists a constant $M > 0$ such that for each $t \geq 0$, $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ and $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg(\lambda)| \leq \frac{3\pi}{4}$ we have $\lambda \in \rho(\mathcal{L}_1)$ and*

$$\|(\lambda - \mathcal{L}_1)^{-1}\mathbf{g}\|_{L^\infty} \leq \frac{M}{|\lambda|}\|\mathbf{g}\|_{L^\infty}, \qquad \|\mathrm{e}^{\mathcal{L}_1 t}\mathbf{g}\|_{L^\infty} \leq M\|\mathbf{g}\|_{L^\infty}.$$

Finally, for each $t \geq 0$, $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > -\varepsilon\gamma$ it holds $\lambda \in \rho(\mathcal{L}_2)$ and

$$\|(\lambda - \mathcal{L}_2)^{-1}\mathbf{g}\|_{L^\infty} \leq \frac{\|\mathbf{g}\|_{L^\infty}}{\operatorname{Re}(\lambda) + \varepsilon\gamma}, \qquad \|\mathrm{e}^{\mathcal{L}_2 t}\mathbf{g}\|_{L^\infty} \leq \mathrm{e}^{-\varepsilon\gamma t}\|\mathbf{g}\|_{L^\infty}.$$

Proof. The operator ∂_ζ generates the strongly continuous translational group on $C_{\mathrm{ub}}(\mathbb{R})$ by [32, Proposition II.2.10.1]. Since translation preserves the L^∞-norm, $\mathrm{e}^{\partial_\zeta t}$ is a group of isometries. Therefore, each $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$ lies in $\rho(\partial_\zeta)$ and it holds $\|(\lambda - \partial_\zeta)^{-1}\mathbf{g}\|_{L^\infty} \leq \operatorname{Re}(\lambda)^{-1}\|\mathbf{g}\|_{L^\infty}$ for $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ by [32, Corollary 3.7]. The bounds on $(\lambda - \mathcal{L}_2)^{-1}$ and $\mathrm{e}^{\mathcal{L}_2 t}$ now readily follow by rescaling space. Moreover, $\mathcal{L}_1$ generates a bounded analytic semigroup $\mathrm{e}^{\mathcal{L}_1 t}$ by [32, Corollary II.4.9] being the square of the operator ∂_ζ. The resolvent estimate on $(\lambda - \mathcal{L}_1)^{-1}$ is stated in the proof of [32, Corollary II.4.9]. □

[6] Indeed, the naive bound $\|(\lambda - \mathcal{L}_0)^{-1}\| \lesssim \frac{1}{\operatorname{Re}\lambda}$, given by the Hille-Yosida theorem, is not strong enough.

Now, we state the high-frequency expansion of the resolvent $(\lambda - \mathcal{L}_0)^{-1}$ obtained in [8].

Proposition 3.2.6. *Assume* (H1), (H2) *and* (D1)-(D3). *Let* $b_0 > 0$. *Then, there exist constants* $C, \varpi_0 > 0$ *such that we have* $b + \mathrm{i}\varpi \in \rho(\mathcal{L}_0)$ *with*

$$(b + \mathrm{i}\varpi - \mathcal{L}_0)^{-1}\mathbf{g} = I^1_{b,\varpi}\mathbf{g} + I^2_{b,\varpi}\mathbf{g} + I^3_{b,\varpi}\mathbf{g} + I^4_{b,\varpi}\mathbf{g},$$

for all $\mathbf{g} = (g_1, g_2)^\top \in C_{\mathrm{ub}}(\mathbb{R})$ *and* $b, \varpi \in \mathbb{R}$ *with* $-\frac{3}{4}\varepsilon\gamma \leq b \leq b_0$ *and* $|\varpi| \geq \varpi_0$, *where we denote*

$$I^1_{b,\varpi}\mathbf{g} = \begin{pmatrix} (\mathrm{i}\varpi - \mathcal{L}_1)^{-1}g_1 \\ (b + \mathrm{i}\varpi - \mathcal{L}_2)^{-1}g_2 \end{pmatrix}, \qquad I^2_{b,\varpi}\mathbf{g} = \begin{pmatrix} (\mathrm{i}\varpi - \mathcal{L}_1)^{-1}(b + \mathrm{i}\varpi - \mathcal{L}_2)^{-1}g_2 \\ -\varepsilon(b + \mathrm{i}\varpi - \mathcal{L}_2)^{-1}(\mathrm{i}\varpi - \mathcal{L}_1)^{-1}g_1 \end{pmatrix}$$

and

$$I^3_{b,\varpi}\mathbf{g} = \begin{pmatrix} 0 \\ -\varepsilon(b + \mathrm{i}\varpi - \mathcal{L}_2)^{-1}(\mathrm{i}\varpi - \mathcal{L}_1)^{-1}(b + \mathrm{i}\varpi - \mathcal{L}_2)^{-1}g_2 \end{pmatrix},$$

and the residual operator $I^4_{b,\varpi} \colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ *obeys the estimate*

$$\left\| I^4_{b,\varpi}\mathbf{g} \right\|_{L^\infty} \leq C|\varpi|^{-\frac{3}{2}} \|\mathbf{g}\|_{L^\infty}.$$

Proof. This result was proved in [8, Lemma B.4] for $\mathbf{g} \in C^\infty(\mathbb{R})$, which immediately yields the statement by density of $C^\infty(\mathbb{R})$ in $C_{\mathrm{ub}}(\mathbb{R})$. □

3.3. Semigroup decomposition and linear estimates

In this section, we decompose the C_0-semigroup generated by the linearization $\mathcal{L}_0$ of (3.3) about the wave train ϕ_0 and establish corresponding estimates. To this end, we employ the complex inversion formula (3.20) of the C_0-semigroup. We first deform and partition the integration contour in (3.20). The high-frequency contribution of the deformed integration contour lies fully in the open left-half plane. Thus, exponential decay of the associated part of the C_0-semigroup can be obtained with the aid of the high-frequency resolvent expansion established in Proposition 3.2.6.

For low frequencies, we employ the resolvent decomposition obtained in Proposition 3.2.3 leading to a critical and residual low-frequency contribution of the contour integral. On the one hand, we shift the contour fully into the open left-half plane to render exponential decay of the residual low-frequency contribution. On the other hand, we relate the critical low-frequency contribution to its Floquet-Bloch representation by shifting the integration contour onto the critical spectral curve. This allows us to gather the relevant estimates on this critical part of the semigroup from [84].

3.3.1. Inverse Laplace representation

We start by showing that $\mathcal{L}_0$ generates a C_0-semigroup on $C_{\mathrm{ub}}(\mathbb{R})$ and represent its action by the complex inversion formula.

Proposition 3.3.1. *Assume* (H1). *Let $k \in \mathbb{N}_0$. The operator $\mathcal{L}_0 \colon D(\mathcal{L}_0) \subset C^k_{\mathrm{ub}}(\mathbb{R}, \mathbb{C}^2) \to C^k_{\mathrm{ub}}(\mathbb{R}, \mathbb{C}^2)$ with domain $D(\mathcal{L}_0) = C^{k+2}_{\mathrm{ub}}(\mathbb{R}, \mathbb{C}) \times C^{k+1}_{\mathrm{ub}}(\mathbb{R}, \mathbb{C})$ generates a strongly continuous semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ on $C^k_{\mathrm{ub}}(\mathbb{R}, \mathbb{C}^2)$. Moreover, there exists $\eta > 0$ such that the integration contour Γ^R_0, which is depicted in Figure 3.1 and connects $\eta - \mathrm{i}R$ to $\eta + \mathrm{i}R$, lies in the resolvent set $\rho(\mathcal{L}_0)$ and the inverse Laplace representation*

$$\mathrm{e}^{\mathcal{L}_0 t}\mathbf{g} = \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma^R_0} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}\, \mathrm{d}\lambda \tag{3.30}$$

holds for any $\mathbf{g} \in D(\mathcal{L}_0)$ and $t > 0$, where the limit in (3.30) *is taken with respect to the C^k_{ub}-norm.*

Proof. The operator $\mathcal{L}_0$ is a bounded perturbation of the diagonal diffusion-advection operator $L_0 = D\partial_{\zeta\zeta} + c_0\partial_\zeta$ on $C^k_{\mathrm{ub}}(\mathbb{R}, \mathbb{C}^2)$ with dense domain $D(L_0) = C^{k+2}_{\mathrm{ub}}(\mathbb{R}, \mathbb{C}) \times C^{k+1}_{\mathrm{ub}}(\mathbb{R}, \mathbb{C})$. The first component of L_0 is sectorial by [73, Corollary 3.1.9] and thus generates an analytic semigroup, which is strongly continuous by [73, p. 34]. On the other hand, the second component of L_0 generates the strongly continuous translational semigroup on $C^k_{\mathrm{ub}}(\mathbb{R})$ by [32, Proposition II.2.10.1]. Since $\mathcal{L}_0$ is a bounded perturbation of L_0, $\mathcal{L}_0$ also generates a C_0-semigroup by [32, Theorem III.1.3]. The inverse Laplace representation, given by the complex inversion formula (3.30), follows from [6, Proposition 3.12.1]. □

We note that standard semigroup theory provides sufficient control on the short-time behavior of the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$. To distinguish between short- and long-time behavior, we introduce a smooth temporal cut-off function $\chi \colon [0, \infty) \to \mathbb{R}$ satisfying $\chi(t) = 0$ for $t \in [0, 1]$ and $\chi(t) = 1$ for $t \in [2, \infty)$ and obtain the following short-time bound.

Lemma 3.3.2. *Assume* (H1). *Consider $\mathcal{L}_0$ as an operator on $C_{\mathrm{ub}}(\mathbb{R})$. There exist constants $C, \alpha > 0$ such that*

$$\|(1 - \chi(t))\mathrm{e}^{\mathcal{L}_0 t}\mathbf{g}\| \leq C\mathrm{e}^{-\alpha t}$$

holds for $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ and $t \geq 0$.

Proof. This follows immediately from [32, Proposition I.5.5], Proposition 3.3.1 and the fact that $1 - \chi$ vanishes on $[2, \infty)$. □

Next, we deform the integration contour Γ^R_0 in (3.30) using Cauchy's integral theorem and analyticity of the resolvent $\lambda \mapsto (\lambda - \mathcal{L}_0)^{-1}$ on $\rho(\mathcal{L}_0)$.

Proposition 3.3.3. *Assume* (H1) *and* (D1)-(D2). *Consider $\mathcal{L}_0$ as an operator on $C_{\mathrm{ub}}(\mathbb{R})$ and let $\eta > 0$ be as in Proposition 3.3.1. For each $\varpi_0 > 0$ sufficiently large the integration contours Γ^R_1 and Γ^R_3, which are depicted in Figure 3.1 and connect $\mathrm{i}\varpi_0 - \frac{3}{4}\varepsilon\gamma$ to $\mathrm{i}R - \frac{3}{4}\varepsilon\gamma$ and $-\mathrm{i}R - \frac{3}{4}\varepsilon\gamma$ to $-\mathrm{i}\varpi_0 - \frac{3}{4}\varepsilon\gamma$, respectively, as well as the rectangular integration contour Γ_2, which connects $-\mathrm{i}\varpi_0 - \frac{3}{4}\varepsilon\gamma$ via $-\mathrm{i}\varpi_0 + \frac{\eta}{2}$ and $\mathrm{i}\varpi_0 + \frac{\eta}{2}$ to $\mathrm{i}\varpi_0 - \frac{3}{4}\varepsilon\gamma$, lie in the resolvent set $\rho(\mathcal{L}_0)$. Moreover, we have*

$$\begin{aligned}\mathrm{e}^{\mathcal{L}_0 t}\mathbf{g} = {} & \frac{\chi(t)}{2\pi\mathrm{i}} \int_{\Gamma_2} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}\, \mathrm{d}\lambda + \lim_{R\to\infty} \frac{\chi(t)}{2\pi\mathrm{i}} \int_{\Gamma^R_1 \cup \Gamma^R_3} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}\, \mathrm{d}\lambda \\ & + (1 - \chi(t))\mathrm{e}^{\mathcal{L}_0 t}\mathbf{g}\end{aligned} \tag{3.31}$$

for $\mathbf{g} \in D(\mathcal{L}_0)$ and $t \geq 0$.

Proof. Let $\mathbf{g} \in D(\mathcal{L}_0)$ and $t > 0$. Let $R > \varpi_0$. Let Γ_0^R be as in Proposition 3.3.1. Let Γ_4^R and Γ_5^R be the integration contours depicted in Figure 3.1 connecting $-\mathrm{i}R + \eta$ to $-\mathrm{i}R - \frac{3}{4}\varepsilon\gamma$ and $\mathrm{i}R - \frac{3}{4}\varepsilon\gamma$ to $\mathrm{i}R + \eta$, respectively. Let Γ^R be the closed contour consisting of $-\Gamma_0^R$, Γ_1^R, Γ_2, Γ_3^R, Γ_4^R and Γ_5^R, so that Γ^R is oriented clockwise, cf. Figure 3.1. By Assumption (D1) and Proposition 3.2.6 Γ^R, as well as its interior, lies in $\rho(\mathcal{L}_0)$, provided $\varpi_0 > 0$ is large enough. Moreover, the map $\rho(\mathcal{L}_0) \to C_{\mathrm{ub}}(\mathbb{R})$ given by $\lambda \mapsto \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}$ is analytic. Hence, Cauchy's integral theorem yields

$$0 = \int_{\Gamma^R} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}\,\mathrm{d}\lambda. \tag{3.32}$$

We express the contribution of the complex line integral over $\Gamma_4^R \cup \Gamma_5^R$ as

$$\int_{\Gamma_4^R\cup\Gamma_5^R} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}\,\mathrm{d}\lambda = \int_{\Gamma_4^R\cup\Gamma_5^R} \frac{\mathrm{e}^{\lambda t}}{\lambda}\Big((\lambda - \mathcal{L}_0)^{-1}\mathcal{L}_0\mathbf{g} + \mathbf{g}\Big)\,\mathrm{d}\lambda. \tag{3.33}$$

Lemma 3.2.5 and Proposition 3.2.6 yield an R-independent constant $C > 0$ such that we have the bound $\|(\lambda - \mathcal{L}_0)^{-1}\|_{\mathcal{B}(C_{\mathrm{ub}}(\mathbb{R}))} \leq C$ for $\lambda \in \Gamma_4^R \cup \Gamma_5^R$. Since the length of $\Gamma_4^R \cup \Gamma_5^R$ can be bounded by an R-independent constant $M > 0$, we find that (3.33) implies

$$\left\|\lim_{R\to\infty} \int_{\Gamma_4^R\cup\Gamma_5^R} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}\,\mathrm{d}\lambda\right\|_{L^\infty} \leq \lim_{R\to\infty} \mathrm{e}^{\eta t} M \frac{C\|\mathcal{L}_0\mathbf{g}\|_{L^\infty} + \|\mathbf{g}\|_{L^\infty}}{R} = 0.$$

Combining the latter with Proposition 3.3.1 and identity (3.32), we arrive at (3.31), which concludes the proof. □

3.3.2. Estimates on the high-frequency part

We utilize the resolvent expansion obtained in Proposition 3.2.6 to establish exponential decay of the high-frequency part of the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$, which corresponds to the complex line integrals over the contours Γ_1^R and Γ_3^R in the inverse Laplace representation (3.31) of the semigroup.

Proposition 3.3.4. *Assume* (H1) *and* (D1)-(D2). *Consider $\mathcal{L}_0$ as an operator on $C_{\mathrm{ub}}(\mathbb{R})$. For each $\varpi_0 > 0$ sufficiently large there exist constants $C, \alpha > 0$ such that the operator $S_e^1(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ given by*

$$S_e^1(t)\mathbf{g} = \chi(t) \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_1^R\cup\Gamma_3^R} \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}\,\mathrm{d}\lambda$$

for $\mathbf{g} \in D(\mathcal{L}_0)$ and $t \geq 0$ obeys the estimate

$$\|S_e^1(t)\mathbf{g}\|_{L^\infty} \leq C\mathrm{e}^{-\alpha t}\|\mathbf{g}\|_{L^\infty} \tag{3.34}$$

for $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ and $t \geq 0$.

Proof. Let $\mathbf{g} = (g_1, g_2)^\top \in D(\mathcal{L}_0)$ and $t \geq 0$. We abbreviate $b_1 = -\frac{3}{4}\varepsilon\gamma$. Employing the high-frequency resolvent expansion from Proposition 3.2.6, we arrive, provided $\varpi_0 > 0$ is sufficiently large, at the decomposition

$$\begin{aligned} S_e^1(t)\mathbf{g} &= \chi(t) \lim_{R\to\infty} \frac{1}{2\pi} \left(\int_{-R}^{-\varpi_0} + \int_{\varpi_0}^{R}\right) \mathrm{e}^{\mathrm{i}\varpi t + b_1 t}(b_1 + \mathrm{i}\varpi - \mathcal{L}_0)^{-1}\mathbf{g}\,\mathrm{d}\varpi \\ &= \mathrm{e}^{b_1 t}(S_1(t)\mathbf{g} + S_2(t)\mathbf{g} + S_3(t)\mathbf{g} + S_4(t)\mathbf{g}), \end{aligned} \tag{3.35}$$

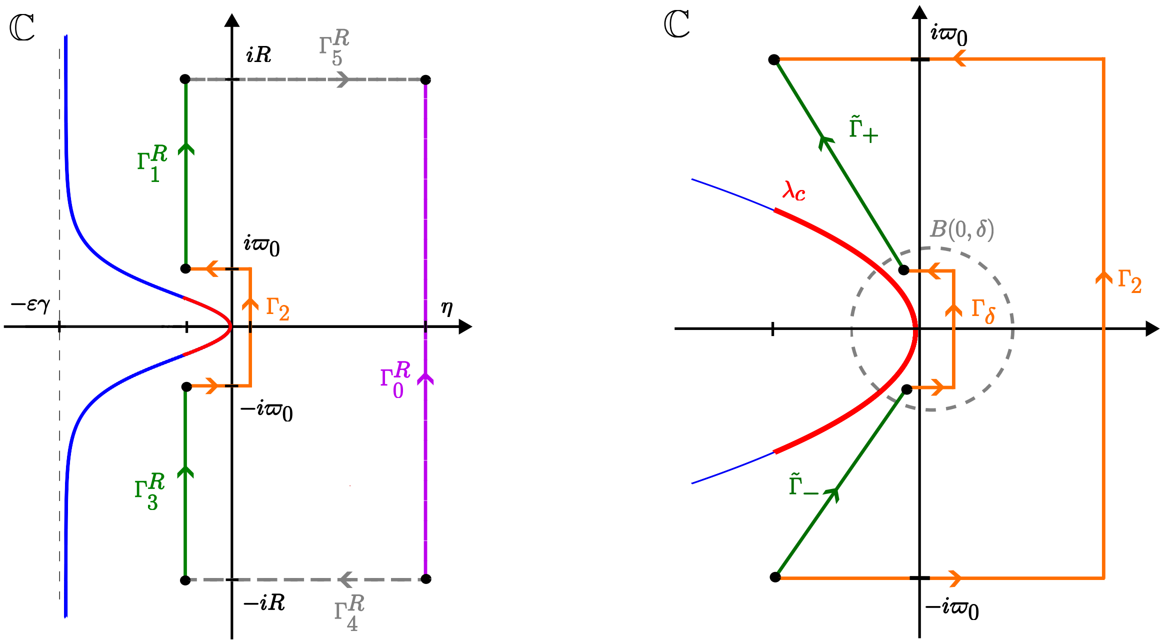

Figure 3.1.: The spectrum of the linearization $\mathcal{L}_0$ of system (3.3) about the wave train ϕ_0 (depicted in blue and red) touches the origin in a quadratic tangency. It asymptotes to the line $\mathrm{Re}(\lambda) = -\varepsilon\gamma$. The red part of the spectrum is the critical curve $\{\lambda_c(\xi) : \xi \in \mathbb{R} \cap V_1\}$ established in Proposition 3.1.3. Left panel: the original contour Γ_0^R used in the inverse Laplace representation (3.30) of the C_0-semigroup $e^{\mathcal{L}_0 t}$, together with the deformed contour $\Gamma_4^R \cup \Gamma_3^R \cup \Gamma_2 \cup \Gamma_1^R \cup \Gamma_5^R$. The contributions of the inverse Laplace integral over Γ_4^R and Γ_5^R vanish as $R \to \infty$, cf. Proposition 3.3.3. Right panel: a zoom-in on the contour Γ_2, as well as its deformation $\widetilde{\Gamma}_- \cup \Gamma_\delta \cup \widetilde{\Gamma}_+$ used in the proof of Proposition 3.3.6. The rectangular contour Γ_δ lies in the ball $B(0,\delta)$, is reflection symmetric in the real axis and connects points $-i\eta_2 - \eta_1$ to $i\eta_2 - \eta_1$ with $\eta_{1,2} > 0$.

where we denote

$$S_j(t)\mathbf{g} = \chi(t) \lim_{R\to\infty} \frac{1}{2\pi} \left(\int_{-R}^{-\varpi_0} + \int_{\varpi_0}^{R} \right) e^{i\varpi t} I^j_{b_1,\varpi}\mathbf{g}\, d\varpi, \qquad j = 1, \ldots, 4.$$

The estimate on $I^4_{b_1,\varpi}\mathbf{g}$ in Proposition 3.2.6 readily provides $\mathbf{g}$- and t-independent constants $C_{1,2} > 0$ such that

$$\|S_4(t)\mathbf{g}\|_{L^\infty} \leq C_1 \int_{\varpi_0}^{\infty} \varpi^{-\frac{3}{2}} \|\mathbf{g}\|_{L^\infty}\, d\varpi \leq C_2 \|\mathbf{g}\|_{L^\infty}. \tag{3.36}$$

We relate the leading-order contributions $S_1(t), S_2(t), S_3(t)$ to (convolutions of) the C_0-semigroups $T_1(t) := e^{\mathcal{L}_1 t}$ and $T_2(t) := e^{(\mathcal{L}_2 - b_1)t}$ using [6, Proposition 3.12.1] and Corollary 3.A.2. To this end, we define an R-independent contour $\check{\Gamma}_2$, which connects $-i\varpi_0$ to $i\varpi_0$ and lies in $\Sigma := \{\lambda \in \mathbb{C} \setminus \{0\} : -\frac{1}{4}\varepsilon\gamma < \mathrm{Re}(\lambda) < \frac{1}{8}\varepsilon\gamma, |\arg(\lambda)| < \frac{3\pi}{4}\}$. Moreover, let $\check{\Gamma}_4^R$ and $\check{\Gamma}_5^R$ be the lines connecting $-iR + \frac{1}{4}\varepsilon\gamma$ with $-iR$ and connecting iR with $iR + \frac{1}{4}\varepsilon\gamma$, respectively. Using that the maps $\Sigma \to C_{\mathrm{ub}}(\mathbb{R})$ given by $\lambda \mapsto (\lambda - \mathcal{L}_1)^{-1}\mathbf{g}$ and $\lambda \mapsto (\lambda + b_1 - \mathcal{L}_2)^{-1}$ are holomorphic by Lemma 3.2.5, Cauchy's integral theorem yields

$$\begin{aligned}\frac{\chi(t)}{2\pi}\left(\int_{-R}^{-\varpi_0}+\int_{\varpi_0}^{R}\right)\mathrm{e}^{\mathrm{i}\varpi t}I_{b_1,\varpi}^{j}\mathbf{g}\,\mathrm{d}\varpi &= \frac{\chi(t)}{2\pi\mathrm{i}}\int_{\frac14\varepsilon\gamma-\mathrm{i}R}^{\frac14\varepsilon\gamma+\mathrm{i}R}\mathrm{e}^{\lambda t}I_{b_1,\lambda}^{j}\mathbf{g}\,\mathrm{d}\lambda\\ &\quad-\frac{\chi(t)}{2\pi\mathrm{i}}\int_{\check\Gamma_2\cup\check\Gamma_4^R\cup\check\Gamma_5^R}\mathrm{e}^{\lambda t}I_{b_1,\lambda}^{j}\mathbf{g}\,\mathrm{d}\lambda.\end{aligned} \tag{3.37}$$

We note that the length of the contours $\check\Gamma_2,\check\Gamma_4^R,\check\Gamma_5^R\subset\Sigma\subset\rho(\mathcal{L}_1)\cap\rho(\mathcal{L}_2-b_1)$ can be bounded by an R-independent constant. So, using the resolvent estimates from Lemma 3.2.5, we establish a t-, R- and $\mathbf{g}$-independent constant $C_3>0$ such that

$$\left\|\chi(t)\int_{\check\Gamma_2\cup\check\Gamma_4^R\cup\check\Gamma_5^R}\mathrm{e}^{\lambda t}I_{b_1,\lambda}^{j}\mathbf{g}\,\mathrm{d}\lambda\right\|_{L^\infty}\leq C_3\mathrm{e}^{\frac14\varepsilon\gamma t}\|\mathbf{g}\|_{L^\infty} \tag{3.38}$$

for $j=1,2,3$.

Lemma 3.2.5 implies that $g_1\in D(\mathcal{L}_1)$, $g_2\in D(\mathcal{L}_2-b_1)$ and the semigroups $T_1(t)$ and $T_2(t)$ are strongly continuous and exponentially bounded with growth bounds $\varpi_0(T_1)\leq 0$ and $\varpi_0(T_2)\leq-\frac14\varepsilon\gamma$. Hence, an application of [6, Proposition 3.12.1] and Corollary 3.A.2 yields

$$\begin{aligned}\frac{\chi(t)}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\frac14\varepsilon\gamma-\mathrm{i}R}^{\frac14\varepsilon\gamma+\mathrm{i}R}\mathrm{e}^{\lambda t}I_{b_1,\lambda}^{1}\mathbf{g}\,\mathrm{d}\lambda&=\chi(t)\begin{pmatrix}T_1(t)g_1\\ T_2(t)g_2\end{pmatrix},\\ \frac{\chi(t)}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\frac14\varepsilon\gamma-\mathrm{i}R}^{\frac14\varepsilon\gamma+\mathrm{i}R}\mathrm{e}^{\lambda t}I_{b_1,\lambda}^{2}\mathbf{g}\,\mathrm{d}\lambda&=\chi(t)\begin{pmatrix}(T_1*T_2)(t)g_2\\ -\varepsilon(T_2*T_1)(t)g_1\end{pmatrix},\end{aligned}$$

and

$$\frac{\chi(t)}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\frac14\varepsilon\gamma-\mathrm{i}R}^{\frac14\varepsilon\gamma+\mathrm{i}R}\mathrm{e}^{\lambda t}I_{b_1,\lambda}^{3}\mathbf{g}\,\mathrm{d}\lambda=\chi(t)\begin{pmatrix}0\\ -\varepsilon(T_2*T_1*T_2)(t)g_2\end{pmatrix}.$$

By [32, Theorem C.17], the convolutions T_1*T_2, T_2*T_1 and $T_2*T_1*T_2$ are strongly continuous and exponentially bounded with growth bounds being at most $\max\{\varpi_0(T_1),\varpi_0(T_2)\}\leq 0$. Therefore, we find a t- and $\mathbf{g}$-independent constant $C_4>0$ such that

$$\left\|\frac{\chi(t)}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\frac14\varepsilon\gamma-\mathrm{i}R}^{\frac14\varepsilon\gamma+\mathrm{i}R}\mathrm{e}^{\lambda t}I_{b_1,\lambda}^{j}\mathbf{g}\,\mathrm{d}\lambda\right\|\leq C_4\|\mathbf{g}\|_{L^\infty}$$

for $j=1,2,3$. Combining the latter with the decompositions (3.35) and (3.37) and the estimates (3.36) and (3.38), we arrive at (3.34) with $\alpha=\frac12\varepsilon\gamma>0$ by density of $D(\mathcal{L}_0)$ in $C_{\mathrm{ub}}(\mathbb{R})$. □

Remark 3.3.5. *Comparing the proof of Proposition 3.3.4 with the high-frequency analysis of the semigroup in [8, Appendix B.2], we find that the identification of the critical high-frequency part of the semigroup as convolutions of the the heat and translation semigroups simplifies the analysis significantly. In particular, it is no longer necessary to compute the inverse Laplace transform of the leading-order terms of the Neumann-series expansion of the resolvent explicitly for a test function* $\mathbf{g}$.

3.3.3. Isolating the critical low-frequency part

We wish to employ the decomposition of the resolvent $(\lambda - \mathcal{L}_0)^{-1}$ for $|\lambda|$ sufficiently small established in Proposition 3.2.3 to isolate the critical low-frequency part of the semigroup. To this end, we deform the contour Γ_2 in the inverse Laplace representation (3.31) of the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$, so that its part in the right-half plane is contained in the ball $B(0,\delta)$, where Proposition 3.2.3 applies, cf. Figure 3.1. The remainder of the deformed contour lies in the open left-half plane, away from the spectrum of $\mathcal{L}_0$ and, thus, the associated complex line integrals are exponentially decaying.

Proposition 3.3.6. *Assume* (H1) *and* (D1)-(D3). *Consider $\mathcal{L}_0$ as an operator on $C_{\mathrm{ub}}(\mathbb{R})$. Let $\varpi_0 > 0$. For each $\delta > 0$ sufficiently small there exist constants $C, \alpha > 0$, a linear operator $S_e^2(t) \colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ and a rectangular contour Γ_δ, which is reflection symmetric in the real axis, lies in $B(0,\delta)$ strictly to the right of $\sigma(\mathcal{L}_0)$ and connects points $-\mathrm{i}\eta_2 - \eta_1$ and $\mathrm{i}\eta_2 - \eta_1$ with $\eta_{1,2} > 0$, such that we have the decomposition*

$$\frac{\chi(t)}{2\pi\mathrm{i}} \int_{\Gamma_2} \mathrm{e}^{\lambda t} (\lambda - \mathcal{L}_0)^{-1} \mathbf{g} \, \mathrm{d}\lambda = \frac{\chi(t)}{2\pi\mathrm{i}} \int_{\Gamma_\delta} \mathrm{e}^{\lambda t} (\lambda - \mathcal{L}_0)^{-1} \mathbf{g} \, \mathrm{d}\lambda + S_e^2(t)\mathbf{g}, \tag{3.39}$$

for each $\mathbf{g} \in D(\mathcal{L}_0)$ and $t \geq 0$ and the estimate

$$\|S_e^2(t)\mathbf{g}\|_{L^\infty} \leq C\mathrm{e}^{-\alpha t} \|\mathbf{g}\|_{L^\infty} \tag{3.40}$$

holds for $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ and $t \geq 0$.

Proof. Let $\mathbf{g} \in D(\mathcal{L}_0)$ and $t \geq 0$. By Proposition 3.1.3, there exist constants $a \in \mathbb{R}$ and $b, \delta_0 > 0$ such that the spectrum of $\mathcal{L}_0$ in the ball $B(0,\delta_0)$ lies on or to the left of the parabola $\{\mathrm{i}a\xi - b\xi^2 : \xi \in \mathbb{R}\}$. Take $\delta \in (0,\delta_0)$. By Assumption (D1), there exists a constant $\varrho > 0$ such that the spectrum of $\mathcal{L}_0$ in the compact set $K_0 = \{\lambda \in \mathbb{C} : |\operatorname{Im}(\lambda)| \leq 2\varpi_0, |\operatorname{Re}(\lambda)| \leq \gamma\} \setminus B(0,\delta)$ lies to the left of the line $\operatorname{Re}(\lambda) = -\varrho$. Furthermore, the contour Γ_2 lies in the resolvent set of $\mathcal{L}_0$ by Proposition 3.3.3. We conclude that there exist points $-\eta_1 \pm \mathrm{i}\eta_2$ with $\eta_{1,2} > 0$ lying in $B(0,\delta)$ strictly to the right of $\sigma(\mathcal{L}_0)$, as well as contours $\widetilde{\Gamma}_-$, connecting the lower end point $-\mathrm{i}\varpi_0 - \frac{3}{4}\varepsilon\gamma$ of Γ_2 to the point $-\mathrm{i}\eta_2 - \eta_1$, and $\widetilde{\Gamma}_+$, connecting $\mathrm{i}\eta_2 - \eta_1$ to the upper end point $\mathrm{i}\varpi_0 - \frac{3}{4}\varepsilon\gamma$ of Γ_2, such that $\widetilde{\Gamma}_-$ and $\widetilde{\Gamma}_+$ are both contained in the resolvent set $\rho(\mathcal{L}_0)$ and in the open left-half plane. Hence, there exists a rectangular contour Γ_δ, which connects $-\mathrm{i}\eta_2 - \eta_1$ to $\mathrm{i}\eta_2 - \eta_1$, is reflection symmetric in the real axis and lies in $B(0,\delta)$, strictly to the right of $\sigma(\mathcal{L}_0)$. Since the map $\rho(\mathcal{L}_0) \to C_{\mathrm{ub}}(\mathbb{R})$ given by $\lambda \mapsto \mathrm{e}^{\lambda t}(\lambda - \mathcal{L}_0)^{-1}\mathbf{g}$ is analytic, Cauchy's integral theorem yields (3.39) with

$$S_e^2(t)\mathbf{g} = \frac{\chi(t)}{2\pi\mathrm{i}} \left(\int_{\widetilde{\Gamma}_-} + \int_{\widetilde{\Gamma}_+} \right) \mathrm{e}^{\lambda t} (\lambda - \mathcal{L}_0)^{-1} \mathbf{g} \, \mathrm{d}\lambda.$$

The analytic map $\rho(\mathcal{L}_0) \to C_{\mathrm{ub}}(\mathbb{R}), \lambda \mapsto (\lambda - \mathcal{L}_0)^{-1}$ is bounded on the compact sets $\widetilde{\Gamma}_\pm \subset \rho(\mathcal{L}_0)$, which lie in the open left-half plane. Thus, the estimate (3.40) follows by density of $D(\mathcal{L}_0)$ in $C_{\mathrm{ub}}(\mathbb{R})$. □

We can now identify the critical part of the remaining complex line integral in (3.39) by employing the low-frequency decomposition of the resolvent obtained in Proposition 3.2.3 and using the identity (3.27) derived in Lemma 3.2.4.

Proposition 3.3.7. *Assume* (H1), (H2) *and* (D1)-(D3). *Consider* $\mathcal{L}_0$ *as an operator on* $C_{\mathrm{ub}}(\mathbb{R})$. *For each* $\delta > 0$ *sufficiently small there exist constants* $C, \alpha > 0$ *and a linear operator* $S_e^3(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ *such that for each* $\mathbf{g} \in D(\mathcal{L}_0)$, $\zeta \in \mathbb{R}$ *and* $t \geq 0$ *we have the decomposition*

$$\begin{aligned}\frac{\chi(t)}{2\pi\mathrm{i}} \int_{\Gamma_\delta} \mathrm{e}^{\lambda t} (\lambda - \mathcal{L}_0)^{-1} \mathbf{g} \,\mathrm{d}\lambda[\zeta] = \frac{\chi(t)}{2\pi\mathrm{i}} \int_\zeta^\infty \int_{\Gamma_\delta} \mathrm{e}^{\lambda t + \nu_c(\lambda)(\zeta - \bar\zeta)} \Psi(\zeta,\lambda) \tilde\Psi(\bar\zeta,\lambda)^* \,\mathrm{d}\lambda \,\mathbf{g}(\bar\zeta) \,\mathrm{d}\bar\zeta \\ + \Big(S_e^3(t)\mathbf{g}\Big)[\zeta].\end{aligned} \tag{3.41}$$

Moreover, the estimate

$$\|S_e^3(t)\mathbf{g}\|_{L^\infty} \leq C\mathrm{e}^{-\alpha t} \|\mathbf{g}\|_{L^\infty} \tag{3.42}$$

holds for $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ *and* $t \geq 0$.

Proof. Provided $\delta > 0$ is sufficiently small, identity (3.41) follows readily from Fubini's theorem, Proposition 3.2.3 and Lemma 3.2.4 by setting

$$S_e^3(t)\mathbf{g} = \frac{\chi(t)}{2\pi\mathrm{i}} \int_{\Gamma_\delta} \mathrm{e}^{\lambda t} S_e^0(\lambda)\mathbf{g} \,\mathrm{d}\lambda$$

for $t \geq 0$ and $\mathbf{g} \in D(\mathcal{L}_0)$, where $S_e^0\colon B(0,\delta) \to \mathcal{B}(C_{\mathrm{ub}}(\mathbb{R}))$ is the analytic map from Proposition 3.2.3, obeying the estimate

$$\|S_e^0(\lambda)\mathbf{g}\|_{L^\infty} \leq C_0 \|\mathbf{g}\|_{L^\infty} \tag{3.43}$$

for some $\mathbf{g}$- and λ-independent constant $C_0 > 0$. Now let $\widetilde\Gamma_\delta$ be the straight line connecting the end points $\pm\mathrm{i}\eta_2 - \eta_1$ of Γ_δ. Then, $\widetilde\Gamma_\delta$ lies both in $B(0,\delta)$ and in the open left-half plane. By Cauchy's integral theorem and analyticity of S_e^0, we infer

$$S_e^3(t)\mathbf{g} = \frac{\chi(t)}{2\pi\mathrm{i}} \int_{\widetilde\Gamma_\delta} \mathrm{e}^{\lambda t} S_e^0(\lambda)\mathbf{g} \,\mathrm{d}\lambda$$

for $\mathbf{g} \in D(\mathcal{L}_0)$ and $t \geq 0$. Taking norms in the latter, using that the compact contour $\widetilde\Gamma_\delta$ lies in the open left-half plane and applying the bound (3.43) readily yields the estimate (3.42) by density of $D(\mathcal{L}_0)$ in $C_{\mathrm{ub}}(\mathbb{R})$. □

3.3.4. Floquet-Bloch representation of the critical low-frequency part

Except for the integral appearing on the right-hand side of (3.41) representing its critical low-frequency part, the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ is exponentially decaying by Propositions 3.3.3, 3.3.4, 3.3.6 and 3.3.7 and Lemma 3.3.2. The following result recovers, up to some exponentially decaying terms, the same Floquet-Bloch representation for the critical low-frequency part of the semigroup as in [84].

The main idea is to exploit that the integral

$$\int_{\Gamma_\delta} \mathrm{e}^{\lambda t + \nu_c(\lambda)(\zeta - \bar\zeta)} \Psi(\zeta,\lambda) \tilde\Psi(\bar\zeta,\lambda)^* \,\mathrm{d}\lambda$$

possesses an integrand, which is analytic in λ on $B(0,\delta)$ for each $\zeta,\bar{\zeta} \in \mathbb{R}$ and $t \geq 0$, cf. Proposition 3.1.3 and Lemma 3.2.4. This *pointwise* analyticity[7] allows us to shift (part of) the integration contour Γ_δ onto the critical spectral curve $\lambda_c(\xi)$, see Figure 3.2. Via the identities $\nu_c(\lambda_c(\xi)) = \mathrm{i}\xi$ and (3.28), obtained in Proposition 3.2.2 and Lemma 3.2.4, respectively, we then arrive at the desired Floquet-Bloch representation from [84]. We show that the remainder terms are exponentially decaying by using pointwise estimates obtained through integration by parts, essentially following the same strategy as in [51, Lemma A.1].

Proposition 3.3.8. *Assume* (H1), (H2) *and* (D1)-(D3). *For each $\delta > 0$ sufficiently small there exist constants $\xi_0, C, \alpha > 0$, a linear operator $S_e^4(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ and a smooth cut-off function $\rho\colon \mathbb{R} \to \mathbb{R}$ such that for each $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$, $\zeta \in \mathbb{R}$ and $t \geq 0$ we have*

$$\begin{aligned}\frac{\chi(t)}{2\pi\mathrm{i}} \int_\zeta^\infty \int_{\Gamma_\delta} \mathrm{e}^{\lambda t + \nu_c(\lambda)(\zeta - \bar{\zeta})} \Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^* \,\mathrm{d}\lambda\, \mathbf{g}(\bar{\zeta})\,\mathrm{d}\bar{\zeta} \\ = \frac{\chi(t)}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(\xi) \mathrm{e}^{\lambda_c(\xi)t + \mathrm{i}\xi(\zeta-\bar{\zeta})} \Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^* \,\mathrm{d}\xi\, \mathbf{g}(\bar{\zeta})\,\mathrm{d}\bar{\zeta} + \Big(S_e^4(t)\mathbf{g}\Big)[\zeta].\end{aligned}$$

Moreover, ρ is supported on the interval $(-\xi_0, \xi_0) \subset V_1 \cap \mathbb{R}$ and satisfies $\rho(\xi) = 1$ for $\xi \in [-\frac{1}{2}\xi_0, \frac{1}{2}\xi_0]$. Finally, for each $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$ and $t \geq 0$ it holds

$$\|S_e^4(t)\mathbf{g}\|_{L^\infty} \leq C\mathrm{e}^{-\alpha t}\|\mathbf{g}\|_{L^\infty}.$$

Proof. First, we note that Propositions 3.1.3 and 3.2.2 imply $\nu_c'(0) = -c_g^{-1} \neq 0$ and $\lambda_c'(0) = -\mathrm{i}c_g \neq 0$. So, using Proposition 3.1.3, we can take $\delta > 0$ so small that Proposition 3.2.2 and Lemma 3.2.4 apply, it holds $\nu_c'(\lambda) \neq 0$ for all $\lambda \in \overline{B(0,\delta)}$, and each point in $\sigma(\mathcal{L}_0) \cap B(0,\delta)$ lies on the curve $\{\lambda_c(\xi) : \xi \in V_1 \cap \mathbb{R}\}$. In addition, there exists, again by Proposition 3.1.3, $\xi_0 > 0$ such that we have $[-\xi_0, \xi_0] \subset V_1 \cap \mathbb{R}$, it holds $\mathrm{sgn}(\mathrm{Im}(\lambda_c(\pm\xi_0))) = \pm 1$, each point on the curve $\lambda_c([-\xi_0,\xi_0])$ lies in the ball $B(0,\delta)$ and on the rightmost boundary $\{z \in \sigma(\mathcal{L}_0) : z + w \in \rho(\mathcal{L}_0) \text{ for all } w > 0\}$ of the spectrum of $\mathcal{L}_0$, and $\lambda_c'(\xi)$ is nonzero for each $\xi \in [-\xi_0, \xi_0]$. We let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function, which is supported on $(-\xi_0, \xi_0)$ and satisfies $\rho(\xi) = 1$ for $\xi \in [-\frac{1}{2}\xi_0, \frac{1}{2}\xi_0]$.

Our approach is to deform the contour Γ_δ into a new contour consisting of a smooth curve $\Gamma_- \subset B(0,\delta) \cap \{z \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$ which connects the lower endpoint $-\eta_1 - \mathrm{i}\eta_2$ of Γ_δ to $\lambda_c(-\xi_0)$ and satisfies $\Gamma_- \setminus \{\lambda_c(-\xi_0)\} \subset \rho(\mathcal{L}_0)$, the smooth curve $\Gamma_c \subset B(0,\delta)$ which connects $\lambda_c(-\xi_0)$ to $\lambda_c(\xi_0)$ and is parameterized by λ_c, and a smooth curve $\Gamma_+ \subset B(0,\delta) \cap \{z \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$ which connects the point $\lambda_c(\xi_0)$ to the upper endpoint $-\eta_1 + \mathrm{i}\eta_2$ of Γ_δ and satisfies $\Gamma_+ \setminus \{\lambda_c(\xi_0)\} \subset \rho(\mathcal{L}_0)$, see Figure 3.2. We note that the contours $\Gamma_\pm$ exist, because the points $-\eta_1 \pm \mathrm{i}\eta_2$ lie in the open left-half plane strictly to the right of $\sigma_0(\mathcal{L}_0) \cap B(0,\delta)$, it holds $\mathrm{sgn}(\mathrm{Im}(\lambda_c(\pm\xi_0))) = \pm 1$, and each point on the curve $\lambda_c([-\xi_0,\xi_0])$ lies in the ball $B(0,\delta)$ and on the rightmost boundary $\{z \in \sigma(\mathcal{L}_0) : z + w \in \rho(\mathcal{L}_0) \text{ for all } w > 0\}$ of the spectrum of $\mathcal{L}_0$, which lies in $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\} \cup \{0\}$ by assumption (D1).

We choose parameterizations $\lambda_\pm\colon [0,1] \to \mathbb{C}$ of the curves $\Gamma_\pm$ satisfying $\lambda_\pm'(\xi) \neq 0$ for $\xi \in [0,1]$. Since ν_c and $\Psi(\zeta,\cdot)\tilde{\Psi}(\bar{\zeta},\cdot)^*$ are analytic and it holds

$$\mathrm{i}\Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^* = \Psi(\zeta,\lambda_c(\xi))\tilde{\Psi}(\bar{\zeta},\lambda_c(\xi))^*\lambda_c'(\xi)$$

for each $\zeta,\bar{\zeta} \in \mathbb{R}$ and $\xi \in (-\xi_0,\xi_0)$ by Proposition 3.2.2 and Lemma 3.2.4, Cauchy's integral

[7] See [8, Section 5.1] for further discussion on pointwise and L^p-analyticity.

theorem implies

$$\begin{aligned}\int_{\Gamma_\delta} \mathrm{e}^{\lambda t+\nu_c(\lambda)(\zeta-\bar{\zeta})}\Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^*\,\mathrm{d}\lambda &= \left(\int_{\Gamma_-}+\int_{\Gamma_c}+\int_{\Gamma_+}\right)\mathrm{e}^{\lambda t+\nu_c(\lambda)(\zeta-\bar{\zeta})}\Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^*\,\mathrm{d}\lambda \\ &= \mathrm{i}\int_{\mathbb{R}}\rho(\xi)\mathrm{e}^{\lambda_c(\xi)t+\mathrm{i}\xi(\zeta-\bar{\zeta})}\Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^*\,\mathrm{d}\xi + I_+ + I_- + I_c\end{aligned} \tag{3.44}$$

where we denote

$$\begin{aligned}I_\pm &= \int_{\Gamma_\pm}\mathrm{e}^{\lambda t+\nu_c(\lambda)(\zeta-\bar{\zeta})}\Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^*\,\mathrm{d}\lambda,\\ I_c &= \mathrm{i}\int_{-\xi_0}^{\xi_0}(1-\rho(\xi))\mathrm{e}^{\lambda_c(\xi)t+\mathrm{i}\xi(\zeta-\bar{\zeta})}\Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^*\,\mathrm{d}\xi\end{aligned}$$

for $\zeta,\bar{\zeta}\in\mathbb{R}$ and $t\geq 0$. Using again $\mathrm{i}\Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^* = \Psi(\zeta,\lambda_c(\xi))\tilde{\Psi}(\bar{\zeta},\lambda_c(\xi))^*\lambda_c'(\xi)$, we infer

$$\mathrm{i}\int_{\mathbb{R}}\rho(\xi)\mathrm{e}^{\lambda_c(\xi)t+\mathrm{i}\xi(\zeta-\bar{\zeta})}\Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^*\,\mathrm{d}\xi = I_0 - I_c, \tag{3.45}$$

where we denote

$$I_0 = \int_{\Gamma_c}\mathrm{e}^{\lambda t+\nu(\lambda)(\zeta-\bar{\zeta})}\Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^*\,\mathrm{d}\lambda$$

for $\zeta,\bar{\zeta}\in\mathbb{R}$ and $t\geq 0$. All in all, (3.44) and (3.45) yield the decomposition

$$\begin{aligned}\mathrm{i}\int_{\mathbb{R}}\rho(\xi)\mathrm{e}^{\lambda_c(\xi)t+\mathrm{i}\xi(\zeta-\bar{\zeta})}\Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^*\,\mathrm{d}\xi &= \mathbf{1}_{[\zeta,\infty)}(\bar{\zeta})\int_{\Gamma_\delta}\mathrm{e}^{\lambda t+\nu(\lambda)(\zeta-\bar{\zeta})}\Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^*\,\mathrm{d}\lambda \\ &\quad - \mathbf{1}_{[\zeta,\infty)}(\bar{\zeta})(I_+ + I_- + I_c) + \mathbf{1}_{(-\infty,\zeta]}(\bar{\zeta})(I_0 - I_c)\end{aligned} \tag{3.46}$$

for $\zeta,\bar{\zeta}\in\mathbb{R}$ and $t\geq 0$. We will use integration by parts to establish pointwise approximations of $I_\pm$, I_0 and I_c, which yield integrability in space and exponential decay in time of $\mathbf{1}_{[\zeta,\infty)}(\bar{\zeta})(I_+ + I_- + I_c)$ and of $\mathbf{1}_{(-\infty,\zeta]}(\bar{\zeta})(I_0 - I_c)$. This then readily leads to the desired result.

Pointwise approximations of $I_\pm$ for $\zeta\leq\bar{\zeta}$. We wish to factor out the space-integrable quotient $(1+(\zeta-\bar{\zeta})^2)^{-1}$ by establishing pointwise approximations of I_+ and $(\zeta-\bar{\zeta})^2 I_+$. Recalling $\nu_c'(\lambda)\neq 0$ for all $\lambda\in B(0,\delta)$, abbreviating $\Psi_1(\zeta,\bar{\zeta},\lambda) = \Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^*/\nu_c'(\lambda)$ and using integration by parts and Proposition 3.2.2, we rewrite

$$\begin{aligned}(\zeta-\bar{\zeta})^2 I_+ &= \int_0^1(\zeta-\bar{\zeta})\mathrm{e}^{\lambda_+(\xi)t}\Psi_1(\zeta,\bar{\zeta},\lambda_+(\xi))\partial_\xi\left(\mathrm{e}^{\nu(\lambda_+(\xi))(\zeta-\bar{\zeta})}\right)\mathrm{d}\xi \\ &= \left[(\zeta-\bar{\zeta})\mathrm{e}^{\lambda_+(\xi)t+\nu(\lambda_+(\xi))(\zeta-\bar{\zeta})}\Psi_1(\zeta,\bar{\zeta},\lambda_+(\xi))\right]_{\xi=0}^1 \\ &\qquad - \int_0^1\partial_\xi\left((\zeta-\bar{\zeta})\mathrm{e}^{\lambda_+(\xi)t}\Psi_1(\zeta,\bar{\zeta},\lambda_+(\xi))\right)\mathrm{e}^{\nu(\lambda_+(\xi))(\zeta-\bar{\zeta})}\,\mathrm{d}\xi \\ &= (\zeta-\bar{\zeta})\mathrm{e}^{(-\eta_1+\mathrm{i}\eta_2)t+\nu(-\eta_1+\mathrm{i}\eta_2)(\zeta-\bar{\zeta})}\Psi_1(\zeta,\bar{\zeta},-\eta_1+\mathrm{i}\eta_2) \\ &\quad - \int_0^1\partial_\xi\left((\zeta-\bar{\zeta})\mathrm{e}^{\lambda_+(\xi)t}\Psi_1(\zeta,\bar{\zeta},\lambda_+(\xi))\right)\mathrm{e}^{\nu(\lambda_+(\xi))(\zeta-\bar{\zeta})}\,\mathrm{d}\xi \\ &\quad - (\zeta-\bar{\zeta})\mathrm{e}^{\lambda_c(\xi_0)t+\mathrm{i}\xi_0(\zeta-\bar{\zeta})}\Psi_1(\zeta,\bar{\zeta},\lambda_c(\xi_0)) \\ &=: II_+ + III_+ - (\zeta-\bar{\zeta})\mathrm{e}^{\lambda_c(\xi_0)t+\mathrm{i}\xi_0(\zeta-\bar{\zeta})}\Psi_1(\zeta,\bar{\zeta},\lambda_c(\xi_0)).\end{aligned}$$

Abbreviating $\Psi_2(\zeta,\bar\zeta,\lambda) = \Psi_1(\zeta,\bar\zeta,\lambda)/\nu_c'(\lambda)$ and $\Psi_3(\zeta,\bar\zeta,\lambda) = \partial_\lambda\Psi_1(\zeta,\bar\zeta,\lambda)/\nu_c'(\lambda)$ and integrating by parts once again, we arrive at

$$\begin{aligned} III_+ &= -\int_0^1 \mathrm{e}^{\lambda_+(\xi)t}\Big(t\Psi_2(\zeta,\bar\zeta,\lambda_+(\xi)) + \Psi_3(\zeta,\bar\zeta,\lambda_+(\xi))\Big)\partial_\xi\Big(\mathrm{e}^{\nu(\lambda_+(\xi))(\zeta-\bar\zeta)}\Big)\,\mathrm{d}\xi \\ &= \int_0^1 \partial_\xi\Big(\mathrm{e}^{\lambda_+(\xi)t}\Big(t\Psi_2(\zeta,\bar\zeta,\lambda_+(\xi)) + \Psi_3(\zeta,\bar\zeta,\lambda_+(\xi))\Big)\Big)\mathrm{e}^{\nu(\lambda_+(\xi))(\zeta-\bar\zeta)}\,\mathrm{d}\xi \\ &\quad - \Big[\mathrm{e}^{\lambda_+(\xi)t+\nu(\lambda_+(\xi))(\zeta-\bar\zeta)}\Big(t\Psi_2(\zeta,\bar\zeta,\lambda_+(\xi)) + \Psi_3(\zeta,\bar\zeta,\lambda_+(\xi))\Big)\Big]_{\xi=0}^1 \\ &= \int_{\Gamma_+} \mathrm{e}^{\lambda t+\nu(\lambda)(\zeta-\bar\zeta)}\Big(t^2\Psi_2(\zeta,\bar\zeta,\lambda) + t\Big(\partial_\lambda\Psi_2(\zeta,\bar\zeta,\lambda) + \Psi_3(\zeta,\bar\zeta,\lambda)\Big) + \partial_\lambda\Psi_3(\zeta,\bar\zeta,\lambda)\Big)\,\mathrm{d}\lambda \\ &\quad - \Big[\mathrm{e}^{\lambda_+(\xi)t+\nu(\lambda_+(\xi))(\zeta-\bar\zeta)}\Big(t\Psi_2(\zeta,\bar\zeta,\lambda_+(\xi)) + \Psi_3(\zeta,\bar\zeta,\lambda_+(\xi))\Big)\Big]_{\xi=0}^1. \end{aligned}$$

We establish pointwise estimates on the contributions I_+, II_+ and III_+. Here, we use the following facts which follow with the aid of Proposition 3.2.2 and Lemma 3.2.4. First, since the curves $\lambda_\pm$ lie in the open left-half plane and the points $-\eta_1 \pm \mathrm{i}\eta_2$ lie strictly to the right of $\sigma(\mathcal{L}_0)$, there exists a constant $\eta_0 > 0$ such that $\mathrm{Re}(\nu(-\eta_1 \pm \mathrm{i}\eta_2)) \geq \eta_0$ and $\mathrm{Re}(\lambda_\pm(\xi)) \leq -\eta_0$ for all $\xi \in [0,1]$. Second, since the curves $\lambda_\pm$ lie to the right of $\sigma(\mathcal{L}_0)$, it holds $\mathrm{Re}(\nu(\lambda_\pm(\xi))) \geq 0$ for all $\xi \in [0,1]$. Third, the functions $\Psi_i(\zeta,\bar\zeta,\lambda)$ as well as their derivatives with respect to λ are bounded on $\mathbb{R}\times\mathbb{R}\times B(0,\delta)$ for $i = 1,2,3$. Thus, we establish the following pointwise bounds

$$|II_+| \lesssim |\zeta-\bar\zeta|\mathrm{e}^{-\eta_1 t+\eta_0(\zeta-\bar\zeta)}, \qquad |I_+|, |III_+| \lesssim (1+t+t^2)\mathrm{e}^{-\eta_0 t}$$

for $t \geq 0$ and $\zeta,\bar\zeta \in \mathbb{R}$ with $\zeta - \bar\zeta \leq 0$. All in all, we conclude

$$\left|I_+ + \frac{(\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(\xi_0)t+\mathrm{i}\xi_0(\zeta-\bar\zeta)}}{1+(\zeta-\bar\zeta)^2}\Psi_1(\zeta,\bar\zeta,\lambda_c(\xi_0))\right| \lesssim \frac{(1+t+t^2)\mathrm{e}^{-\eta_0 t} + |\zeta-\bar\zeta|\mathrm{e}^{-\eta_1 t+\eta_0(\zeta-\bar\zeta)}}{1+\Big(\zeta-\bar\zeta\Big)^2} \tag{3.47}$$

for $t \geq 0$ and $\zeta,\bar\zeta \in \mathbb{R}$ with $\zeta - \bar\zeta \leq 0$

Analogously, one finds

$$\left|I_- - \frac{(\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(-\xi_0)t-\mathrm{i}\xi_0(\zeta-\bar\zeta)}}{1+(\zeta-\bar\zeta)^2}\Psi_1(\zeta,\bar\zeta,\lambda_c(-\xi_0))\right| \lesssim \frac{(1+t+t^2)\mathrm{e}^{-\eta_0 t} + |\zeta-\bar\zeta|\mathrm{e}^{-\eta_1 t+\eta_0(\zeta-\bar\zeta)}}{1+\Big(\zeta-\bar\zeta\Big)^2}, \tag{3.48}$$

for $t \geq 0$ and $\zeta,\bar\zeta \in \mathbb{R}$ with $\zeta - \bar\zeta \leq 0$.

Pointwise approximation of I_0 for $\zeta \geq \bar\zeta$. Recalling that the integrand of I_0 is analytic in λ on $B(0,\delta)$, we can apply Cauchy's integral theorem to deform the contour Γ_c to a line $\widetilde\Gamma_c$ connecting the point $\lambda_c(-\xi_0)$ to $\lambda_c(\xi_0)$. We parameterize the line by a curve $\lambda_0\colon [0,1] \to \mathbb{C}$ satisfying $\lambda_0'(\xi) \neq 0$ for all $\xi \in [0,1]$, see Figure 3.2. We proceed similarly as before and factor out the quotient $(1+(\zeta-\bar\zeta)^2)^{-1}$, which is integrable in space. Thus, using integration by parts and Proposition 3.2.2, we rewrite

$$\begin{aligned} (\zeta-\bar\zeta)^2 I_0 &= \int_0^1 (\zeta-\bar\zeta)\mathrm{e}^{\lambda_0(\xi)t}\Psi_1(\zeta,\bar\zeta,\lambda_0(\xi))\partial_\xi\Big(\mathrm{e}^{\nu(\lambda_0(\xi))(\zeta-\bar\zeta)}\Big)\,\mathrm{d}\xi \\ &= (\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(\xi_0)t+\mathrm{i}\xi_0(\zeta-\bar\zeta)}\Psi_1(\zeta,\bar\zeta,\lambda_c(\xi_0)) - (\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(-\xi_0)t-\mathrm{i}\xi_0(\zeta-\bar\zeta)}\Psi_1(\zeta,\bar\zeta,\lambda_c(-\xi_0)) \\ &\quad - \int_0^1 \partial_\xi\Big((\zeta-\bar\zeta)\mathrm{e}^{\lambda_0(\xi)t}\Psi_1(\zeta,\bar\zeta,\lambda_0(\xi))\Big)\mathrm{e}^{\nu(\lambda_0(\xi))(\zeta-\bar\zeta)}\,\mathrm{d}\xi. \end{aligned}$$

Using integration by parts once again, we establish

$$\begin{aligned}
II_0 &:= -\int_0^1 \partial_\xi\Big((\zeta-\bar\zeta)\mathrm{e}^{\lambda_0(\xi)t}\Psi_1(\zeta,\bar\zeta,\lambda_0(\xi))\Big)\mathrm{e}^{\nu(\lambda_0(\xi))(\zeta-\bar\zeta)}\,\mathrm{d}\xi\\
&= -\int_0^1 \mathrm{e}^{\lambda_0(\xi)t}\Big(t\Psi_2(\zeta,\bar\zeta,\lambda_0(\xi))+\Psi_3(\zeta,\bar\zeta,\lambda_0(\xi))\Big)\partial_\xi\Big(\mathrm{e}^{\nu(\lambda_0(\xi))(\zeta-\bar\zeta)}\Big)\,\mathrm{d}\xi\\
&= \int_{\widetilde\Gamma_c} \mathrm{e}^{\lambda t+\nu(\lambda)(\zeta-\bar\zeta)}\Big(t^2\Psi_2(\zeta,\bar\zeta,\lambda)+t\Big(\partial_\lambda\Psi_2(\zeta,\bar\zeta,\lambda)+\Psi_3(\zeta,\bar\zeta,\lambda)\Big)+\partial_\lambda\Psi_3(\zeta,\bar\zeta,\lambda)\Big)\,\mathrm{d}\lambda\\
&\quad -\Big[\mathrm{e}^{\lambda_0(\xi)t+\nu(\lambda_0(\xi))(\zeta-\bar\zeta)}\Big(t\Psi_2(\zeta,\bar\zeta,\lambda_0(\xi))+\Psi_3(\zeta,\bar\zeta,\lambda_0(\xi))\Big)\Big]_{\xi=0}^1.
\end{aligned}$$

Since $\widetilde\Gamma_c$ is a straight line in $B(0,\delta)$ lying to the left of $\sigma(\mathcal{L}_0)$, it holds $\mathrm{Re}(\lambda_0(\xi)) \leq \mathrm{Re}(\lambda_c(\pm\xi_0)) \leq -\eta_0$ and $\mathrm{Re}(\nu(\lambda_0(\xi))) \leq 0$ for all $\xi \in [0,1]$ by Proposition 3.2.2. Hence, we obtain the following pointwise bounds

$$|I_0|, |II_0| \lesssim (1+t+t^2)\mathrm{e}^{-\eta_0 t},$$

for $t \geq 0$ and $\zeta,\bar\zeta \in \mathbb{R}$ with $\zeta-\bar\zeta \geq 0$. We conclude that

$$\begin{aligned}
&\left|I_0 - \frac{\zeta-\bar\zeta}{1+(\zeta-\bar\zeta)^2}\Big(\mathrm{e}^{\lambda_c(\xi_0)t+\mathrm{i}\xi_0(\zeta-\bar\zeta)}\Psi_1(\zeta,\bar\zeta,\lambda_c(\xi_0)) - \mathrm{e}^{\lambda_c(-\xi_0)t-\mathrm{i}\xi_0(\zeta-\bar\zeta)}\Psi_1(\zeta,\bar\zeta,\lambda_c(-\xi_0))\Big)\right|\\
&\qquad\lesssim \frac{(1+t+t^2)\mathrm{e}^{-\eta_0 t}}{1+\Big(\zeta-\bar\zeta\Big)^2},
\end{aligned}\tag{3.49}$$

for $t \geq 0$ and $\zeta,\bar\zeta \in \mathbb{R}$ with $\zeta-\bar\zeta \geq 0$.

Pointwise approximation of I_c. Again our approach is to factor out the quotient $(1+(\zeta-\bar\zeta)^2)^{-1}$. Recalling $\mathrm{i}\Phi_\xi(\zeta)\tilde\Phi_\xi(\bar\zeta)^* = \Psi(\zeta,\lambda_c(\xi))\tilde\Psi(\bar\zeta,\lambda_c(\xi))^*\lambda_c'(\xi)$ for $\xi \in [-\xi_0,\xi_0]$, using integration by parts and applying Proposition 3.2.2, we rewrite

$$\begin{aligned}
(\zeta-\bar\zeta)^2 I_c &= \int_{-\xi_0}^{\xi_0} (1-\rho(\xi))(\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(\xi)t}\Psi_1(\zeta,\bar\zeta,\lambda_c(\xi))\partial_\xi\Big(\mathrm{e}^{\nu(\lambda_c(\xi))(\zeta-\bar\zeta)}\Big)\,\mathrm{d}\xi\\
&= (\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(\xi_0)t+\mathrm{i}\xi_0(\zeta-\bar\zeta)}\Psi_1(\zeta,\bar\zeta,\lambda_c(\xi_0)) - (\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(-\xi_0)t-\mathrm{i}\xi_0(\zeta-\bar\zeta)}\Psi_1(\zeta,\bar\zeta,\lambda_c(-\xi_0))\\
&\quad - \int_{-\xi_0}^{\xi_0} \partial_\xi\Big((1-\rho(\xi))(\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(\xi)t}\Psi_1(\zeta,\bar\zeta,\lambda_c(\xi))\Big)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}\,\mathrm{d}\xi.
\end{aligned}$$

Abbreviating $\tilde\Psi_2(\zeta,\bar\zeta,\xi) = (1-\rho(\xi))\Psi_1(\zeta,\bar\zeta,\lambda_c(\xi))$ and integrating by parts once again, we establish

$$\begin{aligned}
II_c &:= -\int_{-\xi_0}^{\xi_0} \partial_\xi\Big((\zeta-\bar\zeta)\mathrm{e}^{\lambda_c(\xi)t}\tilde\Psi_2(\zeta,\bar\zeta,\xi)\Big)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}\,\mathrm{d}\xi\\
&= \mathrm{i}\int_{-\xi_0}^{\xi_0} \mathrm{e}^{\lambda_c(\xi)t}\Big(\lambda_c'(\xi)t\tilde\Psi_2(\zeta,\bar\zeta,\xi)+\partial_\xi\tilde\Psi_2(\zeta,\bar\zeta,\xi)\Big)\partial_\xi\Big(\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}\Big)\,\mathrm{d}\xi\\
&= \int_{-\xi_0}^{\xi_0} \frac{\mathrm{e}^{\lambda_c(\xi)t+\mathrm{i}\xi(\zeta-\bar\zeta)}}{\mathrm{i}}\Big(\Big(\big(\lambda_c'(\xi)t\big)^2+\lambda_c''(\xi)t\Big)\tilde\Psi_2(\zeta,\bar\zeta,\xi)\\
&\qquad\qquad + 2\lambda_c'(\xi)t\partial_\xi\tilde\Psi_2(\zeta,\bar\zeta,\xi)+\partial_\xi^2\tilde\Psi_2(\zeta,\bar\zeta,\xi)\Big)\mathrm{d}\xi\\
&\quad + \mathrm{i}\Big[\mathrm{e}^{\lambda_c(\xi)t+\mathrm{i}\xi(\zeta-\bar\zeta)}\Big(\lambda_c'(\xi)t\tilde\Psi_2(\zeta,\bar\zeta,\xi)+\partial_\xi\tilde\Psi_2(\zeta,\bar\zeta,\xi)\Big)\Big]_{\xi=-\xi_0}^{\xi_0}.
\end{aligned}$$

In order to obtain pointwise estimates on I_c and II_c, we note that there exists $\eta_c > 0$ such that $\mathrm{Re}(\lambda_c(\pm\xi)) \leq -\eta_c$ for all $\xi \in [\frac{1}{2}\xi_0, \xi_0]$ by Proposition 3.1.3. Therefore, recalling that $1-\rho(\xi)$ vanishes on $[-\frac{1}{2}\xi_0, \frac{1}{2}\xi_0]$, we obtain

$$|I_c|, |II_c| \lesssim (1+t+t^2)\mathrm{e}^{-\eta_c t},$$

for $t \geq 0$ and $\zeta, \bar{\zeta} \in \mathbb{R}$. We conclude

$$\begin{aligned}&\left|I_c - \frac{\zeta-\bar{\zeta}}{1+(\zeta-\bar{\zeta})^2}\left(\mathrm{e}^{\lambda_0(\xi_0)t+\mathrm{i}\xi_0(\zeta-\bar{\zeta})}\Psi_1(\zeta,\bar{\zeta},\lambda_0(\xi_0)) - \mathrm{e}^{\lambda_0(-\xi_0)t-\mathrm{i}\xi_0(\zeta-\bar{\zeta})}\Psi_1(\zeta,\bar{\zeta},\lambda_0(-\xi_0))\right)\right|\\ &\quad\lesssim \frac{(1+t+t^2)\mathrm{e}^{-\eta_c t}}{1+\left(\zeta-\bar{\zeta}\right)^2},\end{aligned} \tag{3.50}$$

for $t \geq 0$ and $\zeta, \bar{\zeta} \in \mathbb{R}$.

Conclusion. Denote $\widetilde{\eta} := \min\{\eta_0/2, \eta_c/2, \eta_1\} > 0$. Recalling the decomposition (3.46) and applying the estimates (3.47), (3.48), (3.49) and (3.50), we find the desired bound

$$\begin{aligned}&\left|\mathrm{i}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(\xi)\mathrm{e}^{\lambda_c(\xi)t+\mathrm{i}\xi(\zeta-\bar{\zeta})}\Phi_\xi(\zeta)\widetilde{\Phi}_\xi(\bar{\zeta})^*\,\mathrm{d}\xi\,\mathbf{g}(\bar{\zeta})\,\mathrm{d}\bar{\zeta}\right.\\ &\qquad\left.-\int_\zeta^\infty\int_{\Gamma_\delta} \mathrm{e}^{\lambda t+\nu(\lambda)(\zeta-\bar{\zeta})}\Psi(\zeta,\lambda)\tilde{\Psi}(\bar{\zeta},\lambda)^*\,\mathrm{d}\lambda\,\mathbf{g}(\bar{\zeta})\,\mathrm{d}\bar{\zeta}\right|\\ &\quad\lesssim \|\mathbf{g}\|_{L^\infty}\left(\int_{\mathbb{R}}\frac{(1+t+t^2)\mathrm{e}^{-2\widetilde{\eta}t}}{1+\bar{\zeta}^2}\,\mathrm{d}\bar{\zeta} + \int_{-\infty}^0 \frac{|\bar{\zeta}|\mathrm{e}^{\eta_0\bar{\zeta}-\eta_1 t}}{1+\bar{\zeta}^2}\,\mathrm{d}\bar{\zeta}\right) \lesssim \|\mathbf{g}\|_{L^\infty}\mathrm{e}^{-\widetilde{\eta}t},\end{aligned}$$

for $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$, $\zeta \in \mathbb{R}$ and $t \geq 0$. □

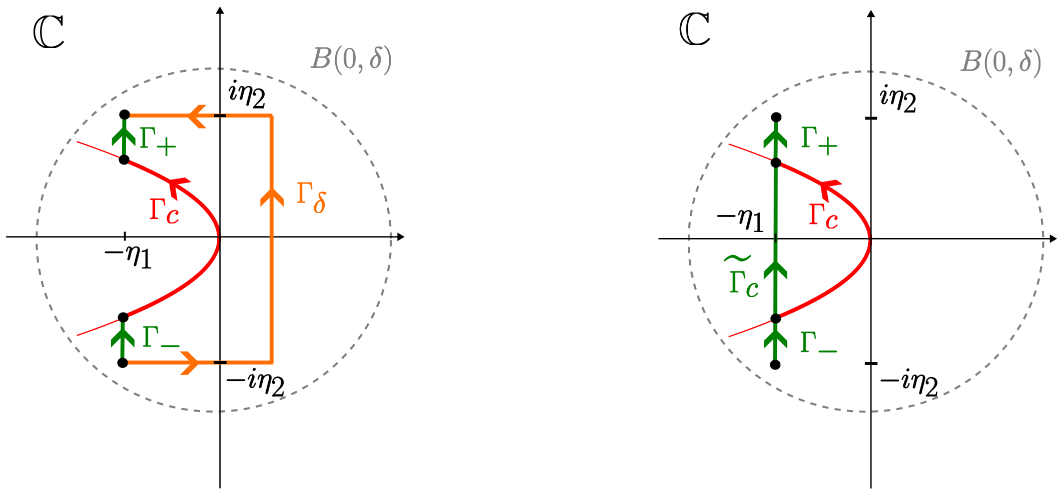

Figure 3.2.: In the proof of Proposition 3.3.8 we relate the Floquet-Bloch representation of the critical part of the semigroup, corresponding to an inverse Laplace integral over Γ_c, with the aid of Cauchy's integral theorem to complex line integrals over Γ_δ, Γ_- and Γ_+ for $\zeta \leq \bar{\zeta}$ (left panel) and over $\widetilde{\Gamma}_c$ for $\zeta \geq \bar{\zeta}$ (right panel). Here, Γ_c lies on the critical spectral curve $\{\lambda_c(\xi) : \xi \in \mathbb{R} \cap V_1\}$ established in Proposition 3.1.3.

3.3.5. Linear estimates

By Propositions 3.3.3, 3.3.4, 3.3.6, 3.3.7 and 3.3.8, the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ decomposes for $t \geq 0$ as

$$\mathrm{e}^{\mathcal{L}_0 t} = S_c(t) + S_e(t),$$

where the operator $S_c(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ given by

$$(S_c(t)\mathbf{g})[\zeta] = \frac{\chi(t)}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(\xi) \mathrm{e}^{\lambda_c(\xi)t + \mathrm{i}\xi(\zeta - \bar{\zeta})} \Phi_\xi(\zeta) \widetilde{\Phi}_\xi(\bar{\zeta})^* \,\mathrm{d}\xi\, \mathbf{g}(\bar{\zeta})\,\mathrm{d}\bar{\zeta} \tag{3.51}$$

corresponds to the critical low-frequency part of the semigroup and

$$S_e(t) = (1 - \chi(t))\mathrm{e}^{\mathcal{L}_0 t} + S_e^1(t) + S_e^2(t) + S_e^3(t) + S_e^4(t) \tag{3.52}$$

is the exponentially decaying residual. The Floquet-Bloch representation (3.51) of the critical part of the semigroup is identical to the one obtained in the stability analysis [84] of wave trains in reaction-diffusion systems against C_{ub}-perturbations. Thus, the further decomposition of $S_c(t)$, as well as the proofs of the associated L^∞-estimates, can be taken verbatim from [84]. On the other hand, estimates on the terms comprising $S_e(t)$ were obtained in Lemma 3.3.2 and Propositions 3.3.4, 3.3.6, 3.3.7 and 3.3.8. In the final result of this section, we collect these results and state the decomposition of the semigroup and associated estimates needed for our nonlinear stability analysis.

Theorem 3.3.9. *Assume* (H1), (H2) *and* (D1)-(D3). *Let* $j, l \in \mathbb{N}_0$. *There exist constants* $C, \alpha > 0$ *such that the semigroup* $\mathrm{e}^{\mathcal{L}_0 t}$ *decomposes as*

$$\mathrm{e}^{\mathcal{L}_0 t} = (\phi_0' + \partial_k \phi(\cdot, 1)\partial_\zeta) S_p(t) + S_r(t) + S_e(t), \tag{3.53}$$

where the operators $S_e(t), S_r(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ *obey the estimates*

$$\|S_e(t)\mathbf{g}\|_{L^\infty} \leq C\mathrm{e}^{-\alpha t}\|\mathbf{g}\|_{L^\infty}, \qquad \|S_r(t)\mathbf{g}\|_{L^\infty} \leq C\frac{\|\mathbf{g}\|_{L^\infty}}{1+t} \tag{3.54}$$

for $t \geq 0$ *and* $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$. *In addition,* $S_p(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ *satisfies* $S_p(t) = 0$ *for* $t \in [0,1]$ *and the map* $t \mapsto S_p(t)\mathbf{g}$ *lies in* $C^i\big([0,\infty), C^k_{\mathrm{ub}}(\mathbb{R})\big)$ *for any* $i, k \in \mathbb{N}_0$ *with*

$$\big\|(\partial_t + c_g\partial_\zeta)^j \partial_\zeta^l S_p(t)\mathbf{g}\big\|_{L^\infty} \leq C\frac{\|\mathbf{g}\|_{L^\infty}}{(1+t)^{j+\frac{l}{2}}} \tag{3.55}$$

for $t \geq 0$ *and* $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$. *We have the further decomposition*

$$\partial_\zeta^m S_p(t)\mathbf{g} = \partial_\zeta^m \mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)t}\Big(\widetilde{\Phi}_0^*\mathbf{g}\Big) + \partial_\zeta^m \widetilde{S}_r(t)\mathbf{g}, \tag{3.56}$$

where the operator $\partial_\zeta^m \widetilde{S}_r(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ *obeys the estimate*

$$\big\|\partial_\zeta^m \widetilde{S}_r(t)\mathbf{g}\big\|_{L^\infty} \leq C(1+t)^{-\frac{1}{2}} t^{-\frac{m}{2}} \|\mathbf{g}\|_{L^\infty} \tag{3.57}$$

for $m = 0, 1$, $t > 0$ *and* $\mathbf{g} \in C_{\mathrm{ub}}(\mathbb{R})$.

Finally, there exist a bounded operator $A_h\colon L^2_{\mathrm{per}}((0,T),\mathbb{R}^2) \to C(\mathbb{R},\mathbb{R})$ *such that it holds*

$$\mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)t}\Big(v\widetilde{\Phi}_0^*\mathbf{g}\Big) = \mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)t}\Big(\langle\widetilde{\Phi}_0, \mathbf{g}\rangle_{L^2(0,T)} v - A_h(\mathbf{g})\partial_\zeta v\Big) + \partial_\zeta \mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)t}(A_h(\mathbf{g})v), \tag{3.58}$$

for $\mathbf{g} \in L^2_{\mathrm{per}}((0,T),\mathbb{R}^2)$, $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$ *and* $t > 0$.

Proof. The decomposition $e^{\mathcal{L}_0 t} = S_c(t) + S_e(t)$, where $S_e(t)$ is given by (3.52) and $S_c(t)$ is given by (3.51), follows from Propositions 3.3.3, 3.3.4, 3.3.6, 3.3.7 and 3.3.8. The desired bound (3.54) on $S_e(t)$ can be derived by combining Lemma 3.3.2 and Propositions 3.3.4, 3.3.6, 3.3.7 and 3.3.8. Moreover, it has been shown in [84, Section 3.3] that $S_c(t)$ decomposes as $S_c(t) = (\phi_0' + \partial_k\phi(\cdot,1)\partial_\zeta)S_p(t) + S_r(t)$, where $S_p(t), S_r(t)\colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}(\mathbb{R})$ are operators obeying the estimates (3.54) and (3.55). Moreover, $S_p(t)$ satisfies $S_p(t) = 0$ for $t \in [0,1]$ and the map $t \mapsto S_p(t)\mathbf{g}$ lies in $C^i([0,\infty), C^k_{\mathrm{ub}}(\mathbb{R}))$ for any $i, k \in \mathbb{N}_0$. Finally, the decomposition (3.56), the estimates (3.57) and the identity (3.58) can be found in [84, Section 3.5]. □

3.4. Nonlinear iteration scheme and nonlinear estimates

In this section, we set up the nonlinear iteration scheme and state associated nonlinear estimates, which will be employed in the upcoming section to prove our nonlinear stability result, Theorem 3.1.4. To this end, we consider a diffusively spectrally stable wave-train solution $\mathbf{u}_0(x,t) = \phi_0(x - c_0 t)$ to (3.2), i.e., we assume that Hypotheses (H1), (H2) and (D1)-(D3) are satisfied, and an initial perturbation $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$. We wish to control the perturbation $\widetilde{\mathbf{w}}(t) = \mathbf{u}(t) - \phi_0$ over time, where $\mathbf{u}(t)$ is the solution to (3.3) with initial condition $\mathbf{u}(0) = \phi_0 + \mathbf{w}_0$. The perturbation $\widetilde{\mathbf{w}}(t)$ satisfies equation (3.17). Theorem 3.3.9 shows that the bounds on full semigroup $e^{\mathcal{L}_0 t}$ are too weak to close a nonlinear iteration argument using the Duhamel formulation of (3.17).

As explained in §3.1.3, this leads us to consider the inverse-modulated perturbation $\mathbf{w}(t)$ given by (3.18). We derive a quasilinear equation for $\mathbf{w}(t)$, establish L^∞-bounds on the nonlinearity and define a suitable phase modulation $\psi(t)$ compensating for the most critical terms in the Duhamel formulation of $\mathbf{w}(t)$. We then infer, as in [84], that $\psi(t)$ satisfies a perturbed viscous Hamilton-Jacobi equation, whose most critical nonlinear term cannot be controled through L^∞-estimates, but can be eliminated with the aid of the Cole-Hopf transform. We formulate an equation for the Cole-Hopf variable and state L^∞-bounds on the nonlinearity.

Lastly, we control regularity in the quasilinear iteration scheme by relying on forward-modulated damping estimates. We obtain an equation for the modified forward-modulated perturbation $\mathring{\mathbf{z}}(t)$ given by (3.22), establish norm equivalences between $\mathring{\mathbf{z}}(t)$ and the residual

$$\mathbf{z}(t) = \mathbf{w}(t) - \partial_k\phi(\cdot;1)\psi_\zeta(t), \tag{3.59}$$

and we derive a nonlinear damping estimate for $\mathring{\mathbf{z}}(t)$ using uniformly local Sobolev norms.

3.4.1. The unmodulated perturbation

The unmodulated perturbation $\widetilde{\mathbf{w}}(t)$ satisfies the semilinear equation (3.17), whose nonlinearity $\widetilde{\mathcal{N}}\colon C^1_{\mathrm{ub}}(\mathbb{R}) \to C^1_{\mathrm{ub}}(\mathbb{R})$ is readily seen to be continuously Fréchet differentiable. On the other hand, regarding $\mathcal{L}_0$ as an operator on $C^1_{\mathrm{ub}}(\mathbb{R})$ with dense domain $C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$, Proposition 3.3.1 yields that $\mathcal{L}_0$ generates a C_0-semigroup on $C^1_{\mathrm{ub}}(\mathbb{R})$. Hence, local existence and uniqueness of a classical solution to (3.17) follows by standard results, e.g. [79, Theorem 6.1.5], from semigroup theory.

Proposition 3.4.1. *Assume* (H1). *Let* $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$. *Then, there exists a maximal time* $T_{\max} \in (0,\infty]$ *such that* (3.17) *admits a unique classical solution*

$$\widetilde{\mathbf{w}} \in C\big([0,T_{\max}), C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,T_{\max}), C^1_{\mathrm{ub}}(\mathbb{R})\big),$$

with initial condition $\widetilde{\mathbf{w}}(0) = \mathbf{w}_0$. *Moreover, if* $T_{\max} < \infty$, *then we have*

$$\limsup_{t \uparrow T_{\max}} \|\widetilde{\mathbf{w}}(t)\|_{C^1_{\mathrm{ub}}} = \infty.$$

3.4.2. The inverse-modulated perturbation

Using that $\mathbf{u}(t)$ and ϕ_0 solve (3.3), one finds that the inverse-modulated perturbation $\mathbf{w}(t)$, given by (3.18), obeys the quasilinear equation

$$(\partial_t - \mathcal{L}_0)[\mathbf{w} + \phi_0' \psi] = \mathcal{N}(\mathbf{w}, \psi, \partial_t \psi) + (\partial_t - \mathcal{L}_0)[\psi_\zeta \mathbf{w}] \tag{3.60}$$

with nonlinearity

$$\mathcal{N}(\mathbf{w}, \psi, \psi_t) = \mathcal{Q}(\mathbf{w}, \psi) + \partial_\zeta \mathcal{R}(\mathbf{w}, \psi, \psi_t),$$

where

$$\mathcal{Q}(\mathbf{w}, \psi) = \left(F(\phi_0 + \mathbf{w}) - F(\phi_0) - F'(\phi_0)\mathbf{w}\right)(1 - \psi_\zeta)$$

is quadratic in $\mathbf{w}$ and

$$\mathcal{R}(\mathbf{w}, \psi, \psi_t) = (c_0 \psi_\zeta - \psi_t)\mathbf{w} + D\left(\frac{(\mathbf{w}_\zeta + \phi_0' \psi_\zeta)\psi_\zeta}{1 - \psi_\zeta} + (\mathbf{w}\psi_\zeta)_\zeta\right).$$

contains all linear terms in $\mathbf{w}$. We refer to [8, Appendix E] for a detailed derivation of (3.60).

It is relatively straightforward to verify the relevant nonlinear bound.

Lemma 3.4.2. *Assume* (H1). *Then, we have*

$$\|\mathcal{N}(\mathbf{w}, \psi, \psi_t)\|_{L^\infty} \lesssim \|\mathbf{w}\|_{L^\infty}^2 + \|(\psi_\zeta, \psi_t)\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} \left(\|\mathbf{w}\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} + \|\psi_\zeta\|_{L^\infty}\right)$$

for $\mathbf{w} = (u, v) \in C^2_{\mathrm{ub}}(\mathbb{R}) \times C^1_{\mathrm{ub}}(\mathbb{R})$ *and* $(\psi, \psi_t) \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^1_{\mathrm{ub}}(\mathbb{R})$ *satisfying* $\|u\|_{L^\infty}, \|\psi_\zeta\|_{L^\infty} \leq \frac{1}{2}$.

Inspired by earlier works [55, 56], we implicitly define the phase modulation by the integral equation

$$\psi(t) = S_p(t)\mathbf{w}_0 + \int_0^t S_p(t - s)\mathcal{N}(\mathbf{w}(s), \psi(s), \partial_t \psi(s))\, \mathrm{d}s. \tag{3.61}$$

Recalling from Theorem 3.3.9 that $S_p(0) = 0$, we find that $\psi(t)$ vanishes at $t = 0$. Thus, integrating (3.60) yields the Duhamel formulation

$$\mathbf{w}(t) + \phi_0' \psi(t) = \mathrm{e}^{\mathcal{L}_0 t}\mathbf{w}_0 + \int_0^t \mathrm{e}^{\mathcal{L}_0(t-s)}\mathcal{N}(\mathbf{w}(s), \psi(s), \partial_t \psi(s))\, \mathrm{d}s + \psi_\zeta(t)\mathbf{w}(t). \tag{3.62}$$

Writing the left-hand side of (3.62) as $\mathbf{w}(t) + \phi_0' \psi(t) = \mathbf{z}(t) + (\phi_0' + \partial_k \phi(\cdot; 1)\partial_\zeta)\psi(t)$, where $\mathbf{z}(t)$ is given by (3.59), and recalling the semigroup decomposition (3.53), we observe that by defining the phase modulation by (3.61), the term $(\phi_0' + \partial_k \phi(\cdot; 1)\partial_\zeta)\psi(t)$ compensates for the critical, slowest decaying, contributions on the right-hand side of (3.62). Indeed, we arrive at the Duhamel formulation

$$\mathbf{z}(t) = (S_r(t) + S_e(t))\mathbf{w}_0 + \int_0^t (S_r(t-s) + S_e(t-s))\mathcal{N}(\mathbf{w}(s), \psi(s), \partial_t\psi(s))\,\mathrm{d}s + \psi_\zeta(t)\mathbf{w}(t), \tag{3.63}$$

for the residual $\mathbf{z}(t)$, where $S_r(t) + S_e(t)$ exhibits stronger decay than $\mathrm{e}^{\mathcal{L}_0 t}$, cf. Theorem 3.3.9.

Local existence of the phase modulation $\psi(t)$ can be obtained by applying a standard contraction mapping argument to the integral equation (3.61), where one employs Proposition 3.4.1 and expresses the inverse-modulated perturbation as

$$\mathbf{w}(\zeta, t) = \widetilde{\mathbf{w}}(\zeta - \psi(\zeta, t), t) + \phi_0(\zeta - \psi(\zeta, t)) - \phi_0(\zeta), \tag{3.64}$$

to obtain a fixed point problem in $\psi(t)$ and its derivatives. This leads to the following result, whose proof is identical to [84, Proposition 4.4].

Proposition 3.4.3. *Assume* (H1). *Let* $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$. *Fix* $j, l, m \in \mathbb{N}_0$. *For* $\widetilde{\mathbf{w}}$ *and* $T_{\max}$ *as in Proposition 3.4.1, there exists a maximal time* $\tau_{\max} \in (0, T_{\max}]$ *such that equation* (3.61), *with* $\mathbf{w}$ *given by* (3.64), *possesses a solution*

$$\psi \in C\big([0, \tau_{\max}), C^{2+m}_{\mathrm{ub}}(\mathbb{R})\big) \cap C^{1+j}\big([0, \tau_{\max}), C^l_{\mathrm{ub}}(\mathbb{R})\big),$$

satisfying $\psi(t) = 0$ *for all* $t \in [0, \tau_{\max})$ *with* $t \leq 1$.

Moreover, we have $\|(\psi(t), \partial_t\psi(t))\|_{C^2_{\mathrm{ub}} \times C_{\mathrm{ub}}} < \frac{1}{2}$ *for all* $t \in [0, \tau_{\max})$. *Finally, if* $\tau_{\max} < T_{\max}$, *then*

$$\limsup_{t \uparrow \tau_{\max}} \|(\psi(t), \partial_t\psi(t))\|_{C^2_{\mathrm{ub}} \times C_{\mathrm{ub}}} = \frac{1}{2}.$$

The existence and regularity of the inverse-modulated perturbation $\mathbf{w}(t)$ and the residual $\mathbf{z}(t)$ now follow immediately from (3.64) and (3.59), respectively, upon applying Propositions 3.1.2, 3.4.1 and 3.4.3 and using the uniform continuity of functions in $C_{\mathrm{ub}}(\mathbb{R})$.

Corollary 3.4.4. *Assume* (H1) *and* (D3). *Let* $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$. *For* $\widetilde{\mathbf{w}}$ *as in Proposition 3.4.1 and* ψ *and* $\tau_{\max}$ *as in Proposition 3.4.3, the inverse-modulated perturbation* $\mathbf{w}$, *defined by* (3.18), *and the residual* $\mathbf{z}$, *defined by* (3.59), *obey*

$$\mathbf{w}, \mathbf{z} \in C\big([0, \tau_{\max}), C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})\big).$$

Moreover, their Duhamel formulations (3.62) *and* (3.63) *hold for* $t \in [0, \tau_{\max})$.

3.4.3. Derivation of the perturbed viscous Hamilton-Jacobi equation

The estimates in Theorem 3.3.9, in combination with (3.61) and (3.63), show that, at least on the linear level, the derivative $\partial_\zeta^j \partial_t^l \psi(t)$ of the phase modulation decays at rate $t^{-(j+l)/2}$ for $j, l \in \mathbb{N}_0$, whereas the residual $\mathbf{z}(t)$ and

$$\tilde{\psi}(t) = \partial_t\psi(t) + c_g\psi_\zeta(t),$$

decay at rate t^{-1}. Therefore, after substituting

$$\mathbf{w}(t) = \mathbf{z}(t) + \partial_k\phi(\cdot;1)\psi_\zeta(t), \qquad \partial_t\psi(t) = \tilde{\psi}(t) - c_g\psi_\zeta(t), \tag{3.65}$$

in the nonlinearity $\mathcal{N}(\mathbf{w},\psi,\psi_t)$ one finds that the nonlinear terms exhibiting the slowest decay are of Burgers'-type, i.e. of the form $\mathbf{f}\psi_\zeta(t)^2$ with coefficient $\mathbf{f} \in L^2_{\mathrm{per}}(0,T)$.

The decay rates of the principal part $S_p(t)$ of the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ are not strong enough to control these most critical nonlinear terms through iterative estimates on the equation (3.61) for the phase modulation. As outlined in §3.1.3, we address this issue by proceeding as in [84]. That is, we show that $\psi(t)$ obeys a perturbed viscous Hamilton-Jacobi equation and subsequently apply the Cole-Hopf transform to this equation to eliminate the critical ψ_ζ^2-contributions.

To derive a viscous Hamilton-Jacobi equation for $\psi(t)$, we first isolate the ψ_ζ^2-contributions in the nonlinearity $\mathcal{N}(\mathbf{w},\psi,\psi_t)$ of (3.61). We do so by reexpressing $\mathbf{w}(t)$ and $\partial_t\psi(t)$ through (3.65) wherever necessary. Thus, recalling $c_0 + c_g = \omega'(1)$ from Proposition 3.1.3, we arrive at

$$\mathcal{N}(\mathbf{w}(s),\psi(s),\partial_t\psi(s)) = \mathbf{f}_p\psi_\zeta(s)^2 + \mathcal{N}_p(\mathbf{z}(s),\mathbf{w}(s),\psi(s),\tilde{\psi}(s)), \tag{3.66}$$

with T-periodic coefficient

$$\mathbf{f}_p = \frac{1}{2}F''(\phi_0)(\partial_k\phi(\cdot;1),\partial_k\phi(\cdot;1)) + \omega'(1)\partial_{\zeta k}\phi(\cdot;1) + D(\phi_0'' + 2\partial_{\zeta\zeta k}\phi(\cdot;1))$$

and residual nonlinearity

$$\mathcal{N}_p(\mathbf{z},\mathbf{w},\psi,\tilde{\psi}) = \mathcal{Q}_p(\mathbf{z},\mathbf{w},\psi) + \partial_\zeta\mathcal{R}_p(\mathbf{z},\mathbf{w},\psi,\tilde{\psi}),$$

where we denote

$$\begin{aligned}
\mathcal{Q}_p(\mathbf{z},\mathbf{w},\psi) &= (F(\phi_0+\mathbf{w}) - F(\phi_0) - F'(\phi_0)\mathbf{w})\psi_\zeta + F(\phi_0+\mathbf{w}) - F(\phi_0) - F'(\phi_0)\mathbf{w}\\
&\quad - \frac{1}{2}F''(\phi_0)(\mathbf{w},\mathbf{w}) + \frac{1}{2}F''(\phi_0)(\mathbf{z},\mathbf{w}) + \frac{1}{2}\psi_\zeta F''(\phi_0)(\mathbf{z},\partial_k\phi(\cdot;1))\\
&\quad + 2\psi_\zeta\psi_{\zeta\zeta}(\omega'(1)\partial_k\phi(\cdot;1) + D(\phi_0' + 2\partial_{\zeta k}\phi(\cdot;1))),\\
\mathcal{R}_p(\mathbf{z},\mathbf{w},\psi,\tilde{\psi}) &= -\tilde{\psi}\mathbf{w} + \omega'(1)\psi_\zeta\mathbf{z}\\
&\quad + D\left(\frac{(\mathbf{w}_\zeta + \phi_0'\psi_\zeta)\psi_\zeta^2}{1-\psi_\zeta} + 2\mathbf{z}_\zeta\psi_\zeta + \mathbf{w}\psi_{\zeta\zeta} + 2\partial_k\phi(\cdot;1)\psi_\zeta\psi_{\zeta\zeta}\right).
\end{aligned}$$

We establish an L^∞-estimate on the residual nonlinearity.

Lemma 3.4.5. *Assume* (H1) *and* (D3). *Then, we have*

$$\begin{aligned}
\|\mathcal{N}_p(\mathbf{z},\mathbf{w},\psi,\tilde{\psi})\|_{L^\infty} \lesssim{}& \left(\|\mathbf{w}\|_{L^\infty} + \|\psi_\zeta\|_{C^1_{\mathrm{ub}}}\right)\left(\|\mathbf{w}\|_{L^\infty}^2 + \|\mathbf{z}\|_{C^2_{\mathrm{ub}}}\right)\\
&+ \left(\|\tilde{\psi}\|_{C^1_{\mathrm{ub}}} + \|\psi_{\zeta\zeta}\|_{C^1_{\mathrm{ub}}}\right)\|\mathbf{w}\|_{C^1_{\mathrm{ub}}}\\
&+ \left(\|\psi_{\zeta\zeta}\|_{C^1_{\mathrm{ub}}} + \|\mathbf{w}\|_{C^2_{\mathrm{ub}}}\|\psi_\zeta\|_{L^\infty} + \|\psi_\zeta\|_{L^\infty}^2\right)\|\psi_\zeta\|_{C^1_{\mathrm{ub}}}
\end{aligned}$$

for $\mathbf{z},\mathbf{w} \in C^2_{\mathrm{ub}}(\mathbb{R})$ *and* $(\psi,\tilde{\psi}) \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^1_{\mathrm{ub}}(\mathbb{R})$ *satisfying* $\|\mathbf{w}\|_{L^\infty}, \|\psi_\zeta\|_{L^\infty} \leq \frac{1}{2}$.

Next, we substitute the decompositions (3.56) of the propagtor $S_p(t)$ and (3.66) of the nonlinearity $\mathcal{N}(\mathbf{w}(s),\psi(s),\partial_t\psi(s))$ into (3.61) and use (3.58) to reexpress $\mathrm{e}^{\left(d\partial_\zeta^2-c_g\partial_\zeta\right)t}\left(\widetilde{\Phi}_0^*\mathbf{f}_p\psi_\zeta^2\right)$. All in all, we arrive at

$$\begin{aligned}\psi(t) = r(t) + \mathrm{e}^{\left(d\partial_\zeta^2-c_g\partial_\zeta\right)t}\left(\widetilde{\Phi}_0^*\mathbf{w}_0\right) + \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2-c_g\partial_\zeta\right)(t-s)}\left(\nu\psi_\zeta(s)^2 - A_h(\mathbf{f}_p)\partial_\zeta\left(\psi_\zeta(s)^2\right)\right)\mathrm{d}s\\ + \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2-c_g\partial_\zeta\right)(t-s)}\left(\widetilde{\Phi}_0^*\mathcal{N}_p(\mathbf{z}(s),\mathbf{w}(s),\psi(s),\tilde{\psi}(s))\right)\mathrm{d}s,\end{aligned} \tag{3.67}$$

where we denote

$$\nu = \langle\widetilde{\Phi}_0,\mathbf{f}_p\rangle_{L^2(0,T)}$$

and

$$\begin{aligned}r(t) = \widetilde{S}_r(t)\mathbf{w}_0 + \int_0^t \widetilde{S}_r(t-s)\left(\mathbf{f}_p\psi_\zeta(s)^2\right)\mathrm{d}s + \partial_\zeta\int_0^t \mathrm{e}^{\left(d\partial_\zeta^2-c_g\partial_\zeta\right)(t-s)}\left(A_h(\mathbf{f}_p)\psi_\zeta(s)^2\right)\mathrm{d}s\\ + \int_0^t \widetilde{S}_r(t-s)\mathcal{N}_p(\mathbf{z}(s),\mathbf{w}(s),\psi(s),\tilde{\psi}(s))\,\mathrm{d}s.\end{aligned} \tag{3.68}$$

Since $\psi(0)$ vanishes identically by Proposition 3.4.3, setting $t=0$ in (3.67) yields

$$r(0) = -\widetilde{\Phi}_0^*\mathbf{w}_0. \tag{3.69}$$

Moreover, following the computations in [29, Section 4.2], one finds that the coefficient ν in (3.67) equals $-\frac{1}{2}\omega''(1)$. Thus, with the aid of Proposition 3.1.3, we arrive at the expression (3.15) for ν. Since $\widetilde{S}_r(t)$ and $\partial_\zeta\mathrm{e}^{\left(d\partial_\zeta^2-c_g\partial_\zeta\right)t}$ decay at rate $t^{-\frac{1}{2}}$ as operators on $C_{\mathrm{ub}}(\mathbb{R})$, we find that $r(t)$ captures, at least on the linear level, the decaying contributions in (3.67), cf. Theorem 3.3.9.

Finally, applying the convective heat operator $\partial_t - d\partial_\zeta^2 + c_g\partial_\zeta$ to (3.67), we arrive at the perturbed viscous Hamilton-Jacobi equation

$$\left(\partial_t - d\partial_\zeta^2 + c_g\partial_\zeta\right)(\psi - r) = \nu\psi_\zeta^2 + G(\mathbf{z},\mathbf{w},\psi,\tilde{\psi}) \tag{3.70}$$

with nonlinear residual

$$G(\mathbf{z},\mathbf{w},\psi,\tilde{\psi}) = \widetilde{\Phi}_0^*\mathcal{N}_p(\mathbf{z},\mathbf{w},\psi,\tilde{\psi}) - A_h(\mathbf{f}_p)\partial_\zeta\left(\psi_\zeta^2\right).$$

Indeed, modulo the higher-order terms r and $G(\mathbf{w},\mathbf{z},\psi,\tilde{\psi})$ equation (3.70) coincides with the Hamilton-Jacobi equation (3.14). Regarding (3.70) as an inhomogeneous parabolic equation, regularity properties of $\psi(t) - r(t)$, and thus of $r(t)$, can be readily deduced from standard analytic semigroup theory.

Corollary 3.4.6. *Assume* (H1) *and* (D3). *Let* $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$. *For* ψ *and* $\tau_{\max}$ *as in Proposition 3.4.3 and for* $\mathbf{w}$ *and* $\mathbf{z}$ *as in Corollary 3.4.4, the residual* r*, given by* (3.68)*, obeys*

$$r \in C\big([0,\tau_{\max}),C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,\tau_{\max}),C_{\mathrm{ub}}(\mathbb{R})\big).$$

Proof. Moving $r(t)$ to the left-hand side, we can regard (3.67) as the mild formulation of the inhomogeneous problem (3.70) for $\psi(t) - r(t)$ with inhomogeneity $t \mapsto \nu\psi_\zeta(t)^2 + G(\mathbf{z}(t),\mathbf{w}(t),\psi(t),\tilde{\psi}(t))$, which lies in $C\big([0,\tau_{\max}),C^1_{\mathrm{ub}}(\mathbb{R})\big)$ by Proposition 3.4.3 and Corollary 3.4.4. It is well-known that $d\partial_\zeta^2 - c_g\partial_\zeta$ is a sectorial operator on $C_{\mathrm{ub}}(\mathbb{R})$ with domain $C^2_{\mathrm{ub}}(\mathbb{R})$, cf. [73, Corollary 3.1.9]. Therefore, since the initial condition $\psi(0) - r(0) = \widetilde{\Phi}_0^*\mathbf{w}_0$ lies in the domain $C^2_{\mathrm{ub}}(\mathbb{R})$ and $C^1_{\mathrm{ub}}(\mathbb{R})$ is an intermediate space between $C_{\mathrm{ub}}(\mathbb{R})$ and the domain $C^2_{\mathrm{ub}}(\mathbb{R})$, it follows from [73, Propositions 2.1.1 and 2.1.4 and Theorem 4.3.8] that $t \mapsto \psi(t) - r(t)$ lies in $C\big([0,\tau_{\max}),C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,\tau_{\max}),C_{\mathrm{ub}}(\mathbb{R})\big)$. Invoking Proposition 3.4.3 then yields the result. □

3.4.4. Application of the Cole-Hopf transform

We apply the Cole-Hopf transform to remove the critical nonlinear term $\nu\psi_\zeta^2$ in (3.70). That is, we introduce the new variable

$$y(t) = \mathrm{e}^{\frac{\nu}{d}(\psi(t)-r(t))} - 1, \tag{3.71}$$

which satisfies

$$y \in C\big([0,\tau_{\max}), C_{\mathrm{ub}}^2(\mathbb{R})\big) \cap C^1\big([0,\tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \tag{3.72}$$

by Proposition 3.4.3 and Corollary 3.4.6. It is readily seen that $y(t)$ is a solution of the convective heat equation

$$\Big(\partial_t - d\partial_\zeta^2 + c_g\partial_\zeta\Big)y = 2\nu r_\zeta y_\zeta + \frac{\nu}{d}\Big(\nu r_\zeta^2 + G(\mathbf{z},\mathbf{w},\psi,\tilde{\psi})\Big)(y+1) \tag{3.73}$$

with initial condition

$$y(0) = \mathrm{e}^{\frac{\nu}{d}\widetilde{\Phi}_0^*(\psi(0)-r(0))} - 1 = \mathrm{e}^{\frac{\nu}{d}\widetilde{\Phi}_0^*\mathbf{w}_0} - 1, \tag{3.74}$$

cf. Proposition 3.4.3 and (3.69).

Recalling that $\psi(t)$ vanishes identically for $t \in [0,1]$ by Proposition 3.4.3, the Cole-Hopf variable $y(t)$ can be expressed in terms of the residual $r(t)$ through

$$y(t) = \mathrm{e}^{-\frac{\nu}{d}r(t)} - 1 \tag{3.75}$$

for $t \in [0,\tau_{\max})$ with $t \leq 1$. On the other hand, the Duhamel formulation of (3.73) reads

$$\begin{aligned} y(t) &= \mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)(t-1)}\left(\mathrm{e}^{\frac{\nu}{d}\widetilde{\Phi}_0^*\mathbf{w}_0} - 1\right) \\ &\quad + \int_1^t \mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)(t-s)}\mathcal{N}_c(r(s),y(s),\mathbf{z}(s),\mathbf{w}(s),\psi(s),\tilde{\psi}(s))\,\mathrm{d}s \end{aligned} \tag{3.76}$$

for $t \in [0,\tau_{\max})$ with $t \geq 1$, where the nonlinearity is given by

$$\mathcal{N}_c(r,y,\mathbf{z},\mathbf{w},\psi,\tilde{\psi}) = 2\nu r_\zeta y_\zeta + \frac{\nu}{d}\Big(\nu r_\zeta^2 + G(\mathbf{z},\mathbf{w},\psi,\tilde{\psi})\Big)(y+1).$$

We use (3.75) for short-time control on $y(t)$ (rather than its Duhamel formulation) in the upcoming nonlinear argument. The reason is that we use a temporal weight $\sqrt{s}\sqrt{1+s}$ on $r_\zeta(s)$, so that the obtained bound on $r_\zeta(s)^2$ is nonintegrable and blows up as $1/s$ as $s \downarrow 0$. We refer to the proof of Theorem 3.1.4 and Remark 3.5.2 for further details.

With the aid of Lemma 3.4.5, we obtain the following nonlinear estimate.

Lemma 3.4.7. *Assume* (H1) *and* (D3). *It holds*

$$\begin{aligned} \|\mathcal{N}_c(r,y,\mathbf{w},\mathbf{z},\psi,\tilde{\psi})\|_{L^\infty} \lesssim\ & (\|r_\zeta\|_{L^\infty} + \|y_\zeta\|_{L^\infty})\|r_\zeta\|_{L^\infty} \\ & + \Big(\|\mathbf{w}\|_{L^\infty} + \|\psi_\zeta\|_{C_{\mathrm{ub}}^1}\Big)\Big(\|\mathbf{w}\|_{L^\infty}^2 + \|\mathbf{z}\|_{C_{\mathrm{ub}}^2}\Big) \\ & + \Big(\|\tilde{\psi}\|_{C_{\mathrm{ub}}^1} + \|\psi_{\zeta\zeta}\|_{C_{\mathrm{ub}}^1}\Big)\|\mathbf{w}\|_{C_{\mathrm{ub}}^1} \\ & + \Big(\|\psi_{\zeta\zeta}\|_{C_{\mathrm{ub}}^1} + \|\mathbf{w}\|_{C_{\mathrm{ub}}^2}\|\psi_\zeta\|_{L^\infty} + \|\psi_\zeta\|_{L^\infty}^2\Big)\|\psi_\zeta\|_{C_{\mathrm{ub}}^1} \end{aligned}$$

for $r,y \in C_{\mathrm{ub}}^1(\mathbb{R})$, $\mathbf{z},\mathbf{w} \in C_{\mathrm{ub}}^2(\mathbb{R})$, $(\psi,\tilde{\psi}) \in C_{\mathrm{ub}}^3(\mathbb{R}) \times C_{\mathrm{ub}}^1(\mathbb{R})$ *with* $\|y\|_{L^\infty}, \|\mathbf{w}\|_{L^\infty}, \|\psi_\zeta\|_{L^\infty} \leq \frac{1}{2}$.

3.4.5. Forward-modulated damping

The modified forward-modulated perturbation $\mathring{\mathbf{z}}(t)$ is given by (3.22), where the T-periodic continuation $\phi(\cdot;k)$ of the wave train ϕ_0 with respect to the wavenumber k is defined for k in the neighborhood $[1-r_0, 1+r_0]$ by Proposition 3.1.2. Combining the latter with Propositions 3.4.1 and 3.4.3, we find that the forward-modulated perturbation $\mathring{\mathbf{z}}(t)$ is well-defined as long as $t \in [0, \tau_{\max})$ is such that $\|\psi_\zeta(t)\|_{L^\infty} < r_0$. Its regularity then follows readily from Propositions 3.1.2, 3.4.1 and 3.4.3.

Corollary 3.4.8. *Assume* (H1) *and* (D3). *Let* $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$. *For* $r_0 > 0$ *as in Proposition 3.1.2,* $\widetilde{\mathbf{w}}$ *as in Proposition 3.4.1, and* ψ *and* $\tau_{\max}$ *as in Proposition 3.4.3, we have*

$$\widetilde{\tau}_{\max} = \sup\{t \in [0, \tau_{\max}) : \|\psi_\zeta(s)\|_{L^\infty} < r_0 \text{ for all } s \in [0,t]\} > 0$$

and the modified forward-modulated perturbation $\mathring{\mathbf{z}}(t)$*, given by* (3.22)*, is well-defined for* $t \in [0, \widetilde{\tau}_{\max})$ *and satisfies*

$$\mathring{\mathbf{z}} \in C\big([0, \widetilde{\tau}_{\max}), C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0, \widetilde{\tau}_{\max}), C^1_{\mathrm{ub}}(\mathbb{R})\big).$$

Using that the wave train $\mathbf{u}_k(x,t) = \phi(kx - \omega(k)t; k)$ is a solution to (3.1) and $\mathbf{u}(t)$ solves (3.3), one obtains the equation

$$\partial_t \mathring{\mathbf{z}} = D\mathring{\mathbf{z}}_{\zeta\zeta} + c_0 \mathring{\mathbf{z}}_\zeta + F'(0)\mathring{\mathbf{z}} + \mathring{\mathcal{Q}}(\mathring{\mathbf{z}}, \psi) + \mathring{\mathcal{R}}(\psi, \tilde{\psi}, \partial_t \psi) \tag{3.77}$$

for the modified forward-modulated perturbation, where

$$\begin{aligned}\mathring{\mathcal{Q}}(\mathring{\mathbf{z}}, \psi) &= F(\mathring{\mathbf{z}} + \phi(\beta(\psi))) - F(\phi(\beta(\psi))) - F'(0)\mathring{\mathbf{z}} \\ &= \begin{pmatrix} (\phi_1(\beta(\psi))(2 + 2\mu - 3\mathring{z}_1) - 3\phi_1(\beta(\psi))^2 + (1 + \mu - \mathring{z}_1)\mathring{z}_1)\mathring{z}_1 \\ 0 \end{pmatrix}\end{aligned}$$

is the nonlinearity in $\mathring{\mathbf{z}} = (\mathring{z}_1, \mathring{z}_2)$,

$$\begin{aligned}\mathring{\mathcal{R}}(\psi, \tilde{\psi}, \psi_t) = D\Big[&\phi_{yy}(\beta(\psi))\Big((1 + \psi_\zeta(1 + \psi_\zeta) + \psi\psi_{\zeta\zeta})^2 - (1 + \psi_\zeta)^2\Big) + \phi_{kk}(\beta(\psi))\psi_{\zeta\zeta}^2 \\ &+ 2\phi_{yk}(\beta(\psi))\psi_{\zeta\zeta}(1 + \psi_\zeta(1 + \psi_\zeta) + \psi\psi_{\zeta\zeta}) \\ &+ \phi_y(\beta(\psi))(\psi_{\zeta\zeta}(1 + 3\psi_\zeta) + \psi\psi_{\zeta\zeta\zeta}) + \phi_k(\beta(\psi))\psi_{\zeta\zeta\zeta}\Big] \\ &+ \phi_k(\beta(\psi))(c_0\psi_{\zeta\zeta} - \psi_{\zeta t}) \\ &+ \phi_y(\beta(\psi))\Big(c_0 + \omega'(1)\psi_\zeta - \omega(1 + \psi_\zeta) - \tilde{\psi} \\ &\qquad + c_0\Big(\psi_\zeta^2 + \psi\psi_{\zeta\zeta}\Big) - \psi_t\psi_\zeta - \psi\psi_{\zeta t}\Big)\end{aligned}$$

is the $\mathring{\mathbf{z}}$-independent residual and we used

$$\beta(\psi)(\zeta, t) = (\zeta + \psi(\zeta,t)(1 + \psi_\zeta(\zeta,t)); 1 + \psi_\zeta(\zeta,t))$$

to abbreviate the argument of the profile function $\phi(y;k) = (\phi_1(y;k), \phi_2(y;k))$ and its derivatives. We refer to Appendix 3.B for further details on the derivation of (3.77).

We proceed with deriving a nonlinear damping estimate for the modified forward-modulated perturbation $\mathring{\mathbf{z}}(t)$, which will be employed in the nonlinear stability argument to control regularity. A nonlinear damping estimate in $H^3(\mathbb{R}) \times H^2(\mathbb{R})$ for the "classical" forward-modulated perturbation $\mathring{\mathbf{w}}(t)$, given by (3.21), was established in [8, Proposition 8.6]. Here, we extend the method in [8] to nonlocalized perturbations by relying on the embedding of the uniformly local Sobolev space $H^1_{\mathrm{ul}}(\mathbb{R})$ in $C_{\mathrm{ub}}(\mathbb{R})$, see [94, Lemma 8.3.11].

The equation (3.77) for $\mathring{\mathbf{z}}(t)$ has a similar structure as the one for $\mathring{\mathbf{w}}(t)$ derived in [8]. That is, the second derivative $\partial_{\zeta\zeta}\mathring{z}_1$ yields damping in the first component of (3.77) and the term $-\varepsilon\gamma\mathring{z}_2$ yields damping in the second component. Since (3.77) is semilinear, all other linear and nonlinear terms can be controlled by these damping terms.

All in all, we arrive at the following result.

Proposition 3.4.9. *Assume* (H1) *and* (D3). *Fix $R>0$. Let $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$. Let $\psi(t)$ be as in Proposition 3.4.3 and let $\mathring{\mathbf{z}}(t)$ and $\widetilde{\tau}_{\max}$ be as in Corollary 3.4.8. There exist $\mathbf{w}_0$- and t-independent constants $C,\alpha>0$ such that the nonlinear damping estimate*

$$\begin{aligned}\|\mathring{\mathbf{z}}(t)\|_{C^2_{\mathrm{ub}}\times C^1_{\mathrm{ub}}} \leq C\Bigg(\|\mathring{\mathbf{z}}(t)\|_{L^\infty} + \|\mathring{\mathbf{z}}(t)\|_{L^\infty}^{\frac{1}{5}}\Bigg[\mathrm{e}^{-\alpha t}\|\mathbf{w}_0\|^2_{C^3_{\mathrm{ub}}\times C^2_{\mathrm{ub}}} + \int_0^t \mathrm{e}^{-\alpha(t-s)}\Big(\|\mathring{z}_1(s)\|^2_{L^\infty} \\ + \|\psi_{\zeta\zeta}(s)\|^2_{C^4_{\mathrm{ub}}} + \|\partial_s\psi_\zeta(s)\|^2_{C^3_{\mathrm{ub}}} + \|\tilde{\psi}(s)\|^2_{C^3_{\mathrm{ub}}} \\ + \|\psi_\zeta(s)\|^2_{L^\infty}\Big(\|\psi_\zeta(s)\|^2_{L^\infty} + \|\partial_s\psi(s)\|^2_{L^\infty}\Big)\Big)\,\mathrm{d}s\Bigg]^{\frac{2}{5}}\Bigg)\end{aligned} \tag{3.78}$$

holds for all $t \in [0,\widetilde{\tau}_{\max})$ with

$$\sup_{0\leq s\leq t}\Big(\|\mathring{z}_1(s)\|_{C^1_{\mathrm{ub}}} + \|\psi(s)\|_{C^3_{\mathrm{ub}}}\Big) \leq R. \tag{3.79}$$

Proof. Fix a constant $R>0$ and set

$$\vartheta = \frac{1}{2}\min\left\{1, \frac{\varepsilon\gamma}{2|c_0|+1}\right\}.$$

We start by relating the $(C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}})$-norm of $\mathring{\mathbf{z}}(t)$ to a uniformly local Sobolev norm. First, we define the window function $\varrho\colon \mathbb{R}\to\mathbb{R}$ given by

$$\varrho(\zeta) = \frac{2}{2+\zeta^2},$$

which is positive, smooth and L^1-integrable, and satisfies

$$|\varrho'(\zeta)| \leq \varrho(\zeta) \leq 1 \tag{3.80}$$

for all $\zeta\in\mathbb{R}$. Next, we apply the Gagliaro-Nirenberg interpolation inequality, while noting that $\varrho \in W^{k+1,1}(\mathbb{R}) \cap W^{k+1,\infty}(\mathbb{R})$ and $\varrho(0)=1$, to infer

$$\begin{aligned}\|\partial_\zeta^k z\|_{L^\infty} &= \sup_{y\in\mathbb{R}}\|\varrho(\vartheta(\cdot+y))\partial_\zeta^k z\|_{L^\infty} \lesssim \|z\|_{C^{k-1}_{\mathrm{ub}}} + \sup_{y\in\mathbb{R}}\big\|\partial_\zeta^k\big(\varrho(\vartheta(\cdot+y))z\big)\big\|_{L^\infty} \\ &\lesssim \|z\|_{C^{k-1}_{\mathrm{ub}}} + \sup_{y\in\mathbb{R}}\big\|\partial_\zeta^{k+1}\big(\varrho(\vartheta(\cdot+y))z\big)\big\|^{\frac{4}{5}}_{L^2}\big\|\varrho(\vartheta(\cdot+y))z\big\|^{\frac{1}{5}}_{L^{\frac{1}{2-k}}} \\ &\lesssim \|z\|_{C^{k-1}_{\mathrm{ub}}} + \|z\|^{\frac{1}{5}}_{L^\infty}\sup_{y\in\mathbb{R}}\big\|\partial_\zeta^{k+1}\big(\varrho(\vartheta(\cdot+y))z\big)\big\|^{\frac{4}{5}}_{L^2} \\ &\lesssim \|z\|_{C^{k-1}_{\mathrm{ub}}} + \|z\|^{\frac{4}{5}}_{C^k_{\mathrm{ub}}}\|z\|^{\frac{1}{5}}_{L^\infty} + \|z\|^{\frac{1}{5}}_{L^\infty}\sup_{y\in\mathbb{R}}\big\|\varrho(\vartheta(\cdot+y))\partial_\zeta^{k+1}z\big\|^{\frac{4}{5}}_{L^2}\end{aligned}$$

for $z \in C^{k+1}_{\mathrm{ub}}(\mathbb{R})$ and $k = 1, 2$. Hence, interpolating between $C^k_{\mathrm{ub}}(\mathbb{R})$ and $C_{\mathrm{ub}}(\mathbb{R})$, applying Young's inequality and rearranging terms, we arrive at

$$\|z\|_{C^k_{\mathrm{ub}}} \lesssim \|z\|_{L^\infty} + \|z\|_{L^\infty}^{\frac{1}{5}} \sup_{y\in\mathbb{R}} \|\varrho(\vartheta(\cdot + y))\partial_\zeta^{k+1} z\|_{L^2}^{\frac{4}{5}}$$

for $z \in C^{k+1}_{\mathrm{ub}}(\mathbb{R})$ and $k = 1, 2$. Combining the latter with (3.80) and recalling Corollary 3.4.8, yields

$$\|\mathring{\mathbf{z}}(t)\|_{C^2_{\mathrm{ub}}\times C^1_{\mathrm{ub}}} \lesssim \|\mathring{\mathbf{z}}(t)\|_{L^\infty} + \|\mathring{\mathbf{z}}(t)\|_{L^\infty}^{\frac{1}{5}} \sup_{y\in\mathbb{R}} E_y(t)^{\frac{2}{5}} \tag{3.81}$$

for $t \in [0, \widetilde{\tau}_{\max})$, where we denote

$$E_y(t) = \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big(\upsilon\Big|\partial_\zeta^3 \mathring{z}_1(\zeta, t)\Big|^2 + \Big|\partial_\zeta^2 \mathring{z}_2(\zeta, t)\Big|^2\Big)\,\mathrm{d}\zeta, \qquad \upsilon := \frac{\varepsilon\gamma}{4} > 0$$

for $y \in \mathbb{R}$. The estimate (3.81) provides the desired relationship between the $(C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}})$-norm of $\mathring{\mathbf{z}}(t)$ and the family of energies $E_y(t)$, which are associated with the norm on the uniformly local Sobolev space $H^3_{\mathrm{ul}}(\mathbb{R}) \times H^2_{\mathrm{ul}}(\mathbb{R})$ with dilation parameter ϑ, cf. [94, Section 8.3.1].

Our next step is to derive an inequality for the energies $E_y(t)$. In order to be able to differentiate $E_y(t)$ with respect to t, we restrict ourselves for the moment to initial conditions $\mathbf{w}_0 \in C^5_{\mathrm{ub}}(\mathbb{R}) \times C^4_{\mathrm{ub}}(\mathbb{R})$. With these two additional degrees of regularity one derives, analogously as in Proposition 3.4.1, that $\widetilde{\mathbf{w}} \in C\big([0, T_{\max}), C^5_{\mathrm{ub}}(\mathbb{R}) \times C^4_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0, T_{\max}), C^3_{\mathrm{ub}}(\mathbb{R})\big)$. Combining the latter with Propositions 3.1.2 and 3.4.3 yields $\mathring{\mathbf{z}} \in C\big([0, \widetilde{\tau}_{\max}), C^5_{\mathrm{ub}}(\mathbb{R}) \times C^4_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0, \widetilde{\tau}_{\max}), C^3_{\mathrm{ub}}(\mathbb{R})\big)$.

Let $y \in \mathbb{R}$ and let $t \in [0, \widetilde{\tau}_{\max})$ be such that (3.79) holds. Using (3.77) and

$$F'(0) = \begin{pmatrix} -\mu & -1 \\ \varepsilon & -\varepsilon\gamma \end{pmatrix},$$

while noting that the second component of $\mathring{\mathcal{Q}}(\mathring{\mathbf{z}}, \psi)$ vanishes, we compute

$$\frac{1}{2}\partial_s E_y(s) = I + II + III + IV, \tag{3.82}$$

where

$$\begin{aligned} I &= \upsilon \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big\langle \partial_\zeta^3 \mathring{z}_1(\zeta, s), \partial_\zeta^5 \mathring{z}_1(\zeta, s) + c_0 \partial_\zeta^4 \mathring{z}_1(\zeta, s) - \partial_\zeta^3 \mathring{z}_2(\zeta, s) - \mu \partial_\zeta^3 \mathring{z}_1(\zeta, s)\Big\rangle\,\mathrm{d}\zeta, \\ II &= \frac{c_0}{2} \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\partial_\zeta \Big|\partial_\zeta^2 \mathring{z}_2(\zeta, s)\Big|^2\,\mathrm{d}\zeta \\ &\quad + \varepsilon \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big\langle \partial_\zeta^2 \mathring{z}_2(\zeta, s), \partial_\zeta^2 \mathring{z}_1(\zeta, s) - \gamma \partial_\zeta^2 \mathring{z}_2(\zeta, s)\Big\rangle\,\mathrm{d}\zeta, \end{aligned}$$

are the contributions from the linear terms, and

$$\begin{aligned} III &= \upsilon \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\left\langle \begin{pmatrix} \partial_\zeta^3 \mathring{z}_1(\zeta, s) \\ 0 \end{pmatrix}, \partial_\zeta^3 \Big(\mathring{\mathcal{Q}}(\mathring{\mathbf{z}}(\zeta, s), \psi(\zeta, s)) \right. \\ &\qquad \left. + \mathring{\mathcal{R}}(\psi(\zeta, s), \tilde{\psi}(\zeta, s), \partial_s \psi(\zeta, s))\Big)\right\rangle \mathrm{d}\zeta, \\ IV &= \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\left\langle \begin{pmatrix} 0 \\ \partial_\zeta^2 \mathring{z}_2(\zeta, s) \end{pmatrix}, \partial_\zeta^2 \mathring{\mathcal{R}}(\psi(\zeta, s), \tilde{\psi}(\zeta, s), \partial_s \psi(\zeta, s))\right\rangle \mathrm{d}\zeta \end{aligned}$$

are the nonlinear contributions for $s \in [0,t]$. Integrating by parts, we rewrite

$$\begin{aligned} I = &-\upsilon \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big(\left|\partial_\zeta^4 \mathring{z}_1(\zeta,s)\right|^2 + \mu\left|\partial_\zeta^3 \mathring{z}_1(\zeta,s)\right|^2 - \left\langle \partial_\zeta^4 \mathring{z}_1(\zeta,s), \partial_\zeta^2 \mathring{z}_2(\zeta,s)\right\rangle\Big)\,\mathrm{d}\zeta \\ &- \upsilon\vartheta \int_{\mathbb{R}} \varrho'(\vartheta(\zeta+y))\left\langle \partial_\zeta^3 \mathring{z}_1(\zeta,s), \partial_\zeta^4 \mathring{z}_1(\zeta,s) - \partial_\zeta^2 \mathring{z}_2(\zeta,s)\right\rangle \mathrm{d}\zeta \\ &+ c_0\upsilon \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left\langle \partial_\zeta^3 \mathring{z}_1(\zeta,s), \partial_\zeta^4 \mathring{z}_1(\zeta,s)\right\rangle \mathrm{d}\zeta \end{aligned}$$

and

$$\begin{aligned} II = &-\varepsilon\gamma \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^2 \mathring{z}_2(\zeta,s)\right|^2 \mathrm{d}\zeta - \frac{c_0\vartheta}{2}\int_{\mathbb{R}} \varrho'(\vartheta(\zeta+y))\left|\partial_\zeta^2 \mathring{z}_2(\zeta,s)\right|^2 \mathrm{d}\zeta \\ &+ \varepsilon \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left\langle \partial_\zeta^2 \mathring{z}_2(\zeta,s), \partial_\zeta^2 \mathring{z}_1(\zeta,s)\right\rangle \mathrm{d}\zeta \end{aligned}$$

for $s \in [0,t]$. Applying Young's inequality to the latter, while using (3.80) and $4|c_0|\vartheta \leq \varepsilon\gamma$, yields a t- and $\mathbf{w}_0$-independent constant $C_1 > 0$ such that

$$\begin{aligned} I \leq &-\frac{\upsilon}{4}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^4 \mathring{z}_1(\zeta,s)\right|^2 \mathrm{d}\zeta + \upsilon \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^2 \mathring{z}_2(\zeta,s)\right|^2 \mathrm{d}\zeta \\ &+ C_1 \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^3 \mathring{z}_1(\zeta,s)\right|^2 \mathrm{d}\zeta \end{aligned} \tag{3.83}$$

and

$$II \leq -\frac{3\varepsilon\gamma}{4}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^2 \mathring{z}_2(\zeta,s)\right|^2 \mathrm{d}\zeta + C_1 \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^2 \mathring{z}_1(\zeta,s)\right|^2 \mathrm{d}\zeta \tag{3.84}$$

for $s \in [0,t]$. Similarly, employing Young's inequality, while using that (3.79) holds and ρ is L^1-integrable, we establish a t- and $\mathbf{w}_0$-independent constant $C_2 > 0$ such that

$$\begin{aligned} III \leq C_2\bigg(&\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big(\left|\partial_\zeta^3 \mathring{z}_1(\zeta,s)\right|^2 + \left|\partial_\zeta^2 \mathring{z}_1(\zeta,s)\right|^2 + |\partial_\zeta \mathring{z}_1(\zeta,s)|^2\Big)\,\mathrm{d}\zeta \\ &+ \|\mathring{z}_1(s)\|_{L^\infty}^2 + \|\psi_{\zeta\zeta}(s)\|_{C^4_{\mathrm{ub}}}^2 + \|\partial_s\psi_\zeta(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\tilde{\psi}(s)\|_{C^3_{\mathrm{ub}}}^2 \\ &+ \|\psi_\zeta(s)\|_{L^\infty}^2\Big(\|\psi_\zeta(s)\|_{L^\infty}^2 + \|\partial_s\psi(s)\|_{L^\infty}^2\Big)\bigg) \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} IV \leq \frac{\varepsilon\gamma}{4}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^2 \mathring{z}_2(\zeta,s)\right|^2 \mathrm{d}\zeta + C_2\Big(&\|\psi_{\zeta\zeta}(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\partial_s\psi_\zeta(s)\|_{C^2_{\mathrm{ub}}}^2 + \|\tilde{\psi}(s)\|_{C^2_{\mathrm{ub}}}^2 \\ &+ \|\psi_\zeta(s)\|_{L^\infty}^2\Big(\|\psi_\zeta(s)\|_{L^\infty}^2 + \|\partial_s\psi(s)\|_{L^\infty}^2\Big)\Big) \end{aligned} \tag{3.86}$$

for $s \in [0,t]$. Applying the estimates (3.83), (3.84), (3.85) and (3.86) to (3.82) and using that $\upsilon = \varepsilon\gamma/4$, we obtain a t- and $\mathbf{w}_0$-independent constant $C_3 > 0$ such that

$$\begin{aligned} \frac{1}{2}\partial_s E_y(s) \leq -\frac{\varepsilon\gamma}{4}E_y(s) - \frac{\upsilon}{4}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\left|\partial_\zeta^4 \mathring{z}_1(\zeta,s)\right|^2 \mathrm{d}\zeta + C_3\Big(&\|\mathring{z}_1(s)\|_{L^\infty}^2 + \|\psi_{\zeta\zeta}(s)\|_{C^4_{\mathrm{ub}}}^2 \\ + \|\partial_s\psi_\zeta(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\tilde{\psi}(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\psi_\zeta(s)\|_{L^\infty}^2\Big(\|\psi_\zeta(s)\|_{L^\infty}^2 + \|\partial_s\psi(s)\|_{L^\infty}^2\Big) \\ + \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big(\left|\partial_\zeta^3 \mathring{z}_1(\zeta,s)\right|^2 + \left|\partial_\zeta^2 \mathring{z}_1(\zeta,s)\right|^2 + |\partial_\zeta \mathring{z}_1(\zeta,s)|^2\Big)\,\mathrm{d}\zeta\Big) \end{aligned} \tag{3.87}$$

for $s \in [0,t]$.

We control the term on the last line of (3.87) by deriving an interpolation inequality. To this end, we take $k \in \mathbb{N}$, $\eta \in (0, \frac{1}{4})$ and $a_1, \ldots, a_k > 0$. Integration by parts, Young's inequality, and the estimate (3.80) yield

$$\begin{aligned}
&\sum_{j=1}^{k} a_j \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left|\partial_\zeta^j z(\zeta)\right|^2 \mathrm{d}\zeta \\
&= -\sum_{j=1}^{k} a_j \int_{\mathbb{R}} \Big(\varrho(\vartheta(\zeta+y)) \left\langle \partial_\zeta^{j+1} z(\zeta), \partial_\zeta^{j-1} z(\zeta) \right\rangle \mathrm{d}\zeta + \vartheta \varrho'(\vartheta(\zeta+y)) \left\langle \partial_\zeta^{j} z(\zeta), \partial_\zeta^{j-1} z(\zeta) \right\rangle \Big) \mathrm{d}\zeta \\
&\leq \sum_{j=1}^{k} \frac{a_j}{2} \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left(\eta \left|\partial_\zeta^{j+1} z(\zeta)\right|^2 + \vartheta \left|\partial_\zeta^{j} z(\zeta)\right|^2 + \left(\frac{1}{\eta} + \vartheta\right) \left|\partial_\zeta^{j-1} z(\zeta)\right|^2 \right) \mathrm{d}\zeta
\end{aligned}$$

for $z \in C_{\mathrm{ub}}^{k+1}(\mathbb{R})$. Setting $a_0 = 0 = a_{k+1}$, using $\vartheta \leq \frac{1}{2}$ and rearranging terms in the latter, we arrive at the interpolation inequality

$$\begin{aligned}
&\sum_{j=1}^{k} \left(\frac{3}{4} a_j - \frac{\eta}{2} a_{j-1} - \frac{1}{2} a_{j+1} \left(\frac{1}{\eta} + \frac{1}{2} \right) \right) \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left|\partial_\zeta^j z(\zeta)\right|^2 \mathrm{d}\zeta \\
&\qquad \leq \frac{\eta}{2} a_k \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left|\partial_\zeta^{k+1} z(\zeta)\right|^2 \mathrm{d}\zeta + \frac{1}{2} a_1 \left(\frac{1}{\eta} + \frac{1}{2} \right) \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) |z(\zeta)|^2 \mathrm{d}\zeta
\end{aligned} \tag{3.88}$$

for $z \in C_{\mathrm{ub}}^{k+1}(\mathbb{R})$. Next, we fix $k = 3$ and solve the linear system

$$\frac{3}{4} a_j - \frac{\eta}{2} a_{j-1} - \frac{1}{2} a_{j+1} \left(\frac{1}{\eta} + \frac{1}{2} \right) = 1, \qquad j = 1,2,3,$$

yielding the solution

$$a_1 = \frac{4(4 - 2\eta^3 + 9\eta^2 + 10\eta)}{3\eta^2(1-4\eta)}, \qquad a_2 = \frac{4(2 + 2\eta^2 + 4\eta)}{\eta(1-4\eta)}, \qquad a_3 = \frac{4(5 + 4\eta^2 + 4\eta)}{3(1-4\eta)}.$$

where we have $a_1, a_2, a_3 > 0$ since $\eta < \frac{1}{4}$. Thus, taking these values for a_1, a_2, a_3 in (3.88), we find

$$\begin{aligned}
\sum_{j=1}^{3} \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left|\partial_\zeta^j z(\zeta)\right|^2 \mathrm{d}\zeta \leq{}& \frac{2\eta(5 + 4\eta^2 + 4\eta)}{3(1-4\eta)} \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left|\partial_\zeta^4 z(\zeta)\right|^2 \mathrm{d}\zeta \\
&+ \frac{(\eta+2)(4 - 2\eta^3 + 9\eta^2 + 10\eta)}{3\eta^3(1-4\eta)} \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) |z(\zeta)|^2 \mathrm{d}\zeta
\end{aligned}$$

for $z \in C_{\mathrm{ub}}^4(\mathbb{R})$. So, taking $\eta \in (0, \frac{1}{4})$ so small that

$$\frac{2\eta(5 + 4\eta^2 + 4\eta)}{3(1-4\eta)} \leq \frac{\upsilon}{4C_3},$$

we establish a constant $C_4 > 0$ such that

$$\sum_{j=1}^{3} \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left|\partial_\zeta^j z(\zeta)\right|^2 \mathrm{d}\zeta \leq \frac{\upsilon}{4C_3} \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y)) \left|\partial_\zeta^4 z(\zeta)\right|^2 \mathrm{d}\zeta + C_4 \|z\|_{L^\infty}^2 \tag{3.89}$$

for $z \in C_{\mathrm{ub}}^4(\mathbb{R})$.

We apply the interpolation identity (3.89) to (3.87) and deduce

$$\partial_s E_y(s) \le -\frac{\varepsilon\gamma}{2} E_y(s) + C_5\Big(\|\mathring{z}_1(s)\|_{L^\infty}^2 + \|\psi_{\zeta\zeta}(s)\|_{C^4_{\mathrm{ub}}}^2 + \|\partial_s\psi_\zeta(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\tilde{\psi}(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\psi_\zeta(s)\|_{L^\infty}^2 \Big(\|\psi_\zeta(s)\|_{L^\infty}^2 + \|\partial_s\psi(s)\|_{L^\infty}^2 \Big)\Big)$$

for $s \in [0,t]$, where $C_5 > 0$ is a t- and $\mathbf{w}_0$-independent constant. Multiplying the latter inequality with $\mathrm{e}^{\frac{\varepsilon\gamma}{2}s}$ and integrating, we acquire

$$E_y(t) \le \mathrm{e}^{-\frac{\varepsilon\gamma}{2}t} E_y(0) + C_5 \int_0^t \mathrm{e}^{-\frac{\varepsilon\gamma}{2}(t-s)} \Big(\|\mathring{z}_1(s)\|_{L^\infty}^2 + \|\psi_{\zeta\zeta}(s)\|_{C^4_{\mathrm{ub}}}^2 + \|\partial_s\psi_\zeta(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\tilde{\psi}(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\psi_\zeta(s)\|_{L^\infty}^2 \Big(\|\psi_\zeta(s)\|_{L^\infty}^2 + \|\partial_s\psi(s)\|_{L^\infty}^2 \Big)\Big) \mathrm{d}s.$$

Lastly, using that there exists a $\mathbf{w}_0$-independent constant $C_6 > 0$ such that

$$E_y(0) \le C_6 \|\mathbf{w}_0\|_{C^3_{\mathrm{ub}} \times C^2_{\mathrm{ub}}}^2$$

and plugging the latter estimate into (3.81), we arrive at (3.78).

In order to extend our result to the general case $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$ we argue as in the proof of [8, Proposition 8.6]. That is, we approximate the initial condition $\mathbf{w}_0$ in $C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$ by a sequence $(\mathbf{w}_{0,n})_{n\in\mathbb{N}}$ in $C^5_{\mathrm{ub}}(\mathbb{R}) \times C^4_{\mathrm{ub}}(\mathbb{R})$. By continuity of solutions with respect to initial data and the fact that (3.78) only depends on the $(C^3_{\mathrm{ub}} \times C^2_{\mathrm{ub}})$-norm of $\mathring{\mathbf{z}}(t)$, the desired result follows by approximation. We refer to [8] for further details. □

Remark 3.4.10. *In addition to the fact that we extend the proof of the nonlinear damping estimate in [8, Proposition 8.6] to nonlocalized perturbations by employing an energy associated with uniformly local Sobolev norms, our analysis deviates from the one in [8] in another important way: rather than using the bound $\|\partial_\zeta^k w\|_{L^\infty} \le \|\partial_\zeta^k w\|_{H^1}$, we employ the Gagliardo-Nirenberg interpolation inequality*

$$\|\partial_\zeta^k w\|_{L^\infty} \le \|\partial_\zeta^{k+1} w\|_{L^2}^{\frac{4}{5}} \|w\|_{L^{\frac{1}{2-k}}}^{\frac{1}{5}},$$

for $w \in H^{k+1}(\mathbb{R}) \cap L^1(\mathbb{R})$ and $k = 1,2$. This leads to the additional factor $\|\mathring{\mathbf{z}}(t)\|_{L^\infty}^{1/5}$ in the nonlinear damping estimate (3.78)*, enabling us to only require that the L^∞-norm of the initial perturbation $\mathbf{w}_0$ is small (and its $(C^3_{\mathrm{ub}} \times C^2_{\mathrm{ub}})$-norm is bounded) in our nonlinear stability result, Theorem 3.1.4. We expect that a similar approach can be adopted to relax the smallness condition on initial data in [8].*

It has been argued in [105, Corollary 5.3] that, as long as $\|\psi_\zeta(t)\|_{L^\infty}$ stays sufficiently small, the Sobolev norms of the forward- and inverse-modulated perturbation $\mathring{\mathbf{w}}(t)$ and $\mathbf{w}(t)$ are equivalent modulo Sobolev norms of $\psi_\zeta(t)$ and its derivatives. We extend this result by proving norm equivalence of the modified forward-modulated perturbation $\mathring{\mathbf{z}}(t)$ and the residual $\mathbf{z}(t)$ (up to controllable errors in $\psi_\zeta(t)$ and its derivatives).

Lemma 3.4.11. *Let $\psi(t)$ be as in Proposition 3.4.3, let $\mathbf{z}(t)$ be as in Corollary 3.4.4 and let $\mathring{\mathbf{z}}(t)$ and $\tilde{\tau}_{\max}$ be as in Corollary 3.4.8. Then, there exists a constant $C > 0$ such that*

$$\|\mathbf{z}(t)\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} \le C\Big(\|\mathring{\mathbf{z}}(t)\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} + \|\psi_{\zeta\zeta}(t)\|_{C^1_{\mathrm{ub}}} + \|\psi_\zeta(t)\|_{L^\infty}^2 \Big), \tag{3.90}$$

and

$$\left\|\mathring{\mathbf{z}}(t)\right\|_{L^\infty} \leq C\Big(\|\mathbf{z}(t)\|_{L^\infty} + \|\psi_{\zeta\zeta}(t)\|_{L^\infty} + \|\psi_\zeta(t)\|_{L^\infty}^2\Big) \tag{3.91}$$

for any $t \in [0, \widetilde{\tau}_{\max})$.

Proof. Inserting $\mathbf{w}(\zeta,t) = \mathbf{u}(\zeta - \psi(\zeta,t), t) - \phi_0(\zeta)$ into (3.59) and using (3.22) to reexpress $\mathbf{u}(\zeta - \psi(\zeta,t), t)$, we arrive at

$$\mathbf{z}(\zeta,t) = \mathring{\mathbf{z}}(a(\zeta,t),t) - \phi_0(\zeta) - \phi_k(\zeta;1)\psi_\zeta(\zeta,t) + \phi(b(\zeta,t); c(\zeta,t)) \tag{3.92}$$

for $\zeta \in \mathbb{R}$ and $t \in [0, \widetilde{\tau}_{\max})$, where we abbreviate

$$a(\zeta,t) = \zeta - \psi(\zeta,t), \qquad b(\zeta,t) = \zeta + \psi(\zeta - \psi(\zeta,t),t)(1 + \psi_\zeta(\zeta - \psi(\zeta,t),t)) - \psi(\zeta,t)$$

and

$$c(\zeta,t) = 1 + \psi_\zeta(\zeta - \psi(\zeta,t),t).$$

Differentiating the latter with respect to ζ yields

$$\begin{aligned}\mathbf{z}_\zeta(\zeta,t) = {} & \mathring{\mathbf{z}}_\zeta(a(\zeta,t),t)a_\zeta(\zeta,t) - \phi_0'(\zeta) - \phi_{k\zeta}(\zeta;1)\psi_\zeta(\zeta,t) - \phi_k(\zeta;1)\psi_{\zeta\zeta}(\zeta,t) \\ & + \phi_\zeta(b(\zeta,t);c(\zeta,t))b_\zeta(\zeta,t) + \phi_k(b(\zeta,t);c(\zeta,t))c_\zeta(\zeta,t)\end{aligned} \tag{3.93}$$

and

$$\begin{aligned}\mathbf{z}_{\zeta\zeta}(\zeta,t) = {} & \mathring{\mathbf{z}}_{\zeta\zeta}(a(\zeta,t),t)a_\zeta(\zeta,t)^2 + \mathring{\mathbf{z}}_\zeta(a(\zeta,t),t)a_{\zeta\zeta}(\zeta,t) - \phi_0''(\zeta) - \phi_{k\zeta\zeta}(\zeta;1)\psi_\zeta(\zeta,t) \\ & - 2\phi_{k\zeta}(\zeta;1)\psi_{\zeta\zeta}(\zeta,t) - \phi_k(\zeta;1)\psi_{\zeta\zeta\zeta}(\zeta,t) + \phi_{\zeta\zeta}(b(\zeta,t);c(\zeta,t))b_\zeta(\zeta,t)^2 \\ & + 2\phi_{k\zeta}(b(\zeta,t);c(\zeta,t))b_\zeta(\zeta,t)c_\zeta(\zeta,t) + \phi_{kk}(b(\zeta,t);c(\zeta,t))c_\zeta(\zeta,t)^2 \\ & + \phi_\zeta(b(\zeta,t);c(\zeta,t))b_{\zeta\zeta}(\zeta,t) + \phi_k(b(\zeta,t);c(\zeta,t))c_{\zeta\zeta}(\zeta,t)\end{aligned} \tag{3.94}$$

for $\zeta \in \mathbb{R}$ and $t \in [0, \widetilde{\tau}_{\max})$.

Next, we use Taylor's theorem to bound

$$\begin{aligned}|b(\zeta,t) - \zeta| &\leq |\psi(\zeta - \psi(\zeta,t),t) - \psi(\zeta,t) + \psi_\zeta(\zeta,t)\psi(\zeta,t)| \\ &\qquad + |\psi_\zeta(\zeta - \psi(\zeta,t),t)\psi(\zeta - \psi(\zeta,t),t) - \psi_\zeta(\zeta,t)\psi(\zeta,t)| \\ &\lesssim \|\psi(t)\|_{L^\infty}\Big(\|\psi_{\zeta\zeta}(t)\|_{L^\infty}\|\psi(t)\|_{L^\infty} + \|\psi_\zeta(t)\|_{L^\infty}^2\Big), \\ |c(\zeta,t) - 1 - \psi_\zeta(\zeta,t)| &\leq \|\psi(t)\|_{L^\infty}\|\psi_{\zeta\zeta}(t)\|_{L^\infty},\end{aligned} \tag{3.95}$$

and

$$\begin{aligned}|b_\zeta(\zeta,t) - 1| &\leq |\psi_\zeta(\zeta - \psi(\zeta,t),t)(1 - \psi_\zeta(\zeta,t)) - \psi_\zeta(\zeta,t)| \\ &\qquad + \Big|\psi_{\zeta\zeta}(\zeta - \psi(\zeta,t),t)\psi(\zeta - \psi(\zeta,t),t) + \psi_\zeta(\zeta - \psi(\zeta,t),t)^2\Big||1 - \psi_\zeta(\zeta,t)| \\ &\lesssim \Big(\|\psi(t)\|_{L^\infty}\|\psi_{\zeta\zeta}(t)\|_{L^\infty} + \|\psi_\zeta(t)\|_{L^\infty}^2\Big)(1 + \|\psi_\zeta(t)\|_{L^\infty})\end{aligned} \tag{3.96}$$

for $\zeta \in \mathbb{R}$ and $t \in [0, \widetilde{\tau}_{\max})$. Recall from Proposition 3.1.2 that $\phi\colon \mathbb{R} \times [1 - r_0, 1 + r_0] \to \mathbb{R}^2$ is smooth. So, applying Taylor's theorem and estimate (3.95), while recalling from

Proposition 3.4.3 and Corollary 3.4.8 that $\|\psi(t)\|_{C^2_{\mathrm{ub}}} < \frac{1}{2}$ and $\|\psi_\zeta(t)\|_{L^\infty} < r_0$, we infer the bounds

$$\begin{aligned}\left|(\partial_\zeta^j \phi)(b(\zeta,t); c(\zeta,t)) - (\partial_\zeta^j \phi)(\zeta; c(\zeta,t))\right| &\leq |b(\zeta,t)-\zeta| \sup_{|k-1|\leq r_0} \left\|\partial_\zeta^{j+1}\phi(\cdot;k)\right\|_{L^\infty} \\ &\lesssim \|\psi_{\zeta\zeta}(t)\|_{L^\infty} + \|\psi_\zeta(t)\|_{L^\infty}^2, \\ \left|(\partial_\zeta^j \phi)(\zeta; c(\zeta,t)) - (\partial_\zeta^j \phi)(\zeta; 1+\psi_\zeta(\zeta,t))\right| &\leq |c(\zeta,t) - 1 - \psi_\zeta(\zeta,t)| \\ &\quad \cdot \sup_{|k-1|\leq r_0} \left\|\partial_\zeta^j \phi_k(\cdot;k)\right\|_{L^\infty} \lesssim \|\psi_{\zeta\zeta}(t)\|_{L^\infty}\end{aligned} \tag{3.97}$$

and

$$\begin{aligned}\left|(\partial_\zeta^j \phi)(\zeta; 1+\psi_\zeta(\zeta,t)) - \partial_\zeta^j \phi_0(\zeta) - (\partial_\zeta^j \phi_k)(\zeta;1)\psi_\zeta(\zeta,t)\right| &\lesssim |\psi_\zeta(\zeta,t)|^2 \left\|\partial_\zeta^j \phi_{kk}(\cdot;1)\right\|_{L^\infty} \\ &\lesssim \|\psi_\zeta(t)\|_{L^\infty}^2\end{aligned} \tag{3.98}$$

for $\zeta \in \mathbb{R}$, $t \in [0, \widetilde{\tau}_{\max})$ and $j = 0,1,2$. Using again $\|\psi(t)\|_{C^2_{\mathrm{ub}}} < \frac{1}{2}$, we obtain

$$\|a(\cdot,t)\|_{C^2_{\mathrm{ub}}} \lesssim 1, \qquad \|c_\zeta(\cdot,t)\|_{C^1_{\mathrm{ub}}}, \|b_{\zeta\zeta}(\cdot,t)\|_{L^\infty} \lesssim \|\psi_{\zeta\zeta}(t)\|_{C^1_{\mathrm{ub}}} \tag{3.99}$$

for $\zeta \in \mathbb{R}$ and $t \in [0, \widetilde{\tau}_{\max})$.

Finally, applying the bounds (3.96), (3.97), (3.98) and (3.99) to (3.92), (3.93) and (3.94), while recalling that ϕ is smooth, one readily infers (3.90). Similarly, applying (3.97) and (3.98) to (3.92), we establish

$$\left\|\mathring{\mathbf{z}}(a(\cdot,t),t)\right\|_{L^\infty} \lesssim \|\mathbf{z}(t)\|_{L^\infty} + \|\psi_{\zeta\zeta}(t)\|_{L^\infty} + \|\psi_\zeta(t)\|_{L^\infty}^2 \tag{3.100}$$

for $t \in [0, \widetilde{\tau}_{\max})$. Since we have $\|\psi_\zeta(t)\|_{L^\infty} < \frac{1}{2}$, it holds $a_\zeta(\zeta,t) \geq \frac{1}{2}$ for all $\zeta \in \mathbb{R}$ and the function $a(\cdot,t)\colon \mathbb{R} \to \mathbb{R}$ is bijective for each $t \in [0, \widetilde{\tau}_{\max})$. Consequently, we have $\left\|\mathring{\mathbf{z}}(a(\cdot,t),t)\right\|_{L^\infty} = \left\|\mathring{\mathbf{z}}(\cdot,t)\right\|_{L^\infty}$ for each $t \in [0, \widetilde{\tau}_{\max})$, which yields (3.91) upon invoking (3.100). □

3.5. Nonlinear stability argument

We prove our nonlinear stability result, Theorem 3.1.4, by applying the linear bounds, obtained in Theorem 3.3.9, and the nonlinear bounds, established in Lemmas 3.4.2, 3.4.5 and 3.4.7, to iteratively estimate the phase modulation $\psi(t)$, the residuals $\mathbf{z}(t)$ and $r(t)$, and the Cole-Hopf variable $y(t)$ through their respective Duhamel formulations (3.61), (3.63), (3.68) and (3.76). We control regularity in the scheme via the nonlinear damping estimate in Proposition 3.4.9.

Proof of Theorem 3.1.4. Take $\mathbf{w}_0 \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^2_{\mathrm{ub}}(\mathbb{R})$ with $\|\mathbf{w}_0\|_{C^3_{\mathrm{ub}} \times C^2_{\mathrm{ub}}} < K$. Propositions 3.4.1 and 3.4.3, Corollaries 3.4.4 and 3.4.6, and identity (3.72) yield that the template function $\eta\colon [0, \tau_{\max}) \to \mathbb{R}$ given by

$$\eta(t) = \eta_1(t) + \eta_2(t)^5,$$

with

$$\begin{aligned}\eta_1(t) = \sup_{0 \leq s \leq t} \Bigg[\|\psi(s)\|_{L^\infty} + \|y(s)\|_{L^\infty} + \sqrt{s}\,\|y_\zeta(s)\|_{L^\infty} + \sqrt{1+s}\left(\frac{\|r(s)\|_{L^\infty} + \sqrt{s}\,\|r_\zeta(s)\|_{L^\infty}}{\log(2+s)}\right. \\ \left. + \|\psi_\zeta(s)\|_{L^\infty}\right) + \frac{1+s}{\log(2+s)}\left(\|\mathbf{z}(s)\|_{L^\infty} + \|\psi_{\zeta\zeta}(s)\|_{C^4_{\mathrm{ub}}} + \|\tilde{\psi}(s)\|_{C^4_{\mathrm{ub}}}\right)\Bigg],\end{aligned}$$

and

$$\eta_2(t) = \sup_{0 \le s \le t} \|\widetilde{\mathbf{w}}(t)\|_{C^1_{\mathrm{ub}}}$$

is well-defined, positive, monotonically increasing and continuous, where we recall $\tilde{\psi}(t) = \partial_t \psi(t) + c_g \psi_\zeta(t)$. In addition, if $\tau_{\max} < \infty$, then we have

$$\lim_{t \uparrow \tau_{\max}} \eta(t) \ge \frac{1}{2}. \tag{3.101}$$

We refer to Remarks 3.5.1 and 3.5.2 for motivation for the choice of template function.

Approach. Let $r_0 > 0$ be the constant from Proposition 3.1.2. As usual in nonlinear iteration arguments, our goal is to prove a nonlinear inequality for the template function $\eta(t)$. Specifically, we show that there exists a constant $C > 1$ such that for all $t \in [0, \tau_{\max})$ with $\eta(t) \le \frac{1}{2}\min\{1, r_0\}$ we have the key inequality

$$\eta(t) \le C\Big(E_0 + \eta(t)^{\frac{6}{5}}\Big), \tag{3.102}$$

where we denote $E_0 := \|\mathbf{w}_0\|_{L^\infty}$. We note that by interpolation there exists an E_0-independent constant $C_0 > 0$ such that it holds $\|\mathbf{w}_0\|_{C^1_{\mathrm{ub}}} \le C_0\sqrt{E_0}$ as long as $E_0 \le 1$. So, recalling that $\psi(0)$ vanishes identically by Proposition 3.4.3 and using (3.69) and (3.74), we find an E_0-independent constant $C_* > 0$ such that $\eta(0) \le C_* E_0$ as long as $E_0 \le 1$. Subsequently, we set

$$M_0 = 2\max\{C, C_*\} > 2, \qquad \epsilon_0 = \min\left\{\frac{1}{M_0^6}, \frac{\min\{1, r_0\}}{2M_0}\right\} < 1.$$

Assuming that (3.102) holds, we claim that, provided $E_0 \in (0, \epsilon_0)$, we have $\eta(t) \le M_0 E_0$ for all $t \in [0, \tau_{\max})$. To prove the claim we, argue by contradiction and assume that there exists a $t \in [0, \tau_{\max})$ with $\eta(t) > M_0 E_0$. Since η is continuous and $\eta(0) \le C_* E_0 < M_0 E_0$, there must exist $t_0 \in (0, \tau_{\max})$ with $\eta(t_0) = M_0 E_0 \le \frac{1}{2}\min\{1, r_0\}$. Thus, applying (3.102) and using $E_0 < \epsilon_0$, we arrive at

$$\eta(t_0) \le C E_0 \left(1 + M_0^{\frac{6}{5}} E_0^{\frac{1}{5}}\right) < 2 C E_0 \le M_0 E_0,$$

which contradicts $\eta(t_0) = M_0 E_0$. Therefore, it must hold $\eta(t) \le M_0 E_0$ for all $t \in [0, \tau_{\max})$. Since $M_0 > 2$, we have $M_0 E_0 < \frac{1}{2}$, which implies $\tau_{\max} = \infty$ by (3.101), i.e., $\mathbf{u}(t) = \widetilde{\mathbf{w}}(t) + \phi_0$ is a global solution to (3.3) satisfying (3.9) by Proposition 3.4.1.

Our next step is thus to establish the key inequality (3.102). The estimates (3.10)-(3.13) and (3.16) then follow readily by employing applying Lemma 3.4.11 and using that $\eta(t) \le M_0 E_0$ holds for all $t \ge 0$.

Bounds on $\mathbf{w}(t)$ and $\partial_t\psi(t)$. Let $t \in [0, \tau_{\max})$ with $\eta(t) \le \frac{1}{2}\min\{1, r_0\}$. We bound $\mathbf{w}(s) = \mathbf{z}(s) + \partial_k \phi(\cdot; 1)\psi_\zeta(s)$ and $\partial_t \psi(s) = \tilde{\psi}(s) - c_g \psi_\zeta(s)$ as

$$\begin{aligned} \|\mathbf{w}(s)\|_{L^\infty} &\lesssim \|\mathbf{z}(s)\|_{L^\infty} + \|\psi_\zeta(s)\|_{L^\infty} \lesssim \frac{\eta_1(t)}{\sqrt{1+s}}, \\ \|\partial_t \psi(s)\|_{L^\infty} &\lesssim \|\tilde{\psi}(s)\|_{L^\infty} + \|\psi_\zeta(s)\|_{L^\infty} \lesssim \frac{\eta_1(t)}{\sqrt{1+s}} \end{aligned} \tag{3.103}$$

for $s \in [0, t]$.

Application of nonlinear damping estimate. Take $t \in [0, \tau_{\max})$ such that $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$. Then, we have $t < \widetilde{\tau}_{\max}$ by Corollary 3.4.8. Moreover, using identity (3.22), $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$ and the fact that $\phi\colon [1-r_0, 1+r_0] \times \mathbb{R} \to \mathbb{R}^2$ is smooth by Proposition 3.1.2, we find a t- and E_0-independent constant $R_0 > 0$ such that $\|\mathring{\mathbf{z}}(\tau)\|_{C^1_{\mathrm{ub}}} \leq R_0$ for $\tau \in [0, t]$. On the other hand, Lemma 3.4.11 implies

$$\|\mathring{\mathbf{z}}(\tau)\|_{L^\infty} \lesssim \eta_1(t)\frac{\log(2+\tau)}{1+\tau}$$

for $\tau \in [0, t]$, where we use that $\eta_1(t) \leq \frac{1}{2}$. Hence, employing the nonlinear damping estimate in Proposition 3.4.9, while using $\eta_1(t) \leq \frac{1}{2}$ and $\|\mathring{\mathbf{z}}(\tau)\|_{C^1_{\mathrm{ub}}} \leq R_0$ for $\tau \in [0, t]$, we arrive at

$$\begin{aligned}\|\mathring{\mathbf{z}}(s)\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} &\lesssim \eta_1(t)\frac{\log(2+s)}{1+s} + \left(\eta_1(t)\frac{\log(2+s)}{1+s}\right)^{\frac{1}{5}}\left(\mathrm{e}^{-\alpha s} + \int_0^s \frac{\log(2+\tau)^2}{\mathrm{e}^{\alpha(s-\tau)}(1+\tau)^2}\,\mathrm{d}\tau\right)^{\frac{2}{5}} \\ &\lesssim \eta_1(t)^{\frac{1}{5}}\frac{\log(2+s)}{1+s}\end{aligned} \tag{3.104}$$

for $s \in [0, t]$. We combine the latter with Lemma 3.4.11 and use $\eta_1(t) \leq \frac{1}{2}$ to obtain

$$\|\mathbf{z}(s)\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} \lesssim \eta_1(t)^{\frac{1}{5}}\frac{\log(2+s)}{1+s} \tag{3.105}$$

for $s \in [0, t]$. Therefore, recalling $\mathbf{w}(s) = \mathbf{z}(s) + \partial_k\phi(\cdot; 1)\psi_\zeta(s)$ and using $\eta_1(t) \leq \frac{1}{2}$, the latter estimate yields

$$\|\mathbf{w}(s)\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} \lesssim \|\mathbf{z}(s)\|_{C^2_{\mathrm{ub}} \times C^1_{\mathrm{ub}}} + \|\psi_\zeta(s)\|_{C^2_{\mathrm{ub}}} \lesssim \frac{\eta_1(t)^{\frac{1}{5}}}{\sqrt{1+s}} \tag{3.106}$$

for $s \in [0, t]$.

Bounds on $\mathbf{z}(t)$, $\psi_{\zeta\zeta}(t)$ and $\tilde{\psi}(t)$. Let $t \in [0, \tau_{\max})$ be such that $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$. We invoke the nonlinear bound in Lemma 3.4.2, employ the estimates (3.103) and (3.106), and use $\eta_1(t) \leq \frac{1}{2}$ to obtain

$$\|\mathcal{N}(\mathbf{w}(s), \psi(s), \partial_t\psi(s))\|_{L^\infty} \lesssim \frac{\eta_1(t)^{\frac{6}{5}}}{1+s} \tag{3.107}$$

for $s \in [0, t]$.

Subsequently, we apply the linear estimates in Theorem 3.3.9 and the nonlinear estimate (3.107) to the Duhamel formulas (3.61) and (3.63) and establish

$$\begin{aligned}\|\mathbf{z}(t)\|_{L^\infty} &\lesssim \left(\frac{1}{1+t} + \mathrm{e}^{-\alpha t}\right)E_0 + \int_0^t \left(\frac{1}{1+t-s} + \mathrm{e}^{-\alpha(t-s)}\right)\frac{\eta_1(t)^{\frac{6}{5}}}{1+s}\,\mathrm{d}s + \frac{\eta_1(t)^2}{1+t} \\ &\lesssim \left(E_0 + \eta_1(t)^{\frac{6}{5}}\right)\frac{\log(2+t)}{1+t}\end{aligned} \tag{3.108}$$

and

$$\left\|(\partial_t + c_g\partial_\zeta)^j\partial_\zeta^l\psi(t)\right\|_{L^\infty} \lesssim \frac{E_0}{1+t} + \int_0^t \frac{\eta_1(t)^{\frac{6}{5}}}{(1+t-s)(1+s)}\,\mathrm{d}s \lesssim \left(E_0 + \eta_1(t)^{\frac{6}{5}}\right)\frac{\log(2+t)}{1+t}, \tag{3.109}$$

for all $t \in [0, \tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$ and $j, l \in \mathbb{N}_0$ with $2 \leq l + 2j \leq 6$, where we used $S_p(0) = 0$ when taking the temporal derivative of (3.61).

Bounds on $r(t)$ and $r_\zeta(t)$. Let $t \in [0,\tau_{\max})$ with $\eta(t) \le \frac{1}{2}\min\{1,r_0\}$. We employ the nonlinear bound in Lemma 3.4.5 and estimates (3.103), (3.105) and (3.106) to establish

$$\|\mathcal{N}_p(\mathbf{z}(s),\mathbf{w}(s),\psi(s),\tilde{\psi})\|_{L^\infty} \lesssim \frac{\log(2+s)}{(1+s)^{\frac{3}{2}}}\eta_1(t)^{\frac{6}{5}}, \tag{3.110}$$

for $s \in [0,t]$, where we used $\eta_1(t) \le \frac{1}{2}$.

We recall the well-known L^∞-estimates on the convective heat semigroup:

$$\left\|\partial_\zeta^m \mathrm{e}^{(d\partial_\zeta^2 - c_g\partial_\zeta)\tau} z\right\|_{L^\infty} \lesssim \tau^{-\frac{m}{2}}\|z\|_{L^\infty}, \qquad \left\|\partial_\zeta \mathrm{e}^{(d\partial_\zeta^2 - c_g\partial_\zeta)\tau} w\right\|_{L^\infty} \lesssim \frac{\|w\|_{C^1_{\mathrm{ub}}}}{\sqrt{1+\tau}} \tag{3.111}$$

for $m = 0,1$, $\tau > 0$, $z \in C_{\mathrm{ub}}(\mathbb{R})$ and $w \in C^1_{\mathrm{ub}}(\mathbb{R})$, cf. [84, Proposition 3.6]. So, using that ∂_ζ commutes with $\mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)(t-s)}$, we estimate

$$\begin{aligned}
&\left\|\partial_\zeta^2 \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)(t-s)}\Big(A_h(\mathbf{f}_p)\psi_\zeta(s)^2\Big)\,\mathrm{d}s\right\|_{L^\infty} \\
&\quad\lesssim \int_0^{\max\{0,t-1\}} \frac{\eta_1(t)^2}{(t-s)(1+s)}\,\mathrm{d}s + \int_{\max\{0,t-1\}}^t \frac{\eta_1(t)^2}{\sqrt{t-s}(1+s)}\,\mathrm{d}s \lesssim \frac{\eta_1(t)^2\log(2+t)}{1+t},
\end{aligned} \tag{3.112}$$

for all $t \in [0,\tau_{\max})$. Thus, applying the linear estimates in (3.111) and in Theorem 3.3.9 and the nonlinear estimates (3.110) to (3.68), we obtain the bounds

$$\|r(t)\|_{L^\infty} \lesssim \frac{E_0}{\sqrt{1+t}} + \int_0^t \frac{\eta_1(t)^{\frac{6}{5}}}{\sqrt{t-s}(1+s)}\,\mathrm{d}s \lesssim \Big(E_0 + \eta_1(t)^{\frac{6}{5}}\Big)\frac{\log(2+t)}{\sqrt{1+t}} \tag{3.113}$$

and, using (3.112),

$$\begin{aligned}
\|r_\zeta(t)\|_{L^\infty} &\lesssim \frac{E_0}{\sqrt{t}\sqrt{1+t}} + \int_0^t \frac{\eta_1(t)^{\frac{6}{5}}}{\sqrt{t-s}\sqrt{1+t-s}(1+s)}\,\mathrm{d}s + \frac{\eta_1(t)^2\log(2+t)}{1+t} \\
&\lesssim \Big(E_0 + \eta_1(t)^{\frac{6}{5}}\Big)\frac{\log(2+t)}{\sqrt{t}\sqrt{1+t}}
\end{aligned} \tag{3.114}$$

for all $t \in [0,\tau_{\max})$ with $\eta(t) \le \frac{1}{2}\min\{1,r_0\}$.

Bounds on $y(t)$ and $y_\zeta(t)$. Applying the estimates (3.113) and (3.114) to (3.75), we derive the short-time bound

$$t^{\frac{m}{2}}\|\partial_\zeta^m y(t)\|_\infty \lesssim t^{\frac{m}{2}}\|\partial_\zeta^m r(t)\|_\infty \lesssim E_0 + \eta_1(t)^{\frac{6}{5}}, \tag{3.115}$$

for $m = 0,1$ and all $t \in [0,\tau_{\max})$ with $t \le 1$ and $\eta(t) \le \frac{1}{2}\min\{1,r_0\}$.

Next, take $t \in [0,\tau_{\max})$ with $t \ge 1$ and $\eta(t) \le \frac{1}{2}\min\{1,r_0\}$. Using the nonlinear bound in Lemma 3.4.7 and the estimates (3.103), (3.105) and (3.106), we infer

$$\|\mathcal{N}_c(r(s),y(s),\mathbf{z}(s),\mathbf{w}(s),\psi(s),\tilde{\psi}(s))\|_{L^\infty} \lesssim \frac{\eta_1(t)^{\frac{6}{5}}\log(2+s)}{(1+s)^{\frac{3}{2}}} \tag{3.116}$$

for $s \in [1,t]$, where we use $\eta_1(t) \le \frac{1}{2}$.

We apply the linear estimates (3.111) and the nonlinear bound (3.116) to the Duhamel formula (3.76) and use (3.115) to establish

$$\left\|\partial_\zeta^m y(t)\right\|_{L^\infty} \leq \frac{\|y(1)\|_{C^m_{\mathrm{ub}}}}{(1+t)^{\frac{m}{2}}} + \int_1^t \frac{\eta_1(t)^{\frac{6}{5}}\log(2+s)}{(t-s)^{\frac{m}{2}}(1+s)^{\frac{3}{2}}}\,\mathrm{d}s \lesssim \frac{E_0+\eta_1(t)^{\frac{6}{5}}}{(1+t)^{\frac{m}{2}}},$$

for $m = 0, 1$ and all $t \in [0, \tau_{\max})$ with $t \geq 1$ and $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$. Combining the latter with the short-time bound (3.115), we arrive at

$$t^{\frac{m}{2}}\|\partial_\zeta^m y(t)\|_{L^\infty} \lesssim E_0 + \eta_1(t)^{\frac{6}{5}}, \tag{3.117}$$

for $m = 0, 1$ and all $t \in [0, \tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$.

Bounds on $\psi(t)$ and $\psi_\zeta(t)$. We start by considering the case $\nu \neq 0$. Through (3.71) we can express $\psi(t)$ in terms of the residual $r(t)$ and the Cole-Hopf variable $y(t)$ as

$$\psi(t) = r(t) + \frac{d}{\nu}\log(y(t)+1),$$

with derivative

$$\psi_\zeta(t) = r_\zeta(t) + \frac{d y_\zeta(t)}{\nu(1+y(t))},$$

for $t \in (0, \tau_{\max})$. We emphasize that, as long as $\eta_1(t) \leq \frac{1}{2}$ and $\nu \neq 0$, the above expressions are well-defined. So, using $\|\partial_\zeta^m \psi(t)\|_{L^\infty} \lesssim \|\partial_\zeta^m r(t)\|_{L^\infty} + \|\partial_\zeta^m y(t)\|_{L^\infty}$, employing the estimates (3.113), (3.114) and (3.117) and recalling the fact that $\psi(s)$ vanishes identically for $s \in [0, \tau_{\max})$ with $s \leq 1$ by Proposition 3.4.3, we establish

$$\|\partial_\zeta^m \psi(t)\|_{L^\infty} \lesssim \frac{E_0+\eta_1(t)^{\frac{6}{5}}}{(1+t)^{\frac{m}{2}}}, \tag{3.118}$$

for $m = 0, 1$ and $t \in [0, \tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$.

Next, we consider the case $\nu = 0$. Recalling that $\psi(s)$ vanishes for $s \in [0, 1]$ by Proposition 3.4.3, we apply the linear estimates in (3.111) and the nonlinear bounds (3.110), (3.113) and (3.114) to (3.67), and deduce

$$\|\psi(t)\|_{L^\infty} \lesssim \left(E_0 + \eta_1(t)^{\frac{6}{5}}\right)\frac{\log(2+t)}{\sqrt{1+t}} + E_0 + \int_0^t \eta_1(t)^{\frac{6}{5}}\frac{\log(2+s)}{(1+s)^{\frac{3}{2}}}\,\mathrm{d}s \lesssim E_0 + \eta_1(t)^{\frac{6}{5}}$$

and

$$\|\psi_\zeta(t)\|_{L^\infty} \lesssim \left(E_0 + \eta_1(t)^{\frac{6}{5}}\right)\frac{\log(2+t)}{\sqrt{t}\sqrt{1+t}} + \frac{E_0}{\sqrt{t}} + \int_0^t \frac{\eta_1(t)^{\frac{6}{5}}\log(2+s)}{\sqrt{t-s}(1+s)^{\frac{3}{2}}}\,\mathrm{d}s \lesssim \frac{E_0+\eta_1(t)^{\frac{6}{5}}}{\sqrt{1+t}}$$

for $t \in [0, \tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1, r_0\}$. That is, (3.118) also holds for $\nu = 0$.

Bounds on $\widetilde{\mathbf{w}}(t)$ and $\mathring{\mathbf{w}}(t)$. Using (3.22), applying the mean value theorem and recalling that ϕ is smooth, we bound the forward-modulated perturbation $\mathring{\mathbf{w}}(t)$, defined by (3.21), as

$$\begin{aligned}\left\|\mathring{\mathbf{w}}(t)\right\|_{L^\infty} &\lesssim \left\|\mathring{\mathbf{z}}(t)\right\|_{L^\infty} + \sup_{\zeta\in\mathbb{R}}\left\|\phi(a(\zeta,t);a_\zeta(\zeta,t)) - \phi_0(a(\zeta,t))\right\| \\ &\qquad + \sup_{\zeta\in\mathbb{R}}\left\|\phi(a(\zeta,t)+\psi(\zeta,t)\psi_\zeta(\zeta,t);a_\zeta(\zeta,t)) - \phi(a(\zeta,t);a_\zeta(\zeta,t))\right\| \\ &\lesssim \left\|\mathring{\mathbf{z}}(t)\right\|_{L^\infty} + \|\psi_\zeta(t)\|_{L^\infty}\sup_{|k-1|\leq r_0}\|\phi_k(\cdot;k)\|_{L^\infty} \\ &\qquad + \|\psi(t)\|_{L^\infty}\|\psi_\zeta(t)\|_{L^\infty}\sup_{|k-1|\leq r_0}\|\phi_\zeta(\cdot;k)\|_{L^\infty} \lesssim \frac{\eta_1(t)}{\sqrt{1+t}}\end{aligned} \tag{3.119}$$

for all $t \in [0,\tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1,r_0\}$, where we abbreviate $a(\zeta,t) = \zeta + \psi(\zeta,t)$. Similarly, we establish

$$\left\|\partial_\zeta^j \widetilde{\mathbf{w}}(t)\right\|_{L^\infty} \lesssim \left\|\partial_\zeta^j \mathring{\mathbf{z}}(t)\right\|_{L^\infty} + \|\psi(t)\|_{L^\infty}\left(1 + \|\psi_\zeta(t)\|_{L^\infty}\right) \sup_{|k-1|\leq r_0} \|\partial_\zeta^j \phi_\zeta(\cdot;k)\|_{L^\infty} \\ + \|\psi_\zeta(t)\|_{L^\infty} \sup_{|k-1|\leq r_0} \|\partial_\zeta^j \phi_k(\cdot;k)\|_{L^\infty} + \|\partial_\zeta^j \psi_\zeta(t)\|_{L^\infty}$$

for $j = 0,1$ and $t \in [0,\tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1,r_0\}$. Hence, combining the latter with (3.104) yields

$$\left\|\widetilde{\mathbf{w}}(t)\right\|_{L^\infty} \leq \eta_1(t), \qquad \left\|\widetilde{\mathbf{w}}(t)\right\|_{C^1_{\mathrm{ub}}} \leq \eta_1(t)^{\frac{1}{5}} \tag{3.120}$$

for $t \in [0,\tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1,r_0\}$.

Proof of key inequality and estimates (3.10)-(3.13)**.** Take $t \in [0,\tau_{\max})$ such that $\eta(t) \leq \frac{1}{2}\min\{1,r_0\}$. By estimate (3.120) there exists a t- and E_0-independent constant $C_2 > 0$ such that

$$\eta_2(t) \leq C_2 \eta_1(t)^{\frac{1}{5}}. \tag{3.121}$$

Employing the estimates (3.108), (3.109), (3.113), (3.114), (3.117) and (3.118), we establish a t- and E_0-independent constant $C_1 > 0$ such that

$$\eta_1(t) \leq C_1\left(E_0 + \eta_1(t)^{\frac{6}{5}}\right). \tag{3.122}$$

Hence, combining (3.121) and (3.122) we acquire

$$\eta(t) = \eta_1(t) + \eta_2(t)^5 \leq \left(1 + C_2^5\right)\eta_1(t) \leq C_1\left(1 + C_2^5\right)\left(E_0 + \eta_1(t)^{\frac{6}{5}}\right) \\ \leq C_1\left(1 + C_2^5\right)\left(E_0 + \eta(t)^{\frac{6}{5}}\right).$$

We conclude that there exists a t- and E_0-independent constant such that the key inequality (3.102) holds for all $t \in [0,\tau_{\max})$ with $\eta(t) \leq \frac{1}{2}\min\{1,r_0\}$. As argued above, this implies, provided $E_0 \in (0,\epsilon_0)$, that $\tau_{\max} = \infty$ and we have $\eta(t) \leq M_0 E_0$ for all $t \geq 0$. The estimates (3.10), (3.11) and (3.12) now follow directly by combining $\eta_1(t) \leq M_0 E_0$ with (3.119) and (3.120), respectively. In addition, $\eta_1(t) \leq M_0 E_0$ and (3.103) yield the estimate (3.13).

Approximation by the viscous Hamilton-Jacobi equation. All that remains is to establish the approximation (3.16). We proceed as in [84] and distinguish between the cases $\nu = 0$ and $\nu \neq 0$. We start with the case $\nu = 0$. Then, (3.14) is a linear convective heat equation. We consider the classical solution $\breve{\psi} \in C\big([0,\infty), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,\infty), C_{\mathrm{ub}}(\mathbb{R})\big)$ of (3.14) with initial condition $\breve{\psi}(0) = \widetilde{\Phi}_0^* \mathbf{w}_0 \in C^2_{\mathrm{ub}}(\mathbb{R})$ given by $\breve{\psi}(t) = \mathrm{e}^{(d\partial_\zeta^2 - c_g\partial_\zeta)t}\widetilde{\Phi}_0^* \mathbf{w}_0$. Recalling that $\psi(t)$ vanishes identically for $t \in [0,1]$ by Proposition 3.4.3, we obtain by (3.111) a t- and E_0-independent constant $M_1 \geq 1$ such that

$$t^{\frac{m}{2}}\left\|\partial_\zeta^m\Big(\psi(t) - \breve{\psi}(t)\Big)\right\|_{L^\infty} = t^{\frac{m}{2}}\left\|\partial_\zeta^m \breve{\psi}(t)\right\|_{L^\infty} \leq \frac{M_1 E_0}{\sqrt{1+t}}, \tag{3.123}$$

holds for $t \in [0,1]$ and $m = 0,1$. For $t \geq 1$, we apply the linear estimates in (3.111) and the nonlinear bounds (3.110) and $\eta_1(t) \leq M_0 E_0$ to (3.67) to establish t- and E_0-independent constants $M_2, M_3 \geq 1$ such that

$$\begin{aligned}\left\|\partial_\zeta^m\Big(\psi(t)-\breve{\psi}(t)\Big)\right\|_{L^\infty} &\leq M_2\left(\left\|\partial_\zeta^m r(t)\right\|_{L^\infty} + \int_0^t \eta_1(t)^{\frac{6}{5}} \frac{\log(2+s)}{(t-s)^{\frac{m}{2}}(1+s)^{\frac{3}{2}}}\,\mathrm{d}s\right)\\ &\leq M_3 \frac{\eta_1(t)}{(1+t)^{\frac{m}{2}}}\left(\eta_1(t)^{\frac{1}{5}} + \frac{\log(2+t)}{\sqrt{1+t}}\right),\end{aligned} \tag{3.124}$$

holds for $m = 0,1$. Estimate (3.16) now follows by combining (3.123) and (3.124) and using $\eta_1(t) \leq M_0 E_0$.

Next, we take $\nu \neq 0$. We consider the solution $\breve{\psi} \in C\big([0,\infty), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big([0,\infty), C_{\mathrm{ub}}(\mathbb{R})\big)$ of (3.14) with initial condition $\breve{\psi}(0) = \widetilde{\Phi}_0^* \mathbf{w}_0$ given by

$$\breve{\psi}(t) = \frac{d}{\nu}\log(1+\breve{y}(t)) \quad \text{with} \quad \breve{y}(t) = \mathrm{e}^{(d\partial_\zeta^2 - c_g\partial_\zeta)t}\left(\mathrm{e}^{\frac{\nu}{d}\widetilde{\Phi}_0^*\mathbf{w}_0} - 1\right),$$

which arises through the Cole-Hopf transform and is well-defined as long as $E_0 = \|\mathbf{w}_0\|_{L^\infty}$ is sufficiently small. Employing Taylor's theorem, Theorem 3.3.9, identities (3.68) and (3.75), and estimates (3.110), (3.111) and $\eta_1(1) \leq M_0 E_0$, while using that

$$0 = S_p(1)\mathbf{w}_0 = \mathrm{e}^{d\partial_\zeta^2 - c_g\partial_\zeta}\widetilde{\Phi}_0^*\mathbf{w}_0 + \widetilde{S}_r(1)\mathbf{w}_0$$

holds by Theorem 3.3.9, we establish an E_0-independent constant $M_4 > 0$ such that

$$\begin{aligned}\|y(1)-\breve{y}(1)\|_{L^\infty} \leq \|y(1) + \tfrac{\nu}{d} r(1)\|_{L^\infty} &+ \left\|\breve{y}(1) - \tfrac{\nu}{d}\mathrm{e}^{d\partial_\zeta^2 - c_g\partial_\zeta}\widetilde{\Phi}_0^*\mathbf{w}_0\right\|_{L^\infty}\\ &+ \tfrac{|\nu|}{d}\left\|r(1) - \widetilde{S}_r(1)\mathbf{w}_0\right\|_{L^\infty} \leq M_4 E_0^{\frac{6}{5}}.\end{aligned} \tag{3.125}$$

Noting that $\breve{y}(t) = \mathrm{e}^{(d\partial_\zeta^2 - c_g\partial_\zeta)(t-1)}\breve{y}(1)$, applying the mean value theorem to (3.71), employing the estimates (3.111) and (3.116) to (3.76), and using (3.125) and $\eta_1(t) \leq M_0 E_0$, we establish

$$\begin{aligned}\left\|\psi(t)-\breve{\psi}(t)\right\|_{L^\infty} &\lesssim \|r(t)\|_{L^\infty} + \|y(t)-\breve{y}(t)\|_{L^\infty} \lesssim \|r(t)\|_{L^\infty} + E_0^{\frac{6}{5}} + \eta_1(t)^{\frac{6}{5}},\\ \left\|\psi_\zeta(t)-\breve{\psi}_\zeta(t)\right\|_{L^\infty} &\lesssim \|r_\zeta(t)\|_{L^\infty} + \|y_\zeta(t)-\breve{y}_\zeta(t)\|_{L^\infty} + \|y(t)-\breve{y}(t)\|_{L^\infty}\|y_\zeta(t)\|_{L^\infty}\\ &\lesssim \|r_\zeta(t)\|_{L^\infty} + \frac{E_0^{\frac{6}{5}} + \eta_1(t)^{\frac{6}{5}}}{\sqrt{1+t}}\end{aligned}$$

for $t \geq 1$. So, using that $\eta_1(t) \leq M_0 E_0$, affords a t- and E_0-independent constant $M_5 > 0$ such that

$$\left\|\partial_\zeta^m\Big(\psi(t)-\breve{\psi}(t)\Big)\right\|_{L^\infty} \leq M_5 \frac{E_0}{(1+t)^{\frac{m}{2}}}\left(E_0^{\frac{1}{5}} + \frac{\log(2+t)}{\sqrt{1+t}}\right),$$

holds for all $t \geq 1$. On the other hand, we establish (3.123) for $t \in [0,1]$ analogously to the case $\nu = 0$. Thus, we obtain (3.16) for $\nu \neq 0$. □

Remark 3.5.1. *Due to the use of forward-modulated damping in the proof of Theorem 3.1.4, it is, in contrast to [84], not necessary to control derivatives of* $\mathbf{z}(t)$ *or* $\widetilde{\mathbf{w}}(t)$ *through iterative estimates on their Duhamel formulas. That is, we find that the template function* $\eta_1(t)$ *in the proof of Theorem 3.1.4 coincides with the one from [84, Theorem 1.3], upon omitting all derivatives of* $\mathbf{z}(t)$ *and* $\widetilde{\mathbf{w}}(t)$*. Nevertheless, in order to apply the nonlinear damping estimate in Proposition 3.4.9, the condition* (3.79) *needs to be fulfilled, which requires control on the* *first derivative of the (forward-modulated) perturbation. For that reason, we introduce the second template function* $\eta_2(t)$ *yielding a priori control on the* C^1_{ub}*-norm of* $\widetilde{\mathbf{w}}(t)$ *and, thus, via* (3.22) *of* $\mathring{\mathbf{z}}(t)$*. We can then a posteriori bound* $\eta_2(t)^5$ *with aid of the nonlinear damping estimate in terms of* $\eta_1(t)$*. Since* $\eta_1(t)$ *obeys the nonlinear key inequality* (3.122)*, the same then follows for the full template function* $\eta(t) = \eta_1(t) + \eta_2(t)^5$*.*

Remark 3.5.2. *The choice of temporal weights in the template function* $\eta(t)$ *in the proof of Theorem 3.1.4 coincides with the one from the proof of [84, Theorem 1.3] and reflects, as explained in [84, Remark 5.1], the linear decay rates of* $\mathbf{z}(t)$*,* $\psi(t)$*,* $y(t)$*,* $\widetilde{\mathbf{w}}(t)$ *and* $r(t)$*, cf. Theorem 3.3.9 and* (3.111)*, up to a logarithmic correction.*

3.6. Discussion and outlook

We discuss the wider applicability of our method to establish nonlinear stability of wave trains against fully nonlocalized perturbations.

3.6.1. Applicability to general semilinear dissipative problems

Our analysis does not rely on the specific structure of the FHN system. As a matter of fact, our approach only requires that the wave train is diffusively spectrally stable, it has nonzero group velocity, the perturbation equation obeys a nonlinear damping estimate and the linearization of the system about the wave train generates a C_0-semigroup on $C_{\mathrm{ub}}(\mathbb{R})$, whose high-frequency component is exponentially damped. As long as these criteria are satisfied, we expect our method to work for general semilinear dissipative problems.

It was already observed in [8] that the same linear terms in the FHN system (3.1), i.e. the term u_{xx} in the first component and the term $-\varepsilon\gamma v$ in the second component, are key to obtaining a nonlinear damping estimate, as well as high-frequency resolvent bounds leading to exponentially damped behavior of the high-frequency part of the semigroup. It has been pointed out in the context of the St. Venant equations in [88] that high-frequency resolvent bounds are equivalent to linear damping estimates, which then yield a nonlinear damping estimate as long as solutions stay small. Therefore, we expect that we can replace the requirements that the high-frequency component of the semigroup is exponentially damped and a nonlinear damping estimate can be derived by the condition that the linearization obeys high-frequency resolvent bounds.

In addition, we expect that it is possible to drop the requirement that the wave train has nonzero group velocity. In the case of zero group velocity the diffusive mode at the origin is *branched*, cf. [8, Section 2.1], i.e., the linear dispersion relation $\lambda_c(\xi)$ has a double root at $\xi = 0$. The fact that the linear dispersion relation $\lambda_c(\xi)$ is no longer locally invertible about $\xi = 0$ poses a technical hurdle in relating the inverse Laplace representation of the low-frequency part of the semigroup to its Floquet-Bloch representation. We anticipate that this challenge can be addressed by unfolding the double root at 0 by working with the spectral parameter $\sigma = \sqrt{\lambda}$ with branch cut along the negative real axis.

3.6.2. Open problems

There are however several prominent examples of semilinear dissipative systems, where nonlinear stability of wave trains against localized perturbations has been established, but where one (or more) of the above requirements are not satisfied, thereby obstructing a straightforward application of our method to extend to fully nonlocalized perturbations. Here, we highlight two of these examples.

The first is the Lugiato-Lefever equation, a damped and forced nonlinear Schrödinger equation arising in nonlinear optics, whose diffusively spectrally stable periodic waves are nonlinear stable against localized perturbations [46]. Here, the principal part of the linearization about the wave is the Schrödinger operator $\mathrm{i}\partial_x^2$, which does not generate a C_0-(semi)group on $C_{\mathrm{ub}}(\mathbb{R})$, cf. [14, Lemma 2.1]. Thus, an extension of our method to this setting necessitates reconsidering the choice of space. Natural candidates are the *modulation spaces* $M^k_{\infty,1}(\mathbb{R})$ on which the Schrödinger operator generates a C_0-group, cf. [70, Proposition 3.8]. These spaces consist of nonlocalized functions as can be seen from the embeddings $C^{k+2}_{\mathrm{ub}}(\mathbb{R}) \hookrightarrow M^k_{\infty,1}(\mathbb{R}) \hookrightarrow C^k_{\mathrm{b}}(\mathbb{R})$ for $k \in \mathbb{N}_0$, cf. [68, Theorem 5.7 and Lemma 5.9]. An application of our method would then require to establish high-frequency damping in modulation spaces, which could be challenging. We refer to [42] for further background on modulation spaces.

A second example are the St. Venant equations, which describe shallow water flow down an inclined ramp and admit viscous roll waves. Nonlinear stability of these periodic traveling waves against localized perturbations has been established in [58, 88]. The St. Venant system is only viscous in one component and therefore, similar to the current analysis for the FHN system, incomplete parabolicity must be addressed. Moreover, due to the presence of an additional conservation law the spectrum of the linearization about the wave train possesses an additional curve touching the imaginary axis at 0, thereby violating the spectral stability assumption (D3). Thus, the leading-order dynamics of perturbations is no longer governed by the scalar viscous Hamilton-Jacobi equation (3.14), but instead by an associated Whitham system describing the interactions between critical modes, cf. [57]. It is an open question of how to handle the most critical nonlinear terms that cannot be controlled through iterative L^∞-estimates on the Duhamel formula as the Cole-Hopf transform is no longer available. However, motivated by the results in [51] on the dynamics of roll waves in the Ginzburg-Landau equation coupled to a conservation law against C_{ub}-perturbations, we do expect that our method yields control of perturbations on exponentially long time scales in the setting of the St. Venant equations and more general semilinear dissipative systems admitting conservation laws.

3.A. Appendix: The Laplace transform and its complex inversion formula

This section is devoted to background material on the vector-valued Laplace transforms. In particular, we prove that the complex inversion formula holds for the Laplace transform of convolutions of semigroups. For an extensive introduction into the topic, we refer to the book [6] of Arendt, Batty, Hieber and Neubrander.

Let X, Y be complex Banach spaces. We denote by $B(X)$ the space of bounded operators mapping from X into X. The *growth bound* $\omega_0(G)$ of a map $G\colon [0,\infty) \to Y$ is given by

$$\omega_0(G) = \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0} \mathrm{e}^{-\omega t}\|G(t)\| < \infty\right\}.$$

If $\omega_0(G) < \infty$, then we say that G is *exponentially bounded*.

For a continuous and exponentially bounded function $F\colon [0,\infty) \to X$, the *Laplace transform* $\mathfrak{L}(F)\colon \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega_0(F)\} \to X$ is given by

$$\mathfrak{L}(F)(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda s} F(s)\,\mathrm{d}s.$$

Strong continuity of an operator-valued map $T\colon [0,\infty) \to B(X)$ entails that for each $x \in X$ the *orbit map* $T_x\colon [0,\infty) \to X$ given by $T_x(t) = T(t)x$ is continuous. For a strongly continuous and exponentially bounded $T\colon [0,\infty) \to B(X)$, the Laplace transform $\mathfrak{L}(T)\colon \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega_0(T)\} \to B(X)$, given by

$$\mathfrak{L}(T)(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)\,\mathrm{d}s,$$

is also well-defined by [6, Proposition 1.4.5]. For a C_0-semigroup $T\colon [0,\infty) \to B(X)$ with infinitesimal generator $A\colon D(A) \subset X \to X$, it is well-known, by [32, Proposition I.5.5 & Theorem II.1.10], that T is exponentially bounded and its Laplace transform is given by the resolvent $\mathfrak{L}(T)(\lambda) = (\lambda - A)^{-1}$ for $\mathrm{Re}(\lambda) > \omega_0(T)$.

Let $S, T\colon [0,\infty) \to B(X)$ be strongly continuous and exponentially bounded. The *convolution* $S * T\colon [0,\infty) \to B(X)$ of S and T is given by

$$(S * T)(t) = \int_0^t S(s)T(t-s)\,\mathrm{d}s.$$

The convolution theorem, cf. [32, Theorem C.17], now states that $S * T$ is also strongly continuous and exponentially bounded with $\omega_0(S * T) \leq \max\{\omega_0(S), \omega_0(T)\}$ and its Laplace transform obeys

$$\mathfrak{L}(S * T)(\lambda) = \mathfrak{L}(S)(\lambda)\mathfrak{L}(T)(\lambda), \tag{3.126}$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \max\{\omega_0(S), \omega_0(T)\}$.

The complex inversion formula of the Laplace transform holds for C_0-semigroups. That is, if T is a C_0-semigroup with infinitesimal operator A, then we have

$$T(t)x = \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\omega - \mathrm{i}R}^{\omega + \mathrm{i}R} \mathrm{e}^{\lambda t} \mathfrak{L}(T)(\lambda)x\,\mathrm{d}\lambda = \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\omega - \mathrm{i}R}^{\omega + \mathrm{i}R} \mathrm{e}^{\lambda t} (\lambda - A)^{-1}x\,\mathrm{d}\lambda$$

for all $t > 0$, $\omega > \omega_0(T)$ and $x \in D(A)$, cf. [6, Proposition 3.12.1].

In Section 3.3, we decompose the C_0-semigroup generated by the linearization $\mathcal{L}_0$ by deforming and partitioning the integration contour of the complex line integral in the inversion formula, alongside decomposing the resolvent operator. It has been shown in [8] that for high frequencies the resolvent can be expanded as a Neumann series, whose leading-order terms can be identified as products of resolvents of simpler, well-understood operators. The formula (3.126) reveals that such products can be recognized as the Laplace transform of a convolution of C_0-semigroups generated by those simpler operators. Indeed, if T and S are C_0-semigroups with infinitesimal operators $A\colon D(A) \subset X \to X$ and $B\colon D(B) \subset X \to X$, respectively, then (3.126) and [32, Theorem II.1.10] yield

$$\mathfrak{L}(S * T)(\lambda) = (\lambda - B)^{-1}(\lambda - A)^{-1},$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \max\{\omega_0(S), \omega_0(T)\}$. Thus, to bound the contour integrals arising in the decomposition of the inverse Laplace transform of the C_0-semigroup $\mathrm{e}^{\mathcal{L}_0 t}$, we wish to show that the inversion formula of the Laplace transform also holds for *convolutions* of C_0-semigroups. As far as we are aware, such a result is not readily stated in the current literature. Therefore, we provide a proof in the upcoming. Our proof relies on the observation that the inversion formula holds for F as long as it is Lipschitz continuous and $F(0) = 0$.

Proposition 3.A.1. *Let X be a complex Banach space. Let $F\colon [0, \infty) \to X$ be Lipschitz continuous. Assume $F(0) = 0$. Then, the complex inversion formula*

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\omega-\mathrm{i}R}^{\omega+\mathrm{i}R} \mathrm{e}^{\lambda t} \mathfrak{L}(F)(\lambda)\,\mathrm{d}\lambda$$

holds for $t > 0$ and $\omega > 0$.

Proof. Since F is Lipschitz continuous, it grows at most linearly and is therefore exponentially bounded with growth bound $\omega_0(F) \leq 0$. Let $t > 0$ and $\omega > 0$. By [6, Theorem 2.3.4], we have

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\omega-\mathrm{i}R}^{\omega+\mathrm{i}R} \mathrm{e}^{\lambda t} \frac{r(\lambda)}{\lambda}\,\mathrm{d}\lambda,$$

where the analytic function $r\colon \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\} \to X$ given by

$$r(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda s}\,\mathrm{d}F(s)$$

is the Laplace-Stieltjes transform of F, cf. [6, Theorem 1.10.6]. We integrate by parts, cf. [6, Formula (1.20)], and arrive at

$$\begin{aligned}\frac{r(\lambda)}{\lambda} &= \lim_{t\to\infty} \int_0^t \frac{\mathrm{e}^{-\lambda s}}{\lambda} \mathrm{d}F(s) = \lim_{t\to\infty} \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda t}F(t) - F(0) - \int_0^t F(s)\,\mathrm{d}\left(\mathrm{e}^{-\lambda s}\right)\right) \\ &= \int_0^\infty F(s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s = \mathfrak{L}(F)(\lambda)\end{aligned}$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, which proves the claim. □

The fact that the complex inversion formula of the Laplace transform holds for convolutions of C_0-semigroups is now a direct consequence of Proposition 3.A.1.

Corollary 3.A.2. *Let X be a complex Banach space. Let $T, S\colon [0, \infty) \to \mathcal{L}(X)$ be C_0-semigroups with infinitesimal generators $A\colon D(A) \subset X \to X$ and $B\colon D(B)\colon X \to X$, respectively. Then, we have*

$$(S * T)(t)x = \frac{1}{2\pi\mathrm{i}} \lim_{R\to\infty} \int_{\omega-\mathrm{i}R}^{\omega+\mathrm{i}R} \mathrm{e}^{\lambda t}(\lambda - B)^{-1}(\lambda - A)^{-1}x\,\mathrm{d}\lambda$$

for $t > 0$, $x \in D(A)$ and $\omega > \max\{\omega_0(S), \omega_0(T)\}$.

Proof. Let $t > 0$, $x \in D(A)$ and $\omega > \max\{\omega_0(S), \omega_0(T)\}$. Take $\max\{\omega_0(S), \omega_0(T)\} < \alpha < \omega$. The rescaled semigroups $\tilde{T}(s) = e^{-\alpha s}T(s)$ and $\tilde{S}(s) = e^{-\alpha s}S(s)$ are generated by $A - \alpha$ and $B - \alpha$, respectively. Moreover, $\tilde{S}$ and $\tilde{T}$ have negative growth bounds $\omega_0(\tilde{S}) = \omega_0(S) - \alpha$ and $\omega_0(\tilde{T}) = \omega_0(T) - \alpha$ and so has their convolution $\tilde{S} * \tilde{T}$.

Since we have $x \in D(A)$, the map $F\colon [0, \infty) \to X$ given by $F(s) = (\tilde{S} * \tilde{T})(s)x$ is differentiable with

$$F'(s) = (\tilde{S} * \tilde{T})(s)(Ax - \alpha x) + \tilde{S}(s)x.$$

Thanks to the fact that $\tilde{S} * \tilde{T}$ and $\tilde{S}$ have negative growth bound, there exists a constant $M > 0$ such that $\|F'(s)\| \leq M(\|Ax\| + \|x\|)$ for all $s \geq 0$. Hence, using the mean value theorem, cf. [6, Proposition 1.2.3], we infer $\|F(s) - F(r)\| \leq M(\|Ax\| + \|x\|)|s - r|$, showing that F is Lipschitz continuous. Since we have in addition $F(0) = 0$, an application of Proposition 3.A.1 yields

$$(\tilde{S} * \tilde{T})(t)x = F(t) = \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\omega}-\mathrm{i}R}^{\tilde{\omega}+\mathrm{i}R} \mathrm{e}^{\lambda t} \mathfrak{L}(F)(\lambda)\, \mathrm{d}\lambda,$$

where we denote $\tilde{\omega} = \omega - \alpha > 0$. On the other hand, with the aid of [32, Theorems II.1.10 and C.17], we compute

$$\mathfrak{L}(F)(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda s}(\tilde{S} * \tilde{T})(s)x\, \mathrm{d}s = \mathfrak{L}(\tilde{S} * \tilde{T})(\lambda)x = (\lambda + \alpha - B)^{-1}(\lambda + \alpha - A)^{-1}x$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Therefore, pulling out the exponential factors and scaling back, we arrive at

$$\begin{aligned}(S * T)(t)x = \mathrm{e}^{\alpha t}(\tilde{S} * \tilde{T})(t)x &= \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\omega-\alpha-\mathrm{i}R}^{\omega-\alpha+\mathrm{i}R} \mathrm{e}^{(\lambda+\alpha)t}(\lambda + \alpha - B)^{-1}(\lambda + \alpha - A)^{-1}x\, \mathrm{d}\lambda \\ &= \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\omega-\mathrm{i}R}^{\omega+\mathrm{i}R} \mathrm{e}^{\lambda t}(\lambda - B)^{-1}(\lambda - A)^{-1}x\, \mathrm{d}\lambda,\end{aligned}$$

which finishes the proof. □

3.B. Appendix: Derivation of equation for $\mathring{\mathbf{z}}(t)$

Assume (H1) and (D3). Let $t \in [0, \tilde{\tau}_{\max})$. Recalling Proposition 3.1.2 and noting that $\|\psi_\zeta(t)\|_{L^\infty} < r_0$, we substitute $k = 1 + \psi_\zeta(\zeta, t)$ and $y = \zeta + \psi(\zeta; t)(1 + \psi_\zeta(\zeta; t))$ in the equation

$$k^2 D\phi_{yy}(y; k) + \omega(k)\phi_y(y; k) + F(\phi(y; k)) = 0$$

for the profile function $\phi(y; k)$ and arrive at

$$(1 + \psi_\zeta(\zeta; t))^2 D\phi_{yy}(\beta(\zeta, t)) + \omega(1 + \psi_\zeta(\zeta, t))\phi_y(\beta(\zeta, t)) + F(\phi(\beta(\zeta, t))) = 0 \tag{3.127}$$

for $\zeta \in \mathbb{R}$, where we abbreviate $\beta(\zeta, t) = \big(\zeta + \psi(\zeta; t)(1 + \psi_\zeta(\zeta; t)); 1 + \psi_\zeta(\zeta, t)\big)$. Using Corollary 3.4.8 and the fact that $\mathbf{u}(\zeta, t)$ solves (3.3), we compute the temporal derivative

$$\mathring{\mathbf{z}}_t = D\mathbf{u}_{\zeta\zeta} + c_0\mathbf{u}_\zeta + F(\mathbf{u}) - (\phi_y \circ \beta)(\psi_t(1 + \psi_\zeta) + \psi\psi_{\zeta t}) - (\phi_k \circ \beta)\psi_{\zeta t}. \tag{3.128}$$

In an effort to reexpress the $\mathbf{u}$-contributions in (3.128) in terms of $\mathring{\mathbf{z}}$, we determining the spatial derivatives of $\mathbf{u}(\zeta, t) = \mathring{\mathbf{z}}(\zeta, t) + \phi(\beta(\zeta, t))$ yielding

$$\begin{aligned}
\mathbf{u}_\zeta &= \mathring{\mathbf{z}}_\zeta + (\phi_y \circ \beta)(1 + \psi_\zeta(1 + \psi_\zeta) + \psi\psi_{\zeta\zeta}) + (\phi_k \circ \beta)\psi_{\zeta\zeta}, \\
\mathbf{u}_{\zeta\zeta} &= \mathring{\mathbf{z}}_{\zeta\zeta} + (\phi_{yy} \circ \beta)(1 + \psi_\zeta(1 + \psi_\zeta) + \psi\psi_{\zeta\zeta})^2 + (\phi_y \circ \beta)(\psi_{\zeta\zeta}(1 + 3\psi_\zeta) + \psi\psi_{\zeta\zeta\zeta}) \\
&\quad + (\phi_{kk} \circ \beta)\psi_{\zeta\zeta}^2 + (\phi_k \circ \beta)\psi_{\zeta\zeta\zeta} + 2(\phi_{yk} \circ \beta)(1 + \psi_\zeta(1 + \psi_\zeta) + \psi\psi_{\zeta\zeta})\psi_{\zeta\zeta}.
\end{aligned}$$

Thus, inserting $\mathbf{u}(\zeta, t) = \mathring{\mathbf{z}}(\zeta, t) + \phi(\beta(\zeta, t))$ into (3.128) and employing (3.127), we arrive at the equation (3.77) for the modified forward-modulated perturbation.

End of Paper

CHAPTER 4

Nonlinear dynamics of reaction-diffusion wave trains under large and fully nonlocalized modulations

This chapter contains the preprint [4] and is a joint work with Björn de Rijk.

Start of Paper

Abstract. We study the dynamics of periodic wave trains in reaction-diffusion systems on the real line under large, fully nonlocalized modulations. We prove that solutions with nearby initial data converge, at an enhanced diffusive rate, to a modulated wave train whose leading-order phase and wavenumber dynamics are governed by an explicit solution to the viscous Hamilton-Jacobi equation. This constitutes a global stability result: such initial data are generally not close to the large-time modulated wave train. In contrast to previous modulational stability results, our analysis does not require that the initial data approach phase shifts of the wave train at spatial infinity. The central methodological advance is a nontrivial extension of the recently developed L^∞-stability theory to accommodate large phase modulations. This framework, based entirely on L^∞-estimates, removes all localization requirements as imposed in the previous literature, allowing us to treat the full range of bounded modulational initial data under minimal regularity assumptions. The main technical contributions include: the strategic use of interpolation inequalities to balance smallness and temporal decay, and a detailed analysis of the linear dynamics under fully nonlocalized modulational data.

Keywords. Reaction-diffusion systems; periodic waves; modulation; global stability; nonlocalized perturbations
Mathematics Subject Classification (2020). 35B10; 35B35; 35B40; 35K57

4.1. Introduction

In this paper, we study the dynamics of modulated wave trains in the reaction-diffusion system

$$\partial_t u = Du_{xx} + f(u), \qquad x \in \mathbb{R},\, t \geq 0,\, u \in \mathbb{R}^n, \tag{4.1}$$

where $n \in \mathbb{N}$, $D \in \mathbb{R}^{n\times n}$ is a symmetric, positive-definite matrix, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinearity. Wave trains are solutions to (4.1) of the form $u_{\mathrm{wt}}(x,t) = \phi_0(k_0 x - \omega_0 t)$ with wavenumber $k_0 \in \mathbb{R} \setminus \{0\}$, temporal frequency $\omega_0 \in \mathbb{R}$, propagation speed $c = \omega_0/k_0$, and 1-periodic profile function $\phi_0(\zeta)$. These periodic traveling waves represent the most fundamental patterns that arise at the onset of a Turing instability and as such play a fundamental role in pattern-forming processes in biology, chemistry, ecology and more; we refer to [30] for further details and references. Under generic conditions, the wave train $u_{\mathrm{wt}}(x,t)$ can be continued in the wavenumber, giving rise to a family of wave trains $u_k(x,t) = \phi(kx - \omega(k)t; k)$, with 1-periodic profile function $\phi(\cdot; k)$ and temporal frequency $\omega(k)$, defined for k near k_0 and satisfying $\omega(k_0) = \omega_0$ and $\phi(\cdot; k_0) = \phi_0$. Here, the function $\omega(k)$, which expresses the frequency in terms of the wavenumber, is referred to as the *nonlinear dispersion relation*; see [29].

We are interested in the dynamics of solutions whose initial data are close to a *modulated wave train*

$$u_{\mathrm{mod}}(x) = \phi_0(k_0 x + \boldsymbol{\gamma}_0(x)),$$

where $\boldsymbol{\gamma}_0 \colon \mathbb{R} \to \mathbb{R}$ denotes a phase modulation. Specifically, we consider initial data of the form

$$u(x,0) = \phi_0(k_0 x + \boldsymbol{\gamma}_0(x)) + \mathring{w}_0(x), \tag{4.2}$$

where $\mathring{w}_0 \colon \mathbb{R} \to \mathbb{R}^n$ is a small perturbation. A natural question is whether the solution $u(x,t)$ to (4.1) with initial condition (4.2) remains close to a modulated wave train for all $t \geq 0$. That is, under what conditions does there exist a modulation function $\boldsymbol{\gamma} \colon \mathbb{R} \times [0,\infty) \to \mathbb{R}$ such that the perturbation

$$\mathring{w}(x,t) = u(x,t) - \phi_0(k_0 x - \omega_0 t + \boldsymbol{\gamma}(x,t)) \tag{4.3}$$

remains small for all time? In this paper, we establish such a *modulational stability* result under standard spectral stability assumptions on the wave train, together with the requirement that $\boldsymbol{\gamma}_0$ is bounded and both $\boldsymbol{\gamma}_0'$ and $\mathring{w}_0$ are uniformly continuous and small in $L^\infty(\mathbb{R})$. In particular, for any bounded function $g_0 \colon \mathbb{R} \to \mathbb{R}$ with uniformly continuous derivative, there exists $\delta > 0$ such that the phase modulation $\boldsymbol{\gamma}_0(x) = g_0(\delta x)$ is admissible.

Our result advances the modulational stability theory for wave-train solutions to (4.1) in two crucial ways. First, in contrast to all prior results [54, 56, 57, 63, 89], we do not require $\boldsymbol{\gamma}_0'$ to be L^1-localized and, in particular, do not assume the existence of asymptotic phase limits $\boldsymbol{\gamma}_\pm = \lim_{x\to\pm\infty} \boldsymbol{\gamma}_0(x)$; see Remark 4.1.1 and Figure 4.1. Second, we remove all localization assumptions on both $\boldsymbol{\gamma}_0'$ and the initial perturbation $\mathring{w}_0$ that are present in the previous literature; see Remark 4.1.2. This allows us to handle a significantly larger class of modulational initial data of the form (4.2) than considered before.

Notably, we do not require $\boldsymbol{\gamma}_0$ to be small in $L^\infty(\mathbb{R})$. To the best of our knowledge, the only other work addressing *large* phase modulations of wave-train solutions to (4.1) (while still assuming $\boldsymbol{\gamma}_0 \in L^1(\mathbb{R})$) is [54]. In this context, we also highlight the recently developed stability theory for periodic patterns in *planar* reaction-diffusion systems [75]. These patterns are intrinsically planar in the sense that they are nontrivial in *all* spatial directions. Building on techniques from [56, 57], the analysis in [75] still relies on decay induced by localization, yet it allows for large, and even unbounded, phase modulations. The reason is that localization leads to stronger temporal decay in higher spatial dimensions, permitting a relaxation of the localization requirement on the initial phase modulation: instead of assuming $\boldsymbol{\gamma}_0' \in L^1(\mathbb{R})$ as in [56, 57], it suffices to require that $\Delta\boldsymbol{\gamma}_0 \in L^1(\mathbb{R}^2)$. This condition imposes only that $\boldsymbol{\gamma}_0$ has bounded mean oscillation (up to additive constants), allowing for unbounded $\boldsymbol{\gamma}_0$; see also §4.7.3.

Our result is an *asymptotic* modulational stability result in the sense that the perturbation $\mathring{w}(t)$ in (4.3) decays over time at a diffusive rate $t^{-\frac{1}{2}}$. Moreover, the modulation $\boldsymbol{\gamma}(t)$, which governs the phase dynamics in (4.3), remains bounded with small derivative, which also decays at rate $t^{-\frac{1}{2}}$. As in [54, 57, 89], we show that $\boldsymbol{\gamma}(t)$ is well-approximated by the solution $\breve{\gamma}(t)$ to the viscous Hamilton-Jacobi equation

$$\partial_t \breve{\gamma} = \frac{d}{k_0^2}\,\breve{\gamma}_{xx} + \omega'(k_0)\,\breve{\gamma}_x - \frac{1}{2}\omega''(k_0)\,\breve{\gamma}_x^2, \tag{4.4}$$

with initial condition $\breve{\gamma}(0) = \boldsymbol{\gamma}_0$. Here, the coefficient $d > 0$ is an explicit Melnikov-type integral; see (4.11).

The first rigorous justification of equation (4.4) as an effective description of the dynamics of slowly modulated wave trains on long time intervals was provided in [29]. Our result, along with those in [54, 57, 89], confirms that this description remains valid globally.

It is not difficult to show that the solution $\breve{\gamma}(t)$ to (4.4) does not, in general, remain close to its initial condition $\boldsymbol{\gamma}_0$, and the same holds for $\boldsymbol{\gamma}(t)$ by approximation; see Remark 4.1.5. Thus, our modulational stability result is *global* in the sense that it provides a complete description of the dynamics of solutions $u(t)$ with initial data of the form (4.2), even though $u(t)$ typically does not remain close to $u(0)$ over time.

It was already observed in [29] that phase and wavenumber modulations are intimately connected. For the modulated wave train described in (4.3), the local wavenumber, i.e. the number of waves per unit interval near position x, is given by $k_0 + \boldsymbol{\gamma}_x(x,t)$. This suggests that incorporating a modulation of the wavenumber may yield a more accurate approximation of the solution $u(t)$. We show that the refined residual

$$\mathring{y}(x,t) := u(x,t) - \phi\Big(k_0 x - \omega_0 t + \boldsymbol{\gamma}(x,t)\Big(1 + \tfrac{1}{k_0}\boldsymbol{\gamma}_x(x,t)\Big);\, k_0 + \boldsymbol{\gamma}_x(x,t)\Big) \tag{4.5}$$

indeed decays in $L^\infty(\mathbb{R})$ at an enhanced rate $t^{-1}\log(1+t)$. This refined asymptotic description of the dynamics of $u(t)$, accounting for both phase and wavenumber modulation, was previously established in [57], but only for *small* initial phase modulations $\boldsymbol{\gamma}_0$ and with weaker decay rates in $L^\infty(\mathbb{R})$; see Table 4.1. Here, we obtain it for the first time in the setting of large initial phase modulations.

While previous modulational stability results rely on localization-induced decay, employing renormalization group techniques [54, 89], iterative L^1-H^k estimates [56, 57], or pointwise Green's function bounds [63], our analysis follows a different approach. We extend the recently developed L^∞-based stability theory of [84] to accommodate large modulational initial data. This method eliminates the need for any localization assumptions by leveraging decay generated by smoothing properties of the critical part of the semigroup to close the nonlinear argument. Further details on the strategy underlying our nonlinear stability analysis are provided in §4.1.2.

Remark 4.1.1. The assumption that $\boldsymbol{\gamma}_0 \in C^1(\mathbb{R})$ is bounded is equivalent to the condition

$$\sup_{R\in\mathbb{R}}\left|\int_0^R \boldsymbol{\gamma}_0'(x)\,\mathrm{d}x\right| < \infty. \tag{4.6}$$

In this work, we thus extend previous modulational stability results [54, 56, 57, 63, 89], which relied on the stronger assumption $\|\boldsymbol{\gamma}_0\|_{L^1} = \int_{\mathbb{R}} |\boldsymbol{\gamma}_0'(x)|\,\mathrm{d}x < \infty$, to the weaker condition (4.6).

However, this assumption still excludes initial data with nonzero asymptotic wavenumber shifts. Whether modulational stability persists under such wavenumber offsets remains an open question; see §4.7.3.

Remark 4.1.2. The modulational stability results [56, 57, 63, 89] consider initial data (4.2), where γ_0' is small in $L^1(\mathbb{R})$. This readily implies that $\|\gamma_0\|_\infty$ is small and that the asymptotic limits $\lim_{x\to\pm\infty}\gamma_0(x) = \gamma_\pm$ exist. While [54] still requires γ_0' to be L^1-localized, the smallness assumption on $\|\gamma_0'\|_{L^1}$ is dropped, thereby allowing for large *phase offsets* $\gamma_d = \gamma_+ - \gamma_-$. See Table 4.1 for further details.

	γ_0 and $\mathring{w}_0$	decay of $\gamma_x(t)$ and $\mathring{w}(t)$	decay of $\mathring{y}(t)$
[89]	$\left\|\rho^2\mathring{w}_0\right\|_{H^2} + \left\|\rho^2\gamma_0'\right\|_{H^2} \ll 1$	$t^{-\frac{1}{2}+\alpha}$	
[56, 57]	$\|\mathring{w}_0\|_{L^1\cap H^3} + \|\gamma_0'\|_{L^1\cap H^3} \ll 1$	$t^{-\frac{1}{2}}$	$t^{-\frac{3}{4}}\log(1+t)$
[63]	$\|\mathring{w}_0\|_{H^2} < \infty$ and $\left\|\rho^{\frac{3}{2}}\mathring{w}_0\right\|_\infty + \left\|\rho^{\frac{3}{2}}\gamma_0'\right\|_{W^{1,\infty}} \ll 1$	$\left(1+\|x-at\|+\sqrt{t}\right)^{-\frac{3}{2}} + (1+t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{\|x-at\|^2}{Ct}}$	
[54]	$\left\|\rho^2\mathring{w}_0\right\|_{H^2} + \left\|\rho^2\gamma_0'\right\|_{H^2} < \infty$ and $\|\widehat{\gamma}_0'(1+\|\cdot\|)\|_{L^1} + \|\mathring{w}_0\|_{H^2} \ll \gamma_d$	$t^{-\frac{1}{2}+\alpha}$	
Thm. 4.1.3	$\|\gamma_0\|_\infty < M$ and $\|\mathring{w}_0\|_\infty + \|\gamma_0'\|_\infty \ll 1$	$t^{-\frac{1}{2}}$	$t^{-1}\log(1+t)$

Table 4.1.: This table compares previously established modulational stability results with our main result, Theorem 4.1.3, for solutions $u(t)$ to (4.1) with initial condition (4.2). The second column presents the assumptions on the initial modulation γ_0 and perturbation $\mathring{w}_0$, where ρ is the algebraic weight $\rho(x) = \sqrt{1+x^2}$, $\widehat{\cdot}$ denotes the Fourier transform, and $M > 0$ is an arbitrary constant fixed a priori. The third column contains the pointwise decay rates obtained for $\gamma_x(t)$ and $\mathring{w}(t)$ in (4.3). Here, $C > 0$ and $a \in \mathbb{R}$ are constants, and $\alpha \in (0, \frac{1}{2})$ is an arbitrary parameter fixed a priori. The final column presents the pointwise decay rates obtained for the refined residual defined in (4.5). We note that the inequality $\|\gamma_0'\|_{L^1} \leq \|\rho^{-\frac{3}{2}}\|_{L^1}\|\rho^{\frac{3}{2}}\gamma_0'\|_\infty$ and the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ imply that $\gamma_0' \in L^1(\mathbb{R})$ in [54, 56, 57, 63, 89].

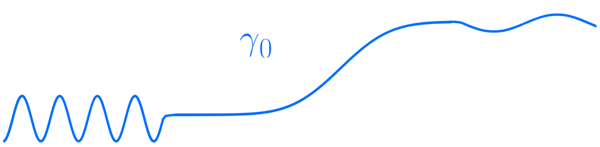

Figure 4.1.: Example of an initial phase modulation $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with nonlocalized derivative.

4.1.1. Statement of main result

Before stating the main result, we formulate the necessary hypotheses, which concern the existence and spectral stability of the underlying wave train. We emphasize that these assumptions are standard in the nonlinear stability literature of wave trains; see, for example, [54–57, 62, 63, 84, 89, 90, 92]. We begin with the existence hypothesis:

(H1) There exist a wavenumber $k_0 \in \mathbb{R} \setminus \{0\}$ and a temporal frequency $\omega_0 \in \mathbb{R}$ such that (4.1) admits a wave-train solution $u_{\mathrm{wt}}(x,t) = \phi_0(k_0 x - \omega_0 t)$, where the profile function $\phi_0 \colon \mathbb{R} \to \mathbb{R}^n$ is nonconstant, smooth and 1-periodic.

Thus, switching to the co-moving frame $\zeta = k_0 x - \omega_0 t$, we find that ϕ_0 is a stationary solution to

$$\partial_t u = k_0^2 D u_{\zeta\zeta} + \omega_0 u_\zeta + f(u). \tag{4.7}$$

Next, we turn to the required spectral assumptions. To this end, we consider the linearization of (4.7) about the wave train ϕ_0:

$$\mathcal{L}_0 u = k_0^2 D u_{\zeta\zeta} + \omega_0 u_\zeta + f'(\phi_0(\zeta))u.$$

We view $\mathcal{L}_0$ as a 1-periodic differential operator acting on the space $C_{\mathrm{ub}}(\mathbb{R})$, with dense domain $D(\mathcal{L}_0) = C^2_{\mathrm{ub}}(\mathbb{R})$. Here, $C^m_{\mathrm{ub}}(\mathbb{R})$, $m \in \mathbb{N}_0$, denotes the Banach space of bounded, uniformly continuous functions that are m times differentiable, with all derivatives up to order m also bounded and uniformly continuous. We equip $C^m_{\mathrm{ub}}(\mathbb{R})$ with the standard $W^{m,\infty}$-norm.

Applying the Floquet-Bloch transform to $\mathcal{L}_0$ yields the family of Bloch operators

$$\mathcal{L}(\xi)u = k_0^2 D(\partial_\zeta + \mathrm{i}\xi)^2 u + \omega_0(\partial_\zeta + \mathrm{i}\xi)u + f'(\phi_0(\zeta))u,$$

defined on $L^2_{\mathrm{per}}(0,1)$ with domain $D(\mathcal{L}(\xi)) = H^2_{\mathrm{per}}(0,1)$, and parameterized by the Bloch frequency $\xi \in [-\pi,\pi)$. Due to translational invariance, the derivative ϕ_0' of the wave train lies in the kernel of $\mathcal{L}(0)$. As a result, the spectrum of $\mathcal{L}_0$, given by

$$\sigma(\mathcal{L}_0) = \bigcup_{\xi \in [-\pi,\pi)} \sigma(\mathcal{L}(\xi)),$$

necessarily touches the origin. The most stable nondegenerate spectral configuration, known as *diffusive spectral stability*, is then characterized by the following conditions:

(D1) It holds $\sigma(\mathcal{L}_0) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\}$;

(D2) There exists $\theta > 0$ such that for any $\xi \in [-\pi,\pi)$ we have $\mathrm{Re}\, \sigma(\mathcal{L}(\xi)) \leq -\theta\xi^2$;

(D3) 0 is a simple eigenvalue of $\mathcal{L}(0)$.

Examples of reaction-diffusion systems that support diffusively spectrally stable wave trains include the complex Ginzburg-Landau equation [48], the Gierer-Meinhardt system [82], and the Brusselator model [97]. We refer to [84] for further discussion and additional references.

It follows from Hypothesis (D3) that 0 is also a simple eigenvalue of the adjoint operator $\mathcal{L}(0)^*$. We denote by $\widetilde{\Phi}_0 \in H^2_{\mathrm{per}}(0,1)$ the corresponding eigenfunction, normalized such that

$$\langle \widetilde{\Phi}_0, \phi_0' \rangle_{L^2(0,1)} = 1. \tag{4.8}$$

Hypothesis (D3) also implies that the wave train ϕ_0 can be continued in the wavenumber, yielding a family of wave-train solutions to (4.1) of the form $u_k(x,t) = \phi(kx - \omega(k)t; k)$, defined for k near k_0, where the frequency $\omega(k)$ and the 1-periodic profile function $\phi(\cdot;k)$ obey

$$\omega(k_0) = \omega_0, \qquad \phi(\cdot;k_0) = \phi_0, \qquad \langle \widetilde{\Phi}_0, \partial_k \phi(\cdot,k_0) \rangle_{L^2(0,1)} = 0, \tag{4.9}$$

cf. Proposition 4.3.1.

We now present our main result, which shows that solutions to (4.7) with initial data close to a largely modulated wave train converge at an enhanced diffusive rate to a modulated wave train, whose phase and wavenumber dynamics are governed by a solution $\breve{\gamma}(t)$ to the viscous Hamilton-Jacobi equation

$$\partial_t \breve{\gamma} = d\breve{\gamma}_{\zeta\zeta} + a\breve{\gamma}_\zeta + \nu\breve{\gamma}_\zeta^2, \tag{4.10}$$

with coefficients

$$a = \omega_0 - k_0\omega'(k_0), \qquad d = \langle \widetilde{\Phi}_0, D\phi_0' + 2k_0 D\partial_{\zeta k}\phi(\cdot;k_0)\rangle_{L^2(0,1)}, \qquad \nu = -\frac{1}{2}k_0^2\omega''(k_0). \tag{4.11}$$

Theorem 4.1.3. *Assume* (H1) *and* (D1)-(D3). *Fix* $\alpha \in [0, \frac{1}{6})$ *and* $M > 0$. *Then, there exist constants* $K, \varepsilon > 0$ *such that, whenever* $u_0 \in C_{\mathrm{ub}}(\mathbb{R})$ *and* $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ *satisfy*

$$\|\gamma_0\|_\infty \leq M, \qquad E_0 := \|u_0 - \phi_0(\cdot + \gamma_0(\cdot))\|_\infty + \|\gamma_0'\|_\infty < \varepsilon,$$

there exist a scalar function

$$\gamma \in C\big([0,\infty), C^1_{\mathrm{ub}}(\mathbb{R})\big) \cap C^j\big((0,\infty), C^l_{\mathrm{ub}}(\mathbb{R})\big), \qquad j,l \in \mathbb{N}_0$$

with $\gamma(0) = \gamma_0$ *and a unique classical global solution*

$$u \in \mathcal{X} := C\big([0,\infty), C_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0,\infty), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0,\infty), C_{\mathrm{ub}}(\mathbb{R})\big)$$

to (4.7) *with* $u(\cdot,0) = u_0$, *which obey the estimates*

$$\begin{aligned} \|\gamma(t)\|_\infty \leq K, \qquad \|u(t) - \phi_0(\cdot + \gamma(\cdot,t))\|_\infty, \|\gamma_\zeta(t)\|_\infty &\leq \frac{KE_0^\alpha}{(1+t)^{\frac{1}{2}-\alpha}}, \\ \big\|u(t) - \phi\big(\,\cdot + \gamma(\cdot,t)(1+\gamma_\zeta(\cdot,t));\, k_0(1+\gamma_\zeta(\cdot,t))\big)\big\|_\infty &\leq KE_0^\alpha \frac{\log(2+t)}{(1+t)^{1-2\alpha}} \end{aligned} \tag{4.12}$$

for $t \geq 0$, *and*

$$\begin{aligned} \|u(t) - \phi_0(\cdot + \gamma(\cdot,t))\|_{C^1_{\mathrm{ub}}} &\leq KE_0^\alpha \frac{(1+t)^\alpha}{\sqrt{t}}, \\ \big\|u(t) - \phi\big(\,\cdot + \gamma(\cdot,t)(1+\gamma_\zeta(\cdot,t));\, k_0(1+\gamma_\zeta(\cdot,t))\big)\big\|_{C^1_{\mathrm{ub}}}, \|\gamma_{\zeta\zeta}(t)\|_{C^1_{\mathrm{ub}}} &\leq KE_0^\alpha \frac{\log(2+t)}{\sqrt{t}\,(1+t)^{\frac{1}{2}-2\alpha}} \end{aligned} \tag{4.13}$$

for all $t > 0$. *Moreover, there exists a unique classical global solution*

$$\breve{\gamma} \in \mathcal{Y} := C\big([0,\infty), C^1_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0,\infty), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0,\infty), C_{\mathrm{ub}}(\mathbb{R})\big)$$

with initial condition $\breve{\gamma}(0) = \gamma_0$ *to the viscous Hamilton-Jacobi equation* (4.10) *such that we have the approximations*

$$\|\gamma(t) - \breve{\gamma}(t)\|_\infty \leq KE_0^{\frac{2}{3}\alpha}, \qquad \|\gamma_\zeta(t) - \breve{\gamma}_\zeta(t)\|_\infty \leq \frac{KE_0^\alpha}{\sqrt{1+t}} \tag{4.14}$$

for all $t \geq 0$. *In particular, it holds*

$$\|u(t) - \phi_0(\cdot + \breve{\gamma}(\cdot,t))\|_\infty \leq KE_0^{\frac{2}{3}\alpha}, \qquad t \geq 0. \tag{4.15}$$

Remark 4.1.4. We observe that the temporal decay rates presented in Theorem 4.1.3 match those established in the sharp nonlinear stability result for wave trains under C_{ub}-perturbations in [84], confirming their optimality up to a possible logarithmic correction; see [84, Section 6.1]. Since our primary objective was to establish a modulational stability result with sharp temporal decay rates, we did not strive for optimal powers of the E_0-terms in the estimates (4.12), (4.13), (4.14), and (4.15) in Theorem 4.1.3. As the toy example in §4.2 suggests, we expect that these powers may be improved with further technical effort.

Remark 4.1.5. Let $g_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ be any function such that $g_0' \in L^1(\mathbb{R}) \setminus \{0\}$ has mean zero. Set $M = \|g_0\|_\infty$ and fix $\alpha \in (0, \frac{1}{6})$. Let $K, \varepsilon > 0$ be as in Theorem 4.1.3. Clearly, there exists $\delta > 0$ such that the function $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ given by $\gamma_0(\zeta) = g_0(\delta\zeta)$ satisfies $\|\gamma_0\|_\infty \leq M$ and $\|\gamma_0'\|_\infty < \varepsilon$. Let $\breve{\gamma}$ denote the solution to the viscous Hamilton-Jacobi equation (4.10) with initial condition $\breve{\gamma}(0) = \gamma_0$. Then, $\breve{k}(t) := \breve{\gamma}_\zeta(t)$ solves the viscous Burgers equation

$$\partial_t \breve{k} = d\partial_\zeta^2 \breve{k} + a\partial_\zeta \breve{k} + \nu\partial_\zeta(\breve{k}^2).$$

Since $\gamma_0' \in L^1(\mathbb{R})$ has mean zero, it follows from [35, Theorem 2 and Corollary 1] that $\breve{k}(t)$ converges to zero in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as $t \to \infty$. This implies that $\breve{\gamma}(t)$ converges to a spatially constant function $\breve{\gamma}_\infty$ in $C^1_{\mathrm{ub}}(\mathbb{R})$ as $t \to \infty$. In particular, for large times, $\breve{\gamma}(t)$ is not close to its initial condition γ_0 in $C^1_{\mathrm{ub}}(\mathbb{R})$. By the estimates (4.12) and (4.14), the same holds for $\gamma(t)$ and $u(t)$, demonstrating that Theorem 4.1.3 provides a *global* stability result.

Remark 4.1.6. Although the reaction-diffusion system (4.7) admits mild solutions for initial data in $L^\infty(\mathbb{R})$ via standard analytic semigroup theory, the minimal condition ensuring that a solution $u(t)$ converges to its initial condition u_0 in $L^\infty(\mathbb{R})$ as $t \downarrow 0$ is that u_0 is uniformly continuous; see [73, Theorems 3.1.7 and 7.1.2]. This right-continuity of $u(t)$ at $t = 0$ is clearly necessary for our main result to be meaningful, as is the requirement that the initial phase modulation γ_0 is bounded and differentiable, since we rely on the smallness of its derivative. In this sense, the regularity assumptions imposed on the initial data in Theorem 4.1.3 are minimal.

Remark 4.1.7. Estimate (4.15) provides a leading-order description of the solution $u(t)$ that is valid for *all* times. This estimate is fully explicit in terms of the underlying wave train ϕ_0 and the solution $\breve{\gamma}(t)$ to the viscous Hamilton-Jacobi equation (4.10), with initial condition $\breve{\gamma}(0) = \gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$. In contrast, in the other work [54] addressing large phase modulations of wave-train solutions to (4.1), the following estimate was established for fixed $\alpha > 0$:

$$\|u(t) - \phi_0(\cdot + \Phi(\cdot, t))\|_\infty \leq K t^{-\frac{1}{2}+\alpha}, \qquad t \geq 0, \tag{4.16}$$

where Φ is a self-similar front solution to (4.10) and $K > 0$ is a time-independent constant. While the estimate (4.16) does not capture the leading-order behavior of $u(t)$ for short times, it yields near-diffusive decay for large times. We argue that this large-time decay stems from the stronger assumptions imposed on the initial phase modulation γ_0 in [54].

Indeed, even with conditions that are less stringent than those in [54], we expect to recover an estimate matching (4.16). To formulate these conditions, we fix asymptotic limits $\gamma_\pm \in \mathbb{R}$ and consider the associated self-similar front solution

$$\Phi(\zeta, t) = \begin{cases} \gamma_- + \gamma_d \operatorname{erf}\left(\dfrac{\zeta + a(t+1)}{\sqrt{d(t+1)}}\right), & \nu = 0, \\ \gamma_- + \dfrac{d}{\nu} \log\left(1 + \left(\mathrm{e}^{\frac{\nu}{d}\gamma_d} - 1\right) \operatorname{erf}\left(\dfrac{\zeta + a(t+1)}{\sqrt{4d(t+1)}}\right)\right), & \nu \neq 0, \end{cases}$$

to (4.10), where $\gamma_d := \gamma_+ - \gamma_-$ denotes its phase offset, and

$$\mathrm{erf}(\zeta) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\zeta} \mathrm{e}^{-\frac{\bar{\zeta}^2}{4}} \, \mathrm{d}\bar{\zeta}$$

is the Gaussian error function. Thus, under the additional assumptions that γ_0 is twice continuously differentiable with $\gamma_0'' \in L^2(\mathbb{R})$ and satisfies

$$\|\gamma_0 - \Phi\|_{L^1 \cap L^\infty} \leq M,$$

we expect estimate (4.15) in Theorem 4.1.3 to improve to

$$\|u(t) - \phi_0(\cdot + \Phi(t))\|_\infty \leq K \frac{(\log(2+t))^2}{\sqrt{1+t}}, \qquad t \geq 0. \tag{4.17}$$

We note that estimate (4.17) is slightly stronger than (4.16) Since the proof of (4.17) requires substantial additional effort, especially due to the need for tracking L^2-norms in the nonlinear iteration, we defer it to future work.

Remark 4.1.8. Converting to the original (x,t)-variables in Theorem 4.1.3, we obtain a constant $\mathring{K} > 0$ such that, for each $\mathring{w}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\boldsymbol{\gamma}_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|\boldsymbol{\gamma}_0\|_\infty \leq M$ and $E_0 = \|\mathring{w}_0\|_\infty + \frac{1}{k_0}\|\boldsymbol{\gamma}_0'\|_\infty < \varepsilon$, there exist a unique classical global solution $u \in \mathcal{X}$ to the reaction-diffusion system (4.1) with initial condition (4.2) and a phase modulation function $\boldsymbol{\gamma} \in \mathcal{Y}$ with $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$ such that

$$\|\boldsymbol{\gamma}(t)\|_\infty \leq \mathring{K}, \qquad \|\mathring{w}(t)\|, \|\boldsymbol{\gamma}_x(t)\|_\infty \leq \frac{\mathring{K} E_0^\alpha}{(1+t)^{\frac{1}{2}-\alpha}}, \qquad \|\mathring{y}(t)\|_\infty \leq \mathring{K} E_0^\alpha \frac{\log(2+t)}{(1+t)^{1-2\alpha}}$$

for all $t \geq 0$, and

$$\|\mathring{w}(t)\|_{C^1_{\mathrm{ub}}} \leq \mathring{K} E_0^\alpha \frac{(1+t)^\alpha}{\sqrt{t}}, \qquad \|\mathring{y}(t)\|_{C^1_{\mathrm{ub}}} \leq \mathring{K} E_0^\alpha \frac{\log(2+t)}{\sqrt{t}\,(1+t)^{\frac{1}{2}-2\alpha}}$$

for all $t > 0$, where $\mathring{w}(t)$ and $\mathring{y}(t)$ are the residuals given by (4.3) and (4.5), respectively. Moreover, we have the approximations

$$\|\boldsymbol{\gamma}(t) - \breve{\boldsymbol{\gamma}}(t)\|_\infty \leq \mathring{K} E_0^{\frac{2}{3}\alpha}, \qquad \|\boldsymbol{\gamma}_x(t) - \breve{\boldsymbol{\gamma}}_x(t)\|_\infty \leq \frac{\mathring{K} E_0^\alpha}{\sqrt{1+t}}$$

for $t \geq 0$, where $\breve{\boldsymbol{\gamma}} \in \mathcal{Y}$ is the classical solution to the viscous Hamilton-Jacobi equation (4.4) with initial condition $\breve{\boldsymbol{\gamma}}(0) = \boldsymbol{\gamma}_0$.

4.1.2. Strategy of proof

Our proof builds on an extension of the recently developed stability theory in [84], which is based on pure L^∞-estimates and accommodates fully nonlocalized perturbations. A key difference from the setting in [84] is that we cannot exploit smallness of $\|\gamma_0\|_\infty$, which presents a significant challenge. In particular, a linear term of the form $\mathrm{e}^{\mathcal{L}_0 t}(\phi_0' \gamma_0)$ appears in the Duhamel formulation of the *inverse-modulated perturbation*

$$v(\zeta, t) = u(\zeta - \gamma(\zeta,t), t) - \phi_0(\zeta), \tag{4.18}$$

where $\mathrm{e}^{\mathcal{L}_0 t}$ denotes the semigroup on $C_{\mathrm{ub}}(\mathbb{R})$ generated by the linearization $\mathcal{L}_0$. The inverse-modulated perturbation $v(t)$ controls the dynamics of the difference

$$\mathring{v}(\zeta,t) = u(\zeta,t) - \phi_0(\zeta + \gamma(\zeta,t)) \tag{4.19}$$

between the solution and the modulated wave train. It is typically used in nonlinear stability analyses, since its evolution equation exhibits more favorable decay properties compared to the one of the *forward-modulated perturbation* $\mathring{v}(t)$; see [105]. A crucial observation is that the lack of smallness in the Duhamel formulation of $v(t)$ can be precisely quantified using the identity

$$\begin{aligned} \mathrm{e}^{\mathcal{L}_0 t}(\phi_0'\gamma_0) - \phi_0'\gamma_0 &= \int_0^t \mathrm{e}^{\mathcal{L}_0 s}\mathcal{L}_0(\phi_0'\gamma_0)\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{\mathcal{L}_0 s}\Big(k_0^2 D(\partial_\zeta(\phi_0'\gamma_0') + \phi_0''\gamma_0') + \omega_0\phi_0'\gamma_0'\Big)\mathrm{d}s, \end{aligned} \tag{4.20}$$

which follows from standard semigroup theory [73, Proposition 2.1.4] and the fact that ϕ_0' lies in the kernel of $\mathcal{L}(0)$. Since the right-hand side of (4.20) depends only on the *derivative* γ_0', whose L^∞-norm is assumed to be small, the lack of smallness in $\mathrm{e}^{\mathcal{L}_0 t}(\phi_0'\gamma_0)$ manifests solely in the term $\phi_0'\gamma_0$.

Since the linear operator $\mathcal{L}_0$ has spectrum touching the imaginary axis, the semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ does not exhibit decay. To obtain sufficient decay to close a nonlinear argument, it is therefore necessary to decompose $\mathrm{e}^{\mathcal{L}_0 t}$ into a principal part of the form $\phi_0' S_p^0(t)$, where $S_p^0(t)$ enjoys the same bounds and smoothing properties as the heat semigroup $\mathrm{e}^{\partial_\zeta^2 t}$, and a residual part exhibiting faster temporal decay. Specifically, the phase modulation $\gamma(t)$ is chosen in such a way that, for large times, it captures the slowest-decaying contributions associated with $S_p^0(t)$ in the Duhamel formulation of the inverse-modulated perturbation $v(t)$; see §4.5.3. Since the phase modulation enters only via its *derivatives* in this formulation, we may exploit improved decay of $\partial_\zeta^j\partial_t^l S_p^0(t)$ for $j,l \in \mathbb{N}_0$ with $j + l \geq 1$ which arises due to diffusive smoothing.

It is thus essential to characterize the lack of smallness in $S_p^0(t)(\phi_0'\gamma_0)$, analogously to the identity (4.20). Inspired by the Fourier-series arguments in [56], we perform a detailed linear analysis, extending the L^∞-techniques from [51] to the setting of large modulational initial data. We find, similar to (4.20), that $S_p^0(t)(\phi_0'\gamma_0) - \gamma_0$ can be bounded in terms of γ_0'. Crucially, this implies that the same holds for spatial and temporal derivatives of $S_p^0(t)(\phi_0'\gamma_0)$, allowing us to recover smallness in the linear terms arising in the Duhamel representations of *derivatives* of $\gamma(t)$.

There is, however, a caveat: bounding $S_p^0(t)(\phi_0'\gamma_0) - \gamma_0$ and its derivatives in terms of γ_0' yields only weak decay rates and, in some cases, even temporal growth. This is also reflected in identity (4.20), where the time integral on the right-hand side naturally leads to bounds exhibiting growth in t. We address this issue by *interpolating* between two types of bounds: (i) classical bounds on $\mathrm{e}^{\mathcal{L}_0 t}(\phi_0'\gamma_0)$ and $\partial_\zeta^j\partial_t^l S_p^0(t)(\phi_0'\gamma_0)$, used in the stability argument in [84], which provide sufficient decay but not smallness, and (ii) bounds on $\mathrm{e}^{\mathcal{L}_0 t}(\phi_0'\gamma_0) - \phi_0'\gamma_0$ and $\partial_\zeta^j\partial_t^l(S_p^0(t)(\phi_0'\gamma_0) - \gamma_0)$ in terms of γ_0', which provide smallness but not sufficient decay. In §4.2, the core idea behind this interpolation strategy is illustrated by means of a representative toy example.

By carefully balancing smallness and decay, we are able to derive sufficient bounds to close the nonlinear iteration scheme. Heuristically, this is possible because the nonlinear estimates in [84] leave some leeway, i.e., the decay bounds on the nonlinear terms are not critical and can tolerate some weakening.

An important distinction from the analysis in [84] lies in how we control regularity in the nonlinear iteration. In [84], regularity is controlled via estimates on the perturbation $\tilde{v}(t) = u(t) - \phi_0$. However, Theorem 4.1.3 shows that in the current modulational setting, $\tilde{v}(t)$ is no longer small, making it natural to instead consider to consider the forward-modulated perturbation $\mathring{v}(t)$. Unlike the inverse-modulated perturbation $v(t)$, which obeys a quasilinear equation (see §4.5.2), the forward-modulated perturbation satisfies a semilinear equation that does not suffer from a loss of derivatives. However, the obtained decay rates of $\mathring{v}(t)$ are too weak to gain good enough large-time control on derivatives. Therefore, following the approach in [5], we establish a nonlinear damping estimate for the *modified forward-modulated perturbation*

$$\mathring{z}(\zeta, t) = u(\zeta, t) - \phi(\zeta + \gamma(\zeta, t)(1 + \gamma_\zeta(\zeta, t)); k_0(1 + \gamma_\zeta(\zeta, t))) \tag{4.21}$$

in order to control regularity for large times. As indicated by the estimate (4.12), this variable enjoys better decay properties than $\mathring{v}(t)$. Due to the absence of L^2-localization, we derive this energy estimate in uniformly local Sobolev spaces; see [94] for background.

With the regularity control in hand, we are able to close a nonlinear iteration argument, thereby proving Theorem 4.1.3. Further details of the nonlinear iteration scheme are provided in §4.5.

4.1.3. Outline

In §4.2, we present the core idea behind handling large modulational data through interpolation arguments in the context of a toy model. Section 4.3 reviews standard preliminary results on wave trains. In §4.4, we recall the semigroup decomposition from [84] and summarize the corresponding linear estimates established therein. We also carry out a detailed analysis of the linear dynamics arising from large, fully nonlocalized modulational initial data and derive associated estimates. In §4.5, we develop our nonlinear iteration scheme and establish a nonlinear damping estimate to control regularity for large times. Section 4.6 is devoted to the proof of our main result, Theorem 4.1.3. An outlook on future research directions is provided in §4.7. Appendix 4.A contains technical low- and high-frequency estimates required for the linear analysis of large modulational data. Finally, Appendix 4.B provides the proof of the local existence result for the phase modulation.

Notation. Let S be a set, and let $A, B \colon S \to \mathbb{R}$. Throughout the paper, the expression "$A(x) \lesssim B(x)$ for $x \in S$", means that there exists a constant $C > 0$, independent of x, such that $A(x) \leq CB(x)$ holds for all $x \in S$.

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4.2. Heuristic behind handling large modulational data

The goal of this section is to convey the core idea behind our L^∞-based nonlinear iteration argument designed to handle large phase modulations. Motivated by the fact that the leading-order modulational dynamics is given by the viscous Hamilton-Jacobi equation (4.10), we consider

$$\partial_t w = dw_{\zeta\zeta} + aw_\zeta + \nu w_\zeta^2 + \mu w_\zeta^3 \tag{4.22}$$

with $d > 0$ and $a, \mu, \nu \in \mathbb{R}$. Here, the cubic contribution w_ζ^3 plays the role of a higher-order residual term. When $\nu \neq 0$, the quadratic term in (4.22) can be removed via the Cole-Hopf transform

$$v(t) = \mathrm{e}^{\frac{\nu}{d} w(t)},$$

which transforms the equation into

$$\partial_t v = dv_{\zeta\zeta} + av_\zeta + \frac{\mu d^2}{\nu^2}\frac{v_\zeta^3}{v^2}.$$

This motivates restricting attention to the case $\nu = 0$ in (4.22). Moreover, by transitioning to a co-moving frame and rescaling space, we may assume without loss of generality that $a = 0$ and $d = 1$. The resulting simplified toy problem takes the form

$$\partial_t w = w_{\zeta\zeta} + \mu w_\zeta^3 \tag{4.23}$$

with $\mu \in \mathbb{R}$. We use this model to illustrate how our L^∞-based scheme effectively manages large phase modulations. Specifically, under the sole assumption that the initial condition $w_0 \in C_{\mathrm{ub}}^1(\mathbb{R})$ has sufficiently small derivative, we establish global existence of the associated solution $w(t)$ to (4.23) and show that its derivative $w_\zeta(t)$ decays at a near-diffusive rate.

Take $w_0 \in C_{\mathrm{ub}}^1(\mathbb{R})$. Integrating (4.23), we obtain the Duhamel formula

$$w(t) = \mathrm{e}^{\partial_\zeta^2 t} w_0 + \mu \int_0^t \mathrm{e}^{\partial_\zeta^2 (t-s)} w_\zeta(s)^3 \mathrm{d}s. \tag{4.24}$$

By standard local existence theory for semilinear parabolic equations [73], there exist $T \in (0, \infty]$ and a unique maximal solution $w \in C\big([0,T), C_{\mathrm{ub}}^1(\mathbb{R})\big)$ satisfying (4.24). If $T < \infty$, then it holds

$$\sup_{t \uparrow T} \|w(t)\|_{C_{\mathrm{ub}}^1} = \infty. \tag{4.25}$$

The smoothing effect of the heat semigroup leads to the linear L^∞-estimate

$$\left\|\partial_\zeta^j \mathrm{e}^{\partial_\zeta^2 t} v\right\|_\infty \lesssim t^{-\frac{j}{2}} \|v\|_\infty \tag{4.26}$$

for $j = 0, 1$, $v \in C_{\mathrm{ub}}(\mathbb{R})$, and $t > 0$. On the other hand, we may commute the derivative with the semigroup, which gives

$$\left\|\partial_\zeta \mathrm{e}^{\partial_\zeta^2 t} v\right\|_\infty = \left\|\mathrm{e}^{\partial_\zeta^2 t} \partial_\zeta v\right\|_\infty \leq \|\partial_\zeta v\|_\infty \tag{4.27}$$

for $v \in C_{\mathrm{ub}}^1(\mathbb{R})$ and $t \geq 0$. Estimates of the form (4.27), where derivatives are transferred onto the initial data through the linear propagator, are crucial to exploit smallness in the derivative of the (possibly large) initial data; see §4.4.1.

If the initial condition w_0 is small in $C^1_{\mathrm{ub}}(\mathbb{R})$, then the linear behavior dominates, and the solution $w(t)$ to (4.24) remains bounded while its derivative decays diffusively at rate $t^{-\frac{1}{2}}$; see [51, Section 2]. Here, we show that, by sacrificing a small amount of decay, we obtain global-in-time control on solutions to (4.24) with large initial data. To that end, we fix $M > 0$ and $a \in (0, \frac{1}{6})$, take $w_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|w_0\|_{C^1_{\mathrm{ub}}} \leq M$, and introduce the template function $\eta\colon [0,T) \to \mathbb{R}$ defined by

$$\eta(t) = \sup_{0\leq s\leq t} (1+s)^{\frac{1}{2}-a} \left\|\partial_\zeta w(s)\right\|_\infty.$$

Using the linear bound (4.26), we estimate the right-hand side of (4.24) as

$$\|w(t)\|_\infty \lesssim \|w_0\|_\infty + \int_0^t \frac{\eta(t)^3}{(1+s)^{\frac{3}{2}-3a}} \mathrm{d}s \lesssim \|w_0\|_\infty + \eta(t)^3, \tag{4.28}$$

for $t \in [0,T)$. For the spatial derivative of (4.24), we apply (4.27) for $s \in [0,1]$ and use the interpolation inequality

$$\left\|\partial_\zeta \mathrm{e}^{\partial_\zeta^2 s} w_0\right\|_\infty \leq \left\|\partial_\zeta \mathrm{e}^{\partial_\zeta^2 s} w_0\right\|_\infty^{1-2a} \left\|\mathrm{e}^{\partial_\zeta^2 s} w_0'\right\|_\infty^{2a} \lesssim (1+s)^{\frac{1}{2}-a} \|w_0\|_\infty^{1-2a} \|w_0'\|_\infty^{2a},$$

for $s \geq 1$, yielding the bound

$$\left\|\partial_\zeta \mathrm{e}^{\partial_\zeta^2 s} w_0\right\|_\infty \lesssim (1+s)^{\frac{1}{2}-a} \|w_0\|_{C^1_{\mathrm{ub}}}^{1-2a} \|w_0'\|_\infty^{2a},$$

for all $s \geq 0$. Altogether, we estimate the spatial derivative of (4.24) as

$$\begin{aligned} \|w_\zeta(s)\|_\infty &\lesssim (1+s)^{\frac{1}{2}-a} \|w_0\|_{C^1_{\mathrm{ub}}}^{1-2a} \|w_0'\|_\infty^{2a} + \int_0^s \frac{\eta(s)^3}{\sqrt{s-r}\,(1+r)^{\frac{3}{2}-3a}} \mathrm{d}r \\ &\lesssim (1+s)^{-\frac{1}{2}+a} \Big(M^{1-2a} \|w_0'\|_\infty^{2a} + \eta(s)^3 \Big), \end{aligned}$$

for $s \in [0,T)$. Taking the supremum in the latter inequality over s, we find a constant $C \geq 1$, independent of t and w_0, such that

$$\eta(t) \leq C \Big(\|w_0'\|_\infty^{2a} + \eta(t)^3 \Big), \tag{4.29}$$

for all $t \in [0,T)$. Now assume $\|w_0'\|_\infty^{4a} < \frac{1}{8C^2}$. If there exists $t \in (0,T)$ with $\eta(t) > 2C\|w_0'\|_\infty^{2a}$, then by continuity of η and the fact that $\eta(0) = \|w_0'\|_\infty < 2C\|w_0'\|_\infty^{2a}$, there must exist $\tau \in [0,t]$ such that $\eta(\tau) = 2C\|w_0'\|_\infty^{2a}$. Applying (4.29) and $\|w_0'\|_\infty^{4a} < \frac{1}{8C^3}$, we obtain

$$\eta(\tau) \leq C \Big(\|w_0'\|_\infty^{2a} + \eta(\tau)^3 \Big) < 2C\|w_0'\|_\infty^{2a},$$

yielding a contradiction. Hence, we must have $\eta(t) \leq 2C\|w_0'\|_\infty^{2a}$ for all $t \in [0,T)$. Combining this with (4.25) and (4.28), we conclude that $T = \infty$. So, the solution is global and enjoys the bounds

$$\|w(t)\|_\infty \lesssim \|w_0\|_\infty + \|w_0'\|_\infty^{6a}, \qquad \|w_\zeta(t)\|_\infty \lesssim \frac{\|w_0'\|_\infty^{2a}}{(1+t)^{\frac{1}{2}-a}}$$

for $t \geq 0$.

We emphasize that the nonlinear iteration argument above relies exclusively on L^∞-estimates and requires smallness solely in the *derivative* of the initial data. This is because the nonlinearity in (4.23) involves only spatial derivatives of w. Analogously, the equations for the inverse- and forward-modulated perturbations, used in the proof of Theorem 4.1.3, depend only on derivatives of the phase modulation $\gamma(t)$.

4.3. Preliminaries

We recall some basic results on wave trains and their dispersion relations from [84]. For a more comprehensive treatment, we refer the reader to [29, Section 4].

The first result asserts that, under the nondegeneracy condition (D3), the wave train can be continued in the wavenumber.

Proposition 4.3.1. *Assume* (H1) *and* (D3). *Then, there exist* $r_0 \in (0, \frac{1}{2})$ *and smooth functions* $\phi\colon \mathbb{R} \times (k_0 - r_0, k_0 + r_0) \to \mathbb{R}$ *and* $\omega\colon (k_0 - r_0, k_0 + r_0) \to \mathbb{R}$, *satisfying* (4.9), *such that*

$$u_k(x,t) = \phi(kx - \omega(k)t; k)$$

is a wave-train solution to (4.1) *for each wavenumber* $k \in (k_0 - r_0, k_0 + r_0)$. *The profile function* $\phi(\cdot; k)$ *has period* 1 *for each* $k \in (k_0 - r_0, k_0 + r_0)$.

The next result provides an expansion of the critical spectral curve of the linearization $\mathcal{L}_0$ in terms of the Bloch frequency parameter ξ. The curve, known as the *linear dispersion relation*, touches the origin in a quadratic tangency. In addition, the result includes an expansion of the associated Bloch eigenfunctions.

Proposition 4.3.2. *Assume* (H1) *and* (D2)-(D3). *Then, there exist a constant* $\xi_0 \in (0, \pi)$ *and an analytic curve* $\lambda_c\colon (-\xi_0, \xi_0) \to \mathbb{C}$ *satisfying*

(i) *The complex number* $\lambda_c(\xi)$ *is a simple eigenvalue of* $\mathcal{L}(\xi)$ *for any* $\xi \in (-\xi_0, \xi_0)$. *An associated eigenfunction* Φ_ξ *of* $\mathcal{L}(\xi)$ *lies in* $H^m_{\mathrm{per}}(0,1)$ *for each* $m \in \mathbb{N}_0$, *satisfies* $\Phi_0 = \phi_0'$, *is analytic in* ξ *and fulfills*

$$\langle \widetilde{\Phi}_0, \Phi_\xi \rangle_{L^2(0,1)} = 1.$$

(ii) *The complex conjugate* $\overline{\lambda_c(\xi)}$ *is a simple eigenvalue of the adjoint* $\mathcal{L}(\xi)^*$ *for any* $\xi \in (-\xi_0, \xi_0)$. *An associated eigenfunction* $\widetilde{\Phi}_\xi$ *lies in* $H^m_{\mathrm{per}}(0,1)$ *for each* $m \in \mathbb{N}_0$, *is smooth in* ξ *and satisfies*

$$\langle \widetilde{\Phi}_\xi, \Phi_\xi \rangle_{L^2(0,1)} = 1.$$

(iii) *The expansions*

$$\left|\lambda_c(\xi) - \mathrm{i}a\xi + d\xi^2\right| \lesssim |\xi|^3, \qquad \left\|\Phi_\xi - \phi_0' - \mathrm{i}k_0\xi\partial_k\phi(\cdot;k_0)\right\|_{H^m(0,1)} \lesssim |\xi|^2$$

hold for $\xi \in (-\xi_0, \xi_0)$ *with coefficients* $a \in \mathbb{R}$ *and* $d > 0$ *given by* (4.11).

4.4. Semigroup decomposition and linear estimates

The linearization $\mathcal{L}_0$ of (4.7) about the wave train ϕ_0 is a densely defined sectorial operator on $C_{\mathrm{ub}}(\mathbb{R})$, with domain $D(\mathcal{L}_0) = C^2_{\mathrm{ub}}(\mathbb{R})$; see [73, Corollary 3.1.9]. In this section, we recall the decomposition of the associated analytic semigroup $\mathrm{e}^{\mathcal{L}_0 t}$ carried out in [84], and summarize the corresponding linear estimates established therein. In addition, we derive novel estimates on modulational data, which are crucial for treating large phase modulations.

The first result from [84] expresses $\mathrm{e}^{\mathcal{L}_0 t}$ as the sum of an explicit principal part, exhibiting the same L^∞-bounds as the heat semigroup $\mathrm{e}^{\partial_\zeta^2 t}$, and a remainder that decays algebraically at rate t^{-1}.

Proposition 4.4.1 ([84, Propositions 3.2 and 3.4]). *Assume* (H1) *and* (D1)-(D3). *Let* $\phi(\cdot;k)$, ξ_0, λ_c, a, *and* $\widetilde{\Phi}_\xi$ *be as in Propositions 4.3.1 and 4.3.2. Fix* $j,l,m \in \mathbb{N}_0$ *and* $\ell_0,\ell_1 \in \{0,1\}$ *with* $\ell_0+\ell_1 \leq 1$. *The semigroup* $\mathrm{e}^{\mathcal{L}_0 t}$ *decomposes as*

$$\mathrm{e}^{\mathcal{L}_0 t} = (\phi_0' + k_0\partial_k\phi(\cdot;k_0)\partial_\zeta)S_p^0(t) + \widetilde{S}(t) \tag{4.30}$$

for $t \geq 0$, *where the principal part satisfies the commutator identities*

$$\partial_\zeta S_p^0(t) - S_p^0(t)\partial_\zeta = S_p^1(t), \qquad \partial_\zeta^2 S_p^0(t) - S_p^0(t)\partial_\zeta^2 = 2\partial_\zeta S_p^1(t) - S_p^2(t) \tag{4.31}$$

and is explicitly given by

$$S_p^i(t)v(\zeta) = \chi(t)\int_{\mathbb{R}} G_p^i(\zeta,\bar{\zeta},t)v(\bar{\zeta})\mathrm{d}\bar{\zeta}, \qquad G_p^i(\zeta,\bar{\zeta},t) = \frac{1}{2\pi}\int_{\mathbb{R}} \rho(\xi)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar{\zeta})}\mathrm{e}^{\lambda_c(\xi)t}\partial_{\bar{\zeta}}^i\widetilde{\Phi}_\xi(\bar{\zeta})^*\mathrm{d}\xi \tag{4.32}$$

for $i = 0,1,2$, *where* $\rho\colon \mathbb{R} \to [0,1]$ *and* $\chi\colon [0,\infty) \to [0,1]$ *are smooth cut-off functions satisfying* $\rho(\xi) = 1$ *for* $|\xi| < \frac{\xi_0}{2}$, $\rho(\xi) = 0$ *for* $|\xi| > \xi_0$, $\chi(t) = 0$ *for* $t \in [0,1]$, *and* $\chi(t) = 1$ *for* $t \in [2,\infty)$. *Moreover, the estimates*

$$\begin{aligned}
\left\|\partial_\zeta^{\ell_0}\mathrm{e}^{\mathcal{L}_0 t}\partial_\zeta^{\ell_1}v\right\|_\infty &\lesssim \left(1+t^{-\frac{\ell_0+\ell_1}{2}}\right)\|v\|_\infty,\\
\left\|\widetilde{S}(t)\partial_\zeta^{\ell_1}v\right\|_\infty &\lesssim (1+t)^{-1}\left(1+t^{-\frac{\ell_1}{2}}\right)\|v\|_\infty,\\
\left\|(\partial_t - a\partial_\zeta)^j\partial_\zeta^l S_p^i(t)\partial_\zeta^m w\right\|_\infty &\lesssim (1+t)^{-\frac{2j+l}{2}}\|w\|_\infty
\end{aligned}$$

hold for all $v \in C_{\mathrm{ub}}^{\ell_1}(\mathbb{R})$, $w \in C_{\mathrm{ub}}^m(\mathbb{R})$, $t > 0$, *and* $i = 0,1,2$.

The second result from [84] connects the principal component of the semigroup to the convective heat equation $\partial_t u = du_{\zeta\zeta} + au_\zeta$, which arises from the quadratic truncation of the low-frequency expansion of the linear dispersion relation in Proposition 4.3.2.

Proposition 4.4.2 ([84, Propositions 3.6 and 3.7]). *Assume* (H1) *and* (D1)-(D3). *Let* $l \in \mathbb{N}_0$, $m \in \{0,1\}$, *and* $i \in \{0,1,2\}$. *There exists a bounded linear operator*

$$A_h\colon L^2_{\mathrm{per}}((0,1),\mathbb{R}^n) \to C(\mathbb{R},\mathbb{R})$$

such that the principal component $S_p^i(t)$ *decomposes as*

$$S_p^i(t) = S_h^i(t) + \widetilde{S}_r^i(t), \tag{4.33}$$

where we denote

$$S_h^i(t)v = \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\left[\left(\partial_\zeta^i\widetilde{\Phi}_0^*\right)v\right]$$

for $v \in C_{\mathrm{ub}}(\mathbb{R})$ *and* $t \geq 0$. *Moreover, we have*

$$S_h^0(t)(gv) = \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\left(\langle\widetilde{\Phi}_0, g\rangle_{L^2(0,1)}v - A_h(g)\partial_\zeta v\right) + \partial_\zeta\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}(A_h(g)v) \tag{4.34}$$

for $g \in L^2_{\mathrm{per}}((0,1),\mathbb{R}^n)$, $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$, *and* $t > 0$. *Finally, the estimates*

$$\left\|\partial_\zeta^l S_h^i(t)v\right\|_\infty \lesssim t^{-\frac{l}{2}}\|v\|_\infty, \qquad \left\|\partial_\zeta^m \widetilde{S}_r^i(t)v\right\|_\infty \lesssim (1+t)^{-\frac{1}{2}}t^{-\frac{m}{2}}\|v\|_\infty$$

and

$$\begin{gathered}\left\|\partial_\zeta^{1+l}\mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)t}w\right\|_\infty \lesssim t^{-\frac{l}{2}}\|w'\|_\infty, \qquad \left\|(\partial_t - a\partial_\zeta)\partial_\zeta^l\mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)t}w\right\|_\infty \lesssim t^{-\frac{1+l}{2}}\|w'\|_\infty,\\ \left\|\partial_\zeta^l\mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)t}v\right\|_\infty \lesssim t^{-\frac{l}{2}}\|v\|_\infty\end{gathered} \tag{4.35}$$

hold for $v \in C_{\mathrm{ub}}(\mathbb{R})$, $w \in C^1_{\mathrm{ub}}(\mathbb{R})$, *and* $t > 0$.

4.4.1. L^∞-ESTIMATES ON MODULATIONAL DATA

Initial phase modulations γ_0 of the wave train ϕ_0 give rise to data of the form $\phi_0'\gamma_0$ in the equation for the inverse-modulated perturbation; see §4.5.2. Such *modulational data* correspond to the linearized approximation of the modulational perturbation $\phi_0(\zeta+\gamma_0(\zeta)) - \phi_0(\zeta) \approx \phi_0'(\zeta)\gamma_0(\zeta)$. Since we allow for large phase modulations γ_0, the nonlinear argument can only exploit smallness in the *derivative* γ_0'. Here, we establish bounds that are designed to estimate the action of linear propagators on modulational data in terms of γ_0'.

Linear estimates on modulational data have also been obtained in [56, 63] under the additional assumption that $\gamma_0' \in L^1(\mathbb{R})$. The approach in [56, 63] involves a detailed analysis in Bloch frequency domain, relying on the Hausdorff-Young inequality to control the action of the propagator in terms of the L^1-norm of γ_0'. Although one could potentially extend this analysis to C_{ub}-data by working with tempered distributions, we avoid such technicalities. Instead, we derive bounds using abstract semigroup theory, a careful decomposition of the principal component $S_p^0(t)$ of the semigroup, and an adaptation of the high- and low-frequency L^∞-bounds from [84, Appendix A] and [51, Appendix A] to the setting of modulational data.

We begin by estimating the action of the full semigroup on modulational data.

Proposition 4.4.3. *Assume* (H1) *and* (D1)-(D3). *Then, the estimates*

$$\begin{aligned}\left\|\mathrm{e}^{\mathcal{L}_0 t}(\phi_0'v) - \phi_0'v\right\|_\infty &\lesssim \sqrt{t(1+t)}\,\|v'\|_\infty,\\ \left\|\mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)t}v - v\right\|_\infty &\lesssim \sqrt{t(1+t)}\,\|v'\|_\infty,\\ \left\|\mathrm{e}^{-\partial_\zeta^4 t}v - v\right\|_\infty &\lesssim t^{\frac{1}{4}}\|v'\|_\infty\end{aligned}$$

hold for all $t > 0$ *and* $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$.

Proof. We employ [73, Proposition 2.1.4] and use that ϕ_0' lies in the kernel of $\mathcal{L}(0)$ to infer

$$\begin{gathered}\mathrm{e}^{\mathcal{L}_0 t}(\phi_0'v) - \phi_0'v = \int_0^t \mathrm{e}^{\mathcal{L}_0 s}\mathcal{L}_0(\phi_0'v)\mathrm{d}s = \int_0^t \mathrm{e}^{\mathcal{L}_0 s}\Big(k_0^2 D\big(\partial_\zeta(\phi_0'v') + \phi_0''v'\big) + \omega_0\phi_0'v'\Big)\mathrm{d}s,\\ \mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)t}v - v = \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)s}(d\partial_\zeta + a)v'\mathrm{d}s, \qquad \mathrm{e}^{-\partial_\zeta^4 t}w - w = \int_0^t \mathrm{e}^{-\partial_\zeta^4 s}\partial_\zeta^3 w'\mathrm{d}s\end{gathered}$$

for $v \in C^2_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$, $w \in C^4_{\mathrm{ub}}(\mathbb{R})$, and $t > 0$. Applying the estimates from Propositions 4.4.1 and 4.4.2 and the bound

$$\|\partial_\zeta^3 \mathrm{e}^{-\partial_\zeta^4 s} z\|_\infty \lesssim s^{-\frac{3}{4}}\|z\|_\infty, \qquad z \in C_{\mathrm{ub}}(\mathbb{R}),\ s > 0$$

to the right-hand sides of the latter, we arrive at

$$\left\|\mathrm{e}^{\mathcal{L}_0 t}(\phi_0' v) - \phi_0' v\right\|_\infty, \left\|\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t} v - v\right\|_\infty \lesssim \int_0^t \left(1 + \frac{1}{\sqrt{s}}\right)\|v'\|_\infty \mathrm{d}s \lesssim \sqrt{t(1+t)}\,\|v'\|_\infty,$$
$$\left\|\mathrm{e}^{-\partial_\zeta^4 t} w - w\right\|_\infty \lesssim \int_0^t s^{-\frac{3}{4}}\|w'\|_\infty \mathrm{d}s \lesssim t^{\frac{1}{4}}\|w'\|_\infty$$

for $v \in C^2_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$, $w \in C^4_{\mathrm{ub}}(\mathbb{R})$, and $t > 0$. Thus, the result follows by density. □

The remaining estimates concern the action of the principal component of the semigroup on modulational data.

Proposition 4.4.4. *Assume* (H1) *and* (D1)-(D3). *Let* $g \in L^2_{\mathrm{per}}(0,1)$. *Fix* $l, m \in \mathbb{N}_0$ *with* $l \geq 1$. *Then, the estimates*

$$\left\|(\partial_t - a\partial_\zeta)^m \partial_\zeta^l S_p^0(t)(gv)\right\|_\infty \lesssim (1+t)^{-\frac{2m+l-1}{2}}\|v'\|_\infty, \tag{4.36}$$
$$\left\|(\partial_t - a\partial_\zeta)^m S_p^0(t)(\phi_0' v) - (\partial_t^m \chi(t))\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t} v\right\|_\infty \lesssim \|v'\|_\infty \tag{4.37}$$

hold for $t \geq 0$ *and* $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$.

Proof. We compute

$$\begin{aligned}&(\partial_t - a\partial_\zeta)^m \partial_\zeta^l \int_{\mathbb{R}} G_p^0(\zeta,\bar\zeta,t) g(\bar\zeta) v(\bar\zeta)\mathrm{d}\bar\zeta \\ &\quad = \frac{\mathrm{i}^l}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\lambda_c(\xi)t}\xi^{l+2m}\rho(\xi)\left(\frac{\lambda_c(\xi) - \mathrm{i}a\xi}{\xi^2}\right)^m \widetilde{\Phi}_\xi(\bar\zeta)^* g(\bar\zeta)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)} v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta\end{aligned}$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$. Using Proposition 4.3.2, we observe that we can apply Lemma 4.A.1 with $\lambda(\xi) = \lambda_c(\xi)$, $m_1 = l + 2m - 1 \geq 0$, $m_2 = 1$, and

$$F(\xi,\zeta,\bar\zeta,t) = \rho(\xi)\left(\frac{\lambda_c(\xi) - \mathrm{i}a\xi}{\xi^2}\right)^m \widetilde{\Phi}_\xi(\bar\zeta)^* g(\bar\zeta).$$

This results in the bound (4.36) upon recalling the representation (4.32) of the principal component of the semigroup and noting that $\chi'(t)$ is supported on $[1,2]$.

We proceed with proving the second estimate. First, applying the product rule to the representation (4.32), we establish

$$\begin{aligned}&\left\|(\partial_t - a\partial_\zeta)^m S_p^0(t)(\phi_0' v) - (\partial_t^m \chi(t))\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t} v\right\|_\infty \\ &\quad \lesssim \left\|\int_{\mathbb{R}} G_p^0(\cdot,\bar\zeta,t)\phi_0'(\bar\zeta)v(\bar\zeta)\mathrm{d}\bar\zeta - \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t} v\right\|_\infty \\ &\qquad + \sum_{\ell=1}^m \left\|(\partial_t - a\partial_\zeta)^\ell \int_{\mathbb{R}} G_p^0(\cdot,\bar\zeta,t)\phi_0'(\bar\zeta)v(\bar\zeta)\mathrm{d}\bar\zeta\right\|_\infty =: J_1 + J_2\end{aligned} \tag{4.38}$$

for $t \geq 1$ and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$. Analogous to the proof of estimate (4.36), we obtain the bound

$$|J_2| \lesssim \|v'\|_\infty$$

for $t \geq 1$ and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$. So, all that remains is to estimate J_1. To that end, we set $\lambda_r(\xi) = \lambda_c(\xi) - \mathrm{i}a\xi + d\xi^2$ and decompose

$$\int_{\mathbb{R}} G^0_p(\zeta,\bar\zeta,t)\phi_0'(\bar\zeta)v(\bar\zeta)\mathrm{d}\bar\zeta - \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t} v = I + II + III + IV \tag{4.39}$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$, where we denote

$$\begin{aligned}
I &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\lambda_c(\xi)t}\xi\rho(\xi)\frac{\widetilde\Phi_\xi(\bar\zeta)^* - \widetilde\Phi_0(\bar\zeta)^*}{\xi}\,\phi_0'(\bar\zeta)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta,\\
II &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\left(\mathrm{i}a\xi - \frac{d}{2}\xi^2\right)t}\xi\rho(\xi)\mathrm{e}^{-\frac{d}{2}\xi^2 t}\,\frac{\mathrm{e}^{\lambda_r(\xi)t}-1}{\xi}\,\widetilde\Phi_0(\bar\zeta)^*\phi_0'(\bar\zeta)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta,\\
III &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\left(\mathrm{i}a\xi - d\xi^2\right)t}\rho(\xi)\Big(\widetilde\Phi_0(\bar\zeta)^*\phi_0'(\bar\zeta) - 1\Big)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta,\\
IV &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta) + \left(\mathrm{i}a\xi - d\xi^2\right)t}(\rho(\xi)-1)v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta,
\end{aligned}$$

In the following, we subsequently bound the contributions I, II, III, and IV. First, we use Proposition 4.3.2 and apply Lemma 4.A.1 to I with $\lambda(\xi) = \lambda_c(\xi)$, $m_1 = 0$, $m_2 = 1$, and

$$F(\xi,\zeta,\bar\zeta,t) = \rho(\xi)\frac{\widetilde\Phi_\xi(\bar\zeta)^* - \widetilde\Phi_0(\bar\zeta)^*}{\xi}\,\phi_0'(\bar\zeta).$$

Thus, we arrive at the bound

$$|I| \lesssim \|v'\|_\infty$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$.

We proceed with bounding II. We note that, by Proposition 4.3.2, there exists a constant $C > 0$ such that $|\lambda_r(\xi)| \leq C|\xi|^3$, $|\lambda_r'(\xi)| \leq C|\xi|^2$, and $|\lambda_r''(\xi)| \leq C|\xi|$ for $\xi \in (-\xi_0,\xi_0)$. Combining this with the identity $|\mathrm{e}^z - 1| \leq \mathrm{e}^{|z|} - 1 \leq |z|\mathrm{e}^{|z|}$ for $z \in \mathbb{C}$, we establish

$$\begin{aligned}
\left|\mathrm{e}^{-\frac{d}{2}\xi^2 t}\,\frac{\mathrm{e}^{\lambda_r(\xi)t}-1}{\xi}\right| &\lesssim \xi^2 t\mathrm{e}^{-\frac{d}{4}\xi^2 t} \lesssim 1,\\
\left|\partial_\xi\left(\mathrm{e}^{-\frac{d}{2}\xi^2 t}\,\frac{\mathrm{e}^{\lambda_r(\xi)t}-1}{\xi}\right)\right| &\lesssim |\xi|t\Big(1+\xi^2 t\Big)\mathrm{e}^{-\frac{d}{4}\xi^2 t} \lesssim \sqrt{t},
\end{aligned}$$

and

$$\left|\partial_\xi^2\left(\mathrm{e}^{-\frac{d}{2}\xi^2 t}\,\frac{\mathrm{e}^{\lambda_r(\xi)t}-1}{\xi}\right)\right| \lesssim t\Big(1+\xi^2 t + \xi^4 t^2\Big)\mathrm{e}^{-\frac{d}{4}\xi^2 t} \lesssim t$$

for $\xi \in (-\xi_0,\xi_0)$ and $t \geq 1$. Hence, applying Lemma 4.A.1 with $\lambda(\xi) = a\mathrm{i}\xi - \frac{d}{2}\xi^2$, $m_1 = 0$, $m_2 = 1$, and

$$F(\xi,\zeta,\bar\zeta,t) = \rho(\xi)\mathrm{e}^{-\frac{d}{2}\xi^2 t}\frac{\mathrm{e}^{\lambda_r(\xi)t}-1}{\xi}\,\widetilde\Phi_0(\bar\zeta)^*\phi_0'(\bar\zeta),$$

we bound

$$|II| \lesssim \|v'\|_\infty.$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$.

To estimate the contribution III, we observe that we can apply Lemma 4.A.1 with $\lambda(\xi) = a\mathrm{i}\xi - d\xi^2$, $m_1 = 0$, $m_2 = 0$, and

$$F(\xi,\zeta,\bar\zeta,t) = \rho(\xi)\Big(\widetilde\Phi_0(\bar\zeta)^*\phi_0'(\bar\zeta) - 1\Big),$$

because it holds $\int_0^1 \widetilde\Phi_0(\bar\zeta)^*\phi_0'(\bar\zeta)\mathrm{d}\bar\zeta = \langle\widetilde\Phi_0,\phi_0'\rangle_{L^2(0,1)} = 1$ by (4.8). Thus, we arrive at the estimate

$$|III| \lesssim \|v'\|_\infty$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$.

Finally, we estimate the contribution IV. Integration by parts in $\bar\zeta$ yields

$$IV = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)+\left(\mathrm{i}a\xi-d\xi^2\right)t}\,\frac{\rho(\xi)-1}{\xi}\,v'(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$. Hence, applying the high-frequency estimate from Lemma 4.A.2 with $F(\xi) = (\rho(\xi)-1)\xi^{-1}$, we find a constant $\mu_0 > 0$ such that

$$|IV| \lesssim \mathrm{e}^{-\mu_0 t}\|v'\|_\infty$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$. Combining (4.38) with (4.39), and the estimates on J_2, I, II, III, and IV, we arrive at the bound (4.37), which concludes the proof. □

4.5. Nonlinear iteration scheme

In this section, we introduce the nonlinear iteration scheme that will be employed in §4.6 to prove our nonlinear stability result, Theorem 4.1.3. To this end, let $u_{\mathrm{wt}}(x,t) = \phi_0(k_0x - \omega_0 t)$ denote a diffusively spectrally stable wave-train solution of (4.1), satisfying assumptions (H1) and (D1)-(D3). Take a perturbation $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and a phase modulation $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$, and consider the solution $u(t)$ to (4.7) with initial condition $u(0) = u_0 \in C_{\mathrm{ub}}(\mathbb{R})$ given by

$$u_0(\zeta) = \phi_0(\zeta + \gamma_0(\zeta)) + \mathring{v}_0(\zeta), \qquad \zeta \in \mathbb{R}. \tag{4.40}$$

Assuming that $\mathring{v}_0$ and γ_0' are sufficiently small in $C_{\mathrm{ub}}(\mathbb{R})$, our aim is to construct a spatiotemporal modulation function $\gamma(t)$ with $\gamma(0) = \gamma_0$ such that the solution $u(t)$ to (4.7) can be written in the form

$$u(\zeta,t) = \phi_0(\zeta + \gamma(\zeta,t)) + \mathring{v}(\zeta,t)$$

where both $\gamma_\zeta(t)$ and the remainder $\mathring{v}(t)$ stay small over time in $C_{\mathrm{ub}}(\mathbb{R})$ and decay at diffusive rates. In particular, this precludes finite-time blow-up and implies that the solution $u(t)$ exists globally in time.

A nonlinear iteration argument cannot be closed by directly estimating the *forward-modulated perturbation* $\mathring{v}(t)$, due to insufficient decay in the nonlinear terms of its evolution equation;

see [105] for a detailed discussion. As outlined in §4.1.2, our approach instead relies on the fact that the L^∞-norm of $\mathring{v}(t)$ is equivalent to that of the *inverse-modulated perturbation* $v(t)$, which is given by (4.18). Taking inspiration from [56], we then choose a phase modulation $\gamma(t)$, which captures the most critical terms in the Duhamel formula for the inverse-modulated perturbation $v(t)$ and satisfies $\gamma(0) = \gamma_0$. As in [57, 84, 89], we find that $\gamma(t)$ obeys a perturbed viscous Hamilton-Jacobi equation. After eliminating its dominant nonlinear term via the Cole-Hopf transformation, an L^∞-based nonlinear iteration argument involving $\gamma(t)$ and $v(t)$ can be closed.

However, due to the quasilinear nature of the evolution equation for the inverse-modulated perturbation $v(t)$, an apparent loss of regularity must be addressed. To control regularity, we distinguish between short and long times. For short times, we estimate the forward-modulated perturbation $\mathring{v}(t)$ iteratively using its Duhamel representation, which is of semilinear nature and does not suffer from a loss of derivatives. This allows us to control regularity of the inverse-modulated perturbation $v(t)$ by relating $\mathring{v}(t)$ and $v(t)$, including their derivatives, through mean-value type estimates; see [105]. However, the obtained decay of $\mathring{v}(t)$ and its derivatives is too slow to provide effective control for large times. To overcome this, we follow the approach in [5] and employ forward-modulated damping estimates. Specifically, we derive a nonlinear damping estimate in uniformly local Sobolev spaces for the *modified forward-modulated perturbation* (4.21), which also satisfies a semilinear equation without derivative loss. This energy estimate provides control over the L^∞-norms of derivatives of $\mathring{z}(t)$ in terms of the L^∞-norm of $\mathring{z}(t)$ itself. To close the nonlinear argument, we then relate $\mathring{z}(t)$ to the residual

$$z(t) = v(t) - k_0 \partial_k \phi(\cdot; 1)\, \gamma_\zeta(t), \tag{4.41}$$

along with their spatial derivatives, again via mean-value type arguments. As a result, derivatives of $v(t)$ are ultimately controlled by the L^∞-norm of $\mathring{z}(t)$ (and hence of $z(t)$). We remark that short-time regularity control through the nonlinear damping estimate on $\mathring{z}(t)$ is not feasible due to the presence of the term $\gamma_{\zeta t}(t)$ in the nonlinearity of the evolution equation for $\mathring{z}(t)$; see §4.5.7. This term exhibits a non-integrable blow-up as $t \downarrow 0$, with L^∞-norm scaling like t^{-1}, thereby precluding control in the short-time regime.

The remainder of this section is organized as follows. We begin by establishing local existence and uniqueness of the solution $u(t)$ to (4.7). We then derive the equation satisfied by the inverse-modulated perturbation $v(t)$ and obtain L^∞-bounds on its nonlinear terms. Next, we introduce the phase modulation $\gamma(t)$ and derive the perturbed viscous Hamilton-Jacobi equation that governs its dynamics. Finally, we formulate the equations for the forward-modulated perturbation $\mathring{v}(t)$ and the modified forward-modulated perturbation $\mathring{z}(t)$, which are used for short- and long-time regularity control, respectively, and we establish a nonlinear damping estimate on $\mathring{z}(t)$.

4.5.1. Local existence and uniqueness of the solution

Since the advection-diffusion operator $L_0 = k_0^2 D \partial_{\zeta\zeta} + \omega_0 \partial_\zeta$ is sectorial on $C^l_{\mathrm{ub}}(\mathbb{R})$ with dense domain $D(L_0) = C^{l+2}_{\mathrm{ub}}(\mathbb{R})$ by [73, Corollary 3.1.9] and the nonlinear map $u \mapsto \partial_u^j f(u)$ is locally Lipschitz continuous on $C^l_{\mathrm{ub}}(\mathbb{R})$ for any $j, l \in \mathbb{N}_0$, local existence and uniqueness of the solution $u(t)$ to the reaction-diffusion system (4.7) follows directly from standard analytic semigroup theory; see [73, Theorem 7.1.5 and Propositions 7.1.8 and 7.1.10].

Proposition 4.5.1. *Assume* (H1). *Let* $u_0 \in C_{\mathrm{ub}}(\mathbb{R})$. *Then, there exists a maximal time* $T_{\max} \in (0, \infty]$ *such that* (4.7) *admits a unique classical solution*

$$u \in C\big([0, T_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0, T_{\max}), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0, T_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big),$$

with initial condition $u(0) = u_0$. *Moreover, the map* $[0, T_{\max}) \to C_{\mathrm{ub}}(\mathbb{R}), t \mapsto \sqrt{t}\, u_\zeta(t)$ *is continuous and, if* $T_{\max} < \infty$, *then we have*

$$\limsup_{t \uparrow T_{\max}} \|u(t)\|_\infty = \infty. \tag{4.42}$$

Finally, for any $j, l \in \mathbb{N}_0$ *we have* $u \in C^j\big((0, T_{\max}), C^l_{\mathrm{ub}}(\mathbb{R})\big)$.

4.5.2. Inverse-modulated perturbation equation

Using that $u(t)$ and ϕ_0 are solutions to (4.7), one finds that the inverse-modulated perturbation, given by (4.18), satisfies the quasilinear equation

$$(\partial_t - \mathcal{L}_0)\big[v + \phi_0'\gamma - \gamma_\zeta v\big] = \mathcal{N}(v, \gamma, \partial_t \gamma), \tag{4.43}$$

where the nonlinearity $\mathcal{N}$ is given by

$$\mathcal{N}(v, \gamma, \gamma_t) = \mathcal{Q}(v, \gamma) + \partial_\zeta \mathcal{R}(v, \gamma, \gamma_t) + \partial_\zeta^2 \mathcal{S}(v, \gamma) \tag{4.44}$$

with

$$\begin{aligned}
\mathcal{Q}(v, \gamma) &= \big(f(\phi_0 + v) - f(\phi_0) - f'(\phi_0)v\big)(1 - \gamma_\zeta),\\
\mathcal{R}(v, \gamma, \gamma_t) &= -\gamma_t v + \omega_0 \gamma_\zeta v + \frac{k_0^2}{1 - \gamma_\zeta} D\left(\gamma_\zeta^2 \phi_0' - \frac{\gamma_{\zeta\zeta} v}{1 - \gamma_\zeta}\right),\\
\mathcal{S}(v, \gamma) &= k_0^2 D\left(2\gamma_\zeta v + \frac{\gamma_\zeta^2 v}{1 - \gamma_\zeta}\right).
\end{aligned} \tag{4.45}$$

We refer to [55, Lemma 4.2] for a detailed derivation of (4.43).

The nonlinearities obey the following L^∞-bounds.

Lemma 4.5.2. *Assume* (H1). *Fix a constant* $C > 0$. *Then, we have*

$$\begin{aligned}
\|\mathcal{Q}(v, \gamma)\|_\infty &\lesssim \|v\|_\infty^2,\\
\|\mathcal{R}(v, \gamma, \gamma_t)\|_\infty &\lesssim \|v\|_\infty \|(\gamma_\zeta, \gamma_t)\|_{C^1_{\mathrm{ub}} \times C_{\mathrm{ub}}} + \|\gamma_\zeta\|_\infty^2,\\
\|\mathcal{S}(v, \gamma)\|_\infty &\lesssim \|v\|_\infty \|\gamma_\zeta\|_\infty,\\
\|\partial_\zeta \mathcal{S}(w, \gamma)\|_\infty &\lesssim \|w\|_{C^1_{\mathrm{ub}}} \|\gamma_\zeta\|_{C^1_{\mathrm{ub}}}
\end{aligned}$$

for $v \in C_{\mathrm{ub}}(\mathbb{R})$, $w \in C^1_{\mathrm{ub}}(\mathbb{R})$, *and* $(\gamma, \gamma_t) \in C^2_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R})$ *satisfying* $\|v\|_\infty \le C$ *and* $\|\gamma_\zeta\|_\infty \le \frac{1}{2}$.

4.5.3. Choice of phase modulation

Assuming that $\gamma(t)$ satisfies $\gamma(0) = \gamma_0$, we arrive at the Duhamel formulation

$$v(t) + \phi_0'\gamma(t) = \mathrm{e}^{\mathcal{L}_0 t}(v_0 + \phi_0'\gamma_0 - \gamma_0' v_0) + \int_0^t \mathrm{e}^{\mathcal{L}_0(t-s)}\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\mathrm{d}s + \gamma_\zeta(t)v(t) \tag{4.46}$$

after integrating (4.43), where we denote

$$v_0(\zeta) = u_0(\zeta - \gamma_0(\zeta)) - \phi_0(\zeta), \qquad \zeta \in \mathbb{R}. \tag{4.47}$$

As in [56, 63], we make a judicious choice for the phase modulation $\gamma(t)$ such that the linear term $\phi_0'\gamma(t)$ on the left-hand side of (4.46) compensates for the slowest decaying nonlinear contributions on the right-hand side of (4.46). This choice must be compatible with the initial condition $\gamma(0) = \gamma_0 \in C_{\mathrm{ub}}^1(\mathbb{R})$. Moreover, to ensure sufficient regularity for the subsequent nonlinear iteration scheme, we require that $\gamma(t)$ is smooth for all $t > 0$. Thus, motivated by the semigroup decomposition (4.30), we define $\gamma(t)$ through the integral equation

$$\begin{aligned}\gamma(t) = S_p^0(t)(v_0 + \phi_0'\gamma_0 + \gamma_0' v_0) + \int_0^t S_p^0(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\mathrm{d}s \\ + (1 - \chi(t))\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0,\end{aligned} \tag{4.48}$$

where $\chi(t)$ is the temporal cut-off function from Proposition 4.4.1.

Before establishing existence of a solution $\gamma(t)$ to (4.48), we argue that our implicit choice for $\gamma(t)$ indeed possesses the required properties. To begin with, we have $\gamma(0) = \gamma_0$, since $S_p^0(t)$ vanishes on $[0, 1]$. Second, since the sectorial operator $-\partial_\zeta^4$ generates an analytic semigroup on $C_{\mathrm{ub}}(\mathbb{R})$, cf. [73, Proposition 2.4.4 and Corollary 3.1.9], and the propagator $S_p^0(t)$ is smoothing by Proposition 4.4.1, it follows that $\gamma(t)$ is smooth for all $t > 0$. The choice of the analytic semigroup $\mathrm{e}^{-\partial_\zeta^4 t}$, rather than the standard heat semigroup $\mathrm{e}^{\partial_\zeta^2 t}$, is motivated by the need to control third-order spatial derivatives of $\gamma(t)$ at short times within our nonlinear argument; see Remark 4.5.3.

Substituting (4.30), (4.41), and (4.48) into (4.46), we obtain the Duhamel formulation

$$\begin{aligned}z(t) = \widetilde{S}(t)(v_0 + \phi_0'\gamma_0 + \gamma_0' v_0) + \int_0^t \widetilde{S}(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\mathrm{d}s + \gamma_\zeta(t)v(t) \\ - (1 - \chi(t))(\phi_0' + k_0\partial_k\phi(\cdot; k_0)\partial_\zeta)\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0\end{aligned} \tag{4.49}$$

for the residual $z(t)$. Recalling that the propagator $\widetilde{S}(t)$ decays at rate t^{-1} as $t \to \infty$, cf. Proposition 4.4.1, we confirm that our choice of $\gamma(t)$ indeed eliminates the slowest decaying contributions on the right-hand side of (4.46).

Remark 4.5.3. Our choice for the analytic semigroup $\mathrm{e}^{-\partial_\zeta^4 t}$ in (4.48) is motivated by the structure of the nonlinear terms in the perturbed viscous Hamilton-Jacobi equation, which governs the leading-order dynamics of the phase modulation $\gamma(t)$; see §4.5.4. In particular, the nonlinearity involves a temporal derivative and up to three spatial derivatives of $\gamma(t)$. To control these terms in the associated Duhamel formulation, it is crucial that the C_{ub}^2-norm of $\gamma_\zeta(t)$, as well as the L^∞-norm of $\gamma_t(t)$, exists for $t > 0$ and exhibits a singularity as $t \downarrow 0$ that remains integrable in time. Since we merely assume $\gamma_0 \in C_{\mathrm{ub}}^1(\mathbb{R})$, this motivates the introduction of $\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$. Indeed, for fixed $j \in \mathbb{N}_0$, standard analytic semigroup theory [73] yields the estimates

$$\left\|\mathrm{e}^{-\partial_\zeta^4 t}\gamma\right\|_\infty \leq \|\gamma\|_\infty, \qquad \left\|\partial_t\partial_\zeta^j \mathrm{e}^{-\partial_\zeta^4 t}\gamma\right\|_\infty \lesssim t^{-\frac{3+j}{4}}\|\gamma'\|_\infty, \qquad \left\|\partial_\zeta^{1+j}\mathrm{e}^{-\partial_\zeta^4 t}\gamma\right\|_\infty \lesssim t^{-\frac{j}{4}}\|\gamma'\|_\infty \tag{4.50}$$

for $t > 0$ and $\gamma \in C^1_{\mathrm{ub}}(\mathbb{R})$, which show that the temporal derivative and the first four spatial derivatives of $\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ permit such integrable blow-up behavior as $t \downarrow 0$. So, the term $\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ is tailored to the demands of our nonlinear argument. Alternatively, one could impose higher regularity on the initial phase modulation γ_0, which allows replacing $\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ with γ_0 in (4.48), as done in [56].

We conclude this subsection by establishing local existence of a solution $\gamma(t)$ to the integral equation (4.48). The contraction-mapping argument is adapted from [84, Proposition 4.4], accommodating large initial data $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$, while still requiring that $\gamma_\zeta(t)$ remains sufficiently small to ensure that the nonlinearity in (4.44) is well-defined.

Proposition 4.5.4. *Assume* (H1) *and* (D3). *Let $r_0 \in (0, \frac{1}{2})$ be as in Proposition 4.3.1. Let $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|\gamma_0'\|_\infty < r_0$. Define $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ by* (4.40) *and* (4.47)*, respectively. For u and $T_{\max}$ as in Proposition 4.5.1, there exists a maximal time $\tau_{\max} \in [1, \max\{1, T_{\max}\}]$ such that* (4.48)*, with v given by* (4.18)*, has a solution*

$$\gamma \in C\big([0,\tau_{\max}), C^1_{\mathrm{ub}}(\mathbb{R})\big) \cap C^j\big((0,\tau_{\max}), C^l_{\mathrm{ub}}(\mathbb{R})\big), \qquad j, l \in \mathbb{N}_0, \tag{4.51}$$

satisfying $\gamma(t) = \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ for $t \in [0,1]$. In addition, it holds $\|\gamma_\zeta(t)\|_\infty < r_0$ for all $t \in [0, \tau_{\max})$. Finally, $\tau_{\max} < T_{\max}$ implies

$$\limsup_{t\uparrow\tau_{\max}} \|\gamma_\zeta(t)\|_\infty = r_0 \tag{4.52}$$

or

$$\limsup_{t\uparrow\tau_{\max}} \|(\gamma(t), \partial_t\gamma(t))\|_{C^2_{\mathrm{ub}}\times C_{\mathrm{ub}}} = \infty. \tag{4.53}$$

Proof. We delegate the proof to Appendix 4.B. □

Local existence and regularity of the inverse-modulated perturbation $v(t)$ and the residual $z(t)$ follow directly from Propositions 4.5.1 and 4.5.4.

Corollary 4.5.5. *Assume* (H1) *and* (D3). *Let $r_0 \in (0, \frac{1}{2})$ be as in Proposition 4.3.1. Let $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|\gamma_0'\|_\infty < r_0$. Define $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ by* (4.40) *and* (4.47)*, respectively. For u as in Proposition 4.5.1 and γ and $\tau_{\max}$ as in Proposition 4.5.4, the inverse-modulated perturbation v, defined by* (4.18)*, and the residual z, defined by* (4.41)*, satisfy*

$$v, z \in C\big([0,\tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0,\tau_{\max}), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0,\tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big).$$

Moreover, the map $[0,\tau_{\max}) \to C_{\mathrm{ub}}(\mathbb{R})$, $t \mapsto \sqrt{t}\,(v_\zeta(t), z_\zeta(t))$ is continuous. Finally, the Duhamel formulations (4.46) *and* (4.49) *are valid for $t \in [0, \tau_{\max})$.*

4.5.4. Derivation of viscous Hamilton-Jacobi equation

Proceeding along the lines of [84], we derive a perturbed viscous Hamilton-Jacobi equation governing the dynamics of the phase modulation $\gamma(t)$.

As a first step, we isolate all contributions involving γ_ζ^2 in the nonlinearity in (4.48). These terms are critical, as they exhibit the slowest decay. As shown in [84, Section 4.3], the nonlinearity (4.44) admits the decomposition

$$\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s)) = k_0^2 f_p \gamma_\zeta(s)^2 + \mathcal{N}_p(z(s),v(s),\gamma(s),\widetilde{\gamma}(s)), \tag{4.54}$$

where we denote

$$\widetilde{\gamma}(t) := \partial_t\gamma(t) - a\gamma_\zeta(t),$$

f_p is the 1-periodic function

$$f_p = \frac{1}{2}f''(\phi_0)(\partial_k\phi(\cdot;k_0),\partial_k\phi(\cdot;k_0)) + \omega'(k_0)\partial_{\zeta k}\phi(\cdot;k_0) + D\big(\phi_0'' + 2k_0\partial_{\zeta\zeta k}\phi(\cdot;k_0)\big),$$

and the residual is given by

$$\mathcal{N}_p(z,v,\gamma,\widetilde{\gamma}) = \mathcal{Q}_p(z,v,\gamma) + \partial_\zeta\mathcal{R}_p(z,v,\gamma,\widetilde{\gamma}) + \partial_{\zeta\zeta}\mathcal{S}_p(z,v,\gamma), \tag{4.55}$$

with

$$\begin{aligned}
&\mathcal{Q}_p(z,v,\gamma)\\
&= (f(\phi_0+v) - f(\phi_0) - f'(\phi_0)v)\gamma_\zeta + f(\phi_0+v) - f(\phi_0) - f'(\phi_0)v - \frac{1}{2}f''(\phi_0)(v,v)\\
&\quad + \frac{1}{2}f''(\phi_0)(z,z) + k_0\gamma_\zeta f''(\phi_0)(z,\partial_k\phi(\cdot;k_0)) + 2k_0^2\omega'(k_0)\gamma_\zeta\gamma_{\zeta\zeta}\partial_k\phi(\cdot;k_0)\\
&\quad + 2k_0^2 D\Big(\gamma_\zeta\gamma_{\zeta\zeta}(\phi_0' + 4k_0\partial_{\zeta k}\phi(\cdot;k_0)) + 2k_0\Big(\gamma_{\zeta\zeta}^2 + \gamma_\zeta\gamma_{\zeta\zeta\zeta}\Big)\partial_k\phi(\cdot;k_0)\Big),\\
&\mathcal{R}_p(z,v,\gamma,\widetilde{\gamma}) = -v\widetilde{\gamma} + k_0\omega_0'(k_0)\gamma_\zeta z + \frac{k_0^2}{1-\gamma_\zeta}D\left(\gamma_\zeta^3\phi_0' - \frac{\gamma_{\zeta\zeta}v}{1-\gamma_\zeta}\right),\\
&\mathcal{S}_p(z,v,\gamma) = k_0^2 D\left(2\gamma_\zeta z + \frac{\gamma_\zeta^2 v}{1-\gamma_\zeta}\right).
\end{aligned}$$

Applying Taylor's theorem, we obtain the following nonlinear estimate.

Lemma 4.5.6. *Assume* (H1) *and* (D3)*. Fix a constant $C > 0$. Then, we have*

$$\begin{aligned}
\|\mathcal{Q}_p(z,v,\gamma)\|_\infty &\lesssim (\|v\|_\infty + \|\gamma_\zeta\|_\infty)\|v\|_\infty^2 + (\|z\|_\infty + \|\gamma_\zeta\|_\infty)\|z\|_\infty + \|\gamma_\zeta\|_{C^1_{\mathrm{ub}}}\|\gamma_{\zeta\zeta}\|_{C^1_{\mathrm{ub}}},\\
\|\mathcal{R}_p(z,v,\gamma,\widetilde{\gamma})\|_\infty &\lesssim \|v\|_\infty(\|\widetilde{\gamma}\|_\infty + \|\gamma_{\zeta\zeta}\|_\infty) + \|z\|_\infty\|\gamma_\zeta\|_\infty + \|\gamma_\zeta\|_\infty^3,\\
\|\mathcal{S}_p(z,v,\gamma)\|_\infty &\lesssim \|v\|_\infty\|\gamma_\zeta\|_\infty^2 + \|z\|_\infty\|\gamma_\zeta\|_\infty
\end{aligned}$$

for $z \in C_{\mathrm{ub}}(\mathbb{R})$, $v \in C_{\mathrm{ub}}(\mathbb{R})$, and $(\gamma,\widetilde{\gamma}) \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R})$ satisfying $\|v\|_\infty \leq C$ and $\|\gamma_\zeta\|_\infty \leq \frac{1}{2}$.

Next, we decompose the phase modulation $\gamma(t)$ into a solution $\widetilde{y}(t)$ solving a perturbed viscous Hamilton-Jacobi equation and a remainder $r(t)$ exhibiting higher-order decay. To this end, we use the commutator identities (4.31) and insert the decomposition (4.33) of the propagator $S_p^i(t)$ and the decomposition (4.54) of the nonlinearity into (4.48). Thus, after using (4.8) and (4.34) to reexpress $\mathrm{e}^{\left(d\partial_\zeta^2 - c_g\partial_\zeta\right)t}(\widetilde{\Phi}_0^* \mathbf{f}_p \gamma_\zeta^2)$ and setting

$$\nu = \langle \widetilde{\Phi}_0, \mathbf{f}_p \rangle_{L^2(0,1)},$$

we arrive at the decomposition

$$\gamma(t) = \widetilde{y}(t) + r(t) \tag{4.56}$$

with

$$\begin{aligned}
\widetilde{y}(t) = {} & S_h^0(t)(v_0 + \gamma_0' v_0) + \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}(\gamma_0 - A_h(\phi_0')\gamma_0') \\
& + \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-s)}\Big(\nu\gamma_\zeta(s)^2 - k_0^2 A_h(f_p)\partial_\zeta\Big(\gamma_\zeta(s)^2\Big)\Big)\mathrm{d}s \\
& + \int_0^t S_h^0(t-s)\mathcal{Q}_p(z(s), v(s), \gamma(s))\mathrm{d}s - \int_0^t S_h^1(t-s)\mathcal{R}_p(z(s), v(s), \gamma(s), \widetilde{\gamma}(s))\mathrm{d}s \\
& + \int_0^t S_h^2(t-s)\mathcal{S}_p(z(s), v(s), \gamma(s))\mathrm{d}s,
\end{aligned} \tag{4.57}$$

and remainder

$$\begin{aligned}
r(t) = {} & \widetilde{S}_r^0(t)(v_0 + \phi_0'\gamma_0 + \gamma_0' v_0) + \partial_\zeta \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}(A_h(\phi_0')\gamma_0) + k_0^2 \int_0^t \widetilde{S}_r^0(t-s)\Big(f_p \gamma_\zeta(s)^2\Big)\mathrm{d}s \\
& + k_0^2 \partial_\zeta \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-s)}\Big(A_h(f_p)\gamma_\zeta(s)^2\Big) + \int_0^t \widetilde{S}_r^0(t-s)\mathcal{Q}_p(z(s), v(s), \gamma(s))\mathrm{d}s \\
& - \int_0^t \widetilde{S}_r^1(t-s)\mathcal{R}_p(z(s), v(s), \gamma(s), \widetilde{\gamma}(s))\mathrm{d}s + \int_0^t \widetilde{S}_r^2(t-s)\mathcal{S}_p(z(s), v(s), \gamma(s))\mathrm{d}s \\
& + \partial_\zeta \int_0^t S_p^0(t-s)\mathcal{R}_p(z(s), v(s), \gamma(s), \widetilde{\gamma}(s))\mathrm{d}s + (1 - \chi(t))\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 \\
& + \partial_\zeta^2 \int_0^t S_p^0(t-s)\mathcal{S}_p(z(s), v(s), \gamma(s))\mathrm{d}s - 2\partial_\zeta \int_0^t S_p^1(t-s)\mathcal{S}_p(z(s), v(s), \gamma(s))\mathrm{d}s.
\end{aligned} \tag{4.58}$$

Invoking the linear estimates from Propositions 4.4.1 and 4.4.2, we observe that $r(t)$ accounts for the contributions in (4.48) that decay on the linear level. On the other hand, applying the convective heat operator $\partial_t - d\partial_\zeta^2 - a\partial_\zeta$ to (4.57), we arrive at the viscous Hamilton-Jacobi equation

$$\Big(\partial_t - d\partial_\zeta^2 - a\partial_\zeta\Big)\widetilde{y} = \nu\widetilde{y}_\zeta^2 + G(r, \widetilde{y}, z, v, \gamma, \widetilde{\gamma}) \tag{4.59}$$

with perturbation

$$\begin{aligned}
G(z, v, \gamma, \widetilde{\gamma}) = {} & 2\nu\widetilde{y}_\zeta r_\zeta + \nu r_\zeta^2 - k_0^2 A_h(f_p)\partial_\zeta(\gamma_\zeta^2) + \widetilde{\Phi}_0^* \mathcal{Q}_p(z, v, \gamma) - \Big(\partial_\zeta \widetilde{\Phi}_0^*\Big)\mathcal{R}_p(z, v, \gamma, \widetilde{\gamma}) \\
& + \Big(\partial_\zeta^2 \widetilde{\Phi}_0^*\Big)\mathcal{S}_p(z, v, \gamma).
\end{aligned}$$

That is, $\widetilde{y}(t)$ is a solution to (4.59) with initial condition

$$\widetilde{y}(0) = \gamma_0 - A_h(\phi_0')\gamma_0' + \widetilde{\Phi}_0^*(v_0 + \gamma_0' v_0) \in C_{\mathrm{ub}}(\mathbb{R}).$$

By expressing the right-hand side of (4.59) as

$$\begin{aligned} F(z, v, \gamma, \widetilde{\gamma}) =& \nu\gamma_\zeta^2 - k_0^2 A_h(f_p)\partial_\zeta(\gamma_\zeta^2) + \widetilde{\Phi}_0^* \mathcal{Q}_p(z, v, \gamma) \\ & - \left(\partial_\zeta \widetilde{\Phi}_0^*\right)\mathcal{R}_p(z, v, \gamma, \widetilde{\gamma}) + \left(\partial_\zeta^2 \widetilde{\Phi}_0^*\right)\mathcal{S}_p(z, v, \gamma), \end{aligned}$$

we can interpret equation (4.59) as an inhomogeneous linear parabolic problem. The inhomogeneity $t \mapsto F(z(t), v(t), \gamma(t), \widetilde{\gamma}(t))$ belongs to $C\big((0, \tau_{\max}), C^1_{\mathrm{ub}}(\mathbb{R})\big) \cap L^1\big((0, \tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big)$ by combining the estimates (4.35) with Proposition 4.5.4 and Corollary 4.5.5. Consequently, regularity properties of $\widetilde{y}(t)$ and $r(t) = \gamma(t) - \widetilde{y}(t)$ follow from analytic semigroup theory, cf. [73, Theorem 4.3.11].

Corollary 4.5.7. *Assume* (H1) *and* (D3). *Let* $r_0 \in (0, \frac{1}{2})$ *be as in Proposition 4.3.1. Let* $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ *and* $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ *with* $\|\gamma_0'\|_\infty < r_0$. *Define* $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ *by* (4.40) *and* (4.47), *respectively. For* γ *and* $\tau_{\max}$ *as in Proposition 4.5.4 and and for* v *and* z *as in Corollary 4.5.5, the Hamilton-Jacobi variable* $\widetilde{y}$, *given by* (4.57) *and the residual* r, *given by* (4.58), *obey*

$$\widetilde{y}, r \in C\big([0, \tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0, \tau_{\max}), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0, \tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big).$$

In addition, the map $[0, \tau_{\max}) \to C_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R}), t \mapsto \sqrt{t}\,\big(\widetilde{y}_\zeta(t), r_\zeta(t)\big)$ *is continuous.*

Remark 4.5.8. The decomposition (4.56) generalizes the corresponding one in [84], where the initial phase modulation was assumed to vanish identically. Indeed, setting $\gamma_0 = 0$ in (4.57) and (4.58) recovers the decomposition used in [84]. Alternatively, by assuming more regular initial data $\mathring{v}_0 \in C^2_{\mathrm{ub}}(\mathbb{R})$, one could employ the simpler decomposition from [5], thereby avoiding the commutator identities (4.31) needed to shift derivatives from the nonlinearity (4.55) onto the propagators.

4.5.5. Application of the Cole-Hopf transform

In the forthcoming nonlinear analysis, we control the remainder $r(t)$ by estimating the right-hand side of (4.58). For short times, a similar approach allows us to bound $\widetilde{y}(t)$ via (4.57). However, for large times, this strategy fails due to the presence of the critical nonlinear term $\nu y_\zeta(s)^2$ in (4.57), which decays at the nonintegrable rate $(1+s)^{-1}$ as $s \to \infty$. To address this issue, we proceed as in [5, 84] and eliminate the problematic nonlinear term $\nu\widetilde{y}_\zeta^2$ in (4.59) by applying the Cole-Hopf transformation. To this end, we introduce the new variable

$$y(t) = \mathrm{e}^{\frac{\nu}{d}\widetilde{y}(t)}, \tag{4.60}$$

which satisfies

$$y \in C\big([0, \tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0, \tau_{\max}), C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0, \tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \tag{4.61}$$

by Corollary 4.5.7. We obtain the convective heat equation

$$\left(\partial_t - d\partial_\zeta^2 - a\partial_\zeta\right)y = 2\nu r_\zeta y_\zeta + \frac{\nu}{d}\left(\nu r_\zeta^2 + G(z, v, \gamma, \widetilde{\gamma})\right)y \tag{4.62}$$

in which critical y_ζ^2-contributions are no longer present. Integrating (4.62), we arrive at the Duhamel formulation

$$y(t) = \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-1)}y(1) + \int_1^t \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-s)}\mathcal{N}_c(r(s), y(s), z(s), v(s), \gamma(s), \widetilde{\gamma}(s))\mathrm{d}s, \tag{4.63}$$

for $t \in [0,\tau_{\max})$ with $t \geq 1$, with nonlinearity

$$\mathcal{N}_c(r,y,z,v,\gamma,\widetilde{\gamma}) = 2\nu r_\zeta y_\zeta + \frac{\nu}{d}\Big(\nu r_\zeta^2 + G(z,v,\gamma,\widetilde{\gamma})\Big)y.$$

Using Lemma 4.5.6, we readily obtain the following nonlinear estimate.

Lemma 4.5.9. *Fix a constant $C > 0$. Then, we have*

$$\begin{aligned}\|\mathcal{N}_c(r,y,z,v,\gamma,\widetilde{\gamma})\|_\infty \lesssim \|r_\zeta\|_\infty \|y_\zeta\|_\infty + \|y\|_\infty \Big(&\|r_\zeta\|_\infty^2 + (\|v\|_\infty + \|\gamma_\zeta\|_\infty)\|v\|_\infty^2 \\ &+ \|\gamma_\zeta\|_{C^1_{\mathrm{ub}}} \|\gamma_{\zeta\zeta}\|_{C^1_{\mathrm{ub}}} + \|v\|_\infty \Big(\|\widetilde{\gamma}\|_\infty + \|\gamma_{\zeta\zeta}\|_\infty + \|\gamma_\zeta\|_\infty^2\Big) \\ &+ (\|z\|_\infty + \|\gamma_\zeta\|_\infty)\|z\|_\infty + \|\gamma_\zeta\|_\infty^3\Big),\end{aligned}$$

for $z \in C_{\mathrm{ub}}(\mathbb{R})$, $v \in C_{\mathrm{ub}}(\mathbb{R})$, $r, y \in C^1_{\mathrm{ub}}(\mathbb{R})$ and $(\gamma,\widetilde{\gamma}) \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R})$ with $\|v\|_\infty \leq C$ and $\|\gamma_\zeta\|_\infty \leq \frac{1}{2}$.

4.5.6. Forward-modulated perturbation

Since the equation (4.43) for the inverse-modulated perturbation $v(t)$ is quasilinear, it presents an apparent loss of derivatives that must be addressed in the nonlinear argument. In order to control regularity for *short* times, we iteratively estimate the forward-modulated perturbation $\mathring{v}(t)$ via its Duhamel representation, which is semilinear and does not suffer from a loss of derivatives. As a first step, we note that the following existence and regularity properties of $\mathring{v}(t)$ follow directly from Propositions 4.5.1 and 4.5.4.

Corollary 4.5.10. *Assume* (H1) *and* (D3). *Let $r_0 \in (0,\frac{1}{2})$ be as in Proposition 4.3.1. Let $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|\gamma_0'\|_\infty < r_0$. Define $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ by* (4.40) *and* (4.47)*, respectively. For u as in Proposition 4.5.1, and γ and $\tau_{\max}$ as in Proposition 4.5.4, the forward-modulated perturbation $\mathring{v}(t)$, given by* (4.19)*, is well-defined for $t \in [0,\tau_{\max})$ and satisfies*

$$\mathring{v} \in C\big([0,\tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \cap C^j\big((0,\tau_{\max}), C^l_{\mathrm{ub}}(\mathbb{R})\big)$$

for any $j, l \in \mathbb{N}_0$.

Differentiating $\mathring{v}(t)$ with respect to time and using the facts that ϕ_0 and $u(t)$ solve (4.7) and that $\gamma(t) = \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ for $t \in [0,1]$ by Proposition 4.5.4, we derive the semilinear equation

$$\partial_t \mathring{v} = k_0^2 D \mathring{v}_{\zeta\zeta} + \omega_0 \mathring{v}_\zeta + A[\gamma_0]\mathring{v} + \mathring{\mathcal{N}}(\mathring{v},\gamma,\partial_t\gamma) \tag{4.64}$$

valid for $t \in [0,1]$, with spatiotemporal coefficient

$$A[\gamma_0](\zeta,t) := f'(\phi_0(\zeta + \Gamma(\zeta,t))), \qquad \Gamma(\zeta,t) := \Big(\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0\Big)[\zeta]$$

and nonlinearity given by

$$\begin{aligned}\mathring{\mathcal{N}}(\mathring{v},\gamma,\gamma_t) = f(\mathring{v} + \phi_0(\kappa(\gamma))) - f(\phi_0(\kappa(\gamma))) - f'(\phi_0(\kappa))\mathring{v} + \phi_0'(\kappa(\gamma))(\omega_0\gamma_\zeta - \gamma_t) \\ + k_0^2 D(\phi_0'(\kappa(\gamma))\gamma_{\zeta\zeta} + \phi_0''(\kappa(\gamma))\gamma_\zeta(2+\gamma_\zeta)),\end{aligned}$$

with shorthand notation

$$\kappa(\gamma)(\zeta,t) = \zeta + \gamma(\zeta,t).$$

Differentiating (4.64) with respect to space, we obtain an evolution equation for $\mathring{w}(t) := \mathring{v}_\zeta(t)$, reading

$$\partial_t \mathring{w} = k_0^2 D \mathring{w}_{\zeta\zeta} + \omega_0 \mathring{w}_\zeta + A[\gamma_0]\mathring{w} + \mathring{\mathcal{N}}_1(\mathring{v},\gamma,\partial_t\gamma) \tag{4.65}$$

valid for $t \in [0,1]$, with nonlinearity

$$\mathring{\mathcal{N}}_1(\mathring{v},\gamma,\gamma_t) = (1+\gamma_\zeta) f''(\phi_0(\kappa(\gamma)))(\phi_0'(\kappa(\gamma)),\mathring{v}) + \partial_\zeta \mathring{\mathcal{N}}(\mathring{v},\gamma,\gamma_t).$$

We now state short-time bounds on the temporal Green's function associated with the linearized equation

$$\partial_t \mathring{v} = k_0^2 D \mathring{v}_{\zeta\zeta} + \omega_0 \mathring{v}_\zeta + A[\gamma_0]\mathring{v}. \tag{4.66}$$

These bounds have been established in [104], using Levi's parametrix method.

Proposition 4.5.11. *Assume* (H1)*. There exist constants $C, M > 0$ and $t_* \in (0,1]$ such that for each $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ the temporal Green's function $G\colon \mathbb{R}\times(0,t_*]\times\mathbb{R}\times(0,t_*] \to \mathbb{R}^{n\times n}$ associated with* (4.66) *is continuously differentiable in its second and fourth coordinate and twice continuously differentiable in its first and third coordinate. The solution to the linearized equation* (4.66) *with initial condition $\mathring{v}(s) = \mathring{v}_0$ is given by*

$$v(\zeta,t) = \int_{\mathbb{R}} G(\zeta,t;\bar\zeta,s)\mathring{v}_0(\bar\zeta)\mathrm{d}\bar\zeta$$

for all $\zeta \in \mathbb{R}$ and any $s,t \in [0,t_*]$ with $0 \le s < t$. Moreover, the Green's function enjoys the pointwise bounds

$$\left|\partial_\zeta^j G(\zeta,t;\bar\zeta,s)\right| \le C(t-s)^{-\frac{1+j}{2}} \mathrm{e}^{-\frac{(\zeta-\bar\zeta)^2}{M(t-s)}}, \qquad j = 0,1,2$$

for all $\zeta,\bar\zeta \in \mathbb{R}$ and $s,t \in (0,t_*]$ with $0 \le s < t$.

Proof. We first observe that the coefficient $A[\gamma_0]$ in (4.66) is uniformly bounded on $\mathbb{R}\times[0,1]$, with a bound that is independent of γ_0. Since γ_0 belongs to the interpolation space $C^1_{\mathrm{ub}}(\mathbb{R})$ of class $J_{\frac14}$ between $C_{\mathrm{ub}}(\mathbb{R})$ and the domain $C^4_{\mathrm{ub}}(\mathbb{R})$ of the sectorial operator $-\partial_\zeta^4$, it follows from [73, Corollary 2.2.3 and Proposition 2.2.4] that the map $[0,1] \to C_{\mathrm{ub}}(\mathbb{R})$, $t \mapsto A[\gamma_0](\cdot,t)$ is Hölder continuous with exponent $\alpha \in (0,\frac14)$. Moreover, for each fixed $t \in [0,1]$, we have $A[\gamma_0](\cdot,t) \in C^1_{\mathrm{ub}}(\mathbb{R})$, so the map is also Hölder continuous in space with exponent 2α. Thus, the claim follows directly from [104, Proposition 11.3]. □

Integrating (4.64) and (4.65) yields the Duhamel formulas

$$\mathring{v}(\zeta,t) = \int_{\mathbb{R}} G(\zeta,t;\bar\zeta,0)\mathring{v}_0(\bar\zeta)\mathrm{d}\bar\zeta + \int_0^t \int_{\mathbb{R}} G(\zeta,t;\bar\zeta,s)\mathring{\mathcal{N}}(\mathring{v}(\zeta,s),\gamma(\zeta,s),\partial_t\gamma(\zeta,s))\mathrm{d}s \tag{4.67}$$

for $\zeta \in \mathbb{R}$ and $t \in (0,t_*]$, and

$$\mathring{v}_\zeta(\zeta,t) = \int_{\mathbb{R}} G\Big(\zeta,t;\bar\zeta,\tfrac{t_*}{2}\Big)\mathring{v}_\zeta\Big(\bar\zeta,\tfrac{t_*}{2}\Big)\mathrm{d}\bar\zeta + \int_{\frac{t_*}{2}}^t \int_{\mathbb{R}} G(\zeta,t;\bar\zeta,s)\mathring{\mathcal{N}}_1(\mathring{v}(\zeta,s),\gamma(\zeta,s),\partial_t\gamma(\zeta,s))\mathrm{d}s \tag{4.68}$$

for $\zeta \in \mathbb{R}$ and $t \in [\frac{t_*}{2}, t_*]$, where we chose $t = \frac{t_*}{2}$, rather than $t = 0$, as our left integration boundary in (4.68), since $\mathring{v}_\zeta(t)$ is only guaranteed to exist for $t > 0$; see Corollary 4.5.10. The representations (4.67) and (4.68), combined with the pointwise Green's function estimates from Proposition 4.5.11 and the following nonlinear bounds, whose derivation follows directly from Taylor's theorem, yield the short-time regularity control needed for the forthcoming nonlinear analysis; see also Remark 4.5.3.

Lemma 4.5.12. *Assume* (H1)*. Fix a constant $C > 0$. Then, we have*

$$\left\|\mathring{\mathcal{N}}(\mathring{v}, \gamma, \gamma_t)\right\|_\infty \lesssim \|\mathring{v}\|_\infty^2 + \|\gamma_\zeta\|_{C^1_{\mathrm{ub}}} + \|\gamma_t\|_\infty,$$

for $\mathring{v} \in C_{\mathrm{ub}}(\mathbb{R})$ and $(\gamma, \gamma_t) \in C^2_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R})$ satisfying $\|\mathring{v}\|_\infty, \|\gamma_\zeta\|_\infty \leq C$. Moreover, we have

$$\left\|\mathring{\mathcal{N}}_1(\mathring{v}, \gamma, \gamma_t)\right\|_\infty \lesssim \|\mathring{v}\|_{C^1_{\mathrm{ub}}} + \|\gamma_\zeta\|_{C^2_{\mathrm{ub}}} + \|\gamma_t\|_{C^1_{\mathrm{ub}}}$$

for $\mathring{v} \in C^1_{\mathrm{ub}}(\mathbb{R})$ and $(\gamma, \gamma_t) \in C^3_{\mathrm{ub}}(\mathbb{R}) \times C^1_{\mathrm{ub}}(\mathbb{R})$ satisfying $\|\mathring{v}\|_\infty, \|\gamma_\zeta\|_\infty \leq C$.

4.5.7. Forward-modulated damping

In order to control regularity in the forthcoming nonlinear argument for *large* times, we follow the strategy of [5] and establish a nonlinear damping estimate for the modified forward-modulated perturbation $\mathring{z}(t)$ given by (4.21). This damping estimate extends the one in [55] to a pure L^∞-setting. It exploits the dissipative structure of the underlying reaction-diffusion system (4.7) and relies on the embedding of the uniformly local Sobolev space $H^1_{\mathrm{ul}}(\mathbb{R})$ into $C_{\mathrm{ub}}(\mathbb{R})$; see [94, Lemma 8.3.11].

Before proving the nonlinear damping estimate, we first derive an evolution equation for $\mathring{z}(t)$. To this end, we recall that $u_k(x,t) = \phi(kx - \omega(k)t; k)$ is a solution to (4.1) for all wavenumbers $k \in (k_0 - r_0, k_0 + r_0)$ by Proposition 4.3.1. Thus, using that $u(t)$ solves (4.7), we obtain

$$\partial_t \mathring{z} = k_0^2 D \mathring{z}_{\zeta\zeta} + \omega_0 \mathring{z}_\zeta + \mathring{\mathcal{Q}}(\mathring{z}, \gamma) + \mathring{\mathcal{R}}(\gamma, \tilde{\gamma}, \partial_t \gamma), \tag{4.69}$$

with

$$\mathring{\mathcal{Q}}(\mathring{z}, \gamma) = f(\mathring{z} + \phi(\beta(\gamma))) - f(\phi(\beta(\gamma)))$$

and

$$\begin{aligned}
&\mathring{\mathcal{R}}(\gamma, \tilde{\gamma}, \gamma_t) \\
&= k_0^2 D\Big[\phi_{yy}(\beta(\gamma))\Big((1 + \gamma_\zeta(1+\gamma_\zeta) + \gamma\gamma_{\zeta\zeta})^2 - (1+\gamma_\zeta)^2\Big) + k_0^2 \phi_{kk}(\beta(\gamma))\gamma_{\zeta\zeta}^2 \\
&\qquad + 2k_0 \phi_{yk}(\beta(\gamma))\gamma_{\zeta\zeta}(1 + \gamma_\zeta(1+\gamma_\zeta) + \gamma\gamma_{\zeta\zeta}) + \phi_y(\beta(\gamma))(\gamma_{\zeta\zeta}(1 + 3\gamma_\zeta) + \gamma\gamma_{\zeta\zeta\zeta}) \\
&\qquad + k_0 \phi_k(\beta(\gamma))\gamma_{\zeta\zeta\zeta}\Big] + k_0 \phi_k(\beta(\gamma))(\omega_0 \gamma_{\zeta\zeta} - \gamma_{\zeta t}) \\
&\qquad + \phi_y(\beta(\gamma))\Big(\omega_0 + k_0 \omega'(k_0)\gamma_\zeta - \omega(k_0(1+\gamma_\zeta)) - \tilde{\gamma} + \omega_0\Big(\gamma_\zeta^2 + \gamma\gamma_{\zeta\zeta}\Big) - \gamma_t \gamma_\zeta - \gamma\gamma_{\zeta t}\Big),
\end{aligned}$$

where we used

$$\beta(\gamma)(\zeta, t) = (\zeta + \gamma(\zeta,t)(1 + \gamma_\zeta(\zeta,t));\, k_0(1 + \gamma_\zeta(\zeta,t)))$$

to abbreviate the argument of $\phi(y;k)$ and its derivatives. For more details on the derivation of (4.69), we refer to [5, Appendix B].

Since the continuation $\phi(\cdot;k)$ of the wave train ϕ_0 with respect to the wavenumber k is defined for all $k \in (k_0 - r_0, k_0 + r_0)$ by Proposition 4.3.1, it follows from Propositions 4.5.1 and 4.5.4 that $\mathring{z}(t)$ is well-defined for all $t \in [0, \tau_{\max})$. Its regularity properties are also a direct consequence of these propositions and are summarized in the following statement.

Corollary 4.5.13. *Assume* (H1) *and* (D3). *Let $r_0 \in (0, \frac{1}{2})$ be as in Proposition 4.3.1. Let $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|\gamma_0'\|_\infty < r_0$. Define $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ by* (4.40) *and* (4.47), *respectively. For u as in Proposition 4.5.1, and γ and $\tau_{\max}$ as in Proposition 4.5.4, the modified forward-modulated perturbation $\mathring{z}(t)$, given by* (4.21), *is well-defined for $t \in [0, \tau_{\max})$ and satisfies*

$$\mathring{z} \in C\big([0,\tau_{\max}), C_{\mathrm{ub}}(\mathbb{R})\big) \cap C^j\big((0,\tau_{\max}), C^l_{\mathrm{ub}}(\mathbb{R})\big)$$

for any $j, l \in \mathbb{N}_0$.

We are now in position to establish the relevant nonlinear damping estimate. The proof follows the strategy of [5, Proposition 4.9], but differs in that damping is induced by the second derivative across all components, not just in the first component.

Proposition 4.5.14. *Assume* (H1) *and* (D3). *Fix a constant $R > 0$. Let $r_0 \in (0, \frac{1}{2})$ be as in Proposition 4.3.1. Let $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|\gamma_0'\|_\infty < r_0$. Define $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ by* (4.40) *and* (4.47), *respectively. Let $\gamma(t)$ and $\tau_{\max}$ be as in Proposition 4.5.4, let t_* be as in Proposition 4.5.11, and let $\mathring{z}(t)$ be as in Corollary 4.5.13. There exists a $\mathring{v}_0$- and γ_0-independent constant $C > 0$ such that the nonlinear damping estimate*

$$\begin{aligned}\|\mathring{z}(t)\|_{C^1_{\mathrm{ub}}} \leq C\Bigg(&\|\mathring{z}(t)\|_\infty + \bigg(\mathrm{e}^{t_*-t}\|\mathring{z}(t_*)\|^2_{C^2_{\mathrm{ub}}} + \int_{t_*}^t \mathrm{e}^{s-t}\Big(\|\mathring{z}(s)\|^2_\infty + \|\gamma_{\zeta\zeta}(s)\|^2_{C^3_{\mathrm{ub}}}\\ &+ \|\partial_s\gamma_\zeta(s)\|^2_{C^2_{\mathrm{ub}}} + \|\tilde{\gamma}(s)\|^2_{C^2_{\mathrm{ub}}} + \|\gamma_\zeta(s)\|^2_\infty\Big(\|\gamma_\zeta(s)\|^2_\infty + \|\partial_s\gamma(s)\|^2_\infty\Big)\Big)\,\mathrm{d}s\bigg)^{\frac{1}{2}}\Bigg)\end{aligned} \tag{4.70}$$

holds for all $t \in [0, \tau_{\max})$ with $t \geq t_*$ and

$$\sup_{t_* \leq s \leq t}\Big(\|\mathring{z}(s)\|_{C^1_{\mathrm{ub}}} + \|\gamma(s)\|_{C^2_{\mathrm{ub}}}\Big) \leq R. \tag{4.71}$$

Proof. Set $\vartheta = \frac{1}{2}$. We begin by relating the C^1_{ub}-norm of $\mathring{z}(t)$ to a uniformly local Sobolev norm. To this end, introduce the window function $\varrho\colon \mathbb{R} \to \mathbb{R}$ given by

$$\varrho(\zeta) = \frac{2}{2+\zeta^2},$$

which is positive, smooth, and L^1-integrable. It satisfies the inequality

$$|\varrho'(\zeta)| \leq \varrho(\zeta) \leq 1 \tag{4.72}$$

for all $\zeta \in \mathbb{R}$. Applying the Gagliaro-Nirenberg interpolation inequality, we estimate

$$\begin{aligned}\|z_\zeta\|_\infty = \sup_{y\in\mathbb{R}} \|\varrho(\vartheta(\cdot + y))z_\zeta\|_\infty &\lesssim \|z\|_\infty + \sup_{y\in\mathbb{R}} \|\partial_\zeta(\varrho(\vartheta(\cdot + y))z)\|_\infty \\ &\lesssim \|z\|_\infty + \sup_{y\in\mathbb{R}} \|\varrho(\vartheta(\cdot + y))z\|_\infty^{\frac{1}{3}} \left\|\partial_\zeta^2(\varrho(\vartheta(\cdot + y))z)\right\|_2^{\frac{2}{3}} \\ &\lesssim \|z\|_\infty + \|z\|_\infty^{\frac{1}{3}} \left(\|z\|_{C^1_{\mathrm{ub}}}^{\frac{2}{3}} + \sup_{y\in\mathbb{R}} \|\varrho(\vartheta(\cdot + y))z_{\zeta\zeta}\|_2^{\frac{2}{3}}\right)\end{aligned}$$

for $z \in C^1_{\mathrm{ub}}(\mathbb{R})$. Hence, employing Young's inequality and rearranging terms, we obtain the bound

$$\|\mathring{z}(t)\|_{C^1_{\mathrm{ub}}} \lesssim \|\mathring{z}(t)\|_\infty + \sup_{y\in\mathbb{R}} E_y(t)^{\frac{1}{2}}, \tag{4.73}$$

valid for all $t \in (0, \tau_{\max})$, where we define

$$E_y(t) = \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))|\mathring{z}_{\zeta\zeta}(\zeta, t)|^2 \,\mathrm{d}\zeta$$

for $y \in \mathbb{R}$ and $t \in (0, \tau_{\max})$.

We now derive a differential inequality for the energy $E_y(t)$. By Proposition 4.5.4 and Corollary 4.5.13, the function $t \mapsto E_y(t)$ is differentiable on $(0, \tau_{\max})$. Since the diffusion matrix D is positive definite, there exists a constant $d_0 > 0$ such that the coercivity estimate

$$\langle z, Dz\rangle \geq d_0|z|^2 \tag{4.74}$$

holds for all $z \in \mathbb{R}^n$. Fix $y \in \mathbb{R}$ and $t \in [0, \tau_{\max})$ with $t > t_*$ such that (4.71) holds. Using (4.69), we compute

$$\frac{1}{2}\partial_s E_y(s) = I + II \tag{4.75}$$

for $s \in [t_*, t]$, where we denote

$$\begin{aligned} I &= \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big\langle \partial_\zeta^2 \mathring{z}(\zeta, s), k_0^2 D\partial_\zeta^4 \mathring{z}(\zeta, s) + \omega_0 \partial_\zeta^3 \mathring{z}(\zeta, s)\Big\rangle \,\mathrm{d}\zeta, \\ II &= \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big\langle \partial_\zeta^2 \mathring{z}(\zeta, s), \partial_\zeta^2\Big(\mathring{\mathcal{Q}}(\mathring{z}(\zeta, s), \gamma(\zeta, s)) + \mathring{\mathcal{R}}(\gamma(\zeta, s), \tilde{\gamma}(\zeta, s), \partial_s\gamma(\zeta, s))\Big)\Big\rangle \,\mathrm{d}\zeta.\end{aligned}$$

Integrating by parts, we find

$$\begin{aligned} I = &-k_0^2 \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big\langle \partial_\zeta^3 \mathring{z}(\zeta, s), D\partial_\zeta^3 \mathring{z}(\zeta, s)\Big\rangle \,\mathrm{d}\zeta \\ &- k_0^2\vartheta \int_{\mathbb{R}} \varrho'(\vartheta(\zeta + y))\Big\langle \partial_\zeta^2 \mathring{z}(\zeta, s), D\partial_\zeta^3 \mathring{z}(\zeta, s)\Big\rangle \,\mathrm{d}\zeta \\ &+ \omega_0 \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big\langle \partial_\zeta^2 \mathring{z}(\zeta, s), \partial_\zeta^3 \mathring{z}(\zeta, s)\Big\rangle \,\mathrm{d}\zeta.\end{aligned}$$

Applying Young's inequality and the estimates (4.72) and (4.74), we obtain a constant $C_1 > 0$, independent of t, $\mathring{v}_0$, and γ_0, such that

$$I \leq -\frac{d_0 k_0^2}{2} \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big|\partial_\zeta^3 \mathring{z}(\zeta, s)\Big|^2 \,\mathrm{d}\zeta + C_1 \int_{\mathbb{R}} \varrho(\vartheta(\zeta + y))\Big|\partial_\zeta^2 \mathring{z}(\zeta, s)\Big|^2 \,\mathrm{d}\zeta \tag{4.76}$$

for $s \in [t_*, t]$. On the other hand, estimating II using Young's inequality and assumption (4.71), we find a t-, $\mathring{v}_0$-, and γ_0-independent constant $C_2 > 0$ such that

$$\begin{aligned} II \leq C_2\bigg(\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big(\Big|\partial_\zeta^2 \mathring{z}(\zeta,s)\Big|^2 + |\partial_\zeta \mathring{z}(\zeta,s)|^2\Big)\mathrm{d}\zeta + \|\mathring{z}(s)\|_\infty^2 + \|\gamma_{\zeta\zeta}(s)\|_{C^3_{\mathrm{ub}}}^2 \\ + \|\partial_s\gamma_\zeta(s)\|_{C^2_{\mathrm{ub}}}^2 + \|\tilde{\gamma}(s)\|_{C^2_{\mathrm{ub}}}^2 + \|\gamma_\zeta(s)\|_\infty^2\Big(\|\gamma_\zeta(s)\|_\infty^2 + \|\partial_s\gamma(s)\|_\infty^2\Big)\bigg) \end{aligned} \tag{4.77}$$

for $s \in [t_*, t]$. Inserting (4.76) and (4.77) into (4.75), we obtain a t-, $\mathring{v}_0$-, and γ_0-independent constant $C_3 > 0$ such that

$$\begin{aligned} \frac{1}{2}\partial_s E_y(s) \leq -\frac{1}{2}E_y(s) - \frac{d_0k_0^2}{2}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big|\partial_\zeta^3 \mathring{z}(\zeta,s)\Big|^2 \mathrm{d}\zeta + C_3\bigg(\|\mathring{z}(s)\|_\infty^2 + \|\gamma_{\zeta\zeta}(s)\|_{C^3_{\mathrm{ub}}}^2 \\ + \|\partial_s\gamma_\zeta(s)\|_{C^2_{\mathrm{ub}}}^2 + \|\tilde{\gamma}(s)\|_{C^2_{\mathrm{ub}}}^2 + \|\gamma_\zeta(s)\|_\infty^2\Big(\|\gamma_\zeta(s)\|_\infty^2 + \|\partial_s\gamma(s)\|_\infty^2\Big) \\ + \int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big(\Big|\partial_\zeta^2 \mathring{z}(\zeta,s)\Big|^2 + |\partial_\zeta \mathring{z}(\zeta,s)|^2\Big)\,\mathrm{d}\zeta\bigg) \end{aligned} \tag{4.78}$$

for $s \in [t_*, t]$.

To estimate the last line in (4.78), we use the interpolation inequality from [5, Estimate (4.30)], choosing parameters

$$\eta \in (0, \tfrac{1}{4}), \qquad k = 2, \qquad a_0 = 0 = a_3, \qquad a_1 = \frac{8(1+2\eta)}{\eta(5-2\eta)}, \qquad a_2 = \frac{4(3+2\eta)}{5-2\eta},$$

resulting in

$$\begin{aligned} \sum_{j=1}^{2}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big|\partial_\zeta^j z(\zeta)\Big|^2 \mathrm{d}\zeta \leq \frac{2\eta(3+2\eta)}{2\eta-5}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big|\partial_\zeta^3 z(\zeta)\Big|^2 \mathrm{d}\zeta \\ + \frac{2(2+\eta)(1+2\eta)}{\eta^2(5-2\eta)}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))|z(\zeta)|^2 \mathrm{d}\zeta. \end{aligned}$$

Taking $\eta \in (0, \frac{1}{4})$ small enough that

$$\frac{2\eta(3+2\eta)}{2\eta-5} \leq \frac{d_0k_0^2}{2C_3},$$

we obtain a constant $C_4 > 0$ such that

$$\sum_{j=1}^{2}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big|\partial_\zeta^j z(\zeta)\Big|^2 \mathrm{d}\zeta \leq \frac{d_0k_0^2}{2C_3}\int_{\mathbb{R}} \varrho(\vartheta(\zeta+y))\Big|\partial_\zeta^3 z(\zeta)\Big|^2 \mathrm{d}\zeta + C_4\|z\|_\infty^2 \tag{4.79}$$

for $z \in C^3_{\mathrm{ub}}(\mathbb{R})$.

Applying the interpolation inequality (4.79) to (4.78), we conclude that

$$\begin{aligned} \partial_s E_y(s) \leq -E_y(s) + C_5\bigg(\|\mathring{z}(s)\|_\infty^2 + \|\gamma_{\zeta\zeta}(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\partial_s\gamma_\zeta(s)\|_{C^2_{\mathrm{ub}}}^2 + \|\tilde{\gamma}(s)\|_{C^2_{\mathrm{ub}}}^2 \\ + \|\gamma_\zeta(s)\|_\infty^2\Big(\|\gamma_\zeta(s)\|_\infty^2 + \|\partial_s\gamma(s)\|_\infty^2\Big)\bigg) \end{aligned}$$

for $s \in [t_*, t]$, where $C_5 > 0$ is independent of t, $\mathring{v}_0$, and γ_0. Multiplying both sides by e^s and integrating over $s \in [t_*, t]$, we arrive at

$$E_y(t) \le \mathrm{e}^{t_*-t} E_y(t_*) + C_5 \int_{t_*}^t \mathrm{e}^{s-t} \Big(\|\mathring{z}(s)\|_\infty^2 + \|\gamma_{\zeta\zeta}(s)\|_{C^3_{\mathrm{ub}}}^2 + \|\partial_s \gamma_\zeta(s)\|_{C^2_{\mathrm{ub}}}^2 + \|\tilde{\gamma}(s)\|_{C^2_{\mathrm{ub}}}^2 \\ + \|\gamma_\zeta(s)\|_\infty^2 \Big(\|\gamma_\zeta(s)\|_\infty^2 + \|\partial_s \gamma(s)\|_\infty^2 \Big) \Big) \mathrm{d}s.$$

The damping estimate (4.70) now follows by plugging the latter bound into (4.73) and using the fact that there exists a constant $C_6 > 0$ (independent of γ_0 and $\mathring{v}_0$) such that $E_y(t_*) \le C_6 \|\mathring{z}(t_*)\|_{C^2_{\mathrm{ub}}}^2$. □

We conclude this section by recalling the results from [5, Lemma 4.11] and [105, Lemma 5.1], which state that the C^k_{ub}-norms of the forward- and inverse-modulated perturbations $\mathring{v}(t)$ and $v(t)$, as well as those of the modified forward-modulated perturbation $\mathring{z}(t)$ and the residual $z(t)$, are equivalent, up to controllable errors depending on $\gamma_\zeta(t)$ and its derivatives.

Lemma 4.5.15. *Fix a constant $R > 0$. Let $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with $\|\gamma_0'\|_\infty < r_0$. Define $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ by (4.40) and (4.47), respectively. Let $\gamma(t)$ and $\tau_{\max}$ be as in Proposition 4.5.4, let $v(t)$ and $z(t)$ be as in Corollary 4.5.5, let $\mathring{v}(t)$ be as in Corollary 4.5.10, and let $\mathring{z}(t)$ be as in Corollary 4.5.13. Then, we have*

$$\|v(t)\|_\infty \lesssim \|\mathring{v}(t)\|_\infty + \|\gamma_\zeta(t)\|_\infty, \qquad \|\mathring{v}(t)\|_\infty \lesssim \|v(t)\|_\infty + \|\gamma_\zeta(t)\|_\infty,$$

for any $t \in [0, \tau_{\max})$ with

$$\sup_{0 \le s \le t} \|\gamma(s)\|_\infty \le R. \tag{4.80}$$

Moreover, we have

$$\|\mathring{z}(t)\|_\infty \lesssim \|z(t)\|_\infty + \|\gamma_{\zeta\zeta}(t)\|_\infty + \|\gamma_\zeta(t)\|_\infty^2 \\ \|z(t)\|_{C^1_{\mathrm{ub}}} \lesssim \|\mathring{z}(t)\|_{C^1_{\mathrm{ub}}} + \|\gamma_{\zeta\zeta}(t)\|_{C^1_{\mathrm{ub}}} + \|\gamma_\zeta(t)\|_\infty^2$$

for any $t \in (0, \tau_{\max})$ satisfying (4.80).

Proof. The first two inequalities were proved in [105, Lemma 5.1]. The last two inequalities follow directly from the estimates established in the proof of [5, Lemma 4.11]. □

4.6. Nonlinear stability argument

In this section, we prove our main result, Theorem 4.1.3, by completing a nonlinear stability argument based on a quasilinear iteration scheme built around the integral equations (4.48), (4.49), (4.58), and (4.63). Short-time regularity control is obtained via iterative estimates applied to the Duhamel representations (4.67) and (4.68) for the forward-modulated perturbation, while long-time regularity is ensured by the nonlinear damping estimate provided in Proposition 4.5.14.

Proof of Theorem 4.1.3. Let $r_0 \in (0, \frac{1}{2})$ be as in Proposition 4.3.1. Take $\mathring{v}_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and $\gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$ with

$$\|\gamma_0\|_\infty \leq M, \qquad E_0 := \|\mathring{v}_0\|_\infty + \|\gamma_0'\|_\infty < r_0.$$

Define $u_0, v_0 \in C_{\mathrm{ub}}(\mathbb{R})$ by (4.40) and (4.47), respectively.

By Proposition 4.5.1, there exist a maximal time $T_{\max} \in (0, \infty]$ and a unique classical solution $u(t)$ to (4.7) with initial condition $u(0) = u_0$ satisfying (4.51). If $T_{\max} < \infty$, then (4.42) holds. Moreover, Proposition 4.5.4 yields a maximal time $\tau_{\max} \in [1, \max\{1, T_{\max}\}]$ and a solution $\gamma(t)$ to (4.48) satisfying (4.51), $\gamma(t) = \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ for $t \in [0, 1]$, and $\|\gamma_\zeta(t)\|_\infty < r_0$ for all $t \in [0, \tau_{\max})$. Finally, if $\tau_{\max} < T_{\max}$, then we have (4.52) or (4.53).

Our goal is to prove that $\tau_{\max} = T_{\max} = \infty$ and that $u(t)$ and $\gamma(t)$ satisfy the decay estimates (4.12), (4.13), (4.14), and (4.15), where $\breve{\gamma}(t)$ is the classical solution to the viscous Hamilton-Jacobi equation (4.10) with initial condition $\breve{\gamma}(0) = \gamma_0$. To this end, we define a template function, controlling the norms of the phase modulation $\gamma(t)$, the residuals $z(t)$ and $r(t)$, and the Cole-Hopf variable $y(t)$, which are defined by (4.41), (4.58), and (4.60), respectively. We first establish the result for $\alpha \in (0, \frac{1}{6})$. The case $\alpha = 0$ then follows a posteriori.

Template function. Let $\varrho\colon [0, \infty) \to [0, 1]$ be a smooth temporal cutoff function which vanishes on $[0, \frac{t_*}{2}]$ and satisfies $\varrho(t) = 1$ for all $t \in [t_*, \infty)$, where t_* is as in Proposition 4.5.11. In addition, set $\widetilde{\tau}_{\max} = \min\{\tau_{\max}, T_{\max}\}$. By Proposition 4.5.4, Corollaries 4.5.5 and 4.5.7, and identities (4.50) and (4.61), the template function $\eta\colon [0, \widetilde{\tau}_{\max}) \to \mathbb{R}$ given by

$$\begin{aligned}
\eta(t) = \sup_{0 \leq s \leq t} \Bigg[&\frac{(1+s)^{1-2\alpha}}{\log(2+s)} \Big(\|z(s)\|_{L^\infty} + \varrho(s) \Big(\|\gamma_{\zeta\zeta\zeta\zeta}(s)\|_{C^1_{\mathrm{ub}}} + \|\widetilde{\gamma}_\zeta(s)\|_{C^2_{\mathrm{ub}}} \Big) \Big) \\
&+ \frac{\sqrt{s}\,(1+s)^{\frac{1}{2}-2\alpha}}{\log(2+s)} \left(\|r_\zeta(s)\|_\infty + \|z_\zeta(s)\|_\infty + \|\gamma_{\zeta\zeta\zeta}(s)\|_\infty \right) \\
&+ \frac{s^{\frac{3}{4}}(1+s)^{\frac{1}{4}-2\alpha}}{\log(2+s)} \|\widetilde{\gamma}(s)\|_\infty + \frac{s^{\frac{1}{4}}(1+s)^{\frac{3}{4}-2\alpha}}{\log(2+s)} \|\gamma_{\zeta\zeta}(s)\|_\infty \\
&+ \sqrt{s}\,(1+s)^{-\alpha}\|y_\zeta(s)\|_\infty + (1+s)^{\frac{1}{2}-\alpha}\|\gamma_\zeta(s)\|_\infty \Bigg],
\end{aligned}$$

is well-defined, continuous, non-negative, and monotonically increasing, where we recall $\widetilde{\gamma}(t) = \partial_t\gamma(t) - a\gamma_\zeta(t)$.

Approach. Fix $\alpha \in (0, \frac{1}{6})$. Our goal is to show that there exist constants $C, \eta_0 > 0$ such that the estimates

$$\eta(0) \leq C E_0^\alpha, \qquad \eta(t) \leq C\Big(E_0^\alpha + \eta(t)^2\Big), \qquad \|\gamma(t)\|_\infty \leq C \tag{4.81}$$

hold for all $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \eta_0$. Set

$$\varepsilon := \min\left\{\frac{1}{4C^2}, \frac{\eta_0}{2C}\right\}^{\frac{1}{\alpha}}.$$

Then, provided $E_0 \in (0, \varepsilon)$, we find that (4.81) implies that, for any $t \in [0, \widetilde{\tau}_{\max})$ such that $\eta(s) \leq 2CE_0^\alpha$ for all $s \in [0, t]$, we have $\eta(t) \leq \eta_0$ and

$$\eta(t) \leq C\Big(E_0^\alpha + 4C^2 E_0^{2\alpha}\Big) < 2CE_0^\alpha.$$

Hence, by continuity of η and the initial bound $\eta(0) \leq CE_0^\alpha$, it follows that if $E_0 \in (0,\varepsilon)$, then $\eta(t) \leq 2CE_0^\alpha < \eta_0$ for *all* $t \in [0,\widetilde{\tau}_{\max})$. The latter, in combination with the last estimate in (4.81), precludes the blow-up alternatives (4.42), (4.52), and (4.53), thereby implying $\tau_{\max} = T_{\max} = \infty$. We use the obtained global control of $z(t)$ and $\gamma(t)$ to subsequently prove the estimates (4.12) and (4.13), where we make use of the norm equivalences established in Lemma 4.5.15. Finally, we show the bounds (4.14) and (4.15) by proceeding along the lines of [84].

In the following, we will establish the key inequalities (4.81) by bounding the terms in the template function $\eta(t)$ one by one. Since we consider initial data of minimal regularity, cf. Remark 4.1.6, higher-order derivatives may suffer from nonintegrable temporal bounds. To address this, we often distinguish between short-time bounds, which focus on reducing the number of derivatives, and long-time bounds, which ensure sufficient temporal decay.

Bound on v_0. Inserting (4.40) into (4.47), we express

$$v_0(\zeta) = \mathring{v}_0(\zeta - \gamma_0(\zeta)) + \phi_0(\zeta - \gamma_0(\zeta) + \gamma_0(\zeta - \gamma_0(\zeta)) - \phi_0(\zeta).$$

Using the mean-value theorem twice, we obtain the bound

$$\|v_0\|_\infty \leq \|\mathring{v}_0\|_\infty + \|\phi_0'\|_\infty \|\gamma_0\|_\infty \|\gamma_0'\|_\infty \lesssim E_0. \tag{4.82}$$

Interpolation bounds. We establish bounds on the linear terms involving the initial phase modulation γ_0 in the Duhamel formulations of $\gamma(t)$, $z(t)$, $r(t)$, $\widetilde{y}(t)$, and their derivatives. Since γ_0 is bounded, the standard linear estimates from Propositions 4.4.1 and 4.4.2 provide sufficient temporal decay. However, since γ_0 is not necessarily small, they do not guarantee global-in-time smallness. To address this, we turn to the modulational estimates in Propositions 4.4.3 and 4.4.4, which crucially exploit the smallness of the derivative γ_0'. By interpolating between the standard and modulational linear estimates, we obtain both the required temporal decay and global-in-time smallness. For a heuristic overview of this argument in a simplified setting, we refer to §4.2.

We start by bounding the linear term

$$J_1(t) := \widetilde{S}(t)(\phi_0'\gamma_0) - (1-\chi(t))\phi_0' \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$$

in the Duhamel representation (4.49) of $z(t)$. Proposition 4.4.1 and estimate (4.50) yield

$$\|J_1(t)\|_\infty \lesssim (1+t)^{-1} \tag{4.83}$$

for $t \geq 0$. Using (4.30), we express

$$\begin{aligned} &J_1(t) \\ &= \mathrm{e}^{\mathcal{L}_0 t}(\phi_0'\gamma_0) - \phi_0'\gamma_0 - k_0\partial_k\phi(\cdot;k_0)\partial_\zeta S_p^0(t)(\phi_0'\gamma_0) \\ &\quad - \phi_0'\Big(\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 - \gamma_0 + S_p^0(t)(\phi_0'\gamma_0) - \chi(t)\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 + \chi(t)\Big(\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 - \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0\Big)\Big) \end{aligned}$$

Hence, Propositions 4.4.3 and 4.4.4 afford the bound

$$\|J_1(t)\|_\infty \lesssim (1+t)E_0 \tag{4.84}$$

for $t \geq 0$. Interpolating between (4.83) and (4.84) and using $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, we arrive at

$$\|J_1(t)\|_\infty \lesssim (1+t)^{-1+2\alpha} E_0^\alpha \tag{4.85}$$

for $t \geq 0$.

We proceed with estimating the linear term

$$I_{j,l}(t) := (\partial_t - a\partial_\zeta)^j \partial_\zeta^l \Big(S_p^0(t)(\phi_0'\gamma_0) + (1-\chi(t))\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 \Big)$$

in the Duhamel formulation (4.48) of $(\partial_t - a\partial_\zeta)^j \partial_\zeta^l \gamma(t)$ for $j = 0,1$ and $l \in \mathbb{N}_0$ with $l \leq 5$. Applying Proposition 4.4.1, we obtain

$$\left\| (\partial_t - a\partial_\zeta)^j \partial_\zeta^l S_p^0(t)(\phi_0'\gamma_0) \right\|_\infty \lesssim (1+t)^{-\frac{2j+l}{2}} \tag{4.86}$$

for $t \geq 0$, $j = 0,1$, and $l \in \mathbb{N}_0$ with $l \leq 5$. On the other hand, the modulational estimates in Propositions 4.4.3 and 4.4.4 yield

$$\begin{aligned} \left\| (\partial_t - a\partial_\zeta)^j \partial_\zeta^l S_p^0(t)(\phi_0'\gamma_0) \right\|_\infty &\lesssim (1+t)^{-\frac{2j+l-1}{2}} E_0, \\ \left\| (\partial_t - a\partial_\zeta) S_p^0(t)(\phi_0'\gamma_0) - \chi'(t)\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 \right\|_\infty &\lesssim E_0, \\ \chi'(t) \left\| \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 - \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 \right\|_\infty &\lesssim E_0 \end{aligned} \tag{4.87}$$

for $t \geq 0$, $j \in \{0,1\}$, and $l \in \mathbb{N}_0$ with $1 \leq l \leq 5$. Furthermore, we note that

$$\begin{aligned} I_{1,0}(t) = {} & (\partial_t - a\partial_\zeta) S_p^0(t)(\phi_0'\gamma_0) - \chi'(t)\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 + \chi'(t)\left(\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 - \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 \right) \\ & + (1-\chi(t))(\partial_t - a\partial_\zeta)\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 \end{aligned}$$

for $t \geq 0$. Hence, interpolating between (4.86) and (4.87), using (4.50), and recalling $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, we establish

$$\|I_{j,l}(t)\|_\infty \lesssim (1+t)^{-\frac{2j+l}{2} + 2\alpha} E_0^\alpha \tag{4.88}$$

for $t \geq 1$, $j \in \{0,1\}$, and $l \in \mathbb{N}_0$ with $1 \leq \min\{j,l\}$ and $l \leq 5$.

Finally, we bound the linear terms

$$\begin{aligned} J_2(t) &:= \partial_\zeta \Big(\widetilde{S}_r^0(t)(\phi_0'\gamma_0) + \partial_\zeta \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}(A_h(\phi_0')\gamma_0) + (1-\chi(t))\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 \Big), \\ J_3(t) &:= \partial_\zeta \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0, \end{aligned}$$

appearing in the formulas (4.58) and (4.57) for $r_\zeta(t)$ and $\widetilde{y}_\zeta(t)$, respectively. By Propositions 4.4.1 and 4.4.2 we have

$$\sqrt{t}\,\|J_2(t)\|_\infty \lesssim (1+t)^{-\frac{1}{2}}, \qquad \|J_3(t)\|_\infty \lesssim (1+t)^{-\frac{1}{2}} \tag{4.89}$$

for $t \geq 0$. We use (4.8), (4.32), and (4.34) to rewrite

$$J_2(t) = \partial_\zeta \Big(S_p^0(t)(\phi_0'\gamma_0) - \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}(\gamma_0 - A_h(\phi_0')\gamma_0') + (1-\chi(t))\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 \Big).$$

Applying estimates (4.87) and (4.50) and Proposition 4.4.2 to the latter, we arrive at

$$\sqrt{t}\,\|J_2(t)\|_\infty \lesssim \sqrt{1+t}\,E_0, \qquad \|J_3(t)\|_\infty \lesssim E_0, \tag{4.90}$$

for $t \geq 0$. Interpolating between (4.89) and (4.90) and using $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, we obtain

$$\sqrt{t}\,\|J_2(t)\|_\infty \lesssim (1+t)^{-\frac{1}{2}+2\alpha} E_0^\alpha, \qquad \|J_3(t)\|_\infty \lesssim (1+t)^{-\frac{1}{2}+\alpha} E_0^\alpha \tag{4.91}$$

for $t \geq 0$.

Bounds on $v(t)$, $v_\zeta(t)$ and $\partial_t\gamma(t)$. Let $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$. We bound $v(s) = z(s) + \partial_k\phi(\cdot;k_0)\gamma_\zeta(s)$ and $\partial_t\gamma(s) = \widetilde{\gamma}(s) - a\gamma_\zeta(s)$ as

$$\begin{aligned}
\|v(s)\|_\infty &\lesssim \|z(s)\|_\infty + \|\gamma_\zeta(s)\|_\infty \lesssim (1+s)^{-\frac{1}{2}+\alpha}\eta(t),\\
\sqrt{s}\,\|v_\zeta(s)\|_\infty &\lesssim \sqrt{s}\left(\|z_\zeta(s)\|_\infty + \|\gamma_\zeta(s)\|_{C^1_{\mathrm{ub}}}\right) \lesssim (1+s)^\alpha\eta(t),\\
s^{\frac{3}{4}}\,\|\partial_t\gamma(s)\|_\infty &\lesssim s^{\frac{3}{4}}(\|\widetilde{\gamma}(s)\|_\infty + \|\gamma_\zeta(s)\|_\infty) \lesssim (1+s)^{\frac{1}{4}+\alpha}\eta(t)
\end{aligned} \tag{4.92}$$

for $s \in [0,t]$.

Bounds on $r(t)$ and $r_\zeta(t)$. We start with bounding the nonlinear terms in the representation (4.58) for $r(t)$ one by one. To this end, let $t \in (0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$. We employ Lemma 4.5.6 and estimate (4.92) to establish the nonlinear bounds

$$\begin{aligned}
&\|\mathcal{Q}_p(z(s),v(s),\gamma(s))\|_\infty, \|\mathcal{R}_p(z(s),v(s),\gamma(s),\partial_t\gamma(s))\|_\infty, \|\mathcal{S}_p(z(s),v(s),\gamma(s))\|_\infty\\
&\qquad \lesssim \frac{\eta(s)^2\log(2+s)}{s^{\frac{3}{4}}(1+s)^{\frac{3}{4}-3\alpha}}
\end{aligned} \tag{4.93}$$

for $s \in (0,t]$. So, invoking Proposition 4.4.1 and using $0 < \alpha < \frac{1}{6}$, we obtain

$$\begin{aligned}
\left\|\partial_\zeta^{m+1}\int_0^t S_p^0(t-s)\mathcal{R}_p(z(s),v(s),\gamma(s),\widetilde{\gamma}(s))\mathrm{d}s\right\|_\infty &\lesssim \int_0^t \frac{\eta(s)^2\log(2+s)}{(1+t-s)^{\frac{1+m}{2}}s^{\frac{3}{4}}(1+s)^{\frac{3}{4}-3\alpha}}\mathrm{d}s\\
&\lesssim \frac{\eta(t)^2}{(1+t)^{\frac{1+m}{2}}}
\end{aligned} \tag{4.94}$$

and, analogously,

$$\left\|\partial_\zeta^{m+2-i}\int_0^t S_p^i(t-s)\mathcal{S}_p(z(s),v(s),\gamma(s))\mathrm{d}s\right\|_\infty \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{1+m}{2}}} \tag{4.95}$$

for all $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$ and $i, m \in \{0,1\}$. Furthermore, Proposition 4.4.2 and estimate (4.93) yield

$$\begin{aligned}
t^{\frac{m}{2}}\left\|\partial_\zeta^m\int_0^t \widetilde{S}_r^0(t-s)\mathcal{Q}_p(z(s),v(s),\gamma(s))\mathrm{d}s\right\|_\infty &\lesssim \int_0^t \frac{t^{\frac{m}{2}}\eta(s)^2\log(2+s)}{(t-s)^{\frac{m}{2}}\sqrt{1+t-s}\,s^{\frac{3}{4}}(1+s)^{\frac{3}{4}-3\alpha}}\mathrm{d}s\\
&\lesssim \frac{\eta(t)^2}{\sqrt{1+t}}
\end{aligned} \tag{4.96}$$

and, analogously,

$$\begin{aligned} t^{\frac{m}{2}}\left\|\partial_\zeta^m \int_0^t \widetilde{S}_r^1(t-s)\mathcal{R}_p(z(s),v(s),\gamma(s),\widetilde{\gamma}(s))\mathrm{d}s\right\|_\infty &\lesssim \frac{\eta(t)^2}{\sqrt{1+t}},\\ t^{\frac{m}{2}}\left\|\partial_\zeta^m \int_0^t \widetilde{S}_r^2(t-s)\mathcal{S}_p(z(s),v(s),\gamma(s))\mathrm{d}s\right\|_\infty &\lesssim \frac{\eta(t)^2}{\sqrt{1+t}} \end{aligned} \tag{4.97}$$

for $m = 0,1$ and all $t \in [0,\widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$. Moreover, Proposition 4.4.2 affords the bound

$$\left\|\partial_\zeta^m \int_0^t \widetilde{S}_r^0(t-s)\Big(f_p\gamma_\zeta(s)^2\Big)\mathrm{d}s\right\|_\infty \lesssim \int_0^t \frac{\eta(s)^2(1+s)^{2\alpha-1}}{\sqrt{1+t-s}(t-s)^{\frac{m}{2}}}\mathrm{d}s \lesssim \frac{\eta(t)^2(\log(2+t))^m}{(1+t)^{\frac{1+m}{2}-2\alpha}} \tag{4.98}$$

for $m = 0,1$ and all $t \in [0,\widetilde{\tau}_{\max})$. Using Proposition 4.4.2, we subsequently infer

$$\left\|\partial_\zeta \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)(t-s)}\Big(A_h(f_p)\gamma_\zeta(s)^2\Big)\mathrm{d}s\right\|_\infty \lesssim \int_0^t \frac{\eta(s)^2}{\sqrt{t-s}(1+s)^{1-2\alpha}}\mathrm{d}s \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{1}{2}-2\alpha}} \tag{4.99}$$

for all $t \in [0,\widetilde{\tau}_{\max})$. Similarly, exploiting that ∂_ζ commutes with $\mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)(t-s)}$, we obtain

$$\begin{aligned} &\left\|\partial_\zeta^2 \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)(t-s)}\Big(A_h(f_p)\gamma_\zeta(s)^2\Big)\mathrm{d}s\right\|_\infty \\ &\qquad \lesssim \int_0^{\max\{0,t-1\}} \frac{\eta(s)^2}{(t-s)(1+s)^{1-2\alpha}}\mathrm{d}s + \int_{\max\{0,t-1\}}^t \frac{\eta(s)^2}{\sqrt{t-s}(1+s)^{1-2\alpha}}\mathrm{d}s \\ &\lesssim \frac{\eta(t)^2\log(2+t)}{(1+t)^{1-2\alpha}} \end{aligned} \tag{4.100}$$

for all $t \in [0,\widetilde{\tau}_{\max})$.

Next, we establish bounds on the linear term

$$J_*(t) := \widetilde{S}_r^0(t)(v_0+\phi_0'\gamma_0+\gamma_0'v_0) + \mathrm{e}^{\left(d\partial_\zeta^2+a\partial_\zeta\right)t}\partial_\zeta(A_h(\phi_0')\gamma_0) + (1-\chi(t))\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$$

in (4.58). First, we note that

$$\partial_\zeta J_*(t) = J_2(t) + \partial_\zeta \widetilde{S}_r^0(t)(v_0+\gamma_0'v_0)$$

for $t \geq 0$. Hence, using Proposition 4.4.2, the estimates (4.50), (4.82), (4.90) and (4.91), and the facts that $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, we obtain

$$\|J_*(t)\|_\infty \lesssim \frac{1}{\sqrt{1+t}}, \qquad \sqrt{t}\,\|\partial_\zeta J_*(t)\|_\infty \lesssim \sqrt{1+t}\,E_0, \qquad \sqrt{t}\,\|\partial_\zeta J_*(t)\|_\infty \lesssim \frac{E_0^\alpha}{(1+t)^{\frac{1}{2}-2\alpha}} \tag{4.101}$$

for $t \geq 0$.

Combining the nonlinear bounds (4.94), (4.95), (4.96), (4.97), (4.98), (4.99), and (4.100) with the linear bounds (4.101), we estimate the right-hand side of (4.58) by

$$\|r(t)\|_\infty \lesssim \frac{1}{(1+t)^{\frac{1}{2}-2\alpha}} \qquad \sqrt{t}\,\|r_\zeta(t)\|_\infty \lesssim \Big(E_0^\alpha + \eta(t)^2\Big)\frac{\log(2+t)}{(1+t)^{\frac{1}{2}-2\alpha}} \tag{4.102}$$

and obtain

$$\|r(t) - J_*(t)\|_\infty \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{1}{2}-2\alpha}} \tag{4.103}$$

for all $t \in [0,\widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$.

Short-time bounds on $y(t)$ and $y_\zeta(t)$. Recalling that the cut-off function $\chi(t)$ and the propagator $S_p^0(t)$ vanish on $[0,1]$, we find by (4.48), (4.56), and (4.60) that

$$y(t) = \exp\left(\frac{\nu}{d}\left(\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 - r(t)\right)\right) \tag{4.104}$$

for $t \in [0,1]$. Hence, (4.50) and (4.102) yield an E_0-independent constant $K_* > 0$ such that

$$y(\zeta,t) \leq K_*, \qquad 1 \leq K_* y(\zeta,t) \tag{4.105}$$

and

$$\sqrt{t}\,\|y_\zeta(t)\|_\infty \leq K_*\left(E_0^\alpha + \eta(t)^2\right) \tag{4.106}$$

hold for all $\zeta \in \mathbb{R}$ and $t \in [0,\widetilde{\tau}_{\max})$ with $t \leq 1$ and $\eta(t) \leq \frac{1}{2}$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$.

Bounds on $y(t)$ and $y_\zeta(t)$. First, we bound the linear term in (4.63). For this purpose, we assume that $1 \in [0,\widetilde{\tau}_{\max})$ and $\eta(1) \leq \frac{1}{2}$ and decompose (4.104) at $t = 1$ as

$$y(1) = y_1 + y_2$$

with

$$y_1 = \exp\left(\frac{\nu}{d}\left(\mathrm{e}^{-\partial_\zeta^4 1}\gamma_0 - J_*(1)\right)\right)\left(\mathrm{e}^{\frac{\nu}{d}(J_*(1)-r(1))} - 1\right), \qquad y_2 = \exp\left(\frac{\nu}{d}\left(\mathrm{e}^{-\partial_\zeta^4 1}\gamma_0 - J_*(1)\right)\right).$$

Applying the mean value theorem and the estimates (4.50), (4.101), and (4.103), we obtain

$$\|y_1\|_\infty \lesssim \eta(1)^2, \qquad \|y_2\|_\infty \lesssim 1, \qquad \|y_2'\|_\infty \lesssim E_0.$$

Thus, using the latter bounds and estimate (4.105), applying Proposition 4.4.2, and recalling $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, we obtain

$$\begin{aligned}
\left\|\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-1)} y(1)\right\|_\infty &\leq K_*, \\
\left\|\partial_\zeta \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-1)} y(1)\right\|_\infty &\lesssim \frac{\|y_2\|_\infty^{1-2\alpha}\|y_2'\|_\infty^{2\alpha} + \|y_1\|_\infty + \|y_2'\|_\infty}{(1+t)^{\frac{1}{2}-\alpha}} \lesssim \frac{E_0^\alpha + \eta(t)^2}{(1+t)^{\frac{1}{2}-\alpha}}
\end{aligned} \tag{4.107}$$

for all $t \in [0,\widetilde{\tau}_{\max})$ with $t \geq 1$ and $\eta(t) \leq \frac{1}{2}$, where we used that η is monotonically increasing.

We proceed with bounding the nonlinear term in (4.63). Take $t \in [0,\widetilde{\tau}_{\max})$ with $t \geq 1$ and $\eta(t) \leq \frac{1}{2}$. Define

$$Y(t) := \sup_{1 \leq s \leq t} \|y(s)\|_\infty.$$

Lemma 4.5.9 and estimate (4.92) give rise to the nonlinear bound

$$\|\mathcal{N}_c(r(s), y(s), z(s), v(s), \gamma(s), \widetilde{\gamma}(s))\|_\infty \lesssim \eta(s)^2(1 + Y(t))\frac{\log(2+s)}{(1+s)^{\frac{3}{2}-3\alpha}}$$

for $s \in [1,t]$, where we use $\eta(t) \leq \frac{1}{2}$. Combining the latter with Proposition 4.4.2, we find a constant $R_* > 0$ such that

$$\begin{aligned} &\left\|\partial_\zeta^m \int_1^t \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-s)} \mathcal{N}_c(r(s), y(s), z(s), v(s), \gamma(s), \widetilde{\gamma}(s)) \mathrm{d}s\right\|_\infty \\ &\qquad \lesssim \int_1^t \frac{\eta(s)^2(1+Y(t))\log(2+s)}{(t-s)^{\frac{m}{2}}(1+s)^{\frac{3}{2}-3\alpha}} \mathrm{d}s \leq R_* \frac{\eta(t)^2(1+Y(t))}{(1+t)^{\frac{m}{2}}}. \end{aligned} \tag{4.108}$$

for $m = 0,1$ and all $t \in [0, \widetilde{\tau}_{\max})$ with $t \geq 1$ and $\eta(t) \leq \frac{1}{2}$. Taking suprema with respect to t in (4.63) and using the estimates (4.107) and (4.108), we find an E_0-independent constant $C_* > 0$ such that

$$Y(t) \leq C_*\Big(1 + \eta(t)^2(1+Y(t))\Big)$$

for all $t \in [0, \widetilde{\tau}_{\max})$ with $t \geq 1$ and $\eta(t) \leq \frac{1}{2}$. This implies

$$Y(t) \leq 2C_* \tag{4.109}$$

for all $t \in [0, \widetilde{\tau}_{\max})$ with $t \geq 1$ and $\eta(t) \leq \frac{1}{2}\min\{1, 1/\sqrt{C_*}\}$. Moreover, the lower bound in (4.105) and the fact that $\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}$ is a positive operator yield

$$0 \leq \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}(K_* y(1) - 1) = K_* \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t} y(1) - 1 \tag{4.110}$$

for $t \geq 1$. Finally, applying (4.107), (4.108), (4.109), and (4.110) to (4.63) and using the short-time bounds (4.105) and (4.106), we obtain an E_0-independent constant $M_* > 0$ such that

$$y(\zeta,t) \leq M_*, \qquad M_* y(\zeta,t) \geq 1, \qquad \sqrt{t}\,\|y_\zeta(t)\|_\infty \leq M_*\Big(E_0^\alpha + \eta(t)^2\Big)(1+t)^\alpha \tag{4.111}$$

for all $\zeta \in \mathbb{R}$ and $t \in [0, \widetilde{\tau}_{\max})$ with

$$\eta(t) \leq \frac{1}{2}\min\left\{1, \frac{1}{\sqrt{C_*}}, \frac{1}{\sqrt{K_* R_*(1+2C_*)}}\right\} =: \eta_0.$$

Short-time bounds on $\gamma(t)$ and its derivatives. We use that $\gamma(t) = \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ for $t \in [0, \widetilde{\tau}_{\max})$ with $t \leq 1$ by Proposition 4.5.4. Hence, (4.50) yields

$$\|\gamma(t)\|_\infty \lesssim 1, \qquad t^{\frac{j}{4}}\left\|\partial_\zeta^j \gamma_\zeta(t)\right\|_\infty \lesssim E_0, \qquad t^{\frac{3}{4}}\|\widetilde{\gamma}(t)\|_\infty \lesssim E_0 \tag{4.112}$$

for $j = 0,1,2$ and $t \in [0, \widetilde{\tau}_{\max})$ with $t \leq 1$. In addition, it implies

$$\|\gamma_{\zeta\zeta\zeta\zeta}(t)\|_{C^1_{\mathrm{ub}}} \lesssim E_0, \qquad \|\widetilde{\gamma}_\zeta(t)\|_{C^2_{\mathrm{ub}}} \lesssim E_0 \tag{4.113}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $\frac{t_*}{2} \leq t \leq 1$.

Bounds on $\gamma(t)$ and $\gamma_\zeta(t)$. We first consider the case $\nu \neq 0$. Recalling (4.56) and (4.60), we take the spatial derivative of

$$\gamma(t) = r(t) + \frac{d}{\nu}\log(y(t)),$$

yielding

$$\gamma_\zeta(t) = r_\zeta(t) + \frac{d y_\zeta(t)}{\nu y(t)}.$$

We note that by (4.102) and (4.111) the above expressions are well-defined and we deduce

$$\|\gamma(t)\|_\infty \le \|r(t)\|_\infty + \frac{d}{\nu}|\log(M_*)| \lesssim 1$$

and

$$\|\gamma_\zeta(t)\|_\infty \le \|r_\zeta(t)\|_\infty + \frac{dM_*}{\nu}\|y_\zeta(t)\|_\infty \lesssim \frac{E_0^\alpha + \eta(t)^2}{(1+t)^{\frac{1}{2}-\alpha}} \tag{4.114}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $t \ge 1$ and $\eta(t) \le \eta_0$. Combining the latter with the short-time bounds (4.112) yields

$$\|\gamma(t)\|_\infty \lesssim 1, \qquad \|\gamma_\zeta(t)\|_\infty \lesssim \frac{E_0^\alpha + \eta(t)^2}{(1+t)^{\frac{1}{2}-\alpha}}, \qquad \|\gamma_\zeta(0)\|_\infty \lesssim E_0^\alpha \tag{4.115}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \le \eta_0$, where we use $0 \le E_0 \le 1$ and $0 < \alpha < \frac{1}{6}$.

We proceed with the case $\nu = 0$. Recalling (4.56), we establish bounds on $\widetilde{y}(t)$. First, we invoke Proposition 4.4.2 and estimates (4.82) and (4.91) to bound the linear term

$$I_*(t) := S_h^0(t)(v_0 + \gamma_0' v_0) + \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}(\gamma_0 - A_h(\phi_0')\gamma_0')$$

in (4.57) by

$$\|I_*(t)\|_\infty \lesssim 1, \qquad \|\partial_\zeta I_*(t)\|_\infty \lesssim \frac{E_0^\alpha}{(1+t)^{\frac{1}{2}-\alpha}} \tag{4.116}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $t \ge 1$, where we used $0 \le E_0 \le 1$ and $0 < \alpha < \frac{1}{6}$. Next, we bound the nonlinear terms in (4.57). With the aid of Proposition 4.4.2 and estimate (4.93) we establish

$$\left\|\partial_\zeta^m \int_0^t S_h^0(t-s)\mathcal{Q}_p(z(s), v(s), \gamma(s))\mathrm{d}s\right\|_\infty \lesssim \int_0^t \frac{\eta(s)^2 \log(2+s)}{(t-s)^{\frac{m}{2}} s^{\frac{3}{4}} (1+s)^{\frac{3}{4}-3\alpha}}\mathrm{d}s \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{m}{2}}}, \tag{4.117}$$

and, analogously,

$$\begin{gathered}\left\|\partial_\zeta^m \int_0^t S_h^1(t-s)\mathcal{R}_p(z(s), v(s), \gamma(s), \widetilde{\gamma}(s))\mathrm{d}s\right\|_\infty, \left\|\partial_\zeta^m \int_0^t S_h^2(t-s)\mathcal{S}_p(z(s), v(s), \gamma(s))\mathrm{d}s\right\|_\infty, \\ \left\|\partial_\zeta^m \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-s)}\Big(A_h(f_p)\partial_\zeta\Big(\gamma_\zeta(s)^2\Big)\Big)\mathrm{d}s\right\|_\infty \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{m}{2}}},\end{gathered} \tag{4.118}$$

for $m = 0, 1$ and $t \in [0, \widetilde{\tau}_{\max})$ with $t \ge 1$ and $\eta(t) \le \frac{1}{2}$. Applying (4.116), (4.117), and (4.118) to (4.57), we conclude

$$\|\widetilde{y}(t)\|_\infty \lesssim 1, \qquad \|\widetilde{y}_\zeta(t)\|_\infty \lesssim \frac{E_0^\alpha + \eta(t)^2}{(1+t)^{\frac{1}{2}-\alpha}}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $t \ge 1$ and $\eta(t) \le \frac{1}{2}$. Combining the latter with the short-time estimates (4.112) and the estimates (4.102) on $r(t)$ for large times, we bound (4.56), arriving at (4.115) for $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \le \frac{1}{2}$.

Bounds on $\widetilde{\gamma}(t)$, $\gamma_{\zeta\zeta}(t)$, and their derivatives. First, we use Proposition 4.4.1 and the estimates (4.82) and (4.88) to bound the linear term in (4.48) as

$$\left\|(\partial_t - a\partial_\zeta)^j \partial_\zeta^l S_p^0(t)(v_0 + \phi_0'\gamma_0 + \gamma_0' v_0)\right\|_\infty \lesssim (1+t)^{-\frac{2j+l}{2}+2\alpha} E_0^\alpha \tag{4.119}$$

for $t \geq 1$, $j \in \{0,1\}$, and $l \in \mathbb{N}_0$ with $1 \leq \min\{j,l\}$ and $l \leq 5$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$.

Next, we bound the nonlinear terms in (4.48). To this end, let $t \in (0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$. We invoke Lemma 4.5.2 and employ the estimate (4.92) to obtain

$$\|\mathcal{Q}(v(s),\gamma(s))\|_\infty, \|\mathcal{R}(v(s),\gamma(s),\widetilde{\gamma}(s))\|_\infty, \left\|\partial_\zeta^j \mathcal{S}(v(s),\gamma(s))\right\|_\infty \lesssim \frac{\eta(t)^2}{s^{\frac{3}{4}}(1+s)^{\frac{1}{4}-2\alpha}} \tag{4.120}$$

for $s \in (0,t]$ and $j = 0,1$. So, using Proposition 4.4.1 and the facts that $S_p(t)$ vanishes on $[0,1]$ and we have $0 < \alpha < \frac{1}{6}$, we infer

$$\begin{aligned}\left\|(\partial_t - a\partial_\zeta)^j \partial_\zeta^l \int_0^t S_p^0(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))\mathrm{d}s\right\|_\infty &\lesssim \int_0^t \frac{\eta(s)^2}{(1+t-s)s^{\frac{3}{4}}(1+s)^{\frac{1}{4}-2\alpha}}\mathrm{d}s \\ &\lesssim \frac{\eta(t)^2\log(2+t)}{(1+t)^{1-2\alpha}},\end{aligned} \tag{4.121}$$

for all $t \in [0,\widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$ and $j,l \in \mathbb{N}_0$ with $2 \leq l + 2j \leq 5$. Thus, applying the linear bound (4.119) and the nonlinear bound (4.121) to the right-hand side of (4.48), we obtain

$$\|(\gamma_{\zeta\zeta}(t),\widetilde{\gamma}(t))\|_{C^3_{\mathrm{ub}}\times C^3_{\mathrm{ub}}} \lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{1-2\alpha}},$$

for all $t \in [0,\widetilde{\tau}_{\max})$ with $t \geq 1$ and $\eta(t) \leq \frac{1}{2}$. Combining the latter with the short-time bounds (4.112) and (4.113) and using that $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, we arrive at

$$t^{\frac{1}{4}}\|\gamma_{\zeta\zeta}(t)\|_\infty \lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{\frac{3}{4}-2\alpha}}, \qquad \sqrt{t}\,\|\gamma_{\zeta\zeta\zeta}(t)\|_\infty \lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{\frac{1}{2}-2\alpha}}, \tag{4.122}$$

and

$$\begin{aligned} t^{\frac{3}{4}}\|\widetilde{\gamma}(t)\|_\infty &\lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{\frac{1}{4}-2\alpha}}, \\ \varrho(t)\|(\gamma_{\zeta\zeta\zeta\zeta}(t),\widetilde{\gamma}_\zeta(t))\|_{C^1_{\mathrm{ub}}\times C^2_{\mathrm{ub}}} &\lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{1-2\alpha}}\end{aligned} \tag{4.123}$$

for all $t \in [0,\widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$.

Short-time bounds on $\mathring{v}(t)$ and its derivatives. Let $t \in (0,\widetilde{\tau}_{\max})$ with $t \leq 1$ and $\eta(t) \leq \frac{1}{2}$. Using Lemma 4.5.15 and estimates (4.92) and (4.112), we deduce

$$\|\mathring{v}(s)\|_\infty \lesssim \|v(s)\|_\infty + \|\gamma_\zeta(s)\|_\infty \lesssim \frac{\eta(t)}{(1+s)^{\frac{1}{2}-\alpha}} \tag{4.124}$$

for $s \in [0,t]$. Thus, Lemma 4.5.12 and estimates (4.112) and (4.124) yield the nonlinear bound

$$\left\|\mathring{\mathcal{N}}(\mathring{v}(s),\gamma(s),\partial_t\gamma(s))\right\|_\infty \lesssim \frac{E_0^\alpha + \eta(t)^2}{s^{\frac{3}{4}}}$$

for $s \in (0,t]$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$. Hence, applying the latter estimate and the Green's functions bounds in Proposition 4.5.11 to the Duhamel formula (4.67), we establish

$$\begin{aligned} t^{\frac{j}{2}}\left|\partial_\zeta^j \mathring{v}(\zeta,t)\right| &\lesssim \int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(\zeta-\bar\zeta)^2}{Mt}}}{\sqrt{t}} E_0 \,\mathrm{d}\bar\zeta + t^{\frac{j}{2}}\int_0^t\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(\zeta-\bar\zeta)^2}{M(t-s)}}}{(t-s)^{\frac{1+j}{2}} s^{\frac{3}{4}}}\left(E_0^\alpha + \eta(t)^2\right)\mathrm{d}\bar\zeta \mathrm{d}s \\ &\lesssim E_0 + \int_0^t \frac{t^{\frac{j}{2}}\left(E_0^\alpha + \eta(t)^2\right)}{(t-s)^{\frac{j}{2}} s^{\frac{3}{4}}}\mathrm{d}s \lesssim E_0^\alpha + \eta(t)^2 \end{aligned} \tag{4.125}$$

for $j = 0,1$, $\zeta \in \mathbb{R}$, and $t \in [0,\widetilde\tau_{\max})$ with $t \leq t_*$ and $\eta(t) \leq \frac{1}{2}$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$.

To prepare for the application of the nonlinear damping estimate from Proposition 4.5.14, we also derive a bound on the second derivative of $\mathring{v}(t)$ at time $t = t_*$. For this purpose, we assume $t_* \in [0,\widetilde\tau_{\max})$ with $\eta(t_*) \leq \frac{1}{2}$. Applying Lemma 4.5.12 once again, while using the estimates (4.112), (4.113), and (4.125), we arrive at

$$\left\|\mathring{\mathcal{N}}_1(\mathring{v}(s),\gamma(s),\partial_t\gamma(s))\right\|_\infty \lesssim E_0^\alpha + \eta(t)^2$$

for $s \in [\frac{t_*}{2}, t_*]$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$. Combining the latter with Proposition 4.5.11, we bound the spatial derivative of (4.68) as

$$\begin{aligned} |\mathring{v}_{\zeta\zeta}(\zeta,t_*)| &\lesssim \int_{\mathbb{R}} \mathrm{e}^{-\frac{2(\zeta-\bar\zeta)^2}{Mt_*}}\left(E_0^\alpha + \eta(t_*)^2\right)\mathrm{d}\bar\zeta + \int_{\frac{t_*}{2}}^{t_*}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(\zeta-\bar\zeta)^2}{M(t_*-s)}}}{(t_*-s)}\left(E_0^\alpha + \eta(t_*)^2\right)\mathrm{d}\bar\zeta\mathrm{d}s \\ &\lesssim E_0^\alpha + \eta(t_*)^2 \end{aligned} \tag{4.126}$$

for $\zeta \in \mathbb{R}$.

Short-time bound on $z(t)$. Estimates (4.112) and (4.125) and Lemma 4.5.15 yield

$$\|z(t)\|_\infty \lesssim \|v(t)\|_\infty + \|\gamma_\zeta(t)\|_\infty \lesssim \|\mathring{v}(t)\|_\infty + \|\gamma_\zeta(t)\|_\infty \lesssim E_0^\alpha + \eta(t)^2 \tag{4.127}$$

for $t \in [0,\widetilde\tau_{\max})$ with $t \leq t_*$ and $\eta(t) \leq \frac{1}{2}$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$.

Bounds on $z(t)$. Using Proposition 4.4.1 and the estimates (4.50), (4.82), and (4.85), we bound the linear term

$$J_0(t) := \widetilde{S}(t)(v_0 + \phi_0'\gamma_0 + \gamma_0' v_0) + (1-\chi(t))(\phi_0' + k_0\partial_k\phi(\cdot;k_0)\partial_\zeta)\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$$

in (4.49) as

$$\|J_0(t)\|_\infty \lesssim \frac{E_0^\alpha}{(1+t)^{1-2\alpha}} \tag{4.128}$$

for $t \geq 0$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$. On the other hand, using Proposition 4.4.1 and estimates (4.92) and (4.120), we bound the nonlinear term in (4.49) as

$$\begin{aligned}&\left\|\int_0^t \widetilde{S}(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))\mathrm{d}s + \gamma_\zeta(t)v(t)\right\|_\infty \\ &\quad\lesssim \frac{\eta(t)^2}{(1+t)^{1-2\alpha}} + \int_0^t \left(1+\frac{1}{\sqrt{t-s}}\right)\frac{\eta(t)^2}{(1+t-s)s^{\frac{3}{4}}(1+s)^{\frac{1}{4}-2\alpha}}\mathrm{d}s \lesssim \frac{\eta(t)^2\log(2+t)}{(1+t)^{1-2\alpha}}\end{aligned} \tag{4.129}$$

for $t \in [0,\widetilde{\tau}_{\max})$ with $t \geq t_*$ and $\eta(t) \leq \frac{1}{2}$. Combining the short-time bound (4.127) with the long-time estimates (4.128) and (4.129), we arrive at

$$\|z(t)\|_\infty \lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{1-2\alpha}}, \qquad \|z(0)\|_\infty \lesssim E_0^\alpha \tag{4.130}$$

for $t \in [0,\widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$.

Short-time bounds on $\mathring{z}(t)$, $\mathring{z}_\zeta(t)$, and $z_\zeta(t)$. Recalling the identities (4.19) and (4.21) and using that we have $\|\gamma_\zeta(t)\|_\infty < r_0$ by Proposition 4.5.4, we find that the mean-value theorem yields

$$\begin{aligned}&t^{\frac{j}{2}}\left|\|\mathring{z}(t)\|_{C^j_{\mathrm{ub}}} - \|\mathring{v}(t)\|_{C^j_{\mathrm{ub}}}\right| \lesssim t^{\frac{j}{2}}\|\mathring{z}(t)-\mathring{v}(t)\|_{C^j_{\mathrm{ub}}} \\ &\quad\lesssim t^{\frac{j}{2}}\left(\|\gamma_\zeta(t)\|_{C^j_{\mathrm{ub}}} + \|\gamma_\zeta(t)\|_\infty\left(\|\gamma(t)\|_\infty\|\phi_0'\|_{C^j_{\mathrm{ub}}} + \sup_{k\in[k_0-r_0,k_0+r_0]}\|\partial_k\phi(\cdot;k)\|_{C^j_{\mathrm{ub}}}\right)\right)\end{aligned} \tag{4.131}$$

for $j = 0,1,2$ and $t \in [0,\widetilde{\tau}_{\max})$ with $\eta(t) \leq \frac{1}{2}$. Applying the estimates (4.112), (4.125), and (4.126) to (4.131) and using $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, we establish

$$t^{\frac{j}{2}}\|\mathring{z}(t)\|_{C^j_{\mathrm{ub}}} \lesssim E_0^\alpha + \eta(t)^2 \tag{4.132}$$

for $j = 0,1$ and $t \in [0,\widetilde{\tau}_{\max})$ with $t \leq t_*$ and $\eta(t) \leq \frac{1}{2}$, and

$$\|\mathring{z}(t_*)\|_{C^2_{\mathrm{ub}}} \lesssim E_0^\alpha + \eta(t_*)^2 \tag{4.133}$$

provided $t_* \in [0,\widetilde{\tau}_{\max}]$ with $\eta(t_*) \leq \frac{1}{2}$.

Bounds on $\mathring{z}(t)$, $\mathring{z}_\zeta(t)$ and $z_\zeta(t)$. Lemma 4.5.15 and (4.115), (4.122), and (4.130) yield

$$\|\mathring{z}(t)\|_\infty \lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{1-2\alpha}} \tag{4.134}$$

for $t \in [0,\widetilde{\tau}_{\max})$ with $\eta(t) \leq \eta_0$. Combining (4.115), (4.122), (4.123), (4.133), and (4.134) with the nonlinear damping estimate in Proposition 4.5.14, we deduce

$$\begin{aligned}\|\mathring{z}(t)\|_{C^1_{\mathrm{ub}}} &\lesssim \|\mathring{z}(t)\|_\infty + \left(\mathrm{e}^{t_*-t}\left(E_0^\alpha + \eta(t_*)^2\right)^2 + \int_{t_*}^t \frac{\mathrm{e}^{s-t}\left(\left(E_0^\alpha+\eta(t)^2\right)\log(2+s)\right)^2}{(1+s)^{2-4\alpha}}\mathrm{d}s\right)^{\frac{1}{2}} \\ &\lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{1-2\alpha}}\end{aligned} \tag{4.135}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $t \geq t_*$ and $\eta(t) \leq \eta_0$. Linking the short- and long-time estimates (4.132) and (4.135), we thus obtain

$$\sqrt{t}\,\|\mathring{z}_\zeta(t)\|_\infty \lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{\frac{1}{2}-2\alpha}} \tag{4.136}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \eta_0$. Finally, we apply Lemma 4.5.15 and use (4.115), (4.122), and (4.136) to establish

$$\sqrt{t}\,\|z_\zeta(t)\|_\infty \lesssim \left(E_0^\alpha + \eta(t)^2\right)\frac{\log(2+t)}{(1+t)^{\frac{1}{2}-2\alpha}} \tag{4.137}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \eta_0$.

Bounds on $\mathring{v}(t)$, $\mathring{v}_\zeta(t)$. Combining (4.115), (4.122), (4.134), (4.131), and (4.136), we arrive at

$$\|\mathring{v}(t)\|_\infty \lesssim \frac{E_0^\alpha + \eta(t)}{(1+t)^{\frac{1}{2}-\alpha}}, \qquad \sqrt{t}\,\|\mathring{v}_\zeta(t)\|_\infty \lesssim (E_0^\alpha + \eta(t))(1+t)^\alpha \tag{4.138}$$

for $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \eta_0$.

Proof of key estimates. Combining (4.102), (4.111), (4.115), (4.122), (4.123), (4.130), and (4.137), we conclude that there exists a constant $C > 0$, independent of E_0, such that the key estimates (4.81) hold for all $t \in [0, \widetilde{\tau}_{\max})$ with $\eta(t) \leq \eta_0$. Now take $E_0 \in (0, \varepsilon)$. As previously argued, this implies

$$\eta(t) \leq 2CE_0^\alpha < \eta_0, \qquad \|\gamma(s)\|_\infty \leq C \tag{4.139}$$

for all $t \in [0, \widetilde{\tau}_{\max})$, thereby ruling out the blow-up scenarios (4.52) and (4.53), and hence ensuring $\widetilde{\tau}_{\max} = \tau_{\max} = T_{\max}$. So, (4.138) shows that $u(t)$ remains uniformly bounded on $[0, T_{\max})$, excluding the blow-up alternative (4.42) and establishing that $\tau_{\max} = T_{\max} = \infty$. Finally, the bounds (4.134), (4.136), (4.138), and (4.139) yield the estimates (4.12) and (4.13).

Optimal temporal decay rates. It remains to obtain the estimates (4.12) and (4.13) for the case $\alpha = 0$. Here, we make use of the fact that we have established

$$\eta(t) \leq 2CE_0^{\tilde{\alpha}}, \qquad t \geq 0, \tag{4.140}$$

for a fixed $\tilde{\alpha} \in (0, \frac{1}{6})$. First, we apply Proposition 4.4.2 and employ (4.105), (4.108), and (4.140) to bound the right-hand side of (4.63) as

$$\|y_\zeta(t)\|_\infty \lesssim \frac{1}{\sqrt{1+t}}$$

for $t \geq 1$. The latter, together with (4.114) and (4.140), yields

$$\|\gamma_\zeta(t)\|_\infty \lesssim \frac{1}{\sqrt{1+t}} \tag{4.141}$$

for $t \geq 1$. Therefore, Proposition 4.5.2 and estimates (4.92) and (4.140) result in the nonlinear bounds

$$\|\mathcal{Q}(v(s),\gamma(s))\|_\infty, \|\mathcal{R}(v(s),\gamma(s),\widetilde{\gamma}(s))\|_\infty, \left\|\partial_\zeta^j \mathcal{S}(v(s),\gamma(s))\right\|_\infty \lesssim \frac{1}{s^{\frac{3}{4}}(1+s)^{\frac{1}{4}}},$$
$$\|\gamma_\zeta(s)v(s)\|_\infty \lesssim \frac{1}{1+s}$$

for all $s > 0$ and $j = 0, 1$. This shows that the estimates (4.121) and (4.129) hold for $\alpha = 0$ and $t \geq 1$ (with $\eta(t)$ replaced by 1). Applying the latter bounds, estimates (4.50) and (4.82), and Proposition 4.4.1 to (4.48) and (4.49), we arrive at

$$\|z(t)\|_\infty, \|(\gamma_{\zeta\zeta}(t), \widetilde{\gamma}(t))\|_{C^3_{\mathrm{ub}} \times C^3_{\mathrm{ub}}} \lesssim \frac{\log(2+t)}{1+t} \tag{4.142}$$

for $t \geq t_*$. Next, we observe that Lemma 4.5.15 and the estimates (4.115), (4.141), and (4.142) yield

$$\|\mathring{z}(t)\|_\infty \lesssim \frac{\log(2+t)}{1+t} \tag{4.143}$$

for $t \geq t_*$. Combining (4.115), (4.133), (4.141), (4.142), and (4.143) with Proposition 4.5.14, we find that estimate (4.135) also holds for $\alpha = 0$ and $t \geq t_*$ (with $\eta(t)$ replaced by 1). This leads to the bound

$$\|\mathring{z}(t)\|_{C^1_{\mathrm{ub}}} \lesssim \frac{\log(2+t)}{1+t} \tag{4.144}$$

for $t \geq t_*$. Finally, (4.115), (4.131), (4.141), (4.142), and (4.144) yield

$$\|\mathring{v}(t)\|_{C^1_{\mathrm{ub}}} \lesssim \frac{1}{\sqrt{1+t}} \tag{4.145}$$

for $t \geq t_*$. It follows from (4.141), (4.142), (4.143), (4.144), and (4.145) that the estimates (4.12) and (4.13) also hold for the case $\alpha = 0$.

Approximation by the viscous Hamilton-Jacobi equation. We begin with establishing auxiliary bounds for short time. We use Propositions 4.4.3 and 4.4.4 to establish the modulational bounds

$$\begin{aligned} \left\| S_p^0(t)(\phi_0'\gamma_0) - \chi(t)\mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 \right\|_\infty &\lesssim E_0, \\ (1-\chi(t))\left\| \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0 - \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)t}\gamma_0 \right\|_\infty &\lesssim E_0 \end{aligned} \tag{4.146}$$

for $t \geq 0$. On the other hand, given $\alpha \in (0, \frac{1}{6})$, Propositions 4.4.1 and 4.4.2 in combination with the estimates (4.120) and (4.139) yield the nonlinear bounds

$$\begin{aligned} \left\| \int_0^t S_p^0(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\mathrm{d}s \right\|_\infty &\lesssim \int_0^t \frac{\eta(s)^2}{s^{\frac{3}{4}}(1+s)^{\frac{1}{4}-2\alpha}}\mathrm{d}s \lesssim E_0^{2\alpha}(1+t)^{2\alpha} \\ \left\| \partial_\zeta^m \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-s)}\gamma_\zeta(s)^2\mathrm{d}s \right\|_\infty &\lesssim \int_0^t \frac{\eta(s)^2}{(t-s)^{\frac{m}{2}}(1+s)^{1-2\alpha}}\mathrm{d}s \lesssim \frac{E_0^{2\alpha}}{(1+t)^{\frac{m}{2}-2\alpha}} \end{aligned} \tag{4.147}$$

for $t \geq 0$ and $m = 0, 1$.

Having established the auxiliary bounds (4.146) and (4.147), we follow the approach of [84] and distinguish between the cases $\nu = 0$ and $\nu \neq 0$. We begin with the case $\nu \neq 0$. Since $\mathrm{e}^{(d\partial_\zeta^2 + a\partial_\zeta)t}$ is a positive operator, we have the pointwise estimate

$$\mathrm{e}^{-\frac{\nu}{d}M} \leq \breve{y}(t) \leq \mathrm{e}^{\frac{\nu}{d}M}, \qquad \breve{y}(t) := \mathrm{e}^{(d\partial_\zeta^2 + a\partial_\zeta)t}\left(\mathrm{e}^{\frac{\nu}{d}\gamma_0}\right) \tag{4.148}$$

for all $t \geq 0$. One readily verifies that the function $\breve{\gamma} \in \mathcal{Y}$ defined by

$$\breve{\gamma}(t) = \frac{d}{\nu} \log(\breve{y}(t))$$

is a classical global solution to the viscous Hamilton-Jacobi equation (4.10) with initial condition $\breve{\gamma}(0) = \gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$. Hence, it satisfies the Duhamel formula

$$\breve{\gamma}(t) = \mathrm{e}^{(d\partial_\zeta^2 + a\partial_\zeta)t}\gamma_0 + \nu \int_0^t \mathrm{e}^{(d\partial_\zeta^2 + a\partial_\zeta)(t-s)}\left(\breve{\gamma}_\zeta(s)^2\right)\mathrm{d}s \tag{4.149}$$

for $t \geq 0$. Moreover, the pointwise bound (4.148) and Proposition 4.4.2 yield

$$\|\breve{\gamma}(t)\|_\infty \leq M, \qquad \|\breve{\gamma}_\zeta(t)\|_\infty \lesssim E_0, \qquad \|\breve{\gamma}_\zeta(t)\|_\infty \lesssim \frac{1}{\sqrt{1+t}} \tag{4.150}$$

for all $t \geq 0$. We observe that the bounds on $\gamma(t)$ and $\breve{\gamma}(t)$ in (4.12) and (4.150) readily imply the estimates (4.14) and (4.15) in case $\alpha = 0$. Therefore, we focus on the case $\alpha \in (0, \frac{1}{6})$. By Proposition 4.4.2 and the bound (4.150), we obtain

$$\left\|\partial_\zeta^m \int_0^t \mathrm{e}^{\left(d\partial_\zeta^2 + a\partial_\zeta\right)(t-s)}\left(\breve{\gamma}_\zeta(s)^2\right)\mathrm{d}s\right\|_\infty \lesssim \int_0^t \frac{E_0^{2\alpha}}{(t-s)^{\frac{m}{2}}(1+s)^{1-\alpha}}\mathrm{d}s \lesssim \frac{E_0^{2\alpha}}{(1+t)^{\frac{m}{2}-\alpha}} \tag{4.151}$$

for $m = 0,1$ and $t \geq 0$. Applying Proposition 4.4.1, employing estimates (4.82), (4.146), (4.147), and (4.151), using $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$, and inserting the Duhamel formulas (4.48) and (4.149), we arrive at the short-time bound

$$\|\gamma(t) - \breve{\gamma}(t)\|_\infty \lesssim E_0^{2\alpha}(1+t)^{2\alpha} \tag{4.152}$$

for all $t \geq 0$. On the other hand, using Proposition 4.4.2, the identities (4.57) and (4.149), and the estimates (4.82), (4.117), (4.118), (4.139), (4.147), and (4.151), we deduce

$$\left\|\partial_\zeta^m(\widetilde{y}(1) - \breve{\gamma}(1))\right\|_\infty \lesssim E_0^\alpha$$

for $m = 0,1$, where we use $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$. Hence, using the mean-value theorem and the bound (4.150), we obtain

$$\left\|\partial_\zeta^m(y(1) - \breve{y}(1))\right\|_\infty = \left\|\partial_\zeta^m\left(\mathrm{e}^{\frac{\nu}{d}\widetilde{y}(1)} - \mathrm{e}^{\frac{\nu}{d}\breve{\gamma}(1)}\right)\right\|_\infty \lesssim E_0^\alpha$$

for $m = 0,1$. Combining this with the identities (4.63) and $\breve{y}(t) = \mathrm{e}^{(d\partial_\zeta^2 + a\partial_\zeta)(t-1)}\breve{y}(1)$, Proposition 4.4.2, and the estimates (4.108) and (4.139), we infer

$$\left\|\partial_\zeta^m(y(t) - \breve{y}(t))\right\|_\infty \lesssim \frac{E_0^\alpha}{(1+t)^{\frac{m}{2}}} \tag{4.153}$$

for $m = 0,1$ and $t \geq 0$. Finally, applying the mean-value theorem, using the identities (4.60) and (4.56), and the bounds (4.102), (4.111), (4.139), (4.148), and (4.153), we obtain

$$\begin{aligned} \|\gamma(t) - \breve{\gamma}(t)\|_\infty &\lesssim \|r(t)\|_\infty + \|y(t) - \breve{y}(t)\|_\infty \lesssim E_0^\alpha + \frac{1}{(1+t)^{\frac{1}{2}-2\alpha}}, \\ \sqrt{t}\,\|\gamma_\zeta(t) - \breve{\gamma}_\zeta(t)\|_\infty &\lesssim \sqrt{t}\left(\|r_\zeta(t)\|_\infty + \|y_\zeta(t) - \breve{y}_\zeta(t)\|_\infty\right) \lesssim E_0^\alpha \end{aligned} \tag{4.154}$$

for all $t \geq 0$, where we use $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$. Thus, using (4.152) for $E_0^{\frac{4}{3}}(1+t)^2 \leq 1$ and the first bound in (4.154) for $(1+t)^{-2} \leq E_0^{\frac{4}{3}}$, we establish the first inequality in (4.14), where we use $0 < \alpha < \frac{1}{6}$. On the other hand, using (4.112) and (4.150) for short times $t \in [0,1]$ and the lower bound in (4.154) for large times $t \geq 1$, we obtain the second inequality in (4.14). Finally, estimate (4.15) follows from (4.14) with the aid of the mean-value theorem.

We now consider the case $\nu = 0$. Then, the function $\breve{\gamma} \in \mathcal{Y}$ defined by

$$\breve{\gamma}(t) = \mathrm{e}^{(d\partial_\zeta^2 + a\partial_\zeta)t}\gamma_0$$

is a classical global solution to (4.10) with initial condition $\breve{\gamma}(0) = \gamma_0 \in C^1_{\mathrm{ub}}(\mathbb{R})$. Again it follows from Proposition 4.4.2 and the bounds (4.12) that the estimates (4.14) and (4.15) are trivially satisfied in the case $\alpha = 0$, allowing us to focus on the case $\alpha \in (0, \frac{1}{6})$. On the one hand, Proposition 4.4.1, the estimates (4.82), (4.146), and (4.147), and the representation (4.48) yield the short-time bound

$$\|\gamma(t) - \breve{\gamma}(t)\|_\infty \lesssim E_0^{2\alpha}(1+t)^{2\alpha} \tag{4.155}$$

for all $t \geq 0$, where we used $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$. On the other hand, invoking the identities (4.56) and (4.57), applying Proposition 4.4.2, and using the bounds (4.82), (4.102), (4.117), (4.118), and (4.139), we obtain

$$\begin{aligned}\|\gamma(t) - \breve{\gamma}(t)\|_\infty &\lesssim \|r_\zeta(t)\|_\infty + \|\widetilde{y}(t) - \breve{\gamma}(t)\|_\infty \lesssim E_0^\alpha + \frac{1}{(1+t)^{\frac{1}{2}-2\alpha}},\\ \sqrt{t}\,\|\gamma_\zeta(t) - \breve{\gamma}_\zeta(t)\|_\infty &\lesssim \|r_\zeta(t)\|_\infty + \|\widetilde{y}_\zeta(t) - \breve{\gamma}_\zeta(t)\|_\infty \lesssim E_0^\alpha\end{aligned}$$

for $t \in [0,1]$, where we again use $0 \leq E_0 \leq 1$ and $0 < \alpha < \frac{1}{6}$. Similarly as in the case $\nu \neq 0$, we combine the latter with the short-time bounds (4.112), (4.155), and $\|\breve{\gamma}_\zeta(t)\|_\infty \lesssim E_0$ (cf. Proposition 4.4.2). This results in the estimate (4.14), which once more implies (4.15) via the mean-value theorem. □

4.7. Discussion and outlook

We discuss the wider applicability of our method and outline potential directions for future research.

4.7.1. Adaptations to other dissipative systems

We anticipate that the modulational stability framework developed in this paper extends beyond the class of reaction-diffusion systems and applies to diffusively spectrally stable wave trains in other dissipative problems, provided that the linearization about the wave train generates a C_0-semigroup on $C_{\mathrm{ub}}(\mathbb{R})$ whose high-frequency component is damped. A key observation is that the structure of the critical low-frequency component $S_p^0(t)$ of the semigroup is determined by the diffusive spectral stability assumptions (D1)-(D3), rather than the specific structure of the underlying equation. As a result, the linear estimates on modulational data derived in §4.4.1 are expected to carry over to a broader class of systems. This suggests that our method may even extend to certain dissipative *quasilinear* problems, provided that one obtains sufficient regularity control within the nonlinear iteration scheme. Such control may be established via nonlinear damping estimates, as in [88]. Alternatively, if the quasilinear equation is parabolic, one may use pointwise Green's function estimates [52, 104]; see [84, Section 6.6] for further discussion.

An interesting and more delicate challenge arises when additional conservation laws are present, as in the St. Venant equations for shallow water waves [58]. In such cases, the spectrum of the linearization about the wave train possesses an additional critical mode at the origin, thereby violating the spectral assumption (D3). This changes the nature of the leading-order modulational dynamics: instead of being governed by the scalar viscous Hamilton-Jacobi equation (4.10), it is described by a Whitham modulation system that captures interactions among multiple critical modes; see [59]. This precludes a straightforward application of the Cole-Hopf transform and poses a significant obstacle for extending the current analysis.

4.7.2. Extension to multiple spatial dimensions

We conjecture that our modulational stability results extend to roll waves in reaction-diffusion systems in higher spatial dimensions. These roll waves arise by trivially extending a one-dimensional wave train in the transverse directions. We believe that the core techniques employed in our analysis, such as the Cole-Hopf transform and the detailed decomposition of the linear semigroup, remain valid in this higher-dimensional setting. While we do not expect fundamental differences in the modulational behavior of roll waves compared to the one-dimensional case, we do not anticipate that our framework can be lightly adapted to study fully nonlocalized modulations of the planar periodic patterns studied in [75], or their higher-dimensional counterparts. The reason is that the leading-order dynamics of the modulations of such periodic patterns, which are nontrivial in any spatial direction, are governed by systems that differ qualitatively from the viscous Hamilton-Jacobi equation. These may include hyperbolic-parabolic systems, models with cross-diffusion, or anisotropic systems with dispersive effects; see [75] for more details. Developing an L^∞-theory for such periodic patterns therefore remains an open problem for future research.

4.7.3. Unbounded initial phase modulations and wavenumber offsets

We expect that it may be possible to allow initial data $\gamma_0 \in \mathrm{BMO}(\mathbb{R})$ with $\gamma_0' \in C_{\mathrm{ub}}(\mathbb{R})$, where $\mathrm{BMO}(\mathbb{R})$ denotes the space of functions of bounded mean oscillation. This expectation is motivated by the estimate $\|\partial_\zeta \mathrm{e}^{t\partial_\zeta^2} g\|_\infty \lesssim t^{-\frac{1}{2}}\|g\|_{\mathrm{BMO}}$, which enables interpolation on the critical linear terms in the Duhamel formulation. In particular, this suggests that the initial data γ_0 may be spatially unbounded. We note that, for the application of the Cole-Hopf transform in case $\nu \neq 0$, it may be necessary to additionally assume that $\mathrm{e}^{\frac{\nu}{d}\gamma_0} \in \mathrm{BMO}(\mathbb{R})$. This condition still permits γ_0 to be spatially unbounded. Furthermore, restricting to $\nu = 0$, we expect to allow for algebraic growth of γ_0 at rate $|x|^\beta$ as $x \to \pm\infty$ by assuming L^p-localization of the derivative γ_0'. Here, $\beta > 0$ is sufficiently small and $\frac{1}{1-\beta} < p < \infty$.

A further open question is whether wavenumber offsets can be permitted in settings where the phase modulation grows linearly at spatial infinity. For plane wave solutions in the real Ginzburg-Landau equation, this question has been answered affirmatively in [16, 37]. We refer to [84, Section 6.3] for further discussion.

4.A. Appendix: Technical lemmas for linear estimates on modulational data

This appendix is devoted to technical estimates underlying the L^∞-bounds on modulational data in §4.4.1. Our first result is a variant of the low-frequency estimate established in [84, Lemma A.1], addressing the case of a spatially periodic integral kernel. Following the treatment of modulational data in [56, 63], we make use of a Fourier series expansion of the periodic kernel.

Lemma 4.A.1. *Let $m_1 \in \mathbb{N}_0$, $m_2 \in \{0,1\}$, and $\xi_0 \in (0,\pi]$. Let $\lambda \in C((-\xi_0,\xi_0),\mathbb{C})$ and $F \in C(\mathbb{R}^3 \times [1,\infty),\mathbb{C})$. Suppose that there exist constants $C,\mu > 0$ such that*

i) $\lambda'(0) \in \mathrm{i}\mathbb{R}$;

ii) $\mathrm{Re}\,\lambda(\xi) \leq -\mu\xi^2$ *for all* $\xi \in (-\xi_0,\xi_0)$;

iii) $\overline{\mathrm{supp}(F(\cdot,\zeta,\bar\zeta,t))} \subset (-\xi_0,\xi_0)$ *for all* $\zeta,\bar\zeta \in \mathbb{R}$ *and* $t \geq 1$;

iv) *F is twice continuously differentiable in its first argument and 1-periodic in its third argument such that* $\|\partial_\xi^\ell F(\xi,\zeta,\cdot,t)\|_{L^2(0,1)} \leq Ct^{\frac{\ell}{2}}$ *for* $\xi \in (-\xi_0,\xi_0)$, $\zeta \in \mathbb{R}$, $t \geq 1$, *and* $\ell = 0,1,2$;

v) *if* $m_2 = 0$, *then* $\int_0^1 F(\xi,\zeta,\bar\zeta,t)\mathrm{d}\bar\zeta = 0$ *for all* $\xi \in (-\xi_0,\xi_0)$, $\zeta \in \mathbb{R}$, *and* $t \geq 1$.

Let $a \in \mathbb{R}$ be such that $\lambda'(0) = a\mathrm{i}$. Then, the estimate

$$\left\|\int_\mathbb{R}\int_\mathbb{R} \mathrm{e}^{t\lambda(\xi)}\xi^{m_1+m_2}F(\xi,\cdot,\bar\zeta,t)\mathrm{e}^{\mathrm{i}\xi(\cdot-\bar\zeta)}v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta\right\|_\infty \lesssim t^{-\frac{m_1}{2}}\|v'\|_\infty$$

holds for each $t \geq 1$ and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$.

Proof. Expanding the 1-periodic function $F(\xi,\zeta,\cdot,t)$ as a Fourier series and subsequently integrating by parts, we obtain

$$\begin{aligned}
&\int_\mathbb{R}\int_\mathbb{R} \mathrm{e}^{t\lambda(\xi)}\xi^{m_1+m_2}F(\xi,\zeta,\bar\zeta,t)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta \\
&\qquad = \sum_{j\in\mathbb{Z}}\int_\mathbb{R}\int_\mathbb{R} \mathrm{e}^{t\lambda(\xi)}\xi^{m_1+m_2}\left\langle F(\xi,\zeta,\cdot,t),\mathrm{e}^{2\pi\mathrm{i}j\cdot}\right\rangle_{L^2(0,1)}\mathrm{e}^{\mathrm{i}\xi\zeta+(2\pi j-\xi)\mathrm{i}\bar\zeta}v(\bar\zeta)\mathrm{d}\xi\mathrm{d}\bar\zeta \\
&\qquad = \mathrm{i}\int_\mathbb{R}\int_\mathbb{R} \mathrm{e}^{t\lambda(\xi)}\Big(\xi^{m_1+m_2}h_0(\xi,\zeta,\bar\zeta,t) - \xi^{m_1}h_1(\xi,\zeta,\bar\zeta,t)\Big)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)}\mathrm{d}\xi\, v'(\bar\zeta)\mathrm{d}\bar\zeta
\end{aligned} \tag{4.156}$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$, where we denote

$$h_0(\xi,\zeta,\bar\zeta,t) := \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{\left\langle F(\xi,\zeta,\cdot,t),\mathrm{e}^{2\pi\mathrm{i}j\cdot}\right\rangle_{L^2(0,1)}}{2\pi j-\xi}\mathrm{e}^{2\pi\mathrm{i}j\bar\zeta}$$

and where

$$h_1(\xi,\zeta,\bar\zeta,t) := \left\langle F(\xi,\zeta,\cdot,t),1\right\rangle_{L^2(0,1)} = \int_0^1 F(\xi,\zeta,y,t)\mathrm{d}y$$

vanishes identically in case $m_2 = 0$, by hypothesis v). By Hypothesis iv), we have

$$\left\|\partial_\xi^\ell h_1(\xi,\zeta,\cdot,t)\right\|_\infty \leq \left\|\partial_\xi^\ell F(\xi,\zeta,\cdot,t)\right\|_{L^2(0,1)} \lesssim t^{\frac{\ell}{2}} \tag{4.157}$$

for $\xi \in (-\xi_0, \xi_0)$, $\zeta \in \mathbb{R}$, $t \geq 1$, and $\ell = 0, 1, 2$. On the other hand, hypothesis iv) in combination with an application of Hölder's and Bessel's inequality yields

$$\begin{aligned}\|h_0(\xi,\zeta,\cdot,t)\|_\infty &\leq \sum_{j\in\mathbb{Z}\setminus\{0\}} \left|\frac{\langle F(\xi,\zeta,\cdot,t), \mathrm{e}^{2\pi \mathrm{i} j\cdot}\rangle_{L^2(0,1)}}{2\pi j - \xi}\right| \\ &\leq \|F(\xi,\zeta,\cdot,t)\|_{L^2(0,1)} \left(\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi j-\xi)^2}\right)^{\frac{1}{2}} \lesssim 1\end{aligned} \tag{4.158}$$

for $\xi \in (-\xi_0, \xi_0)$, $\zeta \in \mathbb{R}$, and $t \geq 1$. Similarly, computing

$$\partial_\xi^\ell \left(\frac{1}{2\pi j - \xi}\right) = \frac{\ell!}{(2\pi j-\xi)^{\ell+1}}$$

for $j \in \mathbb{Z} \setminus \{0\}$ and using hypothesis iv), we bound

$$\left\|\partial_\xi^\ell h_0(\xi,\zeta,\cdot,t)\right\|_\infty \lesssim \sum_{m=0}^{\ell} \left\|\partial_\xi^{\ell-m} F(\xi,\zeta,\cdot,t)\right\|_{L^2(0,1)} \left(\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi j-\xi)^{2(m+1)}}\right)^{\frac{1}{2}} \lesssim C t^{\frac{\ell}{2}} \tag{4.159}$$

for $\xi \in (-\xi_0, \xi_0)$, $\zeta \in \mathbb{R}$, $t \geq 1$, and $\ell = 1, 2$.

Let $m \in \mathbb{N}_0$ and $\ell \in \{0, 1\}$. Proceeding as in [84, Lemma A.1], we use integration by parts to rewrite the integrand

$$I(\zeta,\bar\zeta,t) := \int_{\mathbb{R}} \mathrm{e}^{t\lambda(\xi)} \xi^m h_\ell(\xi,\zeta,\bar\zeta,t) \mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta)} \mathrm{d}\xi$$

in (4.156) as

$$I(\zeta,\bar\zeta,t) = \left(1 + \frac{(\zeta-\bar\zeta+at)^2}{t}\right)^{-1} \Big(I(\zeta,\bar\zeta,t) + I_2(\zeta,\bar\zeta,t)\Big) \tag{4.160}$$

for $t \geq 1$ and $\zeta, \bar\zeta \in \mathbb{R}$, where we denote

$$I_2(\zeta,\bar\zeta,t) = \frac{1}{t}\int_{\mathbb{R}} \partial_\xi^2 \Big(\mathrm{e}^{t(\lambda(\xi)-\lambda'(0)\xi)} \xi^m h_\ell(\xi,\zeta,\bar\zeta,t)\Big) \mathrm{e}^{\mathrm{i}\xi(\zeta-\bar\zeta+at)} \mathrm{d}\xi.$$

To bound the first integral on the right-hand side of (4.160), we use hypotheses ii) and iii) and estimates (4.157) and (4.158), yielding

$$\left|I(\zeta,\bar\zeta,t)\right| \lesssim \int_{-\xi_0}^{\xi_0} \left|\xi^m \mathrm{e}^{t\lambda(\xi)}\right| \mathrm{d}\xi \lesssim \int_{\mathbb{R}} |\xi|^m \mathrm{e}^{-\mu\xi^2 t} \mathrm{d}\xi \lesssim t^{-\frac{m+1}{2}} \tag{4.161}$$

for $t \geq 1$ and $\zeta, \bar\zeta \in \mathbb{R}$. To bound the second integral on the right-hand side of (4.160), we first observe that, by hypotheses i) and ii), estimates (4.157), (4.158), and (4.159), and the fact that $\lambda \in C^2((-\xi_0, \xi_0), \mathbb{C})$, it holds

$$\begin{aligned}&\left|\partial_\xi^2 \Big(\mathrm{e}^{t(\lambda(\xi)-\lambda'(0)\xi)} \xi^m h_\ell(\xi,\zeta,\bar\zeta,t)\Big)\right| \\ &\quad\lesssim \Big(|\xi|^m t\Big(1 + |\xi|\sqrt{t} + \xi^2 t\Big) + m|\xi|^{m-1}\sqrt{t}\Big(1 + |\xi|\sqrt{t}\Big) + m(m-1)|\xi|^{m-2}\Big) \mathrm{e}^{-\mu\xi^2 t}\end{aligned}$$

for $\xi \in (-\xi_0, \xi_0)$, $t \geq 1$ and $\zeta, \bar{\zeta} \in \mathbb{R}$. Therefore, using hypotheses ii) and iii), we establish

$$\begin{aligned}\left|I_2(\zeta,\bar{\zeta},t)\right| &\lesssim \int_{\mathbb{R}} \mathrm{e}^{-\mu\xi^2 t}\Big(|\xi|^m\Big(1+|\xi|\sqrt{t}+\xi^2 t\Big) \\ &\qquad + \frac{m}{\sqrt{t}}|\xi|^{m-1}\Big(1+|\xi|\sqrt{t}\Big) + \frac{m(m-1)}{t}|\xi|^{m-2}\Big)\mathrm{d}\xi \lesssim t^{-\frac{m+1}{2}}\end{aligned} \tag{4.162}$$

for $\zeta, \bar{\zeta} \in \mathbb{R}$ and $t \geq 1$. Applying the estimates (4.161) and (4.162) to (4.160), we arrive at

$$\left|I(\zeta,\bar{\zeta},t)\right| \leq t^{-\frac{m+1}{2}}\left(1+\frac{(\zeta-\bar{\zeta}+at)^2}{t}\right)^{-1}$$

for $\zeta, \bar{\zeta} \in \mathbb{R}$ and $t \geq 1$. Finally, we use this pointwise bound to estimate the right-hand side of (4.156), yielding

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{t\lambda(\xi)}\xi^{m_1+m_2}F(\xi,\zeta,\bar{\zeta},t)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar{\zeta})}v(\bar{\zeta})\mathrm{d}\xi\mathrm{d}\bar{\zeta}\right| &\lesssim \int_{\mathbb{R}} t^{-\frac{m_1+1}{2}}\left(1+\frac{(\zeta-\bar{\zeta}+at)^2}{t}\right)^{-1}\|v'\|_\infty \mathrm{d}\bar{\zeta} \\ &\lesssim t^{-\frac{m_1}{2}}\|v'\|_\infty\end{aligned}$$

for $\zeta \in \mathbb{R}$, $t \geq 1$, and $v \in C^1_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$, which concludes the proof. □

The second technical estimate is a slight modification of the high-frequency bound established in [51, Lemma A.2].

Lemma 4.A.2. *Let $\xi_0, d > 0$ and $a \in \mathbb{R}$. Let $F \in C^2_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$ be supported on $\mathbb{R} \setminus [-\xi_0, \xi_0]$. Then, there exists a constant $\mu_0 > 0$ such that we have*

$$\left\|\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi(\cdot-\bar{\zeta})+\left(a\mathrm{i}\xi-d\xi^2\right)t}F(\xi)\mathrm{d}\xi\, v(\bar{\zeta})\mathrm{d}\bar{\zeta}\right\|_\infty \lesssim \left(1+\frac{1}{\sqrt{t}}\right)\mathrm{e}^{-\mu_0 t}\|v\|_\infty$$

for $v \in C_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$ and $t > 0$.

Proof. We follow the strategy of the proof of [51, Lemma A.2] and use integration by parts to rewrite the integrand

$$I(\zeta,\bar{\zeta},t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi(\cdot-\bar{\zeta})+\left(a\mathrm{i}\xi-d\xi^2\right)t}F(\xi)\mathrm{d}\xi$$

as

$$I(\zeta,\bar{\zeta},t) = \frac{1}{1+\left(\zeta-\bar{\zeta}+at\right)^2}\left(I(\zeta,\bar{\zeta},t) + \int_{\mathbb{R}} \partial_\xi^2\left(e^{-d\xi^2 t}F(\xi)\right)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar{\zeta}+at)}\mathrm{d}\xi\right) \tag{4.163}$$

for $\zeta, \bar{\zeta} \in \mathbb{R}$ and $t > 0$. We compute

$$\partial_\xi^2\left(\mathrm{e}^{-d\xi^2 t}F(\xi)\right) = \mathrm{e}^{-dt\xi^2}\left(F''(\xi) - 4dt\xi F'(\xi) + 2dtF(\xi)\left(2dt\xi^2-1\right)\right)$$

for $\xi \in \mathbb{R}$ and $t > 0$. We estimate

$$\begin{aligned}\left|I(\zeta,\bar{\zeta},t)\right| &\lesssim \int_{\mathbb{R}\setminus[-\xi_0,\xi_0]} \mathrm{e}^{-\frac{d}{2}\xi^2 t-\frac{d}{2}\xi_0^2 t}\|F\|_\infty \lesssim \frac{1}{\sqrt{t}}\mathrm{e}^{-\frac{d}{2}\xi_0^2 t}\|F\|_\infty, \\ \left|\int_{\mathbb{R}} \partial_\xi^2\left(e^{-d\xi^2 t}F(\xi)\right)\mathrm{e}^{\mathrm{i}\xi(\zeta-\bar{\zeta}+at)}\mathrm{d}\xi\right| &\lesssim \int_{\mathbb{R}\setminus[-\xi_0,\xi_0]} \mathrm{e}^{-\frac{d}{2}\xi^2 t-\frac{d}{2}\xi_0^2 t}\left(1+t+|\xi|t+\xi^2 t^2\right)\|F\|_{C^2_{\mathrm{ub}}}\mathrm{d}\xi \\ &\lesssim \left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\mathrm{e}^{-\frac{d}{2}\xi_0^2 t}\|F\|_{C^2_{\mathrm{ub}}}\end{aligned}$$

for $\zeta, \bar\zeta \in \mathbb{R}$ and $t > 0$. Applying these estimates to the identity (4.163), we arrive at

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} I(\zeta,\bar\zeta,t)\mathrm{d}\xi\, v(\bar\zeta)\mathrm{d}\bar\zeta\right| \lesssim \int_{\mathbb{R}}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\frac{\mathrm{e}^{-\frac{d}{2}\xi_0^2 t}\|v\|_\infty}{1+\left(\zeta-\bar\zeta+at\right)^2}\mathrm{d}\bar\zeta \lesssim \left(1+\frac{1}{\sqrt{t}}\right)\mathrm{e}^{-\frac{d}{4}\xi_0^2 t}\|v\|_\infty$$

for $\zeta \in \mathbb{R}$, $v \in C_{\mathrm{ub}}(\mathbb{R},\mathbb{R})$, and $t > 0$, which yields the desired bound with $\mu_0 = \frac{d}{4}\xi_0^2$. □

4.B. Appendix: Local existence of the phase modulation

We establish existence of a maximally defined solution $\gamma(t)$ to the integral equation (4.48) by employing a contraction-mapping argument.

Proof of Proposition 4.5.4. It follows from standard analytic semigroup theory [73] that the orbit map $\Gamma \in C\big([0,\infty), C^1_{\mathrm{ub}}(\mathbb{R})\big)$ given by $\Gamma(t) = \mathrm{e}^{-\partial_\zeta^4 t}\gamma_0$ enjoys the regularity property $\Gamma \in C^j\big((0,\infty), C^l_{\mathrm{ub}}(\mathbb{R})\big)$ for all $j,l \in \mathbb{N}_0$ and it obeys the estimates (4.50) for all $t > 0$. In particular, it holds $\|\partial_\zeta\Gamma(t)\|_\infty = \|\mathrm{e}^{-\partial_\zeta^4 t}\gamma_0'\|_\infty \le \|\gamma_0'\|_\infty < r_0$ for $t \ge 0$. Since the propagator $S_p^0(t)$ vanishes on $[0,1]$, the function $\Gamma(t)$ solves (4.48) for $t \in [0,1]$. Hence, we have $\tau_{\max} \ge 1$.

Next, assume $\tau_{\max} \le T_{\max}$. As $\tau_{\max} \ge 1$, we must have $T_{\max} \ge 1$. Take $t_0 > \frac{1}{2}$ and $\delta,\eta > 0$ such that $t_0 + \delta < T_{\max}$ and $\eta < r_0$. Let $\check\gamma \in C\big([0,t_0], C^1_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0,t_0], C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0,t_0], C_{\mathrm{ub}}(\mathbb{R})\big)$ be a solution to (4.48) with $\|\check\gamma_\zeta(t)\|_\infty \le r_0 - \eta$ for all $t \in [0,t_0]$. Let $R \ge 1$ be such that $\|(\check\gamma(t), \partial_t\check\gamma(t))\|_{C^2_{\mathrm{ub}}\times C_{\mathrm{ub}}} \le R$ for all $t \in [\frac{1}{2}, t_0]$. We argue that $\check\gamma$ can be extended to a solution $\gamma_{\mathrm{ext}} \in C\big([0,t_0+\delta], C^1_{\mathrm{ub}}(\mathbb{R})\big) \cap C\big((0,t_0+\delta], C^2_{\mathrm{ub}}(\mathbb{R})\big) \cap C^1\big((0,t_0+\delta], C_{\mathrm{ub}}(\mathbb{R})\big)$ to (4.48) which satisfies $\|\partial_\zeta\gamma_{\mathrm{ext}}(t)\|_\infty \le r_0 - \frac{1}{2}\eta$ for all $t \in [0, t_0+\delta]$ and $\|(\gamma_{\mathrm{ext}}(t), \partial_t\gamma_{\mathrm{ext}}(t))\|_{C^2_{\mathrm{ub}}\times C_{\mathrm{ub}}} \le 2R$ for all $t \in [\frac{1}{2}, t_0+\delta]$. To this end, we close a contraction mapping argument in the metric space

$$\mathcal{M} = \Big\{(\gamma,\gamma_t) \in C\big([t_0,t_0+\delta], C^2_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R})\big) : \|\gamma_\zeta(t)\|_\infty \le r_0 - \tfrac{1}{2}\eta \text{ and } \|(\gamma(t),\gamma_t(t))\|_{C^2_{\mathrm{ub}}\times C_{\mathrm{ub}}} \le 2R \text{ for } t \in [t_0,t_0+\delta]\Big\},$$

endowed with the metric from the Banach space $C\big([t_0,t_0+\delta], C^2_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R})\big)$.

Applying the mean-value theorem and Proposition 4.5.1, it follows that $V\colon C_{\mathrm{ub}}(\mathbb{R}) \times [t_0, t_0 + \delta] \to C_{\mathrm{ub}}(\mathbb{R})$ given by

$$V(\gamma,t)[\zeta] = u(\zeta - \gamma(\zeta), t) - \phi_0(\zeta)$$

is continuous in t and fulfills

$$\|V(\gamma,t) - V(\tilde\gamma,t)\|_\infty \le \|u_\zeta(t)\|_\infty\|\gamma - \tilde\gamma\|_\infty$$

for $\gamma,\tilde\gamma \in C_{\mathrm{ub}}(\mathbb{R})$ and $t \in [t_0,t_0+\delta]$. Therefore, setting $\mathcal{K} = \{(\gamma,\gamma_t) \in C^2_{\mathrm{ub}}(\mathbb{R}) \times C_{\mathrm{ub}}(\mathbb{R}) : \|\gamma'\|_\infty \le r_0 - \frac{1}{2}\eta, \|(\gamma,\gamma_t)\|_{C^2_{\mathrm{ub}}\times C_{\mathrm{ub}}} \le 2R\}$ and recalling (4.45), the nonlinear maps $\mathcal{K} \times [t_0,t_0+\delta] \to C_{\mathrm{ub}}(\mathbb{R})$, $(\gamma,\gamma_t,t) \mapsto \mathcal{Q}(V(\gamma,t),\gamma), \mathcal{R}(V(\gamma,t),\gamma,\gamma_t), \mathcal{S}(V(\gamma,t),\gamma)$ are bounded, continuous in t, and Lipschitz continuous in (γ,γ_t).

Since the propagator $S_p^0(t)$ vanishes on $[0,1]$, it must hold $\check\gamma(t) = \Gamma(t)$ for all $t \in [0,1]$. We show that the action of the nonlinearities on $\Gamma(t)$ is well-defined. Thus, applying the estimates (4.50), using that $t \mapsto \|u(t)\|_\infty$ is bounded on $[0,1]$, and noting that $\mathcal{R}(v,\gamma,\gamma_t)$ is linear in γ_t and $\gamma_{\zeta\zeta}$, we establish

$$\|\mathcal{Q}(V(\Gamma(t),t),\Gamma(t))\|_\infty, \|\mathcal{S}(V(\Gamma(t),t),\Gamma(t))\|_\infty \lesssim 1, \qquad \|\mathcal{R}(V(\Gamma(t),t),\Gamma(t),\partial_t\Gamma(t))\|_\infty \leq t^{-\frac{3}{4}}$$

for all $t \in (0,1]$. Combining the latter with (4.44) and Proposition 4.4.1, we arrive at the estimate

$$\int_0^{\frac{1}{2}} S_p^0(t-s)\mathcal{N}(V(\Gamma(s),s),\Gamma(s),\partial_s\Gamma(s))\mathrm{d}s \lesssim \int_0^{\frac{1}{2}} s^{-\frac{3}{4}}\mathrm{d}s \lesssim 1$$

for all $t \geq 0$.

Finally, we observe that, by Proposition 4.4.1, the propagators $\partial_t^\ell S_p^0(t)\partial_\zeta^i \colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}^l(\mathbb{R})$ are t-uniformly bounded and strongly continuous on $[0,\infty)$ for any $i,l,\ell \in \mathbb{N}_0$. A standard contraction mapping argument, cf. [79, Theorem 6.1.2], in the complete metric space $\mathcal{M}$ yields a unique solution $(\gamma,\gamma_t) \in \mathcal{M}$ to the integral system

$$\begin{aligned}\gamma(t) &= S_p^0(t)\big(v_0 + \phi_0'\gamma_0 + \gamma_0'v_0\big) + (1-\chi(t))\Gamma(t)\\ &\quad + \int_0^{\frac{1}{2}} S_p^0(t-s)\mathcal{N}(V(\Gamma(s),s),\Gamma(s),\partial_s\Gamma(s))\mathrm{d}s\\ &\quad + \int_{\frac{1}{2}}^{t_0} S_p^0(t-s)\mathcal{N}(V(\check{\gamma}(s),s),\check{\gamma}(s),\partial_s\check{\gamma}(s))\mathrm{d}s\\ &\quad + \int_{t_0}^{t_0+\delta} S_p^0(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s))\mathrm{d}s,\\ \gamma_t(t) &= \partial_t S_p^0(t)\big(v_0 + \phi_0'\gamma_0 + \gamma_0'v_0\big) - \chi'(t)\Gamma(t) + (1-\chi(t))\partial_t\Gamma(t)\\ &\quad + \int_0^{\frac{1}{2}} \partial_t S_p^0(t-s)\mathcal{N}(V(\Gamma(s),s),\Gamma(s),\partial_s\Gamma(s))\mathrm{d}s\\ &\quad + \int_{\frac{1}{2}}^{t_0} \partial_t S_p^0(t-s)\mathcal{N}(V(\check{\gamma}(s),s),\check{\gamma}(s),\partial_s\check{\gamma}(s))\mathrm{d}s\\ &\quad + \int_{t_0}^{t_0+\delta} \partial_t S_p^0(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s))\mathrm{d}s,\end{aligned}$$

provided $\delta > 0$ is sufficiently small.

By construction, we have $\gamma \in C^1\big([t_0,t_0+\delta],C_{\mathrm{ub}}(\mathbb{R})\big)$ with $\partial_t\gamma(t) = \gamma_t(t)$ for all $t \in [t_0,t_0+\delta]$. Consequently,

$$\gamma_{\mathrm{ext}}(t) = \begin{cases}\check{\gamma}(t), & t \in [0,t_0),\\ \gamma(t), & t \in [t_0,t_0+\delta],\end{cases}$$

defines a solution to (4.48), which extends $\check{\gamma}$. It satisfies the estimate $\|\partial_\zeta\gamma_{\mathrm{ext}}(t)\|_\infty < r_0$ for all $t \in [0,t_0+\delta]$.

As shown in [22, Theorem 4.3.4] and [79, Theorem 6.1.4], this extension procedure yields the existence of the desired maximal solution $\gamma(t)$. Its regularity properties then follow by the fact that the propagators $\partial_t^\ell S_p^0(t)\partial_\zeta^i \colon C_{\mathrm{ub}}(\mathbb{R}) \to C_{\mathrm{ub}}^l(\mathbb{R})$ are t-uniformly bounded and it holds $\Gamma \in C^i\big((0,\infty),C_{\mathrm{ub}}^l(\mathbb{R})\big)$ for all $i,l,\ell \in \mathbb{N}_0$. □

End of Paper

CHAPTER 5

Nonlinear dynamics of periodic Lugiato-Lefever waves against sums of co-periodic and localized perturbations

With slight changes, this chapter is the content of the preprint [2]. We add an appendix where we introduce the space $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$.

Start of Paper

Abstract. In recent years, essential progress has been made in the nonlinear stability analysis of periodic Lugiato-Lefever waves against co-periodic and localized perturbations. Inspired by considerations from fiber optics, we introduce a novel iteration scheme which allows to perturb against sums of co-periodic and localized functions. This unifies previous stability theories in a natural manner.

Keywords. Nonlinear stability; Periodic waves; Lugiato-Lefever equation; Nonlinear Schrödinger equations; Nonlocalized perturbations; Tooth problem
Mathematics Subject Classification (2020). 35B10; 35B35; 35Q55

5.1. Introduction

We study the Lugiato-Lefever equation on the extended real line

$$\partial_t u = -\beta i \partial_x^2 u - (1 + i\alpha)u + i|u|^2 u + F, \quad \beta \in \{-1, 1\}, \quad \alpha \in \mathbb{R}, \quad F > 0, \tag{5.1}$$

for $u : \mathbb{R} \times [0, \infty) \to \mathbb{C}$, which is a model from nonlinear optics [25]. An important observation is that the principal part $-\beta i \partial_x^2 u$ is dispersive while the damping term $-u$ causes energy dissipation. The forcing term F again adds energy to the physical system and allows for pattern formation as predicted by Lugiato and Lefever in [72]. The most fundamental patterns such as pulses, small-amplitude or dnoidal periodic waves are found in [10, 12, 40, 101] and in [28] and [44], respectively. Recently, in [13], the authors have obtained large-amplitude periodic waves.

In this paper, we prove nonlinear L^∞-stability of T-periodic standing waves against initial perturbations from the space $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$[1] under diffusive spectral stability assumptions. These spectral assumptions are only established in [13] and [28].

We emphasize that sums of periodic and localized perturbations are not necessarily localized or periodic and thus our result is a nontrivial unification of the theories [96] (co-periodic) and [46] (localized). For the precise formulation of our main result, we refer to §5.2.5.

[1] The term $\oplus$ denotes the direct sum, that is for every $u \in L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$, we find precisely one $w \in L^2_{\mathrm{per}}(0,T)$ and $v \in L^2(\mathbb{R})$ such that $u = w + v$.

In view of fully nonlocalized perturbations, the recently developed nonlinear stability theory for dissipative semilinear systems [5] is not immediately applicable to all pattern-forming semilinear systems such as the Lugiato-Lefever equation. From this perspective, the present paper is the first to accommodate nonlocalized perturbations and to combine $H^l_{\mathrm{per}}(0,T)$- and H^k-theory. On the other hand, considerations from fiber optics, see [67] and Remark 5.2.1, where combinations of localized and co-periodic effects naturally occur, motivate the investigation of these types of perturbations. Interpretating the Lugiato-Lefever equation as a variant of the cubic nonlinear Schrödinger equation, a related inspiration for this paper comes from the so-called *tooth problem* asking whether solutions of the nonlinear Schrödinger equation with not necessarily small initial data from $L^2_{\mathrm{per}}(\mathbb{R}) \oplus L^2(\mathbb{R})$ exist globally; we refer to [69] and [24] for answers to the tooth problem and background information.

Our central challenge is to develop the right modulational ansatz to make a nonlinear iteration argument close. Inspired by [47], we introduce both a temporal and a localized spatio-temporal phase modulation to capture the critical dynamics of the perturbation induced by translational invariance of (5.1). Moreover, we employ novel L^2-L^∞-estimates in the nonlinear iteration argument to control the interaction between the periodic and localized components of the perturbation. For more details on the strategy of the proof, we refer to §5.2.6. The outlook section §5.6 is devoted to the robustness of our approach as well as its possible extension to periodic wave trains in viscous conservation laws where the handling of fully nonlocalized perturbations is similarly challenging as for the Lugiato-Lefever equation but for different reasons. In case of the Lugiato-Lefever equation, a crucial difficulty towards extending to a fully nonlocalized stability result is to choose a suitable class of perturbations which contains all C^∞-functions. The generic space of perturbations is given by

$$C^m_{\mathrm{ub}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is } m\text{-times differentiable with uniformly continuous derivatives}\},$$

$m \in \mathbb{N}$, whenever studying reaction-diffusion systems [5, 51, 84]. The solutions of (5.1) with initial data in $H^1_{\mathrm{per}}(\mathbb{R}) \oplus H^1(\mathbb{R})$ naturally lie in $C_{\mathrm{ub}}(\mathbb{R})$ due to Sobolev embedding. However, it is shown in [14] that $t \mapsto ||e^{i\partial_x^2 t} u_0||_{L^\infty}$ blows up in finite times for certain $u_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and therefore it is convenient to study other spaces than $C_{\mathrm{ub}}(\mathbb{R})$ to approach a fully nonlocalized stability result for the Lugiato-Lefever equation (5.1). More suitable variants are given by the so-called modulation spaces $M^m_{\infty,1}(\mathbb{R})$, $m \in \mathbb{N}_0$, which are introduced in [42]. We discuss such an extension in §5.6.4.

5.2. Preparation and main result

We reformulate the Lugiato-Lefever equation as a semilinear system with a $\mathbb{C}$-linear part by splitting into real- and imaginary variables. Then, we construct perturbed solutions with initial data in $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$ and derive the associated perturbation equations. At the end of this section, we impose spectral properties and formulate our main result.

5.2.1. Reformulation as real system

As $|u|^2 u$ is not differentiable with respect to $u \in \mathbb{C}$, we introduce $\mathbf{u} := (\mathbf{u}_r, \mathbf{u}_i)^T := (\mathrm{Re}(u), \mathrm{Im}(u))^T : \mathbb{R} \to \mathbb{R}^2$ which transforms (5.1) into the real system

$$\mathbf{u}_t = \mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix} \mathbf{u}_{xx} + \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \mathbf{u}\right) - \mathbf{u} + \mathcal{N}(\mathbf{u}) + \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{5.2}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{N}(\mathbf{u}) = |\mathbf{u}|^2 J\mathbf{u} = \begin{pmatrix} -\mathbf{u}_i^3 - \mathbf{u}_r^2\mathbf{u}_i \\ \mathbf{u}_r\mathbf{u}_i^2 + \mathbf{u}_r^3 \end{pmatrix}.$$

5.2.2. The perturbed solution in $L^2_{\mathrm{PER}}(0,T) \oplus L^2(\mathbb{R})$

We assume the existence of a periodic standing wave.

(H1) There exists a smooth, nonconstant and T-periodic stationary solution $\phi_0 : \mathbb{R} \to \mathbb{C}$ of (5.1).

We set $\phi := (\phi_r, \phi_i)^T := (\mathrm{Re}(\phi_0), \mathrm{Im}(\phi_0))^T : \mathbb{R} \to \mathbb{R}^2$ and construct a solution of (5.2) with initial datum $\phi + \mathbf{w}_0 + \mathbf{v}_0$ in $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$ with $\mathbf{w}_0 \in L^2_{\mathrm{per}}(0,T)$ and $\mathbf{v}_0 \in L^2(\mathbb{R})$ by following the strategy of [69]. We first solve

$$\begin{aligned} \mathbf{w}_t &= \mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}\mathbf{w}_{xx} + \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\mathbf{w}\right) - \mathbf{w} + \mathcal{N}(\mathbf{w}) + \begin{pmatrix} F \\ 0 \end{pmatrix} \\ \mathbf{w}(0) &= \phi + \mathbf{w}_0 \end{aligned} \tag{5.3}$$

in $L^2_{\mathrm{per}}(0,T)$. Then, we transfer from the solution $\mathbf{w}$ of (5.2) to a solution $\mathbf{u}$ of (5.2) with initial datum $\phi + \mathbf{w}_0 + \mathbf{v}_0$ by solving the perturbed problem

$$\begin{aligned} \mathbf{v}_t &= \mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}\mathbf{v}_{xx} - \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\mathbf{v}\right) - \mathbf{v} + \mathcal{N}(\mathbf{v} + \mathbf{w}) - \mathcal{N}(\mathbf{w}) \\ \mathbf{v}(0) &= \mathbf{v}_0. \end{aligned} \tag{5.4}$$

In summary, if we solve (5.3) and (5.4), then $\mathbf{u} = \mathbf{w} + \mathbf{v}$ is a solution of (5.2) with $\mathbf{u}(0) = \phi + \mathbf{w}_0 + \mathbf{v}_0$.

Remark 5.2.1 (Interpretation from fiber optics). *The cubic nonlinear Schrödinger equation on $L^2_{per}(0,T) \oplus L^2(\mathbb{R})$ can be understood as model from nonlinear fiber optics by considering $t \geq 0$ as point on a fiber and $x \in \mathbb{R}$ as time variable, cf. [67]. The stationary periodic solution ϕ is then the signal at any point on the fiber if it is chosen as input signal. Prescribing a signal at the initial point of the fiber by $\phi + L^2_{per}(0,T)$, we ask how the signal as function depending on the time $x \in \mathbb{R}$ looks at the place $t \geq 0$ on the fiber. Adding an $L^2(\mathbb{R})$-perturbation corresponds to temporally limited changes of the input signal. In particular, one may switch off the periodic signal for finitely many times as illustrated in Figure 5.1. Global existence of the perturbed solutions then translates to the observation that the fiber has infinite length while stability of ϕ is interpreted as that the signal stays close to ϕ at any point $t \geq 0$ on the fiber.*

5.2.3. Unmodulated perturbation equations

Given a solution $\mathbf{u}(t) = \mathbf{w}(t) + \mathbf{v}(t)$ of (5.2), we derive the unmodulated perturbation equations by splitting the perturbation as

$$\tilde{\mathbf{u}}(t) = \mathbf{u}(t) - \phi = (\mathbf{w}(t) - \phi) + \mathbf{v}(t) \text{ and setting } \tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi.$$

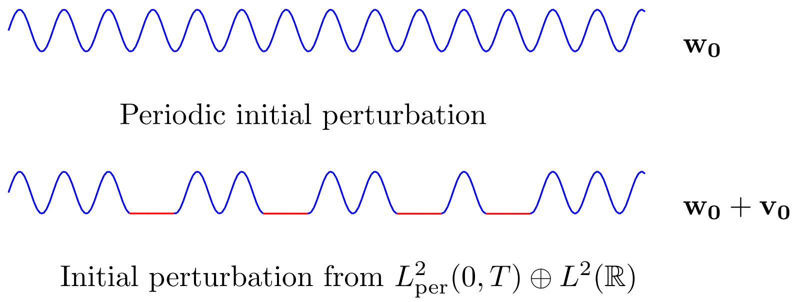

Figure 5.1.: For the sake of illustration, we reduce to the real part of an initial perturbation $\mathbf{w}_0 + \mathbf{v}_0$ with $\mathbf{w}_0 \in L^2_{\rm per}(0,T)$ and $\mathbf{v}_0 \in L^2(\mathbb{R})$. This figure demonstrates that $\mathbf{v}_0$ can in particular be chosen such that $\mathbf{v}_0 + \mathbf{w}_0$ coincides with $\mathbf{w}_0$ except for finitely many periods for which the signal vanishes, which explains the name tooth space for $L^2_{\rm per}(0,T) \oplus L^2(\mathbb{R})$ ("knocked out teeth").

This gives the coupled perturbation system

$$\begin{aligned} \tilde{\mathbf{w}}_t &= \mathcal{L}_0(\phi)\tilde{\mathbf{w}} + \mathcal{R}_1(\phi)(\tilde{\mathbf{w}}) & \mathbf{v}_t &= \mathcal{L}_0(\phi)\mathbf{v} + \mathcal{R}_2(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) \\ \tilde{\mathbf{w}}(0) &= \mathbf{w}_0, & \mathbf{v}(0) &= \mathbf{v}_0, \end{aligned} \tag{5.5}$$

where $\mathcal{L}_0(\phi)$ is the linearization of (5.2) about ϕ, given by

$$\mathcal{L}_0(\phi) = \mathcal{J}\begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta\partial_x^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix} - \mathcal{I}, \tag{5.6}$$

first residual nonlinearity is given by

$$\mathcal{R}_1(\phi)(\tilde{\mathbf{w}}) = \mathcal{N}(\tilde{\mathbf{w}} + \phi) - \mathcal{N}(\phi) - \mathcal{N}'(\phi)\tilde{\mathbf{w}}, \tag{5.7}$$

and the second residual nonlinearity is defined by

$$\mathcal{R}_2(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) = \mathcal{R}_1(\phi)(\mathbf{v} + \tilde{\mathbf{w}}) - \mathcal{R}_1(\phi)(\tilde{\mathbf{w}}) = \mathcal{R}_{2,1}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) + \mathcal{R}_{2,2}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}), \tag{5.8}$$

with

$$\begin{aligned} \mathcal{R}_{2,1}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) &= \mathcal{N}(\mathbf{v} + \tilde{\mathbf{w}} + \phi) - \mathcal{N}(\tilde{\mathbf{w}} + \phi) - \mathcal{N}'(\tilde{\mathbf{w}} + \phi)\mathbf{v}, \\ \mathcal{R}_{2,2}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) &= \mathcal{N}'(\tilde{\mathbf{w}} + \phi)\mathbf{v} - \mathcal{N}'(\phi)\mathbf{v}. \end{aligned}$$

Fix some constant $K > 0$. Then, there exists a constant $C > 0$ such that for $\mathbf{v}, \tilde{\mathbf{w}} \in \mathbb{C}$ with $|\mathbf{v}|, |\tilde{\mathbf{w}}| \leq K$ we have the nonlinear bounds

$$|\mathcal{R}_1(\phi)(\tilde{\mathbf{w}})| \leq C|\tilde{\mathbf{w}}|^2, \quad |\mathcal{R}_{2,1}(\phi)(\tilde{\mathbf{w}}, \mathbf{v})| \leq C|\mathbf{v}|^2, \quad |\mathcal{R}_{2,2}(\phi)(\tilde{\mathbf{w}}, \mathbf{v})| \leq C|\mathbf{v}||\tilde{\mathbf{w}}|. \tag{5.9}$$

For the local wellposedness of (5.5), we refer to §5.4.1.

We also abbreviate $\mathcal{L}_0 = \mathcal{L}_0(\phi)$, $\mathcal{R}_1 = \mathcal{R}_1(\phi)$, $\mathcal{R}_2 = \mathcal{R}_2(\phi)$, $\mathcal{R}_{2,1} = \mathcal{R}_{2,1}(\phi)$ and $\mathcal{R}_{2,2} = \mathcal{R}_{2,2}(\phi)$ whenever ϕ is the original periodic wave profile.

5.2.4. Spectral assumptions on ϕ

Consider the Bloch operators $\mathcal{L}(\xi) = e^{-i\xi\cdot}\mathcal{L}_0 e^{i\xi\cdot}$, $\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})$, posed on $L^2_{\text{per}}(0,T)$ with domain $D(\mathcal{L}(\xi)) = H^2_{\text{per}}(0,T)$. Since $\mathcal{L}(\xi)$ has compact resolvent, its spectrum consists of isolated eigenvalues of finite algebraic multiplicity only.

We introduce the standard *diffusive spectral stability* assumptions, cf. [46, 47, 83, 96].

(D1) We have $\sigma_{L^2}(\mathcal{L}_0) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\}$;

(D2) There exists a constant $\theta > 0$ such that for any $\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})$ we have $\mathrm{Re}\ \sigma_{L^2_{\text{per}}(0,T)}(\mathcal{L}(\xi)) \leq -\theta\xi^2$;

(D3) 0 is a simple eigenvalue of $\mathcal{L}(0)$.

The spectrum of $\mathcal{L}_0$ on $L^2(\mathbb{R})$ is the union of the spectra of the Bloch operators, i.e.,

$$\sigma_{L^2}(\mathcal{L}_0) = \bigcup_{\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})} \sigma_{L^2_{\text{per}}(0,T)}(\mathcal{L}(\xi)). \tag{5.10}$$

The spectrum $\sigma_{L^2}(\mathcal{L}_0)$ is purely essential, see e.g. [39, 64].

We emphasize that the periodic solutions of (5.1) established in [28] and [13] satisfy (H1) and (D1)-(D3).

For the well-known consequences of the diffusive spectral stability assumptions, we refer to [46, Lemma 2.1] and references therein. First, Assumption (D3) together with the translational invariance of (5.1) imply that the kernel of $\mathcal{L}(0)$ is spanned by ϕ'. Therefore, 0 is also a simple eigenvalue of the adjoint operator $\mathcal{L}(0)^*$. By $\widetilde{\Phi}_0 \in H^2_{\text{per}}(0,T)$, we denote the corresponding eigenfunction satisfying

$$\langle \widetilde{\Phi}_0, \phi' \rangle_{L^2(0,T)} = 1.$$

The spectral projection onto $\mathrm{span}\{\phi'\}$ is given by

$$\Pi(0)g = \phi' \langle \widetilde{\Phi}_0, g \rangle_{L^2(0,T)}, \quad g \in L^2_{\text{per}}(0,T)$$

and we again refer to [46, Lemma 2.1] for properties of $\sigma_{L^2}(\mathcal{L}_0)$ and $\sigma_{L^2_{\text{per}}(0,T)}(\mathcal{L}(\xi))$.

Remark 5.2.2 (Spectrum on modulation space). *The identity (5.10) holds on $C_{ub}(\mathbb{R})$ and in contrast $\sigma_{C_{ub}}(\mathcal{L}_0)$ consists entirely of continuous point spectrum, [39, 64]. Since*

$$C^{m+2}_{\text{ub}}(\mathbb{R}) \hookrightarrow M^m_{\infty,1}(\mathbb{R}) \hookrightarrow C^m_{\text{ub}}(\mathbb{R}), \tag{5.11}$$

for any $m \in \mathbb{N}_0$, one finds (5.10) on $M_{\infty,1}(\mathbb{R})$ instead of $L^2(\mathbb{R})$ or $C_{\text{ub}}(\mathbb{R})$ by [32, 2.17 Proposition]. In particular, $\sigma_{M_{\infty,1}}(\mathcal{L}_0)$ also consists entirely of continuous point spectrum. This inspires a linear analysis through the complex inversion formula of the semigroup generated by $\mathcal{L}_0$ on $M_{\infty,1}$. In particular, this allows the possibility to investigate the resolvent $(\lambda - \mathcal{L}_0)^{-1}$ on the modulation space $M_{\infty,1}$ for $\lambda \in \mathbb{C} \setminus \sigma_{M_{\infty,1}}(\mathcal{L}_0)$.

We briefly explain (5.11): from [68], it is known that $C^2_b(\mathbb{R}) \hookrightarrow M_{\infty,1}(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$ and it remains to show that $M_{\infty,1} \hookrightarrow C_{ub}(\mathbb{R})$. This holds due to the observation that $C^\infty(\mathbb{R})$ lies dense in $M_{\infty,1}(\mathbb{R})$, $||f||_{L^\infty} \lesssim ||f||_{M_{\infty,1}}$ and since the uniform limit of uniformly continuous functions is uniformly continuous itself. Similarly, one argues for $m \geq 1$.

5.2.5. Formulation of main result

We are now in the position to state our main result.

Theorem 5.2.3. *Assume* (H1) *and* (D1)-(D3). *There exist constants* $C, \varepsilon > 0$ *such that for initial data* $\mathbf{w}_0 \in H^6_{\mathrm{per}}(0,T)$ *and* $\mathbf{v}_0 \in H^3(\mathbb{R})$ *with*

$$E_0 := ||\mathbf{w}_0 + \mathbf{v}_0||_{H^6_{\mathrm{per}}(0,T) \oplus H^3(\mathbb{R})} < \varepsilon$$

there exist a unique solution

$$\mathbf{u}(t) \in C([0,\infty); C^2_{\mathrm{ub}}(\mathbb{R})) \cap C^1([0,\infty); C_{\mathrm{ub}}(\mathbb{R})) \tag{5.12}$$

of (5.2) with initial condition $\mathbf{u}(0) = \phi + \mathbf{w}_0 + \mathbf{v}_0$, *some smooth function* $\gamma \in C([0,\infty), H^5(\mathbb{R}))$ *and a constant* $\sigma_* \in \mathbb{R}$ *with the properties*

$$||\mathbf{u}(t) - \phi||_{W^{2,\infty}} \leq CE_0 \tag{5.13}$$

and

$$||\mathbf{u}(\cdot,t) - \phi_0(\cdot + \sigma_* + \gamma(\cdot,t))||_{L^\infty} \leq C(1+t)^{-\frac{3}{4}} E_0, \tag{5.14}$$

for all $t \geq 0$.

We briefly discuss the regularity assumptions in Theorem 5.2.3. The assumption $\mathbf{v}_0 \in H^3(\mathbb{R})$ is justified by the fact that the regularity control on $\mathbf{v}$ mainly proceeds along the lines of a standard $L^1 \cap H^k$-nonlinear stability analysis [46, 55] using the nonlinear damping estimate established in [105]. The reason for the regularity assumption on $\mathbf{w}_0$ is that $\mathbf{v}$ is considered as a perturbation of the periodic solution $\mathbf{w}$, which yields expressions in the modulated perturbation equations for $\mathbf{v}$ where $\mathbf{w}$ appears with two spatial derivatives. As we need to control $\mathbf{v}$ in $H^3(\mathbb{R})$, this leads to three more derivatives on $\mathbf{w}$. Therefore, we need to bound the fifth derivative in L^∞ which is covered by assuming $\mathbf{w}_0 \in H^6_{\mathrm{per}}(0,T)$. Since we estimate the H^1-norm of the residual in Section 5.5.2 in order to find an L^∞-estimate (in the spirit of the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$), it suffices to demand $\mathbf{w}_0 \in H^5_{\mathrm{per}}(0,T)$ and $\mathbf{v}_0 \in H^2(\mathbb{R})$ when one only aims to establish pure L^2-estimates on $\mathbf{v}$ and henceforth obtain (5.14) in $L^\infty(\mathbb{R})$ with lower decay rate $(1+t)^{-\frac{1}{2}}$.

Comparing (5.14) to the associated estimate in [46], we loose an algebraic decay factor of $\frac{1}{4}$ and one might ask whether we can compensate this lack of decay by taking the assumption $\mathbf{v}_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$. However, this assumption does not improve the decay rates due to the coupling terms $\mathcal{R}_2(\tilde{\mathbf{w}}, \mathbf{v})$ in (5.5) which cannot be controlled in $L^1(\mathbb{R})$ but only in $L^2(\mathbb{R})$, cf. Remark 5.4.8.

Another interesting contrast to the result in [46] is the constant phase shift arising in the modulational estimate (5.14). The reason for this constant phase shift precisely originates from the fact the we not only enclose localized- but also co-periodic perturbation. This is reflected in designing the modulational approach to exploit the orbital stability result from [96].

Remark 5.2.4 (Uniqueness of solutions). *We also comment on the uniqueness of solutions stated in Theorem 5.2.3. Concerning fully nonlocalized solutions of (5.1), local well-posedness* *of (5.1) with initial data from the modulation space $M^m_{\infty,1}(\mathbb{R})$, $m \in \mathbb{N}_0$, can be established by precisely following the steps from [23, Section 4.2] as the principal linear part of the equation (5.1) generates a C_0-semigroup on $M_{\infty,1}(\mathbb{R})$. That is, there exists a unique (mild) solution as element from $C([0,t), M_{\infty,1}(\mathbb{R}))$, $t > 0$, whenever $\mathbf{u}_0 \in C^2_{ub}(\mathbb{R}) \hookrightarrow M_{\infty,1}(\mathbb{R})$. In particular, this shows that the solution $\mathbf{u}(t)$ of (5.2) with $\mathbf{u}(0) = \mathbf{w}_0 + \mathbf{v}_0$ in Theorem 5.2.3 satisfying (5.12) is unique.*

5.2.6. Strategy of proof

The main task in the proof of Theorem 5.2.3 is to find a suitable way to modulate the perturbations allowing to close a nonlinear argument through iterative estimates on their Duhamel formulae. The construction of the perturbations $\hat{\mathbf{w}}(t)$ and $\mathbf{v}(t)$ shows that $\hat{\mathbf{w}}(t)$ is independent of $\mathbf{v}(t)$. Therefore, we first modulate $\hat{\mathbf{w}}(t)$ by introducing a temporal modulation function $\sigma(t)$. Precisely, we choose $\hat{\mathbf{w}}(x,t) = \mathbf{w}(x+\sigma(t),t) - \phi(x)$. Then, we do not only interpret $\mathbf{v}(t)$ as perturbation of $\mathbf{w}(t)$ but do the same for their modulated variants. This leads to the following inverse-modulated perturbation

$$\hat{\mathbf{v}}(x,t) = \hat{\mathbf{u}}(x,t) - \hat{\mathbf{w}}(x,t) := \mathbf{u}(x-\sigma(t)-\gamma(t),t) - \hat{\mathbf{w}}(x,t) - \phi(x).$$

We arrive at a coupled system which allows to make an a-posteriori choice of the spatio-temporal phase modulation $\gamma : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$. We are then in the position to exploit the exponential decay of $|\sigma_t(t)|$ and $||\hat{\mathbf{w}}(t)||_{L^\infty}$. We also introduce the forward-modulated perturbation

$$\begin{aligned}\mathring{\mathbf{v}}(x,t) &= \mathring{\mathbf{u}}(x,t) - \hat{\mathbf{w}}(x+\sigma(t)+\gamma(x,t),t) \\ &:= \mathbf{u}(x,t) - \phi(x+\sigma(t)+\gamma(x,t),t) - \hat{\mathbf{w}}(x+\sigma(t)+\gamma(x,t),t)\end{aligned}$$

which obeys a semilinear equation and can thus be used to control regularity in the nonlinear iteration. We control regularity by deriving a nonlinear damping estimate on $\mathring{\mathbf{v}}(t)$ and subsequently relating $\mathring{\mathbf{v}}(t)$ to $\hat{\mathbf{v}}(t)$. Finally, we show the following stability estimates: for suitable constants $C, \delta > 0$ and small initial conditions

$$E_0 = E_p + E_l \text{ with } E_p = ||\mathbf{w}_0||_{H^6_{\text{per}}} \text{ and } E_l = ||\mathbf{v}_0||_{H^3},$$

we find

$$|\sigma_t(t)|, ||\hat{\mathbf{w}}(t)||_{H^6_{\text{per}}(0,T)} \leq Ce^{-\delta t}E_p, \quad |\sigma| \leq CE_p, \quad t \geq 0, \tag{5.15}$$

and

$$||\gamma_x(t)||_{H^4}, ||\gamma_t(t)||_{H^3}, ||\mathring{\mathbf{v}}(t)||_{H^3} \leq C(1+t)^{-\frac{1}{2}}E_l, \quad ||\gamma(t)||_{H^5} \leq CE_l, \quad t \geq 0. \tag{5.16}$$

Using L^∞-estimates on the propagators, we arrive at the refined L^∞-estimate

$$||\mathring{\mathbf{v}}(t)||_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}E_l, \quad t \geq 0. \tag{5.17}$$

Applying (5.15) and (5.17), Theorem 5.2.3 follows from the observations

$$||\mathring{\mathbf{u}}(t)||_{L^\infty} \leq ||\hat{\mathbf{w}}(t)||_{L^\infty} + ||\mathring{\mathbf{v}}(t)||_{L^\infty}, \quad t \geq 0,$$

and

$$||\mathbf{u}(t) - \phi||_{W^{2,\infty}} \leq C(||\hat{\mathbf{w}}(t)||_{W^{2,\infty}} + ||\mathring{\mathbf{v}}(t)||_{W^{2,\infty}} + |\sigma(t)| + +||\gamma(t)||_{W^{2,\infty}}), \quad t \geq 0,$$

and Sobolev embedding. The estimate (5.15) can be derived along the lines of the co-periodic stability analysis in [96] and therefore the remaining task of this paper is to prove (5.16) and (5.17).

Notation. Let S be a set, and let $A, B\colon S \to \mathbb{R}$. Throughout the paper, the expression "$A(x) \lesssim B(x)$ for $x \in S$", means that there exists a constant $C > 0$, independent of x, such that $A(x) \leq CB(x)$ holds for all $x \in S$.

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5.3. Linear estimates

We collect well-known facts about the semigroup generated by the linearization $\mathcal{L}_0$ on $L^2_{\text{per}}(0,T)$ and on $L^2(\mathbb{R})$.

5.3.1. Semigroup decomposition and estimates on $L^2_{\text{PER}}(0,T)$

Let $\chi\colon [0,\infty) \to \mathbb{R}$ be a smooth temporal cut-off function satisfying $\chi(t) = 0$ for $t \in [0,1]$ and $\chi(t) = 1$ for $t \in [2,\infty)$. We write

$$\tilde{S}_1(t) = (e^{\mathcal{L}_0 t} - \chi(t)\Pi(0))\mathbf{g}$$

and have the following linear estimate.

Proposition 5.3.1 ([96], [45]). *Assume* (H1) *and* (D1)-(D3). *There exist constants $\delta_0, C > 0$ such that the estimate*

$$||\tilde{S}_1(t)\mathbf{g}||_{H^6_{per}(0,T)} \leq Ce^{-\delta_0 t}||\mathbf{g}||_{H^6_{per}(0,T)}$$

is valid for all $\mathbf{g} \in H^6_{per}(0,T)$.

5.3.2. Semigroup decomposition and estimates on $L^2(\mathbb{R})$

Assume (H1) and (D1)-(D3). Like in [46], we decompose

$$e^{\mathcal{L}_0 t}\mathbf{g} = \tilde{S}_2(t)\mathbf{g} + \phi' s_p(t)\mathbf{g},$$

with $s_p(t) = 0$ for $t \in [0,1]$. We have the following linear estimates.

Proposition 5.3.2. *Assume* (H1) *and* (D1)-(D3). *Let $l, j \in \mathbb{N}_0$ and $k \in \{0,1,2\}$. Then there exists a constant $C_{l,j} > 0$ such that*

$$||\partial_x^l \partial_t^j s_p(t)\partial_x^k \mathbf{g}||_{L^2} \leq C_{l,j}(1+t)^{-\frac{l+j}{2}}||\mathbf{g}||_{L^2}, \quad \mathbf{g} \in L^2(\mathbb{R})$$
$$||\partial_x^l \partial_t^j s_p(t)\mathbf{g}||_{L^2} \leq C_{l,j}(1+t)^{-\frac{1}{4}-\frac{l+j}{2}}||\mathbf{g}||_{L^1}, \quad \mathbf{g} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}),$$

for all $t \geq 0$. Furthermore, there exists a constant $C > 0$ such that

$$||\tilde{S}_2(t)\mathbf{g}||_{L^2} \leq C(1+t)^{-\frac{3}{4}}||\mathbf{g}||_{L^1 \cap L^2}, \quad \mathbf{g} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

and

$$||\tilde{S}_2(t)\mathbf{g}||_{L^2} \leq C(1+t)^{-\frac{1}{2}}||\mathbf{g}||_{L^2}, \quad \mathbf{g} \in L^2(\mathbb{R}),$$

for all $t \geq 0$.

Proof. The first two estimates are precisely shown in [46, Lemma 3.1]. The third one is a consequence of [46, Lemma 3.1 & Lemma 3.2]. Adapting the proof of the estimate $\tilde{S}_c(t)$ in [46, Lemma 3.2], one immediately finds the last estimate. □

Proposition 5.3.3 (L^∞-estimates). *Assume* (H1) *and* (D1)-(D3). *There exist constants* $C, \delta_1 > 0$ *such that*

$$||\tilde{S}_2(t)\mathbf{g}||_{L^\infty} \leq C\Big(e^{-\delta_1 t}||\mathbf{g}||_{H^1} + (1+t)^{-1}||\mathbf{g}||_{L^1\cap L^2}\Big), \quad \mathbf{g} \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}),$$
$$||\tilde{S}_2(t)\mathbf{g}||_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}||\mathbf{g}||_{H^1}, \quad \mathbf{g} \in H^1(\mathbb{R}),$$

and

$$||\partial_x s_p(t)\mathbf{g}||_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}||\mathbf{g}||_{L^2}, \quad \mathbf{g} \in L^2(\mathbb{R}),$$
$$||\partial_x s_p(t)\mathbf{g}||_{L^\infty} \leq C(1+t)^{-1}||\mathbf{g}||_{L^1}, \quad \mathbf{g} \in L^1(\mathbb{R}),$$

for all $t \geq 0$.

Proof. These last two estimates are shown in [46, Lemma 3.2]. The first two estimates are consequences of [46, Lemma 3.1] together with [55, Corollary 3.4] and [56, Proposition 3.1]. □

5.4. Nonlinear iteration scheme

5.4.1. Local existence of the solutions

Using standard semigroup theory, see e.g. [22] or [79], we establish

Proposition 5.4.1. *Let* $\mathbf{w}_0 \in H^6_{per}(0,T)$. *There exist a maximal time* $T_{max} \in (0,\infty]$ *and a unique solution* $\mathbf{w} \in C([0,T_{max}); H^6_{per}(0,T)) \cap C^1([0,T_{max}); H^4_{per}(0,T))$ *of (5.3) with* $\mathbf{w}(0) = \phi + \mathbf{w}_0$. *If* $T_{max} < \infty$, *then*

$$\limsup_{t\uparrow T_{max}} ||\mathbf{w}(t)||_{H^4_{per}(0,T)} = \infty. \tag{5.18}$$

Having the solution $\mathbf{w}$ at hand, the local existence of $\mathbf{v}$ follows.

Proposition 5.4.2. *Let* $\mathbf{w}$ *and* T_{max} *be as in Proposition 5.4.1. Let* $\mathbf{v}_0 \in H^3(\mathbb{R})$. *There exist a maximal time* $\tau_{max} \leq T_{max}$ *and a unique solution* $\mathbf{v} \in C([0,\tau_{max}); H^3(\mathbb{R})) \cap C^1([0,\tau_{max}); H^1(\mathbb{R}))$ *of (5.4) with* $\mathbf{v}(0) = \mathbf{v}_0$. *If* $\tau_{max} < T_{max}$, *then*

$$\limsup_{t\uparrow \tau_{max}} ||\mathbf{v}(t)||_{H^1} = \infty. \tag{5.19}$$

Proof. Due to the embedding $H^1_{\mathrm{per}}(0,T) \hookrightarrow L^\infty(\mathbb{R})$, we have that

$$\mathbf{v} \mapsto \mathcal{N}(\mathbf{v}+\mathbf{w}) - \mathcal{N}(\mathbf{w})$$

is locally Lipschitz-continuous as map from $H^1(\mathbb{R})$ to $H^1(\mathbb{R})$. Moreover,

$$\mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}\partial_x^2 - \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\right) - \mathcal{I}$$

generates a C_0-semigroup on $H^1(\mathbb{R})$ with domain $H^3(\mathbb{R})$. Therefore, Proposition 5.4.2 follows from [79, Theorem 1.4 of Section 6.1] □

5.4.2. Inverse-modulated perturbations

We first modulate $\mathbf{u}(t)$, that is, we consider

$$\mathbf{u}(x-\sigma(t),t) - \phi(x) = (\mathbf{w}(x-\sigma(t),t) - \phi(x)) + \mathbf{v}(x-\sigma(t),t)$$

for some $\sigma : [0,\infty) \to \mathbb{R}$ with $\sigma(0) = 0$ to be defined a-posteriori. Then, we set

$$\hat{\mathbf{w}}(x,t) = \mathbf{w}(x-\sigma(t),t) - \phi(x). \tag{5.20}$$

Motivated by the fact that $\mathbf{v}(t)$ is a perturbation of $\mathbf{w}(t) = \tilde{\mathbf{w}}(t) + \phi$, we subsequently define

$$\hat{\mathbf{v}}(x,t) = \mathbf{u}(x-\sigma(t)-\gamma(x,t),t) - \hat{\mathbf{w}}(x,t) - \phi(x) \tag{5.21}$$

for some $\gamma : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ with $\gamma(\cdot,0) = 0$ to be defined a-posteriori. We find the modulated perturbation equations for $\hat{\mathbf{w}}(t)$,

$$\begin{aligned} (\partial_t - \mathcal{L}_0)(\hat{\mathbf{w}} + \phi'\sigma) &= \mathcal{R}_1(\hat{\mathbf{w}}) - \sigma_t\hat{\mathbf{w}}_x \\ \hat{\mathbf{w}}(0) &= \mathbf{w}_0 - \phi. \end{aligned} \tag{5.22}$$

and the one for $\hat{\mathbf{v}}(t)$,

$$\begin{aligned} &(\partial_t - \mathcal{L}_0)(\hat{\mathbf{v}} + \phi'\gamma - \gamma_x\hat{\mathbf{w}} - \gamma_x\hat{\mathbf{v}}) = \mathcal{R}_3(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) - \sigma_t\hat{\mathbf{v}}_x + (1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}}) + \mathcal{T}(\hat{\mathbf{w}},\gamma) \\ &\hat{\mathbf{v}}(0) = \mathbf{v}_0, \end{aligned} \tag{5.23}$$

where

$$\mathcal{R}_3(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) = \mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) + \partial_x\mathcal{S}(\hat{\mathbf{v}},\gamma) + \partial_x^2\mathcal{P}(\hat{\mathbf{v}},\gamma)$$

and

$$\begin{aligned} \mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) &= (1-\gamma_x)\mathcal{R}_{2,1}(\hat{\mathbf{w}},\hat{\mathbf{v}}), \\ \mathcal{S}(\hat{\mathbf{v}},\gamma) &= -\gamma_t\hat{\mathbf{v}} + \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{v}} - \frac{\gamma_x^2}{1-\gamma_x}\phi'\right), \\ \mathcal{P}(\hat{\mathbf{v}},\gamma) &= -\beta\mathcal{J}\left(\gamma_x + \frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{v}}, \\ \mathcal{T}(\hat{\mathbf{w}},\gamma) &= -\gamma_x\mathcal{R}_1(\hat{\mathbf{w}}) - \partial_x\left(\gamma_t\hat{\mathbf{w}} - \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{w}}\right)\right) - \partial_x^2\left(\beta\mathcal{J}\left(\gamma_x + \frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{w}}\right). \end{aligned}$$

We delegate the derivation of (5.23) to Appendix 5.A. The main observation is that in the nonlinearities of (5.23) any $\hat{\mathbf{w}}$- and σ_t-term is paired with a γ_x, $\hat{\mathbf{v}}$ or γ_t contribution suggesting that we have sufficient control for an L^2-iteration scheme since we expect exponential decay for $||\hat{\mathbf{w}}(t)||_{L^\infty}$ and $|\sigma_t(t)|$ from [96] while we control $\gamma_x(t)$, $\hat{\mathbf{v}}(t)$ and $\gamma_t(t)$ in $H^k(\mathbb{R})$. To this end, we establish the following nonlinear bounds.

Lemma 5.4.3. *Fix a constant $c > 0$ such that $||f||_{L^\infty} \leq \frac{1}{2}$ for all $f \in H^1(\mathbb{R})$ with $||f||_{H^1} \leq c$. There exists a constant $C > 0$ such that*

$$\begin{aligned}
L^1\text{-bound: } ||\mathcal{R}_3(\hat{\mathbf{w}}, \hat{\mathbf{v}}, \gamma)||_{L^1} &\leq C\Big(||\hat{\mathbf{v}}||_{L^2}^2 + ||(\gamma_x, \gamma_t)||_{H^2 \times H^1}(||\hat{\mathbf{v}}||_{H^2} + ||\gamma_x||_{L^2})\Big),\\
L^2\text{-bound: } ||\mathcal{R}_3(\hat{\mathbf{w}}, \hat{\mathbf{v}}, \gamma)||_{L^2} &\leq C\Big(||\hat{\mathbf{v}}||_{H^1}^2 + ||(\gamma_x, \gamma_t)||_{H^2 \times H^1}(||\hat{\mathbf{v}}||_{H^2} + ||\gamma_x||_{L^2})\Big),\\
H^1\text{-bound: } ||\mathcal{R}_3(\hat{\mathbf{w}}, \hat{\mathbf{v}}, \gamma)||_{H^1} &\leq C\Big(||\hat{\mathbf{v}}||_{H^1}^2 + ||(\gamma_x, \gamma_t)||_{H^3 \times H^2}(||\hat{\mathbf{v}}||_{H^3} + ||\gamma_x||_{H^1})\Big),
\end{aligned}$$

and

$$\begin{aligned}
L^2\text{-bounds: } &||\sigma_t \hat{\mathbf{v}}_x||_{L^2} \leq C|\sigma_t|||\hat{\mathbf{v}}||_{H^1}, \quad ||(1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}}, \hat{\mathbf{v}})||_{L^2} \leq C||\hat{\mathbf{v}}||_{L^2}||\hat{\mathbf{w}}||_{H^1_{per}(0,T)},\\
&||\mathcal{T}(\hat{\mathbf{w}}, \gamma)||_{L^2} \leq C||(\gamma_x, \gamma_t)||_{H^2 \times H^1}||\hat{\mathbf{w}}||_{H^3_{per}(0,T)},\\
H^1\text{-bounds: } &||\sigma_t \hat{\mathbf{v}}_x||_{H^1} \leq C|\sigma_t|||\hat{\mathbf{v}}||_{H^2}, \quad ||(1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}}, \hat{\mathbf{v}})||_{H^1} \leq C||\hat{\mathbf{v}}||_{H^1}||\hat{\mathbf{w}}||_{H^2_{per}(0,T)},\\
&||\mathcal{T}(\hat{\mathbf{w}}, \gamma)||_{H^1} \leq C||(\gamma_x, \gamma_t)||_{H^3 \times H^2}||\hat{\mathbf{w}}||_{H^4_{per}(0,T)},
\end{aligned}$$

hold for all $\hat{\mathbf{v}} \in H^3(\mathbb{R})$, $\hat{\mathbf{w}} \in H^4_{per}(0,T)$, $(\gamma_t, \gamma_x) \in H^2(\mathbb{R}) \times H^3(\mathbb{R})$ and $\sigma_t \in \mathbb{R}$ provided

$$||\hat{\mathbf{w}}||_{H^4_{per}(0,T)}, ||\hat{\mathbf{v}}||_{H^1}, ||\gamma_x||_{H^3} \leq c.$$

5.4.3. Modulation in the purely co-periodic setting

The forward-modulated perturbation $\mathring{\mathbf{w}}(x,t) = \mathbf{w}(x,t) - \phi(x + \sigma(t))$ fulfills the semilinear system

$$(\partial_t - \mathcal{L}_0)(\mathring{\mathbf{w}}(t) - \phi'\sigma) = \mathcal{R}_4(\mathring{\mathbf{w}}(t), \sigma(t)) + (\phi'(\cdot + \sigma(t)) - \phi')\sigma_t(t)$$

with

$$\mathcal{R}_4(\mathring{\mathbf{w}}(t), \sigma(t)) = \mathcal{R}_1(\phi(\cdot + \sigma(t)))(\mathring{\mathbf{w}}(t)) - (\mathcal{N}'(\phi) - \mathcal{N}'(\phi(\cdot + \sigma(t))))\mathring{\mathbf{w}}(t).$$

Introducing the temporal modulation function

$$\sigma(t) = \chi(t)\Pi(0)\tilde{\mathbf{w}}_0 + \int_0^t \chi(t-s)\Pi(0)\big(\mathcal{R}_4(\mathring{\mathbf{w}}(s), \sigma(s)) + (\phi'(\cdot + \sigma(s)) - \phi')\sigma_s(s)\big)\, ds \quad (5.24)$$

gives rise to the Duhamel formula

$$\mathring{\mathbf{w}}(t) = \tilde{S}_1(t)\tilde{\mathbf{w}}_0 + \int_0^t \tilde{S}_1(t-s)\big(\mathcal{R}_4(\mathring{\mathbf{w}}(s), \sigma(s)) + (\phi'(\cdot + \sigma(s)) - \phi')\sigma_s(s)\big)\, ds. \quad (5.25)$$

By a standard fixed point argument, we have local existence of σ.

Proposition 5.4.4. *Let $\mathbf{w}$ and T_{max} be as in Proposition 5.4.1. There exists a maximal time $t_{max,\sigma} \leq T_{max}$ such that (5.24) with $\mathring{\mathbf{w}}(x,t) = \mathbf{w}(x,t) - \phi(x + \sigma(t))$ has a unique solution*

$$\sigma \in C^1([0, t_{max,\sigma}); \mathbb{R}) \text{ with } \sigma(0) = 0 \text{ and } |(\sigma(t), \sigma_t(t))| < \frac{1}{2}, \quad t \in [0, t_{max,\sigma}).$$

If $t_{max,\sigma} < \tau_{max}$, then $\limsup_{t \uparrow t_{max,\sigma}} |(\sigma(t), \sigma_t(t))| \geq \frac{1}{2}$.

Furthermore, the following nonlinear bound holds.

Lemma 5.4.5. *Let $K > 0$. There exists a constant $C > 0$ such that*

$$||\mathcal{R}_4(\mathring{\mathbf{w}}, \sigma)||_{H^6_{per}(0,T)} \leq C||\mathring{\mathbf{w}}||_{H^6_{per}(0,T)}\Big(||\mathring{\mathbf{w}}||_{H^6_{per}(0,T)} + |\sigma|\Big),$$
$$||(\phi'(\cdot + \sigma - \phi')\sigma_t||_{H^6_{per}(0,T)} \leq C|\sigma||\sigma_t|,$$

provided

$$|\sigma| + |\sigma_t| + ||\mathring{\mathbf{w}}||_{H^6_{per}(0,T)} \leq K.$$

5.4.4. Choice of the spatial-temporal phase modulation

We have the Duhamel formula for $\hat{\mathbf{v}}(t)$,

$$\begin{aligned}\hat{\mathbf{v}}(t) = e^{\mathcal{L}_0 t}\mathbf{v}_0 - \phi'\gamma(t) + \int_0^t e^{\mathcal{L}_0(t-s)}\Big(&\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s)) - \sigma_s(s)\hat{\mathbf{v}}_x(s) \\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{v}}(s), \hat{\mathbf{w}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))\Big)\, ds \\ +\gamma_x(t)\hat{\mathbf{w}}(t) + \gamma_x(t)\hat{\mathbf{v}}(t),\end{aligned}$$

under the condition that $\gamma(0) = 0$. We make the implicit choice

$$\begin{aligned}\gamma(t) = s_p(t)\mathbf{v}_0 + \int_0^t s_p(t-s)\Big(&\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s)) - \sigma_s(s)\hat{\mathbf{v}}_x(s) \\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{v}}(s), \hat{\mathbf{w}}(s)) \\ &+ \mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))\Big)\, ds\end{aligned} \tag{5.26}$$

which reduces the Duhamel formula for $\hat{\mathbf{v}}(t)$ to

$$\begin{aligned}\hat{\mathbf{v}}(t) =& \tilde{S}_2(t)\mathbf{v}_0 + \gamma_x(t)\hat{\mathbf{w}}(t) + \gamma_x(t)\hat{\mathbf{v}}(t) \\ &+ \int_0^t \tilde{S}_2(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s)) - \sigma_s(s)\hat{\mathbf{v}}_x(s) \\ &\qquad + (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{v}}(s), \hat{\mathbf{w}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))\Big)\, ds.\end{aligned} \tag{5.27}$$

Setting $t = 0$ in (5.26) and using that $s_p(0) = 0$, one indeed verifies that $\gamma(0) = 0$.

Proposition 5.4.6. *Let $\mathbf{w}$ and T_{max} as in Proposition 5.4.1, $\mathbf{v}$, $\mathbf{v}_0$ and τ_{max} as in Proposition 5.4.2 and σ and $t_{max,\sigma}$ as in Proposition 5.4.4. Furthermore, let $0 < c < \frac{1}{2}$ be a constant such that $||f||_{L^\infty} \leq \frac{1}{c}||f||_{H^1}$ for all $f \in H^1(\mathbb{R})$. There exists a maximal time $t_{max,\gamma} \leq \min\{\tau_{max}, t_{max,\sigma}\}$ such that (5.26) with $\hat{\mathbf{w}}(t) = \mathbf{w}(\cdot - \sigma(t)) - \phi$ and $\hat{\mathbf{v}}(t) = \mathbf{u}(\cdot - \sigma(t) - \gamma(\cdot, t), t) - \hat{\mathbf{w}}(t) - \phi$ has a unique solution*

$$\gamma \in C([0, t_{max,\gamma}); H^5(\mathbb{R})) \cap C^1([0, t_{max,\gamma}); H^3(\mathbb{R})) \text{ with } \gamma(0) = 0$$

satisfying

$$||(\gamma(t), \gamma_t(t))||_{H^5\times H^3} < \frac{c}{2}, \quad t \in [0, t_{max,\gamma}). \tag{5.28}$$

If $t_{max,\gamma} < \min\{\tau_{max}, t_{max,\sigma}\}$, then

$$\limsup_{t\uparrow t_{max,2}} ||\gamma(t), \gamma_t(t)||_{H^5\times H^3} \geq \frac{c}{2}. \tag{5.29}$$

Proof. See Appendix 5.C. □

Corollary 5.4.7. *Let* $\mathbf{w}$ *and* T_{max} *as in Proposition 5.4.1 and* $\mathbf{v}$*,* $\mathbf{v}_0$ *and* τ_{max} *as in Proposition 5.4.2. Let* γ*,* σ *and* $t_{max,\sigma}$*,* $t_{max,\gamma}$ *as in Propositions 5.4.4 and 5.4.6. Then, the inverse-modulated perturbation* $\hat{\mathbf{v}} \in C([0, t_{max,2}), L^2(\mathbb{R}))$ *defined by (5.21) satisfies (5.27) with* $\hat{\mathbf{w}}(t) = \mathbf{w}(\cdot - \sigma(t)) - \phi$ *and* $\hat{\mathbf{v}}(t) \in H^3(\mathbb{R})$ *for all* $t \in [0, t_{max,\gamma})$*.*

Proof. Let $t \in [0, t_{\max,\gamma})$. We observe

$$\begin{aligned}\hat{\mathbf{v}}(x,t) &= \mathbf{v}(x - \sigma(t) - \gamma(x,t), t) + \mathbf{w}(x - \sigma(t) - \gamma(x,t), t) - \hat{\mathbf{w}}(x,t) - \phi(x)\\ &= \mathbf{v}(x - \sigma(t) - \gamma(x,t), t) + \mathbf{w}(x - \sigma(t) - \gamma(x,t), t) - \mathbf{w}(x - \sigma(t), t)\end{aligned}$$

yielding on the one hand, with $\mathbf{w}(t) \in H^5_{\text{per}}(0,T) \hookrightarrow W^{4,\infty}(\mathbb{R})$ and the mean value theorem,

$$||\mathbf{w}(x - \sigma(t) - \gamma(x,t), t) - \mathbf{w}(x - \sigma(t), t)||_{H^3} \lesssim ||\gamma(t)||_{H^3}.$$

On the other hand, since $\gamma(t) \in H^4 \hookrightarrow W^{3,\infty}$, $\sup_{s\in[0,t]} ||\gamma_x(s)||_{L^\infty} \leq \frac{1}{2}$ and $\mathbf{v}(t) \in H^3(\mathbb{R})$, we conclude

$$\hat{\mathbf{v}}(t) \in H^3(\mathbb{R})$$

with the help of the chain and substitution rule. □

Remark 5.4.8. *We can now provide some intuition where the decay* $(1+t)^{-\frac{1}{2}}$ *for* $\mathbf{v}$ *and* γ_x *in (5.16) originates from. For this purpose, assume that* $|\sigma_t|$ *admits exponential decay while* $||\hat{\mathbf{v}}||_{L^2} \approx ||\hat{\mathbf{v}}_x||_{L^2}$ *decays at rate* $(1+t)^{-\kappa}$ *with* $\kappa \geq \frac{1}{2}$*. Considering the term* $||\sigma_t \hat{\mathbf{v}}_x||_{L^2} \leq ||\hat{\mathbf{v}}_x||_{L^2} |\sigma_t|$ *in (5.26) and (5.27), this yields the integral*

$$\int_0^t (1+t-s)^{-\frac{1}{2}} (1+s)^{-\kappa} e^{-s}\, ds \lesssim (1+t)^{-\frac{1}{2}}.$$

By noting that $\sigma_t \hat{\mathbf{v}}_x$ *cannot be estimated in any* L^p*-norm with* $1 \leq p < 2$ *due to the lack of localization of* $\hat{\mathbf{v}}$*, we can only close an iterative argument with* $\kappa = \frac{1}{2}$ *at best. This shows that even with the additional assumption* $\mathbf{v}_0 \in L^1(\mathbb{R})$*, the decay rate on* $\hat{\mathbf{v}}$ *cannot be improved.*

5.4.5. Forward-modulation of v and nonlinear damping estimates

We wish to control $||\mathbf{v}||_{H^3}$ in terms of $||\hat{\mathbf{v}}||_{L^2}$, γ_t, γ_x, σ_t and $\hat{\mathbf{w}}$ in the nonlinear iteration argument in order to control regularity, cf. Lemma 5.4.3. For this purpose, we introduce the forward-modulated perturbation $\mathring{\mathbf{v}}$, that is,

$$\begin{aligned}\mathring{\mathbf{v}}(x,t) &= \mathbf{u}(x,t) - \hat{\mathbf{w}}(x + \sigma(t) + \gamma(x,t), t) - \phi(x + \sigma(t) + \gamma(x,t))\\ &= \mathbf{u}(x,t) - \mathbf{w}(x + \gamma(x,t), t)\\ &= \mathbf{v}(x,t) + \mathbf{w}(x,t) - \mathbf{w}(x + \gamma(x,t), t),\end{aligned} \tag{5.30}$$

which satisfies the semilinear system, cf. Appendix 5.B,

$$(\partial_t - \mathcal{L}_0(\mathring{\phi}))\mathring{\mathbf{v}}(t) = \mathcal{R}_2(\mathring{\phi})(\tilde{\mathbf{w}}(\cdot + \gamma(\cdot,t),t), \mathring{\mathbf{v}}(t)) + \mathcal{R}_5(\tilde{\mathbf{w}}(t), \gamma(t), \gamma_t(t)) \tag{5.31}$$

with

$$\mathcal{L}_0(\mathring{\phi}) = \mathcal{J}\begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\mathring{\phi}_1^2 + \mathring{\phi}_2^2 & 2\mathring{\phi}_1\mathring{\phi}_2 \\ 2\mathring{\phi}_1\mathring{\phi}_2 & -\beta\partial_x^2 - \alpha + \mathring{\phi}_1^2 + 3\mathring{\phi}_2^2 \end{pmatrix} - \mathcal{I}, \quad \mathring{\phi}(x,t) = \phi(x+\gamma(x,t)),$$

and

$$\begin{aligned}
&\mathcal{R}_5(\tilde{\mathbf{w}}(t), \gamma(t), \gamma_t(t)) \\
&= -\tilde{\mathbf{w}}_x(\cdot + \gamma(\cdot,t),t)\gamma_t(t) - \phi'(\cdot + \gamma(\cdot,t))\gamma_t(t) \\
&\quad - \beta\mathcal{J}\Big(\tilde{\mathbf{w}}_x(\cdot + \gamma(\cdot,t),t)\gamma_{xx}(t) + \tilde{\mathbf{w}}_{xx}(\cdot + \gamma(\cdot,t),t)(2\gamma_x(t) + \gamma_x(t)^2) \\
&\qquad + \phi'(\cdot + \gamma(\cdot,t))\gamma_{xx}(t) + \phi''(\cdot + \gamma(\cdot,t))(2\gamma_x(t) + \gamma_x(t)^2)\Big).
\end{aligned}$$

Note that $\mathcal{R}_2(\mathring{\phi})$ is as defined in (5.8) and $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$ is the umodulated perturbation of $\mathbf{w}(t)$. We setup the local existence for the forward-modulated perturbation $\mathring{\mathbf{v}}(t)$.

Corollary 5.4.9. *Let $\mathbf{w}$ and T_{max} as in Proposition 5.4.1 and $\mathbf{v}$, and τ_{max} as in (5.4.2). Let γ, σ and $t_{max,\sigma}$, $t_{max,\gamma}$ as in Propositions 5.4.4 and 5.4.6. Then, the forward-modulated perturbation $\mathring{\mathbf{v}} \in C([0,t_{max,\gamma}), H^3(\mathbb{R})) \cap C^1([0,t_{max,\gamma}), H^1(\mathbb{R}))$ defined by (5.30) satisfies (5.31) with $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$.*

Lemma 5.4.10. *Let $j = 1,2,3$. Fix $K > 0$. There exists some $C > 0$ such that for $t \in [0, t_{max,\gamma})$, we obtain*

$$\begin{aligned}
||\partial_x^j \mathcal{R}_5(\tilde{\mathbf{w}}(t), \gamma(t), \gamma_t(t))||_{L^2} &\leq C(||\gamma_x(t)||_{H^{j+1}} + ||\gamma_t(t)||_{H^j}), \\
||\partial_x^j \mathcal{R}_2(\mathring{\phi})(\tilde{\mathbf{w}}(\cdot + \gamma(\cdot,t),t), \mathring{\mathbf{v}}(t))||_{L^2} &\leq C||\mathring{\mathbf{v}}(t)||_{H^j}
\end{aligned}$$

provided

$$\sup_{0\leq s\leq t}\Big(||\tilde{\mathbf{w}}(s)||_{H^6_{per}(0,T)} + ||\mathring{\mathbf{v}}(s)||_{H^3} + ||(\gamma_x(s), \gamma_s(s))||_{H^4\times H^3}\Big) \leq K.$$

Proof. We bound

$$\begin{aligned}
&||\partial_x^j(\beta\mathcal{J}\Big(\tilde{\mathbf{w}}_{xx}(\cdot + \gamma(\cdot,t),t)(2\gamma_x(t) + \gamma_x(t)^2) + \tilde{\mathbf{w}}_x(\cdot + \gamma(\cdot,t),t)\gamma_{xx}(t)\Big) \\
&\quad + \tilde{\mathbf{w}}_x(\cdot + \gamma(\cdot,t),t)\gamma_t(t))||_{L^2} \\
&\lesssim ||\tilde{\mathbf{w}}(t)||_{H^{j+3}_{\text{per}}(0,T)}||\gamma_x(t)||_{H^{j+1}} + ||\tilde{\mathbf{w}}(t)||_{H^{j+2}_{\text{per}}(0,T)}||\gamma_t(t)||_{H^j},
\end{aligned}$$

$j = 1,2,3$, where we used the Sobolev embedding $H^k_{\text{per}}(0,T) \hookrightarrow W^{k-1,\infty}(\mathbb{R})$. Similarly, one proceeds for terms involving ϕ instead of $\tilde{\mathbf{w}}$. Also it is straightforward to check the bound for $\mathcal{R}_2$. □

Proceeding as in [46, 47, 105], we are now in the position to derive a nonlinear damping estimate for $\mathring{\mathbf{v}}$.

Proposition 5.4.11. *Let γ, σ and $t_{max,\sigma}$, $t_{max,\gamma}$ as in Propositions 5.4.4 and 5.4.6. Let $\mathbf{w}$ and T_{max} as in Proposition 5.4.1. Take $\mathbf{v}$, $\mathbf{v}_0$ and τ_{max} as in Proposition 5.4.2. Define $\mathring{\mathbf{v}}$ through (5.30) and set $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$. Fix $K > 0$. There exists a constant $C > 0$ such that*

$$||\mathring{\mathbf{v}}(t)||^2_{H^3} \leq C\Big(e^{-t}||\mathbf{v}_0||^2_{H^3} + ||\mathring{\mathbf{v}}(t)||^2_{L^2} + \int_0^t e^{-(t-s)}\Big(||\mathring{\mathbf{v}}(s)||^2_{L^2} + ||\gamma_x(s)||^2_{H^4} + ||\gamma_t(s)||^2_{H^3}\Big)\,ds\Big) \tag{5.32}$$

for all $t \in [0, t_{max,\gamma})$ *provided*

$$\sup_{0\le s\le t}\Big(||\tilde{\mathbf{w}}(s)||_{H^6_{per}(0,T)} + ||\mathring{\mathbf{v}}(s)||_{H^3} + ||(\gamma_x(s), \gamma_s(s))||_{H^3\times H^2}\Big) \le K. \tag{5.33}$$

Proof. First we use that $\mathbf{v}_0 \in H^5(\mathbb{R})$ gives a solution

$$\mathbf{v} \in C([0, \tau_{\max})); H^5(\mathbb{R})) \cap C^1([0, \tau_{\max}); H^3(\mathbb{R}))$$

of (5.5) arguing analogously as in Proposition 5.4.2. Since $H^5(\mathbb{R})$ is dense in $H^3(\mathbb{R})$ an approximation argument as in [22, Proposition 4.3.7] yields the result for $\mathbf{v}_0 \in H^3(\mathbb{R})$.

Fix $K > 0$. Let $t \in [0, t_{\max,\gamma})$ such that (5.33) holds. The forward-modulated perturbation $\mathring{\mathbf{v}}$ is designed such that the principal part in (5.31) is given by $\partial_t - \mathcal{L}_0(\mathring{\phi})$. This is the reason why we choose the same energies as introduced in [46], [47] and [105], that is,

$$E_j(t) = ||\partial_x^j \mathring{\mathbf{v}}(t)||^2_{L^2} - \frac{1}{2\beta}\langle \mathcal{J} M(\mathring{\phi})\partial_x^{j-1}\mathring{\mathbf{v}}, \partial_x^{j-1}\mathring{\mathbf{v}}\rangle, \quad j = 1, 2, 3,$$

with

$$M(\mathring{\phi}) = 2\begin{pmatrix} -2\mathring{\phi}_r\mathring{\phi}_i & \mathring{\phi}_r^2 - \mathring{\phi}_i^2 \\ \mathring{\phi}_r^2 - \mathring{\phi}_i^2 & 2\mathring{\phi}_r\mathring{\phi}_i \end{pmatrix}.$$

We compute, see [105] and [47],

$$\partial_t E_j(t) = -2E_j(t) + R_1(t) + R_2(t)$$

with

$$|R_1(t)| \le \frac{2}{3}E_j(t) + C_1||\mathring{\mathbf{v}}(t)||^2_{L^2}$$

for some t-independent constant $C_1 > 0$ and

$$\begin{aligned} R_2(t) =& 2\,\mathrm{Re}\,\langle \partial_x^j \mathcal{R}_5(\hat{\mathbf{w}}(t), \gamma(t), \gamma_t(t)), \partial_x^j \mathring{\mathbf{v}}(t)\rangle_{L^2} \\ &- \frac{1}{\beta}\,\mathrm{Re}\,\langle \mathcal{J} M(\mathring{\phi})\partial_x^{j-1}\mathcal{R}_5(\hat{\mathbf{w}}(t), \gamma(t), \gamma_t(t)), \partial_x^{j-1}\mathring{\mathbf{v}}(t)\rangle_{L^2}. \end{aligned}$$

By Lemma 5.4.10, using (5.33), interpolation and Young's inequality, there exists a t-independent constant $C_2 > 0$ such that

$$|R_2(t)| \le \frac{1}{3}E_j(t) + C_2\Big(||\mathring{\mathbf{v}}(t)||^2_{L^2} + ||\gamma_x(t)||^2_{H^{j+1}} + ||\gamma_t(t)||^2_{H^j}\Big).$$

We conclude

$$\partial_t E_j(t) \le -E_j(t) + C_3\Big(||\mathring{\mathbf{v}}(t)||^2_{L^2} + ||\gamma_x(t)||^2_{H^{j+1}} + ||\gamma_t(t)||^2_{H^j}\Big)$$

for some t-independent constant $C_3 > 0$ and $j = 1, 2, 3$. Integrating the latter and using, for some t-independent constant $C_4 > 0$,

$$||\partial_x \mathring{\mathbf{v}}(t)||^2_{L^2} \le E_1(t) + C_4||\mathring{\mathbf{v}}(t)||^2_{L^2}$$

and

$$||\partial_x^j \mathring{\mathbf{v}}(t)||^2_{L^2} \le 2E_j(t) + C_4\Big(||\mathring{\mathbf{v}}(t)||^2_{L^2} + E_{j-1}(t)\Big),$$

$j = 2, 3$, which follow by interpolation and Young's inequality, we arrive at (5.32). ☐

Lemma 5.4.12. *Fix $K > 0$. Let the assumptions be as in the previous corollary. There exists $C > 0$ such that for all $t \in [0, t_{max,\gamma})$ it holds*

$$||\hat{\mathbf{v}}(t)||_{L^2} \leq C(||\mathring{\mathbf{v}}(t)||_{L^2} + ||\gamma_x(t)||_{L^2}), \tag{5.34}$$

$$||\hat{\mathbf{v}}(t)||_{H^3} \leq C(||\mathring{\mathbf{v}}(t)||_{H^3} + ||\gamma_x(t)||_{H^3}) \tag{5.35}$$

and

$$||\mathring{\mathbf{v}}(t)||_{L^\infty} \leq C(||\hat{\mathbf{v}}(t)||_{L^\infty} + ||\gamma_x(t)||_{L^\infty}), \tag{5.36}$$

provided

$$\sup_{0\leq s\leq t}\Big(||\hat{\mathbf{w}}(s)||_{H^5_{per}} + ||\gamma(s)||_{H^4} + |\sigma(s)|\Big) \leq K \text{ and } \sup_{0\leq s\leq t}||\gamma_x(s)||_{L^\infty} \leq \frac{1}{2}.$$

Proof. Let $t \in [0, t_{\max,\gamma})$. We write $A_t(x) = x - \gamma(x,t) - \sigma(t)$ and $B_t(x) = x + \gamma(x,t) + \sigma(t)$. By the inverse function theorem and the fact that $||\gamma_x(t)||_{L^\infty} \leq \frac{1}{2}$, it is easy to check that A_t is invertible with

$$x = A_t(A_t^{-1}(x)) = A_t^{-1}(x) - \gamma(A_t^{-1}(x), t) - \sigma(t)$$

and therefore

$$A_t^{-1}(x) - B_t(x) = \gamma(A_t^{-1}(x), t) - \gamma(x, t).$$

We also compute

$$\begin{aligned} \partial_x(A_t^{-1}(x)) &= \frac{1}{1-\gamma_x(A_t^{-1}(x),t)}, \quad \partial_x^2(A_t^{-1}(x)) = \frac{\gamma_{xx}(A_t^{-1}(x),t)}{(1-\gamma_x(A_t^{-1}(x),t))^3}, \\ \partial_x^3(A_t^{-1}(x)) &= \frac{\gamma_{xxx}(A_t^{-1}(x),t)}{(1-\gamma_x(A_t^{-1}(x),t))^4} + \frac{3\gamma_{xx}(A_t^{-1}(x),t)^2}{(1-\gamma_x(A_t^{-1}(x),t))^5}. \end{aligned} \tag{5.37}$$

Using

$$\gamma(A_t^{-1}(x), t) - \gamma(x,t) = (A_t^{-1}(x) - x)\int_0^1 \gamma_x(x + \theta(A_t^{-1}(x) - x), t)\, d\theta, \tag{5.38}$$

(5.37) and $||\gamma_x(t)||_{L^\infty} \leq \frac{1}{2}$, we can estimate

$$||A_t^{-1} - B_t||_{L^2} \lesssim ||\gamma_x||_{L^2}, \quad ||A_t^{-1} - B_t||_{H^l} \lesssim ||\gamma_x||_{H^l}, \tag{5.39}$$

for $l = 1, 2, 3$. This is shown in [59, Lemma 2.7] and [105, Corollary 5.3]. It additionally follows

$$||\hat{\mathbf{v}}(A_t^{-1}(\cdot), t) - \hat{\mathbf{v}}(\cdot, t)||_{H^3} \lesssim ||\hat{\mathbf{v}}(t)||_{H^3}, \quad ||\mathring{\mathbf{v}}(A_t(\cdot), t) - \mathring{\mathbf{v}}(\cdot, t)||_{L^2} \lesssim ||\mathring{\mathbf{v}}(t)||_{L^2}$$

as for [47, (3.24) & (3.25)]. We observe

$$\hat{\mathbf{v}}(A_t^{-1}(x), t) - \mathring{\mathbf{v}}(x, t) = (\hat{\mathbf{w}}(B_t(x), t) - \hat{\mathbf{w}}(A_t^{-1}(x), t)) + (\phi(A_t^{-1}(x)) - \phi(B_t(x)))$$

and estimate with (5.39) and the mean value theorem,

$$||\hat{\mathbf{v}}(A_t^{-1}(\cdot), t) - \mathring{\mathbf{v}}(\cdot, t)||_{H^3} \lesssim ||A_t^{-1} - B_t||_{H^3} \lesssim ||\gamma_x||_{H^3}.$$

On the other hand,

$$\begin{aligned}\hat{\mathbf{v}}(x,t) - \mathring{\mathbf{v}}(A_t(x),t) = &- \hat{\mathbf{w}}(x,t) + \hat{\mathbf{w}}(x + \gamma(A_t(x),t) - \gamma(x,t),t) \\ &+ (-\phi(x) + \phi(x + \gamma(A_t(x),t) - \gamma(x,t)))\end{aligned}$$

and by and the mean value theorem and with

$$||\gamma(A_t(\cdot),t) - \gamma(\cdot,t)||_{L^2} \lesssim ||\gamma_x(t)||_{L^2},$$

which follows by (5.38), we obtain

$$||\hat{\mathbf{v}}(\cdot,t) - \mathring{\mathbf{v}}(A_t(\cdot),t)||_{L^2} \lesssim ||\gamma_x(t)||_{L^2}.$$

Putting everything together, we arrive at (5.34) and (5.35). The relation (5.36) follows analogously. □

5.5. Nonlinear stability analysis

5.5.1. The nonlinear iteration

We first establish nonlinear stability of $\mathbf{w}$.

Theorem 5.5.1. *Let $\mathbf{w}_0 \in H^6_{per}(0,T)$. Let $\mathbf{w}$, T_{max} be as in Proposition 5.4.1 and let σ and $t_{max,\sigma}$ as in Proposition 5.4.4. There exist $C, \delta_2, \varepsilon_p > 0$ such that for*

$$E_p := ||\mathbf{w}_0||_{H^6_{per}(0,T)} < \varepsilon_p,$$

the functions $\mathbf{w}(t)$ and $\sigma(t)$ exist globally, i.e. $t_{max,\sigma} = T_{max} = \infty$, and

$$|\sigma(t)|, ||\mathbf{w}(t) - \phi||_{H^6_{per}(0,T)} \le CE_p, \quad |\sigma_t(t)|, ||\hat{\mathbf{w}}(t)||_{H^6_{per}(0,T)}, ||\mathring{\mathbf{w}}(t)||_{H^6_{per}(0,T)} \le Ce^{-\delta_2 t}E_p,$$

for all $t \ge 0$.

Proof. The estimates on $|\sigma(t)|$, $|\sigma_t(t)|$ and $||\mathring{\mathbf{w}}(t)||_{H^6_{\mathrm{per}}(0,T)}$ follow by a standard procedure, see [64] and [96], taking Proposition 5.3.1 and Lemma 5.4.5 for (5.25) into account. We observe that

$$||\hat{\mathbf{w}}(t)||_{H^6_{\mathrm{per}}(0,T)} = ||\hat{\mathbf{w}}(\cdot + \sigma(t),t)||_{H^6_{\mathrm{per}}(0,T)} = ||\mathring{\mathbf{w}}(t)||_{H^6_{\mathrm{per}}(0,T)}$$

for all $t \ge 0$, to finish the proof. □

Let $\mathbf{w}_0 \in H^6_{\mathrm{per}}(0,T)$ and ε_p as in Theorem 5.5.1. We set $E_l = ||\mathbf{v}_0||_{H^3}$ and choose $0 < c < \frac{1}{2}$ as in Proposition 5.4.6. Under the assumption $E_p < \varepsilon_p$, we consider the template function $\eta : [0, t_{\mathrm{max},\gamma}) \to \mathbb{R}$ given by

$$\eta(t) = \sup_{0\le s\le t}\Big((1+s)^{\frac{1}{2}}\Big(||\mathring{\mathbf{v}}(s)||_{H^3(\mathbb{R})} + ||(\gamma_x(s),\gamma_s(s))||_{H^4\times H^3}\Big) + ||\gamma(t)||_{L^2}\Big)$$

and show that there exists a constant $C \ge 1$ independent of E_l and E_p such that

$$\eta(t) \le C(E_l + \eta(t)^2 + \eta(t)E_p) \tag{5.40}$$

for every $t \in [0, t_{\max,\gamma})$ with $\eta(t) < \frac{c}{2}$. By Proposition 5.4.6, we have the property: if $t_{\max,\gamma} < \infty$, then

$$\limsup_{t \uparrow t_{\max,\gamma}} \eta(t) \geq \frac{c}{2}. \tag{5.41}$$

Furthermore, η is continuous by Proposition 5.4.6 and Corollaries 5.4.7 and 5.4.9 and monotonically increasing.

Iteration argument. Suppose we have proven (5.40). First we take $E_p < \min\{\varepsilon_p, \frac{1}{2C}\}$ which gives

$$\eta(t) \leq 2C(E_l + \eta(t)^2). \tag{5.42}$$

Now, choose $4C^2 E_l < \frac{c}{2}$. Assuming there exists some $t \in [0, t_{\max,\gamma})$ such that $\eta(t) \geq 4CE_l$, the continuity of η provides a t_0 with $\eta(t_0) = 4CE_l < \frac{c}{2}$. Therefore, (5.42) and $c \in (0, \frac{1}{2})$ imply

$$\eta(t_0) \leq 2C\Big(E_l + (16C^2 E_l)E_l\Big) < 4CE_l.$$

This is a contradiction and we arrive at

$$\sup_{t \in [0, t_{\max,\gamma})} \eta(t) \leq 4CE_l < \frac{c}{2} \tag{5.43}$$

and hence (5.41) cannot hold. We conclude that (5.43) holds with $t_{\max,\gamma} = \tau_{\max} = \infty$. This proves (5.16) and it suffices to justify (5.40). For this purpose, let $t \in [0, t_{\max,\gamma})$ with $\eta(t) < \frac{c}{2}$.

Bound on $\hat{\mathbf{v}}$. We first bound $||\hat{\mathbf{v}}(s)||_{H^3}$ for which we use (5.35), that is,

$$||\hat{\mathbf{v}}(s)||_{H^3} \lesssim (1+s)^{-\frac{1}{2}} \eta(s), \tag{5.44}$$

for $s \in [0, t]$. Together with the nonlinear bounds, Lemma 5.4.3, Proposition 5.3.2 and Theorem 5.5.1, we arrive at

$$\begin{aligned}
||\hat{\mathbf{v}}(s)||_{L^2} &\lesssim (1+s)^{-\frac{1}{2}} E_l + ||\gamma_x(s)||_{L^2} ||\hat{\mathbf{w}}(s)||_{L^\infty} + ||\gamma_x(s)||_{H^1} ||\hat{\mathbf{v}}(s)||_{L^2} \\
&\quad + \int_0^s (1+s-\tau)^{-\frac{3}{4}} ||\mathcal{R}_3(\hat{\mathbf{w}}(\tau), \hat{\mathbf{v}}(\tau), \gamma(\tau))||_{L^1 \cap L^2} \, d\tau \\
&\quad + \int_0^s (1+s-\tau)^{-\frac{1}{2}} \Big(||\sigma_t(\tau) \hat{\mathbf{v}}_x(\tau)||_{L^2} + ||\mathcal{T}(\hat{\mathbf{w}}(\tau), \gamma(\tau))||_{L^2} \\
&\qquad\qquad + ||(1 - \gamma_x(\tau)) \mathcal{R}_{2,2}(\hat{\mathbf{w}}(\tau), \hat{\mathbf{v}}(\tau))||_{L^2} \Big) \, d\tau \\
&\lesssim (1+s)^{-\frac{1}{2}} E_l + (1+s)^{-\frac{1}{2}} \eta(t)^2 + (1+s)^{-\frac{1}{2}} \eta(t) E_p \\
&\quad + \eta(t)^2 \int_0^s (1+s-\tau)^{-\frac{3}{4}} (1+\tau)^{-1} d\tau \\
&\quad + \eta(t) E_p \int_0^s (1+s-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{1}{2}} e^{-\delta_2 \tau} d\tau \\
&\lesssim (1+s)^{-\frac{1}{2}} \Big(E_l + \eta(t)^2 + \eta(t) E_p \Big),
\end{aligned} \tag{5.45}$$

for $s \in [0, t]$, where we used that $\eta(t) \leq \frac{1}{2}$.

Bounds on γ. With (5.44), Proposition 5.3.2, Lemma 5.4.3 and Theorem 5.5.1, we estimate

$$\begin{aligned}
||\partial_s^k\partial_x^l\gamma(s)||_{L^2} &\leq ||\partial_s^k\partial_x^l s_p(s)||_{L^2\to L^2}||\mathbf{v}_0||_{L^2} \\
&\quad + \int_0^s ||\partial_s^k\partial_x^l s_p(s-\tau)||_{L^1\to L^2}||\mathcal{R}_3(\hat{\mathbf{w}}(\tau),\hat{\mathbf{v}}(\tau),\gamma(\tau))||_{L^1}\,d\tau \\
&\quad + \int_0^s ||\partial_s^k\partial_x^l s_p(s-\tau)||_{L^2\to L^2}\Big(||\sigma_t(\tau)\hat{\mathbf{v}}_x(\tau)||_{L^2} + ||\mathcal{T}(\hat{\mathbf{w}}(\tau),\gamma(\tau))||_{L^2} \\
&\qquad + ||(1-\gamma_x(\tau))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(\tau),\hat{\mathbf{v}}(\tau))||_{L^2}\Big)\,d\tau \\
&\lesssim (1+s)^{-\frac{k+l}{2}}E_l + \eta(s)^2\int_0^s (1+s-\tau)^{-\frac{1}{4}-\frac{k+l}{2}}(1+\tau)^{-1}d\tau \\
&\quad + \eta(s)E_p\int_0^s (1+s-\tau)^{-\frac{k+l}{2}}(1+\tau)^{-\frac{1}{2}}e^{-\delta_2\tau}d\tau \\
&\lesssim (E_l+\eta(t)^2+\eta(t)E_p)\begin{cases} 1, & \text{if } l+k=0 \\ (1+s)^{-\frac{1}{2}}, & \text{if } l+k=1 \\ (1+s)^{-1}, & \text{otherwise,}\end{cases}
\end{aligned} \tag{5.46}$$

for every $s\in[0,t]$.

Bounds on $\mathring{\mathbf{v}}$. Invoking (5.45) and (5.46), (5.34) yields

$$||\mathring{\mathbf{v}}(s)||_{L^2} \lesssim (1+s)^{-\frac{1}{2}}\Big(E_l+\eta(t)^2+\eta(t)E_p\Big)$$

for all $0\leq s\leq t$. Finally, with Proposition 5.4.11 and (5.46), we obtain

$$||\mathring{\mathbf{v}}(s)||_{H^3} \lesssim (1+s)^{-\frac{1}{2}}\Big(E_l+\eta(t)^2+\eta(t)E_p\Big),$$

for all $0\leq s\leq t$, using $\eta(t)\leq\frac{1}{2}$.

We have shown the key inequality (5.40).

Remark 5.5.2. *The proof of (5.40) reveals that the nonlinear iteration argument closes as long as $|\sigma_t(t)|$ and $||\hat{\mathbf{w}}(t)||_{H^6_{per}(0,T)}$ decay of order $(1+t)^{-\kappa}$ for some $\kappa>\frac{1}{2}$.*

5.5.2. Refined L^∞-estimates

By Proposition 5.3.3 and the nonlinear bounds in Lemma 5.4.3 and reinserting the L^2-estimates (5.16), we obtain

$$\begin{aligned}
||\gamma_x(t)||_{L^\infty} &\leq ||\partial_x s_p(t)||_{L^2\to L^\infty}||\mathbf{v}_0||_{L^2} + \int_0^t ||\partial_x s_p(t-s)||_{L^1\to L^\infty}||\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s))||_{L^1}\,ds \\
&\quad + \int_0^t ||\partial_x s_p(t-s)||_{L^2\to L^\infty}\Big(||\sigma_t(s)\hat{\mathbf{v}}_x(s)||_{L^2} + ||\mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))||_{L^2} \\
&\qquad + ||(1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s))||_{L^2}\Big)\,ds \\
&\lesssim (1+t)^{-\frac{3}{4}}E_l + E_l\Big(\int_0^t (1+t-s)^{-1}(1+s)^{-1} + (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{2}}e^{-\delta_2 s}\,ds\Big) \\
&\lesssim (1+t)^{-\frac{3}{4}}E_l,
\end{aligned}$$

for all $t \geq 0$. Furthermore, we have

$$\begin{aligned}
||\hat{\mathbf{v}}(t)||_{L^\infty} &\lesssim (1+t)^{-\frac{3}{4}} E_l + \int_0^t (1+t-s)^{-1} ||\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s))||_{L^1 \cap L^2}\, ds \\
&\quad + \int_0^t e^{-\delta_1(t-s)} ||\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s))||_{H^1}\, ds \\
&\quad + \int_0^t (1+t-s)^{-\frac{3}{4}} \Big(||\sigma_t(s)\hat{\mathbf{v}}_x(s)||_{H^1} \\
&\qquad\qquad + ||(1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s))||_{H^1} + ||\mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))||_{H^1} \Big)\, ds \\
&\lesssim (1+t)^{-\frac{3}{4}} E_l + E_l \Big(\int_0^t (1+t-s)^{-1}(1+s)^{-1} + (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{2}} e^{-\delta_2 s}\, ds \Big) \\
&\lesssim (1+t)^{-\frac{3}{4}} E_l,
\end{aligned}$$

for all $t \geq 0$. With the help of (5.36), we deduce

$$||\mathring{\mathbf{v}}(t)||_{L^\infty} \lesssim (1+t)^{-\frac{3}{4}} E_l,$$

for all $t \geq 0$. Finally, we observe that for $\sigma_* := \int_0^\infty \sigma_s(s)\, ds$, we have

$$|\sigma_* - \sigma(t)| \leq \int_t^\infty |\sigma_s(s)|\, ds \lesssim e^{-\delta_1 t} E_p$$

and therefore

$$||\phi(\cdot + \sigma_* + \gamma(\cdot, t), t) - \phi(\cdot + \sigma(t) + \gamma(\cdot, t), t)||_{L^\infty} \lesssim e^{-\delta_1 t} E_p,$$

for all $t \geq 0$. As described in Section 5.2.6, Theorem 5.2.3 follows.

5.6. Discussion

5.6.1. Applicability of the scheme to viscous conservation laws

A satisfying L^∞-stability theory considering C_{ub}-perturbations is established in parabolic reaction-diffusion systems [84] which can be extended beyond the parabolic setting as shown in the context of the FitzHugh-Nagumo system [5]. On the other hand, there are crucial obstacles in establishing pure L^∞-stability results for viscous conservation laws as described in [5, 51]. An interesting ingredient of the L^2-analysis in [59] is that the authors introduce a sum of spatio-temporal modulation functions to capture more than one critical mode arising from the presence of the conservation laws. Therefore, a key difficulty lies in the fact that the critical dynamics is governed by a coupled Whitham system for which one cannot immediately apply the Cole-Hopf transform as done in the scalar case for which this Whitham system reduces to the Burgers' equation [5, 84]. This begs the question of whether we can apply our $L^2_{\text{per}}(0,T) \oplus L^2(\mathbb{R})$-scheme to this setting, cf. [59] and [58]. In order to generate a result like Theorem 5.2.3 in this setting, the question reduces to whether such a sum of spatio-temporal modulation functions is compatibel with our approach. As the $L^2_{\text{per}}(0,T)$-theory can be settled by a standard procedure, we strongly expect that it is relatively straightforward to allow for $L^2_{\text{per}}(0,T) \oplus L^2(\mathbb{R})$-perturbations by precisely following our nonlinear analysis, in particular using the modified versions of inverse- and forward-modulated perturbations in §5.4.2 and §5.4.5, and by respecting the semigroup decomposition and estimates established in [59, pp. 149] and the nonlinear damping estimate in [59, pp. 146].

5.6.2. Uniformly subharmonic plus localized perturbations

Considering the uniformly subharmonic nonlinear stability result for the Lugiato-Lefever equation in [47], the question arises whether we can show a nonlinear stability result involving perturbations $\mathbf{w}_0+\mathbf{v}_0$ with $\mathbf{w}_0 \in H^6_{\mathrm{per}}(0,NT)\cap L^1_{\mathrm{per}}(0,NT)$, $\mathbf{v}_0 \in H^3(\mathbb{R})$ and which is uniform in $N \in \mathbb{N}$. We note that as in [47] the additional condition $L^1_{\mathrm{per}}(0,NT)$ is crucial in [47] to guarantee the uniformity in N.

Suppose (H1) and (D1)-(D3). Now, given a solution $\mathbf{u} = \mathbf{w} + \mathbf{v}$ of (5.1) where $\mathbf{w}$ solves (5.3) and $\mathbf{v}$ solves (5.4) with $\mathbf{u}(0) = \mathbf{w}_0 + \mathbf{v}_0$, we broadly sketch a possible scheme for the claimed stability estimate

$$||\mathbf{u}(t)||_{L^\infty} \lesssim (1+t)^{-\frac{3}{4}}\Big(||\mathbf{w}_0||_{H^6_{\mathrm{per}}(0,NT)\cap L^1_{\mathrm{per}}(0,NT)} + ||\mathbf{v}_0||_{H^3(\mathbb{R})}\Big), \quad t \geq 0, \tag{5.47}$$

uniformly in $N \in \mathbb{N}$. Keeping the main lines of this paper, we might introduce the inverse-modulated perturbations

$$\begin{aligned}\hat{\mathbf{w}}(x,t) &= \mathbf{w}(x-\sigma(t)-\gamma_1(x,t),t) - \phi,\\ \hat{\mathbf{v}}(x,t) &= \mathbf{u}(x-\sigma(t)-\gamma_1(x,t)-\gamma_2(x,t),t) - \hat{\mathbf{w}}(x,t) - \phi(x)\end{aligned}$$

and the forward-modulated perturbations

$$\begin{aligned}\mathring{\mathbf{w}}(x,t) &= \mathbf{w}(x,t) - \phi(x+\sigma(t)+\gamma_1(x,t)),\\ \mathring{\mathbf{v}}(x,t) &= \mathbf{u}(x,t) - \hat{\mathbf{w}}(x+\sigma(t)+\gamma_1(x,t)+\gamma_2(x,t),t) - \phi(x+\sigma(t)+\gamma_1(x,t)+\gamma_2(x,t)),\end{aligned}$$

with modulation functions

$$\sigma : [0,\infty) \to \mathbb{R}, \gamma_1 : [0,\infty) \to L^2_{\mathrm{per}}(0,NT) \text{ and } \gamma_2 : [0,\infty) \to L^2(\mathbb{R}).$$

To derive suitable perturbation equations for $\hat{\mathbf{v}}$ and $\mathring{\mathbf{v}}$, we refer to Appendices 5.A and 5.B. Finally, with the established decay rates [47, Theorem 1.4], that is,

$$||\partial_x\gamma_1(t)||_{L^2_{\mathrm{per}}(0,NT)}, |\sigma_t(t)|, ||\hat{\mathbf{w}}(t)||_{L^2_{\mathrm{per}}(0,NT)} \sim O((1+t)^{-\frac{3}{4}})$$

uniformly in N, one may invoke Remark 5.5.2 and [47, Proposition 3.7]). It remains to prove versions of Proposition 5.4.11 and Lemma 5.4.12 to deduce (5.47).

5.6.3. Nonlocalized phase modulations

An interesting feature of the modulational stability estimate (5.14) is that we capture the most critical dynamics of the periodic wave by a phase modulation from $\mathbb{R} + H^3(\mathbb{R})$. This class of functions covers the simplest nontrivial nonlocalized phase perturbations one could think of. Therefore, an interesting open question is whether we can allow for perturbations from $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$ as well as (L^∞-large) initial modulations $\gamma_0 - \sigma_*$ such that $||\gamma_0'||_{H^3}$ is small to conclude an estimate such as (5.14). We refer to [105] for a nonlinear stability result against localized perturbations in this context, allowing for a nonlocalized initial phase modulation.

5.6.4. Fully nonlocalized perturbations

Local well-posedness results in nonlinear Schrödinger-type equations, possibly with (periodic) potentials, have been established for initial data from the modulation space $M^m_{\infty,1}(\mathbb{R})$, see e.g. [23, 65], [41, pages 245-252] and references therein. At the same time, global in time results are widely open. Due to the additional dissipativity in (5.1) compared to the classical cubic nonlinear Schrödinger equation, we expect the possibility to extend our result to perturbations from $M^m_{\infty,1}(\mathbb{R})$ for sufficiently large $m \in \mathbb{N}$. This might be achieved by conceptually following the lines of [5] through the complex inversion formula of the semigroup, see also Remark 5.2.2. Nevertheless, it turns out that high-frequency damping and regularity control are delicate challenges and it seems that essentially new ideas have to be developed.

5.A. Appendix: Derivation of (5.23)

We write $\hat{\mathbf{u}}(x,t) = \mathbf{u}(x-\sigma(t)-\gamma(x,t),t) - \phi(x)$, insert (5.22) and [47, (3.4)] to obtain

$$\begin{aligned}(\partial_t - \mathcal{L}_0)(\hat{\mathbf{v}}) &= (\partial_t - \mathcal{L}_0)\hat{\mathbf{u}} - (\partial_t - \mathcal{L}_0)\hat{\mathbf{w}}\\ &= -(\partial_t - \mathcal{L}_0)(\phi'\gamma) + (1-\gamma_x)\mathcal{R}_1(\hat{\mathbf{u}}) - \mathcal{R}_1(\hat{\mathbf{w}}) + \sigma_t\hat{\mathbf{w}}_x\\ &\quad + \partial_x\mathcal{S}(\hat{\mathbf{u}},\gamma,\gamma_t,\sigma_t) + \partial_x^2\mathcal{P}(\hat{\mathbf{u}},\gamma) + (\partial_t - \mathcal{L}_0)(\gamma_x\hat{\mathbf{u}}),\end{aligned}$$

where, recalling $\hat{\mathbf{u}} = \hat{\mathbf{v}} + \hat{\mathbf{w}}$,

$$\begin{aligned}\widetilde{\mathcal{S}}(\hat{\mathbf{u}},\gamma,\gamma_t,\sigma_t) &= -\gamma_t\hat{\mathbf{u}} - \sigma_t\hat{\mathbf{u}} + \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{u}} - \frac{\gamma_x^2}{1-\gamma_x}\phi'\right)\\ &= \widetilde{\mathcal{S}}(\hat{\mathbf{v}},\gamma,\gamma_t,\sigma_t) - \gamma_t\hat{\mathbf{w}} - \sigma_t\hat{\mathbf{w}} + \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{w}}\right)\\ \mathcal{P}(\hat{\mathbf{u}},\gamma) &= -\beta\mathcal{J}\left(\gamma_x + \frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{u}} = \mathcal{P}(\hat{\mathbf{v}},\gamma) + \mathcal{P}(\hat{\mathbf{w}},\gamma).\end{aligned}$$

We emphasize that the critical terms $\sigma_t\hat{\mathbf{w}}$ and $(\partial_t - \mathcal{L}_0)(\phi'\sigma)$ cancel out. We arrive at

$$\begin{aligned}(\partial_t - \mathcal{L}_0)(\hat{\mathbf{v}} + \phi'\gamma - \gamma_x\hat{\mathbf{w}} - \gamma_x\hat{\mathbf{v}}) =& \mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) + \partial_x\mathcal{S}(\hat{\mathbf{v}},\gamma) + \partial_x^2\mathcal{P}(\hat{\mathbf{v}},\gamma)\\ &- \sigma_t\hat{\mathbf{v}}_x + (1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}}) + \mathcal{T}(\hat{\mathbf{w}},\gamma)\end{aligned}$$

with

$$\begin{aligned}\mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) &= (1-\gamma_x)(\mathcal{R}_1(\hat{\mathbf{v}}+\hat{\mathbf{w}}) - \mathcal{R}_1(\hat{\mathbf{w}}) - \mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}})) = (1-\gamma_x)\mathcal{R}_{2,1}(\hat{\mathbf{v}},\hat{\mathbf{w}}),\\ \mathcal{S}(\hat{\mathbf{v}},\gamma) &= -\gamma_t\hat{\mathbf{v}} + \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{v}} - \frac{\gamma_x^2}{1-\gamma_x}\phi'\right),\\ \mathcal{T}(\hat{\mathbf{w}},\gamma) &= -\gamma_x\mathcal{R}_1(\hat{\mathbf{w}}) - \partial_x\left(\gamma_t\hat{\mathbf{w}} - \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{w}}\right)\right) - \partial_x^2\left(\beta\mathcal{J}\left(\gamma_x + \frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{w}}\right).\end{aligned}$$

5.B. Appendix: Derivation of (5.31)

We recall that

$$\mathring{\mathbf{v}}(x,t) = \mathbf{u}(x,t) - \mathbf{w}(x+\gamma(x,t),t),$$

set $\mathring{\phi}(x) = \phi(x + \gamma(x,t))$ and use the notions (5.6), (5.7) and (5.8). Furthermore, we write

$$\mathcal{D}(\mathbf{u}) = \mathcal{J}\left(\begin{pmatrix}-\beta & 0\\ 0 & -\beta\end{pmatrix}\mathbf{u}_{xx} + \begin{pmatrix}-\alpha & 0\\ 0 & -\alpha\end{pmatrix}\mathbf{u}\right) - \mathbf{u}.$$

Using that $\mathbf{u}$ and $\mathbf{w}$ are a solutions of (5.2), we derive

$$\begin{aligned}
&(\partial_t - \mathcal{L}_0(\mathring{\phi}))\mathring{\mathbf{v}}(t)\\
&= (\partial_t - \mathcal{D})(\mathbf{u}(t)) - (\partial_t - \mathcal{D})(\mathbf{w})(\cdot + \gamma(\cdot,t),t) - \mathcal{N}'(\mathring{\phi})\mathring{\mathbf{v}}(t) + \mathcal{R}_5(\mathbf{w}(t),\gamma(t),\gamma_t(t))\\
&= \mathcal{N}(\mathring{\mathbf{v}}(t) + \tilde{\mathbf{w}}(\cdot + \gamma(\cdot,t),t) + \mathring{\phi}) - \mathcal{N}(\tilde{\mathbf{w}}(\cdot + \gamma(\cdot,t),t) + \mathring{\phi})\\
&\qquad - \mathcal{N}'(\mathring{\phi})\mathring{\mathbf{v}}(t) + \mathcal{R}_5(\mathbf{w}(t),\gamma(t),\gamma_t(t))\\
&= \mathcal{R}_2(\mathring{\phi})(\tilde{\mathbf{w}}(\cdot + \gamma(\cdot,t),t), \mathring{\mathbf{v}}(t)) + \mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))
\end{aligned}$$

with

$$\begin{aligned}
&\mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))\\
&= -\mathbf{w}_x(\cdot + \gamma(\cdot,t),t)\gamma_t(t)\\
&\qquad - \beta\mathcal{J}\Big(\mathbf{w}_x(\cdot + \gamma(\cdot,t),t)(\gamma_{xx}(t)) + \mathbf{w}_{xx}(\cdot + \gamma(\cdot,t),t)(2\gamma_x(t) + \gamma_x(t)^2)\Big)\\
&= -\tilde{\mathbf{w}}_x(\cdot + \gamma(\cdot,t),t)\gamma_t(t) - \phi'(\cdot + \gamma(\cdot,t),t)\gamma_t(t)\\
&\qquad - \beta\mathcal{J}\Big(\tilde{\mathbf{w}}_x(\cdot + \gamma(\cdot,t),t)(\gamma_{xx}(t)) + \tilde{\mathbf{w}}_{xx}(\cdot + \gamma(\cdot,t),t)(2\gamma_x(t) + \gamma_x(t)^2)\\
&\qquad\qquad + \phi'(\cdot + \gamma(\cdot,t),t)(\gamma_{xx}(t)) + \phi''(\cdot + \gamma(\cdot,t),t)(2\gamma_x(t) + \gamma_x(t)^2)\Big).
\end{aligned}$$

Recall that $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$ denotes the unmodulated perturbation of $\mathbf{w}$.

5.C. Appendix: Proof of Proposition 5.4.6

We recall the Duhamel formula (5.26) given by

$$\begin{aligned}
\gamma(t) = s_p(t)\mathbf{v}_0 + \int_0^t s_p(t-s)\Big(&\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(s)\\
&+ (1 - \gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big)\,ds
\end{aligned}$$

whereas

$$\hat{\mathbf{v}}(x,t) = \mathbf{u}(x - \sigma(t) - \gamma(x,t),t) - \hat{\mathbf{w}}(x,t) - \phi(x).$$

To prevent confusion, we write

$$\hat{\mathbf{v}}(t) = \hat{\mathbf{v}}(\gamma(t),t)$$

and for the sake of readability, we introduce

$$\begin{aligned}
\tilde{\mathcal{N}}(t,\sigma(s),\gamma(s),s) = s_p(t-s)\Big(&\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(s)\\
&+ (1 - \gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big).
\end{aligned}$$

We do a contraction argument. For this purpose, we establish nonlinear bounds.

Lemma 5.C.1. *Fix* $0 \le \tau_1 \le \tau_2 < \min\{\tau_{max}, t_{max,\sigma}\}$. *Let* $j,k \in \mathbb{N}_0$ *and fix* $K > 0$. *Suppose that*

$$\sup_{t\in[0,\tau_2]} (||\mathbf{w}(t)||_{W^{1,\infty}} + |\sigma_t(t)| + ||\mathbf{v}(t)||_{W^{1,\infty}}) \le K.$$

There exist constants $C > 0$ *and* $C_{j,k} > 0$ *such that we have the bounds*

$$\sup_{s\in[\tau_1,\tau_2]} ||\hat{\mathbf{v}}(\gamma_1(s), s) - \hat{\mathbf{v}}(\gamma_2(s), s)||_{L^2} \le C \sup_{s\in[\tau_1,\tau_2]} ||\gamma_1(s) - \gamma_2(s)||_{L^2}, \tag{5.48}$$

and

$$\begin{aligned} \sup_{t\in[\tau_1,\tau_2]} \sup_{s\in[\tau_1,t]} &||\partial_t^j \partial_x^k (\tilde{\mathcal{N}}(t, \sigma(s), \gamma_2(s), s) - \tilde{\mathcal{N}}(t, \sigma(s), \gamma_2(s), s))||_{L^2} \\ &\le C_{j,k} \sup_{s\in[\tau_1,\tau_2]} (||\gamma_1(s) - \gamma_2(s)||_{H^2} + ||\partial_t\gamma_1(s) - \partial_t\gamma_2(s)||_{L^2}), \end{aligned} \tag{5.49}$$

for any $\gamma_1, \gamma_2 \in C([\tau_1, \tau_2]; H^5(\mathbb{R})) \times C^1([\tau_1, \tau_2]; H^3(\mathbb{R}))$ *with*

$$\sup_{t\in[\tau_1,\tau_2]} ||\partial_x\gamma_1(t)||_{L^\infty}, ||\partial_x\gamma_1(t)||_{L^\infty} \le \frac{1}{2}.$$

Proof. Let $t \in [\tau_1, \tau_2]$ and $s \in [\tau_1, t]$. We rewrite

$$\begin{aligned} \hat{\mathbf{v}}(x, \gamma_1(s), s) - \hat{\mathbf{v}}(x, \gamma_2(s), s) = &\mathbf{v}(x - \sigma(s) - \gamma_1(x, s), s) - \mathbf{v}(x - \sigma(s) - \gamma_2(x, s), s) \\ &+ \mathbf{w}(x - \sigma(s) - \gamma_2(x, s), s) - \mathbf{w}(x - \sigma(s) - \gamma_1(x, s), s) \end{aligned}$$

and since $\mathbf{w}(x - \sigma(s) - \gamma(x, s), s) - \phi(x) = \hat{\mathbf{w}}(x - \gamma(x, s), s)$, this yields (5.48) by the mean value theorem. Let $j, k \in \mathbb{N}_0$. Recalling the choice of $\mathcal{R}_3 = \mathcal{Q} + \partial_x\mathcal{S} + \partial_x^2\mathcal{P}$ and the estimates on $s_p(t)$ from Proposition 5.3.2, we obtain

$$\begin{aligned} &||\partial_t^j \partial_x^k s_p(t - s)(\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma_1(s), s), \gamma_1(s)) - \mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma_2(s), s), \gamma_2(s)))||_{L^2} \\ &\quad \le C_{j,k}(||\gamma_1(s) - \gamma_2(s)||_{H^2} + ||\partial_t\gamma_1(s) - \partial_t\gamma_2(s)||_{L^2}). \end{aligned}$$

by taking derivatives on $s_p(t - s)$ and (5.48). Next, with the Cauchy-Schwarz inequality, we find

$$\begin{aligned} &||\partial_t^j \partial_x^k s_p(t - s)\Big((1 - \partial_x\gamma_1(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma_1(s), s)) \\ &\quad - (1 - \partial_x\gamma_2(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma_2(s), s))\Big)||_{L^2} \le C_{j,k}||\gamma_1(s) - \gamma_2(s)||_{H^1}. \end{aligned}$$

Moreover, we have

$$||\partial_t^j \partial_x^k s_p(t - s)\Big(\sigma_t(s)(\hat{\mathbf{v}}_x(\gamma_1(s), s) - \hat{\mathbf{v}}_x(\gamma_2(s), s))\Big)||_{L^2} \le C_{j,k}||\gamma_1(s) - \gamma_2(s)||_{L^2}.$$

Together with

$$\begin{aligned} &||\partial_t^j \partial_x^k s_p(t - s)\Big(\mathcal{T}(\hat{\mathbf{w}}(s), \gamma_1(s)) - \mathcal{T}(\hat{\mathbf{w}}(s), \gamma_1(s))\Big)||_{L^2} \\ &\quad \le C_{j,k}(||\gamma_1(s) - \gamma_2(s)||_{H^2} + ||\partial_t\gamma_1(s) - \partial_t\gamma_2(s)||_{L^2}), \end{aligned}$$

again taking derivatives on $s_p(t - s)$, we arrive at (5.49). □

By the choice of s_p, we immediately have that $\gamma(t) = 0$ for $t \in [0,1]$. We need to justify that we can extend the modulation function γ to a maximal time such that the alternative (5.29) holds.

Proof of Proposition 5.4.6. Let $\tilde{\gamma}$ be a solution of (5.26) on $[0, t_0]$ with some $t_0 > 0$. Lemma 5.C.1 tells us that

$$\gamma \mapsto s_p(t)\mathbf{v}_0 + \int_{t_0}^{t} s_p(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(s) \\ + (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))\Big)\, ds \tag{5.50}$$

defines a contraction on

$$X_{t_0,\tau_0} = \Big\{\gamma \in C([t_0, t_0+\tau_0]; H^5(\mathbb{R})) \cap C^1([t_0, t_0+\tau_0]; H^3(\mathbb{R})) : \\ \sup_{s\in[t_0,t_0+\tau_0]} ||(\gamma(s), \gamma_s(s))||_{H^5\times H^3} < \frac{c}{2}\Big\}$$

for sufficiently small $\tau_0 > 0$. By the Banach fixed point theorem, there exists a unique solution

$$\gamma \in C([t_0, t_0+\tau_0]; H^5(\mathbb{R})) \cap C^1([t_0, t_0+\tau_0]; H^3(\mathbb{R}))$$

of

$$\gamma(t) = s_p(t)\mathbf{v}_0 + \int_{0}^{t_0} s_p(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\tilde{\gamma}(s), s), \tilde{\gamma}(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(\tilde{\gamma}(s), s) \\ + (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\tilde{\gamma}(s), s)) + \mathcal{T}(\hat{\mathbf{w}}(s), \tilde{\gamma}(s))\Big)\, ds \\ + \int_{t_0}^{t} s_p(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma(s), s), \gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(\gamma(s), s) \\ + (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma(s), s)) + \mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))\Big)\, ds \tag{5.51}$$

for $t \in [t_0, t_0 + \tau_0]$.

Since (5.50) defines a contraction on X_{t_1,t_2} for every $t_1 \in (0, \min\{\tau_{\max}, t_{\max,\sigma}\})$ and sufficiently small $t_2 > 0$, we conclude that

$$\gamma(t) = \begin{cases} \tilde{\gamma}(t), & t \in [0, t_0] \\ \gamma(t), & t \in [t_0, t_0+\tau_0] \end{cases} \tag{5.52}$$

is the unique solution of (5.26) on $[0, t_0 + \tau_0]$ such that $\gamma \in X_{t_0,\tau_0}$. Now, there exists a maximal time $t_{\max,\gamma} \in (0, \min\{\tau_{\max}, t_{\max,\sigma}\}]$ such that (5.28) holds. Assume that $t_{\max,\gamma} < \min\{\tau_{\max}, t_{\max,\sigma}\}$. If (5.29) fails, then there exists a unique solution $\tilde{\gamma}$ of (5.26) with $\sup_{t\in[0,t_{\max,\gamma})} ||(\tilde{\gamma}(t), \tilde{\gamma}_t(t))||_{H^5\times H^3} < \frac{c}{2}$. This again allows to solve (5.51) on $X_{t_{\max,\gamma},\tau_0'}$ for some small τ_0' giving a solution of (5.26) via (5.52) on $[0, t_{\max,\gamma} + \tau_0')$ with $||(\gamma(s), \gamma_s(s))||_{H^5\times H^3} < \frac{c}{2}$ for all $t \in [0, t_{\max,\gamma} + \tau_0')$. This contradicts the maximality of $t_{\max,\gamma}$ and we conclude (5.29). □

End of Paper

5.D. Appendix: The spaces $L^2_{\text{PER}}(0,T)$ and $L^2_{\text{PER}}(0,T) \oplus L^2(\mathbb{R})$

Let $l, k \in \mathbb{N}_0$ and $T > 0$. We define, see [94, Section 5.2.2],

$$H^l_{\text{per}}(0,T) = \overline{C^\infty_{\text{per}}(\mathbb{R})} \text{ w.r.t. } ||\cdot||_{H^l([0,T])}$$

where $||u||_{H^l([0,T])} := \sum_{i=0}^l ||\partial_x^i u||_{L^2([0,T])}$ and

$$\begin{aligned} &C^\infty_{\text{per}}(\mathbb{R}) \\ &\quad = \{u : [0,T] \to \mathbb{R}^2 : \exists \tilde{u} \in C^\infty(\mathbb{R};\mathbb{R}^2) \text{ s.t. } u = \tilde{u}_{|[0,T]} \text{ and } \tilde{u}(x+T) = u(x) \text{ for all } x \in \mathbb{R}\}. \end{aligned}$$

The next proposition in particular shows that we may naturally extend an $H^l_{\text{per}}(0,T)$-function to the extended real line.

Proposition 5.D.1. *We have the following properties.*

(i) *We can characterize: $u \in H^l_{per}(0,T)$ if and only if there exists a unique measurable function $\tilde{u} : \mathbb{R} \to \mathbb{R}^2$ s.t. $u = \tilde{u}_{[mT,mT+T]}(\cdot + mT)$ and $\tilde{u}_{[mT,mT+T]}(\cdot + mT) \in H^l_{per}(0,T)$ for all $m \in \mathbb{Z}$.*

(ii) *$H^l_{per}(0,T)$ is a Hilbert space with scalar product $\langle u,v\rangle_{H^l_{per}(0,T)} := \langle u,v\rangle_{H^l([0,T])}$.*

(iii) *If $u \in L^2_{per}(0,T)$, then $u(x) = u(x+mT)$ for all $m \in \mathbb{Z}$ and for almost all $x \in \mathbb{R}$.*

(iv) *Let $l \in \mathbb{N}$. We have $H^l_{per}(0,T) \hookrightarrow C^{l-1}_{ub}(\mathbb{R})$.*

(v) *If $u \in H^1_{per}(0,T)$, then there exists $g \in L^2_{per}(0,T)$ s.t. $\int_0^T \varphi g\,dx = -\int_0^T u\partial_x\varphi\,dx$ for all $\varphi \in C^\infty([0,T])$. We identify $g = \partial_x u$.*

(vi) *We may use integration by parts, that is, for $u, v \in H^1_{per},(0,T)$, we find $\int_0^T u\partial_x v\,dx = -\int_0^T v\partial_x u\,dx$.*

(v) and (vi) analogously hold for $H^m_{per}(0,T)$ for all $m \in \mathbb{N}$.

Proof. Let $\tilde{u}$ as in (i). Then $u = \tilde{u}_{|[0,T]} \in H^l_{\text{per}}(0,T)$. Assume now that $u \in H^l_{\text{per}}(0,T)$ and set $\tilde{u}(x+mT) = u(x)$ for all $x \in [0,T)$ and $m \in \mathbb{Z}$. Let $\widetilde{u_n} \in C^\infty_{\text{per}}$ s.t. $\widetilde{u_n}_{|[0,T]} \to u$ in $H^l([0,T])$ as $n \to \infty$. Let $m \in \mathbb{Z}$. It is straightforward to check that $\widetilde{u_n}_{|[0,T]} = \widetilde{u_n}_{|[mT,mT+T]}(\cdot + mT) \to \tilde{u}_{|[mT,mT+T]}(\cdot + mT)$ in $H^l([0,T])$ as $n \to \infty$. Therefore, $\tilde{u}_{|[mT,mT+T]}(\cdot + mT) \in H^l_{\text{per}}(0,T)$ and $u = \tilde{u}_{|[mT,mT+T]}(\cdot + mT)$. Similarly, The uniqueness is clear by identifying measurable functions only differing on a nullset. The statement (iii) follows directly from (i). By Sobolev embedding, see [15, Theorem 8.8], (iv) and (ii) is simple to check. (v) and (vi) follow directly by density of $C^\infty_{\text{per}}(\mathbb{R})$. □

Lemma 5.D.2. *If $u \in L^2_{per}(0,T) \cap L^2(\mathbb{R})$, then $u = 0$.*

Proof. Assume that $u \in L^2(\mathbb{R})$, that is $|u|^2 \in L^1(\mathbb{R})$. By [94, Lemma 7.3.6], we find $u(x+mT) \to 0$ as $m \to \infty$ for almost all $x \in \mathbb{R}$. If additionally $u \in L^2_{\text{per}}(0,T)$, then together with Proposition 5.D.1 we obtain $u = 0$ almost everywhere. □

We introduce the vector space

$$\begin{aligned} &H^l_{\text{per}}(0,T) \oplus H^k(\mathbb{R}) \\ &\quad := \{u : \mathbb{R} \to \mathbb{R}^2 \text{ measurable} : \exists w \in H^l_{\text{per}}(0,T), \exists v \in H^k(\mathbb{R};\mathbb{R}^2) \text{ s.t. } u = v + w\} \end{aligned}$$

and equip it with the norm

$$||w+v||_{H^l_{\text{per}}(0,T)\oplus H^k(\mathbb{R})} := ||w||_{H^l_{\text{per}}(0,T)} + ||v||_{H^k(\mathbb{R})}. \tag{5.53}$$

Proposition 5.D.3. *We record the following statements.*

(i) *If $u \in H^l_{per}(0,T) \oplus H^k(\mathbb{R})$, then there are unique $w \in H^l_{per}(0,T)$ and one $v \in H^k(\mathbb{R})$ s.t. $u = v + w$. In particular, the norm (5.53) is well-defined.*

(ii) *$H^l_{per}(0,T) \oplus H^k(\mathbb{R})$ is a Hilbert space with the scalar product*

$$\langle v+v', w+w'\rangle_{H^l_{per}(0,T)\oplus H^k(\mathbb{R})} := \langle v, v'\rangle_{H^l_{per}(0,T)} + \langle w, w'\rangle_{H^k(\mathbb{R})}.$$

Proof. Supposing $u = v + w = v' + w'$ with $w, w' \in H^l_{\text{per}}(0,T)$ and $v', v \in H^k(\mathbb{R})$, we have $(v' - v) = -(w' - w)$. It follows that $v' - v \in H^k(\mathbb{R})$ but also $v' - v \in H^l_{\text{per}}(0,T)$. By Lemma 5.D.2, $-(w' - w) = v' - v = 0$ which shows (i). The statement (ii) is an obvious consequence of (i) and the fact that $H^l_{\text{per}}(0,T)$ and $H^k(\mathbb{R})$ are Hilbert spaces themselves. □

APPENDIX A

Uniformity in the Fourier inversion formula with applications to Laplace transforms

This chapter is the content of the preprint [3].

Start of Paper

Abstract. We systematically find conditions which yield locally uniform convergence in the Fourier inversion formula in one and higher dimensions. We apply the gained knowledge to the complex inversion formula of the Laplace transform to extend known results for Banach space-valued functions and, specifically, for C_0-semigroups.

Keywords. Banach space-valued functions; Fourier transform; Laplace transform; complex inversion formula; uniform convergence; local Hölder-continuity; strongly continuous semigroups; Favard spaces
Mathematics Subject Classification (2020). 44A10; 47D06; 26A16

A.1. Introduction

We fix a Banach space X for the entire manuscript.

For an exponentially bounded function $F : \mathbb{R}_+ := [0, \infty) \to X$ with exponential growth bound $\omega_0 \in \mathbb{R}$, i.e. there exists $C > 0$ such that $||F(t)|| \le Ce^{\omega_0 t}$ for all $t \in \mathbb{R}_+$, the Laplace transform is given by

$$\mathcal{L}(F)(\lambda) := \int_0^\infty e^{-\lambda s} F(s)\, ds, \qquad \mathrm{Re}(\lambda) > \omega_0. \tag{A.1}$$

For many decades, mathematicians have been interested in how we can formally and qualitatively invert (A.1). The importance of this question is not only raised by abstract curiosity. Nowadays, the Laplace transform is a standard and important tool to analyze differential equations: having properties of the transformed solution of a differential equation at hand, the validity of a suitable inversion formula may give further information about the solution itself. A prominent realization of this principle is the Hille-Yosida theorem in the context of linear evolution equations.

Let $\omega > \omega_0$. The complex inversion formula is defined by

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\omega-iR}^{\omega+iR} e^{\lambda t} \mathcal{L}(F)(\lambda)\, d\lambda \tag{A.2}$$

for $t \in \mathbb{R}_+$. We are interested under which regularity or localization conditions on F, we can guarantee

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\omega-iR}^{\omega+iR} e^{\lambda t} \mathcal{L}(F)(\lambda)\, d\lambda \text{ locally uniformly in } t \in \mathbb{R}_+. \tag{A.3}$$

In this paper, we consider general Banach space-valued functions and prove

Proposition A.1.1. *If $F \in H_{loc}(\mathbb{R}_+; X)$[1], with $F(0) = 0$, has exponential growth bound $\omega_0 \in \mathbb{R}$, then*

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int\limits_{\omega-iR}^{\omega+iR} e^{\lambda t} \mathcal{L}(s)(\lambda)\, d\lambda \text{ locally uniformly in } t \in \mathbb{R}_+,$$

for any $\omega > \omega_0$.

Before discussing the proof of Proposition A.1.1, we give a short overview of existing results providing answers to (A.3) in the current literature. For strongly continuous functions, the question *under which conditions the complex inversion formula holds and in which sense* was posed more generally and was partially answered by Markus Haase in [43]. Haase considers convolutions of a strongly continuous and a locally integrable scalar function and obtains (A.3) with convergence in operator norm. Another recent result can be found in [5, Appendix A]. The authors show the validity of Laplace inversion for convolutions of two C_0-semigroups as a direct consequence of the upcoming Proposition A.1.3. This observation is then exploited to identify and control the leading-order terms in the Neumann series of high-frequency components of a C_0-semigroup. For merely continuous functions, we cite the result [6, Theorem 4.2.21 b)] which requires introducing the (weaker) Cesàro limit:

$$\mathrm{C} - \lim_{R\to\infty} \int\limits_{\omega-iR}^{\omega+iR} e^{\lambda t} \mathcal{L}(F)(\lambda)\, d\lambda := \lim_{R\to\infty} \frac{1}{R} \int_0^R \int\limits_{\omega-ir}^{\omega+ir} e^{\lambda t} \mathcal{L}(F)(\lambda)\, d\lambda\, dr, \quad t \in \mathbb{R}_+.$$

Proposition A.1.2. *For $F \in C(\mathbb{R}_+; X)$, $F(0) = 0$, with exponential growth bound ω_0, we have*

$$F(t) = \mathrm{C} - \frac{1}{2\pi i} \lim_{R\to\infty} \int\limits_{\omega-iR}^{\omega+iR} e^{\lambda t} \mathcal{L}(F)(\lambda)\, d\lambda \text{ locally uniformly in } t \in \mathbb{R}_+,$$

for any $\omega > \omega_0$.

The next result is, to the author's knowledge, the most general result providing (A.3) in the current literature.

Proposition A.1.3. *If $F \in \mathrm{Lip}(\mathbb{R}_+; X)$, $F(0) = 0$, then*

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int\limits_{\omega-iR}^{w+iR} e^{\lambda t} \mathcal{L}(s)(\lambda)\, d\lambda \text{ locally uniformly in } t \in \mathbb{R}_+,$$

for any $\omega > 0$.

[1]For the definitions of function spaces see Appendix A.A.

The proof of Proposition A.1.3 in [6] relies on [6, Theorem 2.3.4], which is obtained through the Riesz-Stieltjes Representation [6, Theorem 2.1.1]. This representation gives an abstract characterization of $\{F \in \mathrm{Lip}(X) : F(0) = 0\}$. The authors give a second proof of Proposition A.1.3 in [6, page 260], based on the existence of the Cesàro limit, c.f. Proposition A.1.2, together with the property

$$\left\| t \mapsto \int_{\omega-iR}^{\omega+iR} e^{\lambda t} \mathcal{L}(F(t))(\lambda)\, d\lambda - \int_{\omega-iS}^{\omega+iS} e^{\lambda t} \mathcal{L}(F(t))(\lambda)\, d\lambda \right\|_{C([0,a],X)} \to 0 \text{ as } S, R \to \infty, \quad \frac{R}{S} \to 1 \tag{A.4}$$

for all $a > 0$, i.e. F is *feebly oscillating*. Justifying both properties is sufficient to conclude (A.3) by the real Tauberian theorem [6, Theorem 4.2.5].

Remark A.1.4. *An example which is covered by Proposition A.1.1 but not by Proposition A.1.3, can be obtained by considering a Weierstrass function, for example* $G(t) = \sum_{n=1}^{\infty} \frac{\sin(n^2 t)}{n^2}$, *and setting* $F(t) = (1 - e^{-t})G(t)$.

Due to the observation, see Section A.3.1, that for exponentially bounded functions, Laplace inversion in the sense of (A.2) can be traced back to Fourier inversion, we study the Fourier transform first and provide sufficient conditions yielding locally uniform convergence of the Fourier inversion formula in one and higher dimensions. The gained knowledge on the Fourier inversion formula is then exploited to prove Proposition A.1.1. We conclude this paper with an application of Proposition A.1.1 to C_0-semigroups on Favard spaces where we follow the literature and suppose $\omega_0 < 0$.

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A.2. Uniformity in the Fourier inversion formula

A.2.1. Preliminaries

We start with recalling classical results and formulae related to Fourier inversion for Banach space-valued functions. Fix $n \geq 1$ and let $F : \mathbb{R}^n \to X$ be L^1-integrable. Define the Fourier transform

$$\mathcal{F}(F)(k) := \hat{F}(s) := \int_{\mathbb{R}^n} e^{-ik\cdot s} F(s)\, ds, \quad k \in \mathbb{R}^n$$

and the inversion formula is given by

$$\mathcal{F}^{-1}(\hat{F})(x) = \lim_{R\to\infty} S_R(F)(x), \quad x \in \mathbb{R}^n$$

with

$$S_R(F)(x) = \frac{1}{(2\pi)^n} \int_{|k|\le R} e^{ix\cdot k} \hat{F}(k)\, dk, \quad , \quad R \ge 1, \quad x \in \mathbb{R}^n.$$

Analogously to the Laplace transform, we ask for regularity and localization assumptions such that

$$S_R(F)(x) \to F(x) \text{ as } R \to \infty, \text{ locally uniformly in } x \in \mathbb{R}^n. \tag{A.5}$$

If (A.5) holds, we shortly write: *Fourier inversion holds locally uniformly.* A central role in the study of Fourier inversion plays the spherical mean of F at $x \in \mathbb{R}^n$ defined by the surface integral

$$\bar{F}_x(r) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} F(x + r\omega)\, d\omega, \quad r \ge 0,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and ω_{n-1} is its surface measure. Following [80], we rewrite

$$\begin{aligned} S_R(F)(x) &= \int_{\mathbb{R}^n} F(x+z) D_n^R(z)\, dz \\ &= \omega_{n-1} \int_0^\infty r^{n-1} D_n^R(r) \bar{F}_x(r)\, dr \end{aligned} \tag{A.6}$$

with the Dirichlet kernel $D_n^R(r)$ given by[2]

$$D_n^R(z) = \frac{1}{(2\pi)^n} \int_{|\xi|\le R} e^{-iz\cdot\xi}\, d\xi, \quad R \ge 1, \quad z \in \mathbb{R}^n.$$

Note that for a rotation matrix $D \in \mathbb{R}^{n\times n}$, it holds $(Dz)\cdot\xi = z\cdot(D^T\xi)$ and, since D^T is also a rotation matrix, the substitution rule first gives $D_n^R(z) = D_n^R(|z|)$ and then the second identity in (A.6) follows.

To that end, we record the following facts.

Proposition A.2.1 ([80]). (i) *We have*

$$D_1^R(r) = \frac{\sin(Rr)}{\pi r}, \quad D_2^R(r) = \frac{RJ_1(Rr)}{2\pi r}, \quad RJ_1(Rr) = -\frac{d}{dr} J_0(Rr),$$

for $r > 0$ and $R \ge 1$, where J_0 and J_1 denote Bessel functions of first kind. In particular, J_0 and J_1 are smooth and bounded.

(ii) *For every $\delta > 0$,*

$$\int_{-\delta}^{\delta} \frac{\sin(Rs)}{s}\, ds \to \pi, \quad \int_0^{\delta} \frac{\sin(Rs)}{s}\, ds \to \frac{\pi}{2} \text{ as } R \to \infty.$$

(iii) *It holds*

$$J_0(0) = 1, \quad J_0(r) \to 0 \text{ as } r \to \infty.$$

(iv) *For $n \in \mathbb{N}_{\ge 3}$, we have the relation*

$$D_n^R(r) = -\frac{1}{r}\frac{d}{dr} D_{n-2}^R(r), \quad r > 0, \quad R \ge 1.$$

[2] Denote by $\widetilde{D}_R^n$ the Dirichlet kernel given in [80]. Then, the relation $D_n^R(z) = (2\pi)^{-n} \widetilde{D}_R^n((2\pi)^{-1}z)$ holds. This is due to slightly different choices of the Fourier transforms.

(v) *For each $n \in \mathbb{N}$ there exists some $C_n > 0$ such that*

$$|D_n^R(r)| \leq \frac{C_n R^n}{(1+Rr)^{\frac{n+1}{2}}}, \quad r > 0, \quad R \geq 1.$$

We distinguish between odd and even dimensions and study the cases $n = 1$ and $n = 2$ independently. The following formulae are a consequence of an iterative application of Proposition A.2.1 (iv).

Lemma A.2.2. *Let $(\eta_R)_{R\geq 1}$ be a sequence of smooth cut-off functions with $0 \leq \eta_R \leq 1$ being one in the interval $(0, R)$ and zero in $(R+1, \infty)$. We obtain the following identities.*

(i) *Let $n = 2k+1$ with $k \in \mathbb{N}$. Let $B : \mathbb{R}^n \to X$ be L^1-integrable and k-times continuously differentiable. There exist universal constants $C_{0,k}(n), ..., C_{k,k}(n) \in \mathbb{R}$ such that*

$$\int_0^\infty D_n^R(r)B(r)r^{n-1}\,dr = \sum_{j=0}^{k} C_{j,k}(n) \int_0^\infty D_1^R(r) r^j \Big(\frac{d}{dr}\Big)^j (B(r)\eta_R(r))\,dr + \int_0^\infty D_n^R(r)B(r)(1-\eta_R(r))r^{n-1}\,dr.$$

(ii) *Let $n = 2k$ with $k \in \mathbb{N}$. Let $B : \mathbb{R}^n \to X$ be integrable and $(k-1)$-times continuously differentiable. There exist universal constants $C_{0,k-1}(n), ..., C_{k-1,k-1}(n) \in \mathbb{R}$ such that*

$$\int_0^\infty D_n^R(r)B(r)r^{n-1}\,dr = \sum_{j=0}^{k-1} C_{j,k-1}(n) \int_0^\infty D_2^R(r) r^{j+1} \Big(\frac{d}{dr}\Big)^j (B(r)\eta_R(r))\,dr + \int_0^\infty D_n^R(r)B(r)(1-\eta_R(r))r^{n-1}\,dr.$$

Proof. For (i), we refer to [80, pages 141-142]. We prove (ii) for the sake of convenience as we did not find a proof for this formula. For this purpose, we show for every $k \in \mathbb{N}$ with $n = 2k$, that it holds

$$\int_0^\infty D_n^R(r)A(r)r^{n-1}\,dr = \sum_{j=0}^{k-1} C_{j,k-1}(n) \int_0^\infty D_2^R(r) r^{j+1} \Big(\frac{d}{dr}\Big)^j (A(r))\,dr \tag{A.7}$$

for all compactly supported and $(k-1)$-times differentiable $A : (0,\infty) \to X$. For $k = 1$, this is precisely formula (A.6) with $n = 2$. We do an induction argument and therefore suppose that the assertion (A.7) is known for $k-1$ with $k \geq 2$. Let $A : (0, \infty) \to X$ be any compactly supported and $(k-1)$-times differentiable function. There exist universal constants $C_j^1 \in \mathbb{R}$ and $C_j^2 \in \mathbb{R}$ for $j = 0, ..., k-2$ such that

$$\Big(\frac{d}{dr}\Big)^j \Big(r\frac{d}{dr}A(r)\Big) = C_j^1 r \Big(\frac{d}{dr}\Big)^{j+1}(A(r)) + C_j^2 \Big(\frac{d}{dr}\Big)^j (A(r)). \tag{A.8}$$

Integrating by parts once and using Proposition A.2.1, we find

$$\begin{aligned}\int_0^\infty D_n^R(r)A(r)r^{n-1}\,dr &= \int_0^\infty \Big(-\frac{1}{r}\frac{d}{dr}\Big)^{k-1} \Big(D_2^R(r)\Big) A(r) r^{n-1}\,dr \\ &= \int_0^\infty \underbrace{\Big(-\frac{1}{r}\frac{d}{dr}\Big)^{k-2} \Big(D_2^R(r)\Big)}_{=D_{n-2}^R(r)} \Big(r\frac{d}{dr}(A(r)) \;+ (n-2)A(r)\Big) r^{n-3}\,dr.\end{aligned}$$

Applying the induction hypothesis (A.7) to $A(r)$ and $r\frac{d}{dr}(A(r))$ and using (A.8), it holds

$$\int_0^\infty D_n^R(r)A(r)r^{n-1}\,dr = (n-2)\sum_{j=0}^{k-2} C_{j,k-2}(n-2)\int_0^\infty D_2^R(r)r^{j+1}\Big(\frac{d}{dr}\Big)^j(A(r))\,dr$$
$$+\sum_{j=0}^{k-2} C_j^1 C_{j,k-2}(n-2)\int_0^\infty D_2^R(r)r^{j+2}\Big(\frac{d}{dr}\Big)^{j+1}(A(r))\,dr$$
$$+\sum_{j=0}^{k-2} C_j^2 C_{j,k-2}(n-2)\int_0^\infty D_2^R(r)r^{j+1}\Big(\frac{d}{dr}\Big)^j(A(r))\,dr.$$

Since A was chosen arbitrarily, this gives the assertion (A.7) for $n=2k$ with determinable constants $C_{0,k-1}(n),...,C_{k-1,k-1}(n)\in\mathbb{R}$. Splitting

$$\int_0^\infty D_n^R(r)B(r)r^{n-1}\,dr = \int_0^\infty D_n^R(r)(B(r)\eta_R(r))r^{n-1}\,dr$$
$$+\int_0^\infty D_n^R(r)B(r)(1-\eta_R(r))r^{n-1}\,dr,$$

the claimed formula (ii) follows by choosing $A(r)=B(r)\eta_R(r)$ in (A.7). □

Finally, we isolate the part of the inversion formula which tends to zero uniformly for every $x\in\mathbb{R}^n$.

Lemma A.2.3. *For every $n\in\mathbb{N}$, there exists some constant $C_n>0$ such that*

$$\sup_{x\in\mathbb{R}^n}\left|\left|\int_0^\infty D_n^R(r)\bar{F}_x(r)(1-\eta_R(r))r^{n-1}\,dr\right|\right| \le \frac{C_n}{R}||F||_{L^1}$$

for every $R\ge 1$ and $F\in L^1(\mathbb{R};X)$.

Proof. We invoke Proposition A.2.1 (v) to infer

$$\left|\left|\int_0^\infty D_n^R(r)\bar{F}_x(r)(1-\eta_R(r))r^{n-1}\,dr\right|\right| \le \left|\left|\int_R^\infty D_n^R(r)\bar{F}_x(r)r^{n-1}\,dr\right|\right| \le \frac{C_n}{R}\int_R^\infty r^{n-1}|\bar{F}_x(r)|\,dr$$
$$\le \frac{C_n}{R}||F||_{L^1},$$

where we use that $\int_0^\infty r^{n-1}\int_{\mathbb{S}^{n-1}}|F(x+r\omega)|\,d\omega\,dr = ||F||_{L^1}$ for every $x\in\mathbb{R}^n$. We also refer to [80, page 141]. □

We note that in one spatial dimension, i.e. for $n=1$, it holds

$$\bar{F}_x(r)=\frac{1}{2}(F(x+r)+F(x-r)),\quad \omega_0=2. \tag{A.9}$$

Invoking [80, Theorem 2.3.4] and a vector-valued version of the Riemann-Lebesgue lemma, see e.g. [53], one finds that the additional condition $F\in\mathrm{Din}(\mathbb{R};X)$ implies pointwise Fourier inversion. Pinsky also mentions local Hölder-continuity as a sufficient condition for pointwise Fourier inversion in [80, Corollary 2.3.5]. In the upcoming section, we lift this pointwise convergence to locally uniform convergence.

A.2.2. COMPACTNESS CRITERIA AND RESULT ON $\mathbb{R}$

In this section, we consider $n = 1$. The following compactness criteria are sketched in [81, Corollary 2]. For the sake of self-containment, we provide their proofs here.

Lemma A.2.4 (A uniform version of the Riemann-Lebesgue lemma). *Let N be any set. Let $F : N \times \mathbb{R} \to X$. If*

$$\{s \mapsto F(t,s) : t \in N\}$$

is a compact subset of $L^1(\mathbb{R}; X)$, then

$$\int_{\mathbb{R}} \sin(Rs) F(t,s)\, ds \to 0 \text{ as } R \to \infty \text{ uniformly in } t \in N.$$

Proof. We rewrite

$$\int_{\mathbb{R}} F(t,s) \sin(Rs)\, ds = \frac{1}{2i}\Big(\int_{\mathbb{R}} F(t,s) e^{isR}\, ds - \int_{\mathbb{R}} F(t,s) e^{-isR}\, ds\Big). \tag{A.10}$$

Applying the substitution $s \mapsto s + \frac{\pi}{R}$, we arrive at

$$\int_{\mathbb{R}} F(t,s) e^{isR}\, ds = \int_{\mathbb{R}} F\Big(t, s + \frac{\pi}{R}\Big) e^{isR} e^{i\pi}\, ds = -\int_{\mathbb{R}} F\Big(t, s + \frac{\pi}{R}\Big) e^{isR}\, ds.$$

This shows

$$\sup_{t \in N} \Big\| \int_{\mathbb{R}} F(t,s) e^{isR}\, ds \Big\| \le \frac{1}{2} \sup_{t \in N} \int_{\mathbb{R}} \Big\| F(t,s) - F\Big(t, s + \frac{\pi}{R}\Big) \Big\|\, ds.$$

Similarly, one estimates the second integral in (A.10) after taking its complex conjugate. By a Banach space-value version of the Kolmogorov compactness theorem (mimic the proof of [1, 2.32 Theorem]), we arrive at

$$\sup_{t \in N} \int_{\mathbb{R}} \Big\| F(t,s) - F\Big(t, s + \frac{\pi}{R}\Big) \Big\|\, ds \to 0 \text{ as } R \to \infty, \tag{A.11}$$

which finishes the proof. □

Lemma A.2.5. *Let $F : \mathbb{R} \to X$ be L^1-integrable and $a \le b$. Fix $\delta > 0$. Suppose that*[3]

$$\Big\{ s \mapsto \frac{F(t+s) - \mathbf{1}_{(-\delta,\delta)}(s) F(t)}{s} : t \in [a,b] \Big\}$$

is a compact subset of $L^1(\mathbb{R})$. Then,

$$F(t) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} e^{itk} \hat{F}(k)\, dk \text{ uniformly in } t \in [a,b].$$

Proof. Let $t \in [a,b]$. Using (A.6), (A.9) and Proposition A.2.1 (i), we find

$$S_R(F)(t) = \frac{1}{\pi} \int_0^{\infty} \frac{\sin(Rs)}{s} (F(t+s) + F(t-s))\, ds = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(Rs)}{s} F(t+s)\, ds.$$

Invoking Proposition A.2.1 (ii), we arrive at

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} e^{ist} \hat{F}(s)\, ds - F(t) = \lim_{R \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(Rs)}{s} (F(t+s) - \mathbf{1}_{(-\delta,\delta)}(s) F(t))\, ds$$

and the result follows from Lemma A.2.4. □

[3] For $I \subset \mathbb{R}$, we set $\mathbf{1}_I(s) = 1$ for $s \in I$ and $\mathbf{1}_I(s) = 0$ for $s \notin I$.

We are now in the position to consider the special case of locally Hölder-continuous functions. Additionally, a mild localization condition is required which serves to apply dominated convergence. In the subsequent remark, we discuss further conditions and examples.

Proposition A.2.6. *Let $F \in H_{loc}(\mathbb{R}; X)$ be L^1-integrable. Furthermore, suppose that for every $a \leq b$ there exist $\delta > 0$ and $g \in L^1(\mathbb{R})$ such that*

$$|s|^{-1} \sup_{t\in[a,b]} ||F(t+s)|| \leq g(s) \tag{A.12}$$

for all $|s| \geq \delta$. Then:

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{ist} \hat{F}(s)\, ds \text{ locally uniformly in } t \in \mathbb{R}.$$

Proof. Let $a \leq b$. Fix $\delta > 0$. In virtue of Proposition A.2.5, it suffices to show that

$$M = \left\{ s \mapsto \frac{F(t+s) - \mathbf{1}_{(-\delta,\delta)}(s)F(t)}{s} : t \in [a,b] \right\}$$

is compact in $L^1(\mathbb{R})$. So, let $(t_n) \subset [a,b]$. Clearly, by compactness of $[a,b]$, there exists (t_{n_k}) and $t_0 \in [a,b]$ such that $t_{n_k} \to t_0$ as $k \to \infty$. By the Hölder-continuity of F on $[a-\delta, b+\delta]$, there exists some $\alpha \in (0,1]$ and $C > 0$ such that

$$\left\|\frac{F(t_{n_k}+s) - \mathbf{1}_{(-\delta,\delta)}(s)F(t_{n_k})}{s}\right\| \leq C\left(\mathbf{1}_{(-\delta,\delta)}(s)\frac{1}{|s|^{1-\alpha}} + \mathbf{1}_{\{|s|\geq\delta\}}(s)\frac{1}{|s|}\sup_{k\in\mathbb{N}} ||F(t_{n_k}+s)||\right). \tag{A.13}$$

The function on the right hand side admits an integrable majorant thanks to (A.12). On the other hand, for fixed $s \in \mathbb{R} \setminus \{0\}$,

$$\frac{F(t_{n_k}+s) - \mathbf{1}_{(-\delta,\delta)}(s)F(t_{n_k})}{s} \to \frac{F(t_0+s) - \mathbf{1}_{(-\delta,\delta)}(s)F(t_0)}{s} \text{ as } k \to \infty$$

as consequence of continuity of F. By (A.13), the limit function lies also in $L^1(\mathbb{R}; X)$. Dominated convergence now implies

$$\left(s \mapsto \frac{F(t_{n_k}+s) - \mathbf{1}_{(-\delta,\delta)}(s)F(t_{n_k})}{s}\right) \to \left(s \mapsto \frac{F(t_0-s) - \mathbf{1}_{(-\delta,\delta)}(s)F(t_0)}{s}\right) \text{ in } L^1(\mathbb{R}; X)$$

as $k \to \infty$. This shows the compactness of M in $L^1(\mathbb{R}; X)$. □

Remark A.2.7. *We briefly discuss the condition (A.12).*

(i) *If there exist $C > 0$ and $\alpha > 0$ such that*

$$||F(s)|| \leq C(|s|+1)^{-\alpha}, \quad s \in \mathbb{R},$$

then (A.12) holds. Indeed, for $a \leq b$, we find

$$|s|^{-1} \sup_{t\in[a,b]} ||F(t+s)|| \leq C(a,b)(|s|+1)^{-1} \sup_{t\in[a,b]} (|t-s|+1)^{-\alpha} \leq C(a,b)(|s|+1)^{-1-\alpha},$$

for all $|s| \geq \delta$ with $\delta = \max\{|a|,|b|\}+1$. Clearly, the right hand side is an integrable function. As a non-trivial example to this condition, one considers a locally Hölder-continuous and integrable function with values $F(t) = \frac{1}{|k|^{\frac{1}{2}}}$ for $t \in \left[k - \frac{1}{2^{|k|}}, k + \frac{1}{2^{|k|}}\right]$ and $k \in \mathbb{Z} \setminus \{0\}$. Such a function fulfills the decay assumption only with $0 < \alpha \leq \frac{1}{2}$.

(ii) *An example violating the condition in (i) is the function* $F(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-|t-e^n|^2}$, $t \in \mathbb{R}$, *since* $F(e^n) \geq \frac{1}{n^2}$ *for every* $n \in \mathbb{N}$. *The function* F *is clearly Lipschitz-continuous and integrable. Furthermore, since the sum converges locally uniformly, we observe for* $a \leq b$ *and* $t \in [a,b]$ *that*

$$\sup_{t\in[a,b]} |F(t+s)| \leq \sum_{n=0}^{\infty} \frac{1}{n^2} \sup_{t\in[a,b]} e^{-|t+s-e^n|^2} \leq F(b+s) + F(a+s)$$

providing the condition (A.12). Therefore, by Proposition A.2.6, locally uniform Fourier inversion follows. This shows that the decay assumption in (i) is not necessary.

(iii) *The last example teaches that if* $||F||$ *is integrable and eventually monotone, i.e. there exists some* $C > 0$ *such that* $||F(s)||$ *is monotone for* $|s| \geq C$, *then* F *satisfies (A.12).*

(iv) *It would be interesting to find a function that is Dini-continuous and not Hölder-continuous for which locally uniform Fourier inversion fails.*

(v) *As one only needs (A.11) to guarantee uniform convergence, one may replace the compactness condition by the slightly weaker assumption*

$$\sup_{t\in[a,b]} \int_{\mathbb{R}} \left\| \frac{F(t+s) - \mathbf{1}_{(-\delta,\delta)}(s)F(t)}{s} - \frac{F(t+s-h) - \mathbf{1}_{(-\delta,\delta)}(s+h)F(t)}{s+h} \right\| ds \to 0,$$

as $h \downarrow 0$, *in Proposition A.2.5.*

A.2.3. Uniformity in $\mathbb{R}^n$ with odd $n \geq 3$

In this section, let $n = 2k+1$ with $k \in \mathbb{N}$. Let $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to X$ be L^1-integrable and k-times continuously differentiable. With the aid of Lemma A.2.2, we decompose

$$\begin{aligned}
S_R(F) &= I_R(F) + II_R(F) + III_R(F) \text{ with} \\
I_R(F)(x) &= C_{0,k}(n) \int_0^\infty \frac{\sin(Rr)}{\pi r} \bar{F}_x(r)\, dr, \\
II_R(F)(x) &= \sum_{j=1}^{k} C_{j,k}(n) \int_0^\infty \frac{\sin(Rr)}{\pi} r^{j-1} \left(\frac{d}{dr}\right)^j \bar{F}_x(r)\, dr, \\
III_R(F)(x) &= \int_0^\infty D_n^R(r) \bar{F}_x(r)(1-\eta_R(r)) r^{n-1}\, dr \\
&\quad - \sum_{j=0}^{k} C_{j,k}(n) \int_0^\infty \frac{\sin(Rr)}{\pi} r^{j-1} \left(\frac{d}{dr}\right)^j (\bar{F}_x(r)(1-\eta_R(r)))\, dr.
\end{aligned}$$

Since, the constants $C_{j,k}(n)$ are defined iteratively, they can be explicitly determined.

Remark A.2.8 (Determination of $C_{0,k}(n)$). *As described in [80], we use our knowledge on pointwise inversion to show* $C_{0,k}(n) = 2$ *for each* $k \in \mathbb{N}$. *Considering the Schwartz function* $F(x) = e^{-|x|^2}$ *(or any other non-trivial Schwartz function), it is known that pointwise Fourier inversion holds since* F *is in particular Dini-continuous. That is, for every* $x \in \mathbb{R}$, *it holds* $S_R(F)(x) \to F(x)$ *as* $R \to \infty$. *On the other hand,* $r \mapsto \frac{\bar{F}_x(r) - \mathbf{1}_{(-\delta,\delta)}(r)\bar{F}_x(0)}{r}$ *is* L^1*-integrable and therefore by the Riemann-Lebesgue Lemma, we conclude* $\int_0^\infty \frac{\sin(Rr)}{\pi r} \bar{F}_x(r)\, dr \to \frac{1}{2}\bar{F}_x(0) = \frac{1}{2}F(x)$. *Furthermore,* $II_R(F)(x) \to 0$ *as* $R \to \infty$ *since* $r \mapsto r^j \left(\frac{d}{dr}\right)^j \bar{F}_x(r)$ *is* L^1*-integrable for* $j = 1, ..., k$. *Together with Lemma A.2.3, we also find* $III_R(F)(x) \to 0$ *as* $R \to \infty$. *This implies* $F(x) = \frac{1}{2}C_{0,k}(n)F(x)$ *and hence* $C_{0,k}(n) = 2$ *as* $F(x) > 0$.

We formulate the following compactness criterion.

Proposition A.2.9. *Let $K \subset \mathbb{R}^n$ be compact. Let $F : \mathbb{R}^n \to X$ be L^1-integrable and k-times continuously differentiable, and suppose that*

$$\left\{ r \mapsto \frac{\bar{F}_x(r) - \mathbf{1}_{(0,\delta)}(r)F(x)}{r} : x \in K \right\}, \quad \left\{ r \mapsto r^{j-1}\left(\frac{d}{dr}\right)^l \bar{F}_x(r) : x \in K, \quad l = 0, ..., j \right\},$$

$j = 1, ..., k$, are compact subsets of $L^1((0,\infty); X)$ for some $\delta > 0$. Then, Fourier inversion holds uniformly in K.

Proof. Using $\bar{F}_x(0) = F(x)$, we first write

$$I_R(F)(x) - F(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(Rr)}{r}\Big(\bar{F}_x(r) - \mathbf{1}_{(0,\delta)}(r)F(x)\Big)\, dr.$$

By Lemma A.2.4 (extend the functions trivially to $\mathbb{R}$), it holds $I_R(F)(x) - F(x) \to 0$ as $R \to \infty$ uniformly in $x \in K$. Using again Lemma A.2.4, compactness implies $II_R(x) \to 0$ as $R \to$ uniformly in $x \in K$. Finally, by Lemma A.2.3, we find

$$\sup_{x \in K} |III_R(F)(x)| \le C\left(\sum_{j=0}^k \sup_{x\in K}\sum_{l=0}^j \int_R^\infty r^{j-1}\left|\left|\left(\frac{d}{dr}\right)^l \bar{F}_x(r)\right|\right| dr + \frac{1}{R}||F||_{L^1}\right) \tag{A.14}$$

for a suitable constant $C > 0$. The right hand side now converges to 0 as $R \to \infty$ due to Kolmogorov's compactness theorem. □

Proposition A.2.10. *Assume that $F : \mathbb{R}^n \to X$ is L^1-integrable and k-times continuously differentiable. If for every compact $K \subset \mathbb{R}^n$ there exist $g \in L^1(\mathbb{R}^n)$ and $\delta > 0$ such that*

$$\sup_{\alpha\in\mathbb{N}_0^n, |\alpha|\le k}\ \sup_{x\in K} |y|^{-n+k}||\nabla^\alpha F(x+y)|| \le g(y), \tag{A.15}$$

for all $|y| \ge \delta$. Then, Fourier inversion holds locally uniformly.

Proof. Note that, since F is continuously differentiable, it is locally Hölder-continuous. Therefore, there exists a constant $C > 0$ such that

$$\mathbf{1}_{(0,\delta)}(r)\left|\left|\frac{\bar{F}_x(r) - F(x)}{r}\right|\right| \le \mathbf{1}_{(0,\delta)}(r)\frac{1}{r}\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}} ||F(x+r\omega) - F(x)||\, d\omega \le \mathbf{1}_{(0,\delta)}(r)C$$

for every $r \ge 0$ and $x \in K$. Furthermore, we estimate[4], for $r \ge \delta$,

$$\left|\left|\frac{1}{r}\bar{F}_x(r)\right|\right| \le \sup_{x\in K}\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\frac{1}{r}||F(x+r\omega)||\, d\omega \le \frac{C(\delta)}{\omega_{n-1}} r^{n-1}\int_{\mathbb{S}^{n-1}} g(r\omega)\, d\omega \tag{A.16}$$

where the right hand side is integrable in $[\delta, \infty)$ by (A.15). By dominated convergence, argued as in the proof of Proposition A.2.6, we conclude that

$$\left\{ r \mapsto \frac{\bar{F}_x(r) - \mathbf{1}_{(-\delta,\delta)}(r)F(x)}{r} \right\}$$

[4] We denote positive constants only depending on δ by $C(\delta)$.

is compact in $L^1((0,\infty);X)$. Similarly, one obtains the $L^1((0,\infty);X)$-compactness of

$$\left\{r \mapsto r^{j-1}\left(\frac{d}{dr}\right)^l \bar{F}_x(r) : x \in K, \quad l = 0, ..., j\right\}, \quad j = 1, ..., k,$$

with the help of the observation that, for $r \geq 0$,

$$\begin{aligned}\left|\left|r^{j-1}\left(\frac{d}{dr}\right)^l \bar{F}_x(r)\right|\right| &\leq \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \sup_{\alpha \in \mathbb{N}_0^k, |\alpha| \leq l} r^{j-1} ||\nabla^\alpha F(x + r\omega)||\, d\omega \\ &\leq C(\delta)\Big(\mathbf{1}_{[\delta,\infty]}(r) \frac{1}{\omega_{n-1}} r^{n-1} \int_{\mathbb{S}^{n-1}} g(r\omega)\, d\omega \\ &\qquad + \mathbf{1}_{(0,\delta)}(r) \sup_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq l} \sup_{x \in K, y \in \mathbb{S}^{n-1}} ||\nabla^\alpha F(x + ry)||\Big),\end{aligned}$$

for $j = 1, ..., k$ and uniformly in $l = 0, ..., j$ and $x \in K$, where we use the lemma about parameter integrals to pull derivatives into the surface integral. Putting everything together, the previous proposition yields locally uniform Fourier inversion. □

Remark A.2.11 (Comparison of conditions). *For $n = 3$, we compare the conditions of Proposition A.2.10 with the ones from [81, Proposition 4] yielding convergence for fixed $x \in \mathbb{R}^3$. The localization condition is that $|\cdot|^{-2} \nabla F(x - \cdot) \in L^1(\mathbb{R}^3)$. Instead of this pointwise assumption, we need an integrable majorant on $\mathbf{1}_{[\delta,\infty)}(\cdot)|\cdot|^{-2}(||F(x + \cdot)|| + ||\nabla F(x + \cdot)||)$ uniformly for $x \in K$ and for any compact $K \subset \mathbb{R}^3$. The only further regularity assumption in [81, Proposition 4] is that x is a Lebesgue point of F which is covered by the property $F \in C^1(\mathbb{R}^3)$.*

Remark A.2.12. *We consider exemplarily $n = 3$. To guarantee (A.16), it suffices to assume an integrable majorant on $\mathbf{1}_{[\delta,\infty)}(\cdot)|\cdot|^{-3}||F(x + \cdot)||$ uniformly in $x \in K$. On the other hand, writing out (A.14) gives rise to the term $\int_{\mathbb{R}} \bar{F}_x(r)\left(\frac{1-\eta_R(r)}{r} + \eta_R'(r)\right) dr$ for which we used the stronger localization assumption (A.15). We do not see how to relax this assumption to the weaker one required for (A.16).*

A.2.4. Uniformity in $\mathbb{R}^n$ with even $n \geq 2$

As motivation, we consider first the case $n = 2$. Fix $x \in \mathbb{R}^2$. Let $F : \mathbb{R}^2 \to X$ be such that $r \mapsto \bar{F}_x(r)$ is integrable and differentiable with integrable derivative. Using Proposition A.2.1 (i), (iii), the fact that $\omega_1 = 2\pi$ and integrating by parts, we obtain

$$\begin{aligned}S_R(F)(x) &= \int_0^\infty \bar{F}_x(r) R J_1(Rr)\, dr \\ &= \bar{F}_x(0) + \int_0^\infty \frac{d}{dr}(\bar{F}_x(r)) J_0(Rr)\, dr.\end{aligned}$$

Let $K \subset \mathbb{R}^2$ be compact. Since $J_0(Rr) \to 0$ as $R \to \infty$ for any $r > 0$ and $\bar{F}_x(0) = F(x)$, dominated convergence implies $S_R(F)(x) \to F(x)$ uniformly in $x \in K$ whenever there exists some $g \in L^1((0,\infty))$ such that $\sup_{x \in K} ||\frac{d}{dr}\bar{F}_x(r)|| \leq g(r)$. For general even dimensions, we make the following statement.

Proposition A.2.13. *Let $n = 2k$ with $k \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ be compact. Assume $F : \mathbb{R}^n \to X$ is L^1-integrable and k-times continuously differentiable. If there exist $g \in L^1(\mathbb{R}^n)$ and $\delta > 0$ such that*

$$\sup_{\alpha\in\mathbb{N}_0^n,|\alpha|\leq k}\ \sup_{x\in K} |y|^{-n+k}||\nabla^\alpha F(x+y)|| \leq g(y),$$

for all $|y| \geq \delta$, then Fourier inversion holds uniformly in K.

Proof. With the aid of Proposition A.2.1 (i) and Lemma A.2.2, we decompose

$$S_R(F)(x) = I_R(F)(x) + II_R(F)(x) + III_R(F)(x)$$

with

$$\begin{aligned}
I_R(F)(x) &= C_{0,k-1}(n)\int_0^\infty RJ_1(Rr)\bar{F}_x(r)\,dr,\\
II_R(F)(x) &= \sum_{j=1}^{k-1} C_{j,k-1}(n)\int_0^\infty RJ_1(Rr)r^j\Big(\frac{d}{dr}\Big)^j(\bar{F}_x(r)\eta_R(r))\,dr,\\
III_R(F)(x) &= \frac{1}{2\pi}\int_0^\infty D_n^R(r)\bar{F}_x(r)(1-\eta_R(r))r^{n-1}\,dr\\
&\quad - C_{0,k-1}(n)\int_0^\infty RJ_1(Rr)(\bar{F}_x(r)(1-\eta_R(r))\,dr\\
&= III_R^1(F)(x) + III_R^2(F)(x),
\end{aligned}$$

for universal constants $C_{j,k-1}(n)$, $j = 0, ..., k-1$. As in Remark A.2.8, one argues that $C_{0,k-1}(n) = 1$. Furthermore, integrating by parts once more and Proposition A.2.1 (i) yields

$$\begin{aligned}
II_R(F)(x) = \sum_{j=1}^{k-1} C_{j,k-1}(n)\Big(j\int_0^\infty J_0(Rr)r^{j-1}\Big(\frac{d}{dr}\Big)^j(\bar{F}_x(r)\eta_R(r))\,dr\\
+\int_0^\infty J_0(Rr)r^j\Big(\frac{d}{dr}\Big)^{j+1}(\bar{F}_x(r)\eta_R(r))\,dr\Big).
\end{aligned}$$

Arguing as in Theorem A.2.9, one shows, for $r \geq 0$,

$$\begin{aligned}
\sup_{x\in K}\left|\left|r^j\Big(\frac{d}{dr}\Big)^{l+1}(\bar{F}_x(r)\eta_R(r))\right|\right| \leq C(\delta)\Big(\mathbf{1}_{[\delta,\infty]}(r)\frac{1}{\omega_{n-1}}r^{n-1}\int_{\mathbb{S}^{n-1}} g(r\omega)\,d\omega\\
+\mathbf{1}_{(0,\delta)}(r)\sup_{\alpha\in\mathbb{N}_0^n,|\alpha|\leq l}\ \sup_{x\in K,y\in\mathbb{S}^{n-1}}||\nabla^\alpha F(x+ry)||\Big),
\end{aligned}$$

for $l = 0, ..., j$ and $j = 0, ..., k-1$, where the right-hand side is an integrable function on $(0,\infty)$. The introductory considerations for $n = 2$ with $j = 0$ give $I_R(F)(x) \to F(x)$ uniformly in $x \in K$. On the other hand, recalling again that $J_0(Rr) \to 0$ as $R \to \infty$ for any $r > 0$, dominated convergence also implies $II_R(F)(x) \to 0$ as $R \to \infty$ uniformly in $x \in K$. Similarly, it follows $III_R^2(F)(x) \to 0$ as $R \to \infty$ uniformly in $x \in K$. Finally, by Lemma A.2.3, we find $III_R^1(F)(x) \to 0$ as $R \to \infty$ uniformly in $x \in K$. □

Remark A.2.14 (Another uniformity result in higher dimensions). *The result [18, Theorem 4] provides Fourier inversion uniformly in $\mathbb{R}^n$ for $n \geq 2$ whenever $F : \mathbb{R}^n \to \mathbb{R}$ and its derivatives up to the order $\frac{n-1}{2}$ lie in the Lorentz space $L^{\frac{2n}{n-1},1}(\mathbb{R}^n)$. The authors of [18] emphasize that their result may be considered as a version of Dini's test in $\mathbb{R}^n$. On the other hand, Propositions A.2.10 and A.2.13 are rather local analogues of Dini's test in $\mathbb{R}^n$.*

A.3. Applications to Laplace transforms

A.3.1. From Fourier to Laplace inversion

Lemma A.3.1. *Let $0 \le a \le b$ and $R \ge 1$. Let $F : [0,\infty) \to X$ have exponential growth bound $\omega_0 \in \mathbb{R}$. Fix $\omega > \omega_0$. Define the function $\tilde{F} : \mathbb{R} \to X$ by*

$$\tilde{F}(s) = e^{-\omega s} \begin{cases} F(s), & \text{if } s \ge 0 \\ 0 & \text{else.} \end{cases}$$

Then,

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\omega - iR}^{\omega + iR} e^{\lambda t} \mathcal{L}(F)(\lambda)\, d\lambda \text{ uniformly in } t \in [a,b]$$

is equivalent to

$$\tilde{F}(t) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{itk} \mathcal{F}(\tilde{F})(k)\, dk \text{ uniformly in } t \in [a,b].$$

Proof. Let $t \in [a,b]$ and $R \ge 1$. Rewriting

$$\begin{aligned} \frac{1}{2\pi i} \int_{\omega - iR}^{\omega + iR} e^{\lambda t} \mathcal{L}(F)(\lambda)\, d\lambda &= \frac{1}{2\pi i} \int_{-iR}^{iR} e^{(\lambda+\omega)t} \int_0^\infty e^{-(\lambda+\omega)s} F(s)\, ds\, d\lambda \\ &= \frac{1}{2\pi} e^{\omega t} \int_{-R}^{R} e^{ikt} \int_0^\infty e^{-iks} e^{-\omega s} F(s)\, ds\, dk \\ &= e^{\omega t} \frac{1}{2\pi} \int_{-R}^{R} e^{itk} \mathcal{F}(\tilde{F})(k)\, dk \end{aligned}$$

shows the claim. □

As immediate consequence, we can translate Proposition A.2.6 in terms of the Laplace transform and prove our main result:

Proof of Proposition A.1.1. Let $\omega > \omega_0$ and $\tilde{F}$ as in Lemma A.3.1. Then $||\tilde{F}(s)|| \le e^{(\omega_0 - \omega)s}$ for $s > 0$ and $\tilde{F}(s) = 0$ for $s \le 0$ which show that $\tilde{F}$ satisfies (A.12). Furthermore, we have $\tilde{F} \in H_{\text{loc}}(\mathbb{R}_+; X)$. This implies the assertion by Proposition A.2.6 and Lemma A.3.1. □

A.3.2. Application to C_0-semigroups

Let $A : D(A) \subset X \to X$ be the generator of a C_0-semigroup $S(t)$ with exponential growth bound $\omega_0 \in \mathbb{R}$. We recall the following well-known result on the complex inversion formula which is a consequence of Proposition A.1.3, see also [32, 5.15 Corollary].

Corollary A.3.2. *If $\omega > \omega_0$ and $x \in D(A)$, then*

$$S(t)x = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\omega - iR}^{\omega + iR} e^{\lambda t} (\lambda - A)^{-1} x\, d\lambda \text{ locally uniformly in } t \in (0,\infty) \tag{A.17}$$

in X-norm.

Using now Proposition A.1.1, we may extend this result to Favard spaces. For this purpose, let $\alpha \in (0,1]$. By following [32, Section 5 b], we impose the condition $\omega_0 < 0$ and subsequently define the Favard space

$$F_\alpha := \{x \in X : ||x||_{F_\alpha} < \infty\}, \quad ||x||_{F_\alpha} := \sup_{t>0} \frac{1}{t^\alpha} ||S(t)x - x||_X. \tag{A.18}$$

Note that $D(A) \subset F_\alpha \subset F_{\alpha'} \hookrightarrow X$ for any $0 < \alpha' \leq \alpha \leq 1$.[5] The assumption $\omega_0 < 0$ guarantees the latter embedding. The abstract Hölder space is given by

$$X_\alpha := \{x \in F_\alpha : \lim_{t \downarrow 0} ||S(t)x - x||_{F_\alpha} = 0\}.$$

With [32, 5.15 Theorem], we have following properties: both spaces X_α and F_α are Banach spaces with respect to $||\cdot||_{F_\alpha}$. Furthermore, the restriction $S_{|F_\alpha} : [0,\infty) \to B(F_\alpha)$ is a semigroup of bounded operators and the restriction $S_{|X\alpha} : [0,\infty) \to B(X_\alpha)$ is a C_0-semigroup with generator $A_{|X_\alpha}$ and domain $D(A_{|X_\alpha}) = X_{\alpha+1}$.

Corollary A.3.3. *Let $\omega_0 < 0$. If $\omega > \omega_0$ and $x \in F_\alpha$, then*

$$S(t)x - e^{\omega_0 t}x = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\omega - iR}^{\omega + iR} e^{\lambda t}\left((\lambda - A)^{-1} - \frac{1}{\lambda - \omega_0}\right) x \, d\lambda \text{ locally uniformly in } t \in [0,\infty) \tag{A.19}$$

and

$$S(t)x = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\omega - iR}^{\omega + iR} e^{\lambda t}(\lambda - A)^{-1} x \, d\lambda \text{ locally uniformly in } t \in (0,\infty) \tag{A.20}$$

in X- and any $F_{\alpha'}$-norm with $\alpha' \in (0,\alpha)$.

Proof. Set $F(t) = S(t)x - e^{\omega_0 t}x$ and note that F has growth bound ω_0, $F(0) = 0$ and

$$\mathcal{L}(F)(\lambda) = (\lambda - A)^{-1}x - \frac{x}{\lambda - \omega_0}.$$

Obviously, $t \mapsto e^{\omega_0 t}x$ is locally Hölder-continuous and we consider $G(t) = S(t)x$. The function G is Hölder-continuous from $[0,\infty)$ to X. In fact, for $t_2 > t_1 \geq 0$, we find

$$||G(t_1) - G(t_2)||_X \leq ||S(t_1)||_{X\to X} ||S(t_2 - t_1)x - x||_X \leq C_1 |t_2 - t_1|^\alpha ||x||_{F_\alpha}$$

for some constant $C_1 > 0$ independent of t_2 and t_1 and using that $\omega_0 < 0$. Now, let $\alpha' \in (0,\alpha)$ and $t_2 > t_1 \geq 0$. With $\omega_0 < 0$, we estimate

$$\begin{aligned}
&||G(t_2) - G(t_1)||_{F_{\alpha'}} \\
&= \sup_{t>0} \frac{1}{t^{\alpha'}} ||S(t_1)(S(t)(S(t_2 - t_1)x - x) - (S(t_2 - t_1)x - x))||_X \\
&\leq C_1 |t_2 - t_1|^{\alpha - \alpha'} \sup_{t_2 - t_1 \geq t > 0} \frac{1}{t^\alpha} ||(S(t_2 - t_1)(S(t)x - x) - (S(t)x - x)||_X \\
&\quad + C_1 |t_2 - t_1|^{\alpha - \alpha'} \sup_{t \geq t_2 - t_1} \frac{1}{(t_2 - t_1)^\alpha} ||S(t)(S(t_2 - t_1)x - x) - (S(t_2 - t_1)x - x)||_X \\
&\leq C_2 |t_2 - t_1|^{\alpha - \alpha'} ||x||_{F_\alpha},
\end{aligned}$$

[5] If X is reflexive, then $D(A) = F^1$, c.f. [32, 5.21 Corollary].

for some constants $C_{1,2} > 0$ independent of t_2 and t_1. Similarly, one estimates

$$||(e^{\omega_0 t_2} - e^{\omega_0 t_1})x||_{F^{\alpha'}} \leq C(a,b)(t_2 - t_1)^{\alpha-\alpha'}||x||_{F^\alpha},$$

for every $0 \leq a \leq t_1 \leq t_2 \leq b$. Therefore, (A.19) follows by Proposition A.1.1 for $\omega > \omega_0$. Since

$$\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\omega-iR}^{\omega+iR} e^{\lambda t}\frac{x}{\lambda-\omega_0}\,d\lambda = e^{\omega_0 t}x, \quad \omega > \omega_0,$$

locally uniformly for $t \in (0,\infty)$, see e.g. [6], (A.20) follows from (A.19). □

Assume that $F_\alpha \neq X_\alpha$. Then, the restriction $S_{|F_\alpha}$ is not strongly continuous on F_α and, on the other hand,

$$t \mapsto \frac{1}{2\pi i}\int_{\omega-iR}^{\omega+iR} e^{\lambda t}\Big((\lambda-A)^{-1} - \frac{1}{\lambda-\omega_0}\Big)x\,d\lambda \tag{A.21}$$

is a continuous mapping from $[0,\infty)$ to F_α for every $R \geq 1$ and $x \in F_\alpha$. Therefore, the convergence in (A.19) cannot hold for every $x \in F_\alpha$ with respect to the F_α-norm.

We justify the continuity of (A.21). For this purpose, we fix $x \in X$ and set $y(\lambda) = (\lambda - A)^{-1}x$ which is in particular a continuous function in λ from $\rho(A)$ to X. Let $\mathrm{Re}(\lambda) = \omega > \omega_0$. We first estimate

$$||S(t)y(\lambda) - y(\lambda)||_X = ||\int_0^t S(s)Ay(\lambda)\,ds||_X \leq C_1 t||Ay(\lambda)||_X$$

for some constant $C_1 > 0$. Furthermore, we use

$$Ay(\lambda) = \lambda y(\lambda) - x = \lambda\int_0^\infty e^{-(\lambda-\omega_0)s}e^{-\omega_0 s}F(s)\,ds + \Big(\frac{\lambda}{\lambda-\omega_0} - 1\Big)x$$

with F as in the proof of Corollary A.3.3 to estimate

$$||Ay(\lambda)||_X \leq C_2\Big(|\lambda| + \frac{|\omega_0|}{|\lambda-\omega_0|}\Big)||x||_X$$

for some ω-dependent constant $C_2 > 0$. We conclude that there exists some ω-dependent constant $C_3 > 0$ such that

$$\begin{aligned}||y(\lambda)||_{F_\alpha} &\leq \sup_{t>0}\Big(\frac{1}{t}||S(t)y(\lambda) - y(\lambda)||_X\Big)^\alpha ||S(t)y(\lambda) - y(\lambda)||_X^{1-\alpha}\\ &\leq C_3||y(\lambda)||_X^{1-\alpha}(1+|\lambda|)^\alpha||x||_X^\alpha.\end{aligned}$$

This yields the integrability of $\lambda \mapsto \Big((\lambda-A)^{-1} - \frac{1}{\lambda-\omega_0}\Big)x$ in F_α on every $\{\omega + ik : k \in [-R,R]\}$ with $R \geq 1$. Thereby, the continuity of (A.21) follows for every $R \geq 1$ by the lemma about parameter integrals. If S is a C_0-group, then there exists $x \in F^\alpha$ such that (A.20) also fails immediately by the same argument.

Oulook. The considerations of this section beg the question of whether we get convergence in F_α-norm for every $x \in X_\alpha$. More generally, one may ask whether for every Banach space X, C_0-semigroup (or C_0-group) S and exponential growth bound $\omega_0 \in \mathbb{R}$, there exists a (largest) non-empty Banach space $Y \hookrightarrow X$ such that for every $x \in Y$ the convergence in (A.19) (or (A.20)) holds in Y-norm. How does X or S have to be specified and how does Y look like? We leave these questions open for future research. As an example of such a specification, we mention: for every C_0-semigroup and if X is a UMD-space, locally uniform convergence of the complex inversion formula is positively answered choosing $Y = X$ as shown in [31].

For another straightforward example, consider a C_0-semigroup S with generator A such that $\ker(A) \neq \emptyset$. By the Hille-Yosida theorem, S has necessarily growth bound $\omega_0 \geq 0$ and one chooses $Y = \ker(A)$ equipped with the norm $||\cdot||_X$ as non-trivial choice for the general question.

A.A. Appendix: Function spaces

Let $I \in \{\mathbb{R}^n, \mathbb{R}_+\}$ with $\mathbb{R}_+ = [0,\infty)$ and $n \in \mathbb{N}$. We introduce the following spaces

$$\begin{aligned}
\mathrm{Lip}(I;X) &:= \{f : I \to X \text{ is Lipschitz-continuous}\},\\
H_{\mathrm{loc}}(I;X) &:= \{f : I \to X \text{ is locally Hölder-continuous}\},\\
\mathrm{Din}(I;X) &:= \{f : I \to X \text{ is Dini-continuous}\},\\
C(I;X) &:= \{f : I \to X \text{ is continuous}\},
\end{aligned}$$

where we call $F : I \to X$ Dini-continuous at point $t_0 \in I$ if there exists some $\delta_0 > 0$ such that

$$\int_{(-\delta_0,\delta_0)^n} \frac{||F(t_0+h) - F(t_0)||}{|h|}\, dh < \infty.$$

The function F is called Dini-continuous if it is Dini-continuous in every point. We call a function $F : I \to X$ locally Hölder-continuous, if for every compact subset K of I there exist an $\alpha_0 = \alpha_0(K) \in (0,1]$ and $C = C(K) > 0$ such that

$$||F(x) - F(y)|| \leq C|x-y|^{\alpha_0} \text{ for all } x, y \in K.$$

Obviously, we have the inclusions

$$\mathrm{Lip}(I;X) \subset H_{\mathrm{loc}}(I;X) \subset \mathrm{Din}(I;X) \subset C(I;X).$$

End of Paper

Bibliography

[1] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305. ISBN: 0-12-044143-8.

[2] Joannis Alexopoulos. *Nonlinear dynamics of periodic Lugiato–Lefever waves against sums of co-periodic and localized perturbations*. CRC 1173 Preprint 2025/17. Karlsruhe Institute of Technology, 2025. DOI: 10.5445/IR/1000181781. URL: https://www.waves.kit.edu/downloads/CRC1173_Preprint_2025-17.pdf.

[3] Joannis Alexopoulos. *Uniformity in the Fourier inversion formula with applications to Laplace transforms*. 2025. arXiv: 2503.23210 [math.FA]. URL: https://arxiv.org/abs/2503.23210.

[4] Joannis Alexopoulos and Björn de Rijk. *Nonlinear dynamics of reaction-diffusion wave trains under large and fully nonlocalized modulations*. CRC 1173 Preprint 2025/43. Karlsruhe Institute of Technology, 2025. DOI: 10.5445/IR/1000183766. URL: https://www.waves.kit.edu/downloads/CRC1173_Preprint_2025-43.pdf.

[5] Joannis Alexopoulos and Björn de Rijk. "Nonlinear stability of periodic wave trains in the FitzHugh-Nagumo system against fully nonlocalized perturbations". In: *preprint arXiv: 2409.17859* (2024). arXiv: 2409.17859 [math.AP]. URL: https://arxiv.org/abs/2409.17859.

[6] Wolfgang Arendt et al. *Vector-valued Laplace transforms and Cauchy problems*. Second. Vol. 96. Monographs in Mathematics. Basel: Birkhäuser/Springer Basel AG, Basel, 2011, pp. xii+539. ISBN: 978-3-0348-0086-0. DOI: 10.1007/978-3-0348-0087-7. URL: https://doi.org/10.1007/978-3-0348-0087-7.

[7] M. Avery et al. "Spectral stability of phase wave trains in the FitzHugh-Nagumo system". In preparation.

[8] Montie Avery et al. "Stability of coherent pattern formation through invasion in the FitzHugh-Nagumo system". In: *Journal of the European Mathematical Society* (2025).

[9] M. Bär et al. "Theoretical modeling of spatiotemporal self-organization in a surface catalyzed reaction exhibiting bistable kinetics". In: *J. Chem. Phys.* 96 (1992), pp. 8595–8604.

[10] I. V. Barashenkov and Yu. S. Smirnov. "Existence and stability chart for the ac-driven, damped nonlinear Schrödinger solitons". In: *Phys. Rev. E* 54 (5 1996), pp. 5707–5725. DOI: 10.1103/PhysRevE.54.5707. URL: https://link.aps.org/doi/10.1103/PhysRevE.54.5707.

[11] D. Barkley. "Simplifying the complexity of pipe flow". In: *Phys. Rev. E* 84 (2011), p. 016309.

[12] Lukas Bengel. "Stability of solitary wave solutions in the Lugiato-Lefever equation". In: *Z. Angew. Math. Phys.* 75.4 (2024), Paper No. 130, 19. ISSN: 0044-2275,1420-9039. DOI: 10.1007/s00033-024-02273-0. URL: https://doi.org/10.1007/s00033-024-02273-0.

[13] Lukas Bengel and Björn de Rijk. "Existence and stability of soliton-based frequency combs in the Lugiato-Lefever equation". In: *preprint arXiv: 2502.03189* (2025). arXiv: 2502.03189 [math.AP]. URL: https://arxiv.org/abs/2502.03189.

[14] J. L. Bona et al. "Dispersive blow-up for nonlinear Schrödinger equations revisited". In: *J. Math. Pures Appl. (9)* 102.4 (2014), pp. 782–811. ISSN: 0021-7824,1776-3371. DOI: 10.1016/j.matpur.2014.02.006. URL: https://doi.org/10.1016/j.matpur.2014.02.006.

[15] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599. ISBN: 978-0-387-70913-0.

[16] J. Bricmont and A. Kupiainen. "Renormalization group and the Ginzburg-Landau equation". In: *Comm. Math. Phys.* 150.1 (1992), pp. 193–208. ISSN: 0010-3616,1432-0916. URL: http://projecteuclid.org/euclid.cmp/1104251790.

[17] J. Bricmont, A. Kupiainen, and G. Lin. "Renormalization group and asymptotics of solutions of nonlinear parabolic equations". In: *Comm. Pure Appl. Math.* 47.6 (1994), pp. 893–922. ISSN: 0010-3640. DOI: 10.1002/cpa.3160470606. URL: https://doi.org/10.1002/cpa.3160470606.

[18] Anthony Carbery and Fernando Soria. "Almost-everywhere convergence of Fourier integrals for functions in Sobolev spaces, and an L^2-localisation principle". In: *Rev. Mat. Iberoamericana* 4.2 (1988), pp. 319–337. ISSN: 0213-2230. DOI: 10.4171/RMI/76. URL: https://doi.org/10.4171/RMI/76.

[19] Paul Carter, Björn de Rijk, and Björn Sandstede. "Stability of traveling pulses with oscillatory tails in the FitzHugh-Nagumo system". In: *J. Nonlinear Sci.* 26.5 (2016), pp. 1369–1444. ISSN: 0938-8974. DOI: 10.1007/s00332-016-9308-7. URL: https://doi.org/10.1007/s00332-016-9308-7.

[20] Paul Carter and Björn Sandstede. "Fast pulses with oscillatory tails in the FitzHugh-Nagumo system". In: *SIAM J. Math. Anal.* 47.5 (2015), pp. 3393–3441. ISSN: 0036-1410. DOI: 10.1137/140999177. URL: https://doi.org/10.1137/140999177.

[21] Paul Carter and Arnd Scheel. "Wave train selection by invasion fronts in the FitzHugh-Nagumo equation". In: *Nonlinearity* 31.12 (2018), pp. 5536–5572. ISSN: 0951-7715. DOI: 10.1088/1361-6544/aae1db. URL: https://doi.org/10.1088/1361-6544/aae1db.

[22] Thierry Cazenave and Alain Haraux. *An introduction to semilinear evolution equations.* Vol. 13. Oxford Lecture Series in Mathematics and its Applications. Translated from the 1990 French original by Yvan Martel and revised by the authors. The Clarendon Press, Oxford University Press, New York, 1998, pp. xiv+186. ISBN: 0-19-850277-X.

[23] Leonid Chaichenets. "Modulation spaces and nonlinear Schrödinger equations". PhD thesis. Karlsruhe Institute of Technology (KIT), 2018. URL: https://doi.org/10.5445/IR/1000088173.

[24] Leonid Chaichenets et al. "Knocking out teeth in one-dimensional periodic nonlinear Schrödinger equation". In: *SIAM J. Math. Anal.* 51.5 (2019), pp. 3714–3749. ISSN: 0036-1410,1095-7154. DOI: 10.1137/19M1249679. URL: https://doi.org/10.1137/19M1249679.

[25] Y.K. Chembo, M. Tlidi D. Gomila, and C.R. Menyuk. "Topical issue: theory and applications of the Lugiato-Lefever equation". In: *Eur. Phys. J. D* 71 (2017).

[26] P. Collet, J.-P. Eckmann, and H. Epstein. "Diffusive repair for the Ginzburg-Landau equation". In: *Helv. Phys. Acta* 65.1 (1992), pp. 56–92. ISSN: 0018-0238.

[27] Jonathan H. P. Dawes. "After 1952: the later development of Alan Turing's ideas on the mathematics of pattern formation". In: *Historia Math.* 43.1 (2016), pp. 49–64. ISSN: 0315-0860,1090-249X. DOI: 10.1016/j.hm.2015.03.003. URL: https://doi.org/10.1016/j.hm.2015.03.003.

[28] Lucie Delcey and Mariana Haragus. "Instabilities of periodic waves for the Lugiato-Lefever equation". In: *Rev. Roumaine Math. Pures Appl.* 63.4 (2018), pp. 377–399. ISSN: 0035-3965.

[29] A. Doelman et al. "The dynamics of modulated wave trains". In: *Mem. Amer. Math. Soc.* 199.934 (2009), pp. viii+105. ISSN: 0065-9266. DOI: 10.1090/memo/0934. URL: http://dx.doi.org/10.1090/memo/0934.

[30] Arjen Doelman. "Pattern formation in reaction-diffusion systems—an explicit approach". In: *Complexity science*. World Sci. Publ., Hackensack, NJ, 2019, pp. 129–182.

[31] A. Driouich and O. El-Mennaoui. "On the inverse Laplace transform for C_0-semigroups in UMD-spaces". In: *Arch. Math. (Basel)* 72.1 (1999), pp. 56–63. ISSN: 0003-889X,1420-8938. DOI: 10.1007/s000130050303. URL: https://doi.org/10.1007/s000130050303.

[32] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*. Vol. 194. Graduate Texts in Mathematics. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Berlin Heidelberg: Springer-Verlag, New York, 2000, pp. xxii+586. ISBN: 0-387-98463-1.

[33] E. G. Eszter. "An Evans function analysis of the stability of periodic travelling wave solutions of the FitzHugh-Nagumo system". PhD thesis. University of Massachusetts, 1999.

[34] R. FitzHugh. "Impulses and physiological states in theoretical models of nerve membrane". In: *Biophysical journal* 1.6 (1961), p. 445.

[35] Heinrich Freistühler and Denis Serre. "L^1 stability of shock waves in scalar viscous conservation laws". In: *Comm. Pure Appl. Math.* 51.3 (1998), pp. 291–301. ISSN: 0010-3640. DOI: 10.1002/(SICI)1097-0312(199803)51:3<291::AID-CPA4>3.3.CO;2-S. URL: https://doi.org/10.1002/(SICI)1097-0312(199803)51:3%3C291::AID-CPA4%3E3.3.CO;2-S.

[36] Hiroshi Fujita. "On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$". In: *J. Fac. Sci. Univ. Tokyo Sect. I* 13 (1966), pp. 109–124. ISSN: 0368-2269.

[37] Thierry Gallay and Alexander Mielke. "Diffusive mixing of stable states in the Ginzburg-Landau equation". In: *Comm. Math. Phys.* 199.1 (1998), pp. 71–97. ISSN: 0010-3616. DOI: 10.1007/s002200050495. URL: https://doi.org/10.1007/s002200050495.

[38] Thierry Gallay and Siniša Slijepčević. "Diffusive relaxation to equilibria for an extended reaction-diffusion system on the real line". In: *J. Evol. Equ.* 22.2 (2022), Paper No. 47, 33. ISSN: 1424-3199. DOI: 10.1007/s00028-022-00804-8. URL: https://doi.org/10.1007/s00028-022-00804-8.

[39] R. A. Gardner. "On the structure of the spectra of periodic travelling waves". In: *J. Math. Pures Appl. (9)* 72.5 (1993), pp. 415–439. ISSN: 0021-7824.

[40] Janina Gärtner and Wolfgang Reichel. "Soliton Solutions for the Lugiato–Lefever Equation by Analytical and Numerical Continuation Methods". In: *Mathematics of Wave Phenomena*. Ed. by Willy Dörfler et al. Cham: Springer International Publishing, 2020, pp. 179–195. ISBN: 978-3-030-47174-3.

[41] Vladimir Georgiev et al., eds. *Advances in harmonic analysis and partial differential equations*. Trends in Mathematics. Birkhäuser/Springer, Cham, 2020, pp. x+317. DOI: 10.1007/978-3-030-58215-9. URL: https://doi.org/10.1007/978-3-030-58215-9.

[42] Karlheinz Gröchenig. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001, pp. xvi+359. ISBN: 0-8176-4022-3. DOI: 10.1007/978-1-4612-0003-1. URL: https://doi.org/10.1007/978-1-4612-0003-1.

[43] Markus Haase. "The complex inversion formula revisited". In: *J. Aust. Math. Soc.* 84.1 (2008), pp. 73–83. ISSN: 1446-7887. DOI: 10.1017/S1446788708000050. URL: https://doi.org/10.1017/S1446788708000050.

[44] Sevdzhan Hakkaev, Milena Stanislavova, and Atanas G. Stefanov. "On the generation of stable Kerr frequency combs in the Lugiato-Lefever model of periodic optical waveguides". In: *SIAM J. Appl. Math.* 79.2 (2019), pp. 477–505. ISSN: 0036-1399,1095-712X. DOI: 10.1137/18M1192767. URL: https://doi.org/10.1137/18M1192767.

[45] Mariana Haragus, Mathew A. Johnson, and Wesley R. Perkins. "Linear modulational and subharmonic dynamics of spectrally stable Lugiato-Lefever periodic waves". In: *J. Differential Equations* 280 (2021), pp. 315–354. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2021.01.028. URL: https://doi.org/10.1016/j.jde.2021.01.028.

[46] Mariana Haragus et al. "Nonlinear modulational dynamics of spectrally stable Lugiato-Lefever periodic waves". In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 40.4 (2023), pp. 769–802. ISSN: 0294-1449,1873-1430. DOI: 10.4171/aihpc/65. URL: https://doi.org/10.4171/aihpc/65.

[47] Mariana Haragus et al. "Nonlinear subharmonic dynamics of spectrally stable Lugiato-Lefever periodic waves". In: *Comm. Math. Phys.* 405.10 (2024), Paper No. 227, 31. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-024-05104-5. URL: https://doi.org/10.1007/s00220-024-05104-5.

[48] Aart van Harten. "Modulated modulation equations". In: *Proceedings of the IUTAM/ISIMM Symposium on Structure and Dynamics of Nonlinear Waves in Fluids (Hannover, 1994)*. Vol. 7. Adv. Ser. Nonlinear Dynam. World Sci. Publ., River Edge, NJ, 1995, pp. 117–130.

[49] S. P. Hastings. "On the existence of homoclinic and periodic orbits for the FitzHugh-Nagumo equations". In: *Quarterly J. Mathematics* 27.1 (1976), pp. 123–134.

[50] Kantaro Hayakawa. "On nonexistence of global solutions of some semilinear parabolic differential equations". In: *Proc. Japan Acad.* 49 (1973), pp. 503–505. ISSN: 0021-4280. URL: http://projecteuclid.org/euclid.pja/1195519254.

[51] Bastian Hilder, Björn de Rijk, and Guido Schneider. "Nonlinear stability of periodic roll solutions in the real Ginzburg-Landau equation against C^m_{ub}-perturbations". In: *Comm. Math. Phys.* 400.1 (2023), pp. 277–314. ISSN: 0010-3616. DOI: 10.1007/s00220-022-04619-z. URL: https://doi.org/10.1007/s00220-022-04619-z.

[52] Peter Howard. "Short-time existence theory toward stability for nonlinear parabolic systems". In: *J. Evol. Equ.* 15.2 (2015), pp. 403–456. ISSN: 1424-3199,1424-3202. DOI: 10.1007/s00028-014-0266-y. URL: https://doi.org/10.1007/s00028-014-0266-y.

[53] Tuomas Hytönen et al. *Analysis in Banach Spaces: Volume I: Martingales and Littlewood-Paley Theory*. English. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer, 2016. ISBN: 978-3-319-48519-5. DOI: 10.1007/978-3-319-48520-1.

[54] Sameer Iyer and Björn Sandstede. "Mixing in reaction-diffusion systems: Large phase offsets". In: *Archive for Rational Mechanics and Analysis* 233.1 (2019), pp. 323–384. DOI: 10.1007/s00205-019-01358-9.

[55] M. A. Johnson and K. Zumbrun. "Nonlinear stability of spatially-periodic traveling-wave solutions of systems of reaction-diffusion equations". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28.4 (2011), pp. 471–483. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2011.05.003. URL: http://dx.doi.org/10.1016/j.anihpc.2011.05.003.

[56] M. A. Johnson et al. "Nonlocalized modulation of periodic reaction diffusion waves: nonlinear stability". In: *Arch. Ration. Mech. Anal.* 207.2 (2013), pp. 693–715. ISSN: 0003-9527. DOI: 10.1007/s00205-012-0573-9. URL: http://dx.doi.org/10.1007/s00205-012-0573-9.

[57] M. A. Johnson et al. "Nonlocalized modulation of periodic reaction diffusion waves: the Whitham equation". In: *Arch. Ration. Mech. Anal.* 207.2 (2013), pp. 669–692. ISSN: 0003-9527. DOI: 10.1007/s00205-012-0572-x. URL: https://doi.org/10.1007/s00205-012-0572-x.

[58] Mathew A. Johnson, Kevin Zumbrun, and Pascal Noble. "Nonlinear stability of viscous roll waves". In: *SIAM J. Math. Anal.* 43.2 (2011), pp. 577–611. ISSN: 0036-1410. DOI: 10.1137/100785454. URL: https://doi.org/10.1137/100785454.

[59] Mathew A. Johnson et al. "Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations". In: *Invent. Math.* 197.1 (2014), pp. 115–213. ISSN: 0020-9910. DOI: 10.1007/s00222-013-0481-0. URL: https://doi.org/10.1007/s00222-013-0481-0.

[60] C. K. R. T. Jones. "Geometric singular perturbation theory". In: *Dynamical systems (Montecatini Terme, 1994)*. Vol. 1609. Lecture Notes in Math. Springer, Berlin, 1995, pp. 44–118. DOI: 10.1007/BFb0095239. URL: http://dx.doi.org/10.1007/BFb0095239.

[61] C. K. R. T. Jones. "Stability of the travelling wave solution of the FitzHugh-Nagumo system". In: *Trans. Amer. Math. Soc.* 286.2 (1984), pp. 431–469. ISSN: 0002-9947. DOI: 10.2307/1999806. URL: http://dx.doi.org/10.2307/1999806.

[62] S. Jung. "Pointwise asymptotic behavior of modulated periodic reaction-diffusion waves". In: *J. Differential Equations* 253.6 (2012), pp. 1807–1861. ISSN: 0022-0396. URL: https://doi.org/10.1016/j.jde.2012.05.014.

[63] S. Jung and K. Zumbrun. "Pointwise nonlinear stability of nonlocalized modulated periodic reaction-diffusion waves". In: *J. Differential Equations* 261.7 (2016), pp. 3941–3963. ISSN: 0022-0396. DOI: 10.1016/j.jde.2016.06.013. URL: http://dx.doi.org/10.1016/j.jde.2016.06.013.

[64] T. Kapitula and K. Promislow. *Spectral and dynamical stability of nonlinear waves*. Vol. 185. Applied Mathematical Sciences. With a foreword by Christopher K. R. T. Jones. Springer New York, NY, 2013, pp. xiv+361. DOI: 10.1007/978-1-4614-6995-7. URL: https://doi.org/10.1007/978-1-4614-6995-7.

[65] Keiichi Kato, Masaharu Kobayashi, and Shingo Ito. "Estimates on modulation spaces for Schrödinger evolution operators with quadratic and sub-quadratic potentials". In: *J. Funct. Anal.* 266.2 (2014), pp. 733–753. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2013.08.017. URL: https://doi.org/10.1016/j.jfa.2013.08.017.

[66] T. Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Reprint of the 1980 edition. Springer-Verlag, Berlin, 1995, pp. xxii+619. ISBN: 3-540-58661-X.

[67] Yuri S. Kivshar and Barry Luther-Davies. "Dark optical solitons: physics and applications". In: *Physics Reports* 298.2 (1998), pp. 81–197. ISSN: 0370-1573. DOI: https://doi.org/10.1016/S0370-1573(97)00073-2. URL: https://www.sciencedirect.com/science/article/pii/S0370157397000732.

[68] Friedrich Klaus. "Nonlinear Schrödinger equations with rough data". PhD thesis. Karlsruhe Institute of Technology (KIT), 2022. URL: https://doi.org/10.5445/IR/1000153893.

[69] Friedrich Klaus and Peer Kunstmann. "Global wellposedness of NLS in $H^1(\mathbb{R})+H^s(\mathbb{T})$". In: *J. Math. Anal. Appl.* 514.2 (2022), Paper No. 126359, 14. ISSN: 0022-247X,1096-0813. DOI: 10.1016/j.jmaa.2022.126359. URL: https://doi.org/10.1016/j.jmaa.2022.126359.

[70] Peer Christian Kunstmann. "Modulation type spaces for generators of polynomially bounded groups and Schrödinger equations". In: *Semigroup Forum* 98.3 (2019), pp. 645–668. ISSN: 0037-1912. DOI: 10.1007/s00233-019-10016-1. URL: https://doi.org/10.1007/s00233-019-10016-1.

[71] Ji Li et al. *Nonlinear Stability of Large-Period Traveling Waves Bifurcating from the Heteroclinic Loop in the FitzHugh-Nagumo Equation.* 2025. arXiv: 2503.21509 [math.AP]. URL: https://arxiv.org/abs/2503.21509.

[72] L. A. Lugiato and R. Lefever. "Spatial dissipative structures in passive optical systems". In: *Phys. Rev. Lett.* 58.21 (1987), pp. 2209–2211. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.58.2209. URL: https://doi.org/10.1103/PhysRevLett.58.2209.

[73] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995, pp. xviii+424. ISBN: 3-7643-5172-1. DOI: 10.1007/978-3-0348-9234-6. URL: http://dx.doi.org/10.1007/978-3-0348-9234-6.

[74] S. Luther et al. "Low-energy control of electrical turbulence in the heart". In: *Nature* 475 (2011), pp. 235–239.

[75] Benjamin Melinand and L. Miguel Rodrigues. "Phase sinks and sources around two-dimensional periodic-wave solutions of reaction-diffusion-advection systems". In: *preprint arXiv:2408.14869* (2024).

[76] A. Mikhailov and G. Ertl. *Chemical complexity: self-organization processes in molecular systems.* The Frontiers Collection. Springer Cham, 2017.

[77] J. D. Murray. *Mathematical biology. I.* Third. Vol. 17. Interdisciplinary Applied Mathematics. An introduction. Springer-Verlag, New York, 2002, pp. xxiv+551. ISBN: 0-387-95223-3.

[78] J. Nagumo, S. Arimoto, and S. Yoshizawa. "An active pulse transmission line simulating nerve axon". In: *Proceedings of the IRE* 50.10 (1962), pp. 2061–2070.

[79] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Vol. 44. Applied Mathematical Sciences. Springer-Verlag, New York, 1983, pp. viii+279. ISBN: 0-387-90845-5. DOI: 10.1007/978-1-4612-5561-1. URL: https://doi.org/10.1007/978-1-4612-5561-1.

[80] Mark A. Pinsky. *Introduction to Fourier analysis and wavelets.* Brooks/Cole Series in Advanced Mathematics. Brooks/Cole, Pacific Grove, CA, 2002, pp. xviii+376. ISBN: 0-534-37660-6.

[81] Mark A. Pinsky and Michael E. Taylor. "Pointwise Fourier inversion: a wave equation approach". In: *J. Fourier Anal. Appl.* 3.6 (1997), pp. 647–703. ISSN: 1069-5869,1531-5851. DOI: 10.1007/BF02648262. URL: https://doi.org/10.1007/BF02648262.

[82] H. van der Ploeg and A. Doelman. "Stability of spatially periodic pulse patterns in a class of singularly perturbed reaction-diffusion equations". In: *Indiana Univ. Math. J.* 54.5 (2005), pp. 1219–1301. ISSN: 0022-2518. DOI: 10.1512/iumj.2005.54.2792. URL: http://dx.doi.org/10.1512/iumj.2005.54.2792.

[83] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, pp. xv+396. ISBN: 0-12-585004-2.

[84] Björn de Rijk. "Nonlinear Stability and Asymptotic Behavior of Periodic Wave Trains in Reaction–Diffusion Systems Against C_{ub}-perturbations". In: *Arch. Ration. Mech. Anal.* 248.3 (2024), Paper No. 36. ISSN: 0003-9527. DOI: 10.1007/s00205-024-01980-2. URL: https://doi.org/10.1007/s00205-024-01980-2.

[85] Björn de Rijk and Björn Sandstede. "Diffusive stability against nonlocalized perturbations of planar wave trains in reaction-diffusion systems". In: *J. Differential Equations* 265.10 (2018), pp. 5315–5351. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2018.07.011. URL: https://doi.org/10.1016/j.jde.2018.07.011.

[86] Björn de Rijk and Guido Schneider. "Global existence and decay in multi-component reaction-diffusion-advection systems with different velocities: oscillations in time and frequency". In: *NoDEA Nonlinear Differential Equations Appl.* 28.1 (2021), Paper No. 2, 38. ISSN: 1021-9722,1420-9004. DOI: 10.1007/s00030-020-00665-5. URL: https://doi.org/10.1007/s00030-020-00665-5.

[87] Björn de Rijk and Guido Schneider. "Global existence and decay in nonlinearly coupled reaction-diffusion-advection equations with different velocities". In: *J. Differential Equations* 268.7 (2020), pp. 3392–3448. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2019.09.056. URL: https://doi.org/10.1016/j.jde.2019.09.056.

[88] L. Miguel Rodrigues and Kevin Zumbrun. "Periodic-coefficient damping estimates, and stability of large-amplitude roll waves in inclined thin film flow". In: *SIAM J. Math. Anal.* 48.1 (2016), pp. 268–280. ISSN: 0036-1410. DOI: 10.1137/15M1016242. URL: https://doi.org/10.1137/15M1016242.

[89] Björn Sandstede et al. "Diffusive mixing of periodic wave trains in reaction-diffusion systems". In: *J. Differential Equations* 252.5 (2012), pp. 3541–3574. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2011.10.014. URL: https://doi.org/10.1016/j.jde.2011.10.014.

[90] Arnd Scheel and Qiliang Wu. "Diffusive stability of Turing patterns via normal forms". In: *J. Dynam. Differential Equations* 27.3-4 (2015), pp. 1027–1076. ISSN: 1040-7294. DOI: 10.1007/s10884-013-9335-0. URL: https://doi.org/10.1007/s10884-013-9335-0.

[91] Guido Schneider. "Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation". In: *Comm. Math. Phys.* 178.3 (1996), pp. 679–702. ISSN: 0010-3616,1432-0916. URL: http://projecteuclid.org/euclid.cmp/1104286771.

[92] Guido Schneider. "Nonlinear diffusive stability of spatially periodic solutions—abstract theorem and higher space dimensions". In: *Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997)*. Vol. 8. Tohoku Math. Publ. Tohoku Univ., Sendai, 1998, pp. 159–167.

[93] Guido Schneider. "Nonlinear stability of Taylor vortices in infinite cylinders". In: *Arch. Rational Mech. Anal.* 144.2 (1998), pp. 121–200. ISSN: 0003-9527. DOI: 10.1007/s002050050115. URL: https://doi.org/10.1007/s002050050115.

[94] Guido Schneider and Hannes Uecker. *Nonlinear PDEs. A dynamical systems approach.* English. Vol. 182. Providence, RI: American Mathematical Society (AMS), 2017, pp. xiii + 575.

[95] Cristina Soto-Treviño. "A geometric method for periodic orbits in singularly-perturbed systems". In: *Multiple-time-scale dynamical systems (Minneapolis, MN, 1997)*. Vol. 122. IMA Vol. Math. Appl. Springer, New York, 2001, pp. 141–202. DOI: 10.1007/978-1-4613-0117-2_6. URL: https://doi.org/10.1007/978-1-4613-0117-2_6.

[96] Milena Stanislavova and Atanas G. Stefanov. "Asymptotic stability for spectrally stable Lugiato-Lefever solitons in periodic waveguides". In: *J. Math. Phys.* 59.10 (2018), pp. 101502, 12. ISSN: 0022-2488,1089-7658. DOI: 10.1063/1.5048017. URL: https://doi.org/10.1063/1.5048017.

[97] Alim Sukhtayev et al. "Diffusive stability of spatially periodic solutions of the Brusselator model". In: *Comm. Math. Phys.* 358.1 (2018), pp. 1–43. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-017-3056-x. URL: https://doi.org/10.1007/s00220-017-3056-x.

[98] J. Swift and P. C. Hohenberg. "Hydrodynamic fluctuations at the convective instability". In: *Phys. Rev. A* 15 (1 1977), pp. 319–328. DOI: 10.1103/PhysRevA.15.319. URL: https://link.aps.org/doi/10.1103/PhysRevA.15.319.

[99] A. M. Turing. "The chemical basis of morphogenesis". In: *Philos. Trans. Roy. Soc. London Ser. B* 237.641 (1952), pp. 37–72. ISSN: 0080-4622.

[100] Hannes Uecker. "Self-similar decay of spatially localized perturbations of the Nusselt solution for the inclined film problem". In: *Arch. Ration. Mech. Anal.* 184.3 (2007), pp. 401–447. ISSN: 0003-9527. DOI: 10.1007/s00205-006-0027-3. URL: https://doi.org/10.1007/s00205-006-0027-3.

[101] S. Wabnitz. "Suppression of interactions in a phase-locked soliton optical memory". In: *Opt. Lett.* 18.8 (1993), pp. 601–603. DOI: 10.1364/OL.18.000601. URL: https://opg.optica.org/ol/abstract.cfm?URI=ol-18-8-601.

[102] Fred B. Weissler. "Existence and nonexistence of global solutions for a semilinear heat equation". In: *Israel J. Math.* 38.1-2 (1981), pp. 29–40. ISSN: 0021-2172. DOI: 10.1007/BF02761845. URL: https://doi.org/10.1007/BF02761845.

[103] E. Yanagida. "Stability of fast travelling pulse solutions of the FitzHugh-Nagumo equations". In: *J. Math. Biol.* 22.1 (1985), pp. 81–104. ISSN: 0303-6812. DOI: 10.1007/BF00276548. URL: http://dx.doi.org/10.1007/BF00276548.

[104] K. Zumbrun and P. Howard. "Pointwise semigroup methods and stability of viscous shock waves". In: *Indiana Univ. Math. J.* 47.3 (1998), pp. 741–871. ISSN: 0022-2518. DOI: 10.1512/iumj.1998.47.1604. URL: http://dx.doi.org/10.1512/iumj.1998.47.1604.

[105] Kevin Zumbrun. "Forward-modulated damping estimates and nonlocalized stability of periodic Lugiato-Lefever waves". In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 41.2 (2024), pp. 497–510. ISSN: 0294-1449,1873-1430. DOI: 10.4171/aihpc/76. URL: https://doi.org/10.4171/aihpc/76.