Linear Programming Bounds for Non-Euclidean Sphere Packings

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Introduction

This thesis is concerned with the connection of two streams of research in the theory of sphere packings and codes. The first is the development of linear programming bounds for invariants of sphere packings. The second is the study of random sphere packings in homogeneous spaces, as initiated by Bowen and Radin for hyperbolic space.

Our main aim is to demonstrate that the study of random sphere packings leads very naturally to linear programming bounds for invariants of sphere packings, once a range of tools from stochastic geometry is adapted to homogeneous spaces. With these methods we obtain general linear programming bounds on sphere packing invariants for a wide range of homogeneous spaces.

As special cases we obtain linear programming bounds for packing invariants of homogeneous spaces for which no such bounds have been available so far. One important example is the linear programming bound on the optimal packing density of hyperbolic space, which was conjectured by Cohn and Zhao.

In the following we first describe the known linear programming bounds for sphere packings in $\mathbb{R}^n$ and for spherical and binary codes. Then we will contrast this with our generalizations of these bounds and give an overview of the methods used to obtain these results.

Codes and Euclidean sphere packings

Let (X, d_X) be a metric space. An *r-sphere packing* P in X is a collection of disjoint open balls in X with radius r. Classically, three examples are of particular importance: the case where X is the n-sphere S^n with the angular distance, the case where X is $\mathbb{F}_2^n$ with the Hamming distance and the case where X is given by Euclidean n-space. The first two cases are of particular practical importance, as they have applications in the field of error correcting codes. In the first case P is also called a *spherical code* with *separation* $2r$ and in the second case P is called a *binary code* with *minimal distance* $2r + 1$.

However, the case $X = \mathbb{R}^n$ is the most notorious, primarily because of the difficulty of determining the invariant

$$\Delta(\mathbb{R}^n, r) := \sup_P \limsup_{R\to\infty} \frac{\lambda(B(0,R) \cap \bigcup P)}{\lambda(B(0,R))},$$

with the supremum taken over all r-sphere packings of $\mathbb{R}^n$. $\Delta(\mathbb{R}^n, r)$ is called the *optimal packing density* of $\mathbb{R}^n$. Note that this invariant does not depend on r. It is known that there always exists an r-sphere packing P such that

$$\Delta(\mathbb{R}^n, r) = D(P) := \lim_{R\to\infty} \frac{\lambda(B(0,R) \cap \bigcup P)}{\lambda(B(0,R))};$$

the number $D(P)$ is then called the *density* of P. Thus we can equivalently define $\Delta(\mathbb{R}^n, r)$ as the supremum of the densities of all r-sphere packings which have a density. For $n = 2$, Fejes Tóth showed in [49] that the so-called honeycomb packing P satisfies $\Delta(\mathbb{R}^2, r) = D(P)$. In dimension 3 Hales showed in [62] that the density of the so-called canonball packing equals $\Delta(\mathbb{R}^3, r)$. Both Fejes Tóth's and Hales' approaches to the problem are geometric and Hales' approach was first suggested by Fejes Tóth.
The only other cases where $\Delta(\mathbb{R}^n, r)$ has been determined are $n = 8$ and $n = 24$. The case $n = 8$ was settled by Viazovska in [96] and the case $n = 24$ by Cohn, Kumar, Miller, Radchenko and Viazovska in [38], following the approach used by Viazovska in the case $n = 8$.
In both of these dimensions it is possible to obtain upper bounds on $\Delta(\mathbb{R}^n, r)$ which agree with densities of specific sphere packings (the E8 packing for $n = 8$ and the Leech lattice packing for $n = 24$). Thus $\Delta(\mathbb{R}^n, r)$ must equal the density of these specific packings. These upper bounds are obtained using the following linear programming bound by Cohn and Elkies:

Theorem (Cohn–Elkies, [36])**.** We have

$$\triangle(\mathbb{R}^n, r) \leq \lambda(B(0,r)) \frac{f(0)}{\widehat{f}(0)}$$

for every f in a certain set $\mathcal{W}(\mathbb{R}^n, r)$ of *witness functions.*

Here an integrable $f : \mathbb{R}^n \to \mathbb{R}$ is in $\mathcal{W}(\mathbb{R}^n, r)$ if

(W1) $f(x) \leq 0$ for $\|x\| \geq 2r$,

(W2) $\widehat{f} \geq 0$ and $\widehat{f}(0) > 0$,

(W3) f and $\widehat{f}$ satisfy certain decay conditions.

The problem of determining $\triangle(\mathbb{R}^n, r)$ is related to an energy minimization problem. A set $P \subset \mathbb{R}^n$ such that $\|x - y\| \geq r$ for all $x \neq y \in P$ is called r-uniformly discrete and we will denote the set of all r-uniformly discrete sets in $\mathbb{R}^n$ by $\mathrm{UD}_r(\mathbb{R}^n)$. For any function $p : [0, \infty) \to [0, \infty)$, which we call a *potential function*, and any point set $P \subset \mathbb{R}^n$ which is r-uniformly discrete for some $r > 0$ the *p-energy* of P is defined as

$$E_p(P) := \liminf_{R\to\infty} \frac{1}{\#(P \cap B(0,R))} \sum_{x\neq y \in B(0,R)\cap P} p(\|x - y\|).$$

As the problem of minimizing $E_p(P)$ is trivial if one allows the set P to be spread arbitrarily "thin", one introduces a quantity $i(P)$, called the *intensity* of P, which obstructs arbitrarily thin spread of P. More precisely, we say that P has intensity $i(P)$ if

$$i(P) = \lim_{R\to\infty} \frac{\#(B(0,R) \cap P)}{\lambda(B(0,R))}.$$

Now the *p-energy minimization problem with intensity* δ is the task of determining

$$E(p,\delta) := \inf_{P} E_p(P),$$

where the infimum is taken over all P with intensity δ.
If we consider the potential $p_r := \chi_{[0,2r)}$, then $E_{p_r}(P) = 0$ if and only if P is $2r$-uniformly discrete. Moreover, if P has intensity δ and $E_{p_r}(P) = 0$, then we can construct an r-sphere packing P^r consisting of the balls of radius r centered at the points in P. Then P^r will have density $D(P^r) = \lambda(B(0,r))\delta$. Thus a solution of the energy minimization problem with intensity δ for every potential p_s, $s > 0$, and every $\delta > 0$ would solve the problem of determining $\Delta(\mathbb{R}^n, r)$.
Recently Cohn, Kumar, Miller, Radchenko and Viazovska have shown in [39] that the E8 lattice in dimension 8 and the Leech lattice in dimension 24 are energy-minimizing for all completely monotonic potential functions. As in the case of sphere packings in dimension 8 and 24, their proof uses a linear programming bound:

Theorem (Cohn–Kumar [37], Cohn–de Courcy-Ireland [34])**.** For $\delta > 0$ and any potential function p we have

$$E(p,\delta) \geq \delta \widehat{f}(0) - f(0)$$

for all integrable, continuous $f : \mathbb{R}^n \to \mathbb{R}$ satisfying

(i) $f(x) \leq p(\|x\|)$ for all $x \in \mathbb{R}^n \setminus \{0\}$,

(ii) $\widehat{f} \geq 0$.

All notions and results in the preceding paragraphs have analogues for spherical and binary codes. More specifically, let X denote either S^n or $\mathbb{F}_2^n$ and let m_X denote either the surface measure on S^n or the counting measure on $\mathbb{F}_2^n$. Let P be a spherical code with angular separation $2r$ or a binary code with minimal distance $2r+1$. Then the density of P is simply given by

$$D(P) = \frac{m_X(B(x_0,r))}{m_X(X)} \#P,$$

where $x_0 \in X$ is an arbitrary point. Thus, if one thinks in terms of error correcting codes, the density measures how many different pieces of information an n-bit code can encode whilst maintaining a given quality (i.e. minimal separation). For the density of binary codes a linear programming bound was obtained by Delsarte in [46] and for the density of spherical codes a linear programming bound was obtained by Kabatjanskiĭ and Levenšteĭn in [67].
In both of these cases the space X can be written as G/K with G a locally compact group and K a compact subgroup; the sphere being given by $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{F}_2^n$ being given by $(\mathbb{F}_2^n \rtimes S_n)/S_n$, where we consider $\mathbb{F}_2^n \rtimes S_n$ as a subgroup of the affine motion group of $\mathbb{F}_2^n$ by sending elements of S_n to permutation matrices. This enables the introduction of a representation theoretic analogue of the Fourier transform for (sufficiently regular) *radial functions* f on S^n or $\mathbb{F}_2^n$, i.e. those functions which only depend on the distance to a chosen fixed basepoint x_0. This transform is called the

spherical transform and denoted by $\widehat{f}$. Now the following linear programming bound holds for either $(G, K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$ or $(G, K) = (\mathbb{F}_2^n \rtimes S_n, S_n)$.

Theorem (Delsarte [46], Kabatjanskiĭ and Levenšteĭn [67])**.** For every r-sphere packing P in X we have

$$D(P) \leq m_X(B(x_0, r)) \frac{f(x_0)}{\widehat{f}(\mathbf{1})},$$

where $\mathbf{1}$ is the trivial character on G and f is in a certain set $\mathcal{W}(X, r)$ of *witness functions.*

Here a radial function $f : X \to \mathbb{R}$ is in $\mathcal{W}(X, r)$ if

(W1) $f(x) \leq 0$ if $d(x, x_0) \geq 2r$,

(W2) $\widehat{f} \geq 0$ and $\widehat{f}(\mathbf{1}) > 0$,

(W3) f and $\widehat{f}$ satisfy certain regularity conditions.

Similarly, one can extend the notions of energy and the linear programming bound for energy to spherical codes and binary codes, see the article [37] by Cohn and Kumar and the article [40] by Cohn and Zhao as well as the references therein. All of this also generalizes to other sufficiently nice compact homogeneous spaces.

Packings in non-Euclidean geometries

If one tries to further generalize these results to non-compact non-Euclidean geometries, several issues with the definition of density we have used above become serious enough to require more attention.

Let us focus on the hyperbolic n-space $\mathbb{H}^n = \mathrm{SO}(n,1)_0/\mathrm{SO}(n)$ for now and let P be an r-sphere packing in $\mathbb{H}^n$. In [25] Böröczky pointed out that the exponential volume growth in hyperbolic space leads to the fact that the quantity

$$\overline{D}(P, x) := \limsup_{R \to \infty} \frac{m_{\mathbb{H}^n}(B(x, R) \cap \bigcup P)}{m_{\mathbb{H}^n}(B(x, R))}$$

has a fundamental dependence on the point $x \in \mathbb{H}^n$, as it is dominated by effects near the boundary of $B(x, R)$. Even if the limes superior is realized as a limit, these effects still occur. This does not happen in Euclidean space, which is why we could use the fixed basepoint $0 \in \mathbb{R}^n$ in our definitions of $\Delta(\mathbb{R}^n, r)$ and $D(P)$.

This issue severely complicates the theory of sphere packings in $\mathbb{H}^n$ and led Bowen and Radin to define a notion of optimal packing density for $\mathbb{H}^n$ in [28, 29] in terms of random sphere packings. We will now describe the basics of their approach in terms of point processes as we formulate it in Chapter 3 . Note that this is equivalent to their original formalism.

The set

$$\mathcal{N}^*_{2r}(\mathbb{H}^n) := \Big\{ \sum_{x \in P} \delta_x \mid P \in \mathrm{UD}_{2r}(\mathbb{H}^n) \Big\}$$

can be equipped with a compact metric such that the group $G := \mathrm{SO}(n,1)_0$ acts on it continuously. A bijection with the set of all r-sphere packings is given by identifying an element $\mu = \sum_{x \in P} \delta_x \in \mathcal{N}^*_{2r}(\mathbb{H}^n)$ with the sphere packing

$$\text{supp}(\mu)^r := \{B(x, r) \mid x \in P\}.$$

Sometimes we will identify μ with P and write $x \in \mu$ instead of $x \in P$ to simplify notation. An *invariant random r-sphere packing* is a random variable $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*_{2r}(\mathbb{H}^n)$ which has a G-invariant distribution. In the terminology common in stochastic geometry Λ is a stationary hard core point process with hard core distance $2r$. The *density* of Λ is defined as

$$D_r(\Lambda) = \mathbb{P}(x \in \bigcup \text{supp}(\Lambda)^r),$$

where the choice of $x \in \mathbb{H}^n$ does not matter because of the G-invariance of the distribution of Λ. Using an ergodic theorem for G-actions, Bowen and Radin now show that

$$D(\Lambda) = \lim_{R \to \infty} \frac{m_{\mathbb{H}^n}(B(x, R) \cap \bigcup \text{supp}(\Lambda_\omega)^r)}{m_{\mathbb{H}^n}(B(x, R))}$$

for almost all $\omega \in \Omega$ and all $x \in \mathbb{H}^n$ (where $m_{\mathbb{H}^n}$ is the hyperbolic volume measure) if the distribution of Λ is G-ergodic. Thus calling $D(\Lambda)$ a density is justified. Now a notion of *probabilistic optimal packing density* of $\mathbb{H}^n$ is defined by Bowen and Radin as

$$\Delta_{\text{prob}}(\mathbb{H}^n, r) := \sup_{\Lambda} D_r(\Lambda),$$

with the supremum taken over all invariant random r-sphere packings (ignoring set theoretic issues).

In this thesis we develop a second formulation of Bowen and Radin's formalism via the notion of a *generically measured* element of $\text{UD}_{2r}(X)$. These are sets which are essentially generic points of some G-ergodic measure μ on $\text{UD}_{2r}(X)$. To any such P we can associate an invariant random r-sphere packing $\Lambda^P : (\text{UD}_{2r}(X), \mu) \to \mathcal{N}^*_{2r}(X),\ Q \mapsto \sum_{x \in Q} \delta_x$ with ergodic distribution. We show that any random invariant r-sphere packing Λ with ergodic distribution can be obtained in this way. We then show that all functions to which Bowen and Radin apply ergodic theorems in their work are in some sense Riemann integrable. This allows us to show that the r-sphere packing P^r associated to P satisfies

$$D_r(\Lambda^P) = \lim_{R \to \infty} \frac{m_{\mathbb{H}^n}(B(x_0, R) \cap \bigcup P^r)}{m_{\mathbb{H}^n}(B(x_0, R))}$$

for all $x \in \mathbb{H}^n$; hence it is meaningful to say that P^r has packing density $D(P^r) := \overline{D}(P^r, x_0)$. This further allows us to show

$$\Delta_{\text{prob}}(\mathbb{H}^n, r) = \sup_{P} D(P^r),$$

where the supremum is taken over all generically measured elements of $\text{UD}_{2r}(X)$.

If $P \in \text{UD}_{2r}(\mathbb{H}^n)$ is a union of finitely many lattice orbits, we call P *weakly-periodic* and note that P is generically measured. In the terminology of Bowen and Radin P^r is called a periodic sphere packing.

Cohn, Lurie and Sarnak obtained a linear programming bound on the density $D(P^r)$ of weakly-periodic P. Note that we again have a representation theoretic notion of spherical transform for radial functions on $\mathbb{H}^n = \mathrm{SO}(n,1)_0/\mathrm{SO}(n)$, i.e. functions $f : \mathbb{H}^n \to \mathbb{R}$ such that $f(x)$ only depends on $d_{\mathbb{H}^n}(x, x_0)$, where $x_0 = e\mathrm{SO}(n)$.

Theorem (Cohn–Lurie–Sarnak, [41]). Let $P \in \mathrm{UD}_{2r}(X)$ be weakly-periodic. Then

$$D(P^r) \leq m_{\mathbb{H}^n}(B(x_0, r)) \frac{f(x_0)}{\widehat{f}(\mathbf{1})}$$

for all radial functions $f\colon \mathbb{H}^n \to \mathbb{R}$ with

(W1) $f(x) \leq 0$ if $d_{\mathbb{H}^n}(x, x_0) \geq 2r$,

(W2) $\widehat{f} \geq 0$, where $\widehat{f}$ denotes the spherical transform of f,

(W3) f is admissible, see [41, pg. 1989].

This led Cohn and Zhao to conjecture in [41] that this bound holds more generally for $\Delta_{\mathrm{prob}}(\mathbb{H}^n, r)$. In this thesis we obtain the following linear programming bound on $\Delta_{\mathrm{prob}}(\mathbb{H}^n, r)$.

Theorem A. *We have*

$$\Delta_{\mathrm{prob}}(\mathbb{H}^n, r) \leq m_{\mathbb{H}^n}(B(x, r)) \frac{f(x_0)}{\widehat{f}(\mathbf{1})},$$

where f is in a set $\mathcal{W}(\mathbb{H}^n, r)$ of witness functions*.*

Here $f \in \mathcal{W}(\mathbb{H}^n, r)$ if and only if

(W1) $f(x) \leq 0$ if $d_{\mathbb{H}^n}(x, x_0) \geq 2r$,

(W2) $\widehat{f} \geq 0$ and $\widehat{f}(\mathbf{1}) > 0$,

(W3) f is a Harish-Chandra L^1-Schwartz function.

We should note that Cohn and Zhao conjectured a slightly bigger function space, namely the space of all continuous, integrable functions instead of the space of Harish-Chandra L^1-Schwartz functions. We also note that the spherical transform $\widehat{f}$ of f can be computed very explicitly, see Remark 1.4.9.

The bound given by Theorem A is actually a special case of a theorem in the following general setup:

- Let G be a Lie group and $K < G$ a compact subgroup such that (G, K) is a Gelfand pair, see Chapter 1. We set $x_0 = eK$ and $\pi : G \to G/K, g \mapsto gx_0$.
- d_X is a G-invariant, proper and continuous metric on $X = G/K$.
- The invariant pointwise ergodic theorem holds for $(C(\mathrm{UD}_{2r}(G/K)), (G_t)_{t>0})$, with $G_t = \pi^{-1}(B(x_0, t))$, see Definition 2.4.2.
- $\mathcal{S}(G, K)$ denotes a Schwartz-like function space, see Definition 1.5.1.

We then call $(G, K, d_X, \mathcal{S}(G,K))$ a *convenient Gelfand pair.* As a volume measure on X we take the unique G-invariant Borel measure m_X on X (see Section 1.1 for the normalization of m_X).

Bowen and Radin's method for handling sphere packing density now directly extends to sphere packings in the homogeneous space $X = G/K$. For all noncompact amenable explicit examples of G we consider, the necessary ergodic theorems go back to work by Calderon in the 1950s. In the semisimple case we rely on recent pointwise ergodic theorems by Gorodnik and Nevo, [56]. In fact, the availability of these new ergodic theorems motivated the start of our investigations into Bowen and Radin's approach to packing density. Finally we should mention that Bowen and Radin's approach recovers the ordinary optimal density bounds the case $G = \mathbb{R}^n$ and $K = \{0\}$ and for compact G.

As Gelfand pairs come with a version of the spherical transform, we can obtain linear programming bounds in this general setup. In this thesis we obtain the following density bound, which implies Theorem A as a special case:

Theorem B. *We have*

$$\Delta_{\text{prob}}(X, r) \leq m_X(B(x,r)) \frac{f(e)}{\widehat{f}(\mathbf{1})},$$

where f is in a set $\mathcal{W}(X,r)$ of witness functions*.*

Here $f : G \to \mathbb{R}$ is in $\mathcal{W}(X,r)$ if and only if

(W1) $f(g) \leq 0$ if $d_X(gx_,x_0) \geq 2r$,

(W2) $\widehat{f} \geq 0$ and $\widehat{f}(\mathbf{1}) > 0$,

(W3) $f \in \mathcal{S}(G,K)$.

This result is very general and implies the previously mentioned linear programming bounds on density by Delsarte, Kabatjanskiĭ and Levenšteĭn, and Cohn and Elkies.

In analogy to Bowen and Radin's definition of density, we define a notion of *energy* of an invariant random r-sphere packing Λ in X as

$$E_p(\Lambda) := \frac{1}{i(\Lambda)} \mathbb{E}\left[\frac{1}{m_X(B(x_0,R))} \sum_{x \in \Lambda \cap B(x_0,R)} \sum_{y \in \Lambda} p(d_X(x,y))\right] - p(0),$$

where it turns out that the choice of $R > 0$ does not matter and $i(\Lambda) = D_r(\Lambda) m_X(B(x_0,r))^{-1}$ is the intensity of Λ as a point process. This definition is based on Björklund and Byléhn's definition of the autocorrelation measure of Λ in [14].

We prove the following energy sampling theorem, showing that $E_p(\Lambda)$ deserves the name energy:

Theorem C. *Let Λ be a random invariant r-sphere packing in X with G-ergodic distribution. Assume that the potential function p is continuous, monotonically decreasing* *and rapidly decaying in relation to the volume growth*[1] *of X. Then*

$$E_p(\Lambda) = \lim_{R\to\infty} \frac{1}{\#(B(x_0,R)\cap\Lambda_\omega)} \sum_{x\in\Lambda_\omega\cap B(x_0,R)} \sum_{x\neq y\in\Lambda_\omega} p(d_X(x,y))$$

for almost all $\omega \in \Omega$, assuming the sequence $(G_t)_{t>0}$ is very convenient, see Definition 2.5.7.

For $\delta > 0$ we define the quantity

$$E(p,\delta) = \inf_{\Lambda} E_p(\Lambda),$$

with the infimum taken over all random invariant r-sphere packings with $i(\Lambda) = \delta$ and $r > 0$ (not fixed!). We obtain the following linear programming bound for $E(p,\delta)$:

Theorem D. *We have*

$$E(p,\delta) \geq \delta \widehat{f}(\mathbf{1}) - f(e),$$

where $f : G \to \mathbb{R}$ satisfies

(i) $f(g) \leq p(d_X(gx_0, x_0))$ *for* $g \in G \setminus K$,

(ii) $\widehat{f} \geq 0$,

(iii) $f \in \mathcal{S}(G,K)$.

Note that this generalizes the Euclidean linear programming bound for the energy.

Methods

We obtain our results by embedding Bowen and Radin's framework for packing density in the wider theory of point processes in homogeneous spaces. For homogeneous spaces of Gelfand pairs (often called commutative spaces), an extension of the usual spectral theory of point processes was developed by Björklund, Hartnick and Pogorzelski for the study of so-called model sets in [20, 21]. Björklund and Byléhn further refined the theory in [14, 15] for questions related to hyperuniformity.

More specifically Björklund, Hartnick and Pogorzelski first define the autocorrelation of locally bounded point processes in $X = G/K$ as a measure on $K\backslash G/K$ and then apply spherical harmonic analysis to this measure. We use the following definition, given by Björklund and Byléhn for locally square integrable point processes Λ in X. The *autocorrelation measure* of Λ is the unique bi-K-invariant measure η_Λ^+ on G such that

$$\eta_\Lambda^+(f) = \mathbb{E}\left[\int_X \int_X f(\sigma(x)^{-1}\sigma(y)) b(x) d\Lambda(y) d\Lambda(x)\right],$$

where $f : G \to \mathbb{C}$ is left and right K-invariant and continuous with compact support, $\sigma : X \to G$ is a Borel section and $b : X \to [0,\infty)$ is measurable with $m_X(b) = 1$ and bounded support. Note the similarity between this measure and our definition of energy.

[1] See Theorem 2.6.4 for the precise condition.

The measure η_Λ^+ and the measure $\eta_\Lambda := \eta_\Lambda^+ - i(\Lambda)^2 m_G$, where m_G is the Haar measure on G, are both positive-definite measures, which allows the definition of an analogue of the spectral measure of a point process in this general context.

For this, one proceeds by first defining a class of functions, called *positive-definite spherical functions*, which serve as replacements for the functions $e_k(x) := \exp(ixk)$, $k \in \mathbb{R}$, in the definition of the Fourier transform. We denote the space of all positive-definite spherical functions by $PS(G,K)$. Then one defines the *spherical transform* $\widehat{f}$ of a function $f \in L^1(G)$ by

$$\widehat{f} : PS(G,K) \to \mathbb{C},\ \omega \mapsto \widehat{f}(\omega) := \int f(g)\omega(g^{-1})dm_G(g).$$

Gelfand pairs are essentially defined as pairs (G,K) for which this procedure works and yields a reasonably well-behaved variant of the Fourier transform. This theory is very well-developed and enters deeply in the representation theory of semisimple Lie groups, see for instance [64]. Now the Godement-Plancherel theorem ensures the existence of unique positive Borel measures $\widehat{\eta}_\Lambda^+$ and $\widehat{\eta}_\Lambda$ on $PS(G,K)$ (appropriately topologized) such that

$$\eta_\Lambda^+(f^* * f) = \widehat{\eta}_\Lambda^+(|\widehat{f}|^2) \quad \text{and} \quad \eta_\Lambda(f^* * f) = \widehat{\eta}_\Lambda(|\widehat{f}|^2)$$

for all $f \in C_c(G,K)$. One can think of these formulas as "Poisson summation" for point processes. In the case of point processes coming from the theory of quasi-crystals, they yield exotic Poisson summation formulas, see [20]. Alternatively one can view $\widehat{\eta}_\Lambda$ as an analogue for the spectral measure of a point process in $\mathbb{R}^n$ for point processes in G/K.

In fact, these formulas are close enough analogues to the Poisson summation formula to allow us to transfer Cohn and Elkies Poisson summation based proof of the linear programming bound on the Euclidean optimal packing density to this general setting. Our proof for the linear programming bound on the energy uses a similar method.

We obtain the sampling theorem for the energy by rewriting the energy of Λ in terms of the autocorrelation measure and rewriting the autocorrelation measure in terms of the Palm measure $\mathbb{P}_\Lambda$ of Λ, as defined by Last in [75]. Then we show the following sampling theorem for the Palm measure, which we can apply to obtain the energy sampling result:

Theorem E. *Let $(G_t)_{t>0}$ be a nice sequence of Borel sets with positive measure in G and assume that a strong enough ergodic theorem*[2] *holds for $(G_t)_{t>0}$. Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(X)$ be a $2r$-uniformly discrete (i.e. hard core), ergodic, G-invariant point process. Then there is a G-invariant conull set $\Omega_0 \subset \Omega$ such that*

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{x\in\Lambda_\omega\cap\pi(G_t)} f(\sigma(x)^{-1}\Lambda_\omega) = \mathbb{P}_\Lambda(f(\Lambda))$$

for every $f \in C(\mathcal{N}_{2r}^*(X))$ which is invariant under the action of K on $\mathcal{N}_{2r}^*(X)$, and every $\omega \in \Omega_0$.

[2] See Theorem 2.5.9 for the precise conditions.

Organization of this thesis

In the **first chapter** we go over the spherical harmonic analysis on Gelfand pairs needed to develop the spectral theory of point processes in commutative spaces to the level required for the linear programming bounds. Only two things are novel in this chapter, our definition of Schwartz-like function spaces and the Bochner theorem for spherical distributions on the Heisenberg group, which we prove in Appendix C. It seems likely that this Bochner theorem is known to experts on the harmonic analysis of the Heisenberg group, but we could not find a suitable reference in the literature. The material in this section corresponds quite closely to the first section of my preprint [98].

In the **second chapter** we go over the point process theory needed to handle invariant random r-sphere packings. The material here is collected from a number of different sources. In this chapter we prove several new results. We prove a sampling theorem for Palm measures of hard core point processes. We also define the energy of hard core stationary point processes and use the Palm sampling theorem to prove our energy sampling theorem. We also introduce the concept of generically measured sets, which has been implicit in the literature on mathematical quasicrystals for some time.

The **third chapter** is concerned with Bowen and Radin's theory of packing density in $\mathbb{H}^n$. We give a survey of their theory in the generality of homogeneous spaces with sufficiently strong ergodic theorems using the language of point processes. Bowen and Radin's showed in [28] that periodic sphere packings in $\mathbb{H}^n$ can only be optimally dense for countably many radii. We give a proof of this result not only for hyperbolic n-space but in the generality of symmetric spaces of noncompact type. This proof is based on an approach suggested to us by Bowen during the writing of my master's thesis under the supervision of my advisor Tobias Hartnick.

In addition we give a second formulation of Bowen and Radin's notion of optimal packing density in terms of generically measured sets. We further use this opportunity to give new proofs for several results of Bowen and Radin. Of note here is our proof of the Voronoi density formula, which we prove via the refined Campbell formula for the Palm measure.

In the **fourth chapter** we use the tools we collected in the previous chapters to prove our general linear programming bounds on density and energy. We also explain some analogies between our approach and Cohn and Elkies use of Poisson summation in the derivation of the density bound in the Euclidean setting. The proof of the density bound contained in this chapter can also be found in Section 5 of my preprint [98].

In the **fifth chapter** we introduce model sets, which give a class of examples for random invariant r-sphere packings in groups (i.e. homogeneous spaces of the form $G/\{e\}$). This class of examples was developed in the generality we cover by Björklund, Hartnick and Pogorzelski [19, 20, 21], though especially in the case of model sets in $\mathbb{R}^n$ a large amount of theory has been developed in the last 50 years, see for instance [7] and the references therein. We obtain a number of results related to the ergodic theory of these examples. More precisely, we are able to give explicit conull sets of generic points. We also use the Palm measure to derive a very explicit method of calculating certain frequencies of patterns occurring in model sets in terms of volumes of so-called acceptance domains. We then give one example demonstrating how some of the methods we present in this chapter can be used to obtain very precise control over a specific family of Euclidean sphere packings.

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1. Harmonic analysis on homogeneous spaces

1.1. Basic conventions

Throughout this thesis, G denotes a fixed unimodular locally compact and second countable topological group and $K \subset G$ a compact subgroup. We fix a left-Haar measure m_G on G and denote by m_K the Haar measure on K, normalized such that $m_K(K) = 1$. For $g \in G$ we let L_g denote the left-multiplication by g and let R_g denote the right-multiplication by g^{-1}. For any function $f : G \to \mathbb{C}$ we denote by $\check{f}$ the function defined by $\check{f}(g) = f(g^{-1})$. We denote by X the homogeneous space $G/K =: X$ with base point $x_0 := eK$ and let π denote the quotient map $\pi : G \to G/K,\ g \mapsto gx_0 = gK$. We will always assume that X is equipped with a G-invariant proper metric d_X inducing the quotient topology.

As G is unimodular, there is a unique G-invariant Radon measure m_X on X such that the Weil disintegration formula

$$\int_G f(g) dm_G(g) = \int_X \int_K f(gk) dm_K(k) dm_X(gK)$$

holds for all $f \in C_c(G)$, i.e. all compactly supported continuous functions on G.

We note that for any G-invariant Radon measure μ on X there exists some constant $C \geq 0$ such that $\mu = C m_X$.

If Y is a topological space, we will denote the Borel σ-algebra of Y by $\mathfrak{B}(Y)$. If Y is locally compact, we will call a measure μ defined on $\mathfrak{B}(Y)$ a Borel measure if $\mu(C) < \infty$ for any compact $C \subset Y$. If Y is in addition separable, then any Borel measure μ on Y is automatically inner and outer regular and thus a Radon measure.

1.2. Gelfand pairs and the spherical transform

We denote by

$$C_c(G, K) := \{f \in C_c(G) : f(k_1 g k_2) = f(g) \text{ for all } k_1, k_2 \in K\},$$

the set of all complex valued bi-K-invariant compactly supported continuous functions. $C_c(G, K)$ is a $*$-algebra with multiplication given by the convolution

$$f_1 * f_2(g) = \int f_1(h) f_2(h^{-1} g) dm_G(h)$$

and involution given by $f^*(g) \coloneqq \overline{f(g^{-1})}$. We denote by $L^1(G,K)$ the closure of $C_c(G,K)$ in $L^1(G)$.

Definition 1.2.1. The pair (G,K) is called a *Gelfand pair* if $C_c(G,K)$ (or equivalently $L^1(G,K)$) is commutative.

Even if we only assume that G is a lcsc group with a compact group K such that $C_c(G,K)$ is commutative and drop our standing assumption of unimodularity, the group G is automatically unimodular.
For $f \in C_c(G)$ the function $f^\natural : G \to \mathbb{C}$, defined by

$$f^\natural(g) \coloneqq \int_K \int_K f(k_1 g k_2) dm_K(k_1) dm_K(k_2)$$

is in $C_c(G,K)$ and is called the *K-periodization* of f.
If $f : G \to \mathbb{C}$ is a bi-K-invariant function on G, then there is a K-invariant function f_K on G/K given by $f_K(gK) \coloneqq f(g)$. If $h : G/K \to \mathbb{C}$ is a K-invariant function, then there is a bi-K-invariant function h^K on G given by $h^K(g) \coloneqq h(gK)$. Note that $(f_K)^K = f$ and $(h^K)_K = h$. $f \mapsto f_K$ and $h \mapsto h^K$ send continuous functions to continuous functions (by a theorem of Gleason on the existence of continuous local sections $G/K \to G$, see [54]), functions with compact support to functions with compact support. If G is a Lie group they also send smooth functions to smooth functions as the quotient map π is a submersion and thus admits smooth local sections, cf. [77, Theorem 4.26].
We now review the notion of the spherical transform of functions on Gelfand pairs. The following is well known and can be found in the books [47], [53], [94] and [100].
Denote by $C(G,K)$ the set of bi-K-invariant continuous functions on G. The convolution algebra $L^1(G,K)$ equipped with the L^1-norm is a commutative Banach algebra with involution. Each character $\phi : L^1(G,K) \to \mathbb{C}$ is of the form

$$\phi(f) = \int f(g)\omega(g^{-1}) dm_G(g), \tag{1.2.1}$$

for some bounded $\omega \in C(G,K)$ satisfying

$$\int_K \omega(g_1 k g_2) dm_K(k) = \omega(g_1)\omega(g_2) \tag{1.2.2}$$

for all $g_1, g_2 \in G$. If $\omega \in C(G,K)$ is bounded and satisfies equation (1.2.2), then it induces a character ϕ_ω of the commutative Banach algebra $L^1(G,K)$ by formula (1.2.1) and is called a bounded spherical function. We denote the set of bounded spherical functions of (G,K) by $BS(G,K)$ and for $f \in L^1(G,K)$ we define

$$G(f) : BS(G,K) \to \mathbb{C}, \quad \omega \mapsto \phi_\omega(f).$$

Note that this is just the Gelfand transform of f, rewritten by parametrizing the Gelfand spectrum of $L^1(G,K)$ by $BS(G,K)$. We equip $BS(G,K)$ with the weak topology induced by the family $\{G(f)\}_{f \in L^1(G,K)}$ of maps. Then $BS(G,K)$ is a locally-compact

Hausdorff space and we have a map

$$G : L^1(G,K) \to C_0(BS(G,K)), \quad f \mapsto G(f).$$

We denote by $PS(G,K)$ the set of positive-definite functions in $BS(G,K)$, i.e. the set of functions $\phi \in BS(G,K)$ such that

$$\int\int \phi(h^{-1}g)\overline{f(h)}f(g)dm_G(h)dm_G(g) \geq 0$$

for all $f \in C_c(G)$.

Definition 1.2.2. We define the *spherical transform* of $f \in L^1(G,K)$ by

$$\widehat{f} := G(f)|_{PS(G,K)} : PS(G,K) \to \mathbb{C}.$$

Note that the spherical transform satisfies $\widehat{f^*} = \overline{\widehat{f}}$ for all $f \in L^1(G,K)$.
The restriction to positive-definite spherical functions comes from the fact that these functions admit a representation theoretic interpretation. If $\pi : G \to U(H)$ is a unitary representation of G such that the subspace

$$H^K := \{x \in H \mid \forall k \in K : \pi(k)x = x\}$$

of *K-fixed vectors* is one-dimensional, π is called *K-spherical*. There is a one to one correspondence between positive-definite functions and irreducible K-spherical unitary representations of G up to unitary equivalence. More precisely, if $\omega \in PS(G,K)$, then there is an irreducible K-spherical unitary representation H_ω and a K-fixed unit vector $x_\omega \in H_\omega^K$ such that $\omega(g) = \langle \pi_\omega(g)x_\omega, x_\omega \rangle$. (π_ω, H_ω) is unique up to unitary equivalence. If on the other hand $\pi : G \to U(G)$ is any irreducible K-spherical unitary representation and $x \in H^K$ is a unit vector, then $g \mapsto \langle \pi(g)x, x \rangle$ is a positive-definite spherical function of (G,K).

1.3. Plancherel transforms of measures and distributions

The Plancherel transform of measures

Definition 1.3.1. Let μ be a Borel measure on G.

(i) μ is called *positive-definite* if

$$\mu(f^* * f) \geq 0 \quad \text{for all } f \in C_c(G).$$

(ii) μ is called *K-spherical* if $\mu(f) = \mu(f^\natural)$ for all $f \in C_c(G)$.

Positive-definite measures are important because they admit spherical transforms by the Plancherel-Godement theorem:

Theorem 1.3.2 (Godement, [55]). *Let μ be a positive-definite Borel measure on G. Then there is a unique regular Borel measure $\widehat{\mu}$ on $PS(G, K)$ such that*

(i) $\widehat{f} \in L^2(\widehat{\mu})$ *for all* $f \in C_c(G, K)$,

(ii) $\mu(f * g^*) = \widehat{\mu}(\widehat{f} \cdot \overline{\widehat{g}})$ *for all* $f, g \in C_c(G, K)$.

A detailed proof of the Plancherel-Godement theorem can be found in [47]. A simple proof of a version for distributions was given by Barker in [8] and can be modified to yield this version for measures. Motivated by the theorem above, we will write $C_c(G, K)^2$ for the complex linear span of $\{f * g \mid f, g \in C_c(G, K)\}$.

The Plancherel transform of distributions

In order to prove estimates involving Schwartz functions, we will later take advantage of the Lie group structure of the Gelfand pairs we consider by using a version of the Plancherel transform for distributions.
For a manifold M let $D(M)$ denote the algebra of differential operators on M, i.e. the algebra generated by the derivations of $C^\infty(M)$ and the maps $D_f : C^\infty(M) \to C^\infty(M),\ g \mapsto fg$ for $f \in C^\infty(M)$.

Definition 1.3.3. Let M be a manifold. A *distribution* T on M is a linear map $T : C_c^\infty(M) \to \mathbb{C}$ such that for every open, relatively-compact $O \subset M$ there are finitely many $D_1, \ldots, D_k \in D(M)$ such that

$$|T[f]| \leq \sum_{i=1}^{k} \|D_i f\|_\infty \quad \text{for all } f \in C_c^\infty(M) \text{ with } \operatorname{supp}(f) \subset O.$$

A more detailed discussion of distributions on manifolds and Lie groups can be found in [64] and in the appendix of [99].

Lemma 1.3.4. *Assume that G is a Lie group and let μ be a Radon measure on G. Then the map*

$$T_\mu : C_c^\infty(G) \to \mathbb{C}, \quad f \mapsto \mu(f)$$

is a distribution on G.

Proof. Let $C \subset M$ be open and relatively-compact and let $f \in C_c^\infty(M)$ with $\operatorname{supp}(f) \subset C$. Then

$$\left| \int f d\mu \right| \leq \mu(C)\|f\|_\infty \leq \mu(\overline{C})\|f\|_\infty. \qquad \square$$

Definition 1.3.5. Assume that G is a Lie group.

(i) A distribution T on G is called *positive-definite*, if

$$T[f^* * f] \geq 0 \quad \text{for all } f \in C_c^\infty(G).$$

(ii) A distribution T on G is called *K-spherical*, if $T[f] = T[f \circ L_k] = T[f \circ R_k]$ for all $k \in K$ and $f \in C_c^\infty(G)$.

Theorem 1.3.6 (Godement, [55])**.** *Assume that G is a Lie group. Let T be a positive-definite distribution on G. Then there is a unique regular Borel measure $\widehat{T}$ on $PS(G,K)$ such that*

(i) $\widehat{f} \in L^2(\widehat{T})$ for all $f \in C_c^\infty(G,K)$,

(ii) $T[f * g^*] = \widehat{T}(\widehat{f} \cdot \overline{\widehat{g}})$ for all $f, g \in C_c^\infty(G,K)$.

A proof of the theorem can be found in [9]. An important special case occurs if the distribution is given by integration against a positive-definite Borel measure. The uniqueness in the Plancherel-Godement theorems for measures and distributions together with Lemma 1.3.4 implies the following:

Corollary 1.3.7. *Assume that G is a Lie group and μ is a positive-definite Radon measure. Then $T_\mu = \mu|_{C_c^\infty(G)}$ is a positive-definite distribution and*

$$\widehat{T_\mu} = \widehat{\mu}.$$

Remark 1.3.8. In all of the examples for Gelfand pairs we will consider, positive-definite distributions extend to tempered distributions (wrt. to an appropriate Schwartz space) by variants of the Bochner-Schwartz theorem. This tempered distribution is no longer necessarily given by integration against μ.

1.4. Examples

1.4.1. Euclidean space

See [94], [100] or [4] for the material of this section.

(i) Consider the pair $G = \mathbb{R}^n$, $K = \{0\}$. Then $X = G/K = \mathbb{R}^n$ and as convolution of functions in $L^1(G,K) = L^1(\mathbb{R}^n)$ is commutative, $(\mathbb{R}^n, \{0\})$ is a Gelfand pair. For each $\xi \in \mathbb{R}^n$, let

$$\phi_\xi : \mathbb{R}^n \to \mathbb{C}, \quad x \mapsto \exp(2\pi i \langle x, \xi \rangle).$$

Then $PS(\mathbb{R}^n, \{0\}) = \{\phi_\xi \mid \xi \in \mathbb{R}^n\}$ and the spherical transform is the ordinary Fourier transform. Let $\mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space on $\mathbb{R}^n$.

Theorem 1.4.1 (Bochner–Schwartz)**.** *If T is a positive-definite distribution on $\mathbb{R}^n$, then T extends uniquely to a tempered distribution $\tilde{T}$ and we have*

$$\tilde{T} f = \widehat{T}(\widehat{f})$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$.

(ii) Consider the pair $G = \mathbb{R}^n \rtimes \mathrm{SO}(n)$, $K = \mathrm{SO}(n)$. Then $X = G/K = \mathbb{R}^n$ and one can show that $(\mathbb{R}^n \rtimes \mathrm{SO}(n), \mathrm{SO}(n))$ is a Gelfand pair. If f is a bi-K-invariant function on $\mathbb{R}^n \rtimes \mathrm{SO}(n)$, then f satisfies

$$f(x, B) = f((0, A)(x, B)(0, B^{-1}A^{-1})) = f(Ax, I_n)$$

for all $x \in \mathbb{R}^n$ and $A, B \in \mathrm{SO}(n)$. Thus there is a function $f_0 : [0, \infty) \to \mathbb{C}$ with $f(x) = f_0(\|x\|)$.

The spherical transform of f is related to the Hankel transform of f_0. More specifically, the spherical functions of $(\mathbb{R}^n \rtimes \mathrm{SO}(n), \mathrm{SO}(n))$ are given by

$$\varphi_\lambda(x, A) := \Gamma\left(\frac{n}{2}\right)\left(\frac{\lambda\|x\|}{2}\right)^{(2-n)/2} J_{(n-2)/2}(\lambda\|x\|) \qquad \text{for } \lambda \geq 0,$$

where J_k denotes the Bessel function of the first kind of order $k \geq 0$. Thus the spherical transform of f is given by

$$\begin{aligned}\widehat{f}(\lambda) &:= \int_{\mathbb{R}^n}\int_{\mathrm{SO}(n)} f(x, A)\varphi_\lambda(-A^{-1}x, A^{-1})dm_{\mathrm{SO}(n)}(A)dx \\ &= (2\pi)^{n/2}\frac{1}{\lambda^{(n-2)/2}}H_f((n-2)/2, \lambda),\end{aligned}$$

where H_f denotes the Hankel transform of $r \mapsto r^{-n/2}f_0(r)$. Note that this is just the Euclidean Fourier transform of f in radial coordinates.

1.4.2. Riemannian symmetric pairs of compact type

Assume in this subsection that (G, K) is a Riemannian symmetric pair with G compact and semisimple (and connected). We denote by $\mathfrak{g}$ the Lie algebra of G, by $\mathfrak{k}$ the Lie algebra of K and by θ the Cartan involution with $G_0^\theta \subset K \subset G^\theta$, where G^θ is the group of fixed points of θ and G_0^θ its connected component.
Let κ denote the Killing form on $\mathfrak{g}$. Then $-\kappa$ defines a scalar product on $\mathfrak{g}$ which extends uniquely to a complex scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}$.
Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition of $\mathfrak{g}$ with respect to θ and choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Let $\Sigma \subset i\mathfrak{a}^*$ denote the set of restricted roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{a}_\mathbb{C}$ and Σ^+ a choice of positive roots.
Denote the universal covering group of G by $\tilde{G}$. The involution θ lifts to an involution $\tilde{\theta}$ on $\tilde{G}$ and $\tilde{K} = \tilde{G}^{\tilde{\theta}}$ is connected.

Theorem 1.4.2 (Helgason, [64])**.** *$PS(\tilde{G}, \tilde{K})$ is in bijection with the set*

$$\Lambda^+(\tilde{G}/\tilde{K}) := \left\{\lambda \in i\mathfrak{a}^* \mid \forall\alpha \in \Sigma^+ : \frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z}^+\right\}.$$

More precisely, there is a bijection between $\Lambda^+(\tilde{G}/\tilde{K})$ and the set of equivalence classes of irreducible $\tilde{K}$-spherical representations sending $\lambda \in \Lambda^+(\tilde{G}/\tilde{K})$ to the equivalence class of irreducible representations with highest weight λ.

For $\lambda \in \Lambda^+(\tilde{G}/\tilde{K})$ let (π_λ, V_λ) denote a fixed irreducible representation with highest weight λ. Let $\Lambda^+(G/K)$ denote the set of $\lambda \in \Lambda^+(\tilde{G}/\tilde{K})$ such that (π_λ, V_λ) descends to a K-spherical representation of G.

Lemma 1.4.3 (Olafsson–Schlichtkrull, [85])**.** *There are $\lambda_1, \dots, \lambda_k \in \Lambda^+(\tilde{G}/\tilde{K})$ such that*

$$\Lambda^+(G/K) = \mathbb{Z}^+\lambda_1 \oplus \cdots \oplus \mathbb{Z}^+\lambda_k \subset i\mathfrak{a}^*$$

with $k = \dim \mathfrak{a}$.

For $\lambda \in \Lambda^+(G/K)$ we denote by ϕ_λ the spherical function associated to (π_λ, V_λ). A K-invariant distribution on $X = G/K$ is an continuous functional T on $C^\infty(X)$ such that

$$T(f) = T(f(k\cdot)) \quad \text{for all } f \in C^\infty(G/K) \text{ and } k \in G.$$

Theorem 1.4.4 (Olafsson–Schlichtkrull, [85])**.** *Let T be a K-invariant distribution on $X = G/K$. Then we have*

$$T(f_K) = \sum_{\lambda \in \Lambda^+(G/K)} \dim(V_\lambda)\widehat{f}(\phi_\lambda)T(\phi_{\lambda^*})$$

for any $f \in C^\infty(G, K)$, where ϕ_{λ^*} is defined by $\phi_{\lambda^*}(g) = \phi_\lambda(g^{-1})$.

Corollary 1.4.5. *Let T be a bi-K-invariant distribution on G. Then*

$$T(f) = \widehat{T}(\widehat{f})$$

for all $f \in C^\infty(G, K)$.

Remark 1.4.6. A particularly interesting case occurs in this family of examples when $G = \mathrm{SO}(n+1)$ and $K = \mathrm{SO}(n)$. In this case G/K can be identified with the n-sphere $\mathbb{S}^n = \partial B(0,1) \subset \mathbb{R}^{n+1}$ as $\mathrm{SO}(n+1)$ acts transitively on $\mathbb{S}^n$ and $e_1 = (1, 0, \dots, 0)^T$ is stabilized by a subgroup isomorphic to $\mathrm{SO}(n)$.
We will quickly review the Haar measure and spherical functions for the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$, see [4] and references therein for more information.
Let $x \in \mathbb{S}^n$ and let $\theta(x)$ denote the angle between x and $e_1 := (1, 0, \dots, 0)^\top$. We can identify bi-$\mathrm{SO}(n)$-invariant functions with $\mathrm{SO}(n)$-invariant functions on $\mathbb{S}^n$. With respect to the usual spherical coordinates these functions only depend on $\cos(\theta(\cdot))$. The integral of a radial measurable function f with respect to the Haar measure is then given by

$$\int f(g)dm_{\mathrm{SO}(n+1)}(g) = 2\frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\pi f_{\mathrm{SO}(n)}(\cos(\theta))\sin(\theta)^{n-1}d\theta$$

The spherical functions on $\mathbb{S}^n$ are of the form

$$\phi_\lambda(\cos(\theta)) = \frac{\lambda!(n-2)!}{(\lambda+n-2)!}C_\lambda^{(\frac{n-2}{2})}(\cos(\theta)) = \frac{\lambda!}{(\frac{n}{2})_\lambda}P_\lambda^{(\frac{n}{2}-1,\frac{n}{2}-1)}(\cos(\theta))$$

where $\lambda \in \mathbb{N}$. Here the $C_k^{(m)}$ denote the Gegenbauer polynomials, the $P_\lambda^{(\alpha,\beta)}$ denote the Jacobi polynomials and $(m)_k$ denotes the falling Pochhammer symbol. They can also be expressed in terms of the Gauss hypergeometric function:

$$\phi_\lambda(\cos(\theta_1)) = {}_2F_1(-\lambda, \lambda+n-1; n/2; \sin(\theta_1)^2).$$

In view of Theorem 1.4.4, the case of finite Gelfand pairs is instructive, even though they are not Riemannian symmetric pairs:

Remark 1.4.7. In the case that (G, K) is a finite Gelfand pair (i.e. G is finite), the theory becomes quite easy. Then $PS(G, K)$ is finite, i.e. there are finitely many spherical function $\omega_1, \ldots, \omega_n$ and the Plancherel theorem states that there are constants $d_1, \ldots, d_n$, given by $d_i = \dim V_{\omega_i}$ (the dimension of the associated irreducible K-spherical representation), such that

$$\sum_{x \in G/K} f_1(x)\overline{f_2(x)} = \frac{1}{\#(G/K)} \sum_{i=1}^{n} d_i \widehat{f_1}(\omega_i)\overline{\widehat{f_2}(\omega_i)}$$

for any K-invariant functions $f_1, f_2 : G/K \to \mathbb{C}$. As any K-spherical measure μ on G can be identified with a K-invariant function on X, we see that

$$\mu(f) = \sum_{x \in G/K} f(x)\mu(x) = \frac{1}{\#(G/K)} \sum_{i=1}^{n} d_i \widehat{f}(\omega_i)\overline{\widehat{\mu}(\omega_i)},$$

reminiscent of Theorem 1.4.4. If μ is positive-definite, Bochner's theorem for Gelfand pairs ([100, Theorem 9.3.4]) implies that there are $\lambda_1, \ldots, \lambda_n \geq 0$ such that

$$\mu = \sum_{i=1}^{n} \lambda_i \omega_i.$$

As $\widehat{\omega_i}(\omega_i) = \frac{\#(G/K)}{d_i}$, the orthogonality relations for matrix coefficients of compact groups imply

$$\mu(f) = \sum_{i=1}^{n} \widehat{f}(\omega_i)\lambda_i,$$

essentially a version the Bochner-Schwartz theorem. More details can be found in [32, Chapter 4.7] and [42, Chapter 9].

1.4.3. Riemannian symmetric pairs of noncompact type

Assume in this subsection that G is a (connected) semisimple Lie group with finite center and no compact factors and choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra of G. The following setup is well-known (see [53] and [70]) and we use the usual notation, i.e.

(i) $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$,

(ii) ρ is the Weyl vector associated to the restricted root space decomposition with respect to $\mathfrak{a}$,

(iii) K is the maximal compact subgroup of G associated to $\mathfrak{k}$,

(iv) κ is the Killing form of $\mathfrak{g}$,

(v) W is the Weyl group of the restricted root system wrt. $\mathfrak{a}$.

(G,K) is a Gelfand pair and G has the Cartan decomposition $G = K\exp(\mathfrak{p})$. The Iwasawa decomposition $G = KAN$ allows us to define

$$H : G = KAN \to \mathfrak{a},\ g = kan \mapsto \exp^{-1}(a).$$

We will assume that the measures m_K, m_A, m_N and $m_{\mathfrak{a}}$ obey the standard normalization, see Section 2.4 of [53]. Using the function H, for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the dual of the complexification of $\mathfrak{a}$, we define the function

$$\varphi_\lambda(g) = \int_K e^{(i\lambda-\rho)(H(gk))} dm_K(k).$$

This formula parametrizes a superset of all bounded spherical functions of (G,K) and is called the *Harish-Chandra parametrization* (see [64] and [53]). It has the following properties:

(i) $\varphi_\lambda = \varphi_\mu$ iff there is some $w \in W$ with $w\lambda = \mu$.

(ii) $\varphi_{-\lambda}(g) = \varphi_\lambda(g^{-1})$.

(iii) If $\varphi_\lambda \in PS(G,K)$, then there is a $w \in W$ with $w\lambda = \overline{\lambda}$.

Define the *tube domains* $\mathscr{F}^\varepsilon := \mathfrak{a}^* + i\varepsilon C$, where C is the closed convex hull of $\{w\rho \mid w \in W\}$. The Helgason-Johnson theorem states that

$$BS(G,K) = \{\varphi_\lambda \mid \lambda \in \mathscr{F}^1\}$$

and using property (i) above, one can identify $BS(G,K)$ with the set $\mathscr{F}^1/W$.
The *Harish-Chandra Ξ-function* is defined as

$$\Xi := \varphi_0$$

and one defines

$$\sigma : G = K\exp(\mathfrak{p}) \to \mathbb{R},\ k\exp(X) \mapsto \sqrt{\kappa(X,X)}.$$

Using Ξ and σ one can define the *Harish-Chandra L^p-Schwartz seminorms*

$$q_{D,E,m,p}(f) := \sup_{g\in G} \frac{|f(D;g;E)|}{(1+\sigma(g))^{-m}\Xi(g)^{-2/p}},$$

where $0 < p \leq 2$, $D, E \in \mathcal{U}(\mathfrak{g})$ and $m \in \mathbb{N}_0$. Here $\mathcal{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ and $f(D;\cdot;E)$ is the function obtained from f by acting on the left by the differential operator D and from the right by the differential operator E.
We define the *Harish-Chandra L^p-Schwartz spaces* for $0 < p \leq 2$ as

$$\mathscr{S}^p(G) := \{f \in C^\infty(G) \mid \forall m \in \mathbb{N}_0 \forall D, E \in \mathcal{U}(\mathfrak{g}) : q_{D,E,m,p}(f) < \infty\}.$$

The spaces $\mathscr{S}^p(G)$ are topologized by the families

$$(q_{D,E,m,p})_{D,E\in\mathcal{U}(\mathfrak{g}),m\in\mathbb{N}_0}$$

of seminorms and we denote by $\mathscr{S}^p(G, K)$ the set of bi-K-invariant functions in $\mathscr{S}^p(G)$. It can be shown that $\mathscr{S}^p(G, K) \subset \mathscr{S}^q(G, K)$ for $p \leq q$.
The *Harish-Chandra transform* of $f \in \mathscr{S}^2(G, K)$ is defined by

$$\mathcal{H}(f)(\lambda) := \int f(g)\varphi_{-\lambda}(g)dm_G(g)$$

and the *Abel transform* of f is defined by

$$\mathcal{A}(f)(H) = e^{\rho(H)} \int_N f(\exp(H)n)dm_N(n).$$

These transforms fit in the following commutative diagram of isomorphism:

$$\begin{array}{ccc} \mathscr{S}^2(G,K) & \xrightarrow{\quad\mathcal{H}\quad} & \mathscr{S}(\mathscr{F}^0)^W \\ & \searrow^{\mathcal{A}} \qquad \nearrow^{\mathcal{F}} & \\ & \mathscr{S}(\mathfrak{a})^W & \end{array}$$

Here $\mathcal{F}$ denotes the Fourier transform (note that any factors of 2π disappear by the standard normalization of the Haar measures)

$$\mathcal{F}(f)(\lambda) := \int_{\mathfrak{a}} f(H)e^{-i\lambda(H)}dm_{\mathfrak{a}}(H)$$

and $\mathscr{S}(\mathfrak{a})^W$ and $\mathscr{S}(\mathscr{F}^0)^W$ denote the Weyl group invariant elements of the ordinary Schwartz spaces on the real vector spaces $\mathfrak{a}$ and $\mathscr{F}^0 = \mathfrak{a}^*$. A theorem by Trombi and Varadarajan characterizes the image of $\mathscr{S}^P(G, K)$ under $\mathcal{H}$ in terms of spaces of functions on tube domains, see for instance Theorem 7.10.9 in [53] for a precise statement. A continuous functional $\mathscr{S}^p(G) \to \mathbb{C}$ is called a *L^p-tempered distribution.*

Theorem 1.4.8 (Spherical Bochner-Schwartz theorem (Barker, [8]))**.** *In the setting above assume that T is a positive-definite distribution on G. Then T has a unique extension to an L^1-tempered distribution $\tilde{T}$ and for all $f \in \mathscr{S}^1(G, K)$ the formula*

$$\tilde{T}f = \int \widehat{f} d\widehat{T}$$

holds.

Note that $\kappa|_{\mathfrak{p}\times\mathfrak{p}}$ is positive-definite. Hence the spaces G/K can be equipped with a natural left-invariant Riemannian metrics induced by the bilinear form κ on $\mathfrak{p} \cong T_{x_0}(G/K)$. The length metric on G/K induced by this Riemannian metric is called the *Cartan-Killing metric.* A wealth of information about the geometry and harmonic analysis on these spaces can be found in the books [65] and [64] by Helgason.

Remark 1.4.9. A particularly interesting case occurs in this family of examples when $G = \mathrm{SO}(n, 1)_0$ and $K = \mathrm{SO}(n)$, $n \geq 2$.

In this case G/K with the Riemannian metric induced by κ is isometric to the hyperbolic n-space $\mathbb{H}^n$. Moreover A is one-dimensional, such that bi-K-invariant functions only depend on $r = d_{\mathbb{H}^n}(gx_0, x_0)$.

Thus the value of a spherical function φ at $g \in G$ only depends on the distance $r = d_{\mathbb{H}^n}(gx_0, x_0)$. Set $\rho = \frac{n-1}{2}$. Then the positive definite spherical functions in radial coordinates are given by

$$\varphi_\lambda(r) = \varphi_\lambda^{(\rho,-\frac{1}{2})}(r) = {}_2F_1\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \frac{n}{2}; -\sinh(r)^2\right)$$

with

$$\lambda \in i[0,\rho] \cup [0,\infty),$$

where ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric function and the $\varphi_\lambda^{(\alpha,\beta)}$ denote the Jacobi functions, see [4]. The spherical transform in radial coordinates is then given by

$$\widehat{f}(\lambda) = 2\frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty f(r)\varphi_\lambda(r)\sinh(r)^{n-1}dr,$$

with the additional sinh-term coming from the Haar measure.

In this case there is also an explicit formula for the Abel transform (see [4]), given in radial coordinates by

$$\mathcal{A}f(r) = \frac{(2\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{|r|}^\infty \sinh(s)(\cosh(s)-\cosh(r))^{\frac{n-3}{2}} f(s)ds.$$

In odd dimensions the inverse of the Abel transform can be obtained by

$$\mathcal{A}^{-1}(f)(r) = (2\pi)^{-\frac{n-1}{2}} \left(-\frac{1}{\sinh(r)}\frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} f(r)$$

and in even dimensions by

$$\mathcal{A}^{-1}(f)(r) = \frac{1}{2^{\frac{n-1}{2}}\pi^{\frac{n}{2}}} \int_{|r|}^\infty \frac{-\frac{\partial}{\partial s}\left(-\frac{1}{\sinh(r)}\frac{\partial}{\partial s}\right)^{n/2-1} g(s)}{\sqrt{\cosh(s)-\cosh(r)}} ds.$$

For $G = \mathrm{SO}(n,1)_0$, $K = \mathrm{SO}(n)$ the spaces $\mathscr{S}^p(G,K)$ can be identified via radial coordinates with the spaces

$$\cosh^{-\frac{n-1}{p}} \mathscr{S}_{\text{even}}(\mathbb{R}),$$

where $\mathscr{S}_{\text{even}}(\mathbb{R})$ denotes the space of even Schwartz functions on $\mathbb{R}$ (see Theorem 6.1 in [73] or 2.28 in [5]).

The image of $\mathscr{S}^p(G,K)$, $1 \leq p \leq 2$, under the Harish-Chandra transform $\mathcal{H}$ is given by the set $\overline{\mathscr{L}}(\mathscr{F}^{1/p-1/2})$ of smooth even functions f on the strip

$$\mathscr{F}^{1/p-1/2} = \left\{ \lambda \in \mathbb{C} \;\middle|\; |\mathrm{Im}(\lambda)| \leq \left(\frac{1}{p} - \frac{1}{2}\right)(n-1) \right\},$$

which are holomorphic on its interior and satisfy

$$\sup_{|\mathrm{Im}(\lambda)| \leq \left(\frac{1}{p}-\frac{1}{2}\right)(n-1)} (1+|\lambda|)^N \left| \left(\frac{d}{d\lambda}\right)^M f(\lambda) \right| < \infty$$

for all $M, N \in \mathbb{N}_0$. The image of $\mathscr{S}^p(G,K)$ under the Abel transform can be identified with the space

$$\cosh^{-(1/p-1/2)(n-1)} \mathscr{S}_{\mathrm{even}}(\mathbb{R})$$

and we have the following commutative diagram of isomorphisms:

$$\begin{array}{ccc} \cosh^{-\frac{n-1}{p}} \mathscr{S}_{\mathrm{even}}(\mathbb{R}) & \xrightarrow{\quad\mathcal{H}\quad} & \overline{\mathscr{L}}(\mathscr{F}^{1/p-1/2}) \\ & \searrow^{\mathcal{A}} \qquad \nearrow_{\mathcal{F}} & \\ & \cosh^{-(1/p-1/2)(n-1)} \mathscr{S}_{\mathrm{even}}(\mathbb{R}) & \end{array}$$

See [73] and [5] for further details and references.
For semisimple groups with $\dim(A) = 1$, similar formulas for the spherical functions hold, see for instance [73] or [100]. Moreover an explicit inversion formula for the Abel transform is known, see [73] and [4].
In the case of $\dim(A) > 1$, one must replace the hypergeometric functions by multivariable analogues. These can be handled in a somewhat explicit way with Dunkl theory, see [4] and the references therein.
In contrast to the Fourier transform, there can be no eigenfunctions of the Harish-Chandra transform (as it maps functions on G to functions on $PS(G,K)$). A partial substitute for eigenfunctions is given in [73, Section 9] by Koornwinder. More specifically Koornwinder gives an orthogonal system of functions in $L^2(G,K)$ which is mapped by the Harish-Chandra transform to an orthogonal system in $L^2(PS(G,S), \widehat{\delta_e})$.

1.4.4. The Heisenberg group

Set $\mathcal{H}_n := \mathbb{R} \times \mathbb{C}^n$ and for $(t,x), (s,y) \in \mathcal{H}_n$ define

$$(t,v) \cdot (s,w) := (ts - \frac{1}{2}\mathrm{Im}\langle v, w\rangle, v + w).$$

Then $(\mathcal{H}_n, \cdot)$ is a nilpotent Lie group, called the *Heisenberg group*. The group $U(n)$ acts via

$$k.(t,v) := (t, kv) \quad \text{for } k \in U(n)$$

by automorphisms on $\mathcal{H}_n$. Thus we can form the semidirect product $\mathcal{H}_n \rtimes U(n)$, called the *Heisenberg motion group*. We note that $\mathcal{H}_n = \mathcal{H}_n \rtimes U(n)/U(n)$ (as manifolds), which is why we will also call $\mathcal{H}_n$ the *Heisenberg space*. We can equip $\mathcal{H}_n$ with the Cygan-Koranyi metric d_{CK}, which is induced by the group norm

$$\|(t,v)\|_{CK} := (t^2 + \|v\|_2^4)^{1/4}$$

on $\mathcal{H}_n$ and is proper, left-invariant and complete.[1] We note that the Heisenberg group can be equipped with a family $(D_r)_{r>0}$ of group automorphisms, defined by

$$D_r : \mathcal{H}_n \to \mathcal{H}_n,\ (t,v) \mapsto (r^2 t, rv).$$

These automorphisms are called *dilations* of $\mathcal{H}_n$ and the Cygan-Koranyi metric is compatible with the family $(D_r)_{r>0}$ in the sense that

$$\|D_r(g)\| = r\|g\| \quad \text{for all } g \in \mathcal{H}_n, r > 0.$$

This implies that

$$D_r(B(g,s)) = B(D_r(g), rs)$$

for all $g \in \mathcal{H}_n$ and $r, s > 0$. For all measurable functions $f, \varphi : \mathcal{H}_n \to \mathbb{C}$ and all $r > 0$ we have

$$\int (f \circ D_r)(x)\varphi(x)dm_{\mathcal{H}_n}(x) = \frac{1}{r^{2n+2}} \int f(x)(\varphi \circ D_{1/r})(x)dm_{\mathcal{H}_n}(x)$$

if the integrals exist. The number $h := 2n + 2$ appearing in this formula is called the *homogeneous dimension* of $\mathcal{H}_n$. Here we should also point out that there is a rich theory of nilpotent Lie groups equipped with dilation structures, see for instance [52].
The pair $(\mathcal{H}_n \rtimes U(n), U(n))$ forms a Gelfand pair. See [93] or [100] for a detailed exposition of the theory of spherical harmonic analysis for $(\mathcal{H}_n \rtimes U(n), U(n))$.

Theorem 1.4.10 (Benson–Jenkins–Ratcliff, [12])**.** *The spherical functions of $(\mathcal{H}_n \rtimes U(n), U(n))$ fall into the following two families:*

(A)

$$\phi_{\lambda,m}(t,v,k) = e^{i\lambda t} L_m^{(n-1)}\left(\frac{1}{2}\lambda\|v\|_2^2\right) e^{-\frac{1}{4}\lambda\|v\|_2^2}$$

for $\lambda > 0$ and $m \in \mathbb{Z}_+$ and $\phi_{\lambda,m} = \overline{\phi_{|\lambda|,m}}$ for $m \in \mathbb{Z}_+$ and $\lambda < 0$.

(B)

$$\eta_\tau(t,v,k) = \frac{2^{n-1}(n-1)!}{(\tau\|v\|_2)^{n-1}} J_{n-1}(\tau\|v\|_2)$$

for $\tau > 0$ and $\eta_0(t,v,k) = 1$ for all $(t,v,k) \in \mathcal{H}_n \rtimes U(n)$.

Here

$$L_m^{(n-1)}(x) = (n-1)! \sum_{j=0}^{m} \binom{m}{j} \frac{(-x)^j}{(j+n-1)!}$$

is the generalized Laguerre polynomial of order $n-1$ normalized to 1 at 0 and J_{n-1} is the Bessel function of order $n-1$.

[1] There are many other proper, left-invariant and complete metrics on $\mathcal{H}_n$.

As the Haar measure $m_{\mathcal{H}_n}$ is given by $m_{\mathbb{R}} \otimes m_{\mathbb{C}^n}$, the spherical transform on $\mathcal{H}_n$ is just given by ordinary integration against the spherical functions with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{C}^n$.
Denote the Lie algebra of the Heisenberg group $\mathcal{H}_n$ by $\mathfrak{h}_n$ and note that $\exp : \mathfrak{h}_n \to \mathcal{H}_n$ is a diffeomorphism. We define the Schwartz space $\mathscr{S}(\mathcal{H}_n)$ by

$$\mathscr{S}(\mathcal{H}_n) := \{f \circ \exp^{-1} \mid f \in \mathscr{S}(\mathfrak{h}_n)\}$$

and set

$$\mathscr{S}(\mathcal{H}_n, U(n)) := \{f \in \mathscr{S}(\mathcal{H}_n) \mid f(t, kv) = f(t, v) \text{ for all } k \in U(n)\}.$$

We define the space of bi-$U(n)$-invariant Schwartz functions on $\mathcal{H}_n \rtimes U(n)$ by

$$\mathscr{S}(\mathcal{H}_n \rtimes U(n), U(n)) = \{f \circ \pi \mid f \in \mathscr{S}(\mathcal{H}, U(n))\}$$

and topologize it such that the canonical map $\mathscr{S}(\mathcal{H}_n, U(n)) \to \mathscr{S}(\mathcal{H}_n \rtimes U(n), U(n))$ is a topological isomorphism.
Due to lack of reference we prove the following theorem in Appendix C:

Theorem 1.4.11. *Let T be a positive-definite distribution on $\mathcal{H}_n \rtimes U(n)$. Then there is a (unique) continuous functional*

$$\tilde{T} : \mathscr{S}(\mathcal{H}_n \rtimes U(n), U(n)) \to \mathbb{C}$$

such that

$$\tilde{T}f = Tf$$

for any $f \in C_c^\infty(\mathcal{H}_n \rtimes U(n), U(n))$. In addition

$$\tilde{T}f = \widehat{T}(\widehat{f})$$

for all $f \in \mathscr{S}(\mathcal{H}_n \rtimes U(n), U(n))$.

1.5. Schwartz-like function spaces

Assume in this subsection that G is a Lie group.

Definition 1.5.1. We say that a topological $*$-subalgebra $\mathcal{S}(G, K) \subset L^1(G, K) \cap C(G)$ (with a possibly finer topology) containing $C_c^\infty(G, K)$ is *Schwartz-like*, if

(i) for every $f \in \mathcal{S}(G, K)$ without compact support there is a sequence $(g_n)_{n \geq 1}$ in $C_c^\infty(G, K)$ such that

   a) $g_n f \in C_c^\infty(G, K)$ for all $n \in \mathbb{N}$,

   b) $g_n f \to f$ in $\mathcal{S}(G, K)$,

   c) g_n takes values in $[0, 1]$,

   d) $g_n(e) = 1$.

(ii) $\widehat{f} \in L^1(\widehat{\mu})$ for all $f \in \mathcal{S}(G,K)$ and K-spherical positive-definite Radon measures μ on G.

(iii) For any K-spherical positive-definite Radon measure μ there is a unique continuous linear functional $T_\mu : \mathcal{S}(G,K) \to \mathbb{C}$ such that

$$T_\mu(f) = \int \widehat{f} d\widehat{\mu}$$

for all $f \in \mathcal{S}(G,K)$ and $T_\mu|_{C_c^\infty(G,K)} = \mu|_{C_c^\infty(G,K)}$.

Proposition 1.5.2. *The following algebras are Schwartz-like:*

(i) $C_c(G,K)^2 = \operatorname{span}\{f * g \mid f, g \in C_c(G,K)\}$ *for a general Lie group Gelfand pair.*

(ii) $\mathscr{S}(\mathbb{R}^n)$.

(iii) $\mathscr{S}^1(G,K)$*, where G is a semisimple Lie group with finite center and no compact factors, K a maximal compact subgroup.*

(iv) $C^\infty(G,K)$ *for Riemannian symmetric pairs (G,K) of compact type.*

(v) $\mathscr{S}(\mathcal{H}_n \rtimes U(n), U(n))$.

(vi) $C(G,K)$ *the set of bi-K-invariant functions $G \to \mathbb{C}$ if (G,K) is a finite Gelfand pair.*

Proof. (i) As every function in $C_c(G,K)^2$ has compact support, Condition (i) holds vacuously. Conditions (ii) and (iii) follow directly from the Godement-Plancherel theorem 1.3.2.

(ii) This follows from the Bochner-Schwartz theorem together with Corollary 1.3.7 and [58, Theorem 1.8.7].

(iii) This follows from Barkers spherical Bochner-Schwartz theorem (Theorem 1.4.8) together with Corollary 1.3.7 and the remarks after [53, Definition 7.8.4] together with [53, Lemma 6.1.7].

(iv) This follows directly from Corollary 1.4.5.

(v) The existence of a sequence $(g_n)_{n\geq 1}$ follows directly from the Euclidean case, by the definition of the Schwartz space. Now Theorem 1.4.11 together with Corollary 1.3.7 implies that $\mathscr{S}(\mathcal{H}_n \rtimes U(n), U(n))$ is Schwartz-like.

(vi) This follows directly by applying Remark 1.4.7. □

2. Point processes in homogeneous spaces

2.1. Spaces of sets and measures

2.1.1. Spaces of sets

Assume that (Y, d_Y) is a second countable and proper metric space and note that (Y, d_Y) is automatically complete. The set $\mathfrak{F}(Y)$ of closed subsets of Y can be equipped with a natural compact (Hausdorff) topology, called the *Chabauty-Fell topology* (see [1, 10, 19, 82, 84]). This topology has a subbasis given by the sets

$$U_V := \{A \in \mathfrak{F}(Y) \mid A \cap V \neq \emptyset\}, \quad U^K := \{A \in \mathfrak{F}(Y) \mid A \cap K = \emptyset\},$$

where V runs over the open subsets of Y and K over the compact subsets of Y. This topology is metrizable. In [13] Biringer gives a particularly nice expression for a metric.

Proposition 2.1.1 ([11, Proposition E.12]). *A sequence $(C_n)_{n\geq 1}$ in $\mathfrak{F}(Y)$ converges to $C \in \mathfrak{F}(Y)$ if and only if*

(i) for every $x \in C$ there are $x_n \in C_n$ such that $x_n \to x$,

(ii) for every sequence $(n_k)_{k\geq 1}$ in $\mathbb{N}$ with $n_k \to \infty$ and $x_{n_k} \in C_{n_k}$ such that $(x_{n_k})_{k\geq 1}$ is convergent to some $x \in Y$ we have $x \in C$.

Definition 2.1.2. (i) Let $r > 0$. We say that $C \subset Y$ is *r-uniformly discrete* if $d_Y(x, y) \geq r$ for all $x \neq y \in C$. We denote the set of r-uniformly discrete subsets of Y by $\mathrm{UD}_r(Y)$.

(ii) For $R \geq 0$, we say that $C \subset Y$ is R-relatively dense if for every $x \in Y$ there is some $y \in C$ with $d_Y(x, y) \leq R$.

(iii) If C is r-uniformly discrete and R-relatively dense, we say that C is (r, R)-Delone. We denote by $\mathrm{Del}_r^R(Y) \subset \mathrm{UD}_r(Y)$ the subset of (r, R)-Delone sets.

Note that r-uniformly discrete sets are also sometimes called r-separated sets. The following lemma follows directly from Proposition 2.1.1, see for instance [21, Proposition 3.2].

Proposition 2.1.3. *The sets $\mathrm{UD}_r(Y)$ and $\mathrm{Del}_r^R(Y)$ are closed subsets of $\mathfrak{F}(Y)$.*

Proposition 2.1.4. *Assume that a lcsc group G acts continuously from the left on Y. Then the natural left G-action on $\mathfrak{F}(Y)$ given by $G \times \mathfrak{F}(Y)$, $(g, A) \mapsto \{gx \mid x \in A\}$ is continuous.*

Proof. Notice that $gU_V = U_{g^{-1}V}$ and $gU^K = U^{g^{-1}K}$. $\square$

Note further that $\mathrm{UD}_r(Y)$ and $\mathrm{Del}_r^R(Y)$ are G-invariant, if the action of G on Y is continuous and d_Y is G-invariant.

2.1.2. Spaces of measures

As (Y, d_Y) is proper, it follows from our definitions that any Borel measure μ on Y is *locally finite* in the sense that $\mu(B) < \infty$ for any bounded Borel set $B \subset Y$. We denote the set of all (positive) Borel measures on Y by $\mathcal{M}(Y)$. We say that $P \subset Y$ is *locally finite*, if $\#(P \cap B) < \infty$ for every bounded Borel set $B \subset Y$ and set

$$\mathcal{N}^*(Y) := \left\{ \sum_{x \in P} \delta_x \mid P \subset Y \text{ locally finite} \right\}.$$

The elements of $\mathcal{N}^*(Y)$ are called *simple point measures*.
We equip the set $\mathcal{M}(Y)$ with the topology of weak-$*$ convergence wrt. to $C_c(Y)$. This topology is commonly also called the weak-# topology (cf. [44, Appendix A2.6]) or vague topology (cf. [69, Chapter 4.1]). By [69, Lemma 4.6], the topological space $\mathcal{M}(Y)$ is a Polish space.
We further note that by [44, Proposition A2.6.II] a sequence $(\mu_n)_{n \geq 1}$ in $\mathcal{M}(Y)$ converges to $\mu \in \mathcal{M}(Y)$ if and only if $\mu_n(B) \to \mu(B)$ for every bounded Borel set $B \subset Y$ with $\mu(\partial B) = 0$ (these sets are often called continuity sets of μ). Moreover, by [44, Theorem 2.6.III] the Borel σ-algebra on $\mathcal{M}(Y)$ is generated by the maps $(\pi_B : \mu \mapsto \mu(B))_{B \in \mathcal{S}}$, where $\mathcal{S}$ is an arbitrary semiring of bounded Borel sets of Y generating the Borel σ-algebra on Y.
Thus all evaluation functionals $\pi_B : \mu \mapsto \mu(B)$, with $B \subset Y$ a Borel set, are measurable, as we can decompose B into countably many pairwise disjoint bounded Borel sets $(B_i)_{i \in \mathbb{N}}$ and $\pi_B = \sum_{i \in \mathbb{N}} \pi_{B_i}$ is measurable as a countable sum of non-negative measurable functions.
Measurability of certain subsets of $\mathcal{M}(Y)$ can easily be determined from the following measurable form of atomic decomposition, see for instance [69, Lemma 1.6]. Note that in [69] Kallenberg works with so-called localized Borel spaces. Taking as a Borel space Y with its Borel σ-algebra and as localizing ring the set of all bounded Borel sets, one obtains the following:

Theorem 2.1.5 (Atomic decomposition)**.** *Any $\mu \in \mathcal{M}(Y)$ has a decomposition*

$$\mu = \alpha(\mu) + \sum_{k \leq \kappa(\mu)} \beta_k(\mu) \delta_{\sigma_k(\mu)}$$

with $\alpha(\mu) \in \mathcal{M}(Y)$ non-atomic, $\kappa(\mu) \in \mathbb{N}_0 \cup \{\infty\}$ and $\beta(\mu) \in \mathbb{R}_{\geq 0}$ and pairwise distinct $\sigma_k(\mu) \in Y$ for all $k \leq \kappa(\mu)$. Moreover we can choose the functions $\alpha : \mathcal{M}(Y) \to \mathcal{M}(Y)$, $\mu \mapsto \alpha(\mu)$, $\kappa : \mathcal{M}(Y) \to \mathbb{N}_0 \cup \{\infty\}$, $\mu \mapsto \kappa(\mu)$, and $(\beta_k, \sigma_k) : \mathcal{M}(Y) \to \mathbb{R}_{\geq 0} \times Y$, $\mu \mapsto (\beta_k(\mu), \sigma_k(\mu))$, $k \in \mathbb{N}_0$, as measurable functions and the decomposition above is unique up to reordering of terms.

This theorem implies that the set $\mathcal{N}^*(Y)$ is a Borel subset of $\mathcal{M}(Y)$. We further define the set

$$\mathcal{N}_r^*(Y) = \left\{ \sum_{x \in P} \delta_x \mid P \in \mathrm{UD}_r(X) \right\}.$$

By [21, Proposition 3.2] the map

$$\delta : \mathrm{UD}_r(Y) \to \mathcal{N}_r^*(Y),\ P \mapsto \delta_P := \sum_{x \in P} \delta_x$$

is a homeomorphism and thus $\mathcal{N}_r^*(Y)$ is compact. The inverse of δ is given by the map

$$\mathrm{supp} : \mathcal{N}_r^*(Y) \to \mathrm{UD}_r(Y), \quad \sum_{x \in P} \delta_x \mapsto P.$$

Lemma 2.1.6. *If G acts continuously on Y, then the action of G on $\mathcal{M}(Y)$ given by*

$$G \times \mathcal{M}(Y) \to \mathcal{M}(Y), \quad (g, \mu) \mapsto (L_g)_* \mu$$

is continuous.

Proof. Observe first that for any bounded Borel set $B \subset Y$ and $g \in G$ there is a compact set $C \subset Y$ with $B \subset C$ and thus $g^{-1}B \subset g^{-1}C$, i.e. $g^{-1}B$ is bounded. Note now that for any $g \in G$ and $\nu \in \mathcal{M}(Y)$ we have $(L_g)_*\nu \in \mathcal{M}(Y)$, as we have $(L_g)_*\nu(B) = \nu(g^{-1}B) < \infty$ for any bounded Borel set $B \subset Y$.
Assume that $((g_n, \mu_n))_{n \geq 1}$ is a convergent sequence in $G \times \mathcal{M}(Y)$ with limit (g, μ) and let $f \in C_c(Y)$. Then

$$\begin{aligned} |(g_n\mu_n)(f) - (g\mu)(f)| &= |\mu_n(f \circ L_{g_n}) - \mu_n(f \circ L_g)| + |\mu_n(f \circ L_g) - \mu(f \circ L_g)| \\ &\leq \mu_n(A_n)\|f \circ L_{g_n} - f \circ L_g\|_\infty + |\mu_n(f \circ L_g) - \mu(f \circ L_g)|, \end{aligned}$$

where $A_n = \mathrm{supp}(f \circ L_{g_n} - f \circ L_g) \subset g_n^{-1}\mathrm{supp}(f) \cup g^{-1}\mathrm{supp}(f)$. As $g_n \to g$ we know that there is a compact $S \subset G$ with $g_n \in S$ for all n and thus $A_n \subset S\mathrm{supp}(f)$ for all n. Note that $S\mathrm{supp}(f)$ is compact and choose $\varphi \in C_c(Y)$ with $\varphi|_{S\mathrm{supp}(f)} = 1$ and $\varphi \geq 0$. Then $\mu_n(A_n) \leq \mu_n(\varphi) \to \mu(\varphi)$ and hence there is a $\lambda > 0$ with $\mu_n(A_n) \leq \lambda$ for all n. Thence

$$\mu_n(A_n)\|f \circ L_{g_n} - f \circ L_g\|_\infty + |\mu_n(f \circ L_g) - \mu(f \circ L_g)| \to 0. \qquad \square$$

Remark 2.1.7. If G acts continuously on Y, then $\mathcal{N}^*(Y)$ is G-invariant. If, in addition, the metric d_Y on Y is G-invariant, then $\mathcal{N}_r^*(Y)$ is G-invariant and the maps δ and supp are G-equivariant.

2.2. Point processes and random sets

Point processes

Definition 2.2.1. (i) Let $(\Omega, \mathbb{P})$ be a probability space. A $\mathcal{N}^*(Y)$-valued random variable $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*(Y)$ is called a *point process (in Y)*.

(ii) If G acts continuously on Y and the distribution $\Lambda_*\mathbb{P}$ of Λ is G-invariant, we say that Λ is *G-stationary*.

(iii) If in addition G acts measurably and probability measure preserving on Ω and the map Λ is G-equivariant, we say that Λ is *G-invariant*.

(iv) We say that Λ is *r-uniformly discrete*, if $\Lambda_\omega \in \mathcal{N}_r^*(Y)$ for every $\omega \in \Omega$.

(v) If, for $1 \leq p \leq \infty$, we have that for every bounded Borel set B the random variable $\Lambda(B)$ is in $L^p(\Omega, \mathbb{P})$, we say that Λ is *locally L^p*. If $p = 1$ we say that Λ is *locally integrable*. If $p = 2$ we say that Λ is *locally square integrable*.

(vi) If Λ is G-stationary and its distribution $\Lambda_*\mathbb{P}$ is G-ergodic, we say that Λ is *ergodic*.

For a given point process Λ we will often denote the support of Λ_ω, $\omega \in \Omega$, by P_ω such that $\Lambda_\omega = \sum_{x \in P_\omega} \delta_x$.
In this thesis we will mainly be interested in G-stationary/G-invariant point processes in X. We will see some examples for point processes in Subsection 3.4.1 and Chapter 5. Our terminology of r-uniformly discrete point processes is non-standard. These processes are more typically called hard-core point processes with hard core radius r in the majority of sources on stochastic geometry.

Random sets

Recall that $\mathfrak{F}(Y)$ denotes the set of closed subsets of the proper second-countable metric space (Y, d_Y) equipped with the Chabauty-Fell topology. Denote the set of open subsets of Y by $\mathfrak{G}(Y)$.

Definition 2.2.2. Let $(\Omega, \mathbb{P})$ be a probability space.

(i) A *random closed set* in Y is a measurable map $C : (\Omega, \mathbb{P}) \to \mathfrak{F}(Y)$.

(ii) A *random open set* in (Y, d_Y) is a map $O : (\Omega, \mathbb{P}) \to \mathfrak{G}(Y)$ such that for every $F \in \mathfrak{F}(Y)$ the set $\{\omega \in \Omega \mid F \subset O_\omega\}$ is measurable.

(iii) A *weak random open set* is a map $U : (\Omega, \mathbb{P}) \to \mathfrak{G}(Y)$ such that for every finite set $E \subset Y$ the set $\{\omega \in \Omega \mid E \subset U_\omega\}$ is measurable.

(iv) A map $S : (\Omega, \mathbb{P}) \to \mathcal{P}(Y)$ is called *weakly measurable*, if $\{\omega \in \Omega \mid S_\omega \cap G \neq \emptyset\}$ is measurable for every $G \in \mathfrak{G}(Y)$.

Random open sets can equivalently be defined as maps $O : (\Omega, \mathbb{P}) \to \mathfrak{G}(Y)$ such that the complement map $\omega \mapsto Y \setminus O_\omega$ is a random closed set, see [82, pg. 76]. Note that a weak random open set is not necessarily a random open set. The definition of weak random open sets is not a standard definition, but turns out to be exactly what we need to define the packing density of a random sphere packing later.

Proposition 2.2.3. *Assume that $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(Y)$, $\omega \mapsto \sum_{x \in P_\omega} \delta_x$ is a $2r$-uniformly discrete point process. Then the map*

$$\operatorname{supp}(\Lambda, r) : (\Omega, \mathbb{P}) \to \mathfrak{G}(Y),\ \omega \mapsto \bigcup_{x \in P_\omega} B(x, r)$$

is a weak random open set.

Proof. Note that we only need to show the measurability of the sets $A_x := \{\omega \in \Omega \mid x \in \operatorname{supp}(\Lambda, r)_\omega\}$. We have $x \in \operatorname{supp}(\Lambda, r)$ if and only if $\Lambda(B(x,r)) > 0$. Thus the sets A_x are measurable by the definition of the σ-algebra on $\mathcal{N}^*_{2r}(Y)$. $\square$

Remark 2.2.4. It does not seem like $\operatorname{supp}(\Lambda, r)$ is generally a random open set. But in certain circumstances one can show with relative ease that it is. If $X = \mathbb{R}^n$ with the Euclidean metric, then the Minkowski sum $\operatorname{supp}(\Lambda) + \overline{B}(0, r)$ is a random closed set (cf. [82, Theorem 1.3.25]) and the interior of this set is given by $\operatorname{supp}(\Lambda, r)$. As the interior of a random closed set is a random open set (cf. [82, Proposition 1.3.36]), one obtains the claim. This proof-strategy already fails for $\mathbb{R}^n$ with the l_1-metric, as here the interior of the union is no longer necessarily the union of the interiors. In more degenerate spaces it might happen that all sets are clopen, such that $B(x,r)$ is no longer the interior of $\overline{B}(x,r)$, also causing this strategy to fail fundamentally.

We also prove that $\operatorname{supp}(\Lambda, r)$ is weakly measurable, though we do not need it in the following.

Proposition 2.2.5. *If* $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*_{2r}(X)$ *is a* $2r$*-uniformly discrete point process, then* $\operatorname{supp}(\Lambda, r)$ *is weakly measurable.*

Proof. For $x \in X$ consider the map $d_x : \mathrm{UD}_{2r}(X) \to [0, \infty),\ P \mapsto \inf_{y \in P} d_X(x, y)$. Note that $d_x(P) \in (a, b)$ iff $P \cap B(x, b) \neq \emptyset$ and $P \cap \overline{B}(x, a) = \emptyset$, where we use that P is locally finite and the infimum in the definition is thus a minimum. Thus $d_x^{-1}((a, b))$ is open in $\mathrm{UD}_{2r}(X)$ by the definition of the Chabauty-Fell topology and hence d_x is continuous.
Now, for $C \subset X$ consider the continuous map

$$\Psi_C : \mathrm{UD}_{2r}(X) \to \prod_{x \in C} [0, \infty),\ P \mapsto (d_x(P))_{x \in C}.$$

If C is compact, we have $C \cap \operatorname{supp}(P^r) = \emptyset$ iff $\inf_{x \in C} d_x(P) \geq r$ iff $\min_{x \in C} d_x(P) \geq r$ (as $x \mapsto d_x(P)$ is continuous) iff $\Psi_C(P) \in \prod_{x \in C} [r, \infty)$. Hence the set

$$S_C := \{P \in \mathrm{UD}_{2r}(X) \mid \operatorname{dist}(P, C) > r\} = \Psi_C^{-1}(\prod_{x \in C} [r, \infty))$$

is closed as the preimage of a closed set.
If $G \subset Y$ is open, we write G as a countable union of compact sets $(C_n)_{n \geq 1}$ and note that

$$\{P \in \mathrm{UD}_{2r}(X) \mid \exists x \in P : G \cap B(x, r) \neq \emptyset\} = (\bigcap_{n \geq 1} S_{C_n})^{\mathsf{c}}.$$

Hence this set measurable. Thus $\delta(\{P \in \mathrm{UD}_{2r}(X) \mid \exists x \in P : B(x,r) \cap G \neq \emptyset\})$ is measurable and this implies the claim, as

$$\{\omega \in \Omega \mid \operatorname{supp}(\Lambda, r)_\omega \cap G \neq \emptyset\} = \Lambda^{-1}(\delta(\{P \in \mathrm{UD}_{2r}(X) \mid \exists x \in P : B(x,r) \cap G \neq \emptyset\})).$$

$\square$

Remark 2.2.6. The proposition above implies that $\overline{\text{supp}(\Lambda, r)}$ is a random closed set. If X is the n-dimensional hyperbolic space with the usual hyperbolic metric, this implies (similar to the Euclidean case in Remark 2.2.4 above) that $\text{supp}(\Lambda, r)$ is a random open set.

2.3. First and second moments

The first moment

Lemma 2.3.1. *Let Λ be a stationary locally integrable point process in X. Then there is a constant $i(\Lambda) \geq 0$ such that*

$$\mathbb{E}(\Lambda(B)) = i(\Lambda) m_X(B)$$

for all Borel sets B.

Proof. The map $B \mapsto \mathbb{E}[\Lambda(B)]$ is a G-invariant Borel measure, as

$$\begin{aligned}
\mathbb{E}[\Lambda(gB)] &= \int_\Omega \Lambda_\omega(gB) d\mathbb{P}(\omega) = \int_\Omega (L_{g^{-1}})_* \Lambda_\omega(B) d\mathbb{P}(\omega) \\
&= \int_{\mathcal{N}^*(X)} (L_{g^{-1}})_* \mu(B) d\Lambda_* \mathbb{P}(\mu) = \int_{\mathcal{N}^*(X)} \mu(B) d\Lambda_* \mathbb{P}(\mu) \\
&= \int \Lambda_\omega(B) d\mathbb{P}(\omega) = E[\Lambda(B)].
\end{aligned}$$

Hence the claim follows from the uniqueness of the measure m_X up to scaling. □

Definition 2.3.2. The constant $i(\Lambda)$ above is called the *intensity* of Λ.

Remark 2.3.3. Note that Lemma 2.3.1 and the other results in this section also apply if $X = G/\{e\}$.

Proposition 2.3.4. *Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*(X)$ be a stationary locally integrable point process in X and assume that $m_X(X) = \infty$. If $i(\Lambda) > 0$, we have $\Lambda_\omega(X) = \infty$ for $\mathbb{P}$-almost every $\omega \in \Omega$.*

Proof. Assume that there is some measurable $A \subset \Omega$ with $\mathbb{P}(A) > 0$ and $\Lambda_\omega(X) < \infty$ for all $\omega \in A$. Then we know that the G-invariant set $\mathcal{N}^*_{\text{fin}}(X) := \{\mu \in \mathcal{N}^*(X) \mid \mu(X) < \infty\}$ has positive $\Lambda_*\mathbb{P}$-measure. Consider the Borel measure m on X defined by

$$m(B) := \frac{1}{\Lambda_* \mathbb{P}(\mathcal{N}^*_{\text{fin}}(X))} \int_{\mathcal{N}^*_{\text{fin}}(X)} \mu(B) d\Lambda_* \mathbb{P}(\mu).$$

As Λ is G-stationary and $\mathcal{N}^*_{\text{fin}}(X)$ is G-invariant, m is a G-invariant probability measure on X. But any G-invariant measure on X must be a multiple of m_X, a contradiction. □

Remark 2.3.5. Let Λ be a stationary locally integrable point process in X with $i(\Lambda) > 0$ and assume that $m_X(X) = \infty$, i.e. that X is noncompact. Then $\Lambda_*\mathbb{P}$ is a G-invariant probability measure concentrated on $\mathcal{N}^*_{\inf}(X) := \{\mu \in \mathcal{N}^*(X) \mid \mu(X) = \infty\}$. By considering the map

$$\Lambda^\infty : (\mathcal{N}^*_{\inf}(X), \Lambda_*\mathbb{P}) \to \mathcal{N}^*(X),\ \mu \mapsto \mu$$

we obtain a G-equivariant point process with the same distribution as Λ. We call Λ^∞ the *canonical infinite model* of Λ.

Proposition 2.3.6. *Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*_r(X)$ be a r-uniformly discrete point process. Then Λ is locally L^∞ and thus locally L^p for all $1 \leq p \leq \infty$. More precisely*

$$\Lambda(B(x_0, R)) \leq \frac{m_X(B(x_0, R + r))}{m_X(B(x_0, r))}$$

for all $R > 0$.

Proof. Let $B \subset X$ be a bounded Borel set and choose $R > 0$ such that $B \subset B(x_0, R)$ Observe that for any $P \in \mathrm{UD}_r(X)$ we have

$$\bigcup_{x \in P \cap B(x_0,R)} B(x, r) \subset B(x_0, R + r),$$

where the first union is disjoint. Thus

$$\#(P \cap B) \leq \#(P \cap B(x_0, R)) \leq \frac{m_X(B(x_0, R + r))}{m_X(B(x_0, r))} =: C.$$

Hence $\Lambda_\omega(B) \leq C$ for all $\omega \in \Omega$, which implies the claim. □

The second moment

The existence of approximate identities in $C_c(G, K)$ (see for instance [21, Remark A.12]) directly implies the following lemma.

Lemma 2.3.7. *The set $\{f * g \mid f, g \in C_c(G, K)\}$ is dense in $C_c(G, K)$. In particular, if μ, ν are two K-spherical Borel measures on G such that $\mu(f * g) = \nu(f * g)$ for all $f, g \in C_c(G, K)$, then $\mu = \nu$.*

Let $\mathcal{L}^\infty_{\mathrm{bnd}}(G) := \{f : G \to \mathbb{C} \mid f \text{ measurable, bounded with bounded support}\}$ and note that this is a convolution algebra with involution given by $f \mapsto f^*$.

Proposition 2.3.8 (Björklund–Byléhn, [14])**.** *Let Λ be a locally square integrable G-stationary point process in X. For all measurable $b : X \to [0, \infty)$ with $m_X(b) = 1$ and bounded support and Borel sections $\sigma : X \to G$, we have*

$$\mathbb{E}\left[\int_X \int_X f^* * g(\sigma(x)^{-1}\sigma(y))b(x)d\Lambda(y)d\Lambda(x)\right] = \mathbb{E}[\overline{\Lambda(f \circ \sigma)}\Lambda(g \circ \sigma)]$$

and

$$\mathbb{E}\left[\int_X \int_X f^* * g(\sigma(x)^{-1}\sigma(y))b(x)d\Lambda(y)d\Lambda(x)\right] - i(\Lambda)^2 m_G(f^* * g) = \mathrm{Cov}(\Lambda(f \circ \sigma), \Lambda(g \circ \sigma))$$

for all right K-invariant $f, g \in \mathcal{L}^\infty_{\mathrm{bnd}}(G)$.

Proof. Let $f, g \in \mathcal{L}^\infty_{\mathrm{bnd}}(G)$ be right K-invariant. We note that

$$\begin{aligned} f^* * g(\sigma(x)^{-1}\sigma(y)) &= \int_G \overline{f(h^{-1})} g(h^{-1}\sigma(x)^{-1}\sigma(y)) dm_G(h) \\ &= \int_G \overline{f(h)} g(h\sigma(x)^{-1}\sigma(y)) dm_G(h) \\ &= \int_G \overline{f(h\sigma(x))} g(h\sigma(y)) dm_G(h). \end{aligned}$$

Thus

$$\begin{aligned} &\mathbb{E}\left[\int_X \int_X f^* * g(\sigma(x)^{-1}\sigma(y)) b(x) d\Lambda(y) d\Lambda(x)\right] \\ &= \mathbb{E}\left[\int_G \int_X \int_X \overline{f(h\sigma(x))} g(h\sigma(y)) b(x) d\Lambda(y) d\Lambda(x) dm_G(h)\right] \\ &= \int_G \mathbb{E}\left[\int_X \int_X \overline{f(h\sigma(x))} g(h\sigma(y)) b(x) d\Lambda(y) d\Lambda(x)\right] dm_G(h) \\ &= \int_G \mathbb{E}\left[\int_X \int_X \overline{f(\sigma(hx))} g(\sigma(hy)) b(x) d\Lambda(y) d\Lambda(x)\right] dm_G(h) \\ &= \int_G \mathbb{E}\left[\int_X \int_X \overline{f(\sigma(x))} g(\sigma(y)) b(h^{-1}x) dh_*\Lambda(y) dh_*\Lambda(x)\right] dm_G(h) \\ &= \int_G \int_\Omega \int_X \int_X \overline{f(\sigma(x))} g(\sigma(y)) b(h^{-1}x) dh_*\mu(x) dh_*\mu(y) d\Lambda_*(\mathbb{P}) \\ &= \int_G \int_\Omega \int_X \int_X \overline{f(\sigma(x))} g(\sigma(y)) b(h^{-1}x) d\mu(x) d\mu(y) d\Lambda_*(\mathbb{P}) \\ &= \int_G \int_\Omega \int_X \int_X \overline{f(\sigma(x))} g(\sigma(y)) b(h^{-1}x) d\Lambda_\omega(y) d\Lambda_\omega(x) d\mathbb{P}(\omega) dm_G(h) \\ &= m_G(b)\mathbb{E}[\Lambda(f \circ \sigma)\Lambda(g \circ \sigma)] = \mathbb{E}[\Lambda(f \circ \sigma)\Lambda(g \circ \sigma)], \end{aligned}$$

where we have used that the right K-invariance of f and g implies $f(h\sigma(x)) = f(\sigma(hx))$ and $g(h\sigma(y)) = g(\sigma(hy))$. Observe that

$$\begin{aligned} \mathrm{Cov}(\Lambda(f \circ \sigma), \Lambda(g \circ \sigma)) &= \mathbb{E}[\overline{\Lambda(f \circ \sigma)}\Lambda(g \circ \sigma)] - \mathbb{E}[\overline{\Lambda(f \circ \sigma)}]\mathbb{E}[\Lambda(g \circ \sigma)] \\ &= \mathbb{E}[\overline{\Lambda(f \circ \sigma)}\Lambda(g \circ \sigma)] - i(\Lambda)^2 m_G(\overline{f}) m_G(g). \end{aligned}$$

As $m_G(f^* * g) = m_G(\overline{f}) m_G(g)$, the claim follows. ☐

Corollary 2.3.9 (Björklund–Byléhn, [14]). *Let Λ be a locally square integrable point process in X. For any choice of b and σ as in Proposition 2.3.8 we can define K-spherical Radon measures η_Λ and η_Λ^+ on G by*

$$\eta_\Lambda^+(f) := \mathbb{E}\left[\int \int f^\natural(\sigma(x)^{-1}\sigma(y)) b(x) d\Lambda(y) d\Lambda(x)\right]$$

and

$$\eta_\Lambda(f) := \mathbb{E}\left[\int\int f^\natural(\sigma(x)^{-1}\sigma(y))b(x)d\Lambda(y)d\Lambda(x)\right] - i(\Lambda)^2 m_G(f)$$

for all $f \in C_c(G)$. These measures are independent of our choice of b and σ.

Proof. This follows directly from Lemma 2.3.7 and Proposition 2.3.8. □

Now the following corollary follows directly from Lemma 1.3.4.

Corollary 2.3.10. *If G is a Lie group, η_Λ^+ and η_Λ induce distributions on G.*

Definition 2.3.11 (Björklund–Hartnick–Pogorzelski, [21], Björklund–Hartnick, [17])**.** Let Λ be a locally square integrable point process in X. The bi-K-invariant Borel measures η_Λ^+ and η_Λ on G are called *autocorrelation measure* and and *reduced autocorrelation measure* of Λ. The distributions T_Λ^+ and T_Λ induced by η_Λ^+ and η_Λ are called *autocorrelation distribution* and *reduced autocorrelation distribution* of Λ.

Remark 2.3.12. As the autocorrelation measure is one of the central technical tools of this thesis, some remarks on its origin are in order. In the case of $G = \mathbb{R}^n$ and $X = \mathbb{R}^n$, this measure is a common technical tool in the theory of point processes and is known under the name reduced second order measure, see for instance [45, Proposition 12.6.III]. In the theory of mathematical quasi-crystals it is used to define the diffraction of quasi-crystals, see for instance [7, Chapter 9]. Motivated by examples from the field of mathematical quasi-crystals, specifically cut-and-project sets, Björklund, Hartnick and Pogorzelski set out in [19, 20, 21] to generalize the ergodic theory and diffraction theory of cut-and-project sets to a large class of locally compact groups. More specifically for a G-stationary locally square integrable point process Λ in X they defined the second correlation measure $\eta_\Lambda^{(2)}$ on $X \times X$ by

$$\int_{X\times X} f_1(x)f_2(y)d\eta_\Lambda^{(2)}(x,y) := \mathbb{E}[\overline{\Lambda(f_1)}\Lambda(f_2)].$$

The G-stationarity of Λ now implies that $\eta_\Lambda^{(2)}$ is invariant under the diagonal action of G on $X \times X$. Using this they show that the measure $\eta_\Lambda^{(2)}$ disintegrates to a measure $\overline{\eta}_\Lambda^+$ on $K\backslash G/K$ satisfying

$$\overline{\eta}_\Lambda^+(f^* * g) = \mathbb{E}[\overline{\Lambda(f\circ\sigma)}\Lambda(g\circ\sigma)]$$

for all right-K-invariant $f, g \in C_c(G)$ (note that $f^* * g$ is bi-K-invariant and thus defines a function on $K\backslash G/K$). This measure then lifts to a K-spherical measure on G. This allows for application of the full toolbox of harmonic analysis on Gelfand pairs to the study of point processes on X.

A relative version of the following statement was observed by Björklund, Hartnick and Pogorzelski in [20] for their measure $\overline{\eta}_\Lambda^+$ on $K\backslash G/K$.

Proposition 2.3.13. *Let Λ be a locally square integrable point process in X and assume that G is a Lie group. The measures η_Λ and η_Λ^+ are positive-definite and thus the distributions T_Λ^+ and T_Λ are positive definite.*

Proof. Let $g \in C_c(G)$ be right-K-invariant. Then the Proposition 2.3.8 implies that

$$\eta_\Lambda^+(g^* * g) \geq 0.$$

If $f \in C_c(G)$, then

$$\begin{aligned}
\eta_\Lambda^+(f^* * f) &= \int_G \int_K \int_K f^* * f(kgk)\, dm_K(k_1)\, dm_K(k_2)\, d\eta_\Lambda^+(g) \\
&= \int_G \int_K \int_K \int_G \overline{f(h^{-1})} f(h^{-1}k_1gk_2)\, dm_G(h)\, dm_K(k_1)\, dm_K(k_2)\, d\eta_\Lambda^+(g) \\
&= \int_G \int_K \int_K \int_G \overline{f(h^{-1}k_1^{-1})} f(h^{-1}gk_2)\, dm_G(h)\, dm_K(k_1)\, dm_K(k_2)\, d\eta_\Lambda^+(g) \\
&= \int_G \int_K \int_K \int_G \overline{f(h^{-1}k_1)} f(h^{-1}gk_2)\, dm_G(h)\, dm_K(k_1)\, dm_K(k_2)\, d\eta_\Lambda^+(g) \\
&= \int_G \int_G \overline{f'(h^{-1})} f'(h^{-1}g)\, dm_G(h)\, d\eta_\Lambda^+(g) \\
&= \int_G (f')^* * (f')(g)\, d\eta_\Lambda^+(g) \\
&= \eta_\Lambda^+((f')^* * (f')) \geq 0,
\end{aligned}$$

where we note that the function f' on G defined by

$$f'(g) = \int_K f(gk) dm_K(k)$$

is right-K-invariant and in $C_c(G)$. The same calculation applied to η_Λ shows that η_Λ is positive-definite. □

This proposition will allow us to use the Godement-Plancherel theorem and the tools of harmonic analysis to study point processes in commutative spaces.

2.4. Ergodic theory of point processes

In this section we will discuss the ergodic theory of uniformly discrete point processes. We do so by studying ergodic measures on $\mathrm{UD}_{2r}(X)$.

2.4.1. Invariant pointwise ergodic theorems

Definition 2.4.1. Let $(G_t)_{t>0}$ be a sequence of subsets of G with positive measures, let $\mathcal{F}$ be a set of measurable functions on $\mathrm{UD}_{2r}(X)$ such that the map

$$\mathrm{Prob}(\mathrm{UD}_{2r}(X)) \to \mathbb{C}^{\mathcal{F}}, \quad \nu \mapsto (\nu(f))_{f \in \mathcal{F}}$$

is injective, and let μ be a G-ergodic probability measure on $\mathrm{UD}_{2r}(X)$.

(i) We say that $P_0 \in \mathrm{UD}_{2r}(X)$ is $(\mu, \mathcal{F}, (G_t)_{t>0})$-*generic*, if

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f(gP_0)dm_G(G) = \mu(f) \quad (f \in \mathcal{F}).$$

(ii) We say that $P_0 \in \mathrm{UD}_{2r}(X)$ is *invariantly* $(\mu, \mathcal{F}, (G_t)_{t>0})$-*generic*, if

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f(ghP_0)dm_G(G) = \mu(f) \quad (f \in \mathcal{F}, h \in G).$$

Definition 2.4.2. We say that the *invariant pointwise ergodic theorem* holds for the tuple $((G_t)_{t>0}, \mathcal{F})$ if for every G-ergodic $\mu \in \mathrm{Prob}(\mathrm{UD}_{2r}(X))$ there is a G-invariant conull set $\Omega_0 \subset \mathrm{UD}_{2r}(X)$ such that every $P \in \Omega_0$ is invariantly $(\mu, \mathcal{F}, (G_t)_{t>0})$-generic.

In the following chapters we will usually assume that an invariant pointwise ergodic theorem holds for $\mathcal{F} = C(\mathrm{UD}_{2r}(X))$ and some sequence $(G_t)_{t>0}$. We now discuss some examples for choices of $G, K, X, d, (G_t)_{t>0}$ which satisfy this requirement. In order to obtain invariant pointwise ergodic theorems from the (usual) pointwise ergodic theorems we have in all of the examples, we make use of the following.

Definition 2.4.3 (Gorodnik–Nevo, [56])**.** A sequence $(G_t)_{t>0}$ of bounded Borel subsets of G is called quasi-uniform if

(QU1) for every $\varepsilon > 0$ there is an open neighborhood O of e in G such that for all sufficiently large $t > 0$,
$$OG_t \subseteq G_{t+\varepsilon},$$

(QU2) for every $\delta > 0$ there is an $\varepsilon > 0$ such that for all sufficiently large $t > 0$,
$$m_G(G_{t+\varepsilon}) \leq (1 + \delta)m_G(G_t).$$

Theorem 2.4.4 (Gorodnik–Nevo, [56, Theorem 5.22])**.** *Let G be a lcsc group acting measurably on a standard Borel space Ω equipped with a G-invariant Borel probability measure μ. Let $p \geq 1$ and suppose that $(G_t)_{t>0}$ is a family of measurable subsets of G with positive measure such that for every $f \in L^p(\Omega, \mu)$ there is a conull set Ω_0 such that*

$$\frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}x)dm_G(g) \to \int_X f d\mu$$

for every $x \in \Omega_0$. Then, if $(G_t)_{t>0}$ is quasi-uniform, Ω_0 can be chosen to be G-invariant.

This result by Gorodnik and Nevo is a generalization of a technique employed by Bowen and Radin in [29] to obtain invariance properties for an ergodic theorem for the isometry group of hyperbolic space.

Proposition 2.4.5. *Assume that $(G_t)_{t>0}$ is a quasi-uniform family of symmetric sets. Let μ be a G-invariant Borel probability measure on* $\mathrm{UD}_{2r}(X)$*. Let $\mathcal{F} \subset C(\mathrm{UD}_{2r}(X))$ be* *a dense subset. If for every $f \in \mathcal{F}$ there is a conull set $\Omega_0 \subset \mathrm{UD}_{2r}(X)$ such that*

$$\frac{1}{m_G(G_t)}\int_{G_t} f(gP)dm_G(g) \to \int_{\mathrm{UD}_{2r}(X)} f d\mu$$

for every $P \in \Omega_0$, then the invariant pointwise ergodic theorem holds for the tuple $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$.

Proof. Note that $C(\mathrm{UD}_{2r}(X))$ is separable as $\mathrm{UD}_{2r}(X)$ is compact and choose a countable dense subset $\{f_n \mid n \in \mathbb{N}\} \subset \mathcal{F}$. For each $n \in \mathbb{N}$ use Theorem 2.4.4 to choose a G-invariant conull set $\Omega_n \subset \mathrm{UD}_{2r}(X)$ such that

$$\frac{1}{m_G(G_t)}\int_{G_t} f_n(gP)dm_G(g) \to \int_X f_n d\mu$$

for every $P \in \Omega_n$. Set $\Omega_0 := \bigcap_{n\in\mathbb{N}} \Omega_n$ and note that Ω_0 is a G-invariant conull set. Now let $f \in C(\mathrm{UD}_{2r}(X))$ and choose a sequence $(n_k)_{k\geq 1}$ such that $f_{n_k} \to f$ uniformly as $k \to \infty$. Given $\varepsilon > 0$ we choose $N \in \mathbb{N}$ such that $\|f - f_{n_k}\|_\infty < \varepsilon$ for all $k \geq N$. Then for any $P \in \Omega_0$,

$$\begin{aligned}
\Delta(t) &:= \left|\int_S f d\mu - \frac{1}{m_G(G_t)}\int_{G_t} f(gP)dm_G(g)\right| \\
&\leq \int_\Omega |f - f_{n_k}|d\mu + \left|\int f_{n_k} d\mu - \frac{1}{m_G(G_t)}\int_{G_t} f_{n_k}(gP)dm_G(g)\right| \\
&\quad + \frac{1}{m_G(G_t)}\int_{G_t} |f_{n_k} - f| dm_G \\
&\leq \varepsilon + \left|\int f_{n_k} d\mu - \frac{1}{m_G(G_t)}\int_{G_t} f_{n_k}(gP)dm_G(g)\right| + \varepsilon.
\end{aligned}$$

By our assumptions there exists $t_0(\varepsilon)$ such that for all $t \geq t_0(\varepsilon)$ the second summand is bounded by ε, and hence $\Delta(t) \leq 3\varepsilon$ for all $t \geq t_0(\varepsilon)$. This shows that $\Delta(t) \to 0$, which shows that $P \in \Omega_0$ satisfies

$$\frac{1}{m_G(G_t)}\int_{G_t} f_n(gP)dm_G(g) \to \int_X f_n d\mu$$

for every $f \in C(\mathrm{UD}_{2r}(X))$ and $P \in \Omega_0$. As Ω_0 is G-invariant, this implies that every $P \in \Omega_0$ is invariantly $(\mu, C(\mathrm{UD}_{2r}(X)), (G_t)_{t>0})$-generic. □

2.4.2. Generically measured sets

It turns out that $2r$-uniformly discrete G-ergodic point processes are completely determined by certain elements of $\mathrm{UD}_{2r}(X)$.

Definition 2.4.6. Let $(G_t)_{t>0}$ be a sequence of measurable subsets of G with finite positive measure.

(i) We say that an element $P \in \mathrm{UD}_{2r}(X)$ is *generically measured* (wrt. $(G_t)_{t>0}$) if there is an ergodic $\mu \in \mathrm{Prob}^G(\mathrm{UD}_{2r}(X))$ such that the set P is invariantly

$(\mu, C(\mathrm{UD}_{2r}(X)), (G_t)_{t>0})$-generic. We denote the set of P in $\mathrm{UD}_{2r}(X)$ $2r$-uniformly discrete which are generically measured wrt. $(G_t)_{t>0}$ by $\mathrm{UD}_{2r}^{gen}(X, (G_t)_{t>0})$.

(ii) Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(X)$ be a $2r$-uniformly discrete G-invariant ergodic point process. We say that $P \in \mathrm{UD}_{2r}(X)$ is *generically measured* for Λ (with respect to $(G_t)_{t>0}$), if P is invariantly $(\mathrm{supp}_*\Lambda_*\mathbb{P}, C(\mathrm{UD}_{2r}(X), (G_t)_{t>0})$-generic.

Lemma 2.4.7. *$P \in \mathrm{UD}_{2r}(X)$ is generically measured (wrt. $(G_t)_{t>0}$) if and only if there is a $2r$-uniformly discrete G-invariant ergodic point process $(\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(X)$ such that P is generically measured for Λ (wrt. $(G_t)_{t>0}$).*

Proof. It is clear that P is generically measured, if P is generically measured for a G-invariant ergodic point process Λ.
Let $P \in \mathrm{UD}_{2r}(X)$ be generically measured (wrt. $(G_t)_{t>0}$). Then there is an ergodic probability measure $\mu \in \mathrm{Prob}^G(\mathrm{UD}_{2r}(X))$ such that P is invariantly $(\mu, C(\mathrm{UD}_{2r}(X)), (G_t)_{t>0})$-generic. Now the map $\Lambda : (\mathrm{UD}_{2r}(X), \mu) \to \mathcal{N}_{2r}^*(X)$, $P \mapsto \sum_{x \in P} \delta_x$ is a G-invariant ergodic point process such that P is generically measured for Λ (wrt. $(G_t)_{t>0}$). □

Invariant pointwise ergodic theorems are not in a convenient form for application to invariant point processes. The definitions above imply the following conveniently formulated ergodic theorem:

Proposition 2.4.8. *Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(X)$ be a $2r$-uniformly discrete G-invariant ergodic point process. If the invariant pointwise ergodic theorem holds for the tuple $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$, then there is a G-invariant conull set $\Omega_0 \subset \Omega$ such that we have*

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f(\Lambda_{g\omega}) dm_G(g) = \int f(\Lambda_\omega) d\mathbb{P}(\omega)$$

for any $f \in C(\mathcal{N}_{2r}^*(X))$ and any $\omega \in \Omega_0$. Moreover, the set $P_\omega \coloneqq \mathrm{supp}(\Lambda_\omega) \in \mathrm{UD}_{2r}(X)$ is generically measured for Λ (wrt. $(G_t)_{t>0}$) for any $\omega \in \Omega_0$.

Proof. Note that for $f \in C(\mathcal{N}_{2r}(X))$ we have $f \circ \delta \in C(\mathrm{UD}_{2r}(X))$ (and that we can obtain any function in $C(\mathrm{UD}_{2r}(X))$ is this way, as δ is a homeomorphism). Apply the invariant pointwise ergodic theorem to the measure $\mathrm{supp}_*\Lambda_*\mathbb{P}$ and the function $f \circ \delta$. Then, if P is invariantly $(\mathrm{supp}_*\Lambda_*\mathbb{P}, (G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$-generic, we have

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f(\delta(gP)) dm_G(g) = \int_{\mathrm{UD}_{2r}(X)} f(\delta(P)) d\mathrm{supp}_*\Lambda_*\mathbb{P}(P).$$

By the invariant pointwise ergodic theorem there is a $\mathrm{supp}_*\Lambda_*\mathbb{P}$-conull set $\Omega_0' \subset \mathrm{UD}_{2r}(X)$ of invariantly $(\mathrm{supp}_*\Lambda_*\mathbb{P}, (G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$-generic elements. Set $\Omega_0 = \Lambda^{-1}(\mathrm{supp}^{-1}(\Omega_0'))$ and note that $\Omega_0 \subset \Omega$ is a G-invariant $\mathbb{P}$-conull set. We further observe that

$$\int_{\mathrm{UD}_{2r}(X)} f(\delta(P)) d\mathrm{supp}_*\Lambda_*\mathbb{P}(P) = \int_{\mathcal{N}_{2r}^*(X)} f(\delta(\mathrm{supp}(\mu))) d\Lambda_*\mathbb{P}(\mu) = \int_\Omega f(\omega) d\mathbb{P}(\omega),$$

as supp and δ are inverse to each other. We also note that for $\omega \in \Omega_0$ we have

$$\begin{aligned}\int_{G_t} f(\Lambda_{g\omega}) dm_G(g) = \int_{G_t} f(g\Lambda_\omega) dm_G(g) = \int_{G_t} f(g\delta(\text{supp}(\Lambda_\omega))) dm_G(g) \\ = \int_{G_t} f(\delta(g\text{supp}(\Lambda_\omega))) dm_G(g) = \int_{G_t} f(\delta(gP_\omega)) dm_G(g)\end{aligned}$$

with $\text{supp}(\Lambda_\omega) =: P_\omega \in \Omega_0'$, as δ is G-equivariant. Hence the claim follows. □

Proposition 2.4.9. *Assume that there is a sequence $(G_t)_{t>0}$ of subsets of G such that an invariant pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\text{UD}_{2r}(X)))$.*

(i) Let $\mathbb{P} \in \text{Prob}^G(\text{UD}_{2r}(X))$ be ergodic. Then $\mathbb{P}$-almost every $P \in \text{UD}_{2r}(X)$ is generically measured wrt. $(G_t)_{t>0}$.

(ii) Let $P \in \text{UD}_{2r}(X)$ be generically measured wrt. $(G_t)_{t>0}$. Then there exists a unique ergodic measure $\mathbb{P}_P \in \text{Prob}^G(\text{UD}_{2r}(X))$ such that P is invariantly $(\mathbb{P}_P, C(\text{UD}_{2r}(X)), (G_t)_{t>0})$-generic.

Proof. (i) This follows directly from the invariant pointwise ergodic theorem.

(ii) The existence of such a measure is clear from the definitions. We still need to check the uniqueness. Thus assume that there are two ergodic measures μ, ν on $\text{UD}_{2r}(X)$ such that P is invariantly $(\mu, C(\text{UD}_{2r}(X)), (G_t)_{t>0})$-generic and invariantly $(\nu, C(\text{UD}_{2r}(X)), (G_t)_{t>0})$-generic. Then for any $f \in C(\text{UD}_{2r}(X))$ we have

$$\mu(f) = \lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f(gP) dm_G(g) = \nu(f).$$

Hence μ and ν agree as functionals on $C(\text{UD}_{2r}(X))$ and thus they are equal as measures by the Riesz-Markov-Kakutani theorem. □

Note that there is a bijection between ergodic probability measures on $\mathcal{N}^*_{2r}(X)$ and ergodic probability measures on $\text{UD}_{2r}(X)$, given by sending an ergodic $\mu \in \text{Prob}^G(\text{UD}_{2r}(X))$ to $\delta_*\mu$.

Definition 2.4.10. Let $P \in \text{UD}_{2r}(X)$ be generically measured. We define the *ergodic $2r$-uniformly discrete point process Λ^P associated to P* by

$$\Lambda^P : (\text{UD}_{2r}(X), \mathbb{P}_P) \to \mathcal{N}^*_{2r}(X),\ Q \mapsto \delta_Q = \sum_{x\in Q} \delta_x.$$

Theorem 2.4.11 (Structure of ergodic uniformly discrete point processes)**.** *Assume that there is a sequence $(G_t)_{t>0}$ of subsets of G such that the invariant pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\text{UD}_{2r}(X)))$. Then for every $2r$-uniformly discrete G-ergodic point process $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*_{2r}(X)$ there is a generically measured $P \in \text{UD}_{2r}(X)$ wrt. $(G_t)_{t>0}$ such that the distributions of Λ^P and Λ are equal.*

Proof. Apply the invariant pointwise ergodic theorem to the ergodic measure $\text{supp}_*(\Lambda_*\mathbb{P})$ on $\text{UD}_{2r}(X)$ to obtain a generically measured $P \in \text{UD}_{2r}(X)$. By (ii) in Proposition 2.4.9 we then have $\mathbb{P}_P = \text{supp}_*\Lambda_*\mathbb{P}$ and thus $\Lambda_*\mathbb{P}_P = \Lambda_*\mathbb{P}$. □

Remark 2.4.12. If G is a compact group, every set is generically measured. If $P \in \mathrm{UD}_{2r}(X)$, then the orbit GP is compact, as the image of the compact set G under a continuous map. Hence $i\colon G/\mathrm{Stab}_G(P) \to GP$, $g\mathrm{Stab}_G(P) \to gP$ is a bijective continuous map between two compact Hausdorff spaces and thus a homoemorphism. Thus any G-invariant Borel measure on GP is a multiple of $\mu_p := i_* m_{G/\mathrm{Stab}_G(P)}$ by the uniqueness properties of $m_{G/\mathrm{Stab}_G(P)}$ and thus ergodic by the transitivity of the G-action on GP. Moreover, if there is some $T > 0$ with $G_t = G$ for all $t > T$, then

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f(ghP)dm_G(g) = \int_G f(gP)dm_G(g) = \int_{GP} f(Q)d\mu_P(Q)$$

for all $f \in C(\mathrm{UD}_{2r}(X))$ by the unimodularity of G and the Weil disintegration theorem. Moreover, if μ is any ergodic measure on $\mathrm{UD}_{2r}(X)$, then μ is supported on a G-orbit by [102, Proposition 2.1.10 and Proposition 2.1.12] and thus of the form $\lambda\mu_P$ for some $P \in \mathrm{UD}_{2r}(X)$ and $\lambda > 0$.

2.4.3. The case of amenable groups

All amenable examples we consider in this thesis satisfy the assumptions of the following theorem.

Theorem 2.4.13. *Assume that there is are $d \in \mathbb{N}$ and $b > 0$ such that $m_X(B(e,t)) = bt^d$ for all $t > 0$ and set $G_t := \pi^{-1}(B(x_0,t))$. Then the invariant ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$.*

Proof. Note that $m_G(G_t) = m_X(B(x_0,t))$ and that $G_t^{-1}G_t \subset G_{2t}$. Further there is a $C > 0$ such that $m_G(G_{2t}) = Cm_G(G_t)$ and $\lim_{t\to\infty} \frac{m_G(G_{t\pm s})}{m_G(G_t)} = 1$ for any $s \geq 0$. Additionally, for $\varepsilon > 0$ we have that $G_{\varepsilon/2}$ is a unit neighborhood with $G_{\varepsilon/2}G_t \subset G_{t+\varepsilon}$. Hence the sequence $(G_t)_{t>0}$ satisfies (QU1). Moreover, if $\delta > 0$, we choose ε such that $(1+\varepsilon)^d \leq 1+\delta$. Then, if $t > 1$,

$$m_G(G_{t+\varepsilon}) = b(t+\varepsilon)^d \leq bt^d(1+\varepsilon)^d \leq (1+\delta)m_G(G_t)$$

and thus (QU2) holds and $(G_t)_{t>0}$ is quasi-uniform. Now Calderon's pointwise ergodic theorem for families of symmetric unit neighborhoods with volume doubling, cf. [31], together with Proposition 2.4.5 imply our claim. □

Example 2.4.14. In the following cases the assumptions of the previous theorem apply:

- $G = \mathbb{R}^n$, $K = \{0\}$ and $X = \mathbb{R}^n$ with the Euclidean distance.
- In the case $G = \mathbb{R}^n \rtimes \mathrm{SO}(n)$, $K = \mathrm{SO}(n)$ and $X = \mathbb{R}^n$ with the Euclidean distance.
- In the case of the Heisenberg motion group $G = \mathcal{H}_n \rtimes \mathrm{U}(n)$, $K = \mathrm{U}(n)$ and $X = \mathcal{H}_n$ (the Heisenberg group) with the Cygan-Koranyi metric.

More generally these assumptions apply to homogeneous Lie groups equipped with homogeneous metrics.

2.4.4. The case of Riemannian symmetric spaces

In the case of symmetric spaces of noncompact type a special case of an ergodic theorem by Gorodnik and Nevo together with Proposition 2.4.5 provides a invariant pointwise ergodic theorem.

Theorem 2.4.15 (Gorodnik-Nevo, [56]). *Let G be a semisimple Lie group with finite center and no compact factors. Let K be a maximal compact subgroup and let $B(x_0, t)$ denote the balls in the symmetric space G/K (equipped with the Cartan-Killing metric). Let (Ω, μ) be a standard Borel space with a pmp ergodic action of G. Then, for $G_t := \pi^{-1}(B(x_0, t))$ and $f \in L^p(\Omega, \mu)$ with $1 < p < \infty$, there is a G-invariant set $\Omega_f \subset \Omega$ of full measure such that*

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}x) dm_G(g) = \int_\Omega f d\mu$$

for every $x \in \Omega_f$. Moreover the sequence $(G_t)_{t>0}$ is quasi-uniform.

We should also mention that this special case of Gorodnik and Nevo's results for Cartan-Killing balls was previously obtained by Margulis, Nevo and Stein in [78].

2.4.5. Function spaces

Definition 2.4.16. We call a function $f \colon \mathrm{UD}_{2r}(X) \to \mathbb{R}$ *invariantly Riemann-integrable*, if there are sequences $(f_n^+)_{n\geq 1}$ and $(f_n^-)_{n\geq 1}$ in $C(\mathrm{UD}_{2r}(X))$ such that

$$f_n^- \leq f_{n+1}^- \leq f \leq f_{n+1}^+ \leq f_n^+ \quad \text{for all } n \in \mathbb{N}$$

and for any G-invariant Borel probability measure μ on $\mathrm{UD}_{2r}(X)$ we have $\lim_{n\to\infty} f_n^-(P) = \lim_{n\to\infty} f_n^+(P) = f(P)$ for μ-almost every $P \in \mathrm{UD}_{2r}(X)$. Denote the set of all invariantly Riemann-integrable functions on $\mathrm{UD}_{2r}(X)$ by $R(\mathrm{UD}_{2r}(X))$. We say that $A \subset \mathrm{UD}_{2r}(X)$ is *invariantly Jordan-measurable*, if χ_A is invariantly Riemann-integrable.

Proposition 2.4.17. *Assume that the invariant pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$. Then the invariant pointwise ergodic theorem also holds for the tuple $((G_t)_{t>0}, R(\mathrm{UD}_{2r}(X)))$.*

Proof. Let μ be a G-ergodic Borel probability measure on $\mathrm{UD}_{2r}(X)$. Let $f \in R(\mathrm{UD}_{2r}(X))$ and choose sequences $(f_n^+)_{n\geq 1}$, $(f_n^-)_{n\geq 1}$ as in Definition 2.4.16. For every $\varepsilon > 0$ there is (by monotone convergence) a $N \in \mathbb{N}$ such that

$$\int f - f_n^- d\mu \leq \varepsilon \quad \text{and} \quad \int f_n^+ - f d\mu \leq \varepsilon$$

for all $n \geq N$. Choose $T > 0$ such that

$$\left| \frac{1}{m_G(G_t)} \int_{G_t} f_n^\pm(gP) dm_G(g) - \int f_n^\pm d\mu \right| \leq \varepsilon$$

for $t > T$. Then

$$\int f d\mu - 2\varepsilon \leq \frac{1}{m_G(G_t)} \int_{G_t} f_n^-(gP) dm_G(g) \leq \frac{1}{m_G(G_t)} \int_{G_t} f(gP) dm_G(g)$$
$$\leq \frac{1}{m_G(G_t)} \int_{G_t} f_n^+(gP) dm_G(g) \leq \int f d\mu + 2\varepsilon$$

and thus for any $\varepsilon > 0$ there is a $T > 0$ such that

$$\left| \frac{1}{m_G(G_t)} \int_{G_t} f(gP) dm_G(g) - \int f d\mu \right| \leq 2\varepsilon$$

for any $t > T$. □

The following proposition will ensure that an important function that appears in the stochastic definition of packing density is invariantly Riemann-integrable. This allows for our reformulation of Bowen and Radin's density framework in terms of generically measured sets.

Proposition 2.4.18. *Assume that $m_X(\overline{B}(x,r) \setminus B(x,r)) = 0$ for all $r > 0$ and $x \in X$. For every $x \in X$ the set*

$$\mathrm{UD}_{2r}(X)_x := \{P \in \mathrm{UD}_{2r}(X) \mid \mathrm{dist}(P,x) < r\}$$

is invariantly Jordan-measurable and in particular $\mu(\partial \mathrm{UD}_{2r}(X)_x) = 0$ for any G-invariant Borel probability measure on $\mathrm{UD}_{2r}(X)$.

Proof. Let μ be a G-invariant Borel probability measure on $\mathrm{UD}_{2r}(X)$ and consider the set $W = \{(x,P) \in X \times \mathrm{UD}_{2r}(X) \mid \mathrm{dist}(P,x) = r\}$. Then the horizontal slices

$$W^P = \{x \in X \mid \mathrm{dist}(P,x) = r\} = \bigcup_{y \in P} \overline{B}(x,r) \setminus B(x,r)$$

are m_X-nullsets. Thus, by Fubini's theorem for σ-finite measures, we have $m_X \otimes \mu(W) = 0$ and thus, again by Fubini's theorem, for m_X-almost every $x \in X$ the vertical slice

$$W^x := \{P \in \mathrm{UD}_{2r}(X) \mid \mathrm{dist}(P,x) = r\}$$

satisfies $\mu(W^x) = 0$. But $gW^x = W^{g^{-1}x}$ and as G acts transitively on X and μ is G-invariant, we see that $\mu(W^x) = 0$ for every $x \in X$. We note that, by the definition of the Chabauty-Fell topology, the set $\mathrm{UD}_{2r}(X)_x$ is open and the set

$$V_x = \{P \in \mathrm{UD}_{2r}(X) \mid \mathrm{dist}(P,x) \leq r\}$$

is closed. As V_x is closed, χ_{V_x} is upper semicontinuous. As $\mathrm{UD}_{2r}(X)_x$ is open, $\chi_{\mathrm{UD}_{2r}(X)_x}$ is lower semicontinuous. By Baire's theorem on semicontinuity there are continuous functions $(f_n^-)_{n \geq 1}$ and $(f_n^+)_{n \geq 1}$ on $\mathrm{UD}_{2r}(X)$ such that $f_n^- \leq f_{n+1}^-$ and $f_{n+1}^+ \leq f_n^+$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} f_n^- = \chi_{\mathrm{UD}_{2r}(X)_x}$ (pointwise) and $\lim_{n \to \infty} f_n^+ = \chi_{V_x}$ (pointwise). Hence the claim follows, as $W^x = V_x \setminus \mathrm{UD}_{2r}(X)_x$ is a μ-nullset. □

2.5. The Palm measure

Associated to each G-invariant point process is a technical gadget, called the Palm measure, which encodes behavior of the point process around a fixed point. Palm measures for point processes in homogeneous spaces were first developed by Rother and Zähle in [88]. We will use a variant of the Palm measure for point processes in homogeneous spaces due to Last, [75].

The Palm measure and the refined Campbell theorem

For $x \in X$ choose $g_x \in G$ with $g_x x_0 = x$ (e.g. $g_x = \sigma(x)$, with σ a Borel section of π). We define

$$\kappa : X \times \mathcal{B}(G) \to \mathbb{R}, \quad \kappa(x, B) := m_K(g_x^{-1} B \cap K) = m_K(\sigma(x)^{-1} B \cap K).$$

Then κ is a transition kernel. Its definition is independent of our choice of g_x (see [75]). We will write

$$\kappa_x(B) := \kappa(x, B).$$

Then

$$\int_G f(g) d\kappa_x(g) = \int_K f(\sigma(x)k) dm_K(k) = \int_K f(g_x k) dm_K(k)$$

for all measurable $f \geq 0$.

Theorem 2.5.1 (Last, [75]). *Let Λ be a locally square integrable G-invariant point process and let $w : X \to [0, \infty)$ be measurable with $m_X(w) = 1$. Then the formula*

$$\mathbb{P}_\Lambda(A) = \int_\Omega \int_X \int_G \chi_A(g^{-1}\omega) w(x) d\kappa_x(g) d\Lambda_\omega(x) d\mathbb{P}(\omega)$$

for $A \subset \Omega$ measurable defines a (finite) measure $\mathbb{P}_\Lambda$ on Ω.

We should point out that Last's result does not require local square integrability (in fact Last works in a general context where $\mathbb{P}$ is not necessarily a finite measure and $\mathbb{P}_\Lambda$ is then also not necessarily finite). As we will only consider locally square integrable processes later, there is no harm in assuming it for all statements in this section and simplifying our assumptions overall.

Definition 2.5.2. The measure $\mathbb{P}_\Lambda$ is called the *Palm measure* of Λ.

Furthermore, Last observed the following:

Lemma 2.5.3. *The measure $\mathbb{P}_\Lambda$ is concentrated on the measurable set $\mathcal{T} = \{\omega \in \Omega \mid \Lambda_\omega(\{x_0\}) = 1\}$ in the sense that $\mathbb{P}_\Lambda(\Omega \setminus \mathcal{T}) = 0$. We call $\mathcal{T}$ the* canonical Ω-transversal *for Λ.*

By abuse of notation we denote the measure $\mathbb{P}_\Lambda|_\mathcal{T}$ again by $\mathbb{P}_\Lambda$. Note that $g^{-1}\omega \in \mathcal{T}$ if and only if $g = g_x k$ with $\Lambda_\omega(x) = 1$ and $k \in K$.

Definition 2.5.4. Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*(X)$ be a locally square integrable G-invariant point process and let $F : \mathcal{T} \times G \to \mathbb{R}_+$ be measurable. The *Palm periodization* of F is defined as

$$TF(\omega) := \int_X \int_G F(g^{-1}\omega, g) d\kappa_x(g) d\Lambda_\omega(x)$$

for $\omega \in \Omega$. If $F : \Omega \times G \to \mathbb{R}_+$ is measurable, we define TF as the Palm periodization of $F|_{\mathcal{T}}$.

Now the following variant of the refined Campbell theorem holds.

Theorem 2.5.5 (Last, [75])**.** *Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*(X)$ be a locally square integrable G-invariant point process in X. Then $\mathbb{P}_\Lambda$ satisfies*

$$\int_\Omega TF(\omega) d\mathbb{P}(\omega) = \int_{\mathcal{T}} \int_G F(\omega, g) dm_G(g) d\mathbb{P}_\Lambda(\omega)$$

for all measurable $F : \mathcal{T} \times G \to \mathbb{R}_+$. Moreover $\mathbb{P}_\Lambda(\Omega) = i(\Lambda)$.

The significance of the Palm measure for this thesis is given by the following proposition, which will allow us to prove a sampling result for the autocorrelation measure.

Proposition 2.5.6. *Let Λ be a locally square integrable G-invariant point process in X and let $f \in \mathcal{L}^\infty_{\mathrm{bnd}}(G)$ be bi-K-invariant with $f \geq 0$. Then*

$$\eta^+_\Lambda(f) = \mathbb{P}_\Lambda(\Lambda(f \circ \sigma)).$$

Proof. We set

$$F : \Omega \times G \to \mathbb{R}_+, \ (\omega, g) \mapsto \int_X f(\sigma(y)) b(\pi(g)) \Lambda_\omega(dy).$$

Then

$$\begin{aligned}
TF(\omega) &= \int_X \int_G F(g^{-1}\omega, g) d\kappa_x(g) d\Lambda_\omega(x) \\
&= \int_X \int_G \int_X f(\sigma(y)) b(\pi(g)) d\Lambda_{g^{-1}\omega}(y) d\kappa_x(g) d\Lambda_\omega(x) \\
&= \int_X \int_K \int_X f(\sigma(y)) b(\pi(\sigma(x)k)) d\Lambda_{(\sigma(x)k)^{-1}\omega}(y) dm_K(k) d\Lambda_\omega(x) \\
&= \int_X \int_K \int_X f(k^{-1}\sigma(x)^{-1}\sigma(y)) b(x) d\Lambda_\omega(y) dm_K(k) d\Lambda_\omega(x) \\
&= \int_X \int_K \int_X f(\sigma(x)^{-1}\sigma(y)) b(x) d\Lambda_\omega(y) dm_K(k) d\Lambda_\omega(x) \\
&= \int_X \int_X f(\sigma(x)^{-1}\sigma(y)) b(x) d\Lambda_\omega(y) d\Lambda_\omega(x).
\end{aligned}$$

Whence

$$\eta^+_\Lambda(f) = \mathbb{E}[TF] = \int_G \int_\Omega F(\omega, g) d\mathbb{P}_\Lambda(\omega) dm_G(g)$$

$$
\begin{aligned}
&= \int_G \int_\Omega f(\sigma(y))b(\pi(g))d\Lambda_\omega(dy)\mathbb{P}_\Lambda(\omega)dm_G(g) \\
&= \int_\Omega f(\sigma(y))d\Lambda_\omega(y)d\mathbb{P}_\Lambda(\omega) \\
&= \mathbb{P}_\Lambda(\Lambda(f\circ\sigma)).
\end{aligned}
$$

□

A sampling theorem for Palm measures

Given two subsets $A, B \subset G$ we write

$$
A_B^- := \bigcap_{g\in B} gA \quad \text{and} \quad A_B^+ := \bigcup_{g\in B} gA.
$$

Definition 2.5.7. We say that the family $(G_t)_{t>0}$ of symmetric, measurable and bi-K-invariant subsets of G is a *very convenient sequence* if there is a decreasing sequence $(U_n)_{n\geq 1}$ of symmetric open pre-compact identity neighborhoods in G and sequences $(\delta_n)_{n\geq 1}$, $(\varepsilon_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$ of positive real numbers such that $\delta_n, \varepsilon_n \to 0$ and

$$
G_{t-\delta_n} \subseteq (G_t)_{U_n}^- \subseteq (G_t)_{U_n}^+ \subseteq G_{t+\delta_n} \quad \text{for all } t \geq t_n
$$

and

$$
1-\varepsilon_n \leq \liminf_{t\to\infty} \frac{m_G(G_{t-\delta_n})}{m_G(G_t)} \quad \text{and} \quad \limsup_{t\to\infty} \frac{m_G(G_{t+\delta_n})}{m_G(G_t)} \leq 1+\varepsilon_n
$$

for all n.

Example 2.5.8. The sequence $(G_t)_{t>0}$ defined by $G_t = \pi^{-1}(B(x_0,t))$ is very convenient if (G, K, d_X) is given by one of the following triples.

(i) $G = \mathbb{R}^n$ and $K = \{0\}$ with the ordinary Euclidean metric.

(ii) $G = \mathcal{H}_n \rtimes U(n)$ and $K = U(n)$ with the Cygan-Koranyi metric.

(iii) G a semisimple Lie group with finite center and no compact factors, K a maximal compact subgroup and d_X the Cartan-Killing distance (this follows from the work of Gorodnik and Nevo, see Chapter 7, in particular Proposition 7.2, and Theorem 3.18 in [56]).

(iv) (G, K) a Riemannian symmetric pair of compact type and d_X the Cartan-Killing distance, as $G_t = G$ for large t.

(v) (G, K) a finite Gelfand pair with any distance satisfying our basic assumptions, as $G_t = G$ for large t (and we can choose $U_n = \{e\}$).

In the first two cases this can be proven by direct calculation.

The following theorem is a generalization of a sampling result by Björklund, Hartnick and Karasik in [18] for Palm measures of uniformly discrete stationary point processes in groups.

Theorem 2.5.9. *Assume that $(G_t)_{t>0}$ is a very convenient sequence and that the invariant pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$. Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(X)$*

be an $2r$-uniformly discrete, ergodic, G-invariant point process. Then there is a G-invariant conull set $\Omega_0 \subset \Omega$ such that

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{x\in \operatorname{supp}(\Lambda_\omega)\cap\pi(G_t)} f(\sigma(x)^{-1}\Lambda_\omega) = \mathbb{P}_\Lambda(f(\Lambda))$$

for every $\omega \in \Omega_0$ and every $f \in C(\mathcal{N}_{2r}^*(X))$, which is invariant under the action of K on $\mathcal{N}_{2r}^*(X)$. Moreover, we can choose Ω_0 such that for every $\omega \in \Omega_0$ the set $P_\omega := \operatorname{supp}(\Lambda_\omega) \in \mathrm{UD}_{2r}(X)$ is generically measured for Λ (wrt. $(G_t)_{t>0}$).

Lemma 2.5.10. *Let Λ be a locally square integrable G-invariant point process in X. Let $F : \Omega \times G \to \mathbb{R}_+$ be measurable and $h \in G$. Then*

$$TF(h\omega) = TF_h(\omega)$$

where $F_h(\omega, g) := F(\omega, hg)$

Proof. For $h \in G$ and $x \in X$ choose $k_{h,x} \in K$ with $\sigma(hx) = h\sigma(x)k_{h,x}$. We have

$$\begin{aligned}
TF(h\omega) &= \int_X \int_G F(g^{-1}h\omega, g)d\kappa_x(g)\Lambda_{h\omega}(dx)\\
&= \int_X \int_G F(g^{-1}h\omega, g)d\kappa_{hx}(g)\Lambda_\omega(dx)\\
&= \int_X \int_K F((\sigma(hx)k)^{-1}h\omega, \sigma(hx)k)dm_K(k)\Lambda_\omega(dx)\\
&= \int_X \int_K F((h\sigma(x)k_{h,x}k)^{-1}h\omega, h\sigma(x)k_{x,h}k)dm_K(k)\Lambda_\omega(dx)\\
&= \int_X \int_K F((h\sigma(x)k)^{-1}h\omega, h\sigma(x)k)dm_K(k)\Lambda_\omega(dx)\\
&= \int_X \int_G F((hg)^{-1}h\omega, hg)d\kappa_x(g)\Lambda_\omega(dx)\\
&= \int_X \int_G F(g^{-1}\omega, hg)d\kappa_x(g)\Lambda_\omega(dx) = TF_h(\omega),
\end{aligned}$$

where we used the G-invariance of Λ in the second equality and the definition of κ in the third and fifth equality. □

Lemma 2.5.11. *Let $f \in C(\mathcal{N}_{2r}^*(X))$ be non-negative and invariant under the action of K on $\mathcal{N}_{2r}^*(X)$ and let $\rho : G \to \mathbb{R}_+$ be continuous with compact support U such that $\pi(U) \subset B(x_0, 2r)$ and $m_G(\rho) = 1$. Then*

$$f_\rho \colon \mathcal{N}_{2r}^*(X) \to \mathbb{R}_+, \quad \mu \mapsto \int_X \int_G \rho(h)f(h^{-1}\mu)d\kappa_x(h)d\mu(x)$$

is continuous.

Proof. Consider the map

$$g\colon X\times\mathcal{N}^*_{2r}(X)\to\mathbb{R}_+,\ (x,\mu)\mapsto\int_K\rho(g_xk)f(k^{-1}g_x^{-1}\mu)dm_K(k)=\int\rho(h)f(h^{-1}\mu)d\kappa_x(h)$$

and note that it is independent from our choice of g_x by the K-invariance of f. By a result of Gleason, [54], on local sections of actions of compact groups (applied to the action of K on G), for each $x\in X$ there is a open neighborhood U_x of x and a continuous local section $\sigma_x:U_x\to G$ of π. Thus we can write g locally as an integral of the continuous function

$$g_x:U_x\times K\times\mathcal{N}^*_{2r}(X)\to\mathbb{R}_+,\ (y,k,\mu)\mapsto\rho(\sigma_x(y)k)f(\sigma_x(y)^{-1}\mu)$$

over the second variable over a compact set. Additionally, for all $(y,k,\mu)\in X\times K\times\mathcal{N}^*_{2r}(X)$, $|g_x(y,k,\mu)|\leq\|\rho\|_\infty\|f\|_\infty$ is an m_K-integrable dominating function which does not depend on k. Thus g is continuous and satisfies $|g(x,\mu)|\leq\|f\|_\infty\int\rho(h)d\kappa_x(h)$, i.e. is bounded by a compactly supported function independent from μ and is thus compactly supported. Let us now consider the map

$$G\colon\mathcal{N}^*_{2r}(X)\times\mathcal{N}^*_{2r}(X)\to\mathbb{R}_+,\ (\mu,\nu)\mapsto\int_X g(x,\mu)d\nu(x)$$

and let $(\mu_n)_{n\geq1}$ be a convergent sequence in $\mathcal{N}^*_{2r}(X)$ with limit $\mu\in\mathcal{N}^*_{2r}(X)$.
Then

$$\begin{aligned}|G(\mu_n,\mu_n)-G(\mu,\mu)|&=\left|\int_X g(x,\mu_n)d\mu_n(x)-\int_X g(x,\mu)d\mu(x)\right|\\&\leq\left|\int_X g(x,\mu_n)d\mu_n(x)-\int_X g(x,\mu)d\mu_n(x)\right|\\&+\left|\int_X g(x,\mu)d\mu_n(x)-\int_X g(x,\mu)d\mu(x)\right|.\end{aligned}$$

The second summand now goes to zero for $n\to\infty$ by the definition of weak-$*$ convergence. For the first summand we observe that g is compactly supported and thus uniformly continuous wrt. to the product metric $d_X\otimes\tilde{d}$ on $X\times\mathcal{N}^*_{2r}(X)$, where we fix some compact metric $\tilde{d}$ on $\mathcal{N}^*_{2r}(X)$ inducing the topology, for instance the Prohorov metric. Hence for any $\varepsilon>0$ there is a $\delta>0$ such that $d_X\otimes\tilde{d}((x,\mu),(y,\nu))=\sqrt{d_X(x,y)^2+\tilde{d}(\mu,\nu)^2}\leq\delta$ implies $|g(x,\mu)-g(y,\nu)|\leq\varepsilon$. Choose $N\in\mathbb{N}$ such that $\tilde{d}(\mu_n,\mu)\leq\delta$ for all $n\geq N$. Then $d\otimes\tilde{d}((x,\mu_n),(x,\mu))\leq\delta$ for all $n\geq N$ and thus $|g(x,\mu_n)-g(x,\mu)|\leq\varepsilon$ for all $n\geq N$. More precisely, g is supported in $\pi(\mathrm{supp}(\rho))\times\mathcal{N}^*_{2r}(X)$. Thus, if $R>0$ is such that $\pi(\mathrm{supp}(\rho))\subset B(x_0,R)$,

$$\left|\int_X g(x,\mu_n)-g(x,\mu)d\mu_n(x)\right|\leq\mu_n(\pi(\mathrm{supp}(\rho)))\varepsilon\leq\frac{m_X(B(x_0,R+2r))}{m_X(B(x_0,r))}\varepsilon$$

by Proposition 2.3.6. Thus the second summand converges to zero and we see that the map $f_\rho:\mathcal{N}^*_{2r}(X)\to\mathbb{R},\ \mu\mapsto f_\rho(\mu)=G(\mu,\mu)$ is continuous. □

Proof of Theorem 2.5.9. For $\omega \in \Omega$ we set $\tilde{f}(\omega) = f(\Lambda_\omega)$. We set

$$\tilde{f}_\rho(\omega) = T(\rho \otimes \tilde{f}) = f_\rho(\Lambda_\omega) = \int_X \int_G \rho(h) f(h^{-1}\Lambda_\omega) d\kappa_x(h) d\Lambda_\omega(x).$$

By Theorem 2.5.5 we have

$$\mathbb{P}(\tilde{f}_\rho) = \mathbb{P}(T(\rho \otimes \tilde{f})) = \int_\Omega \int_G \rho(h) \tilde{f}(\omega) dm_G(h) d\mathbb{P}_\Lambda(\omega) = \mathbb{P}_\Lambda(\tilde{f}).$$

Using Lemma 2.5.10 we obtain

$$\tilde{f}_\rho(g\omega) = \int_X \int_G \rho(hg) \tilde{f}(h^{-1}\omega) d\kappa_x(h) \Lambda_\omega(dx), \quad \text{for all } g \in G,$$

and thus, for every Borel set $B \subset G$,

$$\begin{aligned}
\int_B \tilde{f}_\rho(g\omega) dm_G(g) &= \int_G \int_X \int_G \chi_B(g) \rho(hg) \tilde{f}(h^{-1}\omega) d\kappa_x(h) \Lambda_\omega(dx) dm_G(g) \\
&= \int_G \int_X \int_G \chi_B(h^{-1}g) \rho(g) dm_G(g) \tilde{f}(h^{-1}\omega) d\kappa_x(h) \Lambda_\omega(dx) \\
&= \int_G \int_X \int_G \chi_{gB^{-1}}(h) \rho(g) dm_G(g) \tilde{f}(h^{-1}\omega) d\kappa_x(h) \Lambda_\omega(dx).
\end{aligned}$$

Suppose V is a symmetric unit neighborhood of G, $\operatorname{supp}(\rho) \subset V$ and that B is a precompact symmetric bi-K-invariant Borel set. Then, as ρ is non-negative,

$$\chi_{B_V^-}(h) \leq \int \chi_{gB^{-1}}(h) \rho(g) dm_G(g) \leq \chi_{B_V^+}(h),$$

where

$$B_V^- = \bigcap_{g \in V} gB^{-1} = \bigcap_{g \in V} gB = \bigcap_{g \in V} g^{-1}B$$

and

$$B_V^+ = \bigcup_{g \in V} gB^{-1} = \bigcup_{g \in V} gB = \bigcup_{g \in V} g^{-1}B.$$

Note that both of these sets are right-K-invariant by the bi-K-invariance of B. Hence

$$\int_X \int_G \chi_{B_V^-}(h) \tilde{f}(h^{-1}\omega) d\kappa_x(h) d\Lambda_\omega(x) \leq \int_X \int_G \int_G \chi_{gB^{-1}}(h) \rho(g) dm_G(g) \tilde{f}(h^{-1}\omega) d\kappa_x(h) d\Lambda_\omega(x)$$

and

$$\int_X \int_G \int_G \chi_{gB^{-1}}(h) \rho(g) dm_G(g) \tilde{f}(h^{-1}\omega) d\kappa_x(h) d\Lambda_\omega(x) \leq \int_X \int_G \chi_{B_V^+}(h) \tilde{f}(h^{-1}\omega) d\kappa_x(h) d\Lambda_\omega(x).$$

In the following we write $P_\omega = \operatorname{supp}(\Lambda_\omega)$ and note that

$$x \in \pi(B_V^\pm) \iff g_x \in B_V^\pm$$

by the right-K-invariance of the sets $B_V^{\pm}$. Now

$$\int_X \int_G \chi_{B_V^-}(h)\tilde{f}(h^{-1}\omega)d\kappa_x(h)d\Lambda_\omega(x) = \sum_{x\in P_\omega} \int_G \chi_{B_V^-}(h)\tilde{f}(h^{-1}\omega)d\kappa_x(h)$$
$$= \sum_{x\in P_\omega} \int_G \chi_{B_V^-}(h)\tilde{f}(h^{-1}\omega)d\kappa_{g_x x_0}(h) = \sum_{x\in P_\omega} \int_K \chi_{B_V^-}(g_x h)\tilde{f}(h^{-1}g_x^{-1}\omega)dm_K(h)$$
$$= \sum_{x\in P_\omega} \tilde{f}(g_x^{-1}\omega) \int_K \chi_{B_V^-}(g_x h)dm_K(h) = \sum_{x\in P_\omega} \tilde{f}(g_x^{-1}\omega) \int_K \chi_{B_V^-}(g_x)dm_K(h)$$
$$= \sum_{x\in P_\omega\cap\pi(B_V^-)} \tilde{f}(g_x^{-1}\omega) = \sum_{x\in P_\omega\cap\pi(B_V^-)} f(g_x^{-1}\Lambda_\omega).$$

Here we have used the right-K-invariance of $\chi_{B_V^-}$ in the fifth and in the second to last equality and the left-K-invariance of f in the fourth equality. Similarly

$$\int_X \int_G \chi_{B_V^+}(h)\tilde{f}(h^{-1}\omega)d\kappa_x(h)d\Lambda_\omega(x) = \sum_{x\in P_\omega\cap\pi(B_V^+)} \tilde{f}(g_x^{-1}\omega) = \sum_{x\in P_\omega\cap\pi(B_V^+)} f(g_x^{-1}\Lambda_\omega)$$

Thus

$$\sum_{x\in P_\omega\cap\pi(B_V^-)} f(g_x^{-1}\Lambda_\omega) \leq \int_X \int_G \int_G \chi_{gB^{-1}}(h)\rho(g)dm_G(g)\tilde{f}(h^{-1}\omega)d\kappa_x(h)d\Lambda_\omega(x)$$

and

$$\int_X \int_G \int_G \chi_{gB^{-1}}(h)\rho(g)dm_G(g)\tilde{f}(h^{-1}\omega)d\kappa_x(h)d\Lambda_\omega(x) \leq \sum_{x\in P_\omega\cap\pi(B_V^+)} f(g_x^{-1}\Lambda_\omega),$$

i.e.

$$\sum_{x\in P_\omega\cap\pi(B_V^-)} f(x^{-1}\omega) \leq \int_G \tilde{f}_\rho(g\omega)dm_G(g) \leq \sum_{x\in P_\omega\cap\pi(B_V^+)} f(g_x^{-1}\omega).$$

As $(G_t)_{t>0}$ is a very convenient sequence there exist sequences $(\delta_n)_{n\geq 1}, (t_n)_{n\geq 1}$ of positive real numbers and neighborhoods $(V_n)_{n\geq 1}$ as in the definition of a very convenient sequence. These satisfy $\delta_n \to 0$ and

$$G_{t-\delta_n} \subset (G_t)_{V_n}^- \subset (G_t)_{V_n}^+ \subset G_{t+\delta_n}, \quad \text{for all } t > t_n.$$

Let $(\rho_n)_{n\geq 1}$ be a sequence of non-negative compactly supported continuous functions such that $\mathrm{supp}(\rho_n) \subset V_n$ and $m_G(\rho_n) = 1$ for all n. Then, by the calculation above, setting $B = G_t$, $\rho = \rho_n$ and noting that G_t is symmetric and bi-K-invariant, we see that

$$\sum_{x\in P_\omega\cap\pi(G_{t-\delta_n})} f(g_x^{-1}\Lambda_\omega) \leq \int_{G_t} f_{\rho_n}(g\Lambda_\omega)dm_G(g) \leq \sum_{x\in P_\omega\cap\pi(G_{t+\delta_n})} f(g_x^{-1}\Lambda_\omega)$$

for all $\omega \in \Omega$ and $t > t_n$. Define

$$\psi_+(\omega) \coloneqq \limsup_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{x\in P_\omega\cap\pi(G_t)} f(g_x^{-1}\Lambda_\omega)$$

and

$$\psi_-(\omega) \coloneqq \liminf_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{x\in P_\omega\cap\pi(G_t)} f(g_x^{-1}\Lambda_\omega)$$

for $\omega \in \Omega$. As f_{ρ_n} is continuous by Lemma 2.5.11, we can use the invariant pointwise ergodic theorem/Proposition 2.4.8 to obtain a G-invariant set Ω_0 of full measure such that $\mathrm{supp}(\Lambda_\omega)$ is an invariantly $(\mathrm{supp}_*\Lambda_*\mathbb{P}, (G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$-generic point for every $\omega \in \Omega_0$ and

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f_{\rho_n}(g\Lambda_\omega) dm_G(g) = \mathbb{P}(f_{\rho_n}(\Lambda)) = \mathbb{P}(\tilde{f}_{\rho_n}) = \mathbb{P}_\Lambda(\tilde{f}) = \mathbb{P}_\Lambda(f(\Lambda)).$$

for every $\omega \in \Omega_0$ and every $n \in \mathbb{N}$. We set

$$\psi_n(\omega) \coloneqq \begin{cases} \lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} f_{\rho_n}(g\omega) dm_G(g), & \omega \in \Omega_0 \\ 0, & \text{otherwise} \end{cases}$$

By the definition of a very convenient sequence there is a sequence $(\varepsilon_n)_{n\geq 1}$ of positive real numbers with $\varepsilon_n \to 0$ and

$$1 - \varepsilon_n \leq \liminf_{t\to\infty} \frac{m_G(G_{t-\delta_n})}{m_G(G_t)} \quad \text{and} \quad \limsup_{t\to\infty} \frac{m_G(G_{t+\delta_n})}{m_G(G_t)} \leq 1 + \varepsilon_n.$$

Hence we see that

$$\begin{aligned} \psi_n(\omega) &\leq \liminf_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{x\in P_\omega\cap\pi(G_{t+\delta_n})} f(g_x^{-1}\Lambda_\omega) \\ &= \liminf_{t\to\infty} \frac{m_G(G_{t+\delta_n})}{m_G(G_{t+\delta_n}) m_G(G_t)} \sum_{x\in P_\omega\cap\pi(G_{t+\delta_n})} f(g_x^{-1}\Lambda_\omega) \\ &\leq \limsup_{t\to\infty} \frac{m_G(G_{t+\delta_n})}{m_G(G_t)} \liminf_{t\to\infty} \frac{1}{m_G(G_{t+\delta_n})} \sum_{x\in P_\omega\cap\pi(G_{t+\delta_n})} f(g_x^{-1}\Lambda_\omega) \\ &\leq (1+\varepsilon_n)\psi_-(\omega) \end{aligned}$$

as all terms are positive and similarly

$$(1-\varepsilon_n)\psi_+(\omega) \leq \psi_n(\omega)$$

for all $\omega \in \Omega_0$. As $\psi_n(\omega) = \mathbb{P}(f_{\rho_n}(\Lambda)) = \mathbb{P}_\Lambda(f(\Lambda))$ for all $\omega \in \Omega_0$ and $\varepsilon_n \to 0$ as $n \to \infty$, we conclude that

$$\lim_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{x\in P_\omega\cap\pi(G_t)} f(g_x^{-1}\Lambda_\omega) = \mathbb{P}_\Lambda(f(\Lambda))$$

for all $\omega \in \Omega_0$. □

2.6. The energy of a point process

In this section we define the energy of point sets and of uniformly discrete G-invariant point processes. We further prove a sampling result which shows that these notions are closely related. The energy of point sets was first introduced for arbitrary potential functions by Cohn and Kumar in [37], and generalizes notions of energy for more specific potential functions by Andreev in [2, 3] and Yudin in [101]. Cohn and Kumar prove linear programming bounds on the energy of periodic point sets, which were extended Cohn and Courcy-Ireland in [34] to more general point sets. In [39] Cohn, Kumar, Miller, Radchenko and Viazovska use these linear programming bounds to prove that the E8 lattice is energy minimizing for a large class of potentials, using methods developed by Viazovska in [96] to prove that the E8 sphere packing is optimally dense. We will develop analogues for the linear programming bounds on energy in Chapter 4.

Definition 2.6.1. Let $P \subset X$ be uniformly discrete and let $p : [0,\infty) \to [0,\infty)$.

(i) The *lower p-energy of P* is defined as

$$\underline{E_p}(P) \coloneqq \liminf_{R\to\infty} \frac{1}{\#(P\cap B(x_0,R))} \sum_{x\in P\cap B(x_0,R)} \sum_{x\neq y\in P} p(d_X(x,y)).$$

(ii) The *upper p-energy of P* is defined as

$$\overline{E_p}(P) \coloneqq \limsup_{R\to\infty} \frac{1}{\#(P\cap B(x_0,R))} \sum_{x\in P\cap B(x_0,R)} \sum_{x\neq y\in P} p(d_X(x,y)).$$

(iii) If $\underline{E_p}(P) = \overline{E_p}(P)$ we define the *p-energy of P* by

$$E_p(P) \coloneqq \underline{E_p}(P) = \overline{E_p}(P).$$

We will sometimes call p a *potential function.*

Definition 2.6.2. Let $p : [0,\infty) \to [0,\infty)$ be measurable and set $d_0(g) \coloneqq d_X(gx_0, x_0)$ for $g \in G$.

(i) Let $r > 0$. For a $2r$-uniformly discrete G-stationary point process Λ in X with $i(\Lambda) \neq 0$ we define the *p-energy* of Λ as

$$E_p(\Lambda) \coloneqq \frac{1}{i(\Lambda)} \eta_\Lambda^+(p\circ d_0) - p(0).$$

(ii) For $\delta > 0$ let $\mathrm{PP}(\delta)$ denote the set of uniformly discrete G-stationary point processes in X with intensity δ. We define the *minimal stochastic* (p, δ)*-energy* as

$$E_{\mathrm{stoch}}(p, \delta) := \inf\{E_p(\Lambda) \mid \Lambda \in \mathrm{PP}(\delta)\}.$$

Definition 2.6.3. The *volume growth function* $v_X : [0, \infty) \to [0, \infty)$ of X is defined by

$$v_X(R) := m_X(B(x_0, R)).$$

Theorem 2.6.4 (Energy sampling)**.** *Assume that $G_t := \pi^{-1}(B(x_0, t))$ defines a very convenient sequence and that the invariant pointwise ergodic theorem holds for the tuple $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$. Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*_{2r}(X)$ be a $2r$-uniformly discrete ergodic G-invariant point process. Let $p : [0, \infty) \to [0, \infty)$ be continuous and monotonically decreasing on $(0, \infty)$ such that*

$$\sum_{n \in \mathbb{N}} p(n) v_X(2n) < \infty.$$

Write $P_\omega = \mathrm{supp}(\Lambda_\omega)$. Then there is a G-invariant conull set $\Omega_0 \subset \Omega$ such that

$$E_p(\Lambda) = E_p(P_\omega)$$

for every $\omega \in \Omega_0$ and for every $\omega \in \Omega_0$ the set P_ω is generically measured for Λ wrt. $(G_t)_{t>0}$.

Proof. We first note that

$$i(\Lambda) = \mathbb{P}_\Lambda(\mathbf{1}) = \lim_{t \to \infty} \frac{1}{m_G(G_t)} \sum_{x \in P_\omega \cap m_X(B(x_0, t))} \mathbf{1}(\sigma(x)^{-1}\Lambda_\omega) = \lim_{t \to \infty} \frac{\#(P \cap B(x_0, t))}{m_X(B(x_0, t))}$$

where we note that $\mathbf{1} : \mathcal{N}^*_{2r}(X) \to \mathbb{R},\ \mu \to 1$ is continuous.
Choose a sequence $(g_n)_{n \geq 1}$ in $C_c(G, K)$ such that $g_n|_{\pi^{-1}(B(x_0, n))} \equiv 1$, $g_n(g) = 0$ for $g \in G \setminus \pi^{-1}(B(x_0, n+1))$ and $0 \leq g_n \leq 1$. Let $\mu \in \mathcal{N}^*_{2r}(X)$ and choose $P \in \mathrm{UD}_{2r}(X)$ with $\mu = \sum_{x \in P} \delta_x$. Then, for $n \geq r$,

$$\begin{aligned}
|\mu(g_n p \circ d_0) - \mu(p \circ d_0)| &\leq \sum_{x \in P \cap (X \setminus B(x_0, n))} |(1 - g_n(x)) p \circ d_0(x)| \\
&\leq \sum_{l \geq n} \sum_{y \in P \cap B(x_0, l+1) \setminus B(x_0, l)} |p \circ d_0(x)| \\
&\leq \sum_{l \geq n} p(l+1) \#(P \cap B(x, l+1)) \\
&\leq \sum_{l \geq n} p(l+1) \frac{v_X(l+1+r)}{m_X(B(x_0, r))} \\
&\leq \sum_{l \geq n} p(l+1) \frac{v_X(2(l+1))}{m_X(B(x_0, r))} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Note that this bound does not depend on $\mu \in \mathcal{N}^*_{2r}(X)$. By the definition of the weak-$*$ topology on $\mathcal{N}^*_{2r}(X)$ this implies that the map $\Psi : \mathcal{N}^*_{2r}(X) \to \mathbb{R}$, $\mu \mapsto \mu(p \circ d_0)$ is continuous as the uniform limit of the continuous maps $\Psi_n : \mathcal{N}^*_{2r}(X) \to \mathbb{R}$, $\mu \mapsto \mu(g_n p \circ d_0)$. Then, by monotone convergence and Proposition 2.5.6,

$$\begin{aligned} E_p(\Lambda) &= \frac{1}{i(\Lambda)}\eta^+_\Lambda(p \circ d_0) - p(0) = \lim_{n\to\infty} \frac{1}{i(\Lambda)}\eta^+_\Lambda(g_n p \circ d_0) - p(0) \\ &= \lim_{n\to\infty} \frac{1}{i(\Lambda)}\mathbb{P}(\Lambda(g_n p \circ d_0)) - p(0) \\ &= \frac{1}{i(\Lambda)}\mathbb{P}_\Lambda(\Lambda(p \circ d_0)) - p(0) \\ &= \frac{1}{i(\Lambda)}\mathbb{P}_\Lambda(\Psi(\Lambda)) - p(0) \end{aligned}$$

and we can apply the second part of Theorem 2.5.9. Thus, setting $P_\omega := \operatorname{supp}(\Lambda_\omega)$, we obtain

$$\begin{aligned} \mathbb{P}_\Lambda(\Psi(\Lambda)) &= \lim_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{x\in P_\omega \cap B(x_0,t)} \Psi(\sigma(x)^{-1}\Lambda_\omega) \\ &= \lim_{t\to\infty} \frac{1}{m_X(B(x_0,t))} \sum_{x\in P_\omega \cap B(x_0,t)} \sum_{y\in\sigma(x)^{-1}P_\omega} p(d_X(y,x_0)) \\ &= \lim_{t\to\infty} \frac{1}{m_X(B(x_0,t))} \sum_{x\in P_\omega \cap B(x_0,t)} \sum_{y\in P_\omega} p(d_X(y,x)) \\ &= \lim_{t\to\infty} \frac{1}{m_X(B(x_0,t))} \sum_{x\in P_\omega \cap B(x_0,t)} \sum_{x\neq y\in P_\omega} p(d_X(y,x)) \\ &+ \lim_{t\to\infty} \frac{1}{m_G(B(x_0,t))} \sum_{x\in P\cap\pi(G_t)} p(0) \\ &= \lim_{t\to\infty} \frac{\#(P\cap B(x_0,t))}{m_X(B(x_0,t))} \lim_{t\to\infty} \frac{1}{\#(P\cap B(x_0,t))} \sum_{x\in P_\omega \cap B(x_0,t)} \sum_{x\neq y\in P_\omega} p(d_X(y,x)) \\ &+ \lim_{t\to\infty} \frac{1}{m_X(B(x_0,t))} \sum_{x\in P_\omega \cap B(x_0,t)} p(0) \\ &= i(\Lambda)E_p(P_\omega) + i(\Lambda)p(0) \end{aligned}$$

for almost every $\omega \in \Omega$ and in particular for any ω such that P_ω is generically measured for Λ. □

2.7. Spectral theory of point processes

By Proposition 2.3.13 the measures η_Λ and η^+_Λ as well as the distributions T_Λ and T^+_Λ (if G is a Lie group) are positive-definite. Thus they have Plancherel transforms by Theorem 1.3.2 resp. Theorem 1.3.6. This enables us to make the following definition, a precursor of which (involving the measure $\overline{\eta}_\Lambda^+$ on $K\backslash G/K$) can be found in [20]. Issues with the Plancherel transform of measures on $K\backslash G/K$ and the respective Bochner-Schwartz theorem, see [90] and the references in [86], make the following definition better suited for our purposes.

Definition 2.7.1. Let Λ be a locally L^2 stationary point process in X. Then the Plancherel transform $\widehat{\eta_\Lambda^+}$ of η_Λ^+ is called the *diffraction measure of* Λ.

Remark 2.7.2. In the context of point process theory in $\mathbb{R}^n$ the Plancherel transform of η_Λ is known as the *spectral measure of* Λ. Decay behavior of the spectral measure around the trivial character (i.e. $0 \in \mathbb{R}^n$) has been related to the notions of number rigidity and hyperuniformity, see [74] and [17] and the references therein. See [14] and [15] for related work in hyperbolic space.

If G is a Lie group, then it does not matter if we define the diffraction in terms of the autocorrelation measure or via the autocorrelation distribution:

Proposition 2.7.3. *If G is a Lie group and Λ is a locally L^2 stationary point process in X, then*

$$\widehat{\eta_\Lambda^+} = \widehat{T_\Lambda^+} \quad \text{and} \quad \widehat{\eta_\Lambda} = \widehat{T_\Lambda}.$$

Proof. This follows directly from Corollary 1.3.7. □

Lemma 2.7.4. *If Λ is a locally L^2 stationary point process in X, we have*

$$\widehat{\eta_\Lambda^+} = \widehat{\eta_\Lambda} + i(\Lambda)^2 \delta_{\mathbf{1}},$$

where $\mathbf{1} : G \to \mathbb{C},\ g \mapsto 1$ denotes the trivial character of G.

Proof. We have $\eta_\Lambda^+ = \eta_\Lambda + i(\Lambda)^2 m_G$. The uniqueness of the Plancherel-Godement transform in Theorem 1.3.2 implies that $\widehat{m_G} = \delta_{\mathbf{1}}$, as

$$\delta_{\mathbf{1}}(\widehat{f}) = \widehat{f}(\mathbf{1}) = \int f(g)\mathbf{1}(g^{-1})dm_G(g) = m_G(f)$$

for all $f \in C_c(G, K)^2$. The uniqueness in Theorem 1.3.2 then implies the claim. □

Remark 2.7.5. A very detailed exposition of the spectral theory of point processes on Gelfand pairs can be found in the article [15] by Björklund and Byléhn, dealing with questions of hyperuniformity and number variance, as well as conditions for existence and uniqueness of the spectral measure/diffraction measure. We have used the positive definite lift η_Λ^+ of $\overline{\eta}_\Lambda^+$ from $K\backslash G/K$ to G to define the spectral measure of Λ, following Björklund and Byléhn's approach in [14].
In [15] they show that one can define the spectral measure without lifting the measure to a measure on G by considering the functional induced by $\overline{\eta}_\Lambda^+$ on a large enough function space of bi-K-invariant functions on G. For this they refine the Godement-Plancherel theorem to [15, Theorem 3.2].

We should also mention the following result by Björklund and Byléhn, which refines the statement of Lemma 2.7.4 and gives a generalization of a well-known fact in the Euclidean setting. See [17, Proposition 2.5] for the same statement in the setting of LCA groups.

Proposition 2.7.6 (Björklund-Byléhn, [15, Proposition 3.6])**.** *If Λ is a locally L^2 stationary point process in X, then we have $\eta_\Lambda^+(\{\mathbf{1}\}) = i(\Lambda)^2$.*

Remark 2.7.7. We should point out that there are other ways to construct the K-spherical lift of $\overline{\eta}_\Lambda^+$ to G. One can for example consider the functional $C_c(G) \to \mathbb{C}$, $f \mapsto \overline{\eta}_\Lambda^+(f^\natural)$, where we note that we can evaluate $f^\natural$ on elements of $K \backslash G / K$. Another way of constructing the lift of $\overline{\eta}_\Lambda^+$ can be obtained as follows: In [75] Last constructs the Palm measure of Λ by lifting the invariant point process Λ to an invariant random measure Λ' on G, defined by

$$\Lambda' = \int_X \kappa_x d\Lambda(x).$$

Last then defines the Palm measure of Λ as the Palm measure of Λ'. Now one can define the autocorrelation measure of Λ', see for instance [21, Corollary 4.11]. Taking the K-periodization of this autocorrelation measure, one again obtains a positive-definite K-spherical measure on G, which satisfies the equations in Proposition 2.3.8.

3. Sphere packings in homogeneous spaces

In this chapter, we will describe Bowen and Radin's approach to sphere packings in hyperbolic space, reformulated in the language of point processes and for more general homogeneous spaces.

3.1. The classical notion of packing density

Definition 3.1.1. Let $r > 0$. An r-sphere packing in a metric space (Y, d_Y) is a set P of disjoint open balls of radius r. The set of all r-sphere packings in Y will be denoted by $\mathrm{Pack}(Y, r)$. If $P \in \mathrm{Pack}(Y, r)$ we define $\mathrm{supp}(P) = \bigcup P$.

In this thesis we will be mainly interested in sphere packings in the homogeneous space X. Note that for any $P \in \mathrm{UD}_{2r}(X)$ the set $P^r := \{B(x, r) \mid x \in X\} \in \mathrm{Pack}(X, r)$ is an r-sphere packing.

Definition 3.1.2. Let P be a r-sphere packing in (X, d_X).

(i) The *upper lower density of P at $x \in X$* is defined as

$$D^{\pm}(P, x) := \limsup_{R\to\infty} \frac{m_X(\bigcup\{B \in P \mid B \subset B(x, R)\})}{m_X(B(x, R))}$$

(ii) The *lower density of P at $x \in X$* is defined as

$$\underline{D}(P, x) := \liminf_{R\to\infty} \frac{m_X(B(x, R) \cap \mathrm{supp}(P))}{m_X(B(x, R))}.$$

(iii) The *upper density of P at $x \in X$* is defined as

$$\overline{D}(P, x) := \limsup_{R\to\infty} \frac{m_X(B(x, R) \cap \mathrm{supp}(P))}{m_X(B(x, R))}.$$

(iv) If $\underline{D}(P, x) = \overline{D}(P, x)$, we say that *$P$ has well-defined density at x* and write $D(P, x)$ instead of $\underline{D}(P, x)$ or $\overline{D}(P, x)$.

(v) If P has well-defined density at every $x \in X$ and the function $x \mapsto D(P, x)$ is constant, we say that *P has well-defined density* and write $D(P) := D(P, x_0)$.

Groemer investigated the fundamental properties of this notion of density in [59, 60, 61].

Theorem 3.1.3 (Groemer)**.** *If $x \in X$ and $r > 0$, then there is an r-sphere packings P^+ whose upper lower density at x is*

$$D^{\pm}(P^+, x) = \sup_{P \in \mathrm{Pack}(X,r)} D^{\pm}(P, x).$$

Theorem 3.1.4 (Groemer)**.** *If*

$$\lim_{R \to \infty} \frac{m_X(B(x_0, R+\delta))}{m_X(B(x_0, R))} = 1$$

for every $\delta > 0$, then

(i) $D^{\pm}(P, x) = \overline{D}(P, x)$ for every $x \in X$,

(ii) the map $X \to [0,1]$, $x \mapsto \overline{D}(P, x)$ is constant,

(iii) for any packing P which has well-defined density at some $x \in X$, P has well-defined density at every x and the map $X \to [0,1]$, $x \mapsto D(P, x)$ is constant, i.e. P has well-defined density.

Definition 3.1.5. We define

$$\Delta(\mathbb{R}^n, r) \coloneqq \sup_{x \in \mathbb{R}^n} \sup_{P \in \mathrm{Pack}(\mathbb{R}^n, r)} \overline{D}(P, x).$$

Theorem 3.1.6 (Groemer)**.** *There is an r-sphere packing $P^{\triangle}$ of $\mathbb{R}^n$ with well-defined density such that*

$$D(P^{\triangle}) = \Delta(\mathbb{R}^n, r).$$

Moreover, $P^{\triangle}$ can be chosen such that the functions

$$f_R : X \to \mathbb{R},\ x \mapsto \frac{m_X(B(x, R) \cap \mathrm{supp}(P^{\triangle}))}{m_X(B(x, R))}$$

converge uniformly to the constant function

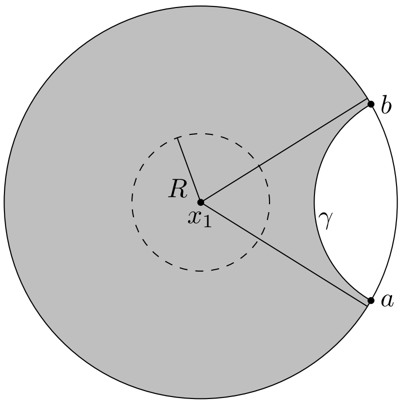
$$f : X \to \mathbb{R},\ x \mapsto \Delta(\mathbb{R}^n, r).$$

Remark 3.1.7. Note that $r \mapsto \Delta(\mathbb{R}^n, r)$ is a constant function. We will later define quantities $\Delta_{\mathrm{prob}}(X, r)$ and $\Delta_{\mathrm{BR}}(X, r)$, extending the definition of $\Delta(\mathbb{R}^n, r)$ to other spaces X, where this is no longer the case.

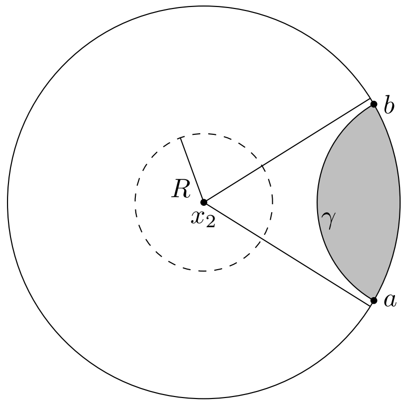
Definition 3.1.8. We say that $P \in \mathrm{Pack}(X, r)$ is *weakly-periodic*, if there is a lattice $\Gamma \leq G$ and $x_1, \ldots, x_n \in X$ such that $P = \{B(\gamma x_i) \mid \gamma \in \Gamma, i \in \{1, \ldots, n\}\}$. If Γ is cocompact, we say that P is *periodic*.

Let P be a periodic r-sphere packing in $\mathbb{R}^n$ with $\Gamma \coloneqq \mathrm{Stab}_{\mathbb{R}^n}(P)$. Then there are finitely many points $x_1, \ldots x_k \in \mathbb{R}^n$ such that $P = \{B(\gamma + x_i, r) \mid i \in \{1, \ldots, k\}, \gamma \in \Gamma\}$ and all of the orbits $\Gamma + x_i$ are pairwise disjoint. Moreover P has density

$$D(P) = \frac{k m_{\mathbb{R}^n}(B(0, r))}{|\Gamma|}.$$



(a) Packing drawn with x_1 in the center



(b) Packing drawn with x_2 in the center

Figure 3.1.: Construction of a packing without well-defined density. The shaded region in (a) equals the shaded area in (b) and contains all of the balls in the packing.

We will later see a generalization of this formula, see Proposition 3.4.4. Now the following folklore result holds, see for instance [36, Appendix A].

Theorem 3.1.9. *We have*

$$\Delta(\mathbb{R}^n, r) = \sup\{D(P) \mid P \in \text{Pack}(\mathbb{R}^n, r) \textit{ periodic } \}.$$

Pathological examples in $\mathbb{H}^2$

While the notion of packing density is reasonably well-behaved in Euclidean space, issues appear in hyperbolic space. We will give several examples which illustrate that there is a fundamental dependence of $D(P, x)$ on the point x. In spaces with polynomial volume growth this phenomenon does not appear by Theorem 3.1.4. As the volume of hyperbolic balls grows asymptotically exponential in the radius, this theorem does not apply to hyperbolic n-space. This issue was first noted by Böröczky in [25], where he gave several examples of packings for which the notion of density is problematic.

The following example is a modification of an example by Bowen and Radin in [29], where they observed that for a half-space $H \subset \mathbb{H}^2$ the limit

$$\limsup_{R\to\infty} \frac{m_{\mathbb{H}^2}(H \cap B(x, R))}{m_{\mathbb{H}^2}(B(x, R))}$$

depends on the point x. It shows that an analogue of Theorem 3.1.4 does not hold in hyperbolic space.

Example 3.1.10 (Half-planes in angular sectors)**.** Let $\Gamma \leq \text{Iso}(\mathbb{H}^2)$ be a cocompact lattice and $y \in \mathbb{H}^2$. Then there is some $r > 0$ such that $P = \{B(\gamma y, r) \mid \gamma \in \Gamma\}$ is an r-sphere packing of $\mathbb{H}^2$. We will see in Proposition 3.4.4 (in conjunction with Proposition

3.3.1) that it has the well-defined density

$$D(P) = \frac{1}{k}\frac{m_{\mathbb{H}^2}(B(x_0,r))}{|\Gamma|},$$

where $|\Gamma|$ denotes the covolume of Γ and $k = \#(\mathrm{Stab}_\Gamma(y))$. Choose points a, b on the boundary of $\mathbb{H}^2$ and let γ be the geodesic in $\mathbb{H}^2$ connecting them. Denote the two connected components of $\mathbb{H}^2 \setminus \gamma(\mathbb{R})$ by H_1 and H_2. Let Q be the sphere packing constructed from P by removing all spheres fully contained in H_1. As depicted in Figure 3.1 we can choose points $x_1, x_2 \in \mathbb{H}^2$ such that H_i is contained in the cone given by the geodesics $\gamma_{a,i}, \gamma_{b,i}$ from x_i to a resp. b with interior angle $\pi/6$. The shaded area in Figures 3.1a and 3.1b is the half-space H_2 and the white area the half-space H_1. Let us denote the cone spanned by $\gamma_{a,i}$ and $\gamma_{b,i}$ by S_i. Suppose for the moment that Γ was chosen such that P is invariant under rotations of angle $\pi/6$ around x_1 and x_2. Then

$$\begin{aligned}\overline{D}(Q,x_1) &= \limsup_{R\to\infty} \frac{m_{\mathbb{H}^2}(\mathrm{supp}(Q)\cap B(x_1,R))}{m_{\mathbb{H}^2}(B(x_1,R))} \\ &\geq \limsup_{R\to\infty} \frac{m_{\mathbb{H}^2}(\mathrm{supp}(P)\cap B(x_1,R)\setminus S_1)}{m_{\mathbb{H}^2}(B(x_1,R))} = \frac{5}{6}D(P)\end{aligned}$$

and

$$\begin{aligned}\overline{D}(Q,x_2) &= \limsup_{R\to\infty} \frac{m_{\mathbb{H}^2}(\mathrm{supp}(Q)\cap B(x_1,R))}{m_{\mathbb{H}^2}(B(x_1,R))} \\ &\leq \limsup_{R\to\infty} \frac{m_{\mathbb{H}^2}(\mathrm{supp}(P)\cap B(x_1,R)\cap S_2)}{m_{\mathbb{H}^2}(B(x_1,R))} = \frac{1}{6}D(P).\end{aligned}$$

Hence $\overline{D}(Q,x_1) > \overline{D}(Q,x_2)$ can not have a well-defined density. The only thing left to see is that it is possible to choose Γ and the points $x_1, x_2, y \in \mathbb{H}^2$ such that P has the rotational symmetries around x_1 and x_2 required for this calculation.
Consider the triangle $\mathcal{P}$ with all interior angles $\pi/6$ (such a triangle exists because $1/6+1/6+1/6 < 1$, see [87, Theorem 3.5.1]). Place a polygon $\mathcal{Q}$ congruent to $\mathcal{P}$ on the geodesic γ' connecting x_1 and x_2 such that one corner is x_1 and another corner lies on the geodesic. Let Γ be the Fuchsian group generated by the reflections at the sides of $\mathcal{Q}$ and choose $y = x_1$. Then Γ will have an infinite amount of points which it stabilizes, namely all corners in the tiling $\mathcal{T}$ of $\mathbb{H}^2$ by $\mathcal{P}$ containing $\mathcal{Q}$. It is easy to see that the geodesic γ contains infinitely many corners and more specifically there is a $d > 0$ such that if $\gamma(t)$ is stabilized by Γ, then $\gamma(t+d)$ is stabilized by Γ. As slightly moving x_2 at most $d/2$ on γ' away from x_1 does not disturb the properties we required above, we can assume that x_2 is a corner of $\mathcal{T}$ and that x_1 and x_2 satisfy all other properties the density calculation above requires.

The example above shows that there are issues with the notion of density in hyperbolic space, as we defined it. Böröczky gave another example which makes it clear that there is also very little hope of defining a notion of density by using tilings associated to packings by giving an example for a packing to which one can not even assign some sort of density on an intuitive level.

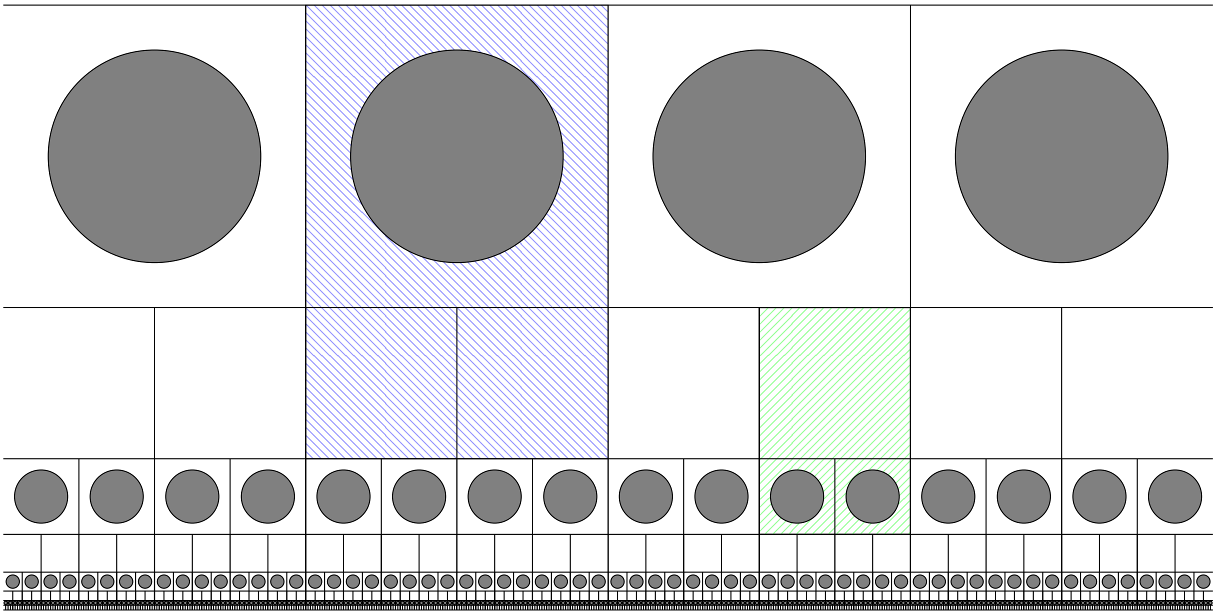


Figure 3.2.: Binary tiling in upper half plane with associated sphere packing and isometric tiles.

Example 3.1.11 (Böröczky's binary tiling)**.** Consider the sphere packing and tiling pictured in Figure 3.2. This tiling and the associated sphere packing were discovered by Böröczky in [25]. More accessible expositions can be found in [29, 51]. The blue and the green tile are isometric. But the relative amount of volume taken up by a sphere in the blue tile is half of the relative volume taken up by a sphere in the green tile. It is not intuitively clear what the packing density of this packing should be. Note that the borders of the tiles consist of geodesics and horocycles. This tiling can be modified such that the tiles are isometric convex pentagons and the same seemingly paradoxical density behaviour occurs.

3.2. Random sphere packings and their density

Definition 3.2.1. A *random invariant r-sphere packing in X* is a $2r$-uniformly discrete G-stationary point process Λ. If the point process Λ is G-invariant, we call Λ a *random equivariant r-sphere packing in X*. If $\Lambda_\omega = \delta_\emptyset$ for every $\omega \in \Omega$, we say that Λ is *trivial*. As in the deterministic case, we define $\Lambda^r \coloneqq \{B(x,r) \mid x \in \operatorname{supp}(\Lambda)\}$.

It might seem more natural to define random invariant r-sphere packings as actual G-stationary random sphere packings, i.e. as random variables $\Lambda : (\Omega, \mathbb{P}) \to \operatorname{Pack}(X,r)$ such that the distribution $\Lambda_*\mathbb{P}$ is G-invariant (where we equip $\operatorname{Pack}(X,r)$ with some appropriate σ-algebra). If $B(x,r) = B(y,r)$ implies $x = y$, then there is a unique map $c\colon \{B(x,r) \mid x \in X\} \to X$ such that $c(B(x,r)) = x$ and $c^*\colon \operatorname{Pack}(X,r) \to \mathrm{UD}_{2r}(X)$, $P \mapsto \{c(B) \mid B \in P\}$ is a G-equivariant map. Then we obtain a random set $c^* \circ \Lambda$ with G-invariant distribution. As $\mathrm{UD}_{2r}(X)$ is G-equivariantly homeomorphic to $\mathcal{N}^*_{2r}(X)$ we obtain a G-stationary point process in X encoding the centers of spheres in Λ. For various reasons it is technically easier to directly work with this point process.

Moreover some results about random invariant r-sphere packings also apply when balls do not have unique centers, if we work with this definition. We do however need to assume in this chapter that $m_X(B(x,r)) = m_X(\overline{B}(x,r))$ for all $x \in X$ and $r > 0$ and will do so from now on.

Definition 3.2.2 (Bowen–Radin, [28])**.** The *density* of a random invariant r-sphere packing Λ in X is defined as

$$D_r(\Lambda) = \mathbb{P}(x_0 \in \operatorname{supp}(\Lambda, r)).$$

Note that this is well-defined by Proposition 2.2.3.

We keep track of r in the notation for the density, as any random invariant r-sphere packing is also a random invariant s-sphere packing for $0 < s < r$. We will actually make use of this fact later on.

Lemma 3.2.3. *Let Λ be a random invariant r-sphere packing in X. Then Λ is locally bounded and for every $x \in X$ we have*

$$D_r(\Lambda) = \mathbb{P}(x \in \operatorname{supp}(\Lambda, r)) = \mathbb{E}[\Lambda(B(x,r))] = i(\Lambda) m_X(B(x,r)).$$

Proof. The fact that Λ is locally L^∞ follows directly from Proposition 2.3.6. For $x \in X$ and $\Lambda_\omega = \sum_{x \in P_\omega} \delta_x$ we have $x \in \operatorname{supp}(\Lambda, r) \iff \exists y \in P_\omega : d_X(y,x) < r \iff \Lambda(B(x,r)) \neq 0 \iff \Lambda(B(x,r)) = 1$. Thus

$$\mathbb{P}(x \in \operatorname{supp}(\Lambda, r)) = \mathbb{P}(\Lambda(B(x,r)) = 1) = \mathbb{E}[\Lambda(B(x,r))] = i(\Lambda) m_X(B(x,r)),$$

as $\Lambda(B(x,r)) \in \{0,1\}$. This implies the claim, as

$$\mathbb{P}(x \in \operatorname{supp}(\Lambda, r)) = i(\Lambda) m_X(B(x_0,r)) = \mathbb{P}(x_0 \in \operatorname{supp}(\Lambda, r)). \qquad \square$$

Observe that for any random invariant r-sphere packing $\Lambda : (\Omega, \ \mathbb{P}) \to \mathcal{N}^*_{2r}(X)$ there is a random invariant r-sphere packing $\Lambda^{cm} : (\mathcal{N}^*_{2r}(X), \ \Lambda_*\mathbb{P}) \to \mathcal{N}^*_{2r}(X), \ \mu \mapsto \mu$, the *canonical measure model* of Λ, with $D_r(\Lambda) = D_r(\Lambda^{cm})$. Moreover if $\mathbb{P}$ is a G-invariant probability measure on $\mathcal{N}^*_{2r}(X)$, there is a canonical point process $\Lambda_{\mathbb{P}} : (\mathcal{N}^*_{2r}(X), \mathbb{P}) \to \mathcal{N}^*_{2r}(X)$ and it is clear that we obtain every canonical measure model of a point process from a G-invariant probability measure on $\mathcal{N}^*_{2r}(X)$. This motivates the following definition.

Definition 3.2.4. The *probabilistic optimal r-sphere packing density* of X is given by

$$\Delta_{\text{prob}}(X,r) \coloneqq \sup\{D_r(\Lambda_{\mathbb{P}}) \mid \mathbb{P} \in \operatorname{Prob}^G(\mathcal{N}^*_{2r}(X))\}.$$

A random invariant r-sphere packing Λ in X is called *optimally dense*, if

$$D_r(\Lambda) = \Delta_{\text{prob}}(X,r).$$

Theorem 3.2.5 (Bowen–Radin, [28, Theorem 1])**.** *There is an invariant G-ergodic probability measure $\mathbb{P}$ on $\mathcal{N}^*_{2r}(X)$ such that $D_r(\Lambda_{\mathbb{P}}) = \Delta_{\text{prob}}(X,r)$.*

Proof. We have

$$\delta(\mathrm{UD}_{2r}(X)_{x_0}) = \left\{ \sum_{x\in P} \delta_x \;\middle|\; P \in \mathrm{UD}_{2r}(X),\ \mathrm{dist}(P, x_0) < r \right\}$$
$$= \{\mu \in \mathcal{N}^*_{2r}(X) \mid \mu(B(x_0, r)) = 1\}.$$

Hence $A_{x_0} \coloneqq \{\mu \in \mathcal{N}^*_{2r}(X) \mid \mu(B(x_0, r)) = 1\}$ is open and for any $\mathbb{P}' \in \mathrm{Prob}^G(\mathcal{N}_{2r}(X))$ we have

$$D_r(\Lambda_{\mathbb{P}'}) = \mathbb{P}'(\Lambda_{\mathbb{P}'}(B(x_0, r)) = 1) = \mathbb{P}'(A_{x_0}).$$

Set $B_x \coloneqq \left\{\sum_{x\in P} \delta_x \;\middle|\; P \in \mathrm{UD}_{2r}(X),\ \mathrm{dist}(P, x_0) \leq r\right\}$ and note that this set is closed by the definition of the Chabauty-Fell topology. By the proof of Proposition 2.4.18 we have that

$$\mathbb{P}'(A_{x_0}) = \mathbb{P}'(B_{x_0}).$$

Choose a sequence $(\mathbb{P}_n)_{n\geq 1}$ in $\mathrm{Prob}^G(\mathcal{N}_{2r}(X))$ with $D_r(\Lambda_{\mathbb{P}_n}) \to \Delta_{\mathrm{prob}}(X, r)$. As the topological space $\mathrm{Prob}^G(\mathcal{N}_{2r}(X))$ is compact (wrt. the weak-$*$ topology), there is a subsequence $(\mathbb{P}_{n_k})_{k\geq 1}$ with a limit $\mathbb{P}_\infty$. As B_{x_0} is closed, $\chi_{B_{x_0}}$ is upper semicontinuous. Hence we can find a decreasing sequence $(f_j)_{j\geq 1}$ in $C(\mathcal{N}^*_{2r}(X))$ with $\lim_{j\to\infty} f_j = \chi_{B_{x_0}}$ pointwise. Thus

$$\mathbb{P}_{n_k}(f_j) \to \mathbb{P}_\infty(f_j) \quad \text{as } (k \to \infty)$$

by the definition of weak-$*$ convergence and

$$\mathbb{P}_\infty(f_j) \to \mathbb{P}_\infty(B_{x_0}) = D_r(\Lambda_{\mathbb{P}_\infty})$$

by dominated convergence. Since

$$\mathbb{P}_{n_k}(f_j) \geq \mathbb{P}_{n_k}(B_{x_0}) = D_r(\Lambda_{\mathbb{P}_{n_k}})$$

we have

$$\mathbb{P}_\infty(f_j) \geq \limsup_{k\to\infty} \mathbb{P}_{n_k}(B_{x_0}) = \lim_{k\to\infty} D_r(\Lambda_{\mathbb{P}_{n_k}}) = \Delta_{\mathrm{prob}}(X, r).$$

Hence

$$D_r(\Lambda_{\mathbb{P}_\infty}) \geq D_r(\Lambda_{\mathbb{P}_{n_k}}) \to \Delta_{\mathrm{prob}}(X, r)$$

and thus $D_r(\Lambda_{\mathbb{P}_\infty}) = \Delta_{\mathrm{prob}}(X, r)$. By ergodic decomposition, cf. [95, Theorem 4.4.], there is a probability measure μ on the set $\mathrm{Prob}^G_e(\mathcal{N}^*_{2r}(X))$ of invariant G-ergodic Borel probability measures on $\mathcal{N}^*_{2r}(X)$ such that

$$\mathbb{P}_\infty(B_{x_0}) = \int \mathbb{P}(B_{x_0}) d\mu(\mathbb{P}).$$

Thus there must be an invariant G-ergodic Borel probability measure $\mathbb{P}$ with

$$D_r(\Lambda_{\mathbb{P}_\infty}) = \mathbb{P}_\infty(B_{x_0}) = \mathbb{P}(B_{x_0}) = D_r(\Lambda_{\mathbb{P}}).$$ □

3.3. Optimal density in terms of generically measured sets

It is also possible to obtain the optimal probabilistic packing density $\triangle_{\text{prob}}(X)$ as a supremum of ordinary packing densities of particularly nicely behaved packings.

Proposition 3.3.1. *Assume that an invariant pointwise ergodic theorem holds for* $((G_t)_{t>0}, C(\text{UD}_{2r}(X)))$*, where* $G_t = \pi^{-1}(B(x_0,t))$*. Let* $P \in \text{UD}_{2r}(X)$ *be generically measured wrt.* $(G_t)_{t>0}$*. Then* P^r *has well-defined density and*

$$D(P^r) = D_r(\Lambda^P).$$

Proof. Note that for any $g \in G$ the set gP is generically measured by definition. We have

$$\begin{aligned} D_r(\Lambda^P) &= \mathbb{P}_P(x_0 \in \text{supp}(\Lambda^P, r)) = \mathbb{P}_P(\{Q \in \text{UD}_{2r}(X) \mid \text{dist}(x_0, \text{supp}(\Lambda^P_Q)) < r\}) \\ &= \mathbb{P}_P(\{Q \in \text{UD}_{2r}(X) \mid \text{dist}(x_0, Q) < r\}) = \int_{\text{UD}_{2r}(X)} \chi_{\text{UD}_{2r}(X)_{x_0}}(Q) d\mathbb{P}_P(Q). \end{aligned}$$

But by Proposition 2.4.18 the function $\chi_{\text{UD}_{2r}(X)_{x_0}}$ is invariantly Riemann-integrable. Thus, by Proposition 2.4.17, we have that

$$\int_{\text{UD}_{2r}(X)} \chi_{\text{UD}_{2r}(X)_{x_0}}(Q) d\mathbb{P}_P(Q) = \lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} \chi_{\text{UD}_{2r}(X)_{x_0}}(h^{-1}gP) dm_G(h).$$

Observe that $h^{-1}gP \in \text{UD}_{2r}(X)_{x_0}$ if and only if $\text{dist}(x_0, h^{-1}gP) < r$ if and only if $\text{dist}(hx_0, gP) < r$ if and only if $hx_0 \in \text{supp}(gP^r)$. Thus

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} \chi_{\text{UD}_{2r}(X)_{x_0}}(h^{-1}gP) dm_G(h) &= \lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{G_t} \chi_{\text{supp}(gP^r)}(hx_0) dm_G(h) \\ &= \lim_{t\to\infty} \frac{1}{m_G(G_t)} \int_{B(x_0,t)} \chi_{\text{supp}(gP^r)}(x) dm_X(x) \\ &= \lim_{t\to\infty} \frac{m_X(B(x_0,t) \cap \text{supp}(gP^r))}{m_X(B(x_0,t))} \\ &= \lim_{t\to\infty} \frac{m_X(B(g^{-1}x_0,t) \cap \text{supp}(P^r))}{m_X(B(x_0,t))} \\ &= D(P^r, g^{-1}x_0). \end{aligned}$$

As G acts transitively on X, the claim follows. □

Recall that $\text{UD}^{gen}_{2r}(X, (G_t)_{t>0})$ denotes the set of all $P \in \text{UD}_{2r}(X)$ which are generically measured wrt. $(G_t)_{t>0}$.

Definition 3.3.2. Set $G_t \coloneqq \pi^{-1}(B(x_0,t))$. The *Bowen-Radin optimal r-sphere packing density of X* is defined as

$$\Delta_{\text{BR}}(X, r) \coloneqq \sup\{D(P^r) \mid P \in \text{UD}^{gen}_{2r}(X, (G_t)_{t>0})\}.$$

If $P \in \mathrm{UD}_{2r}^{gen}(X, (G_t)_{t>0})$ and $D(P^r) = \Delta_{\mathrm{BR}}(X, r)$, we call P^r *optimally dense.*

Theorem 3.3.3. *Assume that there is an invariant pointwise ergodic theorem for the tuple* $((\pi^{-1}(B(x_0, t)))_{t>0},\ X,\ C(\mathrm{UD}_{2r}(X)))$. *Then*

$$\Delta_{\mathrm{BR}}(X, r) = \Delta_{\mathrm{prob}}(X, r).$$

Proof. By Proposition 3.3.1, every generically measured $P \in \mathrm{UD}_{2r}(X)$ has a well-defined density and there is a random invariant r-sphere packing Λ^P with $D_r(\Lambda^P) = D(P^r)$. Hence $\Delta_{\mathrm{BR}}(X, r) \leq \Delta_{\mathrm{prob}}(X, r)$.
By Theorem 3.2.5 there is a random invariant r-sphere packing Λ with ergodic distribution such that $D_r(\Lambda) = \Delta_{\mathrm{prob}}(X, r)$. By Proposition 2.4.9 there is a generically measured $P \in \mathrm{UD}_{2r}(X)$ (wrt. $(\pi^{-1}(B(x_0, t))))_{t>0}$) such that Λ^P and Λ have the same distribution. Now $D(P^r) = D_r(\Lambda^P) = D_r(\Lambda)$ and thus $\Delta_{\mathrm{BR}}(X, r) = \Delta_{\mathrm{prob}}(X, r)$. □

3.4. Density formulas for random invariant sphere packings

3.4.1. Periodic random invariant sphere packings

Definition 3.4.1. (i) Let $P \in \mathrm{UD}_{2r}(X)$ and $\Gamma \leq G$ a lattice. P is called *weakly-periodic* (wrt. to Γ), if there are $x_1, \dots, x_n \in X$ such that $P = \bigcup_{i=1}^n \Gamma x_i$ for all $\gamma \in \Gamma$. If Γ is cocompact, P is called *periodic.*

(ii) $\mu \in \mathcal{N}_{2r}^*(X)$ is called *(weakly)-periodic (wrt. to* Γ*)* if there is a (weakly-)periodic $P \in \mathrm{UD}_{2r}(X)$ (wrt. to Γ) such that $\mu = \delta_P$.

(iii) An invariant random sphere packing $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(X)$ is called *weakly-periodic wrt. to* Γ, if there is a *weakly-periodic* $\mu \in \mathcal{N}_{2r}^*(X)$ wrt. to Γ such that $\Lambda_*\mathbb{P} = q_* m_{G/\Gamma}$, where $q : G/\Gamma \to G.\mu,\ g\Gamma \to g\mu$. In this case we say that Λ is *modeled* on μ. If $P \in \mathrm{UD}_{2r}(X)$ such that $\mu = \delta_P$, we also say that Λ is modeled on P. If Γ is cocompact, we say that Λ is *periodic.*

In degenerate examples it might happen that $P \in \mathrm{UD}_{2r}(X)$ is not (weakly-)periodic, but $P^r \in \mathrm{Pack}(X, r)$ is (weakly-)periodic. For example if X is an ultrametric space. In the case that balls have unique centers, this can not happen.
If G is a semisimple Lie group with finite center and no compact factors or $G = \mathbb{R}^n$ or $G = \mathrm{Iso}(\mathbb{R}^n)$, then any lattice Γ in G has a finite index subgroup Γ' that is torsion free, see Theorem B.3.2.

Lemma 3.4.2. *If* $P \in \mathrm{UD}_{2r}(X)$ *and there is a lattice* $\Gamma \leq G$ *leaving* P *invariant such that* Γ *has a finite index torsion-free subgroup, then there are finitely many* $x_1, \dots, x_n \in X$ *such that* $P = \bigcup_{i=1}^n \Gamma x_i$ *and* $x_i \notin \Gamma x_j$ *for* $i \neq j$.

Proof. Let $\Gamma' \leq \Gamma$ be a torsion-free finite index subgroup and let F be a strict fundamental domain for the action of Γ' on X (F exists by Lemma B.2.6). Assume that $\#(P \cap F) = \infty$ and let $P \cap F = \{x_1, x_2, \dots\}$. As Γ' is torsion free, we know that $\gamma B(x_i, r) \cap B(x_i, r) = \emptyset$ for all i and $\gamma \in \Gamma \setminus \{e\}$. Thus the map $\bigcup_{i=1}^\infty B(x_i, r) \to \Gamma\backslash X$

is injective. By Lemma B.2.6 we can find a strict fundamental domain $F' \subset X$ for the action of Γ' such that $\bigcup_{i=1}^{\infty} B(x_i, r) \subset F'$. Observe that

$$\infty = \sum_{i=1}^{\infty} m_X(B(x_i, r)) = m_X(\bigcup_{i=1}^{\infty} B(x_i, r)) \leq m_X(F') < \infty,$$

a contradiction. □

Lemma 3.4.3. *Assume that X is a Riemannian symmetric space of noncompact type and that $G = \mathrm{Iso}(X)_0$. Then G is semisimple with trivial center and no compact factors. Let $P \in \mathrm{UD}_{2r}(X)$ be periodic. Then $\mathrm{Stab}_G(P)$ is a lattice in G.*

Proof. We know that $\mathrm{Stab}_G(P)$ is cocompact as it contains a cocompact subgroup by the definition of P. Thus we are finished once we know that $\mathrm{Stab}_G(P)$ is discrete. Assume that it is not and let H denote the connected component of the identity in $\mathrm{Stab}_G(P)$. For $h \in H$ choose a 1-parameter subgroup γ_h in H with $\gamma_h(0) = e$ and $\gamma_h(1) = h$. Then, for each $x \in P$, $t \mapsto \gamma(t)x$ defines a continuous map $\mathbb{R} \to P$ and thus we see that $H.x = x$ for every $x \in P$. Let $\exp_0$ denote the Riemannian exponential map at x_0 and set $\overline{P} := \exp_0^{-1}(P)$. If $\overline{P}$ spans a proper subspace U of the tangent space $T_{x_0}X$, then we can choose some $v \in T_{x_0}X$ with $v \perp U$ with respect to the Riemannian metric at x_0. We claim that $d_X(\exp(tv), x) \geq d_X(\exp(tv), x_0) \geq t$ for all $t > 0$ and $x \in P$. If we establish this claim, then it follows that P does not contain the orbit of a cocompact subgroup. As this is a contradiction, there must be a $B \subset P$ such that $\tilde{B} = \exp_0^{-1}(B)$ is a basis of the tangent space and for each $b \in \tilde{B}$ and $f \in H$ we have $f(\exp(b)) = \exp(b)$. Hence, if γ_b denotes the unique geodesic from x_0 to b, we have $f(\gamma_b) = \gamma_b$, as f is an isometry and fixes two points on γ_b. Thus $df(b) = b$ and hence $df = \mathrm{id}$. Thus $f = \mathrm{id}$.

We will conclude our claim from facts about the geometry of CAT(0)-spaces, see Appendix A for additional explanation. Set $q = \exp(tv)$ and let $\triangle(\overline{q}, \overline{x_0}, \overline{x})$ be the Euclidean comparison triangle to $\triangle(q, x_0, x)$. Then by [30, Chapter II Corollary 1A.7], the Alexandrov angle $\angle_{x_0}(q, x) = \pi/2$. By [30, Chapter II 1.7 Proposition (4)], we have $\pi/2 = \angle_{x_0}(q, x) \leq \angle_{\overline{x_0}}(\overline{q}, \overline{x}) \leq \pi$. Thus we see that the comparison triangle has an angle of at least $\pi/2$ at x_0. Thus

$$d_X(q, x_0) \leq d_{\mathbb{E}^2}(\overline{q}, \overline{x_0}) \leq d_{\mathbb{E}^2}(\overline{q}, \overline{x}) = d_X(q, x)$$

and we are finished. □

Proposition 3.4.4. *Let Λ be a random invariant r-sphere packing, weakly-periodic wrt. $\Gamma \leq G$ and modeled on $P = \bigsqcup_{i=1}^{k} \Gamma x_i \in \mathrm{UD}_{2r}(X)$. Set $n_i := \#\mathrm{Stab}_\Gamma(x_i)$. Then*

$$i(\Lambda) = \frac{1}{|\Gamma|} \sum_{i=1}^{k} \frac{1}{n_i}$$

and thus

$$D_r(\Lambda) = \frac{1}{|\Gamma|} \left(\frac{1}{n_1} + \cdots + \frac{1}{n_k} \right) m_X(B(x_0, r)).$$

Proof. We need to calculate $\mathbb{E}[\Lambda(B(x_0, r))]$. Let $F \subset G$ be a strict right fundamental domain for Γ and choose $g_1, \ldots, g_k \in G$ with $g_i x_0 = x_i$. Set $\mu = \delta_P$. We have

$$\begin{aligned}
\mathbb{E}[\Lambda(B(x_0,r))] &= \frac{1}{|\Gamma|}\int_{G/\Gamma} g_*\mu(B(x_0,r))dm_{G/\Gamma}(g\Gamma) = \frac{1}{|\Gamma|}\int_{G/\Gamma} \mu(g^{-1}(B(x_0,r)))dm_{G/\Gamma}(g\Gamma)\\
&= \frac{1}{|\Gamma|}\int_F \mu(g^{-1}B(x_0,r))dm_G(g) = \frac{1}{|\Gamma|}\int_F \sum_{x\in P}\chi_{B(x_0,r)}(gx)dm_G(g)\\
&= \frac{1}{|\Gamma|}\sum_{x\in P}\int_F \chi_{B(x_0,r)}(gx)dm_G(g) = \frac{1}{|\Gamma|}\sum_{i=1}^k\sum_{x\in\Gamma x_i}\int_F \chi_{B(x_0,r)}(gx)dm_G(g)\\
&= \frac{1}{|\Gamma|}\sum_{i=1}^k\frac{1}{n_i}\sum_{\gamma\in\Gamma}\int_F \chi_{B(x_0,r)}(g\gamma x_i)dm_G(g) = \frac{1}{|\Gamma|}\sum_{i=1}^k\frac{1}{n_i}\int_G \chi_{B(x_0,r)}(gx_i)dm_G(g)\\
&= \frac{1}{|\Gamma|}\sum_{i=1}^k\frac{1}{n_i}\int_G \chi_{B(x_0,r)}(gg_ix_0)dm_G(g) = \frac{1}{|\Gamma|}\sum_{i=1}^k\frac{1}{n_i}\int_G \chi_{B(x_0,r)}(gx_0)dm_G(g)\\
&= \frac{1}{|\Gamma|}\sum_{i=1}^k\frac{1}{n_i}m_X(B(x_0,r)).
\end{aligned}$$

□

Remark 3.4.5. This result is often wrongly stated without consideration of the stabilizers. The calculation of the intensity for a single orbit can be found in a very similar form in [14].

Corollary 3.4.6 (Bowen–Radin, [28, Proposition 1])**.** *Let Λ be a random invariant r-sphere packing, weakly-periodic wrt. $\Gamma \leq G$ and modeled on $P = \bigsqcup_{i=1}^k \Gamma x_i \in \mathrm{UD}_{2r}(X)$. Assume that Γ is torsion-free and let $F \subset X$ be a strict fundamental domain for the action of Γ on X. Then*

$$D_r(\Lambda) = \frac{m_X(F \cap \mathrm{supp}(P^r))}{m_X(F)} = \frac{\#(F\cap P)m_X(B(x_0,r))}{m_X(F)}.$$

Proof. We know that $D_r(\Lambda) = \frac{k}{|\Gamma|}m_X(B(x_0,r))$. Thus we need to see that $m_X(F \cap \mathrm{supp}(P^r)) = km_X(B(x_0,r))$. Since Γ is torsion free, we have that $\gamma B(x_i,r)\cap B(x_i,r) = \emptyset$ for $i = 1,\ldots,k$ and $\gamma \in \Gamma\setminus\{e\}$. Thus, by Lemma B.2.6, we can choose a fundamental domain F' such that $B(x_1,r)\cup\cdots\cup B(x_k,r) \subset F'$ and $F' \subset X \setminus \bigcup_{i=1}^k \bigcup_{\gamma\in\Gamma\setminus\{e\}} B(\gamma x_i, r)$. Then $\#(P\cap F') = k$ and thus $\#(P \cap F) = k$. Since the set $\mathrm{supp}(P^r)$ is the union of k disjoint Γ-orbits of balls, we know that $m_X(F\cap \mathrm{supp}(P^r)) = km_X(B(x_0,r))$. □

Lemma 3.4.7. *Assume that the pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$ Let $P \in \mathrm{UD}_{2r}(X)$ be periodic and let Λ be a periodic random invariant r-sphere packing modeled on P. Then P is generically measured for Λ.*

Proof. We know that the support of $\Lambda_*\mathbb{P}$ consists of the single (compact!) orbit $G\delta_P$. By the invariant pointwise theorem there must be a G-invariant set of $P \in \mathrm{UD}_{2r}(X)$ generically measured wrt. $(G_t)_{t>0}$. It follows immediately that this set is given by GP. □

If P is only weakly-periodic, then the same result takes more effort:

Lemma 3.4.8. *Assume that the pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$. Let $P \in \mathrm{UD}_{2r}(X)$ be weakly-periodic and let Λ be a weakly-periodic random invariant r-sphere packing modeled on P. Then P is generically measured for Λ.*

Proof. Without loss of generality we can assume that P is not periodic. Assume for the moment that the orbit closure of $G\delta_P$ is given by $O_P := G\delta_P \cup \{\delta_\emptyset\}$. Then the support of $\Lambda_*\mathbb{P} = q_* m_{G/\Gamma}$ must be a subset of O_P. Note that O_P contains two G-orbits, the obit $G\delta_P$ and $\{\delta_\emptyset\}$. Clearly $\Lambda_*\mathbb{P}(G\delta_P) = 1$ and $\Lambda_*\mathbb{P}(\{\delta_\emptyset\}) = 0$ and thus $\Lambda_*\mathbb{P}$ is ergodic. Hence $\mathrm{supp}_*\Lambda_*\mathbb{P}$ is an ergodic measure on $\overline{GP}$ with $\mathrm{supp}_*\Lambda_*\mathbb{P}(GP) = 1$ and $\mathrm{supp}_*\Lambda_*\mathbb{P}(\emptyset) = 0$. By the invariant ergodic theorem there must be a conull set of $Q \in \mathrm{UD}_{2r}(X)$ which are invariantly $(\mu, C(\mathrm{UD}_{2r}(X)), (G_t)_{t>0})$-generic. Hence there must be a $gP \in GP$ which is invariantly $(\mu, C(\mathrm{UD}_{2r}(X)), (G_t)_{t>0})$-generic and thus P is invariantly $(\mu, C(\mathrm{UD}_{2r}(X)), (G_t)_{t>0})$-generic and thus generically measured wrt. $(G_t)_{t>0}$.
By [16, Proposition 4.4], we have $\emptyset \in \overline{G\delta_P}$. Now assume that $\delta_{P'} \in \overline{G\delta_P} \setminus \{\emptyset\}$. Then there are $(g_n)_{n\geq 1}$ in G with $P_n := g_n P \to P'$. By Proposition 2.1.1 for $y \in P'$ are $y_n \in P_n$ with $y_n \to y$. Then, if $P = \bigcup_{i=1}^{l} \Gamma x_i$ there is some x_i such that $y_n = g_n \gamma_n x_i$, for suitable $\gamma_n \in \Gamma$, for infinitely many n. Hence we find a subsequence $(n_k)_{k\geq 1}$ and $\gamma_{n_k} \in \Gamma$ with $g_{n_k}\gamma_{n_k} x_i \to y$. But then $(g_{n_k}\gamma_{n_k})_{k\geq 1}$ must have a convergent subsequence $(g_{n_{k_l}}\gamma_{n_{k_l}})_{l\geq 1}$ with limit $h \in G$. Then $P' = \lim_{l\to\infty} g_{n_{k_l}} P = \lim_{l\to\infty} g_{n_{k_l}}\gamma_{n_{k_l}} P = hP$. □

Remark 3.4.9 (Density of periodic sets)**.** Note that Lemma 3.4.7 and Lemma 3.4.8 allow us to switch freely between (weakly-)periodic $P \in \mathrm{UD}_{2r}(X)$ and (weakly-)periodic random invariant sphere packings modeled on P, assuming that the invariant pointwise ergodic theorem holds. Moreover we see that for each (weakly-)periodic $P \in \mathrm{UD}_{2r}(X)$ there is exactly one (weakly-)periodic random invariant sphere packing modeled on P by Proposition 2.4.9.
Thus, if the invariant pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$ with $G_t = \pi^{-1}(B(e,t))$, we will freely switch between (weakly-)periodic elements of $\mathrm{UD}_{2r}(X)$, (weakly-)periodic random invariant sphere packings and (assuming balls have unique centers) (weakly-)periodic sphere packings. In particular, by an *optimally dense periodic r-sphere packing*, we will understand a periodic r-sphere packing P with $D(P) = \triangle_{\mathrm{prob}}(X, r)$.

Remark 3.4.10 (Periodic approximation)**.** It is not known if the hyperbolic analogue of Theorem 3.1.9,

$$\Delta_{\mathrm{prob}}(\mathbb{H}^n, r) := \sup\{D(P^r) \mid P \in \mathrm{UD}_{2r}(\mathbb{H}^n) \text{ periodic}\}$$

holds for all $n \in \mathbb{N}$. For $n = 2$ Bowen obtained this result in [27] and conjectured that his approach might extend to $n = 3$.
Thus the linear programming bound by Cohn, Lurie and Sarnak, [41], on the density of periodic sphere packings in $\mathbb{H}^n$ implies a linear programming bound on $\Delta_{\mathrm{prob}}(\mathbb{H}^2, r)$, but not (necessarily) on $\Delta_{\mathrm{prob}}(\mathbb{H}^n, r)$ for $n > 2$.

3.4.2. Density in Voronoi cells

Let $\mu \in \mathcal{N}_{2r}^*(X)$ with $P = \mathrm{supp}(\mu) \in \mathrm{UD}_{2r}(X)$. If $x \in X$ such that there is some $y_x \in P$ with $d_X(x, y_x) < d_X(x, z)$ for all $y_x \neq z \in P$, we define the *Voronoi cell* of μ containing x as

$$V(\mu, x) := \{x' \in X \mid \forall y_x \neq z \in P : d_X(x', y_x) < d_X(x', z)\}.$$

Before stating the main result of this subsection, we need to show that Voronoi cells play well with measurability of point processes.

Lemma 3.4.11. *Let $R > 0$. The set $\mathcal{T}_{2r} := \{\mu \in \mathcal{N}_{2r}^*(X) \mid \mu(\{x_0\}) = 1\}$ is closed in $\mathcal{N}_{2r}^*(X)$ and the set*

$$F = \{(g, \mu) \in G \times \mathcal{T}_{2r} \mid gx_0 \in V(\mu, x_0)\}$$

is open in $G \times \mathcal{T}_{2r}$.

Proof. The closedness of $\mathcal{T}_{2r}$ follows directly from the fact that δ is a homeomorphism and the characterization of convergence wrt. to the Chabauty-Fell topology. Assume that $(g_n, \mu_n)_{n\geq 1}$ is a convergent sequence in $G\times \mathcal{T}_{2r} \setminus F$ with limit (g, μ). Let $P_n, P \in \mathrm{UD}_{2r}(X)$ with $\delta_{P_n} = \mu_n$ and $\delta_P = \mu$. Then, for each $n \geq 1$, there is some $x_n \in P_n \setminus \{x_0\}$ with $d_X(x_0, g_n x_0) \geq d_X(x_n, g_n x_0)$. Then

$$d_X(x_n, g_n x_0) \geq |d_X(x_n, x_0) - d_X(g_n x_0, x_0)|$$

by the reverse triangle inequality and thus

$$2d_X(g_n x_0, x_0) \geq d_X(x_n, g_n x_0) + d_X(g_n x_0, x_0) \geq d_X(x_n, x_0),$$

and the left hand side converges, as $(g_n)_{n\geq 1}$ is convergent. Hence there is some $C > 0$ with $d_X(x_n, x_0) < C$ and thus we find a convergent subsequence $(x_{n_k})_{k\geq 1}$ with limit $x \in X$. By the characterization of convergence in the Chabauty-Fell topology we see that $x \in P$ and

$$d_X(g_{n_k} x_0, x_0) \geq d_X(g_{n_k} x_0, x_{n_k}) \to d_X(gx_0, x).$$

Moreover

$$d_X(x_0, x) = \lim_{k\to\infty} d_X(x_0, x_{n_k}) \geq r$$

and thus $x \neq x_0$. Whence $(g, \mu) \in G \times \mathcal{T}_{2r} \setminus F$ and thus F is open. □

The following was obtained by Bowen and Radin via the mass transport principle. We give a proof using the refined Campbell theorem.

Theorem 3.4.12 (Bowen–Radin, [28, Proposition 3])**.** *Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}_{2r}^*(X)$ be a random invariant r-sphere packing in X and for each $\omega \in \Omega$ let $P_\omega := \mathrm{supp}(\Lambda_\omega)$. If $m_X(V(\Lambda_\omega, x)) < \infty$ for every $x \in P_\omega$ and every ω, then*

$$D_r(\Lambda) = \int_\Omega \sum_{x\in P_\omega} \frac{m_X(B(x_0, r))}{m_X(V(\Lambda_\omega, x))} \mathbf{1}\{x_0 \in V(\Lambda_\omega, x)\} d\mathbb{P}(\omega).$$

Before we prove this, we remark that at most one of the summands in the formula above survives, namely the one corresponding to the Voronoi cell containing x_0, if such a Voronoi cell exists. It might happen that x_0 is exactly on the boundary of two or more Voronoi cells and thus not contained in a Voronoi cell. In that case the sum above is 0. We further note that this is exactly Bowen and Radin's formula expressing $D_r(\Lambda)$ in terms of Voronoi cells, cf. [28, Proposition 3], just written down without the assumption that the probability of hitting the boundary of a Voronoi cell is zero (which might be wrong in our general setting, see Proposition 3.4.14).

Proof. Note that $D_r(\Lambda) = D_r(\Lambda^\infty)$ and that

$$\begin{aligned}&\int_\Omega \sum_{x\in P_\omega} \frac{m_X(B(x_0,r))}{m_X(V(\Lambda_\omega,x))}\mathbf{1}\{x_0\in V(\Lambda_\omega,x)\}d\mathbb{P}(\omega)\\ =&\int_{\mathcal{N}^*_{2r}(X)} \sum_{x\in\operatorname{supp}(\mu)} \frac{m_X(B(x_0,r))}{m_X(V(\mu,x))}\mathbf{1}\{x_0\in V(\mu,x)\}d\Lambda_*\mathbb{P}(\mu)\end{aligned}$$

only depends on the distribution of Λ, which is by definition of the canonical infinite model equal to the distribution of Λ^∞. Hence we can assume without loss of generality that Λ is of the form Λ^∞, that $\Omega = \mathcal{N}^*_{2r}(X)$ and that $\mathcal{T} = \mathcal{T}_{2r}$. Consider the function

$$f : G\times\mathcal{T}\to\mathbb{R},\ (g,\omega)\mapsto \frac{\mathbf{1}\{g^{-1}x_0\in V(\Lambda_\omega,x_0)\}}{m_X(V(\Lambda_\omega,x_0))}$$

and note that $m_X(V(\Lambda_\omega,x)) \geq m_X(B(x,r))$ for any $x\in\operatorname{supp}(\Lambda_\omega)$. Observe that Lemma 3.4.11 above implies that f is measurable, as

$$f(g,\omega) = \chi_F(g^{-1},\Lambda_\omega)\frac{1}{\int\chi_F(g,\Lambda_\omega)dm_G(g)}$$

for every $\omega\in\mathcal{T}$ and $g\in G$. We have that

$$\begin{aligned}\int_{\mathcal{T}}\int_G f(g,\omega)dm_G(g)d\mathbb{P}_\Lambda(\omega) &= \int_{\mathcal{T}}\int_G \frac{\mathbf{1}\{gx_0\in V(\Lambda_\omega,x_0)\}}{m_X(V(\Lambda_\omega,x_0))}dm_G(g)d\mathbb{P}_\Lambda(\omega)\\ &= \int_{\mathcal{T}} \frac{m_X(V(\Lambda_\omega,x_0))}{m_X(V(\Lambda_\omega,x_0))}d\mathbb{P}_\Lambda(\omega) = \mathbb{P}_\Lambda(\mathcal{T}) = i(\Lambda),\end{aligned}$$

where we used the unimodularity of G in the first equality. Apply the refined Campbell formula to f to obtain

$$\begin{aligned}&\int_\Omega\sum_{x\in P_\omega}\int_K f(g_xk,k^{-1}g_x^{-1}\omega)dm_K(k)d\mathbb{P}(\omega)\\ =&\int_\Omega\sum_{x\in P_\omega}\int_K \frac{\mathbf{1}\{k^{-1}g_x^{-1}x_0\in V(k^{-1}g_x^{-1}\Lambda_\omega,x_0)\}}{m_X(V(k^{-1}g_x^{-1}\Lambda_\omega,x_0)))}dm_K(k)d\mathbb{P}(\omega)\\ =&\int_\Omega\sum_{x\in P_\omega}\int_K \frac{\mathbf{1}\{x_0\in V(\Lambda_\omega,g_xkx_0)\}}{m_X(V(\Lambda_\omega,g_xkx_0)))}dm_K(k)d\mathbb{P}(\omega)\end{aligned}$$

$$= \int_\Omega \sum_{x \in P_\omega} \frac{\mathbf{1}\{x_0 \in V(\Lambda_\omega, x)\}}{m_X(V(\Lambda_\omega, x)))} d\mathbb{P}(\omega).$$

Hence

$$i(\Lambda) = \int_\Omega \sum_{x \in P_\omega} \frac{\mathbf{1}\{x_0 \in V(\Lambda_\omega, x)\}}{m_X(V(\Lambda_\omega, x)))} d\mathbb{P}(\omega)$$

The claim now follows by multiplying both sides by $m_X(B(x_0, r))$ and using the formula $D_r(\Lambda) = m_X(B(x_0, r))i(\Lambda)$ from Lemma 3.2.3. □

Remark 3.4.13. We should point out that the formula in the theorem is related to the well-known Palm inversion formula in the Euclidean setting. See for instance [76, Proposition 9.7] or [69, Theorem 5.3] in the Euclidean case and [75, Equation 3.17] for homogeneous spaces.

With the notation from the statement of the theorem above, we also note the following:

Proposition 3.4.14. *If $X \setminus \bigcup_{x \in P_\omega} V(\Lambda_\omega, x)$ is a null-set for every $\omega \in \Omega$, then*

$$1 = \sum_{x \in P_\omega} \mathbf{1}\{x_0 \in V(\Lambda_\omega, x)\}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$.

Proof. For $g \in G$ we have $\sum_{x \in P_\omega} \mathbf{1}\{gx_0 \in V(\Lambda_\omega, x)\} = 0$ if and only if $gx_0 \in X \setminus \bigcup_{x \in P_\omega} V(\Lambda_\omega, x)$. As Λ is G-stationary, we have

$$\begin{aligned} \int_\Omega \sum_{x \in P_\omega} \mathbf{1}\{x_0 \in V(\Lambda_\omega, x)\} d\mathbb{P}(\omega) &= \int_\Omega \sum_{x \in P_\omega} \mathbf{1}\{x_0 \in V(g^{-1}\Lambda_\omega, g^{-1}x)\} d\mathbb{P}(\omega) \\ &= \int_\Omega \sum_{x \in P_\omega} \mathbf{1}\{gx_0 \in V(\Lambda_\omega, x)\} d\mathbb{P}(\omega) \end{aligned}$$

for every $g \in G$. Thus $1 = \int_\Omega \sum_{x \in P_\omega} \mathbf{1}\{x_0 \in V(\Lambda_\omega, x)\} d\mathbb{P}(\omega)$ if and only $1 = \int_\Omega \sum_{x \in P_\omega} \mathbf{1}\{gx_0 \in V(\Lambda_\omega, x)\} d\mathbb{P}(\omega)$ for every $g \in G$ and this is the case if and only if

$$0 = \int_\Omega \mathbf{1}\{gx_0 \in X \setminus \bigcup_{x \in P_\omega} V(\Lambda_\omega, x)\} d\mathbb{P}(\omega).$$

But

$$\int_G \int_\Omega \mathbf{1}\{gx_0 \in X \setminus \bigcup_{x \in P_\omega} V(\Lambda_\omega, x)\} d\mathbb{P}(\omega) dm_G(g) = \int_\Omega m_X(X \setminus \bigcup_{x \in P_\omega} V(\Lambda_\omega, x)) d\mathbb{P}(\omega) = 0$$

and thus $1 = \int_\Omega \sum_{x \in P_\omega} \mathbf{1}\{gx_0 \in V(\Lambda_\omega, x)\} d\mathbb{P}(\omega)$ for almost every $g \in G$ and hence by stationarity for every $g \in G$. As $\sum_{x \in P_\omega} \mathbf{1}\{gx_0 \in V(\Lambda_\omega, x)\} \in \{0, 1\}$ for every $g \in G$ and $\omega \in \Omega$, this implies the claim. □

Remark 3.4.15. Bowen and Radin observed in [28] that Theorem 3.4.12 in conjunction with a result by Böröczky, [24], implies that the so-called tight-simplex packings are optimally dense. Tight-simplex packings are the packings obtained by (face-to-face) tiling $\mathbb{H}^n$ with a regular n-simplex of side length $2r$ and placing spheres of radius r at the vertices of the simplices of this tiling. Radii r for which such a tiling exists are called *tight*. For $n = 2$ there are countably many tight radii. If the dimension of $\mathbb{H}^n$ is higher than 4 no hyperbolic simplex reflection groups exist, see [87, pg. 291].

3.5. Complete Saturation

In this section we collect a few facts about the notion of complete saturation, which will be of importance in the next section. The notion of complete saturation was introduced by Fejes Tóth, Kuperberg and Kuperberg in [50] as a substitute for the notion of optimally dense packing, as it enforces "local optimality" of sphere packings.

Definition 3.5.1. We say that $P \in \mathrm{UD}_{2r}(X)$ is *completely saturated*, if there is no finite subset $F_1 \subset P$ and finite subset $F_2 \subset X$ with $\#F_1 < \#F_2$ such that $(P \setminus F_1) \cup F_2 \in \mathrm{UD}_{2r}(X)$. We call $\mu \in \mathcal{N}^*_{2r}(X)$ *completely saturated* if there is a completely saturated $P \in \mathrm{UD}_{2r}(X)$ with $\mu = \delta_P$.
An r-sphere packing P of X is called *completely saturated* if there is no finite subset $F_1 \subset P$ and finite set F_2 of r-balls in X such that $\#F_1 \leq \#F_2$ and $(P \setminus F_1) \cup F_2$ is an r-sphere packing

In particular, the definition above implies that for any bounded set $B \subset X$ and completely saturated $P \in \mathrm{UD}_{2r}(X)$ the number $\#(B \cap P)$ is determined by $P \setminus B$ and given by

$$N_B(P) := \max\{\#F \mid F \subset \mathrm{UD}_{2r}(X) \text{ finite }, F \cup (P \setminus B) \in \mathrm{UD}_{2r}(X)\}.$$

Definition 3.5.2. Let $\Lambda : (\Omega, \mathbb{P}) \to \mathcal{N}^*_{2r}(X)$ be a random invariant r-sphere packing. We say that Λ is *completely saturated* if Λ_ω is completely saturated for $\mathbb{P}$-almost every $\omega \in \mathbb{P}$.

We mention the following theorem of Bowen, although we will not use it anywhere. Note that it's difficult proof was originally given by Bowen for $X = \mathbb{R}^n$ and $X = \mathbb{H}^n$.

Theorem 3.5.3 (Bowen, [28])**.** *Assume that Λ is random invariant r-sphere packing in $X = \mathbb{R}^n$ or X a symmetric space of noncompact type with $D_r(\Lambda) = \Delta_{\mathrm{prob}}(X, r)$. Then Λ is completely saturated.*

3.6. Periodic sphere packings in symmetric spaces of noncompact type

Throughout this section, X denotes an irreducible Riemannian symmetric space of noncompact type and $G := \mathrm{Iso}(X)_0$. Our aim in this section is a proof of the following theorem:

Theorem 3.6.1. *Let $\mathcal{R}$ denote the set of all radii $r > 0$ for which there exists an optimally dense periodic packing of X by spheres with radius r. Then $\mathcal{R}$ is at most countable.*

This is a direct generalization of a result by Bowen in [28] for hyperbolic space. The main idea of the proof (using Mostow rigidity and the fact that any lattice in G is finitely presented), is also due to Bowen. The approach given here was established during my master's thesis under the supervision of my advisor Tobias Hartnick and is based on a suggestion by Bowen.

3.6.1. Outline of the proof

For every $r \in \mathcal{R}$ we choose an optimally dense periodic sphere packing P_r. We denote by C_r the set of centers of spheres in P_r and by $\Gamma_r = \mathrm{Stab}_G(C_r)$ the symmetry group of P_r (or, equivalently, C_r). Note that Γ_r is a lattice by Lemma 3.4.3. By Theorem B.3.2 we may also fix a finite-index torsion-free normal subgroup $\Gamma'_r \lhd \Gamma_r$ and a strict fundamental domain $F(\Gamma'_r)$ for the action of Γ'_r on X. We observe that the number

$$m_r := \frac{|F(\Gamma'_r) \cap C_r|}{[\Gamma_r : \Gamma'_r]} \in \mathbb{Q} \tag{3.6.1}$$

is independent of the choice of Γ'_r. Indeed, if we replace Γ'_r by a subgroup of index k, then both the numerator and the denominator are multiplied by k. We now denote by $\mathcal{FP}$ the set of isomorphism classes of finitely presented groups and observe that it is countable, since

$$\mathcal{FP} = \bigcup_{n,m \in \mathbb{N}} \{[\mathrm{coker}(f)] \mid f \in \mathrm{Hom}(F_n, F_m) \text{ and } f(F_m) \text{ a normal subgroup}\}.$$

By Theorem B.3.2 we obtain a well-defined map

$$\Phi : \mathcal{R} \to \mathbb{Q} \times \mathcal{FP}, \quad r \mapsto (m_r, [\Gamma_r]), \tag{3.6.2}$$

and thus we have reduced the proof of Theorem 3.6.1 to the following theorem, whose proof will occupy the remainder of this section:

Theorem 3.6.2. *The map Φ from* (3.6.2) *is injective.*

Before we go into details we provide a rough outline of the proof of Theorem 3.6.2. We assume for contradiction that $\Phi(r_0) = \Phi(r_1)$ for some $r_1 > r_0 > 0$. We then abbreviate $P_0 := P_{r_0}$ and $P_1 := P_{r_1}$, and similarly $\Gamma_0 := \Gamma_{r_0}$ and $\Gamma_1 := \Gamma_{r_1}$. Since $\Phi(r_0) = \Phi(r_1)$ we have $\Gamma_0 \cong \Gamma_1$, and we fix an isomorphism $f : \Gamma_0 \to \Gamma_1$. Since $r_1 > r_0$, there is a unique sphere packing P with set of centers C_{r_1} and radius r_0 which is obtained from P_{r_1} by shrinking all spheres. Its symmetry group is Γ_1 (since the symmetries only depend on the positions of the centers). We are going to show:

Lemma 3.6.3. *The packing P is not completely saturated.*

This implies:

Lemma 3.6.4. *The packing P is not optimally dense.*

Lemma 3.6.4 could be deduced from Lemma 3.6.3 and Theorem 3.5.3; however, using the fact that P is periodic there is a much easier way to derive Lemma 3.6.4 directly from Lemma 3.6.3 without appeal to the rather involved Theorem 3.5.3. To conclude the proof of Theorem 3.6.2 we finally show:

Lemma 3.6.5. *The packings P_0 and P have the same density.*

Lemmas 3.6.4 and 3.6.5 taken together imply that P_0 does not have optimal density and thus yield the desired contradiction. For the proof of Lemma 3.6.5 we are going to combine the density formula from Corollary 3.4.6 with a volume rigidity theorem for locally symmetric spaces which follows from Mostow rigidity. In the following two subsections we are going to establish Lemmas 3.6.3, 3.6.4 and 3.6.5; this will finish the proof of Theorem 3.6.2.

3.6.2. Step 1: Exploiting the CAT(0) inequality

The goal of this subsection is to establish Lemma 3.6.3. For this we are going to use the fact that X is a CAT(0) space, see Appendix A. Our spaces X also have the additional property (not shared by general CAT(0) spaces) that every geodesic segment can be extended uniquely to a bi-infinite geodesic (i.e. an isometric embedding $\mathbb{R} \to X$). This implies in particular, that for any two distinct points $q, a \in X$ there exists a unique isometric embedding $\gamma : \mathbb{R} \to X$ with $\gamma(0) = q$ and $\gamma(d_X(a, q)) = a$. Given $s > 0$ we can thus define the point $a_q(s) := \gamma(d_X(a, q) + s)$.

Lemma 3.6.6. *Let $a, b, q \in S$ be three distinct points. Then $d_X(a, b) \leq d_X(a_q(s), b_q(s))$ for all $s \geq 0$.*

Proof. Apply the CAT(0) inequality to the triangle $q, a_q(s), b_q(s)$; since $a \in [q, a_q(s)]$ and $b \in [q, b_q(s)]$ the lemma follows from the corresponding statement in Euclidean geometry. □

This weak version of the CAT(0) inequality is sufficient to deduce the following lemma; roughly speaking, the lemma ensures that if there is a little bit of space between all balls in a sphere packing, the packing cannot be completely saturated since we could just push around enough spheres to make space for a new one. This was originally shown by Bowen and Radin in [28, Lemma 3] and their proof extends directly to CAT(0) spaces, since it only requires Lemma 3.6.6.

Lemma 3.6.7. *If $P \in \mathrm{UD}_{2r}(X)$ and there is a $t > 0$ such that $d_X(x, y) \geq 2r + t$ for all $x, y \in P$, then P is not completely saturated.*

Proof. The strategy is to pick a central point q and a large enough radius R and to push points in $B(q, R)$ away from q "radially", using the assumption that there is a little bit of space between neighboring points. If we choose R correctly, we should have enough space to insert a new point at q.
Hence we start by choosing a $q \in \mathrm{supp}(P^r) \setminus P$. Fix $k \in \mathbb{N}$ such that $kt > 2r \geq (k-1)t$ and set $R := k(2r + t)$.

Given $x \in P$ with $d_X(x, q) < R$, we can find a j_x such that $0 < j_x \leq k$ and

$$R - 2(j_x - 1)(r + t) > d_X(q, x) \geq R - 2j_x(r + t).$$

Define a function $f : P \to X$ by

$$f(x) := \begin{cases} x_q(0) = x, & \text{if } d_X(q, x) \geq R, \\ x_q(j_x t), & \text{if } d_X(q, x) < R. \end{cases}$$

Set $P' := f(P) \cup \{q\}$. We claim that $P' \in \mathrm{UD}_{2r}(X)$ was created from P by replacing the points of P in $B(q, R)$ with a set containing one more point. The only things left to show are that $P' \in \mathrm{UD}_{2r}(X)$ and that f is injective. We are finished if we show $d_X(f(x), f(y)) \geq r$ for all $x \neq y \in P$ and $d_X(f(x), q) \geq r$ for all $x \in P$.

The case $d_X(f(x), q) \geq r$: Let $x \in P$ and choose $0 \leq j \leq k$ with $f(x) = x_q(jt)$. Then

$$d_X(f(x), q) = d_X(x_q(jt), q) = d_X(x, q) + jt \geq R - 2j(r + t) + jt = (k - j)(2r + t)$$

and note that these estimates also hold for $j = 0$. Thus, if $d_X(f(x), q) < r$, then $f(x) = x_q(kt)$. But then

$$d_X(q, f(c)) = d_X(q, c) + kt > kt > 2r.$$

We will reuse the estimate

$$d_X(f(x), q) \geq (k - j)(2r + t) \tag{$*$}$$

we just obtained above in the next case.

The case $d_X(f(x), f(y)) \geq r$: Take now $x, y \in P$. If $d_X(x, q) \geq R$ and $d_X(y, q) \geq R$, then

$$d_X(f(x), f(y)) = d_X(x, y) \geq 2r + t > 2r.$$

Assume now that one of x, y has distance less then R to q and that $d_X(f(x), f(y)) < 2r$. Without loss of generality we can assume that $d_X(x, q) < R$. Write $f(x) = x_q(jt)$ with $1 \leq j \leq k$ and $f(y) = y_q(j't)$ with $0 \leq j' \leq k$. Now

$$\begin{aligned} d_X(f(y), q) &\leq d_X(f(y), f(x)) + d_X(f(x), q) \\ &< 2r + d_X(x, q) + jt \\ &< 2r + R - 2(j - 1)(r + t) + jt \\ &= (k - j + 2)(2r + t) \end{aligned} \tag{$**$}$$

By ($*$) and ($**$) we now obtain

$$(k - j')(2r + t) < (k - j + 2)(2r + t)$$

and thus

$$0 < (j' - j + 2)(2r + t).$$

Hence

$$j < j' + 2.$$

If $j' = 0$, then $j = 1$. If $j' \neq 0$, we have $d_X(q, y) < R$ and thus we get

$$j' < j + 2$$

by switching x and y in the derivation above. Thus

$$j \leq j' + 1 \leq j + 2$$

and therefore we can assume that $j = j'$ or $j = j' + 1$. Now 3.6.6 shows

$$\begin{aligned} 2r + t \leq d_X(x, y) &\leq d_X(x_q(jt), y_q(jt)) \\ &= d_X(f(x), y_q(jt)) \\ &\leq d_X(f(x), f(y)) + d_X(y_q(j't), y_q(jt)) \\ &< 2r + d_X(y_q(j't), y_q(jt)) \\ &\leq 2r + t, \end{aligned}$$

which is a contradiction. □

This now immediately implies Lemma 3.6.3:

Proof of Lemma 3.6.3. Since the packing P obtained from P_1 by shrinking the radius of spheres down from r_1 to r_0, we can apply the previous lemma with $t := 2(r_1 - r_0)$: Since the distance of any two centers of distinct spheres in P is at least $2r_1 = 2r_0 + t$, we deduce that P is not completely saturated. □

3.6.3. Step 2: Lattices with large girth

We now give a short direct argument for Lemma 3.6.4, which avoids the use of Theorem 3.5.3. Instead we are going to use the following much easier lemma in the setting of periodic sphere packings.

Lemma 3.6.8. *Suppose that P is a periodic sphere packing of X. If P is Bowen–Radin optimally dense, then P is completely saturated.*

Proof. Assume that P is not completely saturated and denote by r_P, C_P and Γ_P respectively the radius of balls, the set of centers of balls in P and the symmetry group $\Gamma_P := \mathrm{Stab}_G(C_P)$. By definition, we can then find a packing P' such that $r_P = r_{P'}$, a point $q \in C_{P'}$ such that $q \notin C_P$ and some $R > 0$ such that $C_P \setminus B(q, R) = C_{P'} \setminus B(q, R)$ and $\#(C_P \cap B(q, R)) < \#(C_{P'} \cap B(q, R))$. By Theorem B.3.2.(iii) we find a torsion-free subgroup $\Gamma \subset \Gamma_P$ with $\mathrm{girth}(\Gamma \curvearrowright X) > 4R$. By Lemma B.3.3 this implies that the projection $\pi|_{B(q,R)}$ is injective, hence by Lemma B.2.6 there exists a strict fundamental domain $F \subset S$ of Γ which contains $B(q, R)$. Now let

$$C'' := \bigcup_{g \in \Gamma} g(C_{P'} \cap F).$$

Since P is the same as P' outside of $B(q, R)$, it follows that there is a sphere packing P'' of spheres with radius r_P associated to C''. Since F is the fundamental domain of a finite index subgroup of Γ_P and the relative volume of the packings taken up in the fundamental domain is larger for P'' than P, it follows from Corollary 3.4.6 that P'' has a higher density than for P. Thus P is not optimally dense. □

Lemma 3.6.4 now follows from Lemma 3.6.3 and either Theorem 3.5.3 or (more efficiently) Lemma 3.6.8.

3.6.4. Step 3: Exploiting volume rigidity

In this subsection we establish Lemma 3.6.5 and thereby finish the proof of Theorem 3.6.2. For this we have to show that the sphere packings P_0 and P have the same density. This is a consequence of the classical volume rigidity, due to Mostow:

Theorem 3.6.9 (Volume rigidity). *If M_0 and M_1 are compact orientable locally symmetric spaces with isomorphic fundamental groups whose universal cover is the same irreducible symmetric space S of non-compact type, then they have the same volume.*

Proof. If the dimension of M_0 and M_1 is at least 3, then M_0 and M_1 are isometric by Mostow's strong rigidity theorem (cf. [79]), hence in particular have the same volume. Otherwise M_0 and M_1 are compact orientable hyperbolic surfaces of the same Euler characteristic, hence have the same volume by Gauß-Bonnet. □

Proof of Lemma 3.6.5. We recall that the symmetry groups of P_0 and P are given by Γ_0 and Γ_1 respectively and that we have fixed an isomorphism $f : \Gamma_0 \to \Gamma_1$. Since $\Phi(r_0) = \Phi(r_1)$ we also have $m_{r_0} = m_{r_1}$.
By Theorem B.3.2 we now pick a finite index torsion-free subgroup $\Gamma_0' < \Gamma_0$ and define $\Gamma_1' := f(\Gamma_0')$; by the same theorem we find strict fundamental domains F_0 and F_1 for Γ_0' and Γ_1' in X respectively.
Since Γ_0' and Γ_1' are isomorphic via f, the quotients $M_0 := \Gamma_0' \backslash S$ and $M_1 := \Gamma_1' \backslash S$ are compact orientable locally symmetric manifolds with isomorphic fundamental groups and universal cover S. By Theorem 3.6.9 they thus have the same volume, which is given by

$$m_S(F_0) = m_S(F_1). \tag{3.6.3}$$

On the other hand, since $m_{r_0} = m_{r_1}$ and $[\Gamma_0 : \Gamma_0'] = [\Gamma_1 : \Gamma_1']$ we deduce from (3.6.1), that

$$\#(C_{P_0} \cap F_0) = \#(C_{P_1} \cap F_1) = \#(C(P) \cap F_1). \tag{3.6.4}$$

Combining (3.6.3) and (3.6.4) with Corollary 3.4.6 we finally obtain

$$D(P_0) = \frac{\#(C_{P_0} \cap F_0) \cdot m_X(B(x_0, r_0))}{m_X(F_0)} = \frac{\#(C_P \cap F_1) \cdot m_X(B(x_0, r_0))}{m_X(F_1)} = D(P),$$

i.e. P_0 and P have the same density. □

4. Linear programming bounds

In this chapter we will prove to general linear programming bounds on the density and energy of random sphere packings. These linear programming bounds generalize previously known linear programming bounds. For the density these are due to Cohn and Elkies, [36], for $\mathbb{R}^n$, Delsarte, [46], for finite homogeneous spaces, and Kabatjanskiĭ and Levenšteĭn, [67], for compact two-point homogeneous spaces. Additionally Cohn, Lurie and Sarnak have obtained linear programming bounds for periodic sphere packings in $\mathbb{H}^n$ in [41]. The linear programming bounds for the energy are due to Cohn and Kumar, [37], for periodic point sets in $\mathbb{R}^n$ and for point sets in compact two-point homogeneous spaces and Cohn and Courcy-Ireland, [34], for general point sets in $\mathbb{R}^n$.

4.1. Convenient Gelfand pairs

For the linear programming bounds we will assume that our homogeneous spaces have a rich enough harmonic analysis and ergodic theory. We only impose the ergodic theoretic assumptions to guarantee the interpretability of the notions of energy and density of point processes in terms of energy and density of deterministic point sets/sphere packings.

Definition 4.1.1. (i) Assume that G is a Lie group and $K < G$ is a compact subgroup such that (G, K) is a Gelfand pair.

(ii) Assume that d_X is a G-invariant, proper and continuous metric on $X := G/K$ such that $m_X(B(x,r)) = m_X(\overline{B}(x,r))$ for all $x \in X$ and $r > 0$.

(iii) Assume the invariant pointwise ergodic theorem holds for $((G_t)_{t\geq 1}, C(\mathrm{UD}_{2r}(X)))$, where $G_t := \pi^{-1}(B(x_0,t))$ (see Definition 2.4.2).

(iv) Let $\mathcal{S}(G, K)$ be a choice of Schwartz-like function space for (G, K) (see Definition 1.5.1).

Then we refer to $(G, K, d_X, \mathcal{S}(G, K))$ (or just (G, K)) as a *convenient Gelfand pair*. If the sequence $(G_t)_{t>0}$ is in addition very convenient, then we refer to $(G, K, d, \mathcal{S}(G, K))$ as a *very convenient Gelfand pair*.

Example 4.1.2. Recall that the sequence defined by $G_t := \pi^{-1}(B(x_0, t))$ is very convenient by Example 2.5.8 if

(i) $G = \mathbb{R}^n$, $K = \{0\}$, $X = \mathbb{R}^n$ and d_X is the ordinary Euclidean metric.

(ii) $G = \mathcal{H}_n \rtimes \mathrm{U}(n)$, $K = \mathrm{U}(n)$, $X = \mathcal{H}_n$ and d_X is the Cygan-Koranyi metric.

(iii) G is a semisimple Lie group with no compact factors and finite center, K is a maximal compact subgroup and d_X is the Cartan-Killing metric on the symmetric space $X = G/K$ of noncompact type.

(iv) (G, K) is a Riemannian symmetric pair of compact type, d_X is the Cartan-Killing metric on G/K.

Note that in all of these cases the invariant pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$ (see Theorem 2.4.13 and Theorem 2.4.15). Further note that we have the following Schwartz-like function spaces in these cases:

(i) $\mathscr{S}(\mathbb{R}^n)$ for $G = \mathbb{R}^n$ and $K = \{0\}$.

(ii) $\mathscr{S}^1(G, K)$, where G is a semisimple Lie group with finite center and no compact factors, K a maximal compact subgroup.

(iii) $\mathscr{S}(\mathcal{H}_n \rtimes U(n), U(n))$ for $G = \mathcal{H}_n \rtimes U(n)$ and $K = U(n)$.

(iv) $C^\infty(G, K)$ for (G, K) a symmetric pair of compact type.

Hence we have a very convenient Gelfand pair in each of these cases.

4.2. Density

4.2.1. Cohn and Elkies linear programming bound

Before we come to the proof of our general linear programming bound for the density, we will first sketch Cohn and Elkies proof of the linear programming bound for the Euclidean packing density, see [36]. Their proof is based on a reduction to periodic sphere packings and the Poisson summation formula. Our proof of the general bound is in some sense very similar, but replaces the Poisson summation formula with the Plancherel formula for the autocorrelation distribution. In contrast to Cohn and Elkies approach we do not need a reduction to periodic packings, but bound the density of a general random invariant r-sphere packing.

Cohn and Elkies first observe that a bound on the density of all periodic r-sphere packings implies a bound for $\triangle(\mathbb{R}^n, r)$ by Theorem 3.1.9. Assume now that P is a periodic r-sphere packing of $\mathbb{R}^n$. Then there is a lattice $\Gamma \leq \mathbb{R}^n$ and finitely many points $x_1, \ldots, x_k \in \mathbb{R}^n$ such that $P = \{B(\gamma + x_i, r) \mid 1 \leq i \leq k, \gamma \in \Gamma\}$ and for $i \neq j$ the orbits Γx_i and Γx_j are disjoint. Then

$$D(P) = m_{\mathbb{R}^n}(B(0, r)) \frac{k}{|\Gamma|}.$$

We will now use Poisson summation to bound $k \cdot |\Gamma|^{-1}$. Note that this is just the intensity of the random invariant r-sphere packing modeled on P. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ has the properties

(W1) $f(x) \leq 0$ if $\|x\|_2 \geq 2r$,

(W2) $\widehat{f} \geq 0$ and $\widehat{f}(0) > 0$,

(W3) f is a radial Schwartz function.

Then we will call f a *Euclidean witness function*, see also Definition 4.2.1. Now the

Poisson summation formula states

$$\sum_{\gamma\in\Gamma} f(\gamma+v) = \frac{1}{|\Gamma|}\sum_{t\in\widehat{\Gamma}} \widehat{f}(t)e^{-2\pi i\langle v,t\rangle}$$

for every $v \in \mathbb{R}^n$. Here $\widehat{\Gamma}$ denotes the dual lattice to Γ. Thus

$$\begin{aligned}\sum_{1\leq i,j\leq k}\sum_{\gamma\in\Gamma} f(\gamma+x_i-x_j) &= \frac{1}{|\Gamma|}\sum_{t\in\widehat{\Gamma}}\sum_{1\leq i,j\leq k} \widehat{f}(t)e^{2\pi i\langle x_i-x_j,t\rangle}\\ &= \frac{1}{|\Gamma|}\sum_{t\in\widehat{\Gamma}}\widehat{f}(t)\left(\sum_{1\leq i\leq k} e^{2\pi i\langle x_i,t\rangle}\right)\left(\sum_{1\leq j\leq k} e^{2\pi i\langle -x_j,t\rangle}\right)\\ &= \frac{1}{|\Gamma|}\sum_{t\in\widehat{\Gamma}}\widehat{f}(t)\left|\sum_{1\leq i\leq k} e^{2\pi i\langle x_i,t\rangle}\right|^2 \geq \frac{1}{|\Gamma|}\widehat{f}(0)\left|\sum_{1\leq i\leq k} e^{2\pi i\langle x_i,0\rangle}\right|^2\\ &= \frac{1}{|\Gamma|}\widehat{f}(0)k^2.\end{aligned}$$

Additionally

$$\sum_{i\leq i,j\leq k}\sum_{\gamma\in\Gamma} f(\gamma+x_i-x_j) \leq \sum_{i=1}^{k} f(x_i-x_i) = kf(0).$$

Thus

$$\frac{k}{|\Gamma|} \leq \frac{f(0)}{\widehat{f}(0)}$$

and hence

$$D(P) = m_{\mathbb{R}^n}(B(0,r))\frac{k}{|\Gamma|} \leq m_{\mathbb{R}^n}(B(0,r))\frac{f(0)}{\widehat{f}(0)}.$$

Considering the above argument carefully, we see that we did not use the full strength of the Poisson summation formula. We used that there is a measure η_P associated to P (specifically $\sum_{1\leq i,j\leq k}\sum_{\gamma\in\Gamma}\delta_{\gamma+x_i-x_j}$) which has a "diagonal structure" and that $\widehat{\eta}_P$ is a positive measure with an atom at the trivial character, which encodes the intensity of the random invariant r-sphere packing modeled on P.
In the next section we will see that this behaviour is replicated by the autocorrelation measure of (not necessarily periodic) random invariant r-sphere packings.

4.2.2. A general linear programming bound

We now come to the proof of the general linear programming bounds on density for convenient Gelfand pairs. We first announced these bounds in [97] and gave a full proof in [98].

Definition 4.2.1. Given a convenient Gelfand pair $(G, K, d_X, \mathcal{S}(G,K))$ we define the space $\mathcal{W}(X,r)$ of *witness functions* for $X = G/K$ as the set of functions $f : G \to \mathbb{R}$ such that

(W1) $f(g) \leq 0$ if $d_X(gx_0, x_0) \geq 2r$,

(W2) $\widehat{f} \geq 0$ and $\widehat{f}(\mathbf{1}) > 0$,

(W3) $f \in \mathcal{S}(G, K)$.

Theorem 4.2.2. *Let $(G, K, d, \mathcal{S}(G, K))$ be a convenient Gelfand pair and let Λ be a random invariant r-sphere packing in $X = G/K$. Assume that $f \in \mathcal{W}(X, r)$. Then*

$$i(\Lambda) \leq \frac{f(e)}{\widehat{f}(\mathbf{1})}.$$

Hence

$$\triangle_{\mathrm{BR}}(X, r) \leq m_X(B(x_0, r))\frac{f(e)}{\widehat{f}(\mathbf{1})}.$$

Proof. Lemma 2.7.4 implies that

$$\widehat{\eta_\Lambda^+}(\widehat{f}) = \widehat{\eta}_\Lambda(\widehat{f}) + i(\Lambda)^2\widehat{m_G}(\widehat{f}) \geq i(\Lambda)^2\widehat{f}(\mathbf{1}),$$

where we have used condition (W2) and the fact that $\widehat{\eta}_\Lambda$ is a positive measure by the Godement-Plancherel theorem. Assume first that the support of f is compact. Fix $R > 0$ such that $m_X(B(x_0, R)) = 1$ and set $b \coloneqq \chi_{B(x_0,R)}$. We set $P \coloneqq \mathrm{supp}(\Lambda)$. Then

$$\begin{aligned}
\eta_\Lambda^+(f) &= \mathbb{E}\left[\sum_{y\in P}\sum_{x\in P\cap B(x_0,R)} f(\sigma(x)^{-1}\sigma(y))\right] \leq \mathbb{E}\left[\sum_{x\in P\cap B(x_0,R)} f(\sigma(x)^{-1}\sigma(x))\right] \\
&= \mathbb{E}\left[f(e)\sum_{x\in P\cap B(x_0,R)} 1\right] = f(e)\mathbb{E}\left[\#(P\cap B(x_0,R))\right] = f_n(e)\mathbb{E}\left[\Lambda(B(x_0,R))\right] \\
&= f(e)i(\Lambda)m_X(B(x_0,R)) = f(e)i(\Lambda),
\end{aligned}$$

using property (W1) for the inequality and Lemma 2.3.1 in the second to last equality. Hence $\widehat{\eta_\Lambda^+}(\widehat{f}) \leq f(e)i(\Lambda)$.
If the support of f is non-compact, set $f_n \coloneqq g_n f$, where $(g_n)_{n\geq 1}$ is a sequence in $C_c^\infty(G, K)$ as in Definition 1.5.1. Note that the functions f_n satisfy (W1) and that $f(e) = f_n(e)$ for all n. As f_n is compactly supported, the calculation above implies

$$\widehat{\eta_\Lambda^+}(\widehat{f}) = T_{\eta_\Lambda^+}(f) = \lim_{n\to\infty}\eta_\Lambda^+(f_n) \leq \lim_{n\to\infty} i(\Lambda)f_n(e) = i(\Lambda)f(e).$$

Thus in either case

$$i(\Lambda)f(e) \geq \widehat{\eta_\Lambda^+}(\widehat{f}) \geq i(\Lambda)^2\widehat{f}(\mathbf{1})$$

and we obtain

$$i(\Lambda) \leq \frac{f(e)}{\widehat{f}(\mathbf{1})}$$

as $i(\Lambda) > 0$. Now the bound on $\triangle_{\mathrm{prob}}(X, r)$ follows directly from Proposition 3.2.3. Now Theorem 3.3.3 implies $\triangle_{\mathrm{BR}}(X, r) = \triangle_{\mathrm{prob}}(X, r)$ and the claim follows. □

Remark 4.2.3. In the absence of the invariant pointwise ergodic theorem, the method above still yields estimates for the probabilistic optimal packing density.

Remark 4.2.4. (i) As we pointed out in Remark 3.4.10, it is not known for $n > 2$ whether $\Delta_{\mathrm{BR}}(\mathbb{H}^n, r)$ can be approximated by densities of periodic sphere packings. Thus in the case $X = \mathbb{H}^n$, $n > 2$, Theorem 4.2.2 does not follow from the linear programming bound on densities of periodic sphere packings by Cohn, Lurie and Sarnak, [41].

(ii) There are linear programming bounds related to periodic structures we should point out. In [26] Bourque and Petri develop linear programming bounds on various invariants of compact Riemann surfaces, such as the Kissing number and systole, using a variant of the Selberg trace formula. They are able to adapt Cohn and Elkies approach to numeric optimization of the linear programming bounds as they use a variant of the Selberg trace formula which relates the spectrum of the Laplace operator with the length of geodesics via the ordinary Fourier transform on $\mathbb{R}$.

We also record the following curious observation:

Proposition 4.2.5. *Let G be a (connected) simple Lie group without compact factors and with finite center and K a maximal compact subgroup. Let d be the Cartan-Killing metric and let $G = KAN$ be an Iwasawa decomposition of G as in Subsection 1.4.2. Then $\mathfrak{a}$ with metric $d_{\mathfrak{a}}$ induced by the Killing form and the Haar measure $m_{\mathfrak{a}}$ with the standard normalization is a Euclidean space and $(\mathfrak{a}, \{0\}, d_{\mathfrak{a}}, \mathscr{S}(\mathfrak{a}))$ is a convenient Gelfand pair. If $f \in \mathcal{W}(G/K, r)$, then the Abel transform $\mathcal{A}f$ is in $\mathcal{W}(\mathfrak{a}, r)$.*

Proof. [64, Chapter VI, Exercise B2.(iv)] states that $d_X(x_0, nax_0) \geq d_X(x_0, ax_0)$ for all $a \in A, n \in N$. Hence, if $d_X(ax_0, x_0) > 2r$ we have that $d_X(anx_0, x_0) > 2r$. Thus, for $H \in \mathfrak{a}$,

$$\mathcal{A}f(H) = e^{\rho(H)} \int_N f(\exp(H)n)dm_N(n) \leq 0,$$

if $\sqrt{\kappa(H, H)} = d_X(\exp(H)x_0, x_0) > 2r$. Moreover $\mathcal{F}(\mathcal{A}f) = \mathcal{H}(f) \geq 0$ on $\mathfrak{a}^*$. □

4.3. A linear programming bound for the energy

In this section we prove a lower bound on the energy of a uniformly discrete G-equivariant point process with a fixed intensity. This bound is analogous to the linear programming bounds on the energy of point sets that have been obtained by Cohn and Kumar in [37] for periodic point sets and by Cohn and Courcy-Ireland in [34] for more general point sets. The proof of Cohn and Kumar's linear programming bound uses methods that are reminiscent of the methods used by Cohn and Elkies to prove the Euclidean linear programming bound on packing density. The proof of our bound again uses the idea of replacing Poisson summation with the Godement-Plancherel theorem/the spherical Bochner theorem for distributions.

Theorem 4.3.1. *Assume that $(G, K, d, \mathcal{S}(G,K))$ is a convenient Gelfand pair. Let Λ be a uniformly discrete point process in X with $i(\Lambda) \neq 0$. Assume that $p : [0,\infty) \to [0,\infty)$ is measurable and that $f \in \mathcal{S}(G,K)$ satisfies*

(i) $f(g) \leq p \circ d_0(g)$ for all $g \in G \setminus K$ and

(ii) $\widehat{f} \geq 0$.

Then

$$E_p(\Lambda) \geq i(\Lambda)\widehat{f}(\mathbf{1}) - f(e). \tag{4.3.1}$$

Proof. Assume first that f is compactly supported. Then

$$\begin{aligned}
i(\Lambda)E_p(\Lambda) &= \eta_\Lambda^+(p \circ d_0) - i(\Lambda)p(0)\\
&= \mathbb{E}\left[\int_X \int_X p(d_X((\sigma(x)^{-1}\sigma(y)x_0, x_0))b(x)d\Lambda_\omega(x)d\Lambda_\omega(y)\right] - i(\Lambda)p(0)\\
&= \mathbb{E}\left[\int_X \int_{X\setminus\{y\}} p(d_X(\sigma(x)^{-1}\sigma(y)x_0, x_0))b(x)d\Lambda_\omega(x)d\Lambda_\omega(y)\right]\\
&\quad + \underbrace{\mathbb{E}\left[\int_X p(d_X(x_0, x_0))b(y)d\Lambda_\omega(y)\right]}_{=p(0)i(\Lambda)} - i(\Lambda)p(0)\\
&\geq \mathbb{E}\left[\int_X \int_{X\setminus\{y\}} f(\sigma(x)^{-1}\sigma(y))b(x)d\Lambda_\omega(x)d\Lambda_\omega(y)\right]\\
&= \mathbb{E}\left[\int_X \int_{X\setminus\{y\}} f(\sigma(x)^{-1}\sigma(y))b(x)d\Lambda_\omega(x)d\Lambda_\omega(y)\right] + i(\Lambda)f(e) - i(\Lambda)f(e)\\
&= \mathbb{E}\left[\int_X \int_{X\setminus\{y\}} f(\sigma(x)^{-1}\sigma(y))b(x)d\Lambda_\omega(x)d\Lambda_\omega(y)\right] + f(e)\mathbb{E}\left[\int_X b(x)d\Lambda_\omega(x)\right]\\
&\quad - i(\Lambda)f(e)\\
&= \eta_\Lambda^+(f) - i(\Lambda)f(e).
\end{aligned}$$

By Lemma 2.7.4 we have

$$\eta_\Lambda^+(f) = T_\Lambda^+(f) = \widehat{\eta}_\Lambda^+(\widehat{f}) = i(\Lambda)^2\widehat{f}(\mathbf{1}) + \widehat{\eta}_\Lambda(\widehat{f}) \geq i(\Lambda)^2\widehat{f}(\mathbf{1}),$$

and thus the claim follows in this case. Assume now that $f \in \mathcal{S}(G,K)$ is not compactly supported. As f is in $\mathcal{S}(G,K)$, there is a sequence $(g_n)_{n\geq 1}$ in $C_c^\infty(G,K)$ with $g_n f \to f$ in $\mathcal{S}(G,K)$, $g_n \geq 0$ and $g_n(e) = 1$. Hence by the calculation above

$$i(\Lambda)E_p(\Lambda) \geq \eta_\Lambda^+(fg_n) - i(\Lambda)f(e)g_n(e)$$

and taking $n \to \infty$ we obtain

$$i(\Lambda)E_p(\Lambda) \geq \tilde{T}_\Lambda^+(f) - i(\Lambda)f(e) = \widehat{\eta}_\Lambda^+(\widehat{f}) - i(\Lambda)f(e) \geq i(\Lambda)^2\widehat{f}(\mathbf{1}) - i(\Lambda)f(e). \qquad \square$$

Corollary 4.3.2. *Let $(G, K, d, \mathcal{S}(G,K))$ be a convenient Gelfand pair. Assume that*

$p : [0, \infty) \to [0, \infty)$ *is measurable,* $\delta > 0$ *and that* $f \in \mathcal{S}(G, K)$ *satisfies*

(i) $f(g) \leq p \circ d_0(g)$ *for all* $g \in G \setminus K$ *and*

(ii) $\widehat{f} \geq 0$.

Then

$$E_{\text{stoch}}(p, \delta) \geq \delta \widehat{f}(\mathbf{1}) - f(e).$$

This is an analogue to the linear programming bound, [39, Proposition 1.6], used by Cohn, Kumar, Miller, Radchenko and Viazovska in their proof of the universal optimality of the E8 lattice. Note that their notion of density corresponds to our notion of intensity.

4.4. Discussion of the density bound

There are two related questions in the context of the linear programming bounds that we should address. The first is why these bounds are called linear programming bounds and the second is the construction of concrete functions that give bounds.

The name "linear programming bounds" comes from the case of finite Gelfand pairs (G, K) such that $(G/K, d_X)$ is two-point homogeneous, which was historically the first case where the linear programming bounds (on density) were discovered. As $(G/K, d_X)$ is two-point homogeneous, any bi-K-invariant function on G only depends on the radial distance $d_X(gx_0, x_0)$. In this case the spherical functions are usually given in radial coordinates by a set of orthogonal polynomials $\{P_n\}_{n \geq 0}$, where we assume that $P_0 \equiv 1$ induces the trivial character. Making the ansatz

$$f = \sum_{i=0}^{n} \lambda_i P_i,$$

the conditions on witness functions can be expressed as

(W1) $\sum_{i=0}^{n} \lambda_i P_i(d_X(gx_0, x_0)) \leq 0$ for $d_X(gx_0, x_0) \geq 2r$,

(W2) $\lambda_i \geq 0$ and $\lambda_0 > 0$,

(W3) this third condition is trivial satisfied.

We can further enforce $\lambda_0 = 1$. Then we want to minimize

$$\sum_{i=0}^{n} \lambda_i P_i(0)$$

as

$$\Delta_{\text{BR}}(X, r) \leq m_X(B(x_0, r)) \sum_{i=0}^{n} \lambda_i P_i(0).$$

These conditions specify a genuine linear program on $(\lambda_0, \ldots, \lambda_n)$, as (W1) consists of finitely many linear inequalities.

If $G = \text{SO}(n) \rtimes \mathbb{R}^n$, $K = \text{SO}(n)$ with the Euclidean distance, then this approach no longer works. Cohn and Elkies observed in [36] that one can use Laguerre polynomials to obtain a method of constructing bounds. More precisely, in radial coordinates they make the ansatz

$$f(r) = \sum_{i=0}^{k} \lambda_i L_i^{\alpha}(2\pi r^2) e^{-\pi r^2},$$

where $\alpha = n/2 - 1$. As the functions $L_i^{\alpha}(2\pi r^2)e^{-\pi r^2}$ are $(-1)^i$-eigenfunctions of the Fourier transform (in radial coordinates), one sees that

$$\widehat{f}(r) = \sum_{i=0} (-1)^i \lambda_i L_i^{\alpha}(2\pi r^2) e^{-\pi r^2}.$$

Now one can choose the coefficients such that f and $\widehat{f}$ satisfy the conditions (W1) and (W2) by enforcing single zeros, wherever a sign change is needed, and double zeros, wherever a sign change should be avoided.

If $G = \mathrm{SO}(n,1)_0$, $K = \mathrm{SO}(n)$ with the hyperbolic distance on G/K, this approach fails, as there are no eigenfunctions of the spherical transform in this case. In [73] Koornwinder constructed an orthogonal set of functions $\{r_n\}_{n\geq 0}$ in $L^2(G,K)$ which is mapped to an orthogonal set $\{s_n\}_{n\geq 0}$ of functions in $L^2(PS(G,K), \widehat{\delta_e})$. Both of these sets of orthogonal functions converge in the "zero-curvature limit" to the eigenfunctions of the radial Euclidean Fourier transform described above (up to scaling of the argument). Sadly this family seems unsuitable for application to our linear programming bound, as the functions $\{s_n\}_{n\geq 0}$ seem to be supported on the support of the Plancherel measure $\widehat{\delta_e}$, which does not contain the trivial character.

There are two results that we should point out. The first is the fact that the trivial bound $\Delta_{\mathrm{BR}}(X,r) \leq 1$ can always be recovered:

Proposition 4.4.1. *Let $(G, K, d, \mathcal{S}(G,K))$ be a convenient Gelfand pair. Then for every $\delta > 0$ and $r > 0$ there is a $f \in \mathcal{W}(X,r)$ such that*

$$\Delta_{\mathrm{BR}}(X,r) \leq \frac{f(e)}{\widehat{f}(\mathbf{1})} \leq 1 + \delta.$$

In other words Theorem 4.2.2 recovers the trivial bound

$$\Delta_{\mathrm{BR}}(X,r) \leq 1$$

for all $r > 0$.

Proof. Set $B_t := \pi^{-1}(B(x_0,t))$. For $0 < \varepsilon < r$ choose a real-valued symmetric smooth bump function $\chi_\varepsilon : G \to [0,1]$ in $C_c^\infty(G,K)$ such that $\chi_\varepsilon|_{B_{r-\varepsilon}} \equiv 1$ and $\chi_\varepsilon|_{G\setminus B_r} = 0$. Note that such a function can be obtained from a smooth bump function by symmetrization and K-averaging. Now $\chi_\varepsilon * \chi_\varepsilon^*(g) = 0$ for $g \notin B_{2r}$ and $\widehat{\chi_\varepsilon * \chi_\varepsilon^*} = |\chi_\varepsilon|^2 \geq 0$. Then

$$\frac{\chi_\varepsilon * \chi_\varepsilon^*(e)}{\widehat{\chi_\varepsilon * \chi_\varepsilon^*}(\mathbf{1})} = \frac{\int \chi_\varepsilon(g)^2 dm_G(g)}{\left(\int \chi_\varepsilon(g) dm_G(g)\right)^2} \leq \frac{m_G(B_r)}{m_G(B_{r-\varepsilon})^2} = \frac{m_X(B(x_0,r))}{m_G(B(x_0,r-\varepsilon))^2}.$$

Thus Theorem 4.2.2 implies

$$\Delta_{\mathrm{BR}}(X,r) \leq m_X(B(x_0,r))\frac{m_X(B(x_0,r))}{m_X(B(x_0,r-\varepsilon))^2}$$

for all $0 < \varepsilon < r$. $\square$

The second is a result by Cohn and Zhao, [41], that the linear programming bound, if shown for a sufficiently large function space (for instance $C_c(G,K)$), will always beat the hyperbolic version of the so-called Kabatjanskiĭ–Levenšteĭn bound. This bound is obtained from the linear programming bound on spherical codes in conjunction with a geometric argument, see for instance [41]. Cohn and Zhao show that there is a function $f \in C_c(G,K)$ satisfying (W1) and (W2) such that if $f \in \mathcal{S}(G,K)$, f yields a linear programming bound smaller than the Kabatjanskiĭ–Levenšteĭn bound.

More precisely let $g : [-1,1] \to \mathbb{R}$ be a witness function for the linear programming bound on the density of a spherical code with angular separation θ in S^{n-1} (given in the coordinate $\cos(\theta)$) and let $A^{LP}(n,\theta) = \frac{g(e)}{\widehat{g}(\mathbf{1})}$ be the bound on the size of such a code. They show that

$$m_{\mathbb{H}^n}(B(x_0,r))\frac{f(e)}{\widehat{f}(\mathbf{1})} \leq \sin^{n-2}(\theta/2)A^{LP}(n,\theta), \qquad \theta \in [\pi/3,\pi].$$

Here the function f is obtained by first defining R via $\sinh(R) = \sinh(r)/\sin(\theta/2)$ and setting

$$\tilde{f}(x,y) := \int_{B(x,R)\cap B(y,R)} g(\cos(\angle_z(x,y)))dz.$$

Then they observe that $\tilde{f}$ only depends on $d_{\mathbb{H}^n}(x,y)$ (one can rewrite $\cos(\angle_z(x,y))$ using the hyperbolic law of cosines) and thus defines a radial function f. Sadly the Harish-Chandra L^1-Schwartz space does not seem to contain this f, as f does not appear to be smooth. It might be possible to still make some progress towards the function space required by Cohn and Zhao. To circumnavigate the issue, one could either try to directly approximate f by Schwartz functions, as in [33, Section 4] for the Euclidean case, or by using Harish-Chandra L^1-Schwartz space analogues of Weil-Schwartz envelopes, similar to [35, Proposition 3.5.] in the Euclidean case.

Finally we want to sketch an approach for numeric determination of bounds on $\Delta_{\mathrm{BR}}(\mathbb{H}^3,r)$. This approach is based on using the hyperbolic heat kernel in conjunction with partial integration and was first used to bound the density of periodic packings in $\mathbb{H}^3$ during a REU at Columbia University led by Yakov Kerzhner, though this work is not published. Alex Blumenthal, who participated in this REU, privately communicated to me that these bounds were used to bootstrap bounds for $\mathbb{H}^7$, $\mathbb{H}^{13}$ and $\mathbb{H}^{15}$ by constructing witness functions inductively. The bound given by the witness function they obtained for $\mathbb{H}^3$ is $\Delta_{\mathrm{BR}}(\mathbb{H}^3,1) \lessapprox 0.8369$.

The focus on the space $\mathbb{H}^3 = \mathrm{SO}(3,1)_0/\mathrm{SO}(3)$ is due to the simplicity of the spherical transform in this case, see for instance [4]. The positive-definite spherical functions are given by

$$\varphi_\lambda(t) = \frac{\sin(\lambda t)}{\lambda \sinh(t)}$$

in the radial coordinate t for $\lambda \in (0,\infty) \cup i(0,1]$ and $\varphi_0(t) = \frac{t}{\sinh(t)}$. The spherical transform of a radial function f (given in radial coordinates) is given by

$$\widehat{f}(\lambda) = \frac{1}{4\pi} \int_0^\infty f(t)\varphi_\lambda(t)\sinh(t)^2 dt.$$

Additionally $m_{\mathbb{H}^3}(B(x_0, r)) = \pi(\sinh(2r) - 2r)$. The (scaled) heat kernel at a fixed time is given in radial coordinates by $\frac{t}{\sinh(t)} e^{-t^2}$. One ansatz for a witness function that was tried during the REU is

$$f(t) = p(t) \frac{t}{\sinh(t)} e^{-t^2}$$

with p a polynomial. We can use partial integration to see that

$$\begin{aligned} 4\pi \widehat{f}(\lambda) &= \int_0^\infty p(t) \frac{t}{\sinh(t)} e^{-t^2} \frac{\sin(\lambda t)}{\lambda \sinh(t)} \sinh(t)^2 dt = \int_0^\infty \frac{-p(t)\sin(\lambda t)}{2\lambda} \frac{\partial}{\partial t} e^{-t^2} dt \\ &= \frac{1}{2\lambda} \int_0^\infty p'(t)\sin(\lambda t) e^{-t^2} dt + \int_0^\infty \frac{1}{2} p(t)\cos(\lambda t) e^{-t^2} dt \\ &= \frac{1}{2\lambda} S_\lambda(p'(t)e^{-t^2}) + \frac{1}{2} C_\lambda(p(t)e^{-t^2}). \end{aligned}$$

Here S_λ denotes the (one-sided) sine transform evaluated at λ and C_λ the (one-sided) cosine transform evaluated at λ (note that λ here is a complex argument).
Let us observe a few facts under the assumption that a “nice” system of polynomials exists. If we choose $p(t)$ as a finite linear combination $p = \sum_{n=1}^{l} a_n p_n$ of polynomials $(p_n)_{n \geq 1}$ such that for each $n \geq 1$ the function $t \mapsto p_n(t)e^{-t^2}$ is an eigenfunction of the cosine transform and $p_n'(t)e^{-t^2}$ is an eigenfunction of the sine transform, then it might become feasible to optimize the coefficients a_n to control the zeros and sign change behaviour of f and $\widehat{f}$. In particular, if p_n only contains even powers, then $p_n(\lambda)e^{-\lambda^2}$ is real-valued on the real and the imaginary axis. In this case $\frac{1}{\lambda} p_n'(\lambda)e^{-\lambda^2}$ also only contains even powers and is thus real-valued on the real and the imaginary axis. Hence they would be real-valued on the spectrum of $(\mathrm{SO}(3,1)_0/\mathrm{SO}(3))$.
A related approach is given by choosing for p a linear combination of Hermite polynomials with even index, as the derivative of a Hermite polynomial is again a Hermite polynomial (up to scaling) and the integrals above can be evaluated explicitly if p is a Hermite polynomial of even index, see [57, #7.387 and #7.388] and also [48, (18.17.24) and (18.17.28)].

5. Model sets as examples for invariant random sphere packings

In this chapter we will give an overview of a class of examples for invariant random sphere packings, called model sets, which can be understood in a very explicit way. In this class of examples explicit descriptions of generically measured points, the autocorrelation measure, the Palm measure and the diffraction measure are available. A class of subsets of the transversal, called acceptance domains, can be defined. The Palm measure of an acceptance domain directly measures the frequency of patterns appearing. In Euclidean space, the Heisenberg group and the hyperbolic plane there are also explicit bounds on how the number of patterns of a given size grows.

To illustrate the techniques that can be used when working with model sets, we will use them in conjunction with basic algebraic number theory to give a very explicit description of the density of a specific family of model sets.

5.1. Basic definitions

Initially model sets were developed by Meyer ([80, 81]) as examples of sets for which exotic Poisson summation formulas hold. In the Euclidean case they are commonly used as models for mathematical quasi-crystals, see the monograph [7] and the references therein. In [89] Schlottmann generalized many of the Euclidean results on the dynamics to model sets to locally compact abelian groups, using the so-called "torus parametrization". For general lcsc groups the theory of model sets was developed by Björklund, Hartnick and Pogorzelski in [19, 20, 21], initially generalizing Schlottmanns methods to the non-abelian setting. More or less all results we present in this section are due to them in the generality we present them here.

Definition 5.1.1. Let H be a lcsc group and let $\Gamma \leq G \times H$ be a lattice projecting densely to H and injectively to G. Then the triple (G, H, Γ) is called a *cut-and-project-scheme*. If Γ is cocompact, the cut-and-project-scheme (G, H, Γ) is called *uniform*.

For a given cut-and-project-scheme (G, H, Γ) we will denote the projections from $G \times H$ to G resp. H by π_G resp. π_H. Note that the injectivity of $\pi_G|_\Gamma$ implies that for any $g \in \Gamma_G := \pi_G(\Gamma)$ there is a unique $(g, g^*) \in \Gamma$ projecting to g. Hence we have a map $\tau\colon \Gamma_G \to H,\ g \mapsto g^*$, called the *star-map*.

Definition 5.1.2. A compact set $W \subset H$ is called

(i) Γ-*regular*, if $\partial W \cap \pi_H(\Gamma) = \emptyset$,

(ii) *measurably regular*, if $m_H(\partial W) = 0$, where m_H is the Haar-measure on H,

(iii) *topologically regular*, if $\overline{W^\circ} = W$,

(iv) *aperiodic*, if $\mathrm{Stab}_H(W) = \{e\}$,

(v) *regular*, if it satisfies all of the properties above.

Definition 5.1.3. Let (G, H, Γ) be a cut-and-project-scheme and $W \subset H$ regular. The set $P(G, H, \Gamma, W) \coloneqq \tau^{-1}(W) = \pi_G((G \times W) \cap \Gamma)$ is called a *regular model set* and W is called a *window.* If (G, H, Γ) is uniform, we say that $P(G, H, \Gamma, W)$ is a *uniform regular model set.*

Any lcsc group is metrizable with a proper, left-invariant metric by a result of Struble [91]. See also [43, Theorem 2.B.4] for a textbook account. We will now fix such a metric on G and denote it by d_G. It turns out that regular model sets are uniformly discrete for any such choice of d_G.

Proposition 5.1.4 ([16, Proposition 2.13])**.** *Let $P \subset G$ be a regular model set. Then there is some $r > 0$ such that $P \in \mathrm{UD}_{2r}(G)$. P is R-relatively dense for some $R > 0$ if and only if P is a uniform regular model set.*

We will now explain how one obtains an ergodic $2r$-uniformly discrete point process from a regular model set. From now on we denote the regular model set $P(G, H, \Gamma, W)$ by P_0 and let $r > 0$ such that P_0 is r-uniformly discrete. We denote the punctured orbit closure $\overline{G.P_0} \setminus \{\emptyset\} \subset \mathrm{UD}_{2r}(G)$ by Ω_{P_0}. It turns out that Ω_{P_0} is closely related to the *parameter space* $Y \coloneqq (G \times H)/\Gamma$. This is sometimes called the *torus parametrization* and goes back to Schlottmann, [89], in the abelian case.

Theorem 5.1.5 (Björklund–Hartnick–Pogorzelski, [19, Theorem 3.1])**.** *There exists a unique G-equivariant Borel map $\beta : \Omega_{P_0} \to (G \times H)/\Gamma$ which maps P_0 to $(e, e)\Gamma$. This map additionally satisfies the following:*

(i) Set $Y^{\mathrm{ns}} \coloneqq \{(g, h)\Gamma \in Y \mid \partial(h^{-1}W) \cap \pi_H(\Gamma) = \emptyset\}$ and $\Omega^{\mathrm{ns}}_{P_0} = \beta^{-1}(Y^{\mathrm{ns}})$. Then $\beta|_{\Omega^{\mathrm{ns}}_{P_0}} : \Omega^{\mathrm{ns}}_{P_0} \to Y^{\mathrm{ns}}$ is bijective.

(ii) For $P \in \Omega_{P_0}$ we have $x_0 \in P$ if and only if $\beta(P) = (e, h_p)\Gamma$ for some $h_P \in H$.

(iii) If $x_0 \in P$ and $P \in \Omega^{\mathrm{ns}}_{P_0}$, then $P = \pi_G((G \times h_P^{-1}W) \cap \Gamma)$.

(iv) β is automatically continuous if P_0 is a uniform regular model set.

Theorem 5.1.6 (Björklund–Hartnick–Pogorzelski, [19, Theorem 3.4])**.** *There is a unique G-invariant Borel probability measure μ_{P_0} on the punctured orbit closure*

$$\Omega_{P_0} \subset \mathrm{UD}_{2r}(G).$$

Moreover $\mu_{P_0}(\Omega^{\mathrm{ns}}_{P_0}) = 1$ and $\beta : (\Omega_{P_0}, \mu_{P_0}) \to (Y, \frac{1}{|\Gamma|} m_{(G\times H)/\Gamma})$ is a measurable isomorphism of (measured) G-spaces (and thus induces an isomorphism $\beta^* : L^2(Y, \frac{1}{|\Gamma|} m_{G\times H/\Gamma}) \to L^2(\Omega_{P_0}, \mu_{P_0})$ of unitary G-representations).

Thus from each regular model set we obtain an ergodic $2r$-uniformly discrete point process $\Lambda^{P_0} : (\Omega_{P_0}, \mu_{P_0}) \to \mathcal{N}^*_{2r}(G),\ P \mapsto \delta_P$.

Remark 5.1.7. We note that $\Omega_{P_0} = \overline{GP_0}$ (and thus compact) if and only if P_0 is R-relatively dense for some $R > 0$, which is the case if and only if Γ is cocompact, see [16, Proposition 4.4] and [19, Corollary 2.11].

5.2. Ergodic theory

Theorem 5.1.5 and Theorem 5.1.6 in conjunction with the homogeneous dynamics of $(G \times H)/\Gamma$ allow Björklund, Hartnick and Pogorzelski to obtain relatively fine grained control over invariantly $(\mu, \mathcal{F}, (G_t)_{t>0})$-generic points for a large set $\mathcal{F}$ of functions. More precisely there are two lemmas enabling this control, one dealing with the correspondence of evaluation of functions on Ω_{P_0} and Y and one dealing with the ergodic theory of Y. To state the result on the correspondence of functions, we define the following two maps: For $f \in C_c(G)$ the *periodization* $\mathcal{P}(f)$ fo f is defined as

$$\mathcal{P}(f) : \Omega_{P_0} \to \mathbb{C},\ P \mapsto \delta_P(f) = \Lambda^{P_0}(f)$$

and for $F : G \times H \to \mathbb{C}$ bounded with bounded support the Γ*-periodization* $\mathcal{P}_\Gamma(F)$ of F is defined as

$$\mathcal{P}_\Gamma(F) : Y \to \mathbb{C},\ \mathcal{P}_\Gamma(F)((g,h)\Gamma) = \sum_{\gamma \in \Gamma} F((g,h)\gamma).$$

Lemma 5.2.1 ([19, Lemma 4.12])**.** *If $P \in \Omega_{P_0}^{\mathrm{ns}}$ and $f \in C_c(G)$, then*

$$\mathcal{P}(f)(P) = \mathcal{P}_\Gamma(f \otimes \chi_W)(\beta(P)).$$

Having dealt with the correspondence of evaluation of functions, Björklund, Hartnick and Pogorzelski use the following lemma to show that every $P \in \Omega_{P_0}^{\mathrm{ns}}$ is invariantly $(\mu_{P_0}, \mathcal{F}_\mathcal{P}, (G_t)_{t>0})$-generic, where $\mathcal{F}_\mathcal{P} := \mathcal{P}(C_c(G))$, assuming a sufficiently strong ergodic theorem holds.

Lemma 5.2.2 ([19, Lemma 5.8])**.** *Assume that $(G_t)_{t>0}$ is a quasi-uniform sequence of pre-compact subsets of G with positive measures such that for every $f \in C_c(Y)$ there is a conull set $Y^f \subset Y$ with*

$$\frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}y)\,dm_G(g) \to \int_Y f \frac{1}{|\Gamma|} dm_Y \tag{5.2.1}$$

for all $y \in Y^f$. Then

$$\frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}y)\,dm_G(g) \to \int_Y f \frac{1}{|\Gamma|} dm_Y$$

for every $f \in C_c(Y)$ and every $y \in Y$. Note that this then also holds for all compactly supported Riemann-integrable functions on Y.

Remark 5.2.3. The statement given above differs slightly from [19, Lemma 5.8]. More precisely in [19] it is assumed that the limit (5.2.1) holds for every $f \in L^2(Y, m_Y)$. From the proof of [19, Lemma 5.8] it is immediately apparent that this difference is inconsequential, as this is only used for functions in $C_c(Y)$. We have also clarified the order of quantifiers.

Sadly Lemma 5.2.1 is too weak to imply that every $P \in \Omega_{P_0}^{\mathrm{ns}}$ is generically measured, even if we assume that P_0 is a uniform regular model set. The issue is the following:

The functions in $\mathcal{P}(C_c(G))$ separate points in Ω_{P_0}, but do not form an algebra (as products of periodizations are not necessarily periodizations), hence it does not follow from Stone-Weierstrass that $\mathcal{P}(C_c(G))$ is dense in $C(\Omega_{P_0})$.
Luckily it is possible to prove a stronger version of Lemma 5.2.1, using a trick Michael Björklund shared with me. The idea is to consider periodizations of functions on G^n, i.e. periodizations in multiple variables. Given $f \in C_c(G^n)$ we define

$$\mathcal{P}(f) : \Omega_{P_0} \to \mathbb{C}, \quad P \mapsto \sum_{(g_1,\dots,g_n)\in P^n} f(g_1,\dots,g_n)$$

and for $F : G^n \times H^n \to \mathbb{C}$ bounded with bounded support we define

$$\mathcal{P}_\Gamma(F)((g,h)\Gamma) = \sum_{(\gamma_1,\dots,\gamma_n)\in\Gamma^n} F(g\pi_G(\gamma_1),\dots,g\pi_G(\gamma_n),h\pi_H(\gamma_1),\dots,h\pi_H(\gamma_n)).$$

Lemma 5.2.4. *For every $f \in C_c(G^n)$ the map*

$$\overline{\mathcal{P}}(f) : \mathcal{N}^*_{2r}(G) \to \mathbb{C},\ \mu \mapsto \mu \otimes \cdots \otimes \mu(f)$$

is continuous. Thus $\mathcal{P}(f) \in C(\Omega_{P_0})$ is the restriction of the continuous map $\overline{\mathcal{P}}(f) \circ \delta$ on $\mathrm{UD}_{2r}(G)$.

Proof. For $n > 0$ let $(\mu_i^{(k)})_{k\geq 1}$, $1 \leq i \leq n$, be convergent sequences in $\mathcal{N}^*_{2r}(G)$ with limits $\mu_i \in \mathcal{N}^*_{2r}(G)$. By [69, Lemma 4.1] we have $\bigotimes_{i=1}^n \mu_i^{(k)} \to \bigotimes_{i=1}^n \mu_i$ in the weak-$*$ topology on $\mathcal{M}(G^n)$ if and only if $(\bigotimes_{i=1}^n \mu_i^{(k)})(f_1 \otimes \cdots \otimes f_n) \to (\bigotimes_{i=1}^n \mu_i)(f_1 \otimes \cdots \otimes f_n)$ for all $f_1,\dots,f_n \in C_c(G)$. Thus, if $(\mu_i)_{i\geq 1}$ is a sequence in $\mathcal{N}^*_{2r}(X)$ with limit $\mu \in \mathcal{N}^*_{2r}(X)$, then $\mu_i \otimes \cdots \otimes \mu_i \to \mu \otimes \cdots \otimes \mu$ in the weak-$*$ topology and hence $\mu_i \otimes \cdots \otimes \mu_i(f) \to \mu \otimes \cdots \otimes \mu(f)$. Hence $\overline{\mathcal{P}}(f)$ is continuous. □

Lemma 5.2.5. *For $f \in C_c(G^n)$ and $P \in \Omega^{\mathrm{ns}}_{P_0}$ we have*

$$\mathcal{P}(f)(P) = \mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)(\beta(P)).$$

In addition the function $\mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)$ on Y is Riemann-integrable.

Proof. Let $g \in P$ and set $P' = g^{-1}P$. Then $\beta(P) = g\beta(P') = g(e, h_{P'})\Gamma = (g, h_{P'})\Gamma$. Note that $P' = \tau^{-1}(h_{P'}^{-1}W)$. Thus

$$\begin{aligned}
&\mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)(\beta(P)) \\
&= \sum_{(\gamma_1,\dots,\gamma_n)\in\Gamma^n} f(g\pi_G(\gamma_1),\dots,g\pi_G(\gamma_n))\chi_W(h\pi_H(\gamma_1)) \cdot \cdots \cdot \chi_W(h\pi_H(\gamma_n)) \\
&= \sum_{(g'_1,\dots,g'_n)\in\tau^{-1}(h_{P'}^{-1}W)^n} f(gg'_1,\dots,gg'_n) = \sum_{(g'_1,\dots,g'_n)\in(P')^n} f(gg'_1,\dots,gg'_n) \\
&= \sum_{(g_1,\dots,g_n)\in P^n} f(g_1,\dots,g_n) = \mathcal{P}(f)(P).
\end{aligned}$$

To obtain the Riemann integrability we proceed as follows. As W is compact, χ_W is upper semicontinuous. Hence by Baire's theorem on semicontinuity we can choose

(positive) continuous functions $\varphi_k : H \to \mathbb{R}$, $k \in \mathbb{N}$, with $\varphi_k \geq \varphi_{k+1}$ for all $n \geq 1$ and $\lim_{k\to\infty} \varphi_k = \chi_W$ pointwise. Using Urysohn's lemma we choose a function $\alpha \in C_c(H)$ with $\alpha(H) \subset [0,1]$ and $\alpha|_W = 1$. By considering the products $\alpha\varphi_k$ we again have $\alpha\varphi_k \geq \alpha\varphi_{k+1}$ and $\lim_{k\to\infty} \alpha\varphi_k = \chi_W$ pointwise. Thus we can assume without loss of generality that the functions φ_k are compactly supported.
The set W° is open and thus χ_{W° is lower semicontinuous. Hence, by Baire's theorem on semicontinuity, we can choose continuous functions $\psi_k : H \to \mathbb{R}$ with $\psi_k \leq \psi_{k+1}$ and $\lim_{k\to\infty} \psi_k = \chi_{W^\circ}$ pointwise. By replacing ψ_k with $\frac{1}{2}(|\psi_k| + \psi_k)$ we can further assume $\psi_k \geq 0$ and thus also that ψ_k is compactly supported.
Then $\mathcal{P}_\Gamma(f \otimes \varphi_k \otimes \cdots \otimes \varphi_k) \geq \mathcal{P}(f \otimes \varphi_{k+1} \otimes \cdots \otimes \varphi_{k+1})$ for all $k \geq 1$ and for $(g,h) \in G \times H$ we have

$$\begin{aligned} &\mathcal{P}_\Gamma(f \otimes \varphi_k \otimes \cdots \otimes \varphi_k)((g,h)\Gamma) \\ &= \sum_{(\gamma_1,\ldots,\gamma_n)\in\Gamma^n} f(g\pi_G(\gamma_1),\ldots,g\pi_G(\gamma_n))\varphi_k(h\pi_H(\gamma_1))\cdots\varphi_k(h\pi_H(\gamma_n)). \end{aligned}$$

Note that the right hand side is just the integral of a function with respect to a counting measure. Further we have

$$\mathcal{P}_\Gamma(f \otimes \varphi_k \otimes \cdots \otimes \varphi_k) \leq \#(\mathrm{supp}(f \otimes \varphi_1 \otimes \cdots \otimes \varphi_1) \cap \Gamma^n)\|f\|_\infty \|\varphi_1\|_\infty^n,$$

where $\mathrm{supp}(f \otimes \varphi_1 \otimes \cdots \otimes \varphi_1) \cap \Gamma^n$ is to be understood with the necessary obvious reordering of coordinates. Thus by dominated convergence we have

$$\mathcal{P}_\Gamma(f \otimes \varphi_k \otimes \cdots \otimes \varphi_k)((g,h)\Gamma) \to \mathcal{P}_\gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)((g,h)\Gamma).$$

Similarly one shows $\mathcal{P}_\Gamma(f \otimes \psi_k \otimes \cdots \otimes \psi_k) \geq \mathcal{P}(f \otimes \psi_{k+1} \otimes \cdots \otimes \psi_{k+1})$ for all $k \geq 1$ and

$$\mathcal{P}_\Gamma(f \otimes \psi_k \otimes \cdots \otimes \psi_k) \to \mathcal{P}_\Gamma(f \otimes \chi_{W^\circ} \otimes \cdots \otimes \chi_{W^\circ})$$

pointwise. Now

$$\mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)((g,h)\Gamma) \neq \mathcal{P}_\Gamma(f \otimes \chi_{W^\circ} \otimes \cdots \otimes \chi_{W^\circ})((g,h)\Gamma)$$

implies $(g,h)\Gamma \in (G \times \partial W)/\Gamma$, which is a $m_{(G\times H)/\Gamma}$-nullset. Thus we have shown the Riemann-integrability. ☐

If we now have $f_1,\ldots,f_n \in C_c(G)$, we apply the lemma to $\tilde{f} : G^n \to \mathbb{C},\ (g_1,\ldots,g_n) \to f_1(g_1)\cdot\ldots\cdot f_n(g_n)$. As $\mathcal{P}(f_1)\cdot\ldots\cdot\mathcal{P}(f_n) = \mathcal{P}(\tilde{f})$, this allows us to obtain the whole algebra generated by $\mathcal{P}(C_c(G))$. Thus we have circumnavigated the issue described above. This lemma together with Lemma 5.2.2 therefore implies the following theorem:

Theorem 5.2.6. *Assume that $(G_t)_{t>0}$ is a quasi-uniform sequence of symmetric compact subsets of G with positive measures such that the invariant ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(G)))$. Then for any uniform regular model set P_0 every $P \in \Omega_{P_0}^{\mathrm{ns}}$ is generically measured wrt. $(G_t)_{t>0}$ in the homogeneous space G.*

Proof. Note that Y is compact, as P_0 is a uniform regular model set. Thus $C(Y) = C_c(Y)$. We will first prove that for every $f \in C(Y)$ there is a m_Y-conull set $Y^f \subset Y$

such that

$$\frac{1}{m_G(G_t)}\int_{G_t} f(g^{-1}y)dm_G(g) \to \int f\frac{1}{|\Gamma|}dm_Y$$

for every $y \in Y^f$.

As P_0 is a regular model set, $\beta : \Omega_{P_0} \to Y$ is continuous. Moreover $\beta : \Omega_{P_0}^{\mathrm{ns}} \to Y^{\mathrm{ns}}$ is a bijection between conull sets. Consider the function $f \circ \beta \in C(\Omega_{P_0})$ and note that there is a function $\overline{f} \in C(\mathrm{UD}_{2r}(X))$ with $\overline{f}(P) = f(\beta(P))$ for all $P \in \Omega_{P_0}$ by the Tietze extension theorem (as $\Omega_{P_0} \subset \mathrm{UD}_{2r}(X)$ is a compact subset). By the invariant pointwise ergodic theorem for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$ we can find a μ_{P_0}-conull set $\Omega_{\overline{f}} \subset \mathrm{UD}_{2r}(X)$ such that

$$\frac{1}{m_G(G_t)}\int_{G_t} \overline{f}(g^{-1}P)dm_G(g) \to \int \overline{f}d\mu_{P_0}$$

for every $P \in \Omega_{P_0}$. As $\Omega_{P_0}^{\mathrm{ns}}$ is a μ_{P_0}-conull set, the set $\Omega_{\overline{f}} \cap \Omega_{P_0}^{\mathrm{ns}}$ is a μ_{P_0}-conull set. For any $P \in \Omega_{\overline{f}} \cap \Omega_{P_0}^{\mathrm{ns}}$ and $t > 0$ we have

$$\frac{1}{m_G(G_t)}\int_{G_t} \overline{f}(g^{-1}P)dm_G(g) = \frac{1}{m_G(G_t)}\int_{G_t} f(\beta(P))dm_G(g).$$

Set $Y^f := \beta(\Omega_{\overline{f}} \cap \Omega_{P_0}^{\mathrm{ns}})$. Then Y^f is a m_Y-conull set and for each $y \in Y^f$ we have

$$\lim_{t\to\infty}\frac{1}{m_G(G_t)}\int_{G_t} f(\beta(P))dm_G(g) = \int_{\Omega_{P_0}} f \circ \beta d\mu_{P_0} = \int_Y f dm_Y.$$

Now let $f \in C_c(G^n)$ for some $n \geq 1$. Then, if $P \in \Omega_{P_0}^{\mathrm{ns}}$, Lemma 5.2.5 shows $\mathcal{P}(f)(P) = \mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)(\beta(P))$. Observe further that for any $g \in G$ the set gP is in $\Omega_{P_0}^{\mathrm{ns}}$. Thus

$$\mathcal{P}(f)(gP) = \mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)(g\beta(P))$$

for all $g \in G$. Thus, by the definition of μ_{P_0} and Lemma 5.2.2,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{m_G(G_t)}\int_{G_t} \mathcal{P}(f)(g^{-1}P)dm_G(g)\\
&= \lim_{t\to\infty}\frac{1}{m_G(G_t)}\int_{G_t} \mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)(g^{-1}\beta(P))dm_G(g)\\
&= \int_Y \mathcal{P}_\Gamma(f \otimes \chi_W \otimes \cdots \otimes \chi_W)((g,h)\Gamma)\frac{1}{|\Gamma|}dm_{(G\times H)/\Gamma}((g,h)\Gamma)\\
&= \int_{\Omega_{P_0}} \mathcal{P}(f)(P)d\mu_{P_0}(P),
\end{aligned}$$

as $\mu_{P_0}(\Omega_{P_0}^{\mathrm{ns}}) = 1 = m_{(G\times H)/\Gamma}(Y^{\mathrm{ns}})$. Now we note two facts:

(i) The set $\bigcup_{n\geq 1}\mathcal{P}(C_c(G^n))$ contains the algebra generated by the functions $\mathcal{P}(f)$, $f \in C_c(G)$. This algebra separates points and by the Stone-Weierstrass theorem it is dense in $C(\Omega_{P_0})$.

(ii) Every continuous function on $\mathrm{UD}_{2r}(G)$ restricts to a continuous function on Ω_{P_0}.

Combining these two facts with Proposition 2.4.5 implies the claim. □

Remark 5.2.7. Even if P_0 is a regular model set which is not uniform, it is still evident that Lemma 5.2.2 together with Lemma 1.3.4 allows for fine grained control of the ergodic theory of μ_{P_0}. In particular in all of the classes of examples for G we have considered so far in this thesis, the ergodic theorems are strong enough to show the theorem above in the case of non-uniform regular model sets.

5.3. Palm measures

We will now determine the Palm measure of Λ^{P_0}. The results we present in this section are well-known to experts (see [22, Section 7]), but our approach using Last's version of the Palm measure for point processes in homogeneous spaces is novel.
Recall that we have fixed a regular model set $P_0 \in \mathrm{UD}_{2r}(G)$ and the associated point process Λ^{P_0}. We will now give an explicit description of the Palm measure of Λ^{P_0}. The canonical Ω_{P_0}-transversal for Λ_0 is given by $\mathcal{T} := \{P \in \Omega_{P_0} \mid e \in P\}$.
Our determination of the Palm measure of Λ^{P_0} is based on two proposition. The first one identifies the image of the transversal under the inverse parametrization map β and the second one is a generalization of Lemma 5.2.1.

Proposition 5.3.1. *We have* $\mathcal{T} = \overline{\{g^{-1}P_0 \mid g \in P_0\}}$ *and* $\beta(\mathcal{T}) = (\{e\}\times W)/\Gamma$. *Moreover* $\beta|_{\mathcal{T}} : \mathcal{T} \to (\{e\} \times W)/\Gamma$ *is continuous.*

Proof. For $P \in \mathcal{T}$ we have $\beta(P) = (e, h_P)\Gamma$ by Theorem 5.1.5. Assume for the moment that $\mathcal{T} = \overline{\{g^{-1}P_0 \mid g \in P_0\}}$. For $g \in P_0$ we have

$$\beta(g^{-1}P_0) = (g^{-1}, e)\Gamma = (g^{-1}, e)(g, \tau(g))\Gamma = (e, \tau(g))\Gamma \in (\{e\} \times W)/\Gamma.$$

As $\pi_H(\Gamma)$ is dense in H and W is the closure of its interior, we see that $\beta(\{g^{-1}P_0 \mid g \in P_0\})$ is dense in W. Then, if $w \in W$ and $(g_n)_{n\geq 1}$ is a sequence in P_0 such that $\beta(g_n^{-1}P_0) \to w$, we can choose a subsequence $(g_{n_k})_{k\geq 1}$ such that the sequence $(g_{n_k}^{-1}P_0)_{k\geq 1}$ is convergent (as $\mathcal{T}$ is a compact subset of $\mathrm{UD}_{2r}(G)$). Thus, if $P_w \in \mathcal{T}$ is the limit of $(g_{n_k}^{-1}P_0)_{k\geq 1}$, we see that $g_{n_k}^{-1}P_0 \to P$ and $\beta(g_{n_k}^{-1}P_0) \to w$. As the graph of β is closed, we have $\beta(P) = w$.
If $P \in \mathcal{T}$, then there is a sequence $(g_n)_{n\geq 1}$ with $g_n^{-1}P_0 \to P$ and we can consider the sequence $\beta(g_n^{-1}P_0)$. As W is compact, we can choose a subsequence $(g_{n_k})_{k\geq 1}$ such that $\beta(g_{n_k}^{-1}P_0)$ is convergent (to some element $w_P \in W$). As β has closed graph we see that $\beta(P) = w_P$.
Note that we have actually shown that the map $\mathcal{T} \to (\{e\}\times W)/\Gamma$ is continuous. This is not surprising as any closed graph map between compact Hausdorff spaces is continuous by the closed graph theorem.
Now all that is left to show is $\mathcal{T} = \overline{\{g^{-1}P_0 \mid g \in P_0\}}$. To do this we will use that there is a topology on $\mathfrak{F}(G)$, called the local topology, which is equal to the Chabauty-Fell topology on the subspace Ω_{P_0} by [19, Corollary A.8]. By [19, Appendix A.2] a neighborhood basis of $P \in \Omega_{P_0}$ in the local topology is given by the sets

$$U_{K,V}(P) := \{Q \in \Omega_{P_0} \mid \exists t \in V : tQ \cap K = P \cap K\},$$

where K runs over the compact subsets of G and V runs over the open subsets of G. Let $P \in \mathcal{T}$. As $\{gP_0 \mid g \in G\} = \{g^{-1}P_0 \mid g \in G\}$ is dense in Ω_{P_0}, we can choose $g_n \in G$ with $g_nP_0 \to P$. Hence for any $n \in \mathbb{N}$ there is some $t_n \in B(e,r)$ with $t_ng_nP_0 \cap \overline{B}(e,n) = P \cap \overline{B}(e,n)$ and thus $e \in t_ng_nP_0$, i.e. $(t_ng_n)^{-1} \in P_0$. As $t_n \in B(e,r)$, we can choose a convergent subsequence $(t_{n_k})_{k\geq 1}$ by the properness of the metric on G. Let $t \in \overline{B}(e,r)$ denote its limit. Then $t_{n_k}g_{n_k}P_0 \to tP$. As $t_{n_k}g_{n_k}P \in \mathcal{T}$, we see that $tP \in \mathcal{T}$, as $\mathcal{T}$ is compact. Thus $t^{-1} \in P$ and $t^{-1} \in \overline{B}(e,r)$, i.e. $t^{-1} \in P \cap B(e,2r) = \{e\}$. Hence $t = e$ and $t_{n_k}g_{n_k}P_0 \to P$. Setting $\tilde{g}_k := (t_{n_k}g_{n_k})^{-1}$, we obtain $e \in \tilde{g}_k^{-1}P_0$ and $\tilde{g}_k^{-1}P_0 \to P$ and thus $\overline{\{g^{-1}P_0 \mid g \in P_0\}} = \mathcal{T}$. □

Proposition 5.3.2. *Let $P \in \Omega_{P_0}^{\mathrm{ns}}$ with $\beta(P) = (g_P, h_P)\Gamma$, i.e. $P = g_P\tau^{-1}(h_P^{-1}W)$ and let $f : G \times \mathcal{T} \to [0,\infty)$ be measurable. Then*

$$\sum_{g\in P} f(g, g^{-1}P) = \sum_{\gamma\in\Gamma} f(g_P\pi_H(\gamma), \tau^{-1}((h_P\pi_H(\gamma))^{-1}W))\chi_W(h_P\pi_H(\gamma))$$

Proof. Note first that $x \in \pi_G(\gamma)^{-1}\tau^{-1}(h_P^{-1}W)$ is equivalent to $\pi_G(\gamma)x \in \tau^{-1}(h_P^{-1}W)$ which in turn is equivalent to $\tau(\pi_G(\gamma)x) \in h_P^{-1}W$. This is again equivalent to $\pi_H(\gamma)\tau(x) \in h_P^{-1}W$ which is the case if and only if $\tau(x) \in \pi_H(\gamma)^{-1}h_P^{-1}W$. Thus we have $\pi_G(\gamma)^{-1}\tau^{-1}(h_P^{-1}W) = \tau^{-1}((h_P\pi_H(\gamma))^{-1}W)$ and obtain

$$\begin{aligned}\sum_{g\in P} f(g, g^{-1}P) &= \sum_{g\in P} f(g, g^{-1}g_P\tau^{-1}(h_P^{-1}W))\\ &= \sum_{\gamma\in\Gamma} f(g_P\pi(\gamma), (g_P\pi_G(\gamma))^{-1}g_P\tau^{-1}(h_P^{-1}W))\chi_W(h_P\pi_H(\gamma))\\ &= \sum_{\gamma\in\Gamma} f(g_P\pi(\gamma), \pi_G(\gamma)^{-1}\tau^{-1}(h_P^{-1}W))\chi_W(h_P\pi_H(\gamma))\\ &= \sum_{\gamma\in\Gamma} f(g_P\pi(\gamma), \tau^{-1}((h_P\pi_H(\gamma))^{-1}W))\chi_W(h_P\pi_H(\gamma)).\end{aligned}$$ □

In order to state what the Palm measure of Λ^{P_0} is, we need to make some definitions. Let $H^{\mathrm{ns}} := \{h \in H \mid h^{-1}W \text{ is } \Gamma\text{-regular}\}$ and $W^{\mathrm{ns}} := H^{\mathrm{ns}} \cap W$. Let $i : W \to (G\times H)/\Gamma$, $h \mapsto (e,h)\Gamma$ and note that i is continuous. Let $\mathrm{par} : Y^{\mathrm{ns}} \to \Omega_{P_0}^{\mathrm{ns}}$ denote the inverse of $\beta|_{\Omega_{P_0}^{\mathrm{ns}}} : \Omega_{P_0}^{\mathrm{ns}} \to Y^{\mathrm{ns}}$.

Proposition 5.3.3. *W^{ns} is a conull set in W and the Palm measure of Λ^{P_0} is given by*

$$\nu_{P_0} := \frac{1}{|\Gamma|}\mathrm{par}_* i_* m_H|_{W^{\mathrm{ns}}}.$$

In particular $i(\Lambda^{P_0}) = \frac{m_H(W)}{|\Gamma|}$.

Proof. We will first prove that the measure ν_{P_0} satisfies the refined Campbell theorem. Let $f : G \times \mathcal{T} \to [0,\infty)$ be measurable and let F be a fundamental domain for Γ in

$G\times H$. For $P\in\Omega_{P_0}^{\mathrm{ns}}$ choose $(g_P,h_P)\in G\times H$ with $\beta(P)=(g_p,h_P)\Gamma$. Then

$$\begin{aligned}
&\int_{\Omega_{P_0}}\int_G f(g,g^{-1}\omega)d\Lambda_\omega^{P_0}(g)d\mu_{P_0}(\omega)=\int_{\Omega_{P_0}}\sum_{g\in P}f(g,g^{-1}P)d\mu_{P_0}(P)\\
&=\int_{\Omega_{P_0}^{\mathrm{ns}}}\sum_{\gamma\in\Gamma}f(g_P\pi_G(\gamma),\tau^{-1}((h_P\pi_H(\gamma))^{-1}W))\chi_W(h_P\pi_H(\gamma))d\mu(P)\\
&=\int_{Y^{\mathrm{ns}}}\sum_{\gamma\in\Gamma}f(g\pi_G(\gamma),\tau^{-1}((h\pi_H(\gamma))^{-1}W))\chi_W(h\pi_H(\gamma))\frac{1}{|\Gamma|}dm_Y((g,h)\Gamma)\\
&=\int_{F\cap(G\times H^{\mathrm{ns}})}\sum_{\gamma\in\Gamma}f(g\pi_G(\gamma),\tau^{-1}((h\pi_H(\gamma))^{-1}W))\chi_W(h\pi_H(\gamma))\frac{1}{|\Gamma|}dm_G\otimes m_H((g,h))\\
&=\int_{G\times H^{\mathrm{ns}}}f(g,\tau^{-1}(h^{-1}W))\chi_W(h)\frac{1}{|\Gamma|}dm_H(h)dm_G(g)\\
&=\int_G\int_{W^{\mathrm{ns}}}f(g,\tau^{-1}(h^{-1}W))\frac{1}{|\Gamma|}dm_H(h)dm_G(g)\\
&=\int_G\int_{\mathcal{T}}f(g,P)\frac{1}{|\Gamma|}d(\mathrm{par})_*i_*m_H|_{W^{\mathrm{ns}}}(P)dm_G(g)=\int_G\int_{\mathcal{T}}f(g,P)d\nu(P)dm_G(g).
\end{aligned}$$

If $\mathbb{P}_{\mathcal{T}}$ denotes the Palm measure of Λ^{P_0}, then

$$\begin{aligned}
\int_G\int_{\mathcal{T}}f(g,P)d\nu(P)dm_G(g)&=\int_{\Omega_{P_0}}\int_G f(g,g^{-1}\omega)d\Lambda_\omega^{P_0}(g)d\mu_{P_0}(\omega)\\
&=\int_G\int_{\mathcal{T}}f(g,P)d\mathbb{P}_{\mathcal{T}}(P)dm_G(g)
\end{aligned}$$

and thus, choosing $f(g,\omega)=\chi_{B(e,r)}(g)\chi_A(\omega)$ with $A\subset\mathcal{T}$ measurable, we see that $\nu(A)=\mathbb{P}_{\mathcal{T}}(A)$. Thus $\nu=\mathbb{P}_{\mathcal{T}}$ and ν is the Palm measure of Λ. To see that W^{ns} is conull, we observe first that $W\setminus W^{\mathrm{ns}}\subset H\setminus H^{\mathrm{ns}}$. But $h\in H\setminus H^{\mathrm{ns}}$ if and only if $h^{-1}\partial W\cap\pi_H(\Gamma)\neq\emptyset$, which is equivalent to $h\in\partial W\pi_H(\Gamma)$ (compare to the proof of [19, Lemma 3.6]). But this set is a countable union of nullsets and thus a nullset. □

5.4. Patch frequencies and acceptance domains

In this section we will prove that the frequency with which certain patters appear in P_0 is given by the volume of certain subsets of the window W, called acceptance domains. This will give a fairly complete understanding of the transversal and the Palm measure of Λ^{P_0} in terms of frequencies and patterns, which in turn are controlled by these acceptance domains. In the case of model sets in Euclidean space some results of this section are folklore results. The idea of acceptance domains was first introduced by Julien in [66]. They were later used by Koivusalo and Walton in [71, 72] and by Haynes, Koivusalo and Walton in [63] to address questions related to complexity and linear repetitivity. In [68] Kaiser generalized some of their methods and in particular acceptance domains to lcsc groups. It is this generalization that we will present and use to obtain our results.
As in the previous section $P_0=P(G,H,\Gamma,W)$ is a regular model set.

5.4.1. Basic theory of acceptance domains

In this subsection we will quickly introduce acceptance domains and their basic properties. All results in this subsection are due to Kaiser, [68], in the generality we present, except for the example at the end of the subsection.

Definition 5.4.1. Let $g, h \in P_0$ and $s > 0$. We say that $g \sim_s h$ if $g^{-1}P_0 \cap B(e,s) = h^{-1}P_0 \cap B(e,s)$. If $g \sim_s h$, we say that $g, h \in P_0$ are *s-related*. The set $g^{-1}P_0 \cap B(e,s)$ is called the *centered s-patch* at g.

Definition 5.4.2. Let $s > 0$. The *s-slab* of (G, H, Γ, W) is the set

$$S_s := \{\gamma \in \Gamma \mid \pi_G(\gamma) \in B(e,s),\ \pi_H(\gamma) \in W^{-1}W\}.$$

The *s-slab at $g \in P_0$* is

$$S_s(g) := \{\gamma \in S_s \mid g\pi_G(\gamma) \in P_0\}.$$

Lemma 5.4.3. *For $g, h \in P_0$ we have $g \sim_s h \iff S_s(g) = S_s(h)$.*

Proof. If $g \sim_s h$, then $g^{-1}P_0 \cap B(e,s) = h^{-1}P_0 \cap B(e,s)$. For $\gamma \in S_s(g)$ we have $g\pi_G(\gamma) \in P_0$, i.e. $\pi_G(\gamma) \in g^{-1}P_0$ and by the definition of $S_s(g)$ we also have $\pi_G(\gamma) \in B(e,s)$. Thus $\pi_G(\gamma) \in g^{-1}P_0 \cap B(e,s) = h^{-1}P_0 \cap B(e,s)$. Thus $h\pi_G(\gamma) \in P_0$. In addition there is some $h' \in P_0$ with $\pi_G(\gamma) = h^{-1}h'$ and thus $\pi_H(\gamma) = \tau(\pi_G(\gamma)) = \tau(h)^{-1}\tau(h') \in W^{-1}W$. Thus $\gamma \in S_s(h)$.
Assume now that $S_s(h) = S_s(g)$. If $x \in g^{-1}P_0 \cap B(e,s)$ then $x \in \pi_G(\Gamma)$, $\tau(x) \in W^{-1}W$, $x \in B(e,s)$ and $g\pi_G(x) \in P_0$. Thus $(x, \tau(x)) \in S_s(g) = S_s(h)$ and thus (doing the same in reverse) $x \in h^{-1}P_0 \cap B(e,s)$. □

Hence we see that the slabs encode the possible centered s-patches. We now use this to define a decomposition of W.

Definition 5.4.4. Let $x \in P_0$ and $s > 0$.

(i) The *pre-acceptance domain* of (x,s) is the set

$$A_{x,s} := \tau(\{y \in P_0 \mid x \sim_s y\}).$$

(ii) The *acceptance domain* of (x,s) is the set

$$W_{x,s} := \left(\bigcap_{\gamma \in S_s(x)} W^{\circ}\pi_H(\gamma)^{-1}\right) \cap \left(\bigcap_{\gamma \in S_s \setminus S_s(x)} W^{\mathsf{c}}\pi_H(\gamma)^{-1}\right)$$

Theorem 5.4.5. *If $x \not\sim_s y$, then $W_{x,s} \cap W_{y,s} = \emptyset$. If $x \sim_s y$, then $W_{x,s} = W_{x,s}$.*

Proof. By the lemma above we have $x \sim_s y$ iff $S_s(x) = S_s(y)$ and this implies $W_{x,s} = W_{y,s}$. If $x \not\sim_s y$, then $S_s(x) \neq S_s(y)$. Without loss of generality we can assume that there is some $\gamma \in S_s(x) \setminus S_s(y)$ (otherwise swap x and y). Hence $W_{x,s} \subset W^{\circ}\pi_H(\gamma)^{-1}$ and $W_{y,s} \subset W^{\mathsf{c}}\pi_H(\gamma)^{-1}$. Thus $W_{x,s} \cap W_{y,s} = \emptyset$. □

Lemma 5.4.6. *For all $x \in P_0$ and $s > 0$ the pre-acceptance domain $A_{x,s}$ is a dense subset of $W_{x,s}$.*

Proof. Let $y \in P_0$ with $x \sim_s y$. Then $W_{x,s} = W_{y,s}$ as $S_s(x) = S_s(y)$. Thus we must show $\tau(y) \in W_{y,s}$, i.e. $\tau(y) \in W^\circ \pi_H(\gamma)^{-1}$ for $\gamma \in S_s(y)$ and $\tau(y) \notin W\pi_H(\gamma)^{-1}$ for $\gamma \in S_s \setminus S_s(y)$.
But $\tau(y)\pi_H(\gamma) \in W^\circ$ is equivalent to $y\pi_G(\gamma) \in P_0$ (as W is topologically regular and Γ-regular) and by the definition of the s-slab at y we have $\pi_G(\gamma) \in y^{-1}P_0 \cap B(e,s)$.
Similarly for $\gamma \in S_s \setminus S_s(y)$ we have $\pi_G(\gamma) \notin y^{-1}P_0 \cap B(e,s)$ and thus, by the definition of the s-slab, $\pi_G(\gamma) \notin y^{-1}P_0$. Hence $\tau(y)\pi_H(\gamma) \notin W$ and thus $\tau(y) \notin W\pi_H(\gamma)^{-1}$.
The density of $A_{x,s}$ follows directly from the fact that $\bigcup_{x \in P_0} A_{x,s} = \pi_H(\Gamma) \cap W \subset \bigcup_{x\in P_0} W_{x,s}$ and the fact that there are only finitely many acceptance domains. ☐

Example 5.4.7. For an arbitrary (non-Euclidean) cut-and-project scheme the calculation of the r-slab can be quite difficult, as it boils down to computing all lattice points in a given compact unit neighborhood of $G \times H$. However, for specific families of lattices one can sometimes find relatively simple methods. Consider for instance the Hilbert modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ as a lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ via the diagonal embedding $\mathrm{id} \times \sigma$, with σ the entry-wise Galois conjugation in $\mathbb{Q}(\sqrt{2})$ mapping $a + b\sqrt{2}$ to $a - b\sqrt{2}$. If we have any compact unit neighborhood U in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, we can choose $t > 0$ such that $U \subset \pi^{-1}(B(x_0,t)) \times \pi^{-1}(B(x_0,t))$, where $B(x_0,t)$ denotes the ball in $\mathbb{H}^2$. Thus we can compute all lattice points in U by computing all elements of Γ with $\pi_G(\gamma) \in B(x_0,t)$ and $\pi_H(\sigma(\gamma)) \in B(x_0,t)$. but

$$d\left(\begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}.i, i\right) = d_{\mathbb{H}^2}(e^s i, i) = s = \cosh^{-1}\left(\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}^\top \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}\right)\right).$$

The KAK decomposition of $\mathrm{SL}_2(\mathbb{R})$ now implies that $\mathrm{tr}(g^\top g) = 2\cosh(d_{\mathbb{H}^2}(g.i,i))$ for all $g \in \mathrm{SL}_2(\mathbb{R})$. Thus, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma \in \Gamma$ with $d_{\mathbb{H}^2}(\gamma.i,i) < t$ and $d_{\mathbb{H}^2}(\sigma(\gamma).i,i) < t$, we must have $\mathrm{tr}(\gamma^\top\gamma) < 2\cosh(t) =: T$ and $\mathrm{tr}(\sigma(\gamma)^\top\gamma) < 2\cosh(t) = T$. This is equivalent to $a^2 + b^2 + c^2 + d^2 < T$ and $\sigma(a)^2 + \sigma(b)^2 + \sigma(c)^2 + \sigma(d)^2 < T$. This implies

$$|a| \leq T,\ |b| \leq T,\ |c| \leq T,\ |d| < T$$

and

$$|\sigma(a)| \leq T,\ |\sigma(b)| \leq T,\ |\sigma(c)| \leq T,\ |\sigma(d)| < T.$$

If $j : \mathbb{Q}(\sqrt{2}) \to \mathbb{R} \times \mathbb{R},\ x \mapsto (x, \sigma(x))$, then $j(\mathbb{Z}[\sqrt{2}])$ is a lattice in $\mathbb{R}^2$ and every entry of γ must be contained in $C := j(\mathbb{Z}[\sqrt{2}]) \cap ([-T,T] \times [-T,T])$. Hence we can compute every matrix in $\Gamma \cap \pi^{-1}(B(i,t)) \times \pi^{-1}(B(i,t))$ by testing all possible combinations of points in C as matrix entries.

5.4.2. Acceptance domains, the transversal and patch frequencies

We now make the standing assumption that P_0 is a uniform regular model set and that the pointwise ergodic theorem holds for $((G_t)_{t>0}, C(\mathrm{UD}_{2r}(X)))$ with $(G_t)_{t>0}$ a quasi-uniform sequence of symmetric pre-compact sets. In particular, P_0 is generically measured by Theorem 5.2.6.

Definition 5.4.8. Let $\Pi \subset G$ be finite. If there is some $x \in P_0$ with $x^{-1}P_0 \cap B(e,s) = \Pi$ we call Π a *realized s-pattern* of P_0. Denote the set of realized s-patterns of P_0 by $\mathcal{P}$. Let $\Pi \subset G$ be a realized s-pattern and $(G_t)_{t>0}$ some sequence of Borel sets with positive measure in G. If the limit

$$f(P_0, \Pi, (G_t)_{t>0}) \coloneqq \lim_{t\to\infty} \frac{\#\{g \in P_0 \cap G_t \mid g^{-1}P_0 \cap B(e,s) = \Pi\}}{m_G(G_t)}$$

exists, we call $f(P_0, \Pi, (G_t)_{t>0})$ the *pattern frequency* of Π in P_0 wrt. $(G_t)_{t>0}$.

This is not really a "frequency". To obtain something which behaves like a "frequency" we need to multiply with

$$\lim_{t\to\infty} \frac{m_G(G_t)}{\#(P_0 \cap G_t)} = \frac{1}{i(\Lambda^{P_0})}.$$

This is very similar to the definition of energy, which also uses normalization with $(\#(P_0 \cap G_t))^{-1}$ and not normalization with $m_G(G_t)^{-1}$.
Our aim is the following theorem:

Theorem 5.4.9. *Let Π be a realized s-pattern and let $x \in P_0$ with $x^{-1}P_0 \cap B(e,s) = \Pi$. If the invariant pointwise ergodic theorem holds for $(C_c(\mathrm{UD}_{2r}(G)), (G_t)_{t>0})$, then*

$$f(P_0, \Pi, (G_t)_{t>0}) = m_H(W_{x,s}).$$

Recall that ν denotes the Palm measure of Λ^{P_0}.

Lemma 5.4.10. *Assume that the invariant pointwise ergodic theorem holds for $(C_c(\mathrm{UD}_{2r}(G)), (G_t)_{t>0})$ and let $\Pi \subset G$ be a realized s-pattern. Then*

$$\lim_{t\to\infty} \frac{\#\{g \in P_0 \cap G_t \mid \Pi \subset g^{-1}P_0 \cap B(e,s)\}}{m_G(G_t)} = \nu(\{P \in \mathcal{T} \mid \Pi \subset P\}).$$

Proof. We first note that there are only finitely many realized s-patterns, as the finiteness of the s-slab implies that there are only finitely many acceptance domains and thus only finitely many pre-acceptance domains. By [19, Theorem A.7] for any $P \in \Omega_{P_0}$ and $x \in P$ there is some $y \in P_0$ with $x^{-1}P \cap B(e,s) = y^{-1}P_0 \cap B(e,s)$. Hence the set

$$A \coloneqq \bigcup_{P \in \mathcal{T}} P \cap B(e,s)$$

is finite. Choose $\delta > 0$ such that $d_G(x,y) > 2\delta$ for all $x, y \in A$ and for $x \in A$ let $\varphi_x \in C_c(G)$ with $\varphi_x : G \to [0,1]$, $\mathrm{supp}(\varphi_x) \subset B(x,\delta)$ and $\varphi_x(x) = 1$. Let $\Pi = \{x_1, \ldots, x_n\}$ and set

$$f : G^n \to \mathbb{R},\ (g_1, \ldots, g_n) \mapsto \prod_{i=1}^{n} \varphi_{x_i}(g_i).$$

Then $\sum_{x\in P^n} f(x) = 1$ if and only if $\Pi \subset P$. We can now apply the Palm sampling theorem 2.5.9 to f and obtain

$$\lim_{t\to\infty} \frac{\#\{g \in P_0 \cap G_t \mid \Pi \subset g^{-1}P\}}{m_G(G_t)} = \lim_{t\to\infty} \frac{1}{m_G(G_t)} \sum_{g\in P_0\cap G_t} f(g^{-1}P_0)$$
$$= \nu(f) = \nu(\{P \in \mathcal{T} \mid \Pi \subset P \cap B(e,s)\}). \quad \square$$

Lemma 5.4.11. *Assume that the invariant pointwise ergodic theorem holds for* $(C_c(\mathrm{UD}_{2r}(G)), (G_t)_{t>0})$ *and let* $\Pi \subset G$ *be a realized* s*-pattern of* P_0*. Then*

$$\lim_{t\to\infty} \frac{\#\{g \in P_0 \cap G_t \mid \Pi = g^{-1}P_0 \cap B(e,s)\}}{m_G(G_t)} = \nu(\{P \in \mathcal{T} \mid \Pi = P \cap B(e,s)\}).$$

Proof. Consider the set $\mathcal{P}$ of all realized s-patterns of P_0 and note that this is a finite set on which we define an partial order $\leq$ by inclusion. Note that for a maximal element $\Pi \in \mathcal{P}$ we have

$$\{g \in P_0 \mid \Pi \subset g^{-1}P_0\} = \{g \in P_0 \mid g^{-1}P_0 \cap B(e,s) = \Pi\}$$

and

$$\nu(\{P \in \mathcal{T} \mid \Pi \subset P\}) = \nu(\{P \in \mathcal{T} \mid \Pi = P \cap B(e,s)\})$$

(as the set of realized s-patterns of P is contained in $\mathcal{P}$ by [19, Theorem A.7]). Thus the claim holds for all maximal elements of $\mathcal{P}$. Assume now that $\Pi \in \mathcal{P}$ and that we have shown the claim for all $\Pi' \geq \Pi$. Then

$$\{g \in P_0 \mid \Pi \subset g^{-1}P_0 \cap B(e,s)\} = \bigcup_{\Pi\leq\Pi'} \{g \in P_0 \mid \Pi' = g^{-1}P_0 \cap B(e,s)\}$$

and

$$\{P \in \mathcal{T} \mid \Pi \subset P\} = \bigcup_{\Pi\leq\Pi'} \{P \in \mathcal{T} \mid \Pi' = P \cap B(e,s)\}.$$

Note that both of these unions are disjoint unions. We have

$$f(P_0, \Pi, (G_t)_{t>0}) = \lim_{t\to\infty} \frac{\#\{g \in P_0 \mid \Pi = g^{-1}P_0 \cap B(e,s)\}}{m_G(G_t)}$$
$$= \lim_{t\to\infty} \frac{\#\{g \in P_0 \mid \Pi \subset g^{-1}P_0 \cap B(e,s)\} \setminus \bigcup_{\Pi<\Pi'}\{g \in P_0 \mid \Pi' = g^{-1}P_0 \cap B(e,s)\}}{m_G(G_t)}$$
$$= \nu(\{P \in \mathcal{T} \mid \Pi \subset P\}) - \sum_{\Pi<\Pi'} \nu(\{P \in \mathcal{T} \mid \Pi' = P \cap B(e,s)\})$$
$$= \nu(\{P \in \mathcal{T} \mid \Pi = P \cap B(e,s)\}). \quad \square$$

Lemma 5.4.12. *Let* $\Pi \subset G$ *be a realized* s*-pattern and let* $x \in P_0$ *with* $\Pi = x^{-1}P_0 \cap B(e,s)$*. Then*

$$\nu(\{P \in \mathcal{T} \mid \Pi = P \cap B(e,s)\}) = m_H(W_{x,s}).$$

Proof. Recall that $A_{x,s} \subset W^{\mathrm{ns}}$. We first show $\mathrm{par}(i(A_{x,s})) \subset \{P \in \mathcal{T} \mid \Pi = P \cap B(e,s)\}$.

For $\tau(y) \in A_{x,s}$ we have

$$\mathrm{par}(i(\tau(y))) = \mathrm{par}((e,\tau(y))\Gamma) = \mathrm{par}((e,\tau(y))(y,\tau(y))^{-1}\Gamma) = \mathrm{par}((y,e)\Gamma) = y^{-1}P_0$$

and as $\tau(y) \in A_{x,s}$ we have $y^{-1}P_0 \cap B(e,s) = x^{-1}P_0 \cap B(e,s) = \Pi$. Thus $\mathrm{par}(i(\tau(y))) \in \{P \in \mathcal{T} \mid \Pi = P \cap B(e,s)\}$.
We now note that $A_{x,s}$ is dense in $W_{x,s}$ and $W_{x,s} \subset W^{\mathrm{ns}}$, as $W_{x,s}$ does not contain any right $\pi_H(\Gamma)$-translates of the boundary ∂W. If $h \in W_{x,s}$ then we can choose a sequence $(y_k)_{k\geq 1}$ in P_0 with $\tau(y_k) \in A_{x,s}$ for all $k \geq 1$ and $\tau(y_k) \to h$ as $k \to \infty$.
Then, as $\mathcal{T}$ is compact, the sequence $(\mathrm{par}(i(\tau(y_k))))_{k\geq 1}$ has a convergent subsequence $(\mathrm{par}(i(\tau(y_{k_l}))))_{l\geq 1}$ with limit $P \in \mathcal{T}$. As $\{Q \in \mathcal{T} \mid Q \cap B(e,s) = \Pi\} \subset \mathcal{T}$ is closed, we have $P \cap B(e,s) = \Pi$.
Now $\beta(\mathrm{par}(i(\tau(y_{k_l})))) = i(\tau(y_{k_l})) \to i(h)$ by continuity of i. Thus, as β has closed graph we see that $\beta(P) = i(h)$ and thus, as $i(h) \in Y^{\mathrm{ns}}$, we have $P = \mathrm{par}(i(h))$. Hence $\mathrm{par}(i(W_{x,s})) \subset \{Q \in \mathcal{T} \mid Q \cap B(e,s) = \Pi\}$.
Now choose some $x_\Pi \in P_0$ for each realized s-pattern Π of P_0. Then $W^{\mathrm{ns}} = \bigcup_{\Pi\in\mathcal{P}} W_{x_\Pi,s}$, where the union is disjoint, and thus $\nu(\{Q \in \mathcal{T} \mid Q \cap B(e,s) = \Pi\}) = m_H(W_{x_\Pi,s})$. □

Proof of Theorem 5.4.9. The theorem follows directly from the three previous lemmas. □

5.5. A Euclidean example

Using the results of the previous section we will now demonstrate how one can get a complete understanding of the density behaviour of a specific family of Euclidean regular model sets. We will use a very modest amount of algebraic number theory to obtain the lattice Γ with which this family is build.
We specialize to the situation where $G = \mathbb{R}^n$, $H = \mathbb{R}^k$ and let $W := [-1,1]^k$. For $\Gamma \leq \mathbb{R}^n \times \mathbb{R}^k$ a lattice we define the set of Γ-*regular scalings* by

$$S_\Gamma := \{t \in \mathbb{R} \mid t > 0,\ t\partial W \cap \pi_H(\Gamma) = \emptyset\}.$$

For each $t \in S_\Gamma$ we get a regular model set $P(t) := P(\mathbb{R}^n, \mathbb{R}^k, \Gamma, tW)$. We define a function $r : S_\Gamma \to \mathbb{R}$ by

$$r(t) := \max\{r > 0 \mid P(t) \in \mathrm{UD}_{2r}(\mathbb{R}^n)\} = \frac{1}{2}\min\{\|x-y\|_2 \mid x \neq y \in P(t)\}.$$

As $P(t) \subset P(s)$ for $t \leq s$ in S_Γ, we see that r is monotonically decreasing.
We note that the intensity of $\Lambda^{P(t)}$ is given by $\frac{m_H(tW)}{|\Gamma|}$ and thus

$$D_{r(t)}(\Lambda^{P(t)}) = m_G(B(0,r(t)))\frac{m_H(tW)}{|\Gamma|} \leq 1.$$

Hence $C_n r(t)^n \leq \frac{|\Gamma|}{(2t)^k}$, where $C_n := m_G(B(0,1))$. By rearranging we obtain the following estimate:

Lemma 5.5.1.

$$r(t) \leq \left(\frac{|\Gamma|}{C_n(2t)^k}\right)^{\frac{1}{n}} =: \rho(t).$$

We will use the slab and acceptance domains to study the behaviour of r. For $t \in S_\Gamma$ we define the following slight modification of the slab:

$$S(t,r) := \{\gamma \in \Gamma \mid \pi_H(\gamma) \in t(W^\circ - W^\circ),\ \pi_G(\gamma) \in \overline{B}(e,r)\}.$$

Lemma 5.5.2. *For all $t \in S_\Gamma$ we have*

$$r(t) = \frac{1}{2}\min\{\|\pi_G(\gamma)\|_2 \mid \gamma \in S(t, 2\rho(t))\}.$$

Proof. Assume that $\gamma \in S(t,r)$. Then there are $h, h' \in tW^\circ$ such that $h + \pi_H(\gamma) = h'$. By the density of $\pi_H(\Gamma)$ in H we can find a $\gamma' \in \Gamma$ such that $\pi_H(\gamma') \in tW^\circ$ and $\pi_H(\gamma') + \pi_H(\gamma) \in tW^\circ$ by choosing $\gamma' \in \Gamma$ such that $\pi_H(\gamma')$ is close to h. Thus $\pi_H(\gamma') \in tW^\circ$ and $\pi_H(\gamma') + \pi_H(\gamma) \in tW^\circ$. Hence $\pi_G(\gamma'), \pi_G(\gamma') + \pi_G(\gamma) \in P(t)$ and thus $r(t) \leq \frac{1}{2}\|\pi_G(\gamma')\|_2$.
We can find $x, y \in P(t)$ with $2r(t) = \|x - y\|_2 \leq 2\rho(t)$. Thus $(x, \tau(x)), (y, \tau(y)) \in \Gamma \cap G \times tW^\circ$ and $(x - y, \tau(x) - \tau(y)) \in \overline{B}(e, 2\rho(t)) \times t(W^\circ - W^\circ)$. This implies $r(t) \geq \frac{1}{2}\min\{\|\pi_G(\gamma)\| \mid \gamma \in S(t, 2\rho(t))\}$. □

We will now specialize the situation further and assume that L is a totally real number field with embeddings $\sigma_1, \dots, \sigma_d$ in $\mathbb{R}$, with $d = n + k$. Let $\mathcal{O}_L$ denote the ring of algebraic integers of L and assume that $\Gamma := \{(\sigma_1(\xi), \dots, \sigma_d(\xi)) \mid \xi \in \mathcal{O}_L\}$. Then Γ is a lattice in $\mathbb{R}^d$ with covolume $|\Gamma| = |\triangle_L|$, the absolute value of the discriminant of L.
Let $N(\xi) = \sigma_1(\xi) \cdot \dots \cdot \sigma_d(\xi)$ be the (number field) norm of $\xi \in \mathcal{O}_L$ and note that $|N(\xi)|$ is always a natural number. Set $\xi_i := \sigma_i(\xi)$ for all i. Then as in [17, Example 5.2], we obtain for $\|(\xi_1, \dots, \xi_n)\|_2 \leq s$ from the AM-GM inequality

$$\begin{aligned}
\|(\xi_{n+1}, \dots, \xi_d)\|_2 &\geq \|(\xi_{n+1}, \dots, \xi_d)\|_\infty \geq \frac{1}{k}\|(\xi_{n+1}, \dots, \xi_d)\|_1 \geq |\xi_{n+1} \cdot \dots \cdot \xi_d|^{\frac{1}{k}} \\
&= |N(\xi)|^{\frac{1}{k}} |\xi_1|^{-\frac{1}{k}} \cdot \dots \cdot |\xi_n|^{-\frac{1}{k}} \geq |N(\xi)|^{\frac{1}{k}} \|(\xi_1, \dots, \xi_n)\|_\infty^{-\frac{n}{k}} \\
&\geq \|(\xi_1, \dots, \xi_n)\|_2^{-\frac{n}{k}} \geq s^{-\frac{n}{k}}.
\end{aligned}$$

If we now pick $\gamma, \gamma' \in \Gamma \cap G \times tW$ with $2r(t) = \|\pi_G(\gamma) - \pi_G(\gamma')\|$, then $\|\pi_H(\gamma) - \pi_H(\gamma')\| \geq r(t)^{-\frac{n}{k}}$ and $\|\pi_H(\gamma) - \pi_H(\gamma')\| \geq \operatorname{diam}(W) = 2t$. Thus we have shown the following lemma.

Lemma 5.5.3. *For all $t \in S_\Gamma$ we have $r(t) \geq \left(\frac{1}{2t}\right)^{\frac{k}{n}}$.*

Theorem 5.5.4. *Assume that $k = n$, i.e. that $L \subset \mathbb{R}$ has degree $d = 2n$. Assume further that there is a unit $\varepsilon \in \mathcal{O}_L$ with $\varepsilon > 1$ such that*

(i) $|\sigma_i(\varepsilon)| = \varepsilon$ for all $1 \leq i \leq n$,

(ii) $|\sigma_i(\varepsilon)| = \varepsilon^{-1}$ for all $n + 1 \leq i \leq d$.

Then $r(\varepsilon t) = \frac{1}{\varepsilon} r(t)$ for all $t \in S_\Gamma$.

Proof. Note that $r(t) \leq \rho(t) = C\frac{1}{t}$ for some constant $C > 0$. Define $j_1 : K \to \mathbb{R}^n,\ \xi \mapsto (\sigma_1(\xi), \ldots, \sigma_n(\xi))$ and $j_2 : K \to \mathbb{R}^n,\ \xi \mapsto (\sigma_{n+1}(\xi), \ldots, \sigma_d(\xi))$ such that $\Gamma = \{(j_1(\xi), j_2(\xi)) \mid \xi \in \mathcal{O}_L\}$. We also set $j : \mathcal{O}_L \to \mathbb{R}^d,\ \xi \mapsto (j_1(\xi), j_2(\xi))$ and note that Γ with coordinate-wise addition and multiplication is a ring and j is a ring isomorphism from $\mathcal{O}_L$ to Γ. Now

$$\begin{aligned}
r(\varepsilon t) &= \frac{1}{2}\min\{\|\pi_G(\gamma)\|_2 \mid \gamma \in S(\varepsilon t, \rho(\varepsilon t))\} \\
&= \frac{1}{2}\min\{\|\pi_G(\gamma)\|_2 \mid \gamma \in \Gamma,\ \gamma \in \overline{B}(0, \rho(\varepsilon t)) \times [-\varepsilon t, \varepsilon t]^n\} \\
&= \frac{1}{2}\min\{\|\pi_G(\gamma)\|_2 \mid \gamma \in \Gamma,\ \gamma \in \frac{1}{\varepsilon}\overline{B}(0, \rho(t)) \times \varepsilon[-t, t]^n\} \\
&= \frac{1}{2}\min\{\|\pi_G(\gamma)\|_2 \mid \gamma \in \Gamma,\ j(\varepsilon)\gamma \in \varepsilon\frac{1}{\varepsilon}\overline{B}(0, \rho(t)) \times \frac{1}{\varepsilon}\varepsilon[-t, t]^n\} \\
&= \frac{1}{2}\min\{\|\pi_G(j(\varepsilon^{-1})\gamma)\|_2 \mid \gamma \in \Gamma,\ \gamma \in \overline{B}(0, \rho(t)) \times [-t, t]^n\} \\
&= \frac{1}{2}\min\{\varepsilon^{-1}\|\pi_G(\gamma)\|_2 \gamma \in S(t, \rho(t))\} = \frac{1}{\varepsilon}r(t).
\end{aligned}$$

□

Example 5.5.5. If $L = \mathbb{Q}(\sqrt{p})$, with p a prime, the assumptions of the theorem above are fulfilled. Lattices coming from this construction in conjunction with intervals as windows are very commonly used in the Euclidean theory of model sets to test hypothesis. As is evident, these types of model sets are very special.

Corollary 5.5.6. *In the setting of the theorem above we have for every $t \in S_\Gamma$ that*

$$D_{r(\varepsilon t)}(P(\varepsilon t)) = C_n r(t)^n \frac{m_H([-\varepsilon t, \varepsilon t]^n)}{|\Gamma|} = C_n \frac{r(t)^n}{\varepsilon^n} \frac{\varepsilon^n m_H([-t, t]^n)}{|\Gamma|} = D_{r(t)}(P(t)).$$

If we define $D(t) : S_\Gamma \to \mathbb{R}, t \mapsto D_{r(t)}(P(t))$, then $D \circ \exp : \log(S_\Gamma) \to \mathbb{R}$ is periodic.

Moreover, by calculating all relevant sets $S(s, \rho(s))$, $t \leq s \leq \varepsilon t$, we can determine the complete behaviour of $s \mapsto r(s)$. Notice that this can be done by determining all lattice points in $S(\varepsilon t, \rho(t))$, i.e. finitely many lattice points, as $S(s, \rho(s)) \subset S(\varepsilon t, \rho(t))$ for all $t \leq s \leq \varepsilon t$. Thus we can obtain a complete understanding of the densities of the family $(P(t))_{t \in S_\Gamma}$ of regular model sets with a finite amount of calculations.

A. Basic CAT(0) geometry

In this appendix we summarize the basic CAT(0) geometry we need in this thesis. See [30, Chapter II.1] for comprehensive coverage of everything we explain here.
A *geodesic segment* in a metric space (Y, d_Y) joining $x, y \in Y$ is the image of a path of length $d_Y(x, y)$ in Y starting at x and ending at y. Such a path $\gamma : [a, b] \to X$ will be called a *geodesic path* from x to y, if is is an isometric embedding. If any two points in Y can be joined by a geodesic segment, we call Y a *geodesic metric space*. A *geodesic triangle* in Y consists of three points $x, y, z \in Y$ and choices $[x, y]$, $[y, z]$, $[x, z]$ of geodesic segments joining x and y, y and z, and z and x. It will be denoted by $\triangle(x, y, z)$. For any such geodesic triangle there is a Euclidean triangle with corners $\overline{x}, \overline{y}, \overline{z} \in \mathbb{R}^2$ such that

$$d_{\mathbb{E}^2}(\overline{x}, \overline{y}) = d_Y(x, y) \quad \text{and} \quad d_{\mathbb{E}^2}(\overline{y}, \overline{z}) = d_Y(y, z) \quad \text{and} \quad d_{\mathbb{E}^2}(\overline{x}, \overline{z}) = d_Y(x, z).$$

This Euclidean triangle is called a *comparison triangle* for $\triangle(x, y, z)$.

Definition A.0.1. A geodesic metric space (Y, d_Y) is called a CAT(0)-space, if for any geodesic triangle $\triangle(x, y, z)$ and any $p \in [x, y]$, $q \in [x, z]$ and $\overline{p} \in [\overline{x}, \overline{y}]$, $\overline{q} \in [\overline{x}, \overline{z}]$ such that $d_Y(x, p) = d_{\mathbb{E}^2}(\overline{x}, \overline{p})$ and $d_Y(x, q) = d_{\mathbb{E}^2}(\overline{x}, \overline{q})$ we have

$$d_Y(p, q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q}) \tag{A.0.1}$$

Inequality (A.0.1) is called the *CAT(0)-inequality.*

Proposition A.0.2.

(i) CAT(0)-spaces are uniquely geodesic, i.e. any two points are joined by a unique geodesic path.

(ii) Symmetric spaces of noncompact type are CAT(0)-spaces.

Let γ_y, γ_z be geodesic paths from x to y and x to z. Let $\overline{\angle}_x(\gamma_1(t), \gamma_2(t'))$ denote the angle at $\overline{x}$ in the Euclidean comparison triangle with corners $\overline{x}, \overline{\gamma_1(t)}, \overline{\gamma_2(t)}$ to $\triangle(x, \gamma_y(t), \gamma_z(t'))$. The *Alexandrov angle* between γ_1 and γ_2 at x is defined as

$$\angle(\gamma_1, \gamma_2) = \lim_{\varepsilon \to 0} \sup_{0 < t, t' < \varepsilon} \overline{\angle}_x(\gamma_1(t), \gamma_2(t')).$$

As each geodesic triangle $\triangle(x, y, z)$ comes with an implicit choice of geodesic paths we can define the Alexandrov angle $\angle_x(y, z)$ between $[x, y]$ and $[x, z]$ at x. Now the following characterization of CAT(0)-spaces is available, see [30, Chapter II.1 Proposition 1.7]:

Proposition A.0.3. *A geodesic metric space (Y, d_Y) is a CAT(0)-space if and only if for any geodesic triangle $\triangle(x, y, z)$ in Y we have*

$$\angle_x(y, z) \leq \angle_{\overline{x}}(\overline{y}, \overline{z}),$$

where $\angle_{\overline{x}}(\overline{y}, \overline{z})$ denotes the angle at $\overline{x}$ in the Euclidean comparison triangle to $\triangle(x, y, z)$.

For Riemannian manifolds the Alexandrov angle is of course the ordinary Riemannian angle, cf. [30, Chapter II.1 Corollary 1A.7].

Proposition A.0.4. *Let Y be a smooth Riemannian manifold and let γ_1, γ_2 be geodesics with $\gamma_1(0) = \gamma_2(0)$. Then $\angle(\gamma_1, \gamma_2)$ is the ordinary Riemannian angle between γ_1 and γ_2.*

B. Lattices

B.1. Metrizing quotients

In this appendix we will make use of the following theorem, see [87, Theorem 6.6.1].

Theorem B.1.1. *Let H be a group of isometries of a metric space (Y, d_Y). Then*

$$d_{H\backslash Y}([x],[y]) \coloneqq \operatorname{dist}(Hx, Hy)$$

is a metric on $H\backslash Y$ if and only if each H-orbit in Y is closed.

We will mostly use this theorem in the case that H is a lattice. The case where H is a compact subgroup is also of interest:

Remark B.1.2. Let G be a lcsc group equipped with a left-invariant metric. If K is a compact subgroup, then a version of the theorem above for right-actions implies that G/K is a metric space with a metric given by

$$d_{G/K}(x,y) = \operatorname{dist}(\pi^{-1}(x), \pi^{-1}(y)),$$

with $\pi : G \to G/K$ the quotient map.
If we now start with a proper metric space (Y, d_Y) on which $G \coloneqq \operatorname{Iso}(Y)$ acts transitively such that point stabilizers are compact, then we know that $Y = G/\operatorname{Stab}_G(y)$ as a topological space for any $y \in Y$, where we equip $G = \operatorname{Iso}(Y)$ with the compact-open topology. It seems to be unknown when exactly we can put a left-invariant metric on G such that the metric

$$d(x,y) = \operatorname{dist}(\pi^{-1}(x), \pi^{-1}(y))$$

is equal to d_Y.

B.2. Borel sections and fundamental domains

We will make frequent use of the following selection theorem guaranteeing the existence of Borel sections. It can be found in [102, Corollary A6].

Theorem B.2.1. *Assume that Ω and Ω' are topological spaces, metrizable by a complete separable metric. If $f : \Omega \to \Omega'$ is a Borel measurable map such that for each $\omega \in \Omega'$ the set $f^{-1}(y)$ is a countable union of compact sets, then $f(\Omega)$ is a Borel set and there is a Borel measurable map $f' : f(\Omega) \to \Omega$ such that $f(f'(\omega')) = \omega'$ for all $\omega' \in f(\Omega)$.*

Corollary B.2.2. *Let G be a lcsc group and $K \leq G$ a compact subgroup. Then the quotient map $\pi : G \to G/K$ has a measurable section $\sigma : G/K \to G$.*

In many cases it is possible to write down a Borel section explicitly, for example if G is a semisimple Lie group and has a Cartan decomposition $\exp(\mathfrak{p})K$ or if G is a semidirect product of K and another group. In these cases the section will usually even be continuous. More generally, in the setting of Corollary B.2.2 it was shown by Gleason, [54], that there exist continuous local sections of π (around each point in G/K).

Definition B.2.3. Let Γ be a countable group acting by isometries on a metric space (Y, d_Y). A subset $F \subset Y$ is called a *strict fundamental domain* for the Γ-action if Y decomposes as a disjoint union $Y = \bigsqcup_{\gamma \in \Gamma} \gamma F$.

The following examples shows how the selection theorem can be used to construct strict fundamental domains:

Example B.2.4. Let G be a lcsc group and let $\Gamma < G$ be a discrete subgroup; then Γ is countable, hence the canonical projection $\pi_\Gamma : G \to G/\Gamma$ has countable fibers. By Lemma B.2.1 it thus admits a Borel section σ, and $F := \sigma(G/\Gamma)$ is a strict fundamental domain for the action of Γ on G by right-multiplication. Moreover, Γ is a lattice if and only if $m_G(F) < \infty$ and in this case the unique G-invariant probability measure on G/Γ is given by

$$m_{G/\Gamma} = \frac{(\pi_\Gamma)_* m_G|_F}{m_G(F)}. \tag{B.2.1}$$

Example B.2.5. Let X be a CAT(0) space of the form $X = G/K$ where G is an lcsc group and $K < G$ is compact. If $\Gamma < G$ is a discrete subgroup which contains a torsion element γ, then γ has a fixpoint in X; this implies that the action of Γ on X does not admit a strict fundamental domain. It turns out that, by the following lemma, torsion elements are the only obstruction to the existence of strict fundamental domains.

Lemma B.2.6. *Let G be an lcsc group, $K < G$ a compact subgroup and $\Gamma < G$ a torsion-free discrete subgroup. Then the action of Γ on $X := G/K$ admits a strict fundamental domain. Moreover:*

(i) If $B_{\mathrm{inj}} \subset B_{\mathrm{surj}} \subset X$ are Borel subsets such that the map $\pi_\Gamma : X \to \Gamma\backslash X$ is injective on B_{inj} and surjective on B_{surj}, then there exists a strict fundamental domain F with $B_{\mathrm{inj}} \subset F \subset B_{\mathrm{surj}}$.

(ii) If Γ is a lattice, then $m_X(F) < \infty$ for every strict fundamental domain $F \subset X$.

Proof. Assume first that $B_{\mathrm{inj}} \subset B_{\mathrm{surj}}$ are as in (i), and let $Y := \Gamma\backslash X$ and $Y_0 := \pi_\Gamma(B_{\mathrm{inj}}) \subset Y$. By Lemma B.2.1 the set Y_0 is Borel and, hence $Y_1 := X \setminus X_0$ is Borel as well. By the same lemma there exists a Borel section $\sigma : Y \to B_{\mathrm{surj}}$ of $\pi_\Gamma|_{B_{\mathrm{surj}}}$. We now consider the set $F := B_{\mathrm{inj}} \cup \sigma(Y_1)$, which is a Borel set by Lemma B.2.1. The map π_Γ is injective on B_{inj} and $\sigma(Y_1)$, and we have

$$\pi_\Gamma(B_{\mathrm{inj}}) \cap \pi_\Gamma(\sigma(Y_1)) = Y_0 \cap Y_1 = \emptyset.$$

This implies that $\pi_\Gamma|_F : F \to Y$ is injective. Since Γ is torsion-free, it acts freely on X, hence $F \cap \gamma.F = \emptyset$ for all $\gamma \in \Gamma$. Moreover, $\pi_\Gamma|_F$ is surjective (since $\pi_\Gamma(F) = Y_0 \cup Y_1 = Y$), and hence F is a strict fundamental domain for the Γ-action on X.

This proves (i), and choosing $B_{\text{inj}} := \emptyset$ and $B_{\text{surj}} := X$ we see that a strict fundamental domain F for $\Gamma \curvearrowright X$ exists. Its preimage in G is then a strict fundamental domain for $\Gamma \curvearrowright G$, and hence has finite Haar measure if Γ is a lattice; since K is compact, this implies that $m_X(F) < \infty$. □

B.3. Lattices in semisimple Lie groups

The goal of this section is to establish properties of lattices in isometry groups of symmetric spaces and their finite index subgroups and in particular to establish Theorem B.3.2 below.

Existence of torsion-free subgroups of large girth

We recall the following notions:

Definition B.3.1. Let Γ be a countable group acting by isometries on a metric space (X, d).

(i) Γ is called *residually finite* if there exist subgroups $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \dots$ such that Γ_{n+1} is a finite index normal subgroup of Γ_n and $\bigcap \Gamma_n = \{e\}$.

(ii) The *girth* of the Γ-action on X is defined as

$$\operatorname{girth}(\Gamma \curvearrowright X) := \inf_{x \in X} \inf_{\gamma \in \Gamma \setminus \{e\}} d_X(x, \gamma x).$$

We can now state the main theorem of this appendix, which is a straight forward consequence of theorems of Malcev, Bieberbach, Selberg and Borel–Serre:

Theorem B.3.2. *Let X be either Euclidean space or a Riemannian symmetric space of non-compact type and let $G := \operatorname{Iso}(X)_0$. Then for every lattice $\Gamma < G$ the following hold:*

(i) Γ is countable, finitely-presented and residually finite.

(ii) There exists a finite-index normal subgroup $\Gamma' < \Gamma$ which is torsion-free, and any such Γ' admits a strict fundamental domain $F \subset X$ of finite volume.

(iii) If Γ is uniform, then for every $T > 0$ there exists a finite-index normal subgroup Γ' as in (ii) such that $\operatorname{girth}(\Gamma' \curvearrowright X) > T$.

In view of Lemma B.2.6, the second half of Part (ii) of Theorem B.3.2 follows from the first.

We can state (iii) by saying that uniform lattices in G admit finite-index normal subgroups with strict fundamental domains and large girth. In order to apply this, we will also use the following simple observation:

Lemma B.3.3. *If Γ is torsion-free with girth $\geq 4R$, then the restriction $\pi_\Gamma|_{B(q,R)} : B(q, R) \to \Gamma \backslash X$ is isometric for every $q \in X$. In particular, the injectivity radius of $\Gamma \curvearrowright X$ is uniformly bounded below by R.*

Proof. Let $x, y \in B(q, R)$; then $d_X(x, y) < 2R$. We then have $d_{\Gamma\backslash X}(\Gamma x, \Gamma y) = \inf_{\gamma\in\Gamma} d_X(\gamma x, y)$ by B.1.1. Now for $\gamma \neq e$ we have $d_X(\gamma x, x) \geq \mathrm{girth}(\Gamma \curvearrowright S) \geq 4R$ and hence

$$d_X(\gamma x, y) \geq |d_X(\gamma x, x) - d_X(x, y)| > |4R - 2R| > 2R > d_X(x, y).$$

This implies that $d_{\Gamma\backslash X}(\Gamma x, \Gamma y) = d_X(x, y)$ and hence $\pi_\Gamma|_{B(q,R)}$ is an isometry. □

Theorems of Malcev, Bieberbach, Selberg and Borel–Serre

We now turn to the proof of Theorem B.3.2: Note that Γ is countable since G is second countable and $\Gamma < G$ is a discrete subgroup. Residual finiteness is thus immediate from the following two observations (see [83, 4.8#2] for the former):

Theorem B.3.4 (Malcev)**.** *Every countable linear group is residually finite.*

Proposition B.3.5. *The groups G from Theorem B.3.2 (and hence their subgroups) are linear.*

Proof. This is clear for $\mathrm{Iso}(\mathbb{E}^n)_0 = \mathbb{R}^n \rtimes SO(n)$ and follows from the classification of Riemannian symmetric spaces of non-compact type (combined with the fact that the adjoint group of a connected semisimple Lie group is linear) in the non-Euclidean case. □

The existence of a torsion-free finite index subgroup (and hence a torsion-free finite index *normal* subgroup) is a consequence of Bieberbach's theorem (see [92, Theorem 2.1]) and Selberg's lemma (see [83, Theorem 4.8.2]):

Theorem B.3.6 (Bieberbach)**.** *Let Γ be a lattice in $\mathrm{Iso}(\mathbb{E}^n)_0 = \mathbb{R}^n \rtimes SO(n)$. Then there exists a lattice $\Gamma' < \mathbb{R}^n \cap \Gamma < \mathrm{Iso}(\mathbb{E}^n)_0$ which is normal and of finite index in Γ.*

Lemma B.3.7 (Selberg)**.** *Let X be a symmetric space of non-compact type and let Γ be a lattice in $\mathrm{Iso}(X)_0$. Then there exists a torsion-free finite index normal subgroup $\Gamma' < \Gamma$.*

Note that Bieberbach's theorem immediately implies that in the Euclidean case Γ admits a finitely-presented torsion-free finite index normal subgroup, and hence is finitely-presented itself.

Theorem B.3.8 (Borel–Serre)**.** *If X is a symmetric space of non-compact type and Γ is a lattice in $\mathrm{Iso}(X)_0$, then Γ is of type F_∞, hence in particular finitely presented.*

In the present generality, Theorem B.3.8 was established in [23] using reduction theory. If Γ is uniform or G has Property (T), then much simpler proofs are available. At this point we have established Part (i) and the first half of Part (ii) of Theorem B.3.2.

Large girth

We now turn to the proof of Part (iii) of the Theorem B.3.2. We will need the following consequence of residual finiteness:

Lemma B.3.9. *Let Γ be a countable residually finite group. Then for every finite subset $M \subset \Gamma$ there exists a finite index normal subgroup $\Gamma' < \Gamma$ with $M \cap \Gamma' \subset \{e\}$.*

Proof. Let $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \dots$ with Γ_{n+1} a finite index normal subgroup of Γ_n and $\bigcap \Gamma_n = \{e\}$. There thus exists a map $n : \Gamma \to \mathbb{N}$ such that $\{x\} \cap \Gamma_{n(x)} \subset \{e\}$ for all $x \in \Gamma$; now choose $\Gamma' := \Gamma_n$ with $n := \max\{n(x) \mid x \in M\}$. $\square$

Proof of Theorem B.3.2.(iii). Fix $T > 0$. Γ is cocompact we find a compact subset $L \subset G$ such that $G = \Gamma L$, and we may assume that L contains the basepoint $x_0 = eK \in X$. Since G acts properly on X and since $\Gamma < G$ is discrete, the set

$$M \coloneqq \{\gamma \in \Gamma | d_X(x_0, \gamma.x_0) \leq T - 2\mathrm{diam}(L.o)\}$$

is finite. Since Γ is residually finite by Theorem B.3.2.(i), Lemma B.3.9 allows us to extract a finite index normal subgroup $\Gamma' < \Gamma$ such that $\Gamma' \cap M = \{e\}$. As in the proof of Theorem B.3.2.(ii) we can achieve that Γ' is torsion-free and thus has a strict fundamental domain (of finite measure). We then have

$$d_X(x_0, \gamma'.x_0) > T - 2\mathrm{diam}(L.x_0) \quad \text{for all } \gamma' \in \Gamma' \setminus \{e\}.$$

Since G acts transitively on X and every $g \in G$ can be written as $g = \gamma_0^{-1} l$ for some $\gamma_0 \in \Gamma$ and $l \in L$, we have

$$\begin{aligned} \mathrm{girth}(\Gamma' \curvearrowright S) &= \inf_{g \in G} \inf_{\gamma_1 \in \Gamma' \setminus \{e\}} d_X(g.x_0, \gamma_1 g.x_0) \quad = \quad \inf_{\gamma_0 \in \Gamma} \inf_{l \in L} \inf_{\gamma_1 \in \Gamma' \setminus \{e\}} d_X(\gamma_0^{-1} l.x_0, \gamma_1 \gamma_0^{-1} l.x_0) \\ &= \inf_{\gamma_0 \in \Gamma} \inf_{\gamma_1 \in \Gamma' \setminus \{e\}} \inf_{l \in L} d_X(l.x_0, \gamma_0 \gamma_1 \gamma_0^{-1} l.x_0) \quad = \quad \inf_{\gamma' \in \Gamma' \setminus \{e\}} \inf_{x \in L.o} d_X(x, \gamma'.x) \\ &\geq \inf_{\gamma' \in \Gamma' \setminus \{e\}} d_X(x_=, \gamma'.x_0) + 2\mathrm{diam}(L.x_0) \quad > \quad T, \end{aligned}$$

where we have used that Γ normalizes Γ' $\square$

This concludes the proof of Theorem B.3.2.

C. The spherical Bochner-Schwartz theorem for the Heisenberg group

Recall that $\exp : \mathfrak{h}_n \to \mathcal{H}_n$ is a diffeomorphism and that the Schwartz space $\mathscr{S}(\mathcal{H}_n)$ is defined by

$$\mathscr{S}(\mathcal{H}_n) := \{f \circ \exp^{-1} \mid f \in \mathscr{S}(\mathfrak{h}_n)\}.$$

Recall further that we defined

$$\mathscr{S}(\mathcal{H}_n, U(n)) := \{f \in \mathscr{S}(\mathcal{H}_n) \mid f(t, kv) = f(t, v) \text{ for all } k \in U(n)\}.$$

Our aim in this appendix is a proof of the following theorem.

Theorem C.0.1. *Let T be a positive-definite distribution on $\mathcal{H}_n \rtimes U(n)$. Then there exists a unique Borel measure μ on $PS(\mathcal{H}_n \rtimes U(n), U(n))$ and a continuous functional $\tilde{T}$ on the space $\mathscr{S}(\mathcal{H}_n, U(n))$ with the subspace topology coming from $\mathscr{S}(\mathcal{H}_n)$, such that $\tilde{T}[f] = T[f]$ for all $f \in C_c^\infty(\mathcal{H}_n, U(n)) = C_c^\infty(\mathcal{H}_n) \cap \mathscr{S}(\mathcal{H}_n, U(n))$ and*

$$\tilde{T}[f] = \int \widehat{f} d\mu$$

for all $f \in \mathscr{S}(\mathcal{H}_n, U(n))$. Note that this property uniquely defines $\tilde{T}$ and that the Godement-Plancherel theorem for distributions forces $\mu = \widehat{T}$.

For the proof we will use that the image of $\mathscr{S}(\mathcal{H}_n, U(n))$ under the spherical transform has been characterized by Astengo, Di Blasio and Ricci in [6].

C.1. Schwartz spaces

For $l \in \mathbb{N}$ let $\Sigma \subset \mathbb{R}^l$ be closed and set $I(\Sigma) = \{f \in \mathscr{S}(\mathbb{R}^l) \mid f|_\Sigma = 0\}$. We define $\mathscr{S}(\Sigma) := \mathscr{S}(\mathbb{R}^l)/I(\Sigma)$ and equip $\mathscr{S}(\Sigma)$ with the quotient topology.
Given a seminorm p on a vector space X and a linear subspace M, we define p_M on X/M by

$$p_M([x]) := \inf\{p(x + y) \mid y \in M\}.$$

For $m \in \mathbb{N}_0$ we define the Schwartz seminorms

$$p^k(f) := \sup_{x \in \mathbb{R}^l, |\alpha| \leq k} (1 + \|x\|)^k \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right|.$$

Lemma C.1.1. *The topology on $\mathscr{S}(\Sigma)$ is the topology induced by the family $\{q^k \mid k \in \mathbb{N}_0\}$ of the seminorms $q^k := p^k_{I(\Sigma)}$.*

Proof. This is clear from the definition of quotient topology. □

Remark C.1.2. There is a canonical isomorphism of algebras

$$\mathcal{R} : \mathscr{S}(\Sigma) \to \{f|_\Sigma \mid f \in \mathscr{S}(\mathbb{R}^l)\} =: \mathscr{S}(\mathbb{R}^l)|_\Sigma, \quad [f] \mapsto f|_\Sigma$$

and we will identify these two spaces when convenient (and in particular equip $\mathscr{S}(\mathbb{R}^l)|_\Sigma$ with the topology induced by this isomorphism). Given $f \in \mathscr{S}(\mathbb{R}^l)|_\Sigma$ and $\alpha, \beta \in \mathbb{N}_0^l$ we define

$$\|f\|_k := q^k(\mathcal{R}^{-1}(f))$$

and note that the topology on $\mathscr{S}(\mathbb{R}^l)|_\Sigma$ is induced by the family $\{\|\cdot\|_k\}_{k\in\mathbb{N}_0}$.

C.2. Embeddings of the Gelfand spectrum

From now on set $G := \mathcal{H}_n \rtimes U(n)$ and $K := U(n)$. Before coming to the proof of C.0.1, we give overview over the harmonic analysis on the Heisenberg group, mainly based on [6], covering more ground than we did in Chapter 1.

Let $\mathcal{F}_\lambda$ denote the Fock space consisting of entire functions F on $\mathbb{C}^n$ such that

$$\|F\|^2_{\mathcal{F}_\lambda} := \left(\frac{\lambda}{2\pi}\right)^n \int_{\mathbb{C}^n} |F(z)|^2 e^{-\frac{\lambda}{2}|z|^2} dz < \infty$$

and define the Bargmann representation π_λ of G on $\mathcal{F}_\lambda$ by

$$[\pi_\lambda(t, z)F](w) := e^{i\lambda t} e^{-\frac{\lambda}{2}\langle w,\bar{z}\rangle - \frac{\lambda}{4}|z|^2} F(w + z)$$

and

$$\pi_{-\lambda}(t, z) := \pi_\lambda(-t, z).$$

The space $\mathcal{P}(\mathbb{C}^n)$ of polynomials on $\mathbb{C}^n$ is dense in $\mathcal{F}_\lambda$ and decomposes under the action of K into K-irreducible subspaces

$$\mathcal{P}(\mathbb{C}^n) = \sum_{\alpha\in\Lambda} P_\alpha,$$

with $\Lambda \subset \widehat{K}$. Let $\{v_1^\lambda, \ldots, v^\lambda_{\dim(P_\alpha)}\}$ denote an orthonormal basis of P_α with respect to the Fock scalar product $\langle\cdot,\cdot\rangle_{\mathcal{F}_\lambda}$ on $\mathcal{F}_\lambda$ obtained from the Fock norm $\|\cdot\|_{\mathcal{F}_\lambda}$ by polarization and set

$$\phi_{\lambda,\alpha}(t, z) := \frac{1}{\dim(P_\alpha)} \sum_{j=1}^{\dim(P_\alpha)} \langle \pi_\lambda(t, z)v_j^\lambda, v_j^\lambda\rangle_{\mathcal{F}_\lambda}.$$

We also set

$$\eta_{Kw}(t,z) := \int_K \exp(i\mathrm{Re}(\langle z, kw\rangle))dm_K(k).$$

Then

$$PS(G,K) = \{\eta_{Kw} \mid w \in \mathbb{C}^n\} \cup \{\phi_{\lambda,\alpha} \mid \lambda \in \mathbb{R}^*, \alpha \in \Lambda\}.$$

We note that η_{Kw} only depends on the length $\tau = \|w\|$. By enumerating Λ one can obtain the parametrization of the $U(n)$-spherical functions on the Heisenberg group in Theorem 1.4.10.
Let $\mathbb{D}(G/K)$ denote the algebra of G-invariant differential operators on G/K. Recall from [64] that the spherical functions of the Gelfand pair (G,K) are exactly those functions ω such that ω_K is an eigenfunction of every element of $\mathbb{D}(G/K)$. If $\omega \in PS(G,K)$ and $V \in \mathbb{D}(G/K)$, we denote the eigenvalue of ω_K with respect to V by $\widehat{V}(\omega)$. A differential operator $D \in \mathbb{D}(G/K)$ is called *homogeneous of degree* $m \in \mathbb{C}$, if

$$D(f \circ D_r) = r^m D(f) \circ D_r$$

for all $f \in C^\infty(G/K)$ and $r > 0$.

Theorem C.2.1 (Astengo–Di Blasio–Ricci, [6]). *There are differential operators $V_1, \dots, V_s \in \mathbb{D}(G/K)$ such that*

$$\{V_0 := -i\partial_t, V_1, \dots, V_s\}$$

generate $\mathbb{D}(G/K)$ and

(i) each V_j is homogeneous of degree $2m_j$, with $m_j \in \mathbb{N}$ (and $m_0 = 1$),

(ii) each V_j is formally self-adjoint and $\widehat{V}_j(\phi_{1,\alpha}) \in \mathbb{N}$ for each $\alpha \in \Lambda$,

(iii) $\widehat{V}_j(\eta_{Kw}) = \rho_j(w, \overline{w})$ for every $w \in \mathbb{C}^n$, where ρ_j is a non-negative homogeneous polynomial of degree $2m_j$, non-zero away from the origin,

Each element ϕ of $\mathrm{PS}(G,K)$ is smooth and a common eigenfunction of $V_1, \dots, V_s$ with real eigenvalues. Moreover each $\phi \in PS(G,K)$ is uniquely determined by its eigenvalues $\widehat{V}_0(\phi), \dots, \widehat{V}_s(\phi)$ with respect to $V_0, \dots, V_s$.

By [6] the map

$$\widehat{V} : BS(G,K) \to \mathbb{R}^{s+1},\ \phi \mapsto (\widehat{V}_0(\phi), \dots, \widehat{V}_s(\phi))$$

is well-defined and a homeomorphism onto its image

$$\Sigma_n := \widehat{V}(\mathrm{BS}(G,K)) = \widehat{V}(\mathrm{PS}(G,K)).$$

Σ_n is a closed subset of $\mathbb{R}^{s+1}$ and the Gelfand transform $f \mapsto \widehat{f}$ defines a map

$$\mathcal{G} : \mathcal{S}(\mathcal{H}_n, U(n)) \to \mathrm{Map}(\Sigma_n, \mathbb{C}),\ f \mapsto \widehat{f} \circ \widehat{V}^{-1}.$$

Theorem C.2.2 (Astengo–Di Blasio–Ricci, [6])**.** *The map*

$$\mathcal{G} : \mathscr{S}(\mathcal{H}_n, U(n)) \to \mathscr{S}(\mathbb{R}^{s+1})|_{\Sigma_n}, \quad f \mapsto \mathcal{G}(f)$$

is a topological isomorphism. More specifically, for each $p \in \mathbb{N}_0$ *there exists a* $F_p \in \mathscr{S}(\mathbb{R}^{s+1})$ *and* $q \in \mathbb{N}$*, both depending on* p *such that* $F_p|_{\Sigma_n} = \widehat{f}$ *and* $\|F_p|_{\Sigma_n}\|_p \leq C_p \|f\|_q$ *for* $C_p > 0$.

Remark C.2.3. Note that Theorem C.2.1 and the explicit formula for η_{Kw} implies that

$$\widehat{V}(\eta_{Kw} \circ D_r) = \widehat{V}(\eta_{Krw}) = (0, r^{2m_j}\widehat{V}_1(\eta_{Kw}), \dots, r^{2m_s}\widehat{V}_s(\eta_{Kw})).$$

As

$$\phi_{\lambda,\alpha} = \phi_{1,\alpha} \circ D_{\sqrt{\lambda}}$$

for $\lambda > 0$ and

$$\phi_{\lambda,\alpha} = \overline{\phi_{1,\alpha}} \circ D_{\sqrt{|\lambda|}}$$

for $\lambda < 0$, we see that

$$\widehat{V}(\phi_{\lambda,\alpha}) = (|\lambda|\widehat{V}_0(\phi_{1,\alpha}), |\lambda|^{m_1}\widehat{V}_1(\phi_{1,\alpha}, \dots, |\lambda|^{m_s}\widehat{V}_s(\phi_{1,\alpha})))$$

for all $\lambda \neq 0$ and $\alpha \in \Lambda$. Thus

$$\widehat{V}(\phi \circ D_r) = (r^2\widehat{V}_0(\phi), r^{2m_1}\widehat{V}_1(\phi), \dots, r^{2m_s}\widehat{V}_s(\phi))$$

for any $\phi \in PS(G, K)$ and $r > 0$.

C.3. Proof of the spherical Bochner-Schwartz theorem

Lemma C.3.1. *Let* T *be a positive definite bi-*$U(n)$*-invariant distribution on* $\mathcal{H}_n \rtimes U(n)$ *and let* μ *denote the Godement-Plancherel measure of* T*. Then there exists a positive polynomial* Q *on* $\mathbb{R}^{s+1}$ *such that*

$$\int_{\mathrm{PS}(G,K)} \frac{1}{Q \circ \widehat{V}} \, d\mu < \infty$$

Proof. Let ϕ be a smooth function on $\mathcal{H}_n \rtimes U(n)$ with $B_{1/2} \subset \operatorname{supp}(\phi) \subset B_{1/2+\delta}$ (where the balls are with respect to the Cygan-Koranyi metric) and set $\chi = \phi * \phi^*$. Then $B_1 \subset \operatorname{supp}(\chi)$. Let h denote the homogeneous dimension of $\mathcal{H}_n$ and for $\varepsilon \leq 1$ set $\chi_\varepsilon = \left(\frac{1}{\varepsilon}\right)^h \chi \circ D_{1/\varepsilon}$. It follows that there is some $N \in \mathbb{N}$ and $C > 0$, $C' > 0$ such that

$$\begin{aligned} T(\chi_\varepsilon) &\leq C\|\chi_\varepsilon\|_N \\ &= C \sup\{|\partial^\alpha \chi_\varepsilon(t, z)| \mid (t, z) \in \mathcal{H}_n, \alpha \in \mathbb{N}_0^{2n+1}, |\alpha| \leq N\} \\ &\leq C'\varepsilon^{-h-2N}\|\chi\|_N, \end{aligned}$$

where the family $(\|\cdot\|_N)_{N\in\mathbb{N}}$ of seminorms defined by

$$\|f\|_N := \sup\{|\partial^\alpha f(x)| \mid x \in \mathcal{H}_n, \alpha \in \mathbb{N}_0^{2n+1}, |\alpha| \leq N\}$$

induces the topology on $C_c^\infty(\mathcal{H}_n)$.

We also have for any $R > 0$ that

$$\begin{aligned}
T(\chi_\varepsilon) &= \int_{PS(G,K)} \widehat{\chi}_\varepsilon d\mu = \left(\frac{1}{\varepsilon}\right)^h \int_{PS(G,K)} \int_G \chi \circ D_{1/\varepsilon}(g)\omega(g^{-1})\, dm_G(g)\, d\mu(\omega) \\
&= \left(\frac{1}{\varepsilon}\right)^h \int_{PS(G,K)} \int_G \left(\frac{1}{\varepsilon}\right)^{-h} \chi(g)\omega(D_\varepsilon(g))\, dm_G(g)\, d\mu(\omega) \\
&= \int_{PS(G,K)} \int_G \chi(g)\omega(D_\varepsilon(g))\, dm_G(g)\, d\mu(\omega) \\
&= \int_{PS(G,K)} \widehat{\chi}(\omega \circ D_\varepsilon) d\mu(\omega) \\
&= \int \mathcal{G}(\chi)(\widehat{V}(\omega \circ D_\varepsilon))\, d\mu(\omega) \\
&= \int_{PS(G,K)} \mathcal{G}(\chi)(\varepsilon^2 \widehat{V}_0(\phi), \varepsilon^{2m_1}\widehat{V}_1(\omega), \ldots, \varepsilon^{2m_s}\widehat{V}_s(\omega))\, d\mu(\omega) \\
&= \int_{\Sigma_n} \mathcal{G}(\chi)(\varepsilon^2 x_0, \varepsilon^{2m_1}x_1, \ldots, \varepsilon^{2m_s}x_s)\, d\widehat{V}_*\mu(x_0, \ldots, x_s) \\
&\geq \int_{\left\{x_0^2+\ldots x_s^2 \leq R^2\frac{1}{\varepsilon^2}\right\}} \mathcal{G}(\chi)(\varepsilon^2 x_0, \varepsilon^{2m_1}x_1, \ldots, \varepsilon^{2m_s}x_s)\, d\widehat{V}_*\mu(x_0, \ldots, x_s) \\
&\geq \mu\left(B\left(0, \frac{R}{\varepsilon}\right)\right) \inf\{G(\chi)(\varepsilon^2 x_0, \varepsilon^{2m_1}x_1, \ldots, \varepsilon^{2m_s}x_s) \mid x_0^2 + x_1^2 + \ldots x_s^2 \leq R^2\varepsilon^{-2}\} \\
&\geq \mu\left(B\left(0, \frac{R}{\varepsilon}\right)\right) \inf\{G(\chi)(x_0, x_1, \ldots, x_s) \mid x_0^2 + x_1^2 + \ldots x_s^2 \leq R^2\} \\
&= K\mu\left(B\left(0, \frac{R}{\varepsilon}\right)\right),
\end{aligned}$$

with

$$K := \inf\{\mathcal{G}(\chi)(x) \mid x \in B(0,R)\}.$$

Since

$$\mathcal{G}(\chi) = |\mathcal{G}(\phi)|^2 \geq 0$$

and

$$\mathcal{G}(\chi)(0) = \widehat{\chi}(\mathbf{1}) > 0$$

we can choose $R > 0$, independently from ε, such that K is positive (as $\mathcal{G}(\chi)$ is continuous). Substituting $\frac{r}{R} = (1/\varepsilon)$ and $h = 2n + 2$ we get

$$\mu(B(0,r)) \leq \frac{C'}{K}\|\chi\|_N \left(\frac{r}{R}\right)^{(2n+2+2N)} = Lr^{2(n+1+N)}$$

for some $L > 0$, independent of r. This implies the claim. $\square$

Proof of the Bochner-Schwartz theorem. Consider the map

$$\tilde{T} : \mathscr{S}(\mathcal{H}_n, U(n)) \to \mathbb{C}, \quad f \mapsto \int \widehat{f} d\mu.$$

By Lemma C.3.1 and Theorem C.2.2, this map is well-defined. More precisely, as for any Schwartz function F, any positive polynomial P and any subset $A \subset \mathbb{R}^k$ there is a constant $C > 0$ with

$$\sup_{x \in A} |F(x)P(x)| \leq C,$$

we have

$$|F(x)| \leq \frac{C}{P(x)}.$$

This implies that the integral $\int \widehat{f} d\mu$ is well-defined for any $f \in \mathscr{S}(\mathcal{H}_n, U(n))$, as μ has polynomial growth. Choosing a Schwartz function F_0 and q as in Theorem C.2.2 restricting to $\mathcal{G}(f)$, we see that

$$\begin{aligned} |\tilde{T} f| &\leq \int |\widehat{f}| d\mu = \int |\mathcal{G}(f)| d\widehat{V}_* \mu = \int |F_0| d\widehat{V}_* \mu \\ &\leq \int \frac{\|F_0 Q\|_0}{Q} d\widehat{V}_* \mu \leq \int \frac{1}{Q} d\widehat{V}_* \mu \cdot \|Q\|_0 \cdot C_0 \|f\|_q \end{aligned}$$

and thus we see that $\tilde{T}$ induces a well-defined (continuous) functional on $\mathscr{S}(\mathcal{H}_n, U(n))$ satisfying

$$\tilde{T} f = \int \widehat{f} d\mu$$

for all $f \in \mathscr{S}(\mathcal{H}_n, U(n))$. Moreover

$$\tilde{T}(f^* * f) = T(f^* * f)$$

for all $f \in C_c^\infty(\mathcal{H}_n, U(n))$ by the Godement-Plancherel theorem. As these functions span a dense subset of $C_c^\infty(\mathcal{H}_n, U(n))$ (with respect to the topology on $C_c^\infty(\mathcal{H}_n)$), given a function $g \in C_c^\infty(\mathcal{H}_n, U(n))$, we can choose functions $g_k \in C_c^\infty(\mathcal{H}_n, U(n))$ such that $g_k \to g$ in $C_c^\infty(\mathcal{H}_n)$ and $\tilde{T}(g_k) = T(g_k)$ for all k (as the inclusion $C_c^\infty(\mathcal{H}_n, U(n)) \to \mathscr{S}(\mathcal{H}_n, U(n))$ is continuous). Again by the continuity of the inclusion $C_c^\infty(\mathcal{H}_n) \to \mathscr{S}(\mathcal{H}_n)$, we see that $g_k \to g$ in $\mathscr{S}(\mathcal{H}_n)$ and thus $\tilde{T} g = T g$. Now the density of $C_c^\infty(\mathcal{H}_n, U(n))$ in $\mathscr{S}(\mathcal{H}_n, U(n))$ implies the uniqueness of the extension $\tilde{T}$. $\square$

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